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Dedication

To:

My parents

My siblings

All my family

My teachers

My friends

I dedicate this modest work

Acknowledgments

First and foremost, i praise and thank Allah, the Almighty, for giving me strength and patience to finish this humble work.

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Last but not least, i place on record my sense of thankfulness to one and all who directly or indirectly have lent their hand in this venture.

Abstract

This dissertation has embarked on a comprehensive exploration of various equations, delving deeply into the analysis and application of stochastic differential and difference equations. Our investigation covered a spectrum of equations, focusing particularly on how noise can act as a stabilizing factor. The study of stochastic equations in general needs a full knowledge of probability theory and stochastic calculus including stochastic processes, stochastic integration and stochastic differentiation. After studying stochastic calculus, we briefly discuss an unsolved problem of stabilization by noise for difference equations.

Keywords: Stochastic process; Brownian motion; stochastic differential equation; stochastic difference equation; stochastic perturbations; stability; stabilization by noise.

Résumé

Ce mémoire a entamé une exploration complète de diverses équations, se plongeant profondément dans l'analyse et l'application des équations différentielles et des équations aux différences stochastiques. Notre enquête a couvert un spectre d'équations, en se concentrant particulièrement sur la manière dont le bruit peut agir comme facteur stabilisant. L'étude des équations stochastiques en général nécessite une connaissance complète de la théorie des probabilités et du calcul stochastique, y compris les processus stochastiques, l'intégration stochastique et la différenciation stochastique. Après avoir étudié le calcul stochastique, nous discutons brièvement d'un problème non résolu de stabilisation par bruit pour les équations aux différences.

Mots clés: Processus stochastique; mouvement Brownien; équation différentielle stochastique; équation de différence stochastique; perturbations stochastiques; stabilité; stabilisation par bruit.

ملخص

شرعت هذه الأطروحة في استكشاف شامل لمختلف المعادلات، والتعمق في تحليل وتطبيق المعادلات التفاضلية والفرقية العشوائية. غطى بحثنا مجموعة من المعادلات، مع التركيز بشكل خاص على كيفية عمل الضوضاء كعامل استقرار. تحتاج دراسة المعادلات العشوائية بشكل عام إلى معرفة كاملة بنظرية الاحتمالات وحساب التفاضل والتكامل العشوائي بما في ذلك العمليات العشوائية والتكامل العشوائي والتمايز العشوائي. بعد دراسة حساب التفاضل والتكامل العشوائي، نناقش بإيجاز مشكلة لم يتم حلها وهي االاستقرار عن طريق الضوضاء للمعادلات الفرقية.

الكلمات المفتاحية: العملية العشوائية؛ الحركة البراونية, المعادلة التفاضلية العشوائية, المعادلة الفرقية العشوائية, الاضطرابات العشوائية, الاستقرار, الاستقرار عن طريق الضوضاء.

Contents

Notation

ϕ	The empty set
$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integers
$\mathbb{R}$	The set of all real numbers
ω	Event, outcome of a random experiment
Ω	The set of outcomes
A	A subset in Ω
A^c	The complement of A in Ω
Σ	Sigma-field
$\mathcal{B}$	The Borel sigma-field
$\mathcal{B}(E)$	The Borel sigma-field generated by E
$\mathbb{P}$	Probability measure
$\mathbb{P}(A)$	Probability of the event A
$\mathbb{P}(A \cap B)$	Probability of A and B
$\mathbb{P}(B\|A)$	The conditional probability of B given A
(Ω, Σ)	Measurable space
$(\Omega, \Sigma, \mathbb{P})$	Probability space
X or $X(\omega)$	Random variable
$\mathbb{P}_X$	Law of the random variable X
$f(x)$	Probability density
$\mathbb{E}[X]$	Expectation of X

$\mathbb{E}(X\|\Sigma)$	Conditional expectation of the randam variable X given Σ
$Var(X)$	Variance of X
$Cov(X,Y)$	Covariance of X and Y
$X \sim N(\mu, \sigma^2)$	X is normally distributed with mean μ and variance σ^2
X_t or $X(t,\omega)$	Stochastic process
B_t or $B(t,\omega)$	Brownian motion
T	Index set of the Stochastic process
$\{\Sigma_k\}_{k=0}^{n}$	Filtration
$(\Omega, \Sigma, (\Sigma_t), \mathbb{P})$	Filtered probability space
$:=$	The equal by definition sign
$i.e.$	Abbreviation for id est (= Latin for "that is")
iid	Independent and identically distributed
Π	Partition of the interval $[0,T]$
$V_{\Pi}(B)[0,T]$	The first variation of B_t over $[0,T]$ with respect to Π
$\frac{\partial f}{\partial x}, f_x$	Partial derivative
f'	Derivative
f''	Second derivative
SDE	Stochastic Differential Equation
$SDEs$	Stochastic Differential Equations

Introduction

Every mathematical theory has its roots in real life. However, when it comes to randomness, Stochastic Theory is the one which can do a more accurate modeling of the problem. "Stochastic calculus" is a specific area within stochastic theory, focusing on mathematical techniques and formulas for analyzing processes that undergo random changes. The history of this calculus began in 1942 when Kiyosi Itô published the paper on stochastic processes in the Japanese Journal of Mathematics. Itô began to reconstruct from scratch the concept of stochastic processes and its associated theory of analysis [8, 12]. He was; thus, laying down the groundwork for understanding both stochastic differential and difference equations, which are central to understanding systems influenced by randomness. Both play crucial roles in various fields, including physics, finance, and biology, allowing for the modeling of phenomena where uncertainty is inherent. While stochastic differential and difference equations aim to model dynamical systems under uncertainty, their domains of application can differ due to their foundational time-structuring. SDEs are more suitable for continuous-time models, like the motion of particles or financial models reflecting continuously changing market conditions. Stochastic difference equations are advantageous for scenarios where data is naturally sampled in discrete intervals, such as daily stock prices or population studies with yearly census data [20, 34]. An important element in the study of such systems is their stability. Stability essentially concerns the responsiveness of a system to small perturbations: will the system return to its equilibrium, or will minor changes lead to divergent behavior?

Analyzing the stability of an SDE involves understanding how the system reacts over time, considering both the deterministic part and the randomness. The Lyapunov method, for example, is a common approach to examine stability in SDEs. In 1975, Shaikhet introduced a parametric family of Lyapunov functionals, so that an infinite number of Lyapunov functionals were used simultaneously. This way allowed to get different stability conditions for considered equation using only one Lyapunov functional. At last, in the 1990s the general method of Lyapunov functionals construction was proposed by Kolmanovskii and Shaikhet for stochastic differential equations and developed later consistently for stochastic difference equations [34].

In the other hand, when dealing with systems in real life, whether those are biological, ecological, financial, or mechanical, one might encounter what are deemed to be unstable or unpredictable. These systems are prone to changes that can lead to erratic behavior, which can be difficult to manage or predict. The technique of stabilization by noise comes into play as a sophisticated method wielded by scientists and engineers

to bring order to these erratic systems. In other words, the multiplicative noise now plays the role to stabilize unstable deterministic systems [6]. It involves the intentional introduction of a certain type of randomness or 'noise' to steer these systems towards stability. Surprisingly, under the right conditions, this controlled infusion of noise can render an unstable system more predictable and manageable. This counterintuitive phenomenon has become a pivotal element in the study of stochastic differential equations. However, as far as we know, there existed no papers devoting themselves to the investigation of the analogous problems in a discerte frame work, that is, stabilization by noise for stochastic difference equations.
This dissertation consists of three chapters. The main prerequisite is probability theory: probability measures, random variables, expectation, independence and conditional probability. The only other prerequisite is calculus. This covers limits, series, the notion of continuity, differentiation and the Riemann–Stieltjes integral. Familiarity with differential and difference equations would be a bonus.
A brief introduction to probability is presented in the first chapter. This is mainly to fix terminology and notation. However, conditional expectation is treated in the second chapter, including propositions designed to develop the necessary skills and intuition.

The second chapter contains six sections, in the first one, we come into the first contact with the concept of stochastic processes. Much emphasis is put on four important examples, the Markov chain, the Brownian motion, the Poisson process and Martingales. The second two sections are devoted to both stochastic integration and differentiation. These are carefully introduced and explained with several illustrative examples. The fourth and fifth sections are about stochastic differential and difference equations. In the final section, we review shortly the notion of stochastic stability.

In the third chapter, we discuss an unsolved problem of stabilization by noise for difference equations. Numerical examples are provided to demonstrate the validity of hypothesis about a possibility of stabilization by noise for these equations.

I Basic Concepts from Probability Theory

The fundamental concepts of probability theory have roots in measure theory. Like any branch of mathematics, probability theory has its own terminology and its own tools [24]. We start this chapter by defining some of this terms and reviewing basic ideas from probability and measure-theoretic probability.

I.1 Probability Space

The modern theory of probability stems from the work of A. N. Kolmogorov published in 1933. Kolmogorov associates a random experiment with probability space, which is a triplet $(\Omega, \Sigma, \mathbb{P})$, consisting of the set of outcomes Ω, a sigma-field Σ with Boolean algebra properties and a probability measure $\mathbb{P}$. In this section, each of these elements will be discussed in more detail [5].

Definition I.1.1 *[10] Let Ω be a nonempty set, Σ a collection of subsets of Ω and $\mathbb{P} : \Sigma \to \mathbb{R}$ a function. The triple $(\Omega, \Sigma, \mathbb{P})$ is called a probability space, if the following conditions hold:*

1. $\Omega \in \Sigma$;
2. *If sets* $A_1, A_2, \ldots \in \Sigma$, *then* $\bigcup_{i=1}^{\infty} A_i \in \Sigma$;
3. *If* $A, B \in \Sigma$, *then* $A \setminus B \in \Sigma$;
4. *For any* $A \in \Sigma$, $\mathbb{P}(A) \geq 0$;
5. *If* $A_1, A_2, \ldots \in \Sigma$ *are pairwise disjoint, then*
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(A_i\right);$$
6. $\mathbb{P}(\Omega) = 1$.

Remarks I.1.1 *[7, 10, 25, 28]*

- *The elements ω of Ω are called elementary events, while the elements A of Σ are called events.*
- *The event $\Omega \setminus A$ is called the opposite or complementary event to A.*
- *The real number $\mathbb{P}(A)$ is said to be the probability of A.*
- *A collection (set, family) Σ satisfying Conditions $1, 2$ and 3 is called a sigma-field.*
- *The above definition implies that the empty set belongs to a sigma-field $\emptyset \in \Sigma$ and that Σ is closed with respect to countably infinite intersections, i.e.*

$$If \;\; A_1, A_2, A_3 ... \in \Sigma, then \;\; \bigcap_{i=1}^{\infty} A_i \in \Sigma.$$

- *Clearly, any family A of subsets of Ω can be extended to a sigma-field. Among all these extensions, consider all the sigma-fields that contain A and take their intersection, which we call $\mathcal{A}$. $\mathcal{A}$ is indeed a sigma-field. It is the smallest sigma-field containing A, it is also called the sigma-field generated by A.*
- *We say that a sigma-field $\mathcal{A}$ is a sub-sigma-field of Σ if $A \in \Sigma$ for every $A \in \mathcal{A}$.*
- *A function $\mathbb{P}$ defined on a sigma-field and satisfying Conditions $4, 5$ and 6 is called a normalised measure or probability measure.*
- *Instead of saying that a set A belongs to sigma-field Σ we often say that A is Σ-measurable. The pair (Ω, Σ) is called a measurable space.*
- *A function $\varphi : \Omega \to \mathbb{R}$ is called Σ-measurable if $\varphi^{-1}(U) \in \Sigma$, for all open sets $U \in \mathbb{R}$ (or, equivalently, for all Borel sets $U \subset \mathbb{R}$).*

Example I.1.1 *[5] In the case of flipping a coin, the probability space has the following elements: $\Omega = \{H, T\}, \Sigma = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$ and $\mathbb{P}$ defined by $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\{H\}) = \frac{1}{2}, \mathbb{P}(\{T\}) = \frac{1}{2}, \mathbb{P}(\{H, T\}) = 1$.*

Example I.1.2 *[37] (**Borel sigma-fields**) $\mathcal{B}(E)$, the Borel sigma-field on E, is the sigma-field generated by the family of open subsets of E. With slight abuse of notation*

$$\mathcal{B}(E) := \sigma(open\ sets).$$

The sigma-field $\mathcal{B} = \mathcal{B}(\mathbb{R})$ is the most important of all sigma-fields. Every subset of $\mathbb{R}$ which you meet in everyday use is an element of $\mathcal{B}$.

I.2 Conditional Probability

Conditional probabilities correspond to a modified probability model that reflects partial information about the outcome of an experiment. The modified model has a smaller sample space than the original model [38].

Definition I.2.1 *[10] Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $B \in \Sigma$ such that $\mathbb{P}(B) > 0$. Then for each $A \in \Sigma$ we define*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

and call it the conditional probability of A given B.

We think of this as the probability of A if we know that B has occurred. Hence, to compute a conditional probability means to compute a probability given additional information.

Example I.2.1 *[29] Roll a die and observe the number. Let*

$$A = \{odd\ outcome\} \quad and \quad B = \{at\ least\ 4\}$$

What is $\mathbb{P}(A|B)$?
We solve this in two different ways: (1) by using the definition and (2) by intuitive reasoning. Since $\mathbb{P}(A \cap B) = \mathbb{P}(5) = 1/6$ and $\mathbb{P}(B) = 1/2$, the definition gives

$$\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) = (1/6)/(1/2) = 1/3.$$

If we think about this intuitively, to condition on the event B means that we get the additional information that the outcome is at least 4. Since one of these three outcomes is also odd and outcomes are equally likely, the conditional probability of odd is 1/3.

I.3 Random Variables

Definition I.3.1 *[5] Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A random variable X is a function that assigns a numerical value to each state of the world, $X : \Omega \longrightarrow \mathbb{R}$, such that the values taken by X are known to someone who has access to Σ. Another way of saying this is that X is a Σ-measurable function.*

Definition I.3.2 *[7] The law of the random variable X is the probability measure $\mathbb{P}_X$ on the measurable space $(E, \mathcal{B}(E))$ defined by*

$$\mathbb{P}_X(I) = \mathbb{P}(X^{-1}(I)), \quad \forall\, I \in \mathcal{B}(E).$$

Example I.3.1 *Suppose that our experiment consists of tossing two fair coins, so the sample space is the set* $\Omega = \{(T,T),(T,H),(H,T),(H,H)\}$. *Letting* X *denote the number of heads appearing, then* X *is a random variable taking on one of the values* $0,1,2$ *with respective probabilities*

$$\begin{aligned}\mathbb{P}_X(\{0\}) &= \mathbb{P}(X^{-1}(0)) = \mathbb{P}\{(T,T)\} = 1/4,\\ \mathbb{P}_X(\{1\}) &= \mathbb{P}(X^{-1}(1)) = \mathbb{P}\{(T,H),(H,T)\} = 2/4,\\ \mathbb{P}_X(\{2\}) &= \mathbb{P}(X^{-1}(2)) = \mathbb{P}\{(H,H)\} = 1/4.\end{aligned}$$

Of course, $\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)=1$.

I.4 Types of Random Variables

Random variables are classified into discrete and continuous variables. The main difference between the two categories is the type of possible values that each variable can take. In addition, the type of random variable implies the particular method of finding a probability distribution function.

I.4.1 Discrete Random Variables

Definition I.4.1 *[38]* X *is a discrete random variable if the range of* X *is a countable set*

$$S_X = \{x_1, x_2, \ldots\}.$$

Definition I.4.2 *[38] The probability mass function of the discrete random variable* X *is*

$$\mathbb{P}_X(x) = \mathbb{P}[X = x].$$

Example I.4.1 *[32]* (***Poisson Random Variable***) *A random variable* X*, taking on one of the values* $0,1,2,\ldots$ *is said to be a Poisson random variable with parameter* λ*, if for some* $\lambda > 0$,

$$\mathbb{P}(i) = \mathbb{P}(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0,1,\ldots \tag{I.1}$$

Equation I.1 defines a probability mass function since

$$\sum_{i=0}^{\infty}\mathbb{P}(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1.$$

I.4.2 Continuous Random Variables

Definition I.4.3 *[38] A random variable* X *is continuous if the range* S_x *consists of one or more intervals. For each* $x \in S_x, \mathbb{P}[X = x] = 0$.

Definition I.4.4 *[38] The cumulative distribution function of a random variable X is*

$$\mathbb{F}_X(x) = \mathbb{P}\left[X \leq x\right].$$

Example I.4.2 *[32](**Normal Random Variable**) We say that X is a normal random variable (or simply that X is normally distributed) with parameters μ and σ^2 if the density of X is given by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \quad -\infty < x < \infty.$$

Another way of saying that X is normally distributed, is to write

$$X \sim N(\mu, \sigma^2).$$

I.5 Expectation

Suppose that X is a discrete random variable, where $X(\omega) \in \{x_1, x_2, ...\}$ for $\omega \in \Omega$. Let $\mathbb{P}(x)$ be the probability mass function of X. The expectation of X is defined as

$$\mu = \mathbb{E}\left[X\right] = \sum_i x_i \mathbb{P}(x_i).$$

With the fundamental notion of expectation one can define the **variance** as the mean of the square of the difference $X - \mathbb{E}[X]$ i.e. $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$ and the **standard deviation** $\sigma = \sigma(X) = \sqrt{Var(X)}$ for a random variable X. One can also look at the **covariance** $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ of two random variables X and Y [1, 22].
Before considering expectations of continuous random variables, the following example is given.

Example I.5.1 *Consider the mean and variance for the Poisson distributed variable of Example I.4.1. Clearly*

$$\mathbb{E}\left[X\right] = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda$$

and

$$\mathbb{E}\left[X^2\right] = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \lambda^2 \sum_{k=1}^{\infty} \left(\frac{(k-1)\lambda^{k-2}}{(k-1)!} + \frac{\lambda^{k-2}}{(k-1)!} \right) = \lambda^2 + \lambda.$$

Thus, $\mathbb{E}\left[X\right] = Var(X) = \lambda$.

Expectations are defined for continuous random variables in an analogous way as for discrete random variables. Suppose that X is a continuous random variable where

$X(x) = x$ and with probability density $f(x)$. The expectation of X is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

It is useful to note that for constants a and b, $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$[1].

Example I.5.2 *Consider the mean and variance for the normal distributed variable of Example I.4.2. Clearly*

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} = \mu$$

and

$$Var(X) = \mathbb{E}[(X - \mu)^2] = \sigma^2.$$

Thus, $\mathbb{E}[X] = \mu$ *and* $Var(X) = \sigma^2$.

I.6 Independence

The concept of independence is one of the most fundamental in probability theory and statistics. It is this concept that distinguishes probability theory from measure theory. Let us begin from a pair of events. We can agree to say that events A and B are independent if the conditional probability $\mathbb{P}(B|A)$ does not depend on A and $\mathbb{P}(A|B)$ does not depend on B. More precisely, if for any two open intervals $E, F \subset \mathbb{R}$, the events $A = \{\omega,\ X(\omega) \in E\}$ and $B = \{\omega,\ Y(\omega) \in F\}$ are independent, i.e.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B),$$

then X and Y are called independent random variables. This formula makes sense whether or not $P(A)$ and $P(B)$ vanish and its generalisation will be used in the following formal definition of independence[5, 10].

Definition I.6.1 *[10] Let* $(\Omega, \Sigma, \mathbb{P})$ *be a probability space and let events* $A_1, ..., A_n \in \Sigma$. *The events* $A_1, ..., A_n \in \Sigma$ *are independent, if*

$$\mathbb{P}(A_{k_1} \cap ... \cap A_{k_r}) = \mathbb{P}(A_{k_1})\mathbb{P}(A_{k_r}),$$

for any integers $1 \leq k_1 < ... < k_r \leq n$, $1 < r \leq n$.

Example I.6.1 *[29] A card is chosen at random from a deck of cards. Consider the events*

$$A = \{\textit{the card is an ace}\} \quad \textit{and} \quad B = \{\textit{the card is a heart}\}.$$

Are A and B independent?
Let us first solve this by using the definition. We have

$$\mathbb{P}(A) = \frac{4}{52}, \quad \mathbb{P}(B) = \frac{1}{4}, \quad \textit{and} \quad \mathbb{P}(A \cap B) = \frac{1}{52}.$$

And hence

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

so that A and B are independent. Intuitively, the events give no information about each other. The probability of drawing an ace is $\frac{4}{52}$ *and if we are given the information that the chosen card is a heart, the probability of an ace is still* $\frac{4}{52}$*. The proportion of aces is the same in the deck as within the suit of hearts.*

I.7 Convergence of Random Variables

Let $\{X_n\}_{n=1}^{\infty}$ (or simply $\{X_n\}$ for short) denote a sequence of random variables defined on the probability space $(\Omega, \Sigma, \mathbb{P})$. An important question that emerges in the area of probability theory is "Does there exist a limiting random variable X to which the sequence approaches as $n \to \infty$?" Our objective in this section is to characterize the manner in which $(X_n) \to X$ as $n \to \infty$ since, as we shall now see, there are various ways to express convergence of X_n to a limiting random variable X [30].

Definition I.7.1 *[3]*

i) X_n converges to X almost surely (a.s.) if

$$\mathbb{P}\left(\omega; \lim X_n(\omega) = X(\omega)\right) = 1.$$

Denote this by $X_n \xrightarrow{a.s.} X$ or $X_n \to X$ a.s., and we shall write as $-\lim_{n\to\infty} X_n = X$.

ii) X_n converges to X in the mean square if

$$\mathbb{E}[(X_n - X)^2] \to 0.$$

Denote this by $X_n \xrightarrow{2} X$, and we shall write $ms - \lim_{n\to\infty} X_n = X$.

iii) X_n converges in probability to X if for every $\epsilon > 0$,

$$\mathbb{P}\left(|X_n - X| > \epsilon\right) \to 0 \text{ as } n \to \infty.$$

Denote this by $X_n \xrightarrow{\mathbb{P}} X$, and we shall write $\mathbb{P} - \lim_{n\to\infty} X_n = X$.

The mean square convergence is useful when defining the Ito integral. It is worth noting that both almost sure convergence and convergence in mean square imply the convergence in probability [5].

Example I.7.1 *(Convergence in Probability) Imagine you are conducting a series of temperature measurements using different thermometers. Each thermometer n has its own level of*

measurement error X_n, reflecting how far its reading potentially deviates from the true temperature. We assume that as n increases, the thermometers get progressively more accurate, meaning the potential error in the measurement decreases. The sequence of random variables X_n represents the sequence of potential errors in temperature measurements made by each thermometer. Convergence in probability towards $X = 0$ signifies that the likelihood of X_n (the measurement error of the nth thermometer) deviating from 0 by more than any small positive number ϵ becomes increasingly small as n (the thermometer accuracy) increases.
Formally, this means that for any $\epsilon > 0$, no matter how small:

$$P(|X_n - 0| > \epsilon) \to 0 \text{ as } n \to \infty.$$

II Stochastic Calculus

Even if deterministic Calculus is an excellent tool for modeling real life problems, however, when it comes to randomness, Stochastic Calculus is the one which can do a more accurate modeling of the problem. Stochastic calculus can be loosely described as a field of mathematics, that is concerned with infinitesimal calculus on nondifferentiable functions. The need for this calculus comes from the necessity to include unpredictable factors into modelling. This is where probability comes in and the result is a calculus for random functions or stochastic processes [5, 21].

II.1 Stochastic Processes and Their Properties

A stochastic process is a family of random variables $\{X(t), t \in T\}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ and indexed by a parameter t where t varies over a set T. If the set T is discrete, the stochastic process is called discrete. If the set T is continuous, the stochastic process is called continuous. The parameter t usually plays the role of time and the random variables can be discrete-valued or continuous-valued at each value of t. For example, a continuous stochastic process can be discrete-valued [1, 12].

Remark II.1.1 *[18] The stochastic process $X(t, \omega)$ can be expressed as as $X(t)(\omega)$ or simply as $X(t)$ or X_t.*

II.1.1 Discrete Stochastic Processes

In this section, discrete stochastic processes are considered. For these processes, let $T = \{t_0, t_1, t_2, ...\}$ be a set of discrete times. Let the sequence of random variables $X(t_0), X(t_1), X(t_2), ...$ each be defined on the sample space Ω. This sequence may describe, for example, the evolution of a physical, biological, or financial system over the discrete times $t_0, t_1, t_2, ...$[1]. Throughout this section, the discrete stochastic processes that are discussed are Markov chains and discrete martingales.

Markov Chains

The importance of Markov chains comes from two facts: (i) there are a large number of physical, biological, economic, and social phenomena that can be modeled in this way, and (ii) there is a well-developed theory that allows us to do computations [13]. We begin with a famous example, then describe the property that is the defining feature of Markov chains.

Example II.1.1 *[32]* ***(A Random Walk Model)****: A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, ...$ is said to be a random walk if, for some number $0 < p < 1$,*

$$\mathbb{P}_{i,i+1} = p = 1 - \mathbb{P}_{i,i-1}, \quad i = 0, \pm 1, ...$$

The preceding Markov chain is called a random walk for we may think of it as being a model for an individual walking on a straight line who at each point of time either takes one step to the right with probability p or one step to the left with probability $1 - p$.

Definition II.1.1 *[4] Suppose that S is a finite or a countable set. Suppose also that a probability space $(\Omega, \Sigma, \mathbb{P})$ is given. A sequence of random variables X_n, $n \in \mathbb{N}$, is called a Markov chain on S if for all $n \in \mathbb{N}$ and all $s \in S$*

$$\mathbb{P}(X_{n+1} = s | X_0, ..., X_n) = \mathbb{P}(X_{n+1} = s | X_n).$$

The set S is the state space of the Markov chain.

Martingales

A martingale is a stochastic process that generalizes the notion of a fair game. Assume that after n plays of a gambling game your winnings are X. Then, by fair, we mean that your expected future winnings should be X regardless of past history [12]. First of all, let us introduce some basic datum, definitions and properties.
We now take a filtered space $(\Omega, \Sigma, \{\Sigma_n\}, \mathbb{P})$. Here,

- $(\Omega, \Sigma, \mathbb{P})$ is a probability space as usual,
- $\{\Sigma_n : n \geq 0\}$ is a **filtration**, that is, an increasing family of sub-sigma-fields of Σ:

$$\Sigma_0 \subseteq \Sigma_1 \subseteq ... \subseteq \Sigma.$$

The information about ω in Ω aviable to us at time n consists precisely of the values of $X(\omega)$ for all Σ_n measurable functions X [37].

Definition II.1.2 *[4] We say that a sequence of random variables $X_1, X_2, ...$ is adapted to a filtration $\Sigma_1, \Sigma_2, ...$ if X_n is Σ_n-measurable for each $n = 1, 2, ...$*

Definition II.1.3 *[2] Suppose that X is an integrable random variable on $(\Omega, \Sigma, \mathbb{P})$ and that $\mathcal{G}$ is a sigma-field in Σ. There exists a random variable $\mathbb{E}(X|\mathcal{G})$, called the conditional expected value of X given $\mathcal{G}$, having these properties:*

(i) $\mathbb{E}(X|\mathcal{G})$ *is* $\mathcal{G}$*-measurable,*

(ii) $\mathbb{E}(X|\mathcal{G}) < \infty$,

(iii) $\mathbb{E}(X|\mathcal{G})$ *satisfies*

$$\int_G \mathbb{E}(X|\mathcal{G})d\mathbb{P} = \int_G Xd\mathbb{P}, \quad G \in \mathcal{G}.$$

All the equalities and inequalities in the following theorem hold with probability 1.

Theorem II.1.1 *[2, 4] Suppose that X and Y are integrable and $\mathcal{G}$ is a sigma-field on Ω contained in Σ.*

(i) *If $X = a$ with probability 1, then $\mathbb{E}(X|\mathcal{G}) = a$.*

(ii) *if $X \geq 0$, then $\mathbb{E}(X|\mathcal{G}) \geq 0$.*

(iii) *For constants a and b, $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$.*

(iv) *If $X \leq Y$ with probability 1, then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$.*

(v) $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X||\mathcal{G})$.

(vi) *$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$ if X is $\mathcal{G}$-measurable (we assume that the product XY is integrable).*

Definition II.1.4 *[37] A process X is called a* ***martingale*** *(relative to $(\{\Sigma_n\}, \mathbb{P})$) if*

(i) *X is adapted,*

(ii) $\mathbb{E}[|X_n|] < \infty, \ \forall n$,

(iii) $\mathbb{E}[X_n|\Sigma_{n-1}] = X_{n-1}, \ (n \geq 1)$.

Basically, it says that if we have observed the process for $(n-1)$ steps, we at least know that the process in the next step on average will not deviate from the last value X_{n-1} [29].

Example II.1.2 *[12](**Symmetric Random Walk**) The simple symmetric random walk is a martingale. In order to show that, a stochastic process is built as follows. Let $X_1, X_2, \ldots$ be a sequence of iid random variables with*

$$X_k = \begin{cases} +1, & \text{with probablity } 1/2 \\ -1, & \text{with probablity } 1/2 \end{cases}$$

for $k \geq 1$. Set $S_n = X_1 + ... + X_n$, for $n \geq 1$, with $S_0 = 0$. Then, S_n is the random walk's position after n steps. Assuming that all variables are integrable, and that the filtration we are working with is indeed the natural filtration, i.e. $\Sigma_n = \sigma(S_0, X_1, \ldots, X_n)$, $n \geq 1$. Then, we just have to show that

$$\mathbb{E}[S_{n+1}|\Sigma_n] = \mathbb{E}[S_n + X_{n+1}|\Sigma_n] = \mathbb{E}[S_n|\Sigma_n] + \mathbb{E}[X_{n+1}|\Sigma_n] = S_n + \mathbb{E}[X_{n+1}] = S_n.$$

II.1.2 Continuous Stochastic Processes

Now consider a continuous stochastic process $\{X(t), t \in T\}$ defined on the probability space $(\Omega, \Sigma, \mathbb{P})$ where T is an interval in time and the process is defined at all instants of time in the interval. A continuous-time stochastic process is a function $X : T \times \Omega \to \mathbb{R}$ of two variables t and ω and X may be discrete-valued or continuous-valued. In particular, $X(t) = X(t, \cdot)$ is a random variable for each value of $t \in T$ and $X(\cdot, \omega)$ maps the interval T into $\mathbb{R}$ and is called a sample path, a realization, or a trajectory of the stochastic process for each $\omega \in \Omega$. It is interesting that specific knowledge of ω is generally unnecessary but ω is significant as each $\omega \in \Omega$ results in a different trajectory [1]. In dealing with continuous-time stochastic processes, we need two building blocks. One is the continuous-time equivalent of the normal distribution known as Brownian motion or, equivalently, as the Wiener process, and the other one is a Poisson-distributed random process known as Poisson process [17]. Throughout this section, these two continuous stochastic processes are discussed.

Brownian Motion

The Brownian motion process, sometimes called the Wiener process, is one of the most useful stochastic processes in applied probability theory. This phenomenon, named after the English botanist Robert Brown who discovered it, is the motion exhibited by a small particle which is totally immersed in a liquid or gas. Since then, the process has been used beneficially in such areas as statistical testing of goodness of fit, analyzing the price levels on the stock market, and quantum mechanics [32].

Definition II.1.5 *[2, 18] A stochastic process $B(t, \omega)$ is called a Brownian motion or a Wiener process if it satisfies the following conditions:*

1. $\mathbb{P}\{\omega; B(0, \omega) = 0\} = 1$.

2. *For any $0 \le s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, i.e., for any $a < b$,*

$$\mathbb{P}\{a \le B(t) - B(s) \le b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{\frac{-x^2}{2(t-s)}} dx.$$

3. *$B(t, \omega)$ has independent increments, i.e., for any $0 \le t_1 < t_2 < ... < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$, are independent.*

4. *Almost all sample paths of $B(t, \omega)$ are continuous functions, i.e.,*

$$\mathbb{P}\{\omega; B(\cdot, \omega) \text{ is continuous}\} = 1.$$

Remark II.1.2 *[33] A consequence of the independent increments property is that a Brownian motion has no memory.*

See Figure II.1 for a visualation of Brownian sample paths.

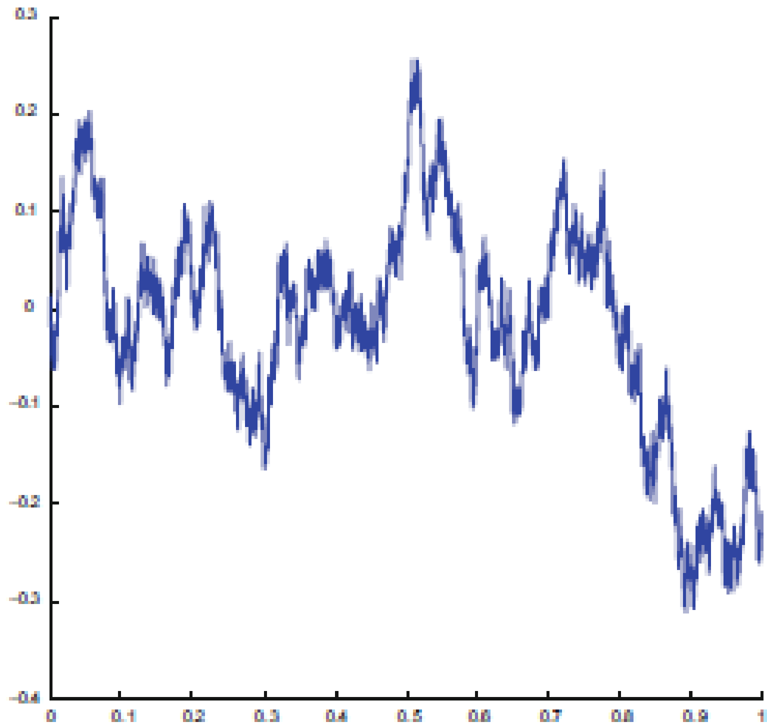

Figure II.1 Simulation of a Brownian sample path on the time interval [0, 1]

Example II.1.3 *[16] When a single drop of ink is placed into a container of still water, the ink will gradually spread out and diffuse through the water, creating a beautiful, swirling pattern that eventually becomes uniform in color. Even without any stirring or external force, the ink molecules move from an area of high concentration (near the drop) to areas of lower concentration until they are evenly distributed throughout the water. This spreading occurs due to the random motion of the water and ink molecules akin to Brownian motion. Each collision between the molecules of ink and water causes the ink to spread out further. While we might not see the individual water molecules moving, their impact on the ink molecules is observed as the spreading pattern of the ink. The randomness inherent in the molecular motion is similar to the principles observed in Brownian motion.*

Poisson Process

The poisson process is largely used to model counts over time and thus might seem to be discrete, it is fundamentally a continuous-time process. It's widely used as a modelling tool in numerous applied probability problems. It not only models many real-world phenomena, but the process allows for tractable mathematical analysis as well [36].

Definition II.1.6 *[5, 21] A Poisson process $N(t)$ is a stochastic process with the following properties.*

1. *The process starts at the origin, $N(0) = 0$.*
2. *(Independence of increments) $N(t) - N(s)$ is independent of the past, that is, of Σ_s, the sigma-field generated by $N(u), u \leq s$.*

3. *(Poisson increments)* $N(t) - N(s), t > s$, *has a Poisson distribution with parameter* $\lambda(t-s)$, *i.e.*

$$\mathbb{P}(N(t) - N(s) = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}.$$

4. *(Step function paths) The paths* $N(t), t \geq 0$, *are increasing functions of t changing only by jumps of of size* 1.

Example II.1.4 *[36] (**A taxi problem**) Group taxis are waiting for passengers at the central railway station. Passengers for those taxis arrive according to a Poisson process with an average of* 20 *passengers per hour. A taxi departs as soon as four passengers have been collected or ten minutes have expired since the first passenger got in the taxi. Suppose you get in the taxi as first passenger. What is the probability that you have to wait ten minutes until the departure of the taxi?*
To answer this questions, we take the minute as time unit so that the arrival rate $\lambda = 1/3$

$$\mathbb{P}\left\{ \textit{less than } 3 \textit{ passengers arrive in } (0, 10) \right\} = \sum_{k=0}^{2} e^{-10/3} \frac{(10/3)^k}{k!} = 0.3528.$$

II.2 Stochastic Integration

Let $B(t, \omega)$ be a Brownian motion. In this section, we will study the very first stochastic integral $\int_a^b f(t, \omega) dB(t, \omega)$ defined by K. Itô in his 1944 paper [19]. The integrand $f(t, \omega)$ is a nonanticipating stochastic process with respect to the filtration Σ_t where $\Sigma_t = \sigma\{B(s); s \leq t\}$ and $\int_a^b \mathbb{E}\left(|f(t)|^2\right) dt < \infty$. The term "nonanticipating" used by Itô is nowadays commonly called "adapted," which we have defined in Definition II.1.2. Before introducing the important notion of Itô stochastic integral. It is worth noting that the major reason for the failure of classical integration methods in such integrals is that the Brownian motion has infinite simple variation ($i.e.$, $V_t(B) = \infty$). Its paths are continuous but nowhere differentiable [18, 23]. A further indication of the irregularity of Brownian sample paths is given by the following facts.

II.2.1 Increments of Brownian Motions

In this subsection, we shall discuss a few basic properties of the increments of a Brownian motion, including its first and quadratic variation which will be useful when computing stochastic integrals.

Proposition II.2.1 *[26] Brownian sample paths do not have bounded variation on any finite interval* $[0, T]$. *This means that*

$$V_\Pi(B)[0, T] = \lim_{|\Pi| \to 0} \sum_{k=1}^{n} |(B(t_k) - B(t_{k-1}))| = \infty,$$

where the sum is taken over all possible partitions $\Pi : 0 = t_0 < ...t_n = T$ *of* $[0, T]$.

Remark II.2.1 *[26] The increment* $\Delta_i B = B(t_i) - B(t_{i-1})$ *on the interval* $[t_{i-1}, t_i]$ *satisfies* $\mathbb{E}[B(t_i) - B(t_{i-1})] = 0$ *and* $\mathbb{E}[(B(t_i) - B(t_{i-1}))^2] = \Delta_i = t_i - t_{i-1}$. *These properties suggest that* $(\Delta_i B)^2$ *is of the order* Δ_i. *In terms of of differentials, we write*

$$(dB(t))^2 = (B(t + dt) - B(t))^2 = dt, \tag{II.1}$$

and in terms of integrals,

$$\int_0^t (dB(s))^2 = \int_0^t ds = t. \tag{II.2}$$

The right-hand side is the quadratic variation of Brownian motion on $[0, t]$.

II.2.2 Itô Stochastic Integral

The Itô integral is defined in a way that is similar to the Riemann integral. The Itô integral is taken with respect to infinitesimal increments of a Brownian motion, $dB(t)$, which are random variables, while the Riemann integral considers integration with respect to the predictable infinitesimal changes dt [5].

Definition II.2.1 *[23] Let* $\{X(t), 0 \leq t \leq T\}$ *be a stochastic process adapted to the filtration generated by the Brownian motion and such that* $\int_0^T \mathbb{E}\left((X(s))^2\right) ds < +\infty$. *The stochastic integral of X is defined as*

$$I(X) = \int_0^t X(s) dB(s) = \lim_{||\Pi_n|| \to 0} \sum_{i=0}^{n-1} X(t_i)(B(t_{i+1}) - B(t_i)),$$

where the convergence is in the quadratic mean and $t_i \in \Pi_n$.

Example II.2.1 *[18] Let* $X(t) = B(t)$. *We shall integrate the process* $B(t)$ *between* 0 *and* T. *Considering an equidistant partition, we take* $t_i = \dfrac{iT}{n}$, $i = 0, 1, ..., n-1$.
The partial sums are given by

$$S_n = \sum_{i=0}^{n-1} B(t_i)(B(t_{i+1}) - B(t_i)).$$

Since

$$xy = \frac{1}{2}\left[(x + y)^2 - x^2 - y^2\right],$$

letting $x = B(t_i)$ *and* $y = B(t_{i+1}) - B(t_i)$ *yields*

$$B(t_i)(B(t_{i+1}) - B(t_i)) = \frac{1}{2}\left[(B(t_{i+1}))^2 - (B(t_i))^2 - (B(t_{i+1}) - B(t_i))^2\right].$$

Then after pair cancelations the sum becomes

$$\begin{aligned} S_n &= \frac{1}{2}\sum_{i=0}^{n-1}(B(t_{i+1}))^2 - \frac{1}{2}\sum_{i=0}^{n-1}(B(t_i))^2 - \frac{1}{2}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 \\ &= \frac{1}{2}(B(t_n))^2 - \frac{1}{2}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2. \end{aligned}$$

Using $t_n = T$, we get

$$S_n = \frac{1}{2}(B(T))^2 - \frac{1}{2}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2.$$

Since the first term on the right side is independent of n, using II.1, we have

$$\begin{aligned} ms - \lim_{n\to\infty} S_n &= \frac{1}{2}(B(T))^2 - \left[ms - \lim_{n\to\infty}\frac{1}{2}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2\right] \\ &= \frac{1}{2}(B(T))^2 - \frac{1}{2}T. \end{aligned}$$

We have now obtained the following explicit formula of a stochastic integral:

$$\int_0^T B(t)dB(t) = \frac{1}{2}(B(T))^2 - \frac{1}{2}T.$$

In a similar way one can obtain

$$\int_a^b B(t)dB(t) = \frac{1}{2}(B(b)^2 - B(a)^2) - \frac{1}{2}(b-a).$$

Remark II.2.2 *[21] If $X(t)$ is a differentiable function (more generally, a function of finite variation), then the stochastic integral $\int_0^T X(t)dB(t)$ can be defined by formally using the integration by parts:*

$$\int_0^T X(t)dB(t) = X(T)B(T) - X(0)B(0) - \int_0^T B(t)dX(t).$$

II.2.3 Properties of the Itô Integral

we will introduce here, without proof, some nice properties of the Itô integral we will use at a later stage.

Proposition II.2.2 *[23] We have*

- *If X is Itô integrable, then*

$$\mathbb{E}\left(\int_0^T X(s)dB(s)\right) = 0$$

and

$$Var\left(\int_0^T X(s)dB(s)\right) = \int_0^T \mathbb{E}X^2(t)dt \quad (\textit{Itô isometry}).$$

- *If X and Y are two Itô integrable processes and a and b two constants, then (linearity)*

$$\int_0^T (aX(t) + bY(t))\, dB(t) = a\int_0^T X(t)dB(t) + b\int_0^T Y(t)dB(t).$$

- *It follows from the linearity property above that*

$$\int_0^T a dB(t) = a\int_0^T dB(t) = aB(T).$$

- *The process $M(t) = M(0) + \int_0^t X(s)dB(s)$ is a martingale with $M(0)$ a constant.*

Corollary II.2.1 *[5] From the first propertie and Itô isometry it follows that the random variable $\int_a^b f(t, B_t)dB_t$ has mean zero and variance $\mathbb{E}\left(\int_a^b f(t, B_t)^2 dt\right)$.*

II.3 Stochastic Differentiation and Itô's Formula

The chain rule in the classic calculus is the formula $\frac{d}{dt}f(g(t)) = f'(g(t))g'(t)$ for differentiable functions f and g. It can be rewritten in the integral form as

$$f(g(t)) - f(g(a)) = \int_a^t f'(g(s))g'(s)ds.$$

On the other hand, the chain rule in the Itô calculus, for the simplest case, states that

$$f(B(t)) = f(B(a)) + \int_a^t f'(B(s))dB(s) + \frac{1}{2}\int_a^t f''(B(s))ds$$

for a Brownian motion $B(t)$ and a twice continuously differentiable function f. This formula is often written in a symbolic differential form as

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt.$$

The appearance of the term $\frac{1}{2}f''(B(t))dt$ is a consequence of the nonzero quadratic variation of the Brownian motion $B(t)$ [18]. In this section, we will study the celebrated Itô formula and introduce some familiar differentiation rules for stochastic calculus.

II.3.1 Stochastic Differentiation

Since most stochastic processes are not differentiable, we will focus on the infinitesimal changes of the process such as dB_t. For a process X_t, the change in the process between t and $t + \Delta t$ is given by $\Delta X_t = X_{t+\Delta t} - X_t$. As $\Delta t \to 0$, we obtain the inifinitesimal change of the process X_t

$$dX_t = X_{t+dt} - X_t,$$

which can be also written as $X_{t+dt} = X_t + dX_t$.
Some Basic differentiation rules from [39] are listed below:

1. The constant multiple rule: If X_t is a stochastic process and c is a constant, then

$$d(cX_t) = cdX_t.$$

2. The sum/difference rule: If X_t and Y_t are two a stochastic processes, then

$$d(X_t \pm Y_t) = dX_t \pm dY_t.$$

3. The product rule: If X_t and Y_t are two a stochastic processes, then

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

4. The quotient rule: If X_t and Y_t are two a stochastic processes, then

$$d(\frac{X_t}{Y_t}) = \frac{Y_t dX_t - X_t dY_t - dX_t dY_t}{Y_t^2} + \frac{X_t}{Y_t^3}(dY_t)^2.$$

The proofs of these rules are easy and we only provide one example of verifying the product rule. The key is to write the definition expression in terms of the incremental form of the process. For example,

$$\begin{aligned} d(X_t Y_t) &= X_{t+dt} Y_{t+dt} - X_t Y_t \\ &= X_{t+dt} Y_{t+dt} - X_{t+dt} Y_t - X_t Y_{t+dt} + X_t Y_t + X_{t+dt} Y_t - X_t Y_t + X_t Y_{t+dt} - X_t Y_t \\ &= (X_{t+dt} - X_t)(Y_{t+dt} - Y_t) + Y_t(X_{t+dt} - X_t) + X_t(Y_{t+dt} - Y_t) \\ &= X_t dY_t + Y_t dX_t + dX_t dY_t. \end{aligned}$$

If X_t is replaced by the deterministic function $f(t)$, the product rule becomes

$$d(f(t)Y_t) = f(t)dY_t + Y_t df(t) + df(t)dY_t.$$

It follows from $dt dB_t = dt^2 = 0$ that $df(t)dY_t = f'(t)dt dY_t = 0$. Thus, the product rule with a deterministic function reduces to

$$d(f(t)Y_t) = f(t)dY_t + Y_t df(t),$$

which is similar to the classical product rule in calculus. Similar result can be obtained from the quotient rule. With the fundamental relation $(dB_t)^2 = dt$ [39].

We can find that some extra term may occur in defferentiation of stochastic processes compared with the deterministic cases. We give a few examples here.

Examples II.3.1 *[5] Find the differentiations of B_t^2 and B_t^3.*
It follows from the product rule that

$$d(B_t^2) = B_t dB_t + B_t dB_t + dB_t dB_t = 2B_t dB_t + dt.$$

$$\begin{aligned} d(B_t^3) &= d(B_t B_t^2) = B_t d(B_t^2) + B_t^2 dB_t + d(B_t^2) dB_t \\ &= B_t(2B_t dB_t + dt) + B_t^2 dB_t + dB_t(2B_t dB_t + dt) \\ &= 2B_t^2 dB_t + B_t dt + B_t^2 dBt + 2B_t(dB_t)^2 + dt dB_t \\ &= 3B_t^2 dB_t + 3B_t dt, \end{aligned}$$

where the last equality follows from the fundamental relations $(dB_t)^2 = dt$ and $dt dB_t = 0$.

II.3.2 Itô's Formula

In this section, we will present an important tool of stochastic calculus from [23] that is Itô formula. This formula can be seen as the stochastic version of a Taylor expansion of $f(X)$ stopped at the second order, where X is a stochastic process. Itô's lemma says that if $f(t, x)$ is a twice differentiable function on both t and x, then

$$f(t, X_t) = f(0, X_0) + \int_0^t f_t(u, X_u) du + \int_0^t f_x(u, X_u) dX_u + \frac{1}{2} \int_0^t f_{xx}(u, X_u)(dX_u)^2,$$

where

$$\frac{\partial f(t,x)}{\partial t} = f_t(t,x), \quad \frac{\partial f(t,x)}{\partial x} = f_x(t,x), \quad \frac{\partial^2 f(t,x)}{\partial x^2} = f_{xx}(t,x),$$

or, in differential form,

$$df(t, X_t) = f_t(t, X_t) dt + f_x(t, X_t) dX_t + \frac{1}{2} f_{xx}(t, X_t)(dX_t)^2.$$

If X_t is the Brownian motion, this simplifies to the following

$$f(t, B_t) = f(0,0) + \int_0^t \left(f_t(u, B_u) + \frac{1}{2} f_{xx}(u, B_u) \right) du + \int_0^t f_x(u, B_u) dB_u$$

or, in differential form,

$$df(t, B_t) = \left(f_t(t, B_t) + \frac{1}{2} f_{xx}(t, B_t) \right) dt + f_x(t, B_t) dB_t.$$

Example II.3.1 *[5] Let $f(t, x) = e^{\alpha t + \beta x}$, then*

$$\frac{\partial f(t,x)}{\partial t} = \alpha f, \quad \frac{\partial f(t,x)}{\partial x} = \beta f, \quad \frac{\partial^2 f(t,x)}{\partial x^2} = \beta^2 f.$$

Then,

$$df(t,B_t) = \left(\alpha f + \frac{1}{2}\beta^2 f\right) dt + \beta f dB_t$$
$$= \left(\alpha + \frac{1}{2}\beta^2\right) f dt + \beta f dB_t.$$

II.4 Stochastic Differential Equations

What is a Stochastic Differential Equation?

Consider the deterministic differential equation

$$dx(t) = a(t,x(t))dt, \quad x(0) = x_0.$$

The easiest way to introduce randomness in this equation is to randomize the initial condition. The solution $x(t)$ then becomes a stochastic process $\{X_t, t \in [0,T]\}$:

$$dX_t = a(t,X_t)dt, \quad X_0(\omega) = Y(\omega).$$

The solution of a differential equation can change quite drastically, even if the change of the initial condition is small. For our purposes, the randomness in the differential equation is introduced via an additional random noise term:

$$dX_t = a(t,X_t)dt + b(t,X_t)dB_t, \quad X_0(\omega) = Y(\omega).$$

Here, as usual, $B = (B_t, t \geq 0)$ denotes Brownian motion, and $a(t,x)$ and $b(t,x)$ are deterministic functions [26].
Stochastic differential equations arise, for example, when the coefficients of ordinary equations are perturbed by White Noise.

Example II.4.1 *[21] Population growth. If $x(t)$ denotes the population density, then the population growth can be described by the ordinary differential equation*

$$dx(t)/dt = ax(t)(1 - x(t)).$$

The growth is exponential with birth rate a, when this density is small, and slows down when the density increases. Random perturbation of the birth rate results in the stochastic differential equation

$$dX(t)/dt = (a + b\xi(t))X(t)(1 - X(t)),$$

or the SDE

$$dX(t) = aX(t)(1 - X(t))dt + bX(t)(1 - X(t))dB(t).$$

The next example demonstrate how to find a solution to an a SDE by using Itô's formula.

Example II.4.2 *[5] Consider the SDE*

$$dX(t) = aX(t)dt + bX(t)dB(t), \quad X(0) = 1.$$

Take $f(x) = \ln(x)$, *then* $f'(x) = 1/x$ *and* $f''(x) = -1/x^2$.

$$\begin{aligned} d(\ln(X(t))) &= \frac{1}{X(t)}dX(t) + \frac{1}{2}\left(-\frac{1}{X^2(t)}\right) b^2 X_2(t)dt \\ &= \frac{1}{X(t)}\left(aX(t)dt + bX(t)dB(t)\right) - \frac{1}{2}b^2 dt \\ &= (a - \frac{1}{2}b^2)dt + bdB_t. \end{aligned}$$

So that $Y(t) = ln(X(t))$ *satisfies*

$$dY(t) = (a - \frac{1}{2}b^2)dt + bdB_t.$$

Its integral representation gives

$$Y(t) = Y(0) + (a - \frac{1}{2}b^2)t + bB_t$$

and

$$X(t) = X(0)e^{(a-\frac{1}{2}b^2)t+bB_t}.$$

II.5 Stochastic Difference Equations

By analogy with the theory of Backward Stochastic Differential Equations, we define Backward Stochastic Difference Equations on spaces related to discrete time, finite state processes [9]. Recently, there has been a great interest in studying stochastic difference equations and systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economy, psychology, sociology, and so forth. A first-order difference equation of the form

$$x_{n+1} = F(n, x_n), \quad n \in \mathbb{N}$$

may be used to describe phenomena that evolve in discrete time, where the size of the each generation is a function of that preceding. But the real world often refuses to conform to such a neat mathematical representation. Unpredictable effects can be included in the form of a sequence of random variables $\{\xi_n\}_{n\in\mathbb{N}}$, and the result is a stochastic difference equation:

$$X_{n+1} = F(n, X_n) + G(n, X_n)\xi_{n+1}, \quad n \in \mathbb{N}. \tag{II.3}$$

The solution of II.3 is a discrete time stochastic process adapted to the natural filtration of $\{\xi_n\}_{n\in\mathbb{N}}$. Such equations also appear naturally as discrete analogues of stochastic differential equations which model various biological and economical systems [11, 31].

Example II.5.1 *[31] Mathematical biology is a good place to look for realworld phenomena that evolve in discrete time (see [27]). Certain species, for example periodic cicadas and fruit flies, reproduce in non-overlapping generations, and the change in biomass from one generation to the next may be represented as a stochastic difference equation of the form*

$$X_{n+1} = X_n \left[N(X_n) + Q(X_n)\xi_{n+1}\right], \quad n \in \mathbb{N}. \tag{II.4}$$

The sequence of random variables $\{\xi_n\}_{n\in\mathbb{N}}$ captures random influences like disease and natural variability in fecundity between generations.

The next example demonstrate how to find a solution to a simple stochastic difference equation by using an iterative solution method.

Example II.5.2 *Let's consider a simple stochastic difference equation:*

$$X_{t+1} = X_t + \xi_t$$

where X_t is the state variable at time t and ξ_t are iid random variables with mean zero and variance σ^2.

1. *Initial Condition: Suppose the initial condition is $X_0 = x_0$.*
2. *First Iteration (t = 0):*
$$X_1 = X_0 + \xi_0.$$
Since $X_0 = x_0$, we have:
$$X_1 = x_0 + \xi_0.$$
3. *Second Iteration (t = 1):*
$$X_2 = X_1 + \xi_1.$$
Substituting X_1:
$$X_2 = (x_0 + \xi_0) + \xi_1$$
$$X_2 = x_0 + \xi_0 + \xi_1.$$
4. *Third Iteration (t = 2):*
$$X_3 = X_2 + \xi_2.$$
Substituting X_2:
$$X_3 = (x_0 + \xi_0 + \xi_1) + \xi_2$$
$$X_3 = x_0 + \xi_0 + \xi_1 + \xi_2.$$
5. *General Solution: Continuing this process, we get the general form for any t:*
$$X_t = x_0 + \sum_{i=0}^{t-1} \xi_i.$$

Properties of the Solution:

- *Mean: Since* ξ_i *are iid random variables with mean zero,*

$$\mathbb{E}[X_t] = x_0 + \sum_{i=0}^{t-1} \mathbb{E}[\xi_i] = x_0.$$

- *Variance: Since* ξ_i *are independent with variance* σ^2*,*

$$Var(X_t) = \sum_{i=0}^{t-1} Var(\xi_i) = \sum_{i=0}^{t-1} \sigma^2 = t\sigma^2.$$

Thus, the solution to the stochastic difference equation $X_{t+1} = X_t + \xi_t$ *with initial condition* $X_0 = x_0$ *is:*

$$X_t = x_0 + \sum_{i=0}^{t-1} \xi_i.$$

This solution describes a random walk starting from x_0*, with iid random variables* ξ_i *causing the state variable to change stochastically over time.*

II.6 Stochastic Stability

In this section, stability of stochastic differential and difference equations is introduced. The concept of stochastic stability encompasses several stability notions, such as almost sure stability, stability in probability, and mean square stability, depending on how rigorously the system's paths remain close to an equilibrium or return to it over time despite randomness [1]. To introduce this topic, it is useful to first review some of the basic stability concepts that we will need in the next chapter.

II.6.1 Equilibrium Points

For equilibruim points, we need to look for points where the system does not change, which means finding fixed points for the deterministic part. A point x^* is called an equilibrium point of the equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{II.5}$$

if

$$x^* = F(x^*, x^*, ..., x^*).$$

That is, $x_n = x^*$ for $n \geq 0$, is a solution of Equation II.5, or equivalently, x is a fixed point of F [15]. Similarly, for stochastic differential equations equilibrium points refer to states of the system where the second term has no effect. Mathematically, the point x^* is an equilibrium point for the stochastic differential equation

$$dX_t = F(t, X_t)dt + G(t, X_t)dB_t$$

if $F(t, x^*) = 0$ for all t [35].

II.6.2 Types of Stability

There are several ways to define stability for these types of equations. Let $(\Omega, \Sigma, \mathbb{P})$ be a basic probability space with the space of events Ω, the sigma-algebra Σ, the probability $\mathbb{P}$ and the expectation $\mathbb{E}$. Consider for the beginning the stochastic differential equation

$$dx(t) = ax(t)dt + \sigma x(t)dB(t) \tag{II.6}$$

where a and σ are constants and $B(t)$ is the standard Brownian motion. Now, we will introduce two following definitions of stability that are used in both the theory of stochastic differential equations and stochastic difference equations.

Definition II.6.1 *[34] The zero solution of Equation II.6 is called stable in probability if for any $\epsilon > 0$ and $\epsilon_1 \in (0,1)$ there exists a $\delta > 0$ such that the solution $x(t)$ of Equation II.6 satisfies the inequality*

$$\mathbb{P}\left\{\sup_{t\geq 0} |x(t)| > \epsilon\right\} < \epsilon_1$$

for any initial value $x(0)$ such that $\mathbb{P}\{|x(0)| < \delta\} = 1$.

Definition II.6.2 *[34] The zero solution of Equation II.6 is called:*

- *mean square stable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbb{E}x^2(t) < \epsilon$, $t \geq 0$, for any initial value $x(0)$ such that $\mathbb{E}x^2(0) < \delta$;*
- *asymptotically mean square stable if it is mean square stable and for each initial value $x(0)$ such that $\mathbb{E}|x(0)|^2 < \infty$ the solution $x(t)$ of Equation II.6 satisfies the condition $\lim_{t\to\infty} \mathbb{E}x^2(t) = 0$.*

II.6.3 Illustrative Example

Let's consider the following simple linear stochastic differential equation (SDE):

$$\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \sigma \mathrm{d}B_t$$

where:

- X_t is the state variable at time t,
- λ is a positive constant which represents the rate of system returning to the equilibrium,
- σ is the intensity of the noise,

- dB_t is the increment of a Brownian motion.

The above SDE is known as the Ornstein-Uhlenbeck process. It models mean-reverting behavior: the term $-\lambda X_t$ tends to pull X_t back toward zero, while $(\sigma \mathrm{d}B_t)$ introduces random fluctuations.

- Stability in Probability: For the Ornstein-Uhlenbeck process, as t tends to infinity, the distribution of X_t converges to a normal distribution with mean 0 and variance $\frac{\sigma^2}{2\lambda}$, regardless of the initial condition X_0. Hence, this process is stable in probability.

- Stability in Mean Square: For our SDE, it is known that the mean-square stability is achieved since $\lambda > 0$, which can be proven through Itô's isometry and other stochastic calculus tools. Specifically, the solution of the SDE converges to zero in mean-square as t tends to infinity.

In terms of mean square stability, we expect the mean square value of X_t i.e., $\mathbb{E}[X_t^2]$ to converge to the variance of the long-term distribution of the process as t approaches infinity, which reflects a stable behavior in the mean square sense. It is important to understand that individual trajectories will still look random and won't converge to a single point.

III Stabilization by Noise for Stochastic Difference Equations

The essence of stabilization by noise is that an unstable deterministic system is stabilized by stochastic perturbations of sufficiently high intensity. The problem is that the effect of "stabilization by noise", well-known already for more than 50 years for stochastic differential equations, still has no analogue for stochastic difference equations. In this chapter, a corresponding hypothesis is formulated and discussed, the truth of which is illustrated and confirmed by numerical simulation of solutions of stochastic linear and nonlinear difference equations. However, a problem of a formal proof of this hypothesis remains open.

III.1 Stabilization by Noise for Stochastic Differential Equations

Here, the problem of "stabilization by noise" is considered, which is a very important problem in the theory of stochastic systems. To explain the essence of this problem, consider the Itô's scalar linear stochastic differential Equation

$$dx(t) = \alpha x(t)dt + \beta x(t)dB(t), \tag{III.1}$$

where α and β are constants and $B(t)$ is the standard Brownian motion.
Before proceeding with the main theorem of this section, we give two Lyapunov type theorems and introduce the following generator connected to III.1

$$Lv(x) = v'(x)\alpha x + \frac{1}{2}v''(x)\beta^2 x^2. \tag{III.2}$$

The operator L is called the generator of III.1 and is defined in the following way:

Definition III.1.1 *[23](Infinitesimal generator of a stochastic process) Given a stochastic process X solution to $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, $X_0 = x$, a differential operator L of the form*

$$(Lv)(x) = \frac{\sigma^2(x)}{2}v''(x) + b(x)v'(x)$$

with v two times differentiable is called the infinitesimal generator of the stochastic process X.

Theorem III.1.1 *[35] Let there exist a twice differentiable function $V(x)$ and positive constants a, b, c, satisfying the conditions*

$$ax^2 \leq V(x) \leq bx^2, \quad LV(x) \leq -cx^2.$$

The zero solution of Equation III.1 is then asymptotically mean square stable.

Theorem III.1.2 *[35] Let there exist a nonnegative twice differentiable function $V(x)$, such that only $V(0) = 0$ and $LV(x) \leq 0$. Then the zero solution of Equation III.1 is stable in probability.*

From Theorems III.1.1 and III.1.2 it follows that the stability investigation of stochastic differential equations can be reduced to construction of appropriate Lyapunov functionals [34].

Corollary III.1.1 *If the condition*

$$2\alpha + \beta^2 < 0 \tag{III.3}$$

holds then the zero solution of Equation III.1 is asymptotically mean square stable.

Theorem III.1.3 *By the condition*

$$0 < 2\alpha < \beta^2 \tag{III.4}$$

the so-called "stabilization by noise" occurs and the zero solution of Equation III.1 becomes stable in probability.

Proof III.1.1 *Using III.4 and the generator III.2 of Equation III.1, for the Lyapunov function*

$$V(x) = |x|^v, \qquad v = 1 - \frac{2\alpha}{\beta^2} \in (0,1),$$

we have

$$LV(x) = v|x|^{v-1}\alpha x + \frac{1}{2}v(v-1)|x|^{v-2}\beta^2 x^2 \leq \alpha v|x|^v \left(1 + (v-1)\frac{\beta^2}{2\alpha}\right) = 0.$$

Via Theorem III.1.2 from the condition $LV(x) \leq 0$, it follows that the zero solution of Equation III.1 is stable in probability. □

III.2 Stabilization by Noise for Stochastic Difference Equations

Despite the fact that for stochastic differential equations the effect of stabilization by noise was established more than 50 years ago, a similar statement for stochastic difference equations has not yet been obtained. In this section, hypothesis about stabilization by noise for stochastic difference equations is considered. This hypothesis is demonstrated and confirmed by numerical simulation of solutions of linear and nonlinear stochastic difference equations.

III.2.1 Linear Stochastic Difference Equation

Consider now the scalar linear stochastic difference Equation

$$x_{n+1} = \alpha_1 x_n + \beta_1 x_n \xi_{n+1}, \quad n \in \{0, 1, ...\}, \tag{III.5}$$

where α_1 and β_1 are constants and ξ_n is a sequence of Σ_n-adapted mutually independent identically distributed random variables such that

$$\mathbb{E}\xi_n = 0, \quad \mathbb{E}\xi_n^2 = 1, \quad n \in \{0, 1, ...\}. \tag{III.6}$$

Let us consider an analogue of the condition III.4 for the linear stochastic difference Equation III.5 by the condition $\alpha_1 > 1$. For this aim, let us represent the difference analogue of Equation III.1 in the form III.5. Let $\Delta > 0$ be the step of discretization,

$$t_n = n\Delta, \quad n \in \{0, 1, ...\}, \quad x_n = x(t_n), \quad B_n = B(t_n).$$

The difference analogue of Equation III.1 then takes the form

$$x_{n+1} - x_n = \alpha x_n \Delta + \beta x_n (B_{n+1} - B_n). \tag{III.7}$$

Note that $\mathbb{E}(B_{n+1} - B_n)^2 = \Delta$.
To construct a difference analogue of the system III.1, put

$$\xi_{n+1} = \frac{1}{\sqrt{\Delta}}(B_{n+1} - B_n)$$

which satisfies the conditions III.6.

$$x_{n+1} = (1 + \alpha\Delta)x_n + \beta\sqrt{\Delta} x_n \xi_{n+1},$$

i.e., in the form III.5 with the coefficients

$$\alpha_1 = 1 + \alpha\Delta, \quad \beta_1 = \beta\sqrt{\Delta}$$

which gives

$$\alpha = \frac{\alpha_1 - 1}{\Delta}, \quad \beta = \frac{\beta_1}{\sqrt{\Delta}},$$

and via III.4, we obtain

$$0 < 2\frac{\alpha_1 - 1}{\Delta} < \frac{\beta_1^2}{\Delta},$$

So, we obtain the following hypothesis:

Hypothesis III.2.1 *If the condition*

$$0 < 2(\alpha_1 - 1) < \beta_1^2$$

holds then the zero solution of Equation III.5 is stable in probability.

Numerical Simulation

To illustrate Hypothesis III.2.1, consider Equation III.5 with $\alpha_1 = 1.05$, $x_0 = 0.2$. In Figures III.1, 100 trajectories of the solution of Equation III.5 are shown with $\beta_1 = 0.2$, while, in Figures III.2, 100 trajectories of the solution of Equation III.5 are shown with $\beta_1 = 0.4$.

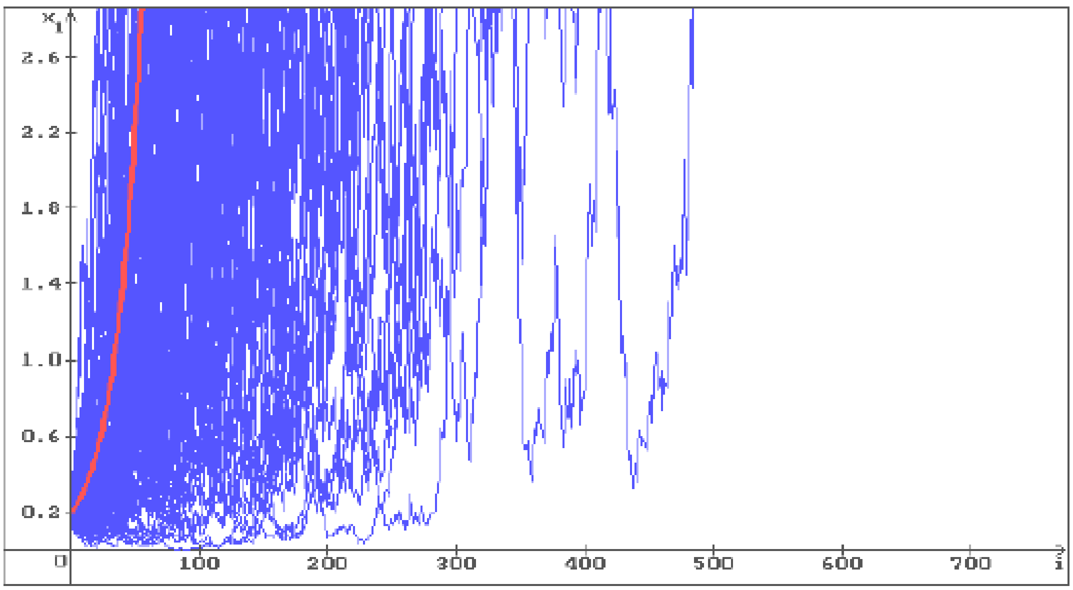

Figure III.1 100 trajectories (blue) of the solution x_i of Equation III.5 with $\alpha_1 = 1.05$, $\beta_1 = 0.2$. The equilibrium $x^* = 0$ is unstable, all trajectories with $x_0 = 0.2$ go to infinity. The red line corresponds to the deterministic case, i.e., $\beta_1 = 0$.

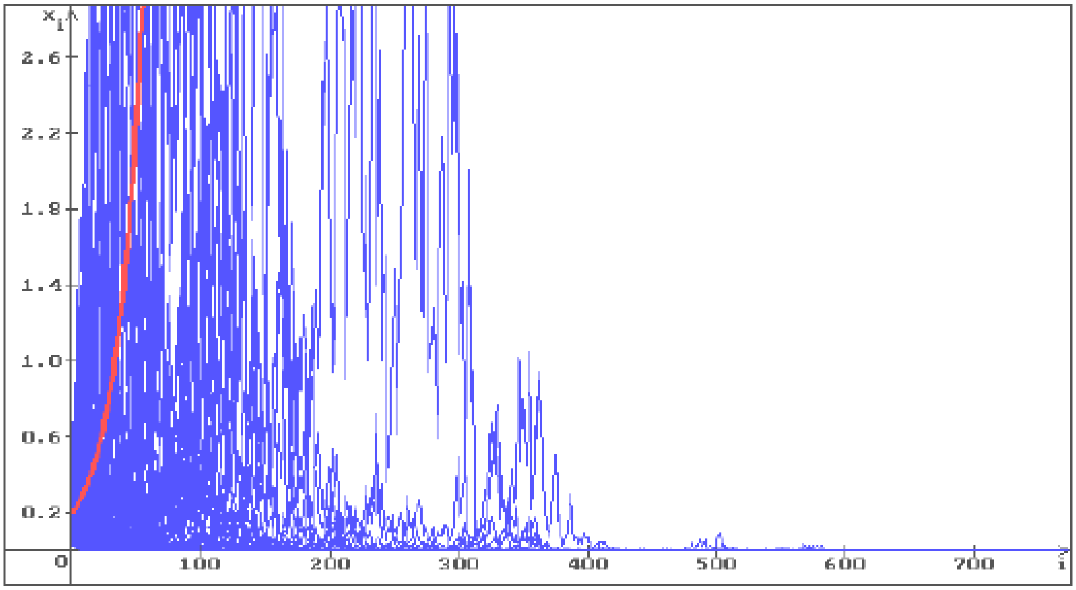

Figure III.2 100 trajectories (blue) of the solution x_i of Equation III.5 with $\alpha_1 = 1.05$, $\beta_1 = 0.4$. Stabilization by noise occurs and all trajectories with $x_0 = 0.2$ converge to $x^* = 0$. The red line corresponds to the deterministic case, i.e., $\beta_1 = 0$.

Thus, the hypothesis about stabilization by noise for stochastic difference equations is confirmed by numerical simulation of solutions of the linear stochastic difference Equation III.5. But still naturally needs to be proven.

III.2.2 Nonlinear Stochastic Difference Equation

As an example of a nonlinear difference equation, consider the bilinear difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + x_{n-1}}, \qquad n \in \{0, 1, ...\}, \tag{III.8}$$

with an initial condition

$$x_n = \varphi_n, \qquad n \in \{-1, 0\}.$$

Here a, b and c are positive parameters.
The following two statements, obtained in [14], for Equation III.8 take the form:

Statement III.2.1 *If $(1-a)(c+1) \neq b$ then the unique equilibrium of Equation III.8 is $x^* = 0$.*

Statement III.2.2 *If $(1-a)(c+1) > b$ then the zero solution of Equation III.8 is locally asymptotically stable.*

In order to find equilibrium points of the previous equation, we solve the following equation

$$x^* = ax^* + \frac{b(x^*)^2}{cx^* + x^*}$$

i. e.,

$$x^* \left(1 - a - \frac{b}{c+1}\right) = 0. \tag{III.9}$$

It is clear that if

$$1 - a - \frac{b}{c+1} \neq 0, \tag{III.10}$$

then $x^* = 0$ only can be the equilibrium of Equation III.8 (that coincides with Statement III.2.1). From the other hand, by the assumption

$$1 - a - \frac{b}{c+1} = 0 \tag{III.11}$$

each $x^* \in \mathbb{R}$ is a solution of Equation III.9 and $x_n = x^*, n \geq -1$, is a solution of Equation III.8. Let us assume that Equation III.8 is exposed to stochastic perturbations that are directly proportional to the deviation of the solution x_n from the equilibrium x^*, i.e., Equation III.8 takes the form of the stochastic difference Equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + x_{n-1}} + \sigma(x_n - x^*)\xi_{n+1}, \quad n \in \{0, 1, ...\}. \tag{III.12}$$

By that the solution $x_n = x^*$ of Equation III.8 is also the solution of Equation III.12.
Here σ is a constant and x^* is an equilibrium point of the equation III.8. It is to be noted that the equilibrium point x^* is a solution of the equation III.12 as well [34].
Substituting $X_n = x^* + y_n$ into III.12, we get

$$x^* + y_{n+1} = a(x^* + y_n) + \frac{b(x^* + y_n)(x^* + y_{n-1})}{c(x^* + y_n) + x^* + y_{n-1}} + \sigma y_n \xi_{n+1}$$

which leads to the equation

$$y_{n+1} = ay_n + (a-1)x^* + \frac{b[(x^*)^2 + (y_n + y_{n-1})x^* + y_n y_{n-1}]}{(c+1)x^* + cy_n + y_{n-1}} + \sigma y_n \xi_{n+1}. \tag{III.13}$$

It is obvious that stability of the zero solution of the equation III.13 is equivalent to stability of the solution x^* of the equation III.12.
Note that the equation III.13 is a nonlinear equation with an order of non- linearity higher than one. From [34], it is established that, within this context, if the zero solution of the following linear approximation

$$Z_{n+1} = \left(a + \frac{b}{(c+1)^2}\right) Z_n + \frac{bc}{(c+1)^2} Z_{n-1} + \sigma Z_{n-1}\xi_{n+1}.$$

or

$$Z_{n+1} = \alpha Z_n + \beta Z_{n-1} + \sigma Z_{n-1}\xi_{n+1} \tag{III.14}$$

where

$$\alpha = a + \frac{b}{(c+1)^2}, \quad \beta = \frac{bc}{(c+1)^2}, \quad \alpha + \beta = 1$$

of III.13 is asymptotically mean square stable, then this condition is sufficient for the stability in probability of the equilibrium x^* of the nonlinear equation III.12.
So, to obtain conditions for stability in probability of the equilibrium x^* of the nonlinear stochastic difference Equation III.12 it is enough to get conditions for asymptotic mean

square stability of the zero solution of the linear stochastic difference Equation III.14 that in the case of $x^* \neq 0$ is the linear part of the nonlinear difference Equation III.13.

Lemma III.2.1 *[34] If*

$$|\alpha| + |\beta| < \sqrt{1 - \sigma^2}, \tag{III.15}$$

then the zero solution of Equation III.14 is asymptotically mean square stable.

Lemma III.2.2 *[34] The inequalities*

$$|\beta| < 1, \quad |\alpha| < 1 - \beta, \quad \sigma^2 < \frac{1+\beta}{1-\beta}\left[(1-\beta)^2 - \alpha^2\right], \tag{III.16}$$

are the necessary and sufficient conditions for asymptotic mean square stability of the zero solution of Equation III.14.

Numerical Simulation

Example III.2.1 *Put in Equation III.12 $a = 0.5$, $b = 0.7$, $c = 1$, $\sigma = 0.5$. The conditions III.15 and III.16 hold. In Figure III.3, 500 trajectories (blue) of the solution of Equation III.11 are shown with the initial conditions $x_{-1} = 0.8$, $x_0 = 1.8$ and the stable equilibrium $x^* = 0$. One can see that all trajectories converge to the stable equilibrium $x^* = 0$. Note that this corresponds to Statement III.2.2. Now place $\sigma = 0.95$ with the same values of all other parameters. The conditions III.15 and III.16 do not hold. In Figure III.4, 500 trajectories (blue) of the solution of Equation III.12 with $x_* = 0$ are shown with the initial conditions $x_{-1} = 0.01$, $x_0 = 0.02$. The zero solution is unstable and the trajectories fill the whole space. The red lines in the both Figures III.3 and III.4 correspond to the deterministic case, i.e., $\sigma = 0$.*

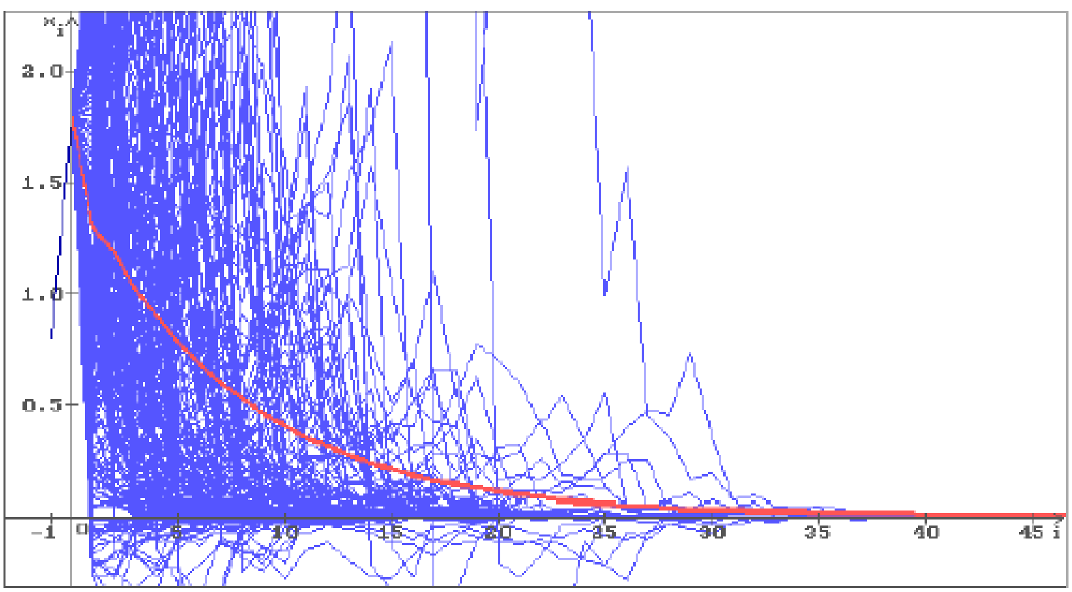

Figure III.3 500 trajectories (blue) of the solution x_n of Equation III.12 with $a = 0.5$, $b = 0.7$, $c = 1$, $\sigma = 0.5$, $x_{-1} = 0.8$, $x_0 = 1.8$ and stable equilibrium $x^* = 0$. The red line corresponds to the deterministic case, i.e., $\sigma = 0$.

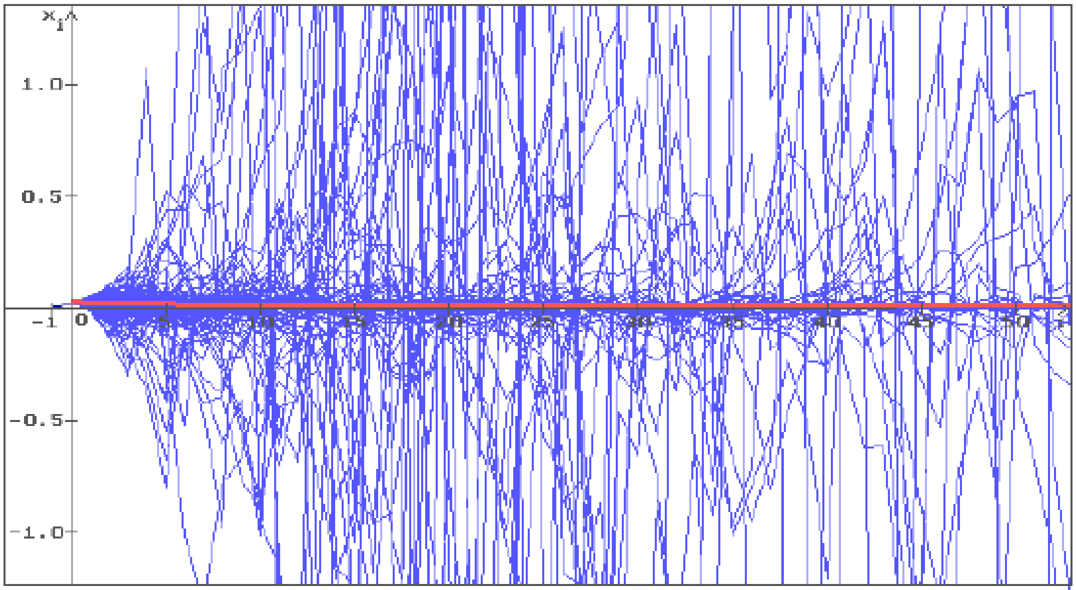

Figure III.4 500 trajectories (blue) of the solution x_n of Equation III.12 with $a = 0.5$, $b = 0.7$, $c = 1$, $\sigma = 0.95$, $x_{-1} = 0.01$, $x_0 = 0.02$ and unstable equilibrium $x^* = 0$. The red line corresponds to the deterministic case, i.e., $\sigma = 0$.

Remark III.2.1 *Note that by the conditions III.15 and III.16 the condition III.10 holds, by which Equation III.8 has the equilibrium $x^* = 0$ only. However, the linear Equation III.14 is obtained by the conditions III.11 and $x^* \neq 0$. Thus, strictly speaking, Equation III.14 cannot be used for studying the zero equilibrium. So, below the equilibrium $x^* \neq 0$ is considered.*

Example III.2.2 *Put $a = 0.65$, $b = 0.7$, $c = 1$, $\sigma = 0.15$, $x^* = 2.7$. Where in $a + \dfrac{b}{c+1} = 1$, therefore, the condition III.11 holds, the conditions III.15 and III.16 do not hold. In Figure III.5, the red straight corresponds to the constant solution $x_n = x^*$, $n \geq -1$, 500 trajectories (blue) of the solution of Equation III.12 are shown with the initial conditions $x_{-1} = 2.6$, $x_0 = 2.5$. The solution $x_n = x^*$ is unstable, so, the trajectories fill the whole space. Put now $\sigma = 0.45$ with the same values of all other parameters. In Figure III.6, one can see that all 500 trajectories converge to the solution $x_n = x^* = 2.7$ of Equation III.12. Putting $\sigma = 0.75$, i.e., increasing once more the level of noise, we obtain (see Figure III.7) that all 500 trajectories converge to the solution $x_n = x^* = 2.7$ of Equation III.12 faster than in Figure III.6. So, the solution $x_n = x^* = 2.7$ of Equation III.12, that is unstable by the small level of noise ($\sigma = 0.15$), becomes stable by increasing the level of noise, i.e.,* ***stabilization by noise*** *occurs.*

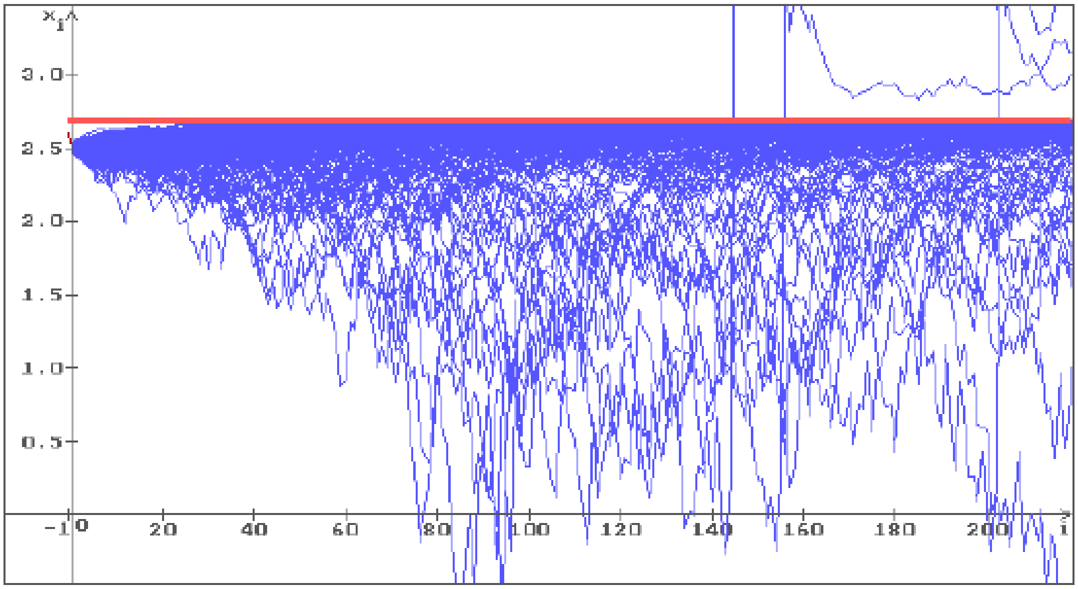

Figure III.5 500 trajectories (blue) of the solution x_n of Equation III.12 with $a = 0.65$, $b = 0.7$, $c = 1$, $\sigma = 0.15$, $x_{-1} = 2.6$, $x_0 = 2.5$ and unstable equilibrium $x^* = 2.7$. The red straight corresponds to the constant solution $x_n = x^*$, $n \geq -1$, $\sigma = 0$.

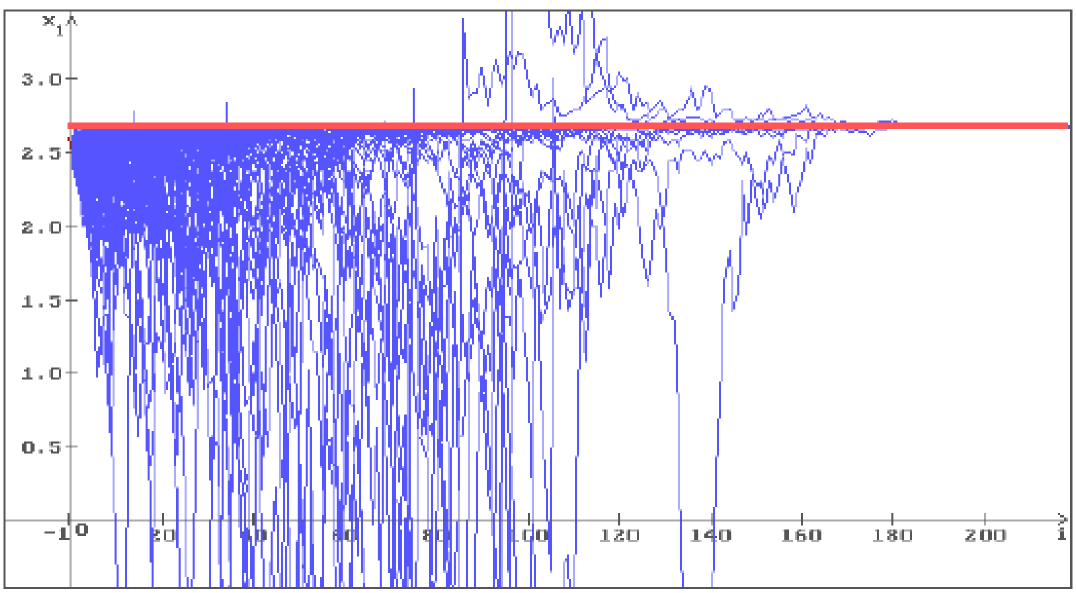

Figure III.6 500 trajectories (blue) of the solution x_n of Equation III.12 with $\sigma = 0.45$ and the same values of all other parameters as in Figure III.5. All trajectories converge to the stable equilibrium $x^* = 2.7$.

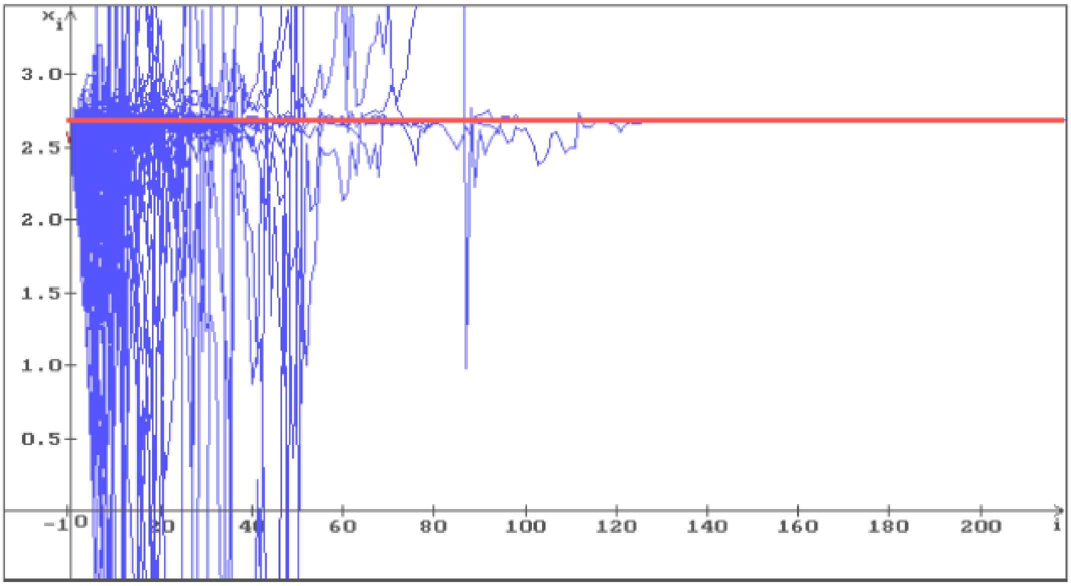

Figure III.7 500 trajectories (blue) of the solution x_n of Equation III.12 with $\sigma = 0.75$ and the same values of all other parameters as in Figure III.5. All trajectories converge to the stable equilibrium $x* = 2.7$ faster than in Figure III.6.

Via these numerical simulations of solutions of linear and nonlinear stochastic difference equations, the hypothesis about a possibility of stabilization by noise is demonstrated and confirmed also for stochastic difference equations. However, a formal proof of this hypothesis remains until now an unsolved problem.

Conclusion

The very important problem of "stabilization by noise", which for more than 50 years has been well known for stochastic differential equations, is also studied here for stochastic difference equations. These equations can be useful in many applications where we assume that there are deterministic changes combined with random ones. It's interesting to observe that while stochastic differential equations and difference equations are significant in various applications across diverse fields including physics, finance, and biology, allowing for the modeling of phenomena where uncertainty is inherent, there appears to be a lack of extensive resources on this subject, particularly related to stabilization by noise. Moving forward, we anticipate further advancements towards a comprehensive solution for this problem especially regarding stabilization by noise for difference equations.

III Bibliography

[1] Allen, E. (2007). Modeling with Itô stochastic differential equations (Vol. 22). Springer Science & Business Media.

[2] Billingsley, P. (2017). Probability and measure. John Wiley & Sons.

[3] Breiman, L. (1992). Probability. Society for Industrial and Applied Mathematics.

[4] Brzezniak, Z., & Zastawniak, T. (2000). Basic stochastic processes: a course through exercises. Springer Science & Business Media.

[5] Calin, O. (2012). An introduction to stochastic calculus with applications to finance. Ann Arbor.

[6] Caraballo, T., Liu, K., & Mao, X. (2001). On stabilization of partial differential equations by noise. Nagoya Mathematical Journal, $161, 155-170$.

[7] Chalasani, P., & Jha, S. (1997). Steven Shreve: Stochastic Calculus and Finance. Lecture Notes, October, $1-343$.

[8] Chung, K. L., & Williams, R. (1990). Introduction to Stochastic Integration. Springer Science & Business Media.

[9] Cohen, S. N., & Elliott, R. J. (2010). A general theory of finite state backward stochastic difference equations. Stochastic Processes and their Applications, $120(4), 442-466$.

[10] Cyganowski, S., Kloeden, O., & Ombach, J. (2001). From elementary probability to stochastic differential equations with MAPLE®. Springer Science & Business Media.

[11] Ding, X., & Zhang, R. (2008). On the difference equation $x_{n+1} = (ax_n + bx_{n-1})e^{-x_n}$. Advances in Difference Equations, $2008, 1-7$.

[12] Dobrow, R. P. (2016). Introduction to stochastic processes with R. John Wiley & Sons.

[13] Durrett, R., & Durrett, R. (1999). Essentials of stochastic processes (Vol. 1). New York: Springer.

[14] Elsayed, E. M. (2008). Qualitative behavior of a rational recursive sequence. Indagationes Mathematicae, 19(2), $189-201$.

[15] Elsayed, E. M. (2011). Behavior of a rational recursive sequences. Studia. Universitatis Babes-Bolyai Mathematica, 56(1), $27-42$.

[16] Givental, A. (2001). Linear algebra and differential equations. American Mathematical Soc.

[17] Hirsa, A., & Neftci, S. N. (2013). An introduction to the mathematics of financial derivatives. Academic press.

[18] Hui-Hsiung, K. (2006). Introduction to stochastic integration.

[19] Itô, K. (1944).109. stochastic integral. Proceedings of the Imperial Academy, 20(8), $519-524$.

[20] Khasminskii, R. (2012). Stochastic stability of differential equations. Springer.

[21] Klebaner, F. C. (2012). Introduction To Stochastic Calculus With Applications. World Scientific Publishing Company.

[22] Knill, O. (1994). Probability and stochastic processes with applications. Havard Web-Based, 5.

[23] Lacu, S. M. (2008). Simulation and inference for stochastic differential equations: with R examples (Vol. 486). New York: Springer.

[24] Laha, R. G., & Rohatgi, V. K. (2020). Probability theory. Courier Dover Publications.

[25] Lord, G. J., Powell, C. E., & Shardlow, T. (2014). An introduction to computational stochastic PDEs (Vol. 50). Cambridge University Press.

[26] Mikosch, T. (1998). Elementary stochastic calculus with finance in view. World scientific.

[27] Murray, J. D. (2002). Mathematical Biology [electronic resource].

[28] Oksendal, B. (2013). Stochastic differential equations: an introduction with applications. Springer Science & Business Media.

[29] Olofsson, P., & Andersson, M. (2012). Probability, statistics, and stochastic processes. John Wiley & Sons.

[30] Panik, M. J. (2017). Stochastic Differential Equations: An Introduction with Applications in Population Dynamics Modeling.

[31] Rodkina, A., & Kelly, C. (2011). Stochastic Difference Equations and Applications.

[32] Ross, S. M. (1997). Introduction to probability models, San Diego. Cal: Academic Press.

[33] Schilling, R. L., & Partzsch, L. (2014). Brownian motion: an introduction to stochastic processes. Walter de Gruyter GmbH & Co KG.

[34] Shaikhet, L. (2011). Lyapunov functionals and stability of stochastic difference equations. Springer Science & Business Media.

[35] Shaikhet, L. (2013). Lyapunov functionals and stability of stochastic functional differential equations. Springer Science & Business Media.

[36] Tijims, H. C. (2003). A first course in stochastic models. John Wiley and sons.

[37] Williams, D. (1991). Probability with martingales. Cambridge university press.

[38] Yates, R. D., & Goodman, D. J. (2014). Probability and stochastic processes: a friendly introduction for electrical and computer engineers. John Wiley & Sons.

[39] Zhang, Z. G. (2023). Fundamentals of stochastic models. CRC Press.