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Mémoire

Présenté en vue de l'obtention du diplôme de
Master en Mathématiques

LA SOLUTION EXACTE DE L'EQUATION FRACTIONNAIRE KAUP-KUPERSHMIDT ET APPROXIMATION DE LA SOLUTION PAR DECOMPOSITION DE LAPLACE ADOMIAN

Option : Analyse numérique et équations aux dérivées partielles (ANEDP)

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At the conclusion of this work, all praise and thanks are due to Allah, who is closer to me than my own jugular vein, for granting me health and well-being to accomplish this task. He blessed me with the courage, patience, willpower, and strength whenever my determination faltered and obstacles blocked my path. All praise is due to Allah, who assisted me at every step towards this lofty goal and guided me to the right path whenever I strayed. Praise be to Allah, always and forever.

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Finally, I extend my deep thanks to all individuals who have helped and supported me, whether near or far, with their simple gestures or substantial contributions.

Dedication

Before everything else, I dedicate this work with sincere intention in the way of Allah Almighty, and I hope that it contributes, even a little, to scientific research.

I would like to dedicate this work to My mother, the symbol of strength and determination, who sacrificed her youth for my education. To the one who challenged everyone for my sake and believed in my abilities?my success is truly your success.

My father, the symbol of patience, who resisted the dawn's sleep to teach me the correct pronunciation of letters. He has never withheld anything from me, whether materially or morally, and has always pushed me to achieve my dreams.

To my siblings, who made my difficult days easier with their jokes.

To my loved ones, friends, and colleagues who care about me, and with whom I have shared moments of joy and sorrow.

My aunt, may God compensate her spiritually and materially for the hard days. She is the symbol of tenderness and care, the support in every step of my life.

To all who bet on my failure and underestimated my determination.

finally, to everyone who reads my work as my first step into the world of research?I hope you benefit from it.

Abstract

In this memory, we develop a method for obtaining analytical solutions for certain Kaup-Kupershmidt equations, using a modified approach known as the new iteration transform method. This technique combines the new integral transformation Kharrat-Toma, with the new iteration method. Additionally, by employing the Laplace Adomian Decomposition Method (LADM), we obtain an approximate solution to the time-fractional Kupershmidt equation, then we compare this approximate solution with the exact one.

Key words : Kaup-Kupershmidt equations, Kharrat-Toma transformation, new iteration method.

Résumé

Dans ce mémoire, nous avons développé une méthode pour obtenir des solutions analytiques pour certaines équations de Kaup-Kupershmidt, en utilisant une approche modifiée connue sous le nom de méthode de transformation par nouvelle itération. Cette technique combine la nouvelle transformation intégrale de Kharrat-Toma avec la méthode de nouvelle itération. De plus, en utilisant la méthode de décomposition de Laplace-Adomian (MDLA), nous avons obtenu une solution approximative de l'équation de Kupershmidt temporelle fractionnaire. Nous avons ensuite comparé cette solution approximative avec la solution exacte.

Mots clés :équations de Kaup-Kupershmidt, transformation de Kharrat-Toma, nouvelle itération.

ملخص

في هذه المذكرة، نقوم بتطوير طريقة للحصول على حلول تحليلية لبعض معادلات كاوب كوبرشميدت باستخدام نهج معدل يُعرف بطريقة تحويل التكرار الجديدة. تجمع هذه التقنية بين تحويل التكامل الجديد لخراط طوما وطريقة التكرار الجديدة. تشير النتائج إلى أن هذه الطريقة فعالة للغاية ومناسبة لدراسة النماذج التفاضلية الكسرية غير الخطية المعقدة. بالإضافة إلى ذلك، باستخدام طريقة تحليل أدوميان لابلاس ، نحصل على حل تقريبي لمعادلة كوبرشميدت الزمنية الكسرية، ثم نقارن هذا الحل التقريبي مع الحل الدقيق.

كلمات مفتاحية: معادلات كوب كوبرشميدث ، تحويل خراط طوما ، التكرار الجديدة ، تحليل أدوميان.

CONTENTS

GENERAL INTRODUCTION

In recent years, many physical phenomena have garnered significant attention from scientists and researchers, such as plasma physics, quantum physics, biophysics, and ocean sciences. Let's focus specifically on those phenomena that occur in the oceans, such as the study of the movement and properties of waves, particularly gravity-capillary waves.

Capillary waves with short wavelengths are the first waves to form on the sea surface when winds blow, in which surface tension is the primary force restoring the wave to its original state [15]. Like the small ripples that appear on the surface of a pond when a stone is thrown .

Gravity waves are generated by a vertical disturbance on the surface of fluids, where gravitational forces work to restore equilibrium, waves have longer wavelengths, more than a few centimeters [21]. At intermediate wavelengths, both gravity and surface tension affect the wave dynamics. This transition occurs at a wavelength of approximately 1.7 centimeters in water, leading to mixed waves called gravity-capillary waves [11], such as phenomena like oil slicks on water also exhibit such waves, the waves generated by moving objects, like the movement of boats or marine creatures, cause disturbances that lead to the formation and propagation of gravity-capillary waves away from the moving object.

The study of gravity-capillary waves deepens our understanding of natural processes and contributes to the advancement of marine technology. This research aids in developing technological and environmental solutions to challenges in aquatic and marine environments. For instance, it helps improve marine and atmospheric prediction models, which enhance the

Figure 1: Capillary waves .

Figure 2: Gravity waves .

ability to respond to environmental disasters and promote maritime safety. Additionally, this knowledge supports the design of ships and marine structures capable of withstanding changing environmental conditions, including strong winds and large waves. Finally, radar remote sensing techniques rely heavily on these waves to extract valuable physical parameters from a distance [10]. Obtaining the necessary information for studying these waves is very difficult[9], largely due to the use of this, is because the physical characteristics of centimeter-scale ocean waves cannot be measured by traditional devices such as buoys, capacitance gauges, or lidar. The issue primarily arises from the disturbance caused by bulky instruments to fine-scale fluid mechanical features. Additionally, the strong sensitivity of these waves to advection by long wave orbital motions and background currents complicates accurate measurement. Therefore, the most appropriate method to retrieve parameters of short ocean waves is to use advanced imaging devices [11].

In a study conducted in[11], it was presented an extended analysis of short wave data collected via a polarimetric camera aboard Research Platform Floating Instrument Platform in the Santa Barbara Channel. Addressed two important aspects: momentum flux in the atmospheric and oceanic envelopes, and the physical characteristics of short waves. Regarding the air-sea momentum flux, the three-dimensional wind velocity vector was sampled at 20 Hz using a Campbell Scientific CSAT sonic anemometer affixed to the end of FLIP's boom. At the same location, a lidar was mounted to retrieve water surface elevation, allowing for the monitoring of long waves on the water surface.

. The physical properties of short-scale ocean waves were obtained using a polarimetric camera and the polarimetric slope sensing (PSS) method developed by Zappa and colleagues.

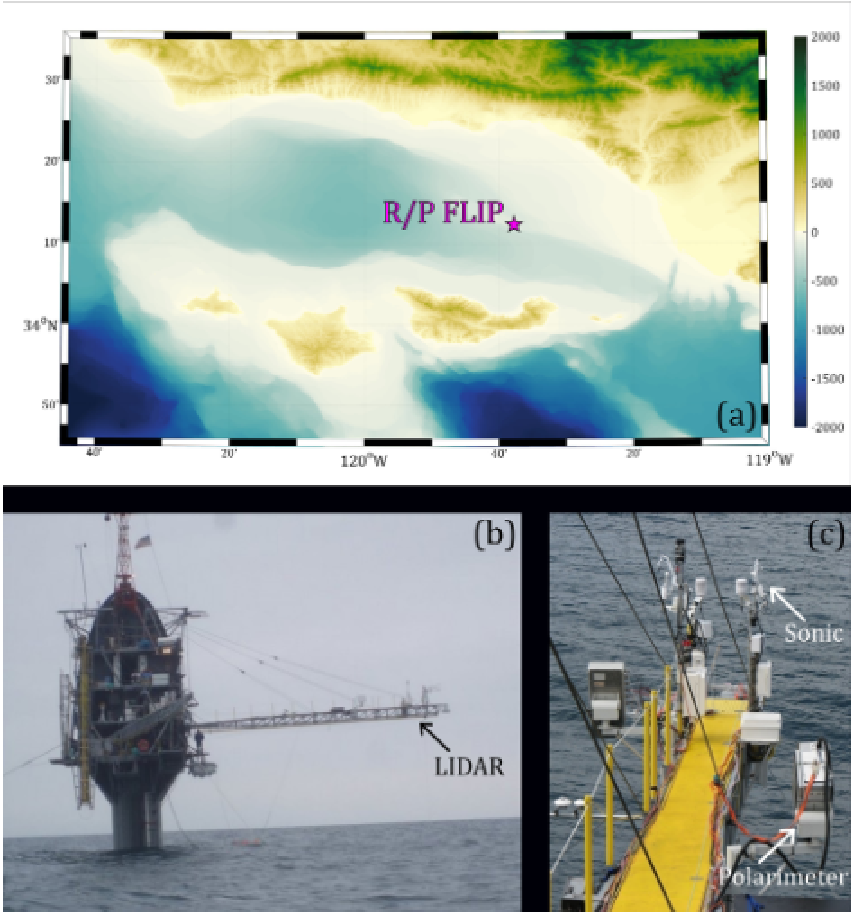

Figure 3: (a) Map of Santa Barbara Channel, with Research Platform Floating Instrument Platform (FLIP) location indicated by the magenta five-pointed star. The color bar indicates displacement of solid Earth from mean sea level. (b) Side view of the fully deployed FLIP. (c) Close-up of the FLIP boom. Bathymetric data used to color (a) were obtained from Divins and Metzger. The arrows in (b) and (c) indicate the position of the Riegl lidar, sonic anemometer, and the polarimetric camera [11]

A total of 62 camera runs were executed, with 16 deemed appropriate for extracting meaningful physical parameters, as the rest were contaminated by sun glitter, preventing the determination of sea surface slopes [11]. The results are presented in Figure4 and Figure5.

This study discovered that sea surface waves, which achieve equilibrium through the forces of gravity and capillarity, hold a unique role in the interaction between air and sea. These waves are notably influenced by their interactions with larger gravity waves, yet their scale is insufficient to make a significant contribution to the total wave energy[11].

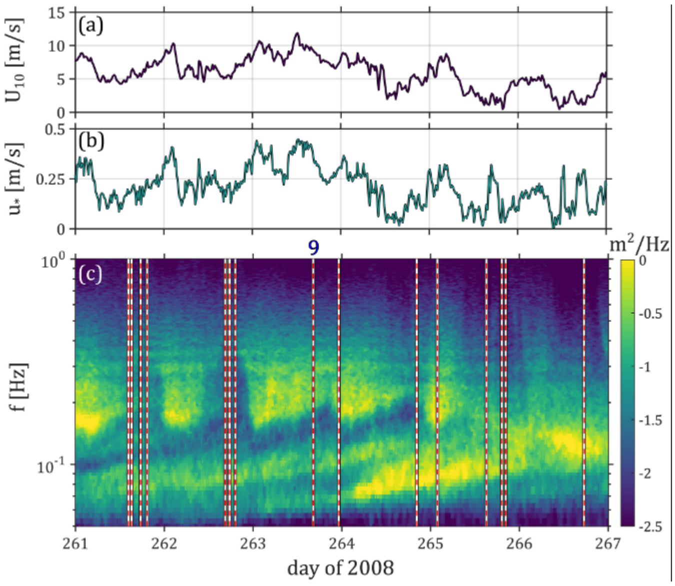

Figure 4: Time series of selected environmental conditions: (a) Ten-meter neutral wind speed and (b) friction velocity u* c wave height frequency spectrogram. The color bar indicates spectral energy density in m2/Hz. The dashed red and white lines mark the instances of polarimetric camera acquisition. Run 9 is labeled with a blue number at the top of the figure.[11]

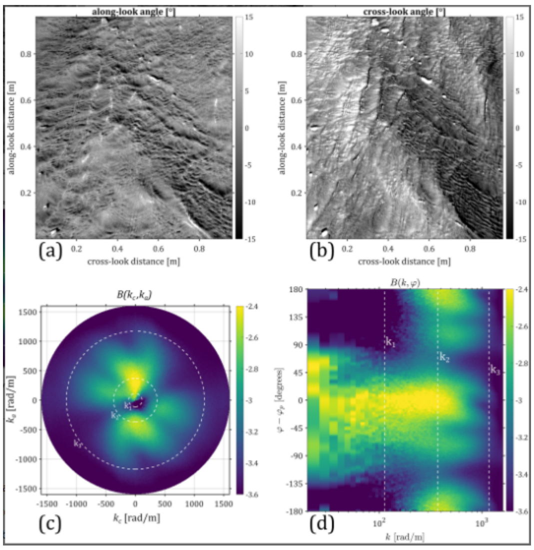

Figure 5: (a) Along-look and (b) cross-look water surface angle maps generated using the polarimetric slope sensing. Note that the gray scale intensity corresponds to the water surface angle in degrees. (c) Two-dimensional Cartesian (cross, along) wave number saturation spectrum, generated from a 10-s long data stream. (d) Same spectrum, represented in terms of wave number k and azimuth angle ϕ , referenced to wind velocity direction. The dashed lines k1, k2, and k3 correspond to 112.7, 371, and 1,173 rad/m, respectively, the boundaries of GC1 and GC2. The properties of the spectrum represented in (c) and (d) are stationary over the course of the full 20-min acquisition [11]

This nonlinear interactions between surface waves such as capillary-gravity waves have been modeled by using a fractional fifth-order partial differential equation (FDP's), called the Kaup Kupershmidt equation.

Kaup presented the famous dispersive classical Kaup Kupershmidt (KK) equation in 1980, and Kupershmidt modified it in 1994 [18], we concerned with the modified (KK) equation in this work, which is applied to analyze the nonlinear dispersive waves and capillary gravity waves [1]. The KK equation has been applied in many branches of physiques like plasma physics, fluid dyna1mics and non-linear optics. The fifth-order (KK) equation has the following form :

$$\boldsymbol{\mathcal{D}_t^\rho u(x,t) + \alpha u u_{xx} + \beta p u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} = 0,} \tag{1}$$

with the initial condition

$$\boldsymbol{u(x,0) = g(x),} \tag{2}$$

Where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are real constants and $\boldsymbol{0 < \rho \leq 1}$ is the parameter symbolizing the order of the fractional-order derivative(Caputo derivative). By considering different values for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, the overload nonlinear fifth-order development model can be scaled down to the fifth-order fractional Kaup Kupershmidt equation[7]. The classical Kaup Kupershmidt equation is known to be integrable at $\boldsymbol{p = \frac{5}{2}}$ and has bilinear representations [19].

The objective of this work is to investigate the solution of the fractional-order Kaup-Kupershmidt equation, using a new iterative transform method to demonstrate the accuracy of the proposed technique, whereas the new method reduces computing costs while increasing convergence speed. We employ the (LADM) to obtain the approximate solution.

The paper is organized as follows: The introduction illustrates the studied phenomenon and its mathematical modeling. Chapter 1 describes some mathematical tools used in the study: Kharrat-Toma transform and Laplace transform. Chapter 2 shows the iterative method with Kharrat-Toma transform and its application on two cases of the Kaup-Kupershmidt equation, to find the exact analytical solution. Chapter 3 explains the(LADM) and its application to the time-fractional (KK) equation, to find the approximate solution, and a comparison of the numerical results with the exact solution. Finally, the conclusion presents an evaluation of the effectiveness of the methods used.

CHAPTER 1

TOOLS MATHEMATICS

In this chapter, we will present some fundamental concepts and tools needed for this study.

1.1 Basic definition

We present some fundamental concepts relevant to this study

Definition 1.1 *[14] Let* $\boldsymbol{\alpha} \in \mathbb{C}$ *with strictly positive real part, the gamma function* $\boldsymbol{\Gamma}$ *is given by :*

$$\boldsymbol{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,} \tag{1.1}$$

An important characteristic of gamma function :

$$\boldsymbol{\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha!,}$$

$$\boldsymbol{\Gamma(1) = 1.}$$

Definition 1.2 *[16] Let* $\boldsymbol{m} \in \mathbb{N}^*$ *and* $\boldsymbol{u} \in \boldsymbol{\mathcal{C}^m_{-1}}$ *and* $\boldsymbol{\Gamma(.)}$ *denotes the Gamma func-*

tion.The Caputo fractional derivative operator of order ρ *is given as :*

$$\begin{cases} \mathcal{D}_t^{\rho} u(x,t) = \dfrac{\partial^{\rho} u(x,t)}{\partial t^{\rho}} = \dfrac{1}{\Gamma(m-\rho)} \displaystyle\int_0^t (t-s)^{m-\rho-1} \dfrac{\partial^m u(x,s)}{\partial s^m} ds, \quad m-1 < \rho < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, \rho = m. \end{cases} \tag{1.2}$$

1.2 Tools mathematics

1.2.1 Kharrat Toma transform

In 2020, Kharrat and Toma proposed a new integral transform namely Kharrata Toma transform, to solve the initial and boundary value problems represented as ordinary differential equations [2]. This transformation has been applied to solve initial value problem[2], partial differential equations and nonlinear fractional differential equations FDE's [13], the transformation proved to be very effective.

Definition 1.3 *[8]. The function f is said that has exponential order on every finite interval in* $[0, +\infty)$ *if there exist a positive number that:*

$$|f(t)| \leq M e^{\alpha t}, \quad M > 0, \quad \alpha > 0, \quad \forall t > 0. \tag{1.3}$$

Definition 1.4 *[8]. Let the function f is piece-wise continuous on* $[0, +\infty)$ *, the Kharrat Toma transform f is given by :*

$$B\Big[f(t)\Big] = G(s) = s^3 \int_0^{\infty} f(t) e^{-t/s^2} dt, t > 0. \tag{1.4}$$

Here s denotes the transform variable

Definition 1.5 *[8].*

If the function f is piecewise continuous on $[0, +\infty)$*and has exponential order, then will be the inverse of the Kharrat Toma transform defined as::*

$$f(t) = B^{-1}\Big[G(s)\Big] = B^{-1}\left[s^3 \int_0^{\infty} f(t) e^{-t/s^2} dt\right]. \tag{1.5}$$

Theorem 1.1 *[8]. [sufficient condition for existence of a Kharrat-Toma transform].* *The Kharrat-Toma transform exist if the function f has exponential order and* $\int_0^b |f(t)|dt$ *exist for any* $b > 0$.

Kharrat-Toma transform of some functions

[8, 13].

$f(t) = B^{-1}\left[G(s)\right]$	$B\left[f(t)\right]$
t	s^7
1	s^5
$\sin(kt)$	$\frac{ks^7}{1+k^2s^4}$
$sh(kt)$	$\frac{ks^7}{1-k^2s^4}$
$\cos(kt)$	$\frac{s^5}{1+k^2s^4}$
$\cosh(kt)$	$\frac{s^5}{1-k^2s^4}$
t^n	$s^{2n+5}n!$
$\frac{t^n}{\Gamma(n+1)}$	s^{2n+5}

Theorem 1.2 (linearity) *[8]. The Kharrata-Toma transform is a linear operator ; If* $c_1, c_2, ...c_n$ *non-zero constants and* $B\left[f_1(t)\right] = G_1(s), ..., B\left[f_n(t)\right] = G_n(s)$ *then*

$$B\left[\sum_{i=1}^{n} c_i f_i(t)\right] = \sum_{i=1}^{n} c_i B\left[f_i(t)\right]. \tag{1.6}$$

Theorem 1.3 (translation) *[8].*

Let $B\left[f(t)\right] = G(s)$ *the Kharrat Toma transform of the function f and* $\alpha > 0$*, then*

$$B\left[f(\alpha t)\right] = \frac{1}{\alpha^2\sqrt{\alpha}} G(\sqrt{\alpha}s). \tag{1.7}$$

Theorem 1.4 (convolution) *[8]. Let* $B\left[f(t)\right] = M(s), B\left[g(t)\right] = N(s)$ *the*

Kharrat Toma transform of the function f and the function g respectively

$$B\Big[f(t) * g(t)\Big] = \frac{1}{s^3} M(s) N(s). \tag{1.8}$$

Theorem 1.5 (Kharrat-Toma Transform of Derivatives) *[8].*

$$\begin{aligned}
B\Big[f'(t)\Big] &= \frac{1}{s^2} G(s) - s^3 f(0), \\
B\Big[f''(t)\Big] &= \frac{1}{s^4} G(s) - s f(0) - s^3 f'(0), \\
B\Big[f^n(t)\Big] &= \frac{1}{s^{2n}} G(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^{(k)}(0), \quad n \geq 1.
\end{aligned}$$

Theorem 1.6 (Kharrat-Toma Transform of $t^n f(t), n \geq 1$) *[8]*
If $B\Big[f(t)\Big] = G(s)$ *then,*

$$\begin{aligned}
B\Big[t f(t)\Big] &= \frac{s^3}{2} \frac{dG(s)}{ds} - \frac{3}{2} s^2 G(s), \\
B\Big[t^2 f(t)\Big] &= \frac{s^6}{4} \frac{d^2 G(s)}{ds^2} - \frac{3}{4} s^5 \frac{dG(s)}{ds} - \frac{3}{4} s^4 G(s)
\end{aligned}$$

Theorem 1.7 (Kharrat-Toma Transform of Caputo fractional derivative) *[13]. Let* $n \in \mathbb{N}$, $\alpha > 0$ *such that* $n - 1 < \alpha \leq n$, *and* $B\Big[f(t)\Big] = G(s)$ *,the Kharrat-Toma Transform of Caputo fractional derivative of function f of order* α *is given by*

$$B\Big[\mathcal{D}_t^\alpha f(t)\Big] = s^{-2\alpha} G(s) - \sum_{k=0}^{n-1} s^{2k-2\alpha+5} f^{(k)}(0). \tag{1.9}$$

1.2.2 Laplace transform

Definition 1.6 *[3]. Let F be a continous function defined fo all* $t \geq 0$*,the Laplace transform of F , is defined by*

$$\mathcal{L}\Big[F(t)\Big] = f(s) = \int_0^\infty F(t) e^{-st} dt. \tag{1.10}$$

Where s is a complex variable

Definition 1.7 *[3]. The inverse Laplace transform is given as*

$$\mathcal{L}^{-1}\Big[f(s)\Big] = F(t). \tag{1.11}$$

Theorem 1.8 *[3]. The important properties of Laplace transform and it's inverse that will be used in this paper are*

1. *If* $F_1(t)$ *and* $F_2(t)$ *are two functions whose Laplace transform exists, then*
   - $\mathcal{L}\Big[aF_1(t) + bF_2(t)\Big] = a\mathcal{L}\Big[F_1(t)\Big] + b\mathcal{L}\Big[F_2(t)\Big],$
   - $\mathcal{L}(t^\alpha) = \Gamma(\alpha + 1)s^{-\alpha-1}, \quad \alpha > 0,$
   - $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}},$ *n a positive integer .*
2. *The inverse Laplace transform is linear, i.e*

$$\begin{aligned} \mathcal{L}^{-1}\Big[af_1(s) + bf_2(s)\Big] &= a\mathcal{L}^{-1}\Big[f_1(s)\Big] + b\mathcal{L}^{-1}\Big[f_2(s)\Big], \\ \mathcal{L}^{-1}\left(\frac{1}{s^\alpha}\right) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0. \end{aligned} \tag{1.12}$$

Definition 1.8 *[3]. The Laplace transform of the Caputo derivative is*

$$\mathcal{L}\Big[D_t^\alpha u(x,t)\Big] = s^\alpha u(x,s) - \sum_{i=0}^{n-1} u^{(i)}(x,0^+)s^{\alpha-1-i}, \quad n-1 < \alpha \leq n. \tag{1.13}$$

CHAPTER 2

ANALYTIC SOLUTION OF KK EQUATION

2.1 Introduction

Many researchs have been introduced to find the analytical or semi analytical solution of the fractional order kaup kupershmidt equations such as, the new iteration transformation technique; this method combines the Yang transform and the new iteration technique [1], the iterative method with Elzaki transform see [7], the homotopy perturbation method (HPM) to obtain the soliton solution of the KK equation [20] and implement the natural decomposition method with the aid of two different fractional derivatives, namely the Atangana Baleanu derivative in Caputo manner (ABC) and Caputo Fabrizio (CF), see [18] to view more details. In this chapter we use the iterative method combines with a new integral transform : Kharrat Toma transform to find the solution of the fractional kaup kupershmidt equation .

2.2 The General Discussion of the Proposed Method

Consider the fractional partial differential equation:

$$\boldsymbol{\mathcal{D}_t^{\rho} u(x,t) + Mu(x,t) + Nu(x,t) = h(x,t), \quad n-1 < \rho \leq n,} \tag{2.1}$$

where $\boldsymbol{n} \in \mathbb{N}$, M is linear and N nonlinear terms and h is a source term. With the initial condition

$$u^k(x,0) = g_k(x), \quad k = 0,1,2,...,n-1 \quad . \tag{2.2}$$

Using the Kharrat Toma transformation of Eq (2.1), we obtain as:

$$B[\mathcal{D}_t^\rho u(x,t)] + B[Mu(x,t) + Nu(x,t)] = B[h(x,t)].$$

Applying the differentiation property given as

$$B[u(x,t)] = s^{2\rho} \sum_{k=0}^{m} s^{2k-2\rho+5} u^k(x,0) + s^{2\rho} B[h(x,t)] - s^{2\rho} B[Mu(x,t) + Nu(x,t)], \tag{2.3}$$

Using the inverse Kharrat Toma of equation (2.3), we have

$$u(x,t) = B^{-1}\Bigg[\bigg(s^{2\rho} \sum_{k=0}^{m} s^{2k-2\rho+5} u^k(x,0) + s^{2\rho} B[h(x,t)]\bigg)\Bigg] - B^{-1}\Big[s^{2\rho} B[Mu(x,t) + Nu(x,t)]\Big]. \tag{2.4}$$

Through the iterative technique, we get

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{2.5}$$

M is a linear operator:

$$M\bigg(\sum_{m=0}^{\infty} u_m(x,t)\bigg) = \sum_{m=0}^{\infty} M(u_m(x,t)), \tag{2.6}$$

and N is the nonlinear operator

$$N\bigg(\sum_{m=0}^{\infty} u_m(x,t)\bigg) = u_0(x,t) + M\bigg(\sum_{k=0}^{m} u_k(x,t)\bigg) - N\bigg(\sum_{k=0}^{m} u_k(x,t)\bigg). \tag{2.7}$$

Substituting (2.6),(2.7), we get the following solution:

$$\begin{aligned} \sum_{m=0}^{\infty} u_m(x,t) &= B^{-1}\Bigg[\bigg(s^{2\rho} \sum_{k=0}^{m} s^{2k-2\rho+5} u^k(x,0) + s^{2\rho} B[h(x,t)]\bigg)\Bigg] \\ &\quad - B^{-1}\Bigg[s^{2\rho} B\Bigg[M\bigg(\sum_{k=0}^{m} u_k(x,t)\bigg) - N\bigg(\sum_{k=0}^{m} u_k(x,t)\bigg)\Bigg]\Bigg]. \end{aligned} \tag{2.8}$$

Applying the iterative method, we get

$$
\begin{aligned}
u_0(x,t) &= B^{-1}\Bigg[\bigg(s^{2\rho}\sum_{k=0}^{m} s^{2k-2\rho+5}u^k(x,0) + s^{2\rho}B[h(x,t)]\bigg)\Bigg],\\
u_1(x,t) &= -B^{-1}\Big[s^{2\rho}B[Mu_0(x,t) + Nu_0(x,t)]\Big]\\
&\vdots\\
u_{m+1}(x,t) &= -B^{-1}\Bigg[s^{2\rho}B\Bigg[-M\bigg(\sum_{k=0}^{m} u_k(x,t)\bigg) - N\bigg(\sum_{k=0}^{m} u_k(x,t)\bigg)\Bigg]\Bigg], m \geq 1.
\end{aligned}
\tag{2.9}
$$

The equations (2.1) and (2.2) provide the series form solution which is defined as

$$
u(x,t) \cong u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_m(x,t), m \in \mathbb{N}. \tag{2.10}
$$

2.3 Main Results

In this part, we investigate the solution of the fractional order kaup kupershmidt equations by using a combination of iterative method and Kharrat Toma transform for the first time.This method is developed for solving fractional differential equation numerically in [13], Ordinary Differential Equations with Initial Boundary Conditions[16], analysis of Cauchy Problems and Diffusion Equations Associated with the Hilfer Prabhakar Fractional Derivative[4], the results proves that this method is effective and very much easier than other integral transforms.

2.3.1 Example 1

Consider the following fractional Kaup Kupershmidt equation which is given as

$$
\mathcal{D}_t^{\rho} u(x,t) = 5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x + u_{xxxxx}, \tag{2.11}
$$

with the initial condition

$$u(x,0) = -2k^2 + \frac{24k^2}{1+e^{kx}} - \frac{24k^2}{(1+e^{kx})^2}. \tag{2.12}$$

Using the Kharrat Toma transform to Eq.(2.11) , we get

$$\begin{aligned}
&B\Big[\mathcal{D}_t^{\rho} u(x,t)\Big] + B\Big[5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x + u_{xxxxx}\Big] = 0,\\
&\frac{1}{s^{2\rho}}B\Big[u(x,t)\Big] = s^{5-2\rho}u(x,0) + B\Big[5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x + u_{xxxxx}\Big],\\
&B\Big[u(x,t)\Big] = s^5 u(x,0) + s^{2\rho}B\Big[5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x + u_{xxxxx}\Big].
\end{aligned} \tag{2.13}$$

Applying the inverse Toma transform of Eq.(2.13), we have

$$u(x,t) = B^{-1}\Big[s^5 u(x,0)\Big] + B^{-1}\Big[s^{2\rho}B\Big[5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x + u_{xxxxx}\Big]\Big]. \tag{2.14}$$

Now, by applying the proposed semi analytical technique, and by use the MAPLE , we obtain

$$\begin{aligned}
u_0(x,t) &= -2k^2 + \frac{24k^2}{1+e^{kx}} - \frac{24k^2}{(1+e^{kx})^2},\\
u_1(x,t) &= B^{-1}\Big[s^{2\rho}B\Big[5u_{(0)}u_{(0)xxx} + \frac{25}{2}u_{(0)x}u_{(0)xx} + 5u_{(0)}^2 u_{(0)x} + u_{(0)xxxxx}\Big]\Big],\\
u_1(x,t) &= -\left(\frac{264e^{kx}(-1+e^{kx})k^7}{(1+e^{kx})^3}\right)\frac{t^{\rho}}{\Gamma(1+\rho)}\\
u_2(x,t) &= B^{-1}\Big[s^{2\rho}B\Big[5u_{(1)}u_{(1)xxx} + \frac{25}{2}u_{(1)x}u_{(1)xx} + 5u_{(1)}^2 u_{(1)x} + u_{(1)xxxxx}\Big]\Big],\\
u_2(x,t) &= \frac{2904e^{kx}(1-4e^{kx}+e^{2kx})k^{12}t^{2\rho}}{(1+e^{kx})^4\Gamma(1+2\rho)}\\
&\vdots\\
u_n(x,t) &= B^{-1}\Big[s^{2\rho}B\Big[5u_{(n)}u_{(n)xxx} + \frac{25}{2}u_{(n)x}u_{(n)xx} + 5u_{(n)}^2 u_{(n)x} + u_{(n)xxxxx}\Big]\Big].
\end{aligned}$$

The series form result is

$$\boldsymbol{u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_n(x,t).} \tag{2.15}$$

Therefore,we have

$$\begin{aligned} u(x,t) = &-2k^2 + \frac{24k^2}{1+e^{kx}} - \frac{24k^2}{(1+e^{kx})^2} - \left(\frac{264e^{kx}(-1+e^{(kx)}k^7}{(1+e^{kx})^3}\right)\frac{t^\rho}{\Gamma(1+\rho)} \\ &+ \frac{2904e^{kx}(1-4e^{kx}+e^{2kx})k^{12}t^{2\rho}}{(1+e^{kx})^4\Gamma(1+2\rho)} + \dots \end{aligned} \tag{2.16}$$

For $\boldsymbol{\rho = 1}$, the exact results of Eq.(2.11) are given by

$$u(x,t) = -2k^2 + \frac{24k^2}{(1+e^{kx+11k^5t})} - \frac{24k^2}{(1+e^{kx+11k^5t})^2}. \tag{2.17}$$

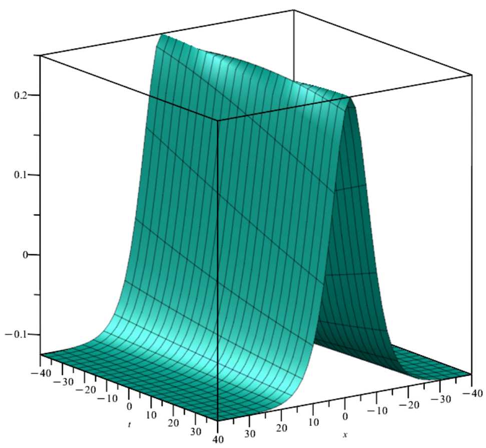

Figure 2.1: The exact solution of example 1, using iterative method with Kharrat Toma transform, by takin $\boldsymbol{\rho = 1}$, and k=0.25 .

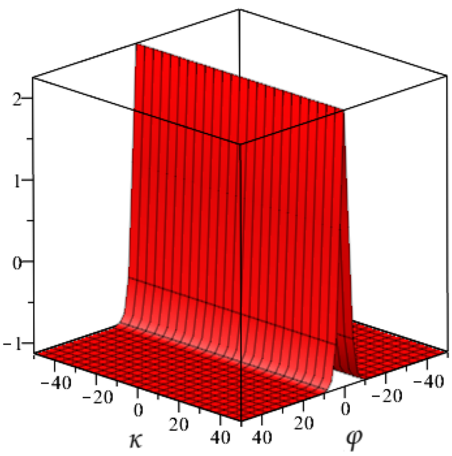

Figure 2.2: The exact solution of example 1, using another method in[18], by takin $\boldsymbol{\rho = 1}$, and k=0.25 .

2.3.2 Example 2

Consider the following fractional Kaup Kupershmidt equation which is given as

$$\mathcal{D}_t^{\rho} u(x,t) = -15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}, \tag{2.18}$$

with the initial condition

$$u(x,0) = \frac{1}{4} w^2 y^2 sech^2 \left(\frac{wxy}{2}\right) + \frac{w^2 y^2}{12}. \tag{2.19}$$

Using the Kharrat Toma transform to Eq.(2.18), we get

$$\begin{aligned}
&B\Big[\mathcal{D}_t^{\rho} u(x,t)\Big] + B\Big[-15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}\Big] = 0,\\
&\frac{1}{s^{2\rho}} B\Big[u(x,t)\Big] = s^{5-2\rho} u(x,0) + B\Big[-15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}\Big],\\
&B\Big[u(x,t)\Big] = s^5 u(x,0) + s^{2\rho} B\Big[-15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}\Big].
\end{aligned} \tag{2.20}$$

Applying the inverse Toma transformation to Eq.(2.20) , we have

$$u(x,t) = B^{-1}\Big[s^5 u(x,0)\Big] + B^{-1}\Big[s^{2\rho} B\Big[-15uu_{xxx} - 15pu_x u_{xx} + 45u^2 u_x + u_{xxxxx}\Big]\Big]. \tag{2.21}$$

Now, by applying the proposed semianalytical technique, and by use the MAPLE we get

$$\begin{aligned}
u_0(x,t) &= \frac{1}{4}w^2y^2sech^2\left(\frac{wxy}{2}\right)+\frac{w^2y^2}{12},\\
u_1(x,t) &= B^{-1}\Big[s^{2\rho}B\Big[-15u_{(0)}u_{(0)xxx}-15pu_{(0)x}u_{(0)xx}+45u_{(0)}^2u_{(0)x}+u_{(0)xxxxx}\Big]\Big],\\
u_1(x,t) &= \frac{w^7y^7}{512}\Big[3843+480p-4(209+60p)\cosh(wxy)\\
&+\cosh(2wxy)sech^6\left(\frac{wxy}{2}\right)\tanh\left(\frac{wxy}{2}\right)\Big]\frac{t^\rho}{\Gamma(1+\rho)},\\
u_2(x,t) &= B^{-1}\Big[s^{2\rho}B\Big[-15u_{(1)}u_{(1)xxx}-15pu_{(1)x}u_{(1)xx}+45u_{(1)}^2u_{(1)x}+u_{(1)xxxxx}\Big]\Big],\\
u_2(x,t) &= \frac{w^{12}y^{12}t^{2\rho}}{524288\Gamma(1+2\rho)}\Big[-733469760p-3947228724+6(148082560p\\
&+777305099+4358400p^2)\cosh(wxy)-20736000p^2\\
&-48(3850520p+18859301+124800p^2)\cosh(2wxy)+46313277\cosh(3wxy)\\
&+10287360p\cosh(3wxy)+345600p^2\cosh(3wxy)-305756\cosh(4wxy)\\
&-87360p\cosh(4wxy)+\cosh(5wxy)sech^{12}\left(\frac{wxy}{2}\right)\Big],\\
&\vdots\\
u_n(x,t) &= B^{-1}\Big[s^{2\rho}B\Big[-15u_{(n)}u_{(n)xxx}-15pu_{(n)x}u_{(n)xx}+45u_{(n)}^2u_{(n)x}+u_{(n)xxxxx}\Big]\Big].
\end{aligned}$$

The series form result is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+...+u_n(x,t). \tag{2.22}$$

Therefore, we have

$$u(x,t) = \frac{1}{4} w^2 y^2 sech^2\left(\frac{wxy}{2}\right) + \frac{w^2y^2}{12} + \frac{w^7y^7}{512}\Big[3843 + 480p - 4(209+60p)\cosh(wxy) + \cosh(2wxy) sech^6\left(\frac{wxy}{2}\right)\tanh\left(\frac{wxy}{2}\right)\Big]\frac{t^\rho}{\Gamma(1+\rho)} + \frac{w^{12}y^{12}t^{2\rho}}{524288\Gamma(1+2\rho)}\Big[-733469760p - 3947228724 + 6(148082560p + 777305099 + 4358400p^2)\cosh(wxy) - 20736000p^2 - 48(3850520p + 18859301 + 124800p^2)\cosh(2wxy) + 46313277\cosh(3wxy) + 10287360p\cosh(3wxy) + 345600p^2\cosh(3wxy) - 305756\cosh(4wxy) - 87360p\cosh(4wxy) + \cosh(5wxy)sech^{12}\left(\frac{wxy}{2}\right)\Big] + \dots \tag{2.23}$$

For $\rho = 1$, the exact results of(2.18) are given by

$$u(x,t) = \frac{1}{4} w^2 y^2 sech^2\left(\frac{y}{2}\left(\frac{-w^5(-8y^2n + 16n^2 + y^4)}{16\Gamma(1+\rho)}t^\rho + wx\right)\right) + \frac{w^2y^2}{12}. \tag{2.24}$$

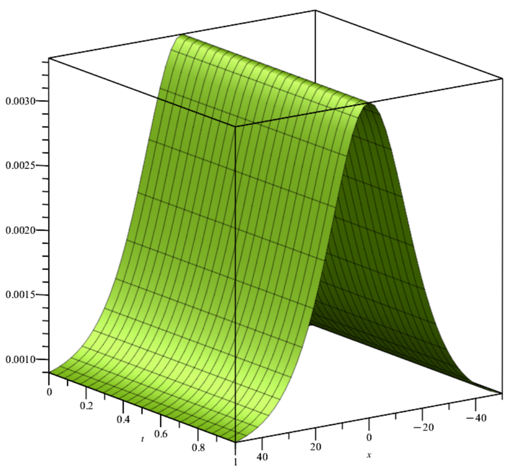

Figure 2.3: The exact solution of example 2, using iterative method with Kharrat Toma transform , by taking $\rho = 1$, and w=1,n=0,y=0.1

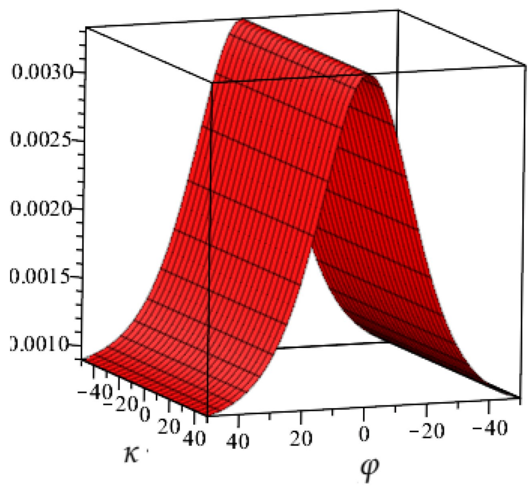

Figure 2.4: The exact solution of example 2, using another method in[18] , by taking $\rho = 1$, and w=1,n=0,y=0.1

2.4 Conclusion

In this chapter, we find the solution of the time fractional kaup kupershmidt equation using new iterative transform method , which are widely used to adress spatial effects in applied sciences. To validate the efficiency of the method, we use two examples, and compare our results with the results of an other method in [18]. This method produced a series of solutions that converge very rapidly in the mathematical model, with easy determinable component, and does not take much of time in the calculate (fast).

We conclude that the proposed method is effective and powerful, highly consistent and suitable for investigating a broad spectrum of fractional order nonlinear mathematical models, thereby enhancing our comprehension of highly nonlinear complex phenomena in related fields of science and engineering .

CHAPTER 3

APPROXIMATE SOLUTION OF KK EQUATION

3.1 Introduction

In the 1980, George Adomian introduced a powerful method to solve nonlinear differential equations. Since then, this method is known as the Adomian decomposition method (ADM) [6]. The solution procedure of this method is an effective approach to broad classes of linear, non-linear, ordinary, or partial differential equations [5], with important applications in different fields of applied mathematics, engineering, physics, biology [14], and chemical reactions. The technique is based on a decomposition of the solution of a nonlinear differential equation in a series functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function[6], provides the solution in a rapidly convergent series with computable terms .
In this chapter, our goal is to obtain the approximate solutions of the time fractional Kupershmidt equation, using the (LADM), and compare the results with the exact solution to prove that the (LADM) algoritm is suitable and efficient for this problem .
Numerous researchers have investigated the general fifth-order KdV equation in various contexts:

$$\boldsymbol{u_t + \omega u_{5x} + \alpha u u_{3x} + \beta u_x u_{2x} + \gamma u^2 u_x = 0.}$$

where $\boldsymbol{\alpha, \beta, \omega}$ and $\boldsymbol{\gamma}$ real constants.This category encompasses the generalized Kaup-Kupershmidt equation:

$$u_t + 20a^2bu_{5x} + 10abuu_{3x} + 25abu_xu_{2x} + bu^2u_x = 0,$$

as the constants $a \neq 0, b \neq 0$ take different values, while, we obtain our equation by taking $a = \frac{1}{10}, b = 5$,
in the form:

$$u_t - u_{5x} - 5uu_{3x} - \frac{25}{2}u_xu_{2x} - 5u^2u_x = 0. \tag{3.1}$$

3.2 The Adomian Decomposition Method Combined with Laplace Transform

The LADM, is essentially the Adomian decomposition method combined with the Laplace transform , the underlying idea of this technique is to identifie and separate the linear and nonlinear parts of a differential equation. Inverting and applying the highest order differential operator that is contained in the linear part of the equation [3] . At this point, the solution is proposed as a sum of an infinite number of components defined by a decomposition series of the form $u = \sum_{n=0}^{\infty} u_n$, then apply the Laplace transformation to the differential equation and express the nonlinear part as a series of Adomian polynomials. Finaly, create an iterative algorithm to find u_n that give the approximate solution of the equation.
Given a partial (or ordinary) differential equation: This part is from [3];

$$Fu(x,t) = h(x,t), \tag{3.2}$$

with initial condition

$$u(x,0) = f(x), \tag{3.3}$$

where F is a differential operator that could, in general, be nonlinear and therefore includes some linear and nonlinear terms. In general EQ could be written as

$$L_tu(x,t) = Ru(x,t) + Nu(x,t) + h(x,t), \tag{3.4}$$

where $L_t = \frac{\partial^\alpha}{\partial t^\alpha}$, $0 < \alpha \leq 1$, R is a linear operator that includes partial derivatives with respect to x , N is a nonlinear operator and h is a non-homogeneous term that is u -independent. The LADM consists of applying Laplace transform first on both sides of equation (3.3), obtaining

$$\mathcal{L}\Big[L_t u(x,t)\Big] = \mathcal{L}\Big[Ru(x,t) + Nu(x,t) + h(x,t)\Big]. \tag{3.5}$$

An equivalent expression to (3.5) is

$$s^{\alpha}u(x,s) - u(x,0)s^{\alpha-1} = \mathcal{L}\Big[Ru(x,t) + Nu(x,t) + h(x,t)\Big]. \tag{3.6}$$

In the homogeneous case, $h(x,t) = 0$, we have

$$u(x,s) - \frac{f(x)}{s} = \frac{1}{s^{\alpha}}\mathcal{L}\Big[Ru(x,t) + Nu(x,t)\Big]. \tag{3.7}$$

Now, applying the inverse Laplace transform:

$$u(x,t) = f(x) + \mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}\Big[Ru(x,t) + Nu(x,t)\Big]\Big]. \tag{3.8}$$

The ADM method proposes a series solution $u(x,t)$ given by,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{3.9}$$

The nonlinear term Nu(x, t) is given by

$$Nu(x,t) = \sum_{n=0}^{\infty} P_n(u_0, u_1, u_2, ..., u_n), \tag{3.10}$$

where $\sum_{n=0}^{\infty} P_n$is the so-called Adomian polynomials sequence established, in general, give us term to term:

$$
\begin{aligned}
P_0 &= N(u_0) \\
P_1 &= u_1 N'(u_0) \\
P_2 &= u_2 N'(u_0) + \frac{1}{2} u_1^2 N''(u_0) \\
P_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N^{(3)}(u_0) \\
&\vdots
\end{aligned}
$$

using (3.8)and (3.9)into Eq.(3.7)we get

$$
\sum_{0}^{\infty} u_n(x,t) = f(x) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\Big[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} P_n(u_0,u_1,u_2,...,u_n)\Big]\right]. \tag{3.11}
$$

We deduce the following recurrence formulas

$$
\begin{cases}
u_0(x,t) = f(x) \\
u_{n+1}(x,t) = f(x) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\Big[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} P_n(u_0,u_1,u_2,...,u_(n)\Big]\right], n = 0,1,2...
\end{cases} \tag{3.12}
$$

Using (3.12) we can obtain an approximate solution of(3.4),using

$$
u(x,t) \approx \sum_{n=0}^{\infty} u_n(x,t), \quad \text{where} \quad \lim_{k\to+\infty}\sum_{n=0}^{k} u_n(x,t) = u(x,t). \tag{3.13}
$$

3.3 Main results

The LADM was used on the time fractional kaup kupershmidt in [3]. In this part we use the same method by taking an other example of time fractional kk equation .

3.3.1 The approximate solution of time-fractional Kupershmidt equation by LADM

The time-fractional Kupershmidt equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{\partial^5 u(x,t)}{\partial x^5} + 5u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} + \frac{25}{2}\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2} + 5u^2(x,t)\frac{\partial u(x,t)}{\partial x}, \tag{3.14}$$

with the initial conditions

$$u(x,0) = -2k^2 + \frac{24k^2}{1+e^{kx}} - \frac{24k^2}{(1+e^{kx})^2}, \tag{3.15}$$

where $0 < \alpha \leq 1$ and $\frac{\partial^\alpha}{\partial t^\alpha} = D_t^\alpha$ the derivatives in the sens of Caputo, comparing (3.14) with Eq. (3.3) we have that $h(x,t) = 0$, L_t and R becomes:

$$L_t = \frac{\partial^\alpha}{\partial t^\alpha}u = D_t^\alpha u, \quad (R) = \frac{\partial^5 u(x,t)}{\partial x^5} = u_{5x}(x,t). \tag{3.16}$$

The nonlinear term are given by

$$\begin{aligned} Nu &= 5u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} + \frac{25}{2}\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2} + 5u^2(x,t)\frac{\partial u(x,t)}{\partial x}. \\ &= 5u(x,t)u_{3x}(x,t) + \frac{25}{2}u_x(x,t)u_{2x}(x,t) + 5u^2(x,t)u_x(x,t). \end{aligned} \tag{3.17}$$

By using the LADM method we obtain recursively

$$\begin{cases} u_0(x,t) = f(x) \\ u_{n+1}(x,t) = f(x) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} P_n(u_0,u_1,u_2,...,u_n)\right]\right], n = 0,1,2... \end{cases} \tag{3.18}$$

From this, we will decompose the nonlinear terms into Adomian polynomials.

$$Nu = N_1 u + N_2 u + N_3 u = \sum_{n=0}^{\infty} P_n(u_0,u_1,u_2,...,u_n). \tag{3.19}$$

Let

$$N_1 u = 5u(x,t)u_{3x}(x,t) = 5\sum_{n=0}^{\infty} u_n \sum_{n=0}^{\infty} u_{n_{3x}} = \sum_{n=0}^{\infty} A_n(u_0,u_1,u_2,...,u_n),$$

$$N_2 u = \frac{25}{2} u_x(x,t) u_{2x}(x,t) = \frac{25}{2} \sum_{n=0}^{\infty} u_{n_x} \sum_{n=0}^{\infty} u_{n_{2x}} = \sum_{n=0}^{\infty} B_n(u_0, u_1, u_2, ..., u_n),$$

$$N_3 u = 5u^2(x,t) u_x(x,t) = 5 \left(\sum_{n=0}^{\infty} u_n \right)^2 \sum_{n=0}^{\infty} u_{n_x} = \sum_{n=0}^{\infty} C_n(u_0, u_1, u_2, ..., u_n). \quad (3.20)$$

Where $P_n = A_n + B_n + C_n$.

Using ADM , Eq (3.9)gives

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (3.21)$$

thus, the Adomian polynomials A_n are in the forms

$$A_0 = 5u_0 u_{0_{3x}}$$
$$A_1 = 5u_1 u_{0_{3x}} + 5u_0 u_{1_{3x}}$$
$$A_2 = 5u_2 u_{0_{3x}} + 5u_1 u_{1_{3x}} + 5u_0 u_{2_{3x}}$$
$$\vdots$$

$$B_0 = \frac{25}{2} u_{0_x} u_{0_{2x}}$$
$$B_1 = \frac{25}{2} u_1 u_{0_{2x}} + \frac{25}{2} u_{0_x} u_{1_{2x}}$$
$$B_2 = \frac{25}{2} u_{2_x} u_{0_{2x}} + \frac{25}{2} u_{1_x} u_{1_{2x}} + \frac{25}{2} u_{0_x} u_{2_{2x}}$$
$$\vdots$$

and

$$C_0 = 5u_0^2 u_{0_x}$$
$$C_1 = 5u_0^2 u_{1_x} + \frac{5}{4} u_1 u_0 u_{0_x}$$
$$C_2 = 5u_0^2 u_{2_x} + 5u_2 u_0 u_{0_x} + 5u_1^2 u_{0_x}$$
$$\vdots$$

Using the LADM method, we recursively obtain

$$\begin{aligned}
u_0(x,t) &= f(x),\\
u_1(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\Big[u_{0_{5x}} + A_0 + B_0 + C_0\Big]\right],\\
u_2(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\Big[u_{1_{5x}} + A_1 + B_1 + C_1\Big]\right],\\
u_3(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\Big[u_{2_{5x}} + A_2 + B_2 + C_2\Big]\right],\\
&\vdots\\
u_{n+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\Big[u_{n_{5x}} + A_n + B_n + C_n\Big]\right].
\end{aligned} \tag{3.22}$$

We used MAPLE for this calculation

$$\begin{aligned}
A_0 &= \frac{240k^7e^{6kx}}{(1+e^{kx})^7} - \frac{5040k^7e^{5kx}}{(1+e^{kx})^7} + \frac{29280k^7e^{4kx}}{(1+e^{kx})^7} - \frac{29280k^7e^{3kx}}{(1+e^{kx})^7} + \frac{5040k^7e^{2kx}}{(1+e^{kx})^7} - \frac{240k^7e^{2kx}}{(1+e^{kx})^7},\\
B_0 &= -\frac{7200k^7e^{5kx}}{(1+e^{kx})^7} + \frac{3600k^7e^{5kx}}{(1+e^{kx})^7} - \frac{3600k^7e^{3kx}}{(1+e^{kx})^7}\frac{7200k^7e^{2kx}}{(1+e^{kx})^7},\\
C_0 &= -\frac{480k^7e^{6kx}}{(1+e^{kx})^7} + \frac{10080k^7e^{5kx}}{(1+e^{kx})^7} - \frac{58560k^7e^{4kx}}{(1+e^{kx})^7} + \frac{58560k^7e^{3kx}}{(1+e^{kx})^7} - \frac{10080k^7e^{2kx}}{(1+e^{kx})^7}\\
&\quad + \frac{480k^7e^{2kx}}{(1+e^{kx})^7}.
\end{aligned}$$

With the above, we have

$$u_0(x,t) = -2k^2 + \frac{24k^2}{1+e^{kx}} - \frac{24k^2}{(1+e^{kx})^2}. \tag{3.23}$$

$$\begin{aligned}
u_1(x,t) = &-\frac{264k^7t^\alpha e^{6kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} - \frac{792k^7t^\alpha e^{5kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} - \frac{528k^7t^\alpha e^{4kx}}{(1+e^{kx})^7\Gamma(\alpha+1)}\\
&+\frac{528k^7t^\alpha e^{3kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} + \frac{792k^7t^\alpha e^{2kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} - \frac{264k^7t^\alpha e^{6kx}}{(1+e^{kx})^7\Gamma(\alpha+1)}.
\end{aligned} \tag{3.24}$$

$$
\begin{aligned}
A_1 = {} & \frac{31680k^{12}e^{kx}}{(1+e^{kx})^8} - \frac{31680k^{12}e^{kx}}{(1+e^{kx})^9} + \frac{889680k^{12}e^{2kx}}{(1+e^{kx})^8} - \frac{2640k^{12}e^{kx}}{(1+e^{kx})^7} - \frac{2346960k^{12}e^{7kx}}{(1+e^{kx})^8} \\
& - \frac{1330560k^{12}e^{9kx}}{(1+e^{kx})^{10}} - \frac{2687520k^{12}e^{4kx}}{(1+e^{kx})^8} + \frac{570240k^{12}e^{6kx}}{(1+e^{kx})^7} + \frac{990000k^{12}e^{5kx}}{(1+e^{kx})^7} \\
& + \frac{2764080k^{12}e^{3kx}}{(1+e^{kx})^8} - \frac{63360k^{12}e^{2kx}}{(1+e^{kx})^7} - \frac{11887920k^{12}e^{6kx}}{(1+e^{kx})^8} + \frac{337920k^{12}e^{4kx}}{(1+e^{kx})^7} \\
& + \frac{16156800k^{12}e^{9kx}}{(1+e^{kx})^{11}} - \frac{41437440k^{12}e^{8kx}}{(1+e^{kx})^{10}} + \frac{36178560k^{12}e^{7kx}}{(1+e^{kx})^9} - \frac{16727040k^{12}e^{9kx}}{(1+e^{kx})^{12}} \\
& - \frac{127258560k^{12}e^{7kx}}{(1+e^{kx})^{11}} + \frac{71913600k^{12}e^{6kx}}{(1+e^{kx})^9} - \frac{50181120k^{12}e^{8kx}}{(1+e^{kx})^{12}} + \frac{129824640k^{12}e^{7kx}}{(1+e^{kx})^{11}} \\
& + \frac{35449920k^{12}e^{5kx}}{(1+e^{kx})^9} - \frac{33454080k^{12}e^{7kx}}{(1+e^{kx})^{12}} + \frac{22049280k^{12}e^{6kx}}{(1+e^{kx})^{11}} + \frac{19768320k^{12}e^{5kx}}{(1+e^{kx})^{10}} \\
& + \frac{33454080k^{12}e^{6kx}}{(1+e^{kx})^{12}} - \frac{92188800k^{12}e^{5kx}}{(1+e^{kx})^{11}} + \frac{66337920k^{12}e^{4kx}}{(1+e^{kx})^{10}} - \frac{15333120k^{12}e^{3kx}}{(1+e^{kx})^9} \\
& - \frac{65767680k^{12}e^{4kx}}{(1+e^{kx})^{11}} + \frac{23664960k^{12}e^{3kx}}{(1+e^{kx})^{10}} - \frac{2344320k^{12}e^{2kx}}{(1+e^{kx})^9} + \frac{16727040k^{12}e^{4kx}}{(1+e^{kx})^{12}} \\
& + \frac{1615680k^{12}e^{2kx}}{(1+e^{kx})^{10}} + \frac{3104640k^{12}e^{8kx}}{(1+e^{kx})^7} + \frac{86296320k^{12}e^{8kx}}{(1+e^{kx})^{11}} - \frac{111672000k^{12}e^{6kx}}{(1+e^{kx})^{10}} \\
& + \frac{50181120k^{12}e^{5kx}}{(1+e^{kx})^{12}} - \frac{11214720k^{12}e^{3kx}}{(1+e^{kx})^{11}} - \frac{16410240k^{12}e^{4kx}}{(1+e^{kx})^9} - \frac{142560k^{12}e^{3}}{(1+e^{kx})^7} \\
& - \frac{14134560k^{12}e^{5kx}}{(1+e^{kx})^8},
\end{aligned}
\tag{3.25}
$$

$$\begin{aligned}
B_1 = {} & \frac{5544000k^{12}e^{9kx}}{(1+e^{kx})^{11}} - \frac{8712000k^{12}e^{8kx}}{(1+e^{kx})^{10}} + \frac{3326400k^{12}e^{7kx}}{(1+e^{kx})^{9}} - \frac{12196800k^{12}e^{9kx}}{(1+e^{kx})^{12}} \\
& - \frac{28987200k^{12}e^{7kx}}{(1+e^{kx})^{10}} + \frac{7128000k^{12}e^{6kx}}{(1+e^{kx})^{9}} - \frac{36590400k^{12}e^{8kx}}{(1+e^{kx})^{12}} + \frac{58132800k^{12}e^{kx}}{(1+e^{kx})^{11}} \\
& + \frac{3168000k^{12}e^{5kx}}{(1+e^{kx})^{9}} - \frac{24393600k^{12}e^{7kx}}{(1+e^{kx})^{12}} + \frac{14889600k^{12}e^{6kx}}{(1+e^{kx})^{11}} + \frac{3484800k^{12}e^{5kx}}{(1+e^{kx})^{10}} \\
& + \frac{24393600k^{12}e^{6kx}}{(1+e^{kx})^{12}} - \frac{37224000k^{12}e^{5kx}}{(1+e^{kx})^{11}} + \frac{14731200k^{12}e^{4kx}}{(1+e^{kx})^{10}} - \frac{1425600k^{12}e^{3kx}}{(1+e^{kx})^{9}} \\
& - \frac{28353600k^{12}e^{4kx}}{(1+e^{kx})^{11}} + \frac{5227200k^{12}e^{3kx}}{(1+e^{kx})^{10}} - \frac{158400k^{12}e^{2kx}}{(1+e^{kx})^{9}} + \frac{12196800k^{12}e^{4kx}}{(1+e^{kx})^{9}} \\
& + \frac{316800k^{12}e^{2kx}}{(1+e^{kx})^{10}} + \frac{35006400k^{12}e^{8kx}}{(1+e^{kx})^{11}} - \frac{26611200k^{12}e^{6kx}}{(1+e^{kx})^{10}} - \frac{1900800k^{12}e^{4kx}}{(1+e^{kx})^{9}} \\
& - \frac{4910400k^{12}e^{3kx}}{(1+e^{kx})^{11}} + \frac{36590400k^{12}e^{5kx}}{(1+e^{kx})^{12}},
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
C_1 = {} & -\frac{126720k^{12}e^{kx}}{(1+e^{kx})^{8}} + \frac{887040k^{12}e^{kx}}{(1+e^{kx})^{9}} - \frac{797280k^{12}e^{2kx}}{(1+e^{kx})^{8}} + \frac{5280k^{12}e^{kx}}{(1+e^{kx})^{7}} \\
& + \frac{17487360k^{12}e^{2kx}}{(1+e^{kx})^{11}} + \frac{760320k^{12}e^{kx}}{(1+e^{kx})^{11}} - \frac{6842880k^{12}e^{2kx}}{(1+e^{kx})^{12}} + \frac{36960k^{12}e^{7kx}}{(1+e^{kx})^{8}} \\
& - \frac{31680k^{12}e^{6kx}}{(1+e^{kx})^{7}} - \frac{79200k^{12}e^{5kx}}{(1+e^{kx})^{7}} + \frac{31680k^{12}e^{3kx}}{(1+e^{kx})^{7}} - \frac{871200k^{12}e^{3kx}}{(1+e^{kx})^{8}} \\
& + \frac{871200k^{12}e^{6kx}}{(1+e^{kx})^{8}} - \frac{42240k^{12}e^{4kx}}{(1+e^{kx})^{7}} + \frac{1974720k^{12}e^{5kx}}{(1+e^{kx})^{8}} - \frac{950400k^{12}e^{7kx}}{(1+e^{kx})^{9}} \\
& - \frac{8173440k^{12}e^{6kx}}{(1+e^{kx})^{9}} - \frac{12925440k^{12}e^{7kx}}{(1+e^{kx})^{11}} + \frac{30412800k^{12}e^{6kx}}{(1+e^{kx})^{10}} - \frac{15206400k^{12}e^{5kx}}{(1+e^{kx})^{9}} \\
& - \frac{43338240k^{12}e^{6kx}}{(1+e^{kx})^{11}} + \frac{37002240k^{12}e^{5kx}}{(1+e^{kx})^{10}} - \frac{5195520k^{12}e^{4kx}}{(1+e^{kx})^{9}} + \frac{20528640k^{12}e^{6kx}}{(1+e^{kx})^{12}} \\
& - \frac{2027520k^{12}e^{4kx}}{(1+e^{kx})^{10}} + \frac{8173440k^{12}e^{3kx}}{(1+e^{kx})^{9}} + \frac{13685760k^{12}e^{5kx}}{(1+e^{kx})^{12}} + \frac{19768320k^{12}e^{4kx}}{(1+e^{kx})^{11}} \\
& + \frac{6272640k^{12}e^{2kx}}{(1+e^{kx})^{9}} - \frac{13685760k^{12}e^{4kx}}{(1+e^{kx})^{12}} + \frac{43338240k^{12}e^{4kx}}{(1+e^{kx})^{11}} - \frac{16220160k^{12}e^{2kx}}{(1+e^{kx})^{10}} \\
& + \frac{6842880k^{12}e^{7kx}}{(1+e^{kx})^{12}} - \frac{37255680k^{12}e^{5kx}}{(1+e^{kx})^{11}} - \frac{20528640k^{12}e^{3kx}}{(1+e^{kx})^{12}} - \frac{30412800k^{12}e^{3kx}}{(1+e^{kx})^{10}} \\
& + \frac{939840k^{12}e^{4kx}}{(1+e^{kx})^{8}} + \frac{7096320k^{12}e^{7kx}}{(1+e^{kx})^{10}} + \frac{31680k^{12}e^{2kx}}{(1+e^{kx})^{7}} - \frac{1520640k^{12}e^{kx}}{(1+e^{kx})^{10}}.
\end{aligned} \tag{3.27}$$

$$u_2 = \frac{2904k^{12}e^{7kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{63360k^{12}e^{6kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} - \frac{786456k^{12}e^{5kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{1347456k^{12}e^{4kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} - \frac{786456k^{12}e^{3kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{63360k^{12}e^{2kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{2904k^{12}e^{kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)}. \tag{3.28}$$

The approximate solution of time-fractional Kupershmidt equation (3.14) with the first three terms is:

$$u(x,t) = \frac{2904k^{12}e^{7kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{63360k^{12}e^{6kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} - \frac{786456k^{12}e^{5kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{1347456k^{12}e^{4kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} - \frac{786456k^{12}e^{3kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{63360k^{12}e^{2kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} + \frac{2904k^{12}e^{kx}t^{2\alpha}}{(1+e^{kx})^8\Gamma(2\alpha+1)} - \frac{264k^7t^\alpha e^{6kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} - \frac{792k^7t^\alpha e^{5kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} - \frac{528k^7t^\alpha e^{4kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} + \frac{528k^7t^\alpha e^{3kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} + \frac{792k^7t^\alpha e^{2kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} + \frac{264k^7t^\alpha e^{kx}}{(1+e^{kx})^7\Gamma(\alpha+1)} - 2k^2 + \frac{24k^2}{1+e^{kx}} - \frac{24k^2}{(1+e^{kx})^2}. \tag{3.29}$$

Set $u(x,t) = u_\alpha(x,t)$ and take in particular $k = 0.25$

$$u_1(x,t) = -0.1250 + \frac{1.5000}{1+e^{0.25x}} - \frac{0.01611328125e^{1.50x}t}{(1+e^{0.25x})^7} - \frac{0.04833984375e^{1.25x}t}{(1+e^{0.25x})^7} - \frac{0.03222656250e^{1.00x}t}{(1+e^{0.25x})^7} + \frac{0.03222656250e^{0.750x}t}{(1+e^{0.25x})^7} + \frac{0.04833984375e^{0.5x}t}{(1+e^{0.25x})^7} + \frac{0.01611328125e^{0.25x}t}{(1+e^{0.25x})^7} + \frac{0.00008654594422e^{1.75x}t^2}{(1+e^{0.25x})^8} + \frac{0.001888275147e^{1.50x}t^2}{(1+e^{0.25x})^8} - \frac{0.02343821526e^{1.25x}t^2}{(1+e^{0.25x})^8} + \frac{0.04015731812e^{x}t^2}{(1+e^{0.25x})^8} - \frac{0.02343821526e^{0.75x}t^2}{(1+e^{0.25x})^8} + \frac{0.001888275147e^{0.50x}t^2}{(1+e^{0.25x})^8} + \frac{0.00008654594422e^{0.25x}t^2}{(1+e^{0.25x})^8} - \frac{1.5000}{(1+e^{0.25x})^2}, \tag{3.30}$$

$$
\begin{aligned}
u_{\frac{1}{2}}(x,t) = & -0.1250 + \frac{1.5000}{1+e^{0.25x}} - \frac{1.5000}{(1+e^{0.25x})^2} - \frac{0.01818189088e^{1.50x}\sqrt{t}}{(1+e^{0.25x})^7} \\
& - \frac{0.05454567263e^{1.25x}\sqrt{t}}{(1+e^{0.25x})^7} - \frac{0.03636378175e^{1.00x}\sqrt{t}}{(1+e^{0.25x})^7} + \frac{0.03636378175e^{0.75x}\sqrt{t}}{(1+e^{0.25x})^7} \\
& + \frac{0.01818189088e^{0.25x}\sqrt{t}}{(1+e^{0.25x})^7} + \frac{0.000173071884e^{1.75x}t}{(1+e^{0.25x})^8} + \frac{0.003776550293e^{1.5x}t}{(1+e^{0.25x})^8} \\
& + \frac{0.08031463624e^{x}t}{(1+e^{0.25x})^8} - \frac{0.04687643052e^{0.75x}t}{(1+e^{0.25x})^8} + \frac{0.003776550293e^{0.5x}t}{(1+e^{0.25x})^8} \\
& + \frac{0.05454567263e^{0.50x}\sqrt{t}}{(1+e^{0.25x})^7} + \frac{0.0001730918884e^{0.25x}t}{(1+e^{0.25x})^8} - \frac{0.04687643052e^{1.25x}t}{(1+e^{0.25x})^8},
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
u_{\frac{3}{4}}(x,t) = & -0.1250 + \frac{1.5000}{1+e^{0.25x}} - \frac{1.5000}{(1+e^{0.25x})^2} - \frac{0.01753230143e^{1.50x}t^{0.75}}{(1+e^{0.25x})^7} \\
& - \frac{0.05259690428e^{1.25x}t^{0.75}}{(1+e^{0.25x})^7} - \frac{0.03506460286e^{1.00x}t^{0.75}}{(1+e^{0.25x})^7} \\
& + \frac{0.01753230143e^{0.25x}t^{0.75}}{(1+e^{0.25x})^7} + \frac{0.0001302088540e^{1.75x}t^{1.5}}{(1+e^{0.25x})^8} \\
& - \frac{0.03526292509e^{1.25x}t^{1.5}}{(1+e^{0.25x})^8} - \frac{0.06041690824e^{x}t^{1.5}}{(1+e^{0.25x})^8} \\
& - \frac{0.03526292509e^{0.75x}t^{1.5}}{(1+e^{0.25x})^8} + \frac{0.002840920450e^{0.5x}t^{1.5}}{(1+e^{0.25x})^8} \\
& + \frac{0.0350646028e^{0.75x}t^{0.75}}{(1+e^{0.25x})^7} + \frac{0.002840920450e^{1.5x}t^{1.5}}{(1+e^{0.25x})^8} \\
& - \frac{0.0001302088540e^{0.25x}t^{1.5}}{(1+e^{0.25x})^8} + \frac{0.05259690428e^{1.25x}t^{0.75}}{(1+e^{0.25x})^7}.
\end{aligned}
\tag{3.32}
$$

	t=1			t=3			t=5		
x	U_{LADM}	U_{Exact}	Error	U_{LADM}	U_{Exact}	Error	U_{LADM}	U_{Exact}	Error
-5	0.136184	0.136203	0.0000187155	0.139124	0.139291	0.0001678617	0.141908	0.142373	0.0004646477
-4	0.171357	0.171378	0.0000211513	0.174091	0.174281	0.0001896680	0.176631	0.177156	0.0005249252
-3	0.203076	0.203094	0.0000185508	0.205397	0.205563	0.0001662636	0.207523	0.207983	0.0004599023
-2	0.228413	0.228424	0.0000113526	0.230110	0.230211	0.0001015993	0.231649	0.231930	0.0002806444
-1	0.244680	0.244684	$3.4375*\mathbf{10}^{(-6)}$	0.245558	0.245588	0.0000306171	0.246326	0.246410	0.0000841954
0	0.249989	0.249989	0	0.249902	0.249902	$1.71*\mathbf{10}^{(-8)}$	0.249729	0.249729	$1.298*\mathbf{10}^{(-7)}$
1	0.243694	0.243697	$3.4747*\mathbf{10}^{(-6)}$	0.242599	0.242630	0.0000316176	0.241394	0.241483	0.0000888255
2	0.226558	0.226569	0.0000114177	0.224545	0.224649	0.0001033521	0.222375	0.222664	0.0002887561
3	0.200559	0.200578	0.0000186296	0.197848	0.198016	0.0001683732	0.194941	0.195411	0.0004696670
4	0.168429	0.168450	0.0000212274	0.165307	0.165499	0.0001917341	0.161991	0.162526	0.0005344906
5	0.133090	0.133109	0.0000187803	0.129842	0.130012	0.0001695934	0.126439	0.126911	0.0004726596

Table 3.1: A comparison between approximate solution and exact solution in (2.17), for t=1, t=3, t=5

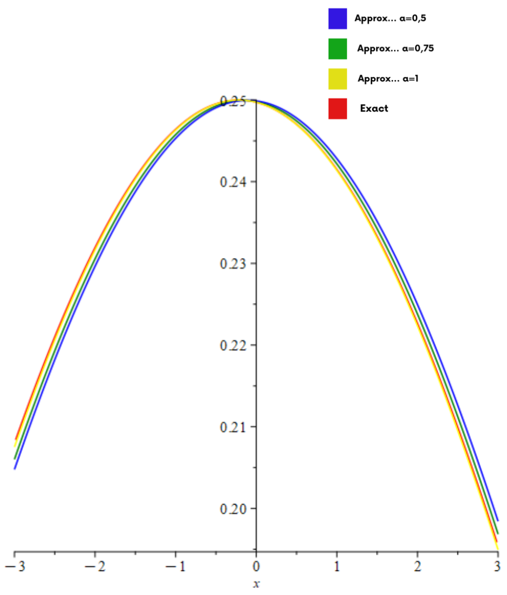

Figure 3.1: Exact and U_{LADM} for t=5,k=0.25 .

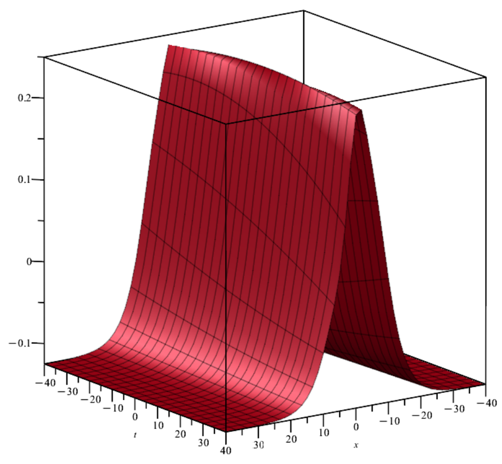

Figure 3.2: The approximate solution U_{LADM}, by takin $\rho = 1$ and k=0.25.

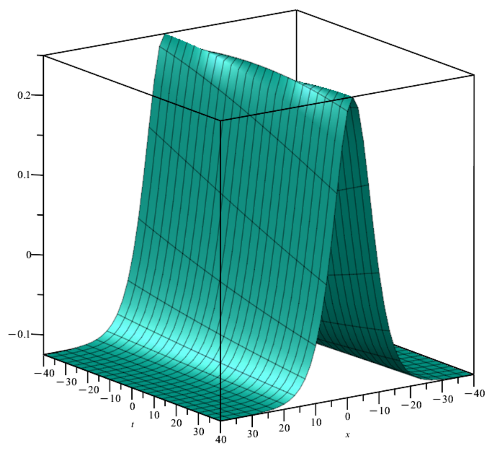

Figure 3.3: The exact solution, by takin $\rho = 1$ and k=0.25 .

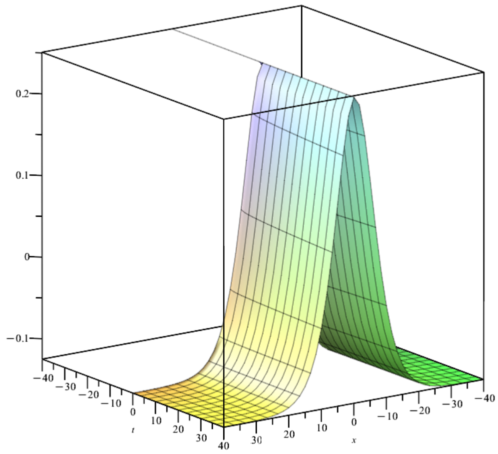

Figure 3.4: The approximate solution $\boldsymbol{U_{LADM}}$, by takin $\boldsymbol{\rho} = \dfrac{\mathbf{1}}{\mathbf{2}}$ and k=0.25 .

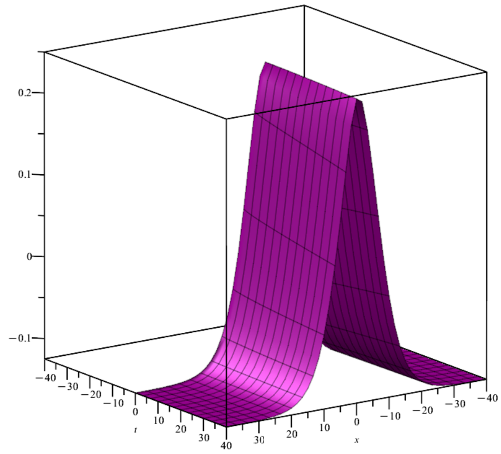

Figure 3.5: The exact solution, by takin $\boldsymbol{\rho} = \dfrac{\mathbf{3}}{\mathbf{4}}$ and k=0.25 .

3.4 Conclusion

In this chapter, we explore two phases in the study of the Kupershmidt equation. Initially, by employing the LADM method, which combines ADM with the Laplace transform, we derive approximate solutions for the time-fractional Kupershmidt equation. Subsequently, to validate the accuracy and efficiency of our approach, we compare our findings with the exact solution .

Our results indicate that the LADM method is a robust tool, capable of yielding high-quality approximate solutions for nonlinear partial differential equations through straightforward calculations and achieving convergence in just a few terms.

CONCLUSION

In this work, we explored three aspects related to the study of Kaup kupershmidt equation. First, we applied iterative method combined with new integral transform Kharrat Toma transform, to find the exact solution of the (KK) equation, which is used for the first time in this study. To validate the effectiveness and applicability of the proposed method, we examined two distinct cases of our equation, and we compared the obtained results with those of other method (NDM), demonstrating that our approach yields the same results but with simple calculations, less time and gave a series type of solution that exhibits accelerated convergence. Consequently, it is determined that this technique is sufficiently consistent and applicable for solving nonlinear fractional partial differential equations. Secondly, we used the LADM (Adomian decomposition method combined with laplace transform) to find the approximate solution of the time-fractional (KK), since this method has been previously applied to the equation, we use it to another variant of the equation. Thirdly, to demonstrate the reliability and the accuracy of our method, we compared the obtained results with the exact solution in Table1. Finally, we concluded that the LADM is powerful and effective tool for finding approximate solutions to complex nonlinear equations using simple calculations.

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