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NEW INTEGRAL TRANSFORMS: THEORY AND APPLICATIONS

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Abstract In this work we introduce a package of classical integral transforms such as those of Laplace and Fourier, as well as some recent integral transforms, with some mathematical applications in order to solve ordinary differentail equations, partail differentail equations and some fractional problems of Bagley-Trovik.

Keywords: Laplace Transform, Elzaki Transform, NE Transform, Fractional Calculus, Bagley-Trovik problem.

ملخص : في هذا العمل نقدم حزمة من التحويلات التكاملية الكلاسيكية كتحويل لابلاس وفوريي، كما نتطرق لبعض التحويلات التكاملية المستحدثة وتوظيف هذه التحويلات لحل مسائل رياضياتية مختلفة، وتشمل هذه الأخيرة معادلات تفاضلية إعتيادية، معادلات تفاضلية ذات مشتقات جزئية وكذلك معادلات تفاضلية ذات مشتقات ذات رتب كسرية.

الكلمات المفتاحية: تحويل لابلاس، تحويل الزاكي، تحويلNE ... إلخ، معادلات تفاضلية، معادلات تفاضلية ذات مشتقات ذات رتب كسرية.

Contents

Introduction:

The Laplace transform is an essential tool of applied mathematics and electrical engineering. This type of transform was developed in the 19^{th} century by the French mathematician Pierre-Simon Laplace, who lived from 1749 to 1827 [1]. In 1820, Laplace published several of his ideas on operational transforms and their practical applications in his book "Théorie des fonctions analytiques" ("Theory of Analytical Functions"). the laplace transform of a function $(f(x))$is represented by the symbol $(F(s))$and is calculated as follows: $F(s) = L[f(x)] = \int_0^{+\infty} e^{-sx} f(x)dx$. the Fourier transform is considered a powerful mathematical tool with widespread applications in mathematics, engineering, and various scientific fields. This notable mathematical tool was invented by the French mathematician and physicist Jean-Baptiste Joseph Fourier in the 19th century. Joseph Fourier was born in 1768 in Auxerre, France, and made significant contributions to the fields of mathematics and physics. The Fourier transform is a mathematical operation that allows the conversion of a signal, such as a sound or electrical signal, from the time domain to the frequency domain. The legacy of Joseph Fourier's contributions continues to influence modern technology and scientific research.

mathematicians faced numerous challenges in this evolution, including difficulties in handling non-standard conditions and integrals with complex calculations. However, by leveraging advanced mathematical models and high-performance computing techniques, mathematicians were able to develop new integral transforms that surpassed these challenges.among these new transforms are the Ezaki transform, the sumudu transform, and the new integral transform. these transforms were developed to address a variety of mathematical problems, including equations with difficult integrals and complex physical and engineering applications. these transforms provide more accurate and efficient solutions to these problems, contributing to the advancement of mathematical sciences and their applications in various fields.

Fractional differential equations play a significant role in describing various physical and engineering phenomena. Finding analytical solutions to these equations is of big importance, as it helps in understanding the fundamental dynamics of the complex systems they describe.

In fact, Fractional differential equations represent a powerful tool for modeling systems with nonlinear characteristics, allowing for the handling of complex dynamics through the use of non-integer order derivatives. This approach opens new horizons for understanding the behavior of unconventional systems that cannot be accurately described using traditional differential equations. The Bagley-Torvik equation is a prominent example of this type of equation, particularly appearing in applications related to ecology and financial mathematics. In this context, multiple analytical methods are employed to obtain exact solutions. One such method is the Sumudu Transform Method (STM), which has proven effective in solving fractional differential equations.

This work aims to provide the analytical solution for the Bagley-Torvik equation using the Sumudu Transform Method [7]. Fractional derivatives will be defined according to the Caputo definition, which is commonly used in this field due to its accuracy and ease of application. We will begin by presenting some fundamental principles of the Fractional derevatives, which particularly appears in Bagley-Torvik problem to achieve an exact solution. By applying the Sumudu Transform Method, precise analytical solutions can be achieved, contributing to both theoretical and practical understanding of these systems.

Chapter 1

CLASSICAL INTEGRAL TRANSFORM

The Laplace transform [1] is a mathematical tool used in the analysis of dynamic systems and their response to inputs in the frequency domain. it helps to transform differential equations into algebraic equations, making them easier to solve using linear algebra and Fourier analysis. the laplace transform is widely used in fields such as electrical engineering, mechanical engineering, and automatic control.the Fourier transform [1] is the extension of the idea to non-periodic functions by taking the limiting form of Fourier series when the fundamental period is made very large (infinite). Fourier transform finds its applications in astronomy, signal processing, linear time invariant (LTI) systems etc.

1.1 Laplace Transform

Definition 1.1.1 *The Laplace transform can be written as follows* [1]

$$L\{f(x)\} = \int_0^{+\infty} e^{-sx} f(x)dx = F(s) \tag{1.1}$$

where:

$f(x)$*: is the original function.*

s *: is a real number.*

$F(s)$: is the transformed function.

L : is Laplace symbol.

Definition 1.1.2 *(the inverse Laplace transform [1]) Let f be a function such that $L[f(s)] = F(s)$ we say that f is the inverse of the Laplace transform and we write*

$$L^{-1}[f(s)] = F(x) \tag{1.2}$$

where L^{-1} is the inverse operator Laplace transform.

1.1.1 Properties

The main properties that characterize the Laplace transform are:

1- Let $L\{f(x)\} = f(s)$, then $\quad if \quad L\{e^{ax}f(x)\} = f(s-a)$

2- $L\{x^n f(x)\} = (-1)^n \dfrac{dn}{ds^n} f(s)dn$

3- $L\{f(ax)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$

4- $L\left\{\displaystyle\int_0^x f(u)du\right\} = \dfrac{f(s)}{s}$

5- $\displaystyle\int_s^{+\infty} f(u)du = L\left\{\dfrac{f(s)}{s}\right\}$ if : $\displaystyle\lim_{x\to 0} \dfrac{F(x)}{x}$ exsists.

6- $L\{f(x) + g(x)\} = L\{f(x)\} + L\{g(x)\}$

7- $L\{cf(x)\} = cL\{f(x)\}$ where "c" is a constant.

8- $L\left\{\dfrac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0) - ... - sf^{n-2}(0) - f^{n-1}(0)$

9- $f(-t) = F(-s)$

10- $f(\dfrac{t}{a}) = aF(as), a > 0$

11- $f(t)e^{-at} = F(s+a)$

12- $e^{s_0 t} f(t) = F(s - s_0)$

13- $\displaystyle\int_0^t f(t)dt = \dfrac{1}{s}F(s)$

the following table gives Laplace transforms of some usual functions [1]:

$f(x)$	Laplace transforms
$f(x) = a$	$L[f(x)] = \frac{a}{s}$
$f(x) = sin(ax)$	$L[f(x)] = \frac{a}{s+a}$
$f(x) = cos(ax)$	$L[f(x)] = \frac{s}{s+as}$
$f(x) = e^{ax}$	$L[f(x)] = \frac{1}{s-a}$
$f(x) = x^n$	$L[f(x)] = \frac{n!}{s^{n+1}}$
$f(x) = cosh(ax)$	$L[f(x)] = \frac{s}{s^2-a^2}$
$f(x) = sinh(ax)$	$L[f(x)] = \frac{a}{s^2-a^2}$
$f(x) = \sqrt{x}$	$L[f(x)] = \left(\frac{1}{2s}\right)\sqrt{\frac{\pi}{s}}$
$f(x) = \frac{1}{\sqrt{x}}$	$L[f(x)] = \sqrt{\frac{\pi}{s}}$
$f(x) = xcos(ax)$	$L[f(x)] = \frac{(s^2-a^2)}{(s^2+a^2)^2}$
$f(x) = xsin(ax)$	$L[f(x)] = \frac{2as}{(s^2+a^2)^2}$
$f(x) = e^{ax}sin(bx)$	$L[f(x)] = \frac{b}{(s-a)^2+b^2}$
$f(x) = e^{ax}cos(bx)$	$L[f(x)] = \frac{s-a}{(s-a)^2+b^2}$
$f(x) = x^n e^{ax}$	$L[f(x)] = \frac{n!}{(s-a)^{n+1}}$

the folowing table gives inverse Laplace transforms of some functions:

Laplace transforms	Inverse Laplace transforms
$L[a] = \frac{a}{s}$	$L^{-1}\left[\frac{a}{s}\right] = a$
$L[e^{ax}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{ax}$
$L[sin(ax)] = \frac{a}{(s^2+a^2)}$	$L^{-1}\left[\frac{a}{(s^2+a^2)}\right] = sin(ax)$
$L[cos(ax)] = \frac{s}{(s^2+a^2)}$	$L^{-1}\left[\frac{s}{(s^2+a^2)}\right] = cos(ax)$
$L[x^n] = \frac{n!}{s^{n+1}}$	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = x^n$
$L[sinh(ax)] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{a}{s^2+a^2}\right] = sinh(ax)$
$L[cosh(ax)] = \frac{s}{s^2-a^2}$	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = cosh(ax)$
$L[x^n e^{ax}] = \frac{1}{(s-a)^{n+1}}$	$L^{-1}\left[\frac{(s^2-a^2)}{(s-a)^2}\right] = xcos(ax)$
$L[xcos(ax)] = \frac{(s^2-a^2)}{(s^2+a^2)^2}$	$L^{-1}\left[\frac{1}{(s-a)^{n+1}}\right] = x^n e^{ax}$
$L[xsin(ax)] = \frac{2as}{(s^2+a^2)^2}$	$L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = xcos(ax)$

1.1.2 Application

In this section, let us solve a second order ordinary differential equation using laplace transform

Example 1.1.1

$$\begin{cases} x''(t) + 4x'(t) + 3x(t) = 2e^{-2t} \\ x(0) = 0 \\ x'(0) = 1 \end{cases} \tag{1.3}$$

sour

Let's take the Laplace transform on both sides of the fundamental equation of the problem (1.3)*it is well known that the transform of the right-hand side is:*

$$L(2e^{2t}) = \frac{2}{p+2} \tag{1.4}$$

Let $X(p)$ denotes the Laplace transform of the function $x(t)$.using the linearity and initial conditions we get

$$L(x''(t)+4x'(t)+3x(t)) = L(x''(t))+4L(x'(t))+3L(x(t)) \quad (1.5)$$

$$= pL(x'(t)) - x'(0) + 4(pL(x(t)) - x(0)) + 3L(x(t)) \quad (1.6)$$

$$= p^2X(p) - px(0) - x'(0) + 4pX(p) - 4x(0) + 3X(p) \quad (1.7)$$

$$= (p^2+4p+3)X(p) - 1 \quad (1.8)$$

then $p^2+4p+3 = (p+1)(p+3)$, we obtain the following equation

$$(p^2+4p+3)X(p) - 1 = \frac{2}{p+2} \quad (1.9)$$

$$therefore \quad (1.10)$$

$$X(p) = \frac{2}{(p+2)(p+1)(p+3)} + \frac{1}{(p+1)(p+3)} \quad (1.11)$$

a decomposition into simple elements of each rational fraction in p gives:

$$\frac{2}{(p+2)(p+1)(p+3)} = -\frac{2}{p+2} + \frac{1}{p+1} + \frac{1}{p+3} \quad (1.12)$$

$$and \quad (1.13)$$

$$\frac{1}{(p+1)(p+3)} = \frac{\frac{1}{2}}{p+1} - \frac{\frac{1}{2}}{p+3} \quad (1.14)$$

thus

$$X(p) = \frac{2}{p+2} + \frac{\frac{3}{2}}{p+1} + \frac{\frac{1}{2}}{p+3} \quad (1.15)$$

it remains to determine the inverse transform of this expression, by linearity we know that:

$$X(t) = -2L^{-1}(\frac{1}{p+2}) + \frac{3}{2}L^{-1}(\frac{1}{p+1}) + \frac{1}{2}L^{-1}(\frac{1}{p+3}) \quad (1.16)$$

therefore the solution is finally given by,

$$x(t) = -2e^{-2t} + \frac{3}{2}e^{t} + \frac{1}{2}e^{3t}. \quad (1.17)$$

We easily verify that the solution found satisfies the initial conditions, and is solution of the differential equation(1.3)

1.2 Fourier transform

Fourier transform is the conversion of a signal from one time domain (mostly) to another (frequency) domain [1]. The Fourier transform can be written as

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-2\pi i x\xi} f(x)dx \tag{1.18}$$

$$= F[f(x)] \tag{1.19}$$

where ξ is a real number can obtain f through $\hat{f}$ using the inverse Fourier transform as follows

$$F^{-1}[\hat{f}(\xi)] = f(x) \tag{1.20}$$

$$= \int_{-\infty}^{+\infty} e^{2\pi i x\xi} \hat{f}(\xi)d\xi \tag{1.21}$$

Theorem 1.2.1 *(Modulation Theorem) If $\hat{f}(\xi)$ is Fourier transforms of $f(x)$, then*

$$F\{f(x)cosax\} = \frac{1}{2}\hat{f}(\xi + a) + \hat{f}(\xi - a) \tag{1.22}$$

$$F\{f(x)sinax\} = \frac{1}{2}\hat{f}_s(\xi + a) - \hat{f}_s(\xi - a) \tag{1.23}$$

Theorem 1.2.2 *(convolution Theorem) convolution of two functions $f(x)$ and $g(x)$ is defined as*

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(u)g(x-u)du \tag{1.24}$$

If $\hat{f}(\xi)$ and $\hat{g}(\xi)$are Fourier transforms of $f(x)$ and $g(x)$ respctively, then

$$F\{f(x) * g(x)\} = F\{f(x)\}.F\{fg(x)\} = \hat{f}(\xi).\hat{g}(\xi) \tag{1.25}$$

1.2.1 Properties

Let the integrable functions $h(x)$, $g(x)$ and $f(x)$ the Fourier transforms of these functions are represented by the symbols: $\hat{f}(\xi)$, $\hat{g}(\xi)$ and $\hat{h}(\xi)$, respectively.

The Fourier transform satesfy the following properties [1]:

Property 1.2.1 (Linearity) *Assuming that $h(x) = af(x) + bg(x)$ where a and b are complex numbers, then:*

$$\hat{h(\xi)} = a\hat{f(\xi)} + b\hat{g(\xi)} \tag{1.26}$$

Property 1.2.2 *(change of scale) If $\hat{f}(\xi)$ is Fourier transform of $f(x)$, then*

$$F\left[f(ax)\right] = \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right) \tag{1.27}$$

Property 1.2.3 *If $\hat{f}(\xi)$ is Fourier transform of $f(x)$, then*

$$F\left[f(x-a)\right] = e^{\imath\xi a}\hat{f}(\xi) \tag{1.28}$$

In the following table we introduce the Fourier transforms [1] of some given functions

Functions	Fourier transform
$e^{-at}u(t)$	$\frac{a+2}{\jmath\pi f}$
$e^{at}u(-t)$	$\frac{1}{a-\jmath^2\pi f}$
$e^{-a\|t\|}$	$\frac{2a}{a^2+(2\pi f)^2}$
$te^{-at}u(t)$	$\frac{1}{(a+\jmath^2\pi f)^2}$
$t^n e^{-at}u(t)$	$\frac{n!}{(a+2\jmath\pi f)}$

1.2.2 Application

In this section, let us solve a partial differentiel equation equiped with an initial condition using Fourier transform

Example 1.2.1 *Consider the problem*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) \qquad x \in \mathbb{R}, t > 0 \\ u(0,x) = 0 \qquad x \in \mathbb{R} \end{cases} \tag{1.29}$$

using the following Fourier transform

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x)e^{-i\xi x}dx \tag{1.30}$$

its application on both sides of the fundamental equation of the problem gives

$$\frac{\partial \hat{u}}{\partial t}(t,\xi) - \frac{\partial^2 \hat{u}}{\partial x^2}(t,\xi) = \hat{f}(t,\xi) \tag{1.31}$$

using the notation $V(t,\xi)$=$\hat{u}(t,\xi)$ and the initial condition we get

$$V'(t,\xi) + \xi^2 V(t,\xi) = \hat{f}(t,\xi) \tag{1.32}$$

which is a non homogeneous first order ordinary differential equation that we can rewrite as:

$$e^{\xi^2 t} V'(t,\xi) + \xi^2 e^{\xi^2 t} V'(t,\xi) = \hat{f}(t,\xi) e^{\xi^2 t} \tag{1.33}$$

thus

$$\left(V(t,\xi)e^{\xi^2 t}\right)' = e^{\xi^2 t}\hat{f}(t,\xi) \tag{1.34}$$

so

$$\int_0^t \left(V(t,\xi)e^{\xi^2 t}\right)' ds = \int_0^t e^{\xi^2 t}\hat{f}(t,\xi) ds \tag{1.35}$$

which means that

$$\left[V(s,\xi)e^{\xi^2 s}\right]_0^t = \int_0^t e^{\xi^2 t}\hat{f}(t,\xi) ds \tag{1.36}$$

then

$$V(t,\xi)e^{\xi^2 t} - V(0,\xi) = \int_0^t e^{\xi^2 t}\hat{f}(t,\xi) ds \tag{1.37}$$

we know that($V(0,\xi) = 0$), then we get

$$V(t,\xi) = \frac{1}{e^{\xi^2 t}} \int_0^t e^{\xi^2 t}\hat{f}(t,\xi) ds \tag{1.38}$$

which implies

$$\hat{u}(t,\xi) = \int_0^t e^{\xi^2 (s-t)}\hat{f}(t,\xi) ds \tag{1.39}$$

after using the inverse Fourier transform, we obtain:

$$F^{-1}(F(u(t,\xi))) = F^{-1}\left(\int_0^t e^{\xi^2(s-t)}\hat{f}(t,\xi)ds\right) \tag{1.40}$$

then

$$u(t,x) = \int_0^t F^{-1}\left(e^{\xi^2(s-t)}\hat{f}(t,\xi)\right)ds \tag{1.41}$$

$$= \int_0^t F^{-1}\left(e^{\xi^2(s-t)}\right) * F^{-1}\left(F(f(s,\xi))\right)ds \tag{1.42}$$

$$= \int_0^t F^{-1}(e^{\xi^2(s-t)}) * (f(s,y))dy \tag{1.43}$$

We have:

$$F(e^{-\alpha x^2}) = \frac{1}{\sqrt{2\alpha}}e^{\left(\frac{-\xi^2}{4\alpha}\right)} \tag{1.44}$$

and

$$\{e^(\frac{-\xi^2}{4\alpha})\} = \sqrt{2\alpha}F(e^{-\alpha x^2}) \tag{1.45}$$

$$F^{-1}\left(\frac{-\xi^2}{e^{4\alpha}}\right) = \sqrt{2\alpha}F\{e^{-\alpha x^2}\} \tag{1.46}$$

setting

$$(s-t) = \frac{-1}{4\alpha} \tag{1.47}$$

$$\Rightarrow \alpha = \frac{1}{4(t-s)} \tag{1.48}$$

$$\tag{1.49}$$

thus

$$F^{-1}(e^{-\xi^2(s-t)}) = \frac{1}{\sqrt{2(t-s)}}e^{\frac{-x^2}{4(t-s)}} \tag{1.50}$$

therefore

$$u(t,x) = \int_0^t \left(\frac{1}{\sqrt{2(t-s)}}e^{\frac{-x^2}{4(t-s)}}\right) f(s,y)ds \tag{1.51}$$

Chapter 2

NEW CLASSES OF INTEGRAL TRANSFORMS

In this chapter, we introduce various recent integral transforms that serve as useful tools in the study of certain mathematical problems.

2.1 Elzaki TRANSFORM

Definition 2.1.1 *The Elzaki transform [8] is a mathematical tool used in signal processing and image processing.it is introduced by the Egyptian mathematician Mostafa Elzaki, in 1962.as follows*

$$E(f(t)) = E(p) \tag{2.1}$$

$$= p\int_0^\infty e^{\left(\frac{-t}{p}\right)} f(t)dt \tag{2.2}$$

its inverse transform has the form

$$f(t) = E^{-1}(E(p)) \tag{2.3}$$

$$= \int_0^\infty e^{st} pE\left(\frac{1}{p}\right) dp \tag{2.4}$$

Remarque 2.1.1 *Elzaki transform is defined for given set A as:*

$$A = \{f(t), \exists M, k_1, k_2 > 0, f(t) \leq Me^{-\left(\frac{t}{k_j}\right)}, if \quad t \in (-1)^j[0, \infty]\} \tag{2.5}$$

Corollary 2.1.1 *let*

$$f(t) = g(t)h(t) \tag{2.6}$$

then

$$E\{f(t)\} = E\{g(p)\}E\{h(t)\} \tag{2.7}$$

Theorem 2.1.1 *Let g(p) is the ELzaki transform [9] of the function f(t), i.e.*

$$[E(f(t)) = g(p)] \tag{2.8}$$

then

$$E[f'(t)] = \frac{g(p)}{p} - pf(0) \tag{2.9}$$

$$E[f''(t)] = \frac{g(p)}{p^2} - f(0) - pf'(0) \tag{2.10}$$

$$E[f^{(n)}(t)] = \frac{g(p)}{p^2} - \sum_{0}^{n-1} p^{2-n+k} f^{(k)}(0) \tag{2.11}$$

Proof.

$$E[f'(t)] = p\int_0^{+\infty} f'(t)e^{\left(\frac{-t}{p}\right)}dt \tag{2.12}$$

integrating by parts to find that

$$E[f'(t)] = \frac{g(P)}{p} - pf(0) \tag{2.13}$$

2- Let $g(t) = f'(t)$, then

$$E[g'(t)] = \frac{1}{p}E[g(t)] - pg(0) \tag{2.14}$$

using (2.9), we get

$$E[f''(t)] = \frac{g(p)}{p^2} - f(0) - pf'(0) \tag{2.15}$$

■

2.1.1 Poperties

Property 2.1.1 *(Linearity) The Elzaki transform is linear. If $F(p)$ is the Elzaki transform of $f(t)$ and $G(s)$ is the Elzaki transform of $g(t)$, then $aF(p) + bG(p)$ is the Elzaki transform of $af(t) + bg(t)$, where a and b are constants.*

Property 2.1.2 *(Time Scaling Property:) If $E(p)$ is the Elzaki transform of $f(t)$, then the Elzaki transform of $f(at)$ is $\frac{1}{a}E\left(\frac{p}{a}\right)$, where a is a constant.*

Property 2.1.3 *(Shift Property) The Elzaki transform of a time-shifted function $f(t-t_0)$ is given by $e^{-pt_0}E(p)$.*

Remarque 2.1.2 *the relationship between Laplace and Elzaki transform of some common functions*

$$L(s) = pE\left(\frac{1}{p}\right) \tag{2.16}$$

the following table contains Elzaki transform for some special functions:

function $f(t)$	Elzaki transform $E(p)$
1	p^2
t	p^3
t^n	$n!p^{n+2}$
e^{at}	$\frac{p^2}{1-ap}$
te^{at}	$\frac{p^3}{(1-ap)^2}$
$\sin(at)$	$\frac{ap^3}{1+a^2p^2}$
$\cos(at)$	$\frac{p^2}{1+a^2p^2}$
$\sinh(at)$	$\frac{ap^3}{1-a^2p^2}$
$\cosh(at)$	$\frac{ap^2}{1-a^2p^2}$

2.1.2 Application

Example 2.1.1 *let solving using Elzaki Transform the one-dimensional heat problem*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t}(x,t) = \alpha^2 \dfrac{\partial^2 u(x,t)}{\partial x^2}, \alpha > 0 \\ u(x,0) = f(x) \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \tag{2.17}$$

let apply the Elzaki transform to both sides of the heat equation, such that $U(x,p) = E\{u(x,t)\}$, *we get:*

$$pU(x,p) - u(x,0) = \alpha^2 E\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} \tag{2.18}$$

since the Elzaki transform of the second spatial derivative is:

$$E\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{\partial^2 U(x,p)}{\partial x^2} \tag{2.19}$$

the transformed PDE becomes:

$$pU(x,p) - f(x) = \alpha^2 \frac{\partial^2 U(x,p)}{\partial x^2} \tag{2.20}$$

Rearrange to get:

$$\frac{\partial^2 U(x,p)}{\partial x^2} - \frac{p}{\alpha^2} U(x,p) = -\frac{f(x)}{\alpha^2} \tag{2.21}$$

which is a second-order linear ordinary differential equation. In fact, its general solution is

$$U(x,p) = A(p)\sinh\left(\frac{\sqrt{p}}{\alpha}x\right) + B(p)\cosh\left(\frac{\sqrt{p}}{\alpha}x\right) + \frac{f(x)}{p} \tag{2.22}$$

furthermore, the boundary condition $u(0,t) = 0$ *gives At* $x = 0$*:*

$$U(0,p) = B(p) = 0 \tag{2.23}$$

and the second condition $u(L,t) = 0$ *gives*

$$U(L,p) = A(p)\sinh\left(\frac{\sqrt{p}}{\alpha}L\right) \tag{2.24}$$

$$= 0 \tag{2.25}$$

since $\sinh\left(\frac{\sqrt{p}}{\alpha}L\right) \neq 0$ *for non-zero* L, $A(p) = 0$ *then*

$$U(x,p) = \frac{f(x)}{p} \tag{2.26}$$

using the inverse Elzaki transform we get inverse Elzaki transform of $\frac{1}{p}$ *is a step function* $H(t)$ *(Heaviside function)*

$$u(x,t) = E^{-1}\left\{\frac{f(x)}{p}\right\} \tag{2.27}$$

$$= f(x)E^{-1}\left\{\frac{1}{p}\right\} \tag{2.28}$$

$$= f(x)H(t) \tag{2.29}$$

where, $H(t)$ *is the heaviside function which represents the inverse Elzaki transform of* $\frac{1}{p}$

2.2 Jafari Integral Transform

Definition 2.2.1 *The Jafari transform is a general integral transform used to solve differential equations by converting them into simple algebraic equations. This transform is particularly useful in applied mathematics, engineering, and various scientific fields due to its ability to simplify complex problems. Mathematically, the Jafari transform of a function* f *is defined as* [5]*:*

$$J\{f(t)\} = h(\sigma)\int_0^\infty f(t)e^{-g(\sigma)t}dt \tag{2.30}$$

where $h(\sigma)$ *and* $g(\sigma)$ *are positive real functions, and the integral exists for some* $g(\sigma)$*.*

Property 2.2.1 *(Linearity:)*

$$J\{af(t) + bg(t)\} = aJ\{f(t)\} + bJ\{g(t)\} \tag{2.31}$$

where a *and* b *are constants, and* $f(t)$ *and* $g(t)$ *are two different functions.*

Theorem 2.2.1 *(Differentiation [5]) let $f(t)$ is differentiable and $p(s)$ and $q(s)$ are positive real functions, then*

$$1 - J\{f'(t),\sigma\} = g(\sigma)J(\sigma) - h(\sigma)f(0). \tag{2.32}$$

$$2 - J\{f''(t),\sigma\} = g^2(\sigma)J\{f(t),\sigma\} - g(\sigma)h(\sigma)f(0) - h(\sigma)f'(0). \tag{2.33}$$

$$3 - J\{f''(t),\sigma\} = g^n(\sigma)J\{f(t),\sigma\} - h(\sigma)\sum_{k=0}^{n-1} g^{n-1-k}(\sigma)f^k(0). \tag{2.34}$$

Proof.

$$J\{f'(t),\sigma\} = h(\sigma)\int_0^\infty f'(t)e^{-g(\sigma)t}dt \tag{2.35}$$

$$= h(\sigma)[e^{-g(\sigma)t}f(t)|_0^\infty + g(\sigma)\int_0^\infty f(t)e^{-g(\sigma)t}dt] \tag{2.36}$$

$$= g(\sigma)J\{f(t),\sigma\} - h(\sigma)f(0) \tag{2.37}$$

to proof (2.35), we assume $\psi(t) = f''(t) = \psi'(t)$ now

$$J\{\psi'(t),\sigma\} = h(\sigma)\int_0^\infty \psi'(t)e^{-g(\sigma)t}dt \tag{2.38}$$

$$= g(\sigma)J\{\psi(t),\sigma\} - h(\sigma)\psi(0) \tag{2.39}$$

$$= g(\sigma)J\{f'(t),\sigma\} - h(\sigma)f'(0) \tag{2.40}$$

$$= g(\sigma)[g(\sigma)J\{f(t),\sigma\} - h(\sigma)f(0)] - h(\sigma)f'(0) \tag{2.41}$$

$$= g^2(\sigma)J\{f(t),\sigma\} - g(\sigma)h(\sigma)f(0) - h(\sigma)f'(0) \tag{2.42}$$

by induction we can prove (2.36) ■

Theorem 2.2.2 *(Convolution) let $f_1(t)$ and $f_2(t)$ have new integral transform $F_1(\sigma)$ and $F_2(\sigma)$.then the new integral transform of the convolution of f_1 and f_2 is*

$$f_1 * f_2 = \int_0^t f_1(t)f_2(t-\tau)d\tau \tag{2.43}$$

$$= \frac{1}{P(\sigma)}F_1(\sigma)F_2(\sigma) \tag{2.44}$$

Proof.

$$J\{f_1 * f_2\} = h(\sigma)\int_0^\infty e^{g(\sigma)t}\int_0^\infty f_1(t)f_2(t-\tau)dt \tag{2.45}$$

$$= h(\sigma)\int_0^\infty f_1(\tau)d\tau\int_0^\infty e^{-g(\sigma)t}f_2(t-\tau)dt \tag{2.46}$$

$$= h(\sigma)\int_0^\infty e^{g(\sigma)t}f_1(\tau)d\tau\int_0^\infty e^{-g(\sigma)t}f_2(t)dt \tag{2.47}$$

$$= \frac{1}{P(\sigma)}F_1(\sigma)F_2(\sigma) \tag{2.48}$$

■

Property 2.2.2 *(Initial Value)*

$$\lim_{t\to 0^+} f(t) = \lim_{p\to\infty} pJ\{f(t)\} \tag{2.49}$$

Property 2.2.3 *(Final Value)*

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pJ\{f(t)\} \tag{2.50}$$

provided that the limits exist and $f(t)$ tends to a finite value as $t \to \infty$

Property 2.2.4 *(Time Shifting) If $J\{f(t)\} = F(p)$, then*

$$J\{f(t-t_0)\} = e^{-pt_0}F(p) \tag{2.51}$$

Property 2.2.5 *(Scaling) If $J\{f(t)\} = F(p)$, then for a scaling factor a:*

$$J\left\{f(at)\right\} = \frac{1}{a}F\left(\frac{p}{a}\right) \tag{2.52}$$

below,we introduce the Jafari transform of some common functions [5]

$v(t)$	Jafari transform $J[v(t)]$
1	$\frac{h(s)}{g(s)}$
e^{at}	$\frac{h(s)}{g(s)-a}$
t^n	$\frac{h(s)n!}{g(s)^{n+1}}$
$sin(at)$	$\frac{ah(s)}{g(s)^2+a^2}$
$cos(at)$	$\frac{g(s)h(s)}{g(s)^2+a^2}$
te^{at}	$\frac{h(s)}{(g(s)-a)^2}$
$cosh(at)$	$\frac{ah(s)}{g(s)^2-a^2}$
$sinh(at)$	$\frac{g(s)h(s)}{g(s)^2-a^2}$

These transforms are useful for solving differential equations and other mathematical problems by simplifying the expressions into simple algebraic forms.

2.2.1 Application

Example 2.2.1 *consider the following third-order ODE*

$$\begin{cases} y'''(t) + 2y''(t) + 2y'(t) + 3y(t) = \sin(t) + cos(t) \\ y(0) = y''(0) \\ y'(0) = 1 \end{cases} \tag{2.53}$$

the Jafari transform J of a function u is defined as :

$$J\{y(t)\} = h(\sigma)\int_0^{\infty} e^{-g(\sigma)t} y(t)dt \tag{2.54}$$

applying the Jafari transform to both sides of the PDE

$$J\left\{y'''(t) + 2y''(t) + 2y'(t) + 3y(t)\right\} = J\left\{\sin(t) + cos(t)\right\} \tag{2.55}$$

$$J\{y'''(t)\} + 2J\{y''(t)\} + 2J\{y'(t)\} + 3J\{y(t)\} = J\{\sin(t)\} + J\{cos(t)\} \tag{2.56}$$

using the property of the Jafari transform for the time derivative:

$$\begin{aligned} g^2(s)J(s) - h(s)(g^2(s)y_0 + g(s)y_1 + y_2) + 2[g^2(s)J(s) \\ -h(s)(g(s)y_0 + y_1)] + 2\{g(s)J(s) - h(s)y_0\} + 3J(s) &= \frac{h(s)}{g^2(s)+1} + \frac{g(s)h(s)}{g^2(s)+1} \end{aligned} \tag{2.57}$$

use the boundary condition

$$[g^3(s) + 2g^2(s) + 2g(s) + 3]J(s)\frac{h(s)}{g^2(s)+1} + \frac{g(s)h(s)}{g^2(s)+1} + h(s)g(s) + 2h(s) \tag{2.58}$$

by simplification, we get

$$J(s) = \frac{h(s)}{g^2(s)+1} \tag{2.59}$$

Finally,apply the inverse Jafari transform :

$$y(t) = J^{-1}\left\{\frac{h(s)}{g^2(t)+1}\right\} = \sin(t) \tag{2.60}$$

2.3 The New Integral Transform

Definition 2.3.1 *the new integral transform of a function f denoted by the operator $N(.)$ is defined [2]*

$$E(s,u) = N\{f(t)\} \tag{2.61}$$

$$= \frac{1}{su}\int_0^\infty e^{\left(\frac{-st}{u}\right)} f(t)dt \tag{2.62}$$

2.3.1 Properties

1. $N\{af(t) + bg(t)\} = aN\{f(t)\} + bN\{g(t)\}$
2. If $N\{f(t)\} = E(s,u)$ exists, then for $a > 0$, $N\{f(at)\} = \frac{1}{a}E\left(\frac{s}{a}, u\right)$
3. If $N\{f(t)\} = E(s,u)$ exists,then for any real $a > 0$ and $U(t-a)$ as the unit step function $N\{f(t-a)U(t-a)\} = e^{-sa}E(s,u)$ where $f(t-a)U(t-a) = 0$ if $0 \leq t < a$ and $f(t-a)U(t-a) = f(t-a)$ if $a \leq t$.
4. If $N\{f(t)\} = E(s,u)$ exists, then for any real a $N\{e^{(at)}f(t)\} = E(s-au, u)$.

Theorem 2.3.1 *(Sufficient conditions) If f is piecewise continuous on $[0,\infty)$ and of exponential order $\frac{1}{k}$, then $N\{f(t)\}$ exists for $\left|\frac{u}{s}\right| < k$.*

Theorem 2.3.2 *([6]) Let $F(s)$ denotes Laplace transform of the function f. Then the new integral transform $E(s,u)$ of $f(t)$ is given by:*

$$E(s,u) = N\{f(t)\} \tag{2.63}$$
$$= \frac{1}{su}F\left(\frac{s}{u}\right) \tag{2.64}$$

Theorem 2.3.3 *Let $E(s,u)$ be the new integral transform of f. Then, the following statements are true [2]*

$$N\{f'(t)\} = \frac{sE(s,u)}{u} - \frac{f(0)}{su} \tag{2.65}$$
$$N\{f''(t)\} = \frac{s^2E(s,u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{su} \tag{2.66}$$
$$N\{f^{(n)}(t)\} = \frac{s^{(n)}E(s,u)}{u^n} - \frac{s^{(n-2)}}{u^n}f(0) - \frac{s^{(n-3)}}{u^{(n-1)}}f'(0)... - \frac{f^{(n-1)}(0)}{su} \tag{2.67}$$
$$N\{t^n\} = \frac{u^n}{s^{n+2}}\Gamma(n+1); \Gamma(n+1) = (n+1)! \tag{2.68}$$

Theorem 2.3.4 *if $N\{f(t)\}$ exists then [2],*

$$N\{t^nf(t)\} = \frac{u^n}{s^n}\frac{d^n}{du^n}u^nE(s,u) \tag{2.69}$$

Below is a table of some common functions and their NE transforms.

$f(t)$	NE Transform $N\{f(t)\}$
1	$\frac{1}{s^2}$
t	$\frac{u}{s^3}$
e^{at}	$\frac{1}{s(s-au)}$
$sin(at)$	$\frac{au}{s(s^2+a^2u^2)}$
$cos(at)$	$\frac{1}{(s^2+a^2u^2)}$
te^{at}	$\frac{1}{su(s+a)^2}$
$cosh(at)$	$\frac{a}{(s^2-a^2)}$
$sinh(at)$	$\frac{a}{s(s^2-a^2)}$

2.3.2 Application

In this section we will solve a mathematical problem using the new integral transform and its properties

Example 2.3.1 *Let us consider the second-order differential equation*

$$\begin{cases} y''(x) + y(x) = 0 \\ y(0) = y'(0) = 1 \end{cases} \tag{2.70}$$

the application of NE transform on both sides of the fundamental equation of the problem (2.70) *gives*

$$\frac{s^2}{u^2}N\{y\} - \frac{1}{su}y'(0) - \frac{1}{u^2}y(0) + N\{y\} = 0 \tag{2.71}$$

using the initail conditions of (2.70), we get

$$\frac{s^2}{u^2}N\{y\} - \frac{1}{su} - \frac{1}{u^2} + N\{y\} = 0 \tag{2.72}$$

which implies that

$$\frac{s^2}{u^2}N\{y\} + N\{y\} = \frac{1}{su} + \frac{1}{u^2} \tag{2.73}$$

then

$$N\{y\}\left(\frac{s^2 + u^2}{u^2}\right) = \frac{u}{su^2} + \frac{s}{su^2} \tag{2.74}$$

therefore

$$N\{y\} = \frac{u + s}{s(s^2 + u^2)} \tag{2.75}$$

using the inverse NE integral transform we get

$$y(t) = \cos\left(\frac{t}{u}\right) + \sin\left(\frac{t}{u}\right) \tag{2.76}$$

since (u) *is a parameter related to the transform, in general applications, we may set* $(u = 1)$ *for simplicity, then*

$$y(t) = \cos(t) + \sin(t) \tag{2.77}$$

2.4 Some Recent Integral Transforms

In the following table we introduce few integral transforms [1]

function integral	transform expression
Hankel transform $f(r)$	$\hat{f}_n(k) = \int_0^{\infty} rJ_n(kr)f(r)dr$
Mellin transform $f(x)$	$\hat{f}(p) = \int_0^{\infty} x^{p-1}f(x)dx$
Hilbert transform $f(t)$	$\hat{f}_H(x) = \frac{1}{\pi}\oint_{-\infty}^{+\infty} \frac{f(t)}{(t-x)}dt$
Tarig transform	$\frac{1}{v}\int_0^{+\infty} e^{\frac{-t}{v^2}} f(t)dt$
Kamal transform $f(k)$	$\int_0^{+\infty} e^{-\frac{t}{v}} f(k)dk$
Shehu transform $f(m)$	$\int_0^{+\infty} e^{-\frac{u}{v}} f(m)dm$
Sawi transform $f(w)$	$\frac{1}{v^2}\int_0^{+\infty} e^{\frac{t}{v}} f(w)dw$
Stieltjes Transform $f(s)$	$S_g[f(s)] = \hat{f}(z,\rho) = \int_0^{\infty} \frac{f(s)}{(s+z)^{\rho}}ds$

Chapter 3

ANALYTICAL SOLUTION OF BAGLEY-TORVIK PROBLEM

3.1 Main principles of fractional calculus

Definition 3.1.1 *The Gamma function $\Gamma(\alpha)$ is the factorial function of real number, except for the non-positive numbers. It is defined for any α which he a real part greater than zero. by the following integral:*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt \qquad \mathbb{R}(\alpha) > 0 \tag{3.1}$$

Remarque 3.1.1 *The gamma function $\Gamma(\alpha)$ plays an important role in the theory of fractional calculus. An other representation of $\Gamma(\alpha)$ is provided by Euler limit*

$$\Gamma(\alpha) = \lim_{N\to+\infty} \frac{N!N^\alpha}{\alpha(\alpha+1)(\alpha+2)...(\alpha+N)} \qquad \alpha > 0 \tag{3.2}$$

Property 3.1.1 *For any positive integer α,*

$$\Gamma(\alpha) = (\alpha - 1)! \tag{3.3}$$

Property 3.1.2 *The Gamma function satisfies the recursive relationship:*

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \tag{3.4}$$

this property is analogous to the recursive property of factorials: $\alpha! = \alpha(\alpha-1)!$.

Property 3.1.3 *The following are the most important properties of the gamma function:*

$$\Gamma(1) = 1 \tag{3.5}$$

$$\Gamma(\frac{1}{2} - n) = \frac{(-4)^n \sqrt{\pi} n!}{(2n)!} \tag{3.6}$$

$$\Gamma(\frac{1}{2} + n) = \frac{(2n)! \sqrt{\pi}}{4^n n!} \tag{3.7}$$

$$\Gamma(-\alpha) = \frac{-\pi \csc(\pi\alpha)}{\Gamma(\alpha+1)} \tag{3.8}$$

$$\Gamma(n\alpha) = \sqrt{\frac{2\pi}{n}} \left[\frac{n\alpha}{\sqrt{2\pi}}\right]^n \prod_{k=0}^{n-1} \Gamma\left(\alpha - \frac{k}{n}\right), n \in \mathbb{N}^+ \tag{3.9}$$

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{sin(\pi\alpha)} \tag{3.10}$$

Definition 3.1.2 *The Beta function* $\beta(x, y)$ *is a special function that is closely related to the Gamma function. It is defined for real numbers* $x > 0$ *and* $y > 0$ *by the following integral:*

$$\beta(x, y) = \int_0^1 q^{x-1}(1-q)^{y-1} dq \qquad p, q > 0 \tag{3.11}$$

Property 3.1.4 *(Symmetry) The Beta function is symmetric*

$$\beta(x, y) = \beta(y, x) \tag{3.12}$$

Remarque 3.1.2 *The Beta function can be expressed in terms of Gamma function as*

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.13}$$

The Beta function is a fundamental function in mathematics, providing a link between the integral calculus and special functions like the Gamma function, and it plays a significant role in various mathematical and applied contexts.

Remarque 3.1.3 *Fractional calculus expands the concept of derivatives and integrals to include non-integer orders. It can perform derivatives of non-integer orders, denoted* *by the formula $D^{\alpha}f(x)$ where α is a real or complex number. Fractional integrals also involve non-integer orders of integration. Fractional derivatives are a generalization of the concept of differentiation to fractional orders. There are several definitions of fractional derivatives.*

Definition 3.1.3 *(Riemann-Liouville Fractional Derivative) The Riemann-Liouville fractional derivative [7] of order α of the function f with respect to the variable t is defined as follows:*

$$D^{\alpha}_{RL}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt \tag{3.14}$$

where n is the smallest integer greater than α and Γ is the Gamma function.

Definition 3.1.4 *(Caputo Fractional Derivative) The Caputo fractional derivative of order α of the function f is defined as*

$$D^{\alpha}_{RL}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt \tag{3.15}$$

where n is the smallest integer greater than α and $f^{(n)}$ denotes the n-th derivative of f.

Definition 3.1.5 *(Abel-Riemann Fractional Derivative) The Abel-Riemann fractional derivative of order $\alpha > 0$ is defined as the inverse of the corresponding A-R fractional integral, it means that*

$$D^{\alpha}_{AR}I^{\alpha}_{AR} = I \tag{3.16}$$

for positive integer m, such that $m-1 < \alpha \leq m$, where I is the identity operator.

$$(D^{m}_{AR}I^{m-\alpha}_{AR})I^{\alpha}_{AR} = D^{m}_{AR}(I^{m-\alpha}_{AR}I^{\alpha}_{AR}) = D^{m}_{AR}I^{m}_{AR} = I \tag{3.17}$$

it means that

$$D^{\alpha}_{AR}f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\dfrac{d^m}{dx^m}\displaystyle\int_a^x \frac{f(t)}{(x-t)^{\alpha+1-m}}dt & m-1<\alpha<m \\ \dfrac{d^m f(x)}{dx^m} & \alpha = m \end{cases} \tag{3.18}$$

3.2 Fractional Integration

We define the fractional integral of the function f as follows:

$$F^{\alpha}[f(t)](s) = \frac{1}{\Gamma(\alpha)} \int_0^s s(s-\xi)^{\alpha-1} f(\xi) d\xi \tag{3.19}$$

Here, α is a non-integer parameter (typically greater than zero).

F^{α} denotes the fractional integral operator.

$\Gamma(\alpha)$ is the gamma function.

$f(t)$ is the function being transformed.

s is the upper limit of integration, and ξ is the integration variable.

Definition 3.2.1 *(Riemann-Liouville integral) The generalization to non-integer α of Riemann-Liouville integral can be written for suitable function $f(x)$, $x \in \mathbb{R}$; as*

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{(\alpha-1)} f(s) ds \qquad \alpha > 0 \tag{3.20}$$

such that $I^0 f(x) = f(x)$.

Property 3.2.1 *The properties of the operator I^{α} can be founded in* [**?**] *for $\beta \geq 0$,$\alpha > 0$ as follows*

$$\begin{cases} I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x) \\ I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x) \end{cases} \tag{3.21}$$

Definition 3.2.2 *(Abel-Riemann fractional integral) The Abel-Riemann fractional integral* [**?**] *of any order $\alpha > 0$, for a function $f(x)$ with $x \in \mathbb{R}^+$ is defined as:*

$$I^{\alpha}_{AR} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{(\alpha-1)} f(\tau) d\tau, x > 0, \alpha > 0 \tag{3.22}$$

$I^0_{AR} = I$ *(Identity operator).*

Fractional calculus has its own set of identities and properties, such as the fractional Leibniz rule, fractional chain rule, and fractional Taylor series expansions, which are essential for manipulating and solving fractional differential equations.

Definition 3.2.3 *The operator D^α of order α in Abel-Riemann (A-R) sense is defined by Podlubny [7] as*

$$D^\alpha u(x) = \begin{cases} \dfrac{d^m}{dx^m} u(x), \alpha = m \\ \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d}{dx^m} \displaystyle\int_0^x \dfrac{1}{(x-t)^{\alpha-m+1} u(t)} dt & m-1 < \alpha < m \end{cases} \tag{3.23}$$

where $m \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}^+$

and

$$D^{-\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt \qquad 0 < \alpha \leq 1. \tag{3.24}$$

Definition 3.2.4 *The A-R fractional order integration operator J^α is described as [4]*

$$J^\alpha u(x) = \frac{1}{\Gamma} \int_0^x (x-t)^{\alpha-1} u(t) dt \qquad t > 0, \alpha > 0 \tag{3.25}$$

then

$$J^\alpha t^\alpha = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} t^{n+\alpha} \tag{3.26}$$

and

$$D^\alpha t^\alpha = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha} \tag{3.27}$$

Definition 3.2.5 *The operator D^α of order α in the Caputo [7] sense is defined as:*

$${}^cD^\alpha u(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \dfrac{u^m(t)}{(x-t)^{\alpha-m+1}} dt & m-1 < \alpha < m \\ \dfrac{d^m u(x)}{dt^m}, \alpha = m \end{cases} \tag{3.28}$$

Definition 3.2.6 [?]

$$D_t^\alpha J_t^\alpha f(t) = f(t) \tag{3.29}$$

$$D_t^\alpha J_t^\alpha f(t) = f(t) - \sum_{k=0}^{m} f^k(0^+) \frac{t^k}{k!} \tag{3.30}$$

for $t > 0$ and $m - 1 < \alpha \geq m$, $m \in \mathbb{N}$.

3.3 Problem Study:

The Bagly-Trovik equation [7] is a fractionnal differential equation that describes the behavoir of viscoelastic systems, it is defined mathematically as [4]:

$$m\frac{d^{\alpha}x(t)}{dt^{\alpha}} + c\frac{d^{\beta}x(t)}{dt^{\beta}} + kx(t) = F(t) \tag{3.31}$$

where m represents the mass.

c: is the damping coefficient.

k: is the stiffness constant.

$x(t)$: represents the displacement of the system at time t.

$F(t)$: is the external force applied to the system.

$\frac{d^{\alpha}x(t)}{dt^{\alpha}}$: denotes the caputo fractional derivative of order α.

$\frac{d^{\beta}x(t)}{dt^{\beta}}$: denotes the caputo fractional derivative of order β.

α and β are fractional orders typically in the range $0 < \alpha, \beta < 1$

Definition 3.3.1 *The Sumudu transform of $y(t)$, is defined in [7] for all real $t \geq 0$, as:*

$$S(y(t)) = F(u) = \frac{1}{u}\int_0^{+\infty} e^{\left(\frac{1}{u}\right)t} y(t)dt. \tag{3.32}$$

Theorem 3.3.1 *If $F(u)$ is the Sumudu Transform of $y(t)$, then the Sumudu Transform of nth order derivative is defined in [3] as:*

$$S\left[\frac{d^n y(t)}{dt^n}\right] = \frac{1}{u}\left[F(u) - \sum_{k=0}^{n-1} u^k \frac{d^k y(t)}{dt^k} \mid_{t=0}\right] \tag{3.33}$$

it means that

$$S\left[\frac{dy(t)}{dt}\right] = \frac{1}{u}\left[F(u) - y(0)\right] \tag{3.34}$$

$$S\left[\frac{dy(t)}{dt}\right] = \frac{1}{u^2}\left[F(u) - y(0) - u\frac{dy(t)}{dt}\mid_{t=0}\right] \tag{3.35}$$

The Sumudu Transform of Caputo fractional derivative is well defined as

$$S[D_t^{\alpha} y(t)] = u^{-\alpha} S[y(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} y^k(0), \qquad m-1 < \alpha \leq m \tag{3.36}$$

Example 3.3.1 *Let us take the following Bagley-Torvik problem*

$$\begin{cases} y^{(3)}(t) + y^{\left(\frac{5}{2}\right)}(t) + y^{\left(\frac{3}{2}\right)}(t) + y(t) = t^3 + 6t \\ y(0) = \frac{dy(t)}{dt}|_{t=0}, t \geq 0 \\ \frac{d^2y(t)}{dt^2}|_{t=0} = 1 \end{cases} \tag{3.37}$$

by taking Sumudu Transform the fundamental equation of 3.37*,we get*

$$S\left[y^3(t)\right] + S\left[y^{\left(\frac{5}{2}\right)}(t)\right] + S\left[y^{\left(\frac{3}{2}\right)}(t)\right] + S\left[y(t)\right] = S\left[t^3\right] + S\left[6t\right] \tag{3.38}$$

therefore

$$\frac{1}{u^3}\left[F(u) - y(0) - u\frac{dy(t)}{dt}|_{t=0} - u^2\frac{d^2y(t)}{dt^2}|_{t=0}\right] + u^{-\left(\frac{5}{2}\right)}[Fu] - \sum_{k=0}^{1} u^{-\left(\frac{3}{2}\right)+k} y^k(0) \tag{3.39}$$

$$+u^{-\left(\frac{3}{2}\right)}[F(u)] - \sum_{k=0}^{1} u^{-\left(\frac{3}{2}\right)+k} y^k(0) + [F(u)] = 6u^2 + 6u$$

then

$$[F(u)]\left(\frac{u^{\left(\frac{1}{2}\right)} + u^{\left(\frac{3}{2}\right)} + u^3}{u^3}\right) = \frac{u^3 + 6(u^2+u)u^3}{u^{\left(\frac{1}{2}\right)} + u^{\left(\frac{3}{2}\right)} + u^3} \tag{3.40}$$

$$\tag{3.41}$$

thus

$$[F(u)] = 6u^7 \tag{3.42}$$

applying inverse Sumudu Transform to (3.40)*,we get*

$$u^{n-1} = \frac{t^{n-1}}{\Gamma(n)}, n > 0 \tag{3.43}$$

if we set $\alpha = 8$ *and* $n = 7$*,we get:* $\Gamma(7) = 360$

by substitution in the equation (3.43) *we find*

$$y(t) = \frac{1}{360}t^7 \tag{3.44}$$

3.4 Conclusion

In this work, we have explored both classical and modern integral transforms, focusing on their theoritical foundations and practical applications. Spicifically, we have examined their utility in solving Bagley-Torvik problem, a fundamental problem in the field of fractional differential equations and viscoelasticity.

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