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Thesis

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Option: ***Applied Functional Analysis***

Existence of Solutions to a fourth order Neutral Differential Equation via the Krasnoselskii Fixed-Point Theorem

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Abstract in English

Existence of Solutions to a fourth order Neutral Differential Equation via the Krasnoselskii Fixed-Point Theorem

In this work, a fourth-order neutral functional differential equation with a time-varying delay is investigated. With the help of the Krasnoselskii fixed point theorem as well as the Green's functions method and some functional analysis tools, we established some sufficient criteria that ensure the existence of at least one positive periodic solution for the proposed equation.

Keywords. Existence, fourth order neutral differential equation, *Green*'s function, Krasnoselskii's fixed point theorem, periodic solution.

Abstract in French

Existence de Solutions d'une Equation Différentielle de type Neutre du quatrième ordre via le théorème du point fixe de Krasnoselskii

Dans ce travail, une équation différentielle fonctionnelle de type neutre du quatrième ordre avec un retard variable en temps est étudiée. À l'aide du théorème du point fixe de Krasnoselskii ainsi que de la méthode des fonctions de Green et de certains outils de l'analyse fonctionnelle, nous avons établi des critères suffisants qui garantissent l'existence d'au moins une solution périodique positive pour l'équation proposée.

Mots-clés. Existence, équation différentielle de type neutre de quatrième ordre , fonction de *Green,* théorème de point fixe de Krasnoselskii, solution périodique.

Abstract in Arabic

وجود الحلول لمعادلة تفاضلية محايدة من الرتبة الرابعة باستخدام نظرية النقطة الثابتة لكراسنوسيلسكي

في هذا العمل، تم دراسة معادلة تفاضلية وظيفية محايدة من الدرجة الرابعة مع تأخير متغير مع الزمن. وبمساعدة نظرية النقطة الثابتة لكراسنوسيلسكي وطريقة دوال غرين وبعض أدوات التحليل الدالي، قمنا بوضع بعض المعايير الكافية التي تضمن وجود حل دوري موجب واحد على الأقل للمعادلة المقترحة.

كلمات مفتاحيه. وجود، معادلة تفاضلية محايدة من الدرجة الرابعة، دالة غرين، نظرية النقطة الثابتة لكراسنوسيلسكي، حل دوري.

Dedication

With all love and gratitude, I dedicate this work to those who have been shining stars in the sky of my life and have contributed to the realization of this work.

To my dear mother, zagda, who has been a warm haven for me, and accompanied me with her hope and prayers. Every kind word and tender touch from you has been a light that illuminated my path.

To my dear father Mouhamed who has been a guiding role model and an exemplar of diligence and hard work. Your advice and boundless support have been the foundation upon which I have built my success.

To the one I love very much and who supported me throughout this work, my fiancé Merouane

To my dear brothers Yassin, Hamza and Yaakoub and Islam and my dear sister Ibtissam

To my dear uncles Abde Elhake, Ibrahim, Aziz and Ali and Moussa

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To those who were my support and companions during my school years, to those who shared with me joy and sadness, success and failure, to those who added colors of joy and challenge to my days. To my beloved friends

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who helped me to produce this modest memoir, especially to my family, my parents, my fiancé, my sister, my brothers and my friends for their moral support and for the many encouragements they gave me.

Acronyms

Abbreviation	Meaning
DDE	Delay differential equation
FDEs	Functional differential equation
NDE	Neutral differential equation
ODE	Ordinary differential equation
SDDE	State-dependent delay differential equation

List of symbols

Sets and numbers

$\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).

$\mathbb{R}^*$: the set of all non-zero real numbers

$\mathbb{R}_+^* = (0, +\infty)$: the set of all non-zero positive real numbers

$\mathbb{R}^n$: n-dimensional real Euclidean space

$[a, b]$: the interval of numbers between a and b, including a and b

(a, b) : an open interval

$[a, +\infty)$: left-closed and right–unbounded interval

$\mathcal{C}\left(\mathbb{R}^{n+1}, (0, +\infty)\right)$ is the space of continuous functions from $\mathbb{R}^{n+1}$ into $(0, +\infty)$

$\mathcal{C}\left([a, b], (0, +\infty)\right)$ is the space of continuous functions from $[a, b]$ into $(0, +\infty)$

$\mathcal{C}^1\left(\mathbb{R}^{n+1}, (0, +\infty)\right)$: space of continuously differentiable functions from $\mathbb{R}^{n+1}$ into $(0, +\infty)$

τ : a delay

w : a period

Functions

$|\cdot|$: absolute value

$\|.\|_{\mathbb{Y}}$: a norm on $\mathbb{Y}$

$\|f\|$: the uniform norm defined by $\sup|f(M)|$

$M^{[n]}(t)$: the composition of the function $M(t)$ with itself n times or the n^{th} iterate of the function $M(t)$

$\sum\limits_{i=1}^{n}$: the summation from index $i=1$ to $i=n$

$\lim\limits_{M\longrightarrow M_0}$: limit as M approaches M_0

$M'(t) := M^{(1)}(t) := \dfrac{dM(t)}{dt}$: the first derivative of the function $M(t)$ with respect to t

$M^{(n)}(t) := \dfrac{d^n M(t)}{dt^n}$: the nth derivative of the function $M(t)$ with respect to t

sup : the supremum

max : the maximum

min : the minimum

$\exp M$: the exponential function of M

$G(t,s)$: the *Green's* function

Other notations will be clarified upon their initial occurrence.

General introduction

Functional differential equations (FDEs) which are equations that contain deviating arguments as functions and some of their derivatives evaluated at different argument values, depend on the present as well as the past or the future state of the system. For this, they are more used for modelling real phenomena than ordinary differential equations (ODE) since they can be deemed as a fundamental pillar in describing specified behaviors or phenomena that depend on both the present and the past or the future state of a system.

Fourth-order delay functional differential equations are ubiquitous in various scientific disciplines. They arise in many real situations including viscoelastic and inelastic flows [46, 49, 51], deformation of structures such as aircraft, building and ships, vibrating motion in bridges [47, 50], soil settlement [25] and electric circuits. The delay can stand for the time taken for the observation to be available for use in control, delayed feedback control of piezoelectric elastic beams, etc.

Problem statement

Delay differential equations (DDEs) are generally not easy to handle, especially those of higher-orders. Despite the fact that there are many approaches which can be used for studying first-order DDEs, they often fail to reach the desired results in investigating higher-order DDEs. This is particularly true for fourth order neutral functional differential equations since it is a pity that there are few published work in this direction.

In the current thesis, we deal with a fourth order neutral functional differential equation (NDEs) with a time varying delay that can describe many natural phenomena. This thesis is motivated by the desire to answer the following fundamental research question:
What are the criteria for the existence of a at least one positive periodic solution?

Objectives

The main goal of this thesis lies in highlighting the practicality of the used technique that combines the fixed point theory and the Green's functions method, to deal with a higher-order neutral differential equation. It is precisely by means of this hybrid technique, that the current work principally probes into the existence of positive periodic solutions for a fourth-order neutral differential equation involving a time-varying delay.

Methodology

The methodology used herein is as follows: the existence of solutions are proved by virtue of a hybrid technique that allows us to study the existence of solutions of a higher-order DDEs easily which are unfortunately not always analyzed or solved easily. the technique employed here combines the application of the Krasnoselskii fixed point theorem with the Green's functions method. The key steps of it lies in

Firstly, we define an appropriate Banach space and a suitable subset of it.

Secondly, we convert the equation at hand into an equivalent integral one whose kernel is a Green's function.

Thirdly, from the obtained integral equation, we construct an integral operator that can be written as a sum of a contraction and a completely continuous operator.

Finally, via the Krasnoselskii fixed point theorem of a sum of two mappings together with the help of Arzelá-Ascoli theorem and some properties of the Green's kernel, we succeed in establishing the existence of at least one positive periodic solution.

Thesis overview

The organization of this manuscript is as follows. In Chapter 1, we introduce certain definitions, concepts, and some preliminary results that will be used to prove the main outcomes. in Chapter 2, we present a quick overview on delay differential equations. We then apply, in Chapter 3, the Krasnoselskii fixed point theorem for showing the existence of at least one positive periodic solution of the following fourth-order neutral differential equation:

$$\frac{d^4}{dt^4}\left(x(t) - g\left(t, x(t-\tau(t))\right)\right) = -a\left(t\right)x(t) + f\left(t, x\left(t-\tau\left(t\right)\right)\right),$$

where $a, \tau \in C\left(\mathbb{R}, (0, \infty)\right)$, $g \in C\left(\mathbb{R} \times [0, \infty), \mathbb{R}\right)$ and $f \in C\left(\mathbb{R} \times [0, \infty), [0, \infty)\right)$ are ω-periodic functions in t. Finally, we draw the conclusion.

CHAPTER 1

Preliminary notions

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In this chapter, we present some notations, definitions and preliminary results that are used in the remainder of the thesis.

1.1 Compact operators

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two normed vector spaces over the same field $\mathbb{F}$.

Definition 1.1 An operator $S : \mathbb{X} \to \mathbb{Y}$ is said to be continuous at a point $x_0 \in \mathbb{X}$ if

$$\lim_{x \to x_0} Sx = Sx_0$$

The continuity at $x_0 \in \mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{X}, (\|x - x_0\|_{\mathbb{X}} < \delta) \Longrightarrow (\|Sx - Sx_0\|_{\mathbb{Y}} < \varepsilon) .$$

If S is continuous at every point of $\mathbb{X}$, then S is said to be continuous on $\mathbb{X}$. The continuity on $\mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \forall x \in \mathbb{X}, \exists \delta > 0, \forall y \in \mathbb{X}, (\|x - y\|_{\mathbb{X}} < \delta) \Longrightarrow (\|Sx - Sy\|_{\mathbb{Y}} < \varepsilon) .$$

Definition 1.2 A map $S : \mathbb{X} \to \mathbb{Y}$ is called Lipschitz continuous if there is a positive constant C such that

$$\forall x, y \in \mathbb{X} : \|Sx - Sy\|_{\mathbb{Y}} \leq C \|x - y\|_{\mathbb{X}} .$$

If $C \in [0, 1[$, S is called a contraction mapping.

Remark 1.1 If $S : \mathbb{X} \to \mathbb{Y}$ then

$$S \text{ is a contraction} \Longrightarrow S \text{ is Lipschitz continuous} \Longrightarrow S \text{ is continuous on } \mathbb{X}.$$

Definition 1.3 A map $S : \mathbb{X} \to \mathbb{Y}$ is said to be compact if and only if S maps bounded sets into relatively compact sets, i.e.,

$$[S \text{ compact}] \Longleftrightarrow \left[\forall M \subset E, (M \text{ bounded}) \Longrightarrow \overline{(S(M))} \text{ compact}\right].$$

Equivalently, S is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, the sequence $(Sx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}$.

1.2 Fixed point theorem

Definition 1.4 Let $\mathbb{X}$ be a vector space over $\mathbb{F}$. A convex subset of $\mathbb{X}$ is a subset $\mathbb{M} \subseteq \mathbb{X}$ such that for every pair of points $x, y \in \mathbb{M}$ and for every $\alpha \in [0,1]$ we have that

$$\alpha x + (1-\alpha)\, y \in \mathbb{M}.$$

Definition 1.5 Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$. A fixed point of a mapping $S : \mathbb{X} \to \mathbb{X}$ of $\mathbb{X}$ into itself is an $x \in \mathbb{X}$ which is mapped onto itself, that is

$$S(x) = x.$$

1.3 Banach's fixed point theorem

One of the very helpful tools which is broadly applicable in proving the existence and uniqueness of solutions, is the well-known Banach fixed point theorem (also known as the contraction mapping theorem or contractive mapping theorem).

Theorem 1.1 *Let $(\mathbb{X}, \|.\|)$ be a Banach space and let $S : \mathbb{X} \to \mathbb{X}$ be a contraction on $\mathbb{X}$. Then S has a unique fixed point $x \in X$ such that*

$$S(x) = x.$$

Theorem 1.2 *If $\mathbb{M}$ is a closed subset of a Banach space $\mathbb{X}$ and $S : \mathbb{M} \to \mathbb{M}$ is a contraction, then S has a unique fixed point in $\mathbb{M}$.*

1.4 Krasnoselskii's fixed point theorem

Theorem 1.3 *Let D be a closed convex nonempty subset of a Banach space$(B, \|.\|)$. Suppose that $\mathcal{F}_2$ and $\mathcal{F}_2$ map D into B such that*

1. $x, y \in D$, implies $\mathcal{F}_1 x + \mathcal{F}_2 y \in D$,
2. $\mathcal{F}_1$ is a contraction mapping,
3. $\mathcal{F}_2$ is completely continuous.

   Then, there exists $z \in D$ with $z = \mathcal{F}_1 z + \mathcal{F}_2 z$.

1.5 Arzelà-Ascoli theorem

Let K be a compact subset of a normed vector space over $\mathbb{F}$ and let $C(K, \mathbb{R})$ be the normed vector space of real valued continuous functions on K with the sup-norm

$$\|f\| = \sup_{x \in \mathbb{X}} |f(x)|.$$

Let $\mathcal{F}$ be a collection of functions in $C(K, \mathbb{R})$.

Definition 1.6 The collection $\mathcal{F}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and $x, y \in \mathbb{X}$ with $\|x - y\| < \delta$ we have $|f(x) - f(y)| < \varepsilon$, i.e.,

$$\forall \varepsilon > 0, \forall x \in \mathbb{X}, \ \exists \delta > 0, \forall y \in \mathbb{X}, \ \ [\|x - y\| < \delta] \Longrightarrow [\forall f \in \mathcal{F}, \ \ |f(x) - f(y)| < \varepsilon].$$

Definition 1.7 The collection $\mathcal{F}$ is said to be uniformly bounded if there is an $M \geq 0$ such that $\|f\| = \sup\limits_{x \in \mathbb{X}} |f(x)| \leq M$ for all $f \in \mathcal{F}$, i.e.,

$$\exists M \geq 0 : \|f\|_\infty = \sup_{x \in \mathbb{X}} |f(x)| \leq M, \ \forall f \in \mathcal{F}.$$

Theorem 1.4 *If $\mathcal{F}$ is a collection of uniformly bounded and equicontinuous functions in $C(K, \mathbb{R})$, then $\mathcal{F}$ is relatively compact in $C(K, \mathbb{R})$.*

1.6 Periodic function

Let f be a function defined on a set I, and let T be a non-zero real constant
.

Definition 1.8 The function f is said to be T-periodic function if

$$f(t+T) = f(t).$$

for all $t \in I$.

Corollary 1.1 *The derivative of a T-periodic function is also a T-periodic function.*

Remark 1.2 The antiderivative of a T-periodic function is not necessarily a T-periodic function for $t \in \mathbb{R}$

1.7 Green's function

The theory of Green's functions is a valuable tool in the analysis of differential equations. Particularly, in solving nonhomogeneous boundary value problems for linear ordinary differential equations where the inverse of the differential operator is an integral operator whose kernel is a Green's function.

Definition 1.9 We will consider two –point n^{th}–order linear boundary value problems of the form

$$\begin{cases} L_n y(t) = \delta(t), & t \in I \equiv [c, d], \\ U_i(y) = \varepsilon_i, & i = 1, ..., m. \end{cases} \tag{1.1}$$

where

$$L_n y(t) = a_0(t)\, y^{(n)}(t) + a_1(t)\, y^{(n-1)}(t) + \dots + a_{n-1}(t)\, y^{(1)}(t) + a_n(t)\, y(t), \tag{1.2}$$

and

$$U_i(y) = \sum_{j=1}^{n-1} \left(\alpha_j^i y^{(j)}(c) + \beta_j^i y^{(j)}(d)\right), i = \overline{1,m},\ m \le n, \tag{1.3}$$

where α_j^i, β_j^i and γ_i real constants for all $i = \overline{1,m}$ and $j = \overline{1,n-1}$, δ and a_k are continuous real functions for all $k = \overline{0,n}$, and $a_0(t) \neq 0$ for all $t \in I$.

We say that G is a Green's function for problem (1.1) if it satisfies the following properties:

$(G1)$ is defined on $I \times I$.

$(G2)$ For $k = \overline{0, n-2}$, the partial derivatives $\dfrac{\partial^k G}{\partial t^k}$ exist and they are continuous on $I \times I$.

$(G3)$ $\dfrac{\partial^{k-1} G}{\partial t^{k-1}}$ and $\dfrac{\partial^k G}{\partial t^k}$ exist and they are continuous on the triangles $c \le s < t \le d$ and $c \le t < s \le d$.

$(G4)$ For each $t \in (c,d)$ there exist the lateral limits

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^+\right) \text{ and } \frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^+\right),$$

(i.e., the limits when $(t,s) \to (t,t)$ with $s > t$ or with $s < t$) and, moreover

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^+\right) - \frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^+\right) = -\frac{1}{a_0(t)}.$$

$(G5)$ For each $s \in (c,d)$, the function $t \to G(t,s)$ is a solution of the differential equation $L_n y = 0$ on $t \in [c,s)$, and $t \in (s,d]$. That is,

$$a_0(t) \frac{\partial^n G}{\partial t^n}(t,s) + a_1(t) \frac{\partial^{n-1} G}{\partial t^{n-1}}(t,s) + \dots + a_{n-1}(t) \frac{\partial G}{\partial t}(t,s) + a_n(t)\, G(t,s) = 0,$$

on both intervals.

$(G6)$ For each $s \in (c, d)$, the function $t \rightarrow G(t, s)$ satisfies the boundary conditions

$$\sum_{j=0}^{n-1} \left(\alpha_j^i \frac{\partial^j G}{\partial t^j}(c, s) + \beta_j^i \frac{\partial^j G}{\partial t^j}(d, s) \right) = 0, \; i = \overline{1, m}.$$

Theorem 1.5 *Let us suppose that the homogeneous problem of problem (1.1) has only the trivial solution. Then there exists a unique Green's function, $G(t, s)$ associated to problem (1.1). Moreover, the unique solution is given by the expression*

$$y\,((t)) = \int_c^d G(t, s)\delta\,(s)\,ds.$$

CHAPTER 2

Delay functional differential equations

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The aim of this chapter is to provide the necessary background for understanding the notion of delay differential equations.

2.1 Introduction

Delay differential equations (DDEs) are a class of differential equations that involve delays or memory effects in their formulations. Something like

$$x^{(m)}(t) = f\left(t, x^{(m_1)}(t-\tau_1), ..., x^{(m_k)}(t-\tau_k)\right),$$

where τ_i, $i = \overline{1,k}$ are the delays and f is a given function. The presence of the terms $x(t-\tau_i)$ indicates that the state $x(t)$ at time t depends on its state at some previous times $t-\tau_i$, where $i = \overline{1,k}$. Such equations have been widely used in modeling physical and biological phenomena that exhibit time delays in their dynamics. For instance, they are commonly used to model the dynamics of populations with time delays in their reproduction and the spread of infectious diseases with incubation periods. The meaning of the memory or the time-delay differs from one model to another. For instance, it can be related to the incubation period of an infectious disease in epidemiology, the time between initiation of cellular production in the bone marrow and release of mature cells into the blood in the production of blood cells in hematology, the transit time or the duration of a cellular transformation in the dynamics of cell populations, the time of gestation, development, the juvenile phase, life cycle or the period of maturation in population dynamics of certain human, animal and plant species, a time lag that often arises in feedback loops involving sensors and actuators in engineering and also an information lag in economic dynamics, to name a few.

2.2 A Brief History

The first reference about a delay differential equation goes back to the XVIIIth century and is due to J. Bernoulli (1728). As the (Latin) title of the paper shows, he considered a weighted stretched vibrating cord with distributed masses on it. He finally led to

$$y' = y\,(t-1)\,,$$

but he ignored it since he thought that there were several mistakes in deducing the equation.

In 1908, during the international conference of mathematicians, Picard emphasized the significance of accounting for hereditary effects when constructing models of physical systems with the following statement:

"Nous pouvons rêver d'équations fonctionnelles plus compliquées que les équations classiques parce qu'elles renfermeront en outre des intégrales prises entre un temps passé très éloigné et le temps actuel, qui apporteront la part de l'hérédité".

Since then, equations of this kind have drawn a lot of attention from scientific researchers which led to publish a great amount of works especially in the fifties that saw an explosion of scientific activities in this direction. To be honest, the thirties of the last century have paved the way to this explosion of research and pushed forward the frontiers of knowledge to improve or to increase the number of publications on this topic. For example, the works and the interesting obtained results of Volterra who wrote in his works on the role of hereditary effects on models for the interaction of species, have contributed in developing both the theoretical ecology and the theory of delay differential equations and their applications.

During the fifties, there have been a sustained attention by scholars as well as a lot of activities and developments in this direction that led to a collection of publications done by Myshkis (1951), Krasovskii (1959), Bellman, Cooke (1963), and Halanay (1966) in which these scholars have gave a clear picture of the topic.

They have witnessed massive movement in recent years. It now occupies a prime position in all fields of engineering and science due to their wide-spread applications in many branches of science, such as life sciences, physical sciences, economical sciences and engineering.

2.3 Classification

Functional differential equations with delays can be classified as linear or nonlinear, autonomous or non-autonomous, etc. Here, we are interested in giving a classification of them according to the most known types of delays cited in the literature where we distinguish two main classes, the first is called "delay differential equations" and the other "delay differential equations of neutral type".

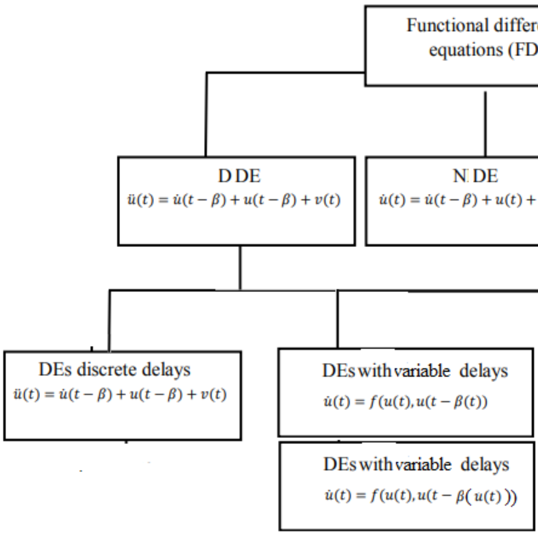
The delay τ can be constant, we call such equation, a DDE with a discrete or constant delay.

If τ_i depends on the time,$\tau = \tau_i(t)$, we are talking about a DDE with a time-dependent or time varying delay.

If τ_i depends on the state,$\tau = \tau(x(t))$, we are talking about a DDE with a state-dependent delay.

If τ_i appears in the highest derivative of the equation, we are talking about a DDE of neutral type.

There are other types of DDEs such DDEs with distributed delays.



Classification of functional differential equations

It is worth noting here that ordinary differential equation can be regarded as a special case and a starting point of delay differential equation. The substantial differences between the two can be summarized as follows:

Delay Differential Equations	**Ordinary Differential Equations**
Supposed to take into account the history of the past due to the influence of the changes on the system is not instantaneous (Hale,1993)	Supposed to take into account the principle of causality due to the influence of the changes on the system is instantaneous (Hale,1993)
Depends on initial function to define a unique solution	Depends on initial value to define a unique solution
Give a system that is infinite dimensional	Give a system that is finite dimensional
Analytical theory is well less developed	Analytical theory is well developed (Lumb,2004)

2.4 Existence and uniqueness results

In this section, we present some classic results on existence and uniqueness of solutions.

2.4.1 Delay differential equations

Given a number $\tau \geq 0$, $\mathcal{C}([a,b],\mathbb{R}^n)$, the *Banach* space of continuous functions defined on $[a,b]$ with values in $\mathbb{R}^n$ is provided with the norm of uniform convergence. If $[a,b] = [-\tau, 0]$, we put $C = C([-\tau,0],\mathbb{R}^n)$ and we denote the norm of an element $\Phi \in C$ by

$$\|\varphi\| = \sup_{-\tau \leq t \leq 0} |\varphi(t)| .$$

If $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in \mathcal{C}([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then for $t \in [t_0, t_0 + A]$, we define $x_t \in \mathcal{C}$ by

$$x_t(s) = x(t+s),$$

for all $s \in [-\tau, 0]$.

Definition 2.1 [29] If $\mathfrak{D}$ is a subset of $\mathbb{R} \times \mathcal{C}$, $f : \mathfrak{D} \longrightarrow \mathbb{R}^n$ is a given function and here represents the derivative on the right, the equation

$$\dot{x}(t) := f(t, x_t), \tag{2.1}$$

where

$$x_t(s) := x(t+s), \ \ s \in [-\tau, 0], \tag{2.2}$$

is a delay functional differential equation on $\mathfrak{D}$ denoted and the number τ is called the delay.

It is clear that the case $\tau = 0$ corresponds to the case of ordinary differential equations. It is obvious that an initial condition appropriate to the time $t = t_0$ requires the determination of the function x over the entire interval $[t_0 - \tau, t_0]$.

$$x(t) = \psi(t), \; t \in [t_0 - \tau, t_0], \tag{2.3}$$

where $\psi : [t_0 - \tau, t_0] \longrightarrow \mathbb{R}^n$ is a given function assumed to be continuous called the initial condition of the delay equation (2.1) Thus, the equation (2.1) can be written in the form

$$\begin{cases} x'(t) := f(t, x_t), & t \geq t_0 \\ x(t) = \psi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{2.4}$$

where is ψ a given continuous function on the interval $[t_0 - \tau, t_0]$.

Definition 2.2 [29] Given $\psi \in C$ and $t_0 \in \mathbb{R}$, a solution of the equation (2.1) is a function denoted $x(t)$ such that $x(t) = \psi(t)$ if $t \in [t_0 - \tau; t_0]$ and satisfying (2.1) if $t \in [t_0, t_0 + A]$ with $A > 0$. Such a function $x(t)$ is called the solution of (2.1) through $(t_0, \psi)$and it is often denoted by

$$x(t) = x\left(t_0, \psi, f\right). \tag{2.5}$$

Lemma 2.1 *[29]Let $\psi \in \mathcal{C}$, $t_0 \in \mathbb{R}$ and $f\,(t, \psi)$ a continuous function. Finding solutions of the equation (2.1) though (t_0, ψ) is equivalent to solving*

$$\begin{cases} x_{t_0} = \psi, \\ x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u)du, \; t \geq t_0. \end{cases}$$

Theorem 2.1 *(Existence) [29] For equation (2.1), suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and $f \in \mathcal{C}(\Omega, \mathbb{R}^n)$ is a continuous mapping on Ω. if $(t_0, \psi) \in \Omega$, then there exists a solution of the equation (2.1) passing through (t_0, ψ).*

Definition 2.3 [29] The function $f(t,\varphi)$ is said to be a Lipschitz mapping with respect to φ on a compact K of $\mathbb{R}\times\mathcal{C}$ if there is a constant $k>0$ such that for all $(t,\psi_i)\in K,\ i=1,2$, one has

$$|f(t,\psi_1)-f(t,\psi_2)|\leq k\ \ |\psi_1-\psi_2|\,. \tag{2.6}$$

Theorem 2.2 *[29] Suppose that Ω is an open subset of $\mathbb{R}\times\mathcal{C}$, $f:\Omega\longrightarrow\mathbb{R}^n$ is continuous and $f(t,\psi)$ is a Lipschitz mapping with respect to ψ on any compact subset Ω. If $(t_0,\psi)\in\Omega$, then there is a unique solution of the equation (2.1) passing through (t_0,ψ).*

2.4.2 Neutral differential equation

Definition 2.4 [30] Suppose that Ω is an open subset of $\mathbb{R}\times\mathcal{C}$ of elements (t,ψ). A function $D:\Omega\to\mathbb{R}^n$ is said to be atomic at a point β of Ω. if D, its first and second derivatives are continuous in the sense of Fréchet with respect to ψ and D_ψ, is atomic in β of Ω.

Definition 2.5 [30] Suppose that Ω is an open subset of $\mathbb{R}\times\mathcal{C}$, $D:\Omega\to\mathbb{R}^n$ and $f:\Omega\to\mathbb{R}^n$ are two given continuous functions with D is atomic in zero .The relation

$$\frac{d}{dt}D(t,x_t)=f(t,x_t), \tag{2.7}$$

is said to be a differential equation of neutral type.

Theorem 2.3 *[30] (Existence) If Ω is an open subset of $\mathbb{R}\times\mathcal{C}$ and $(t_0,\psi)\in\Omega$, then there is a solution of the equation (2.7) passing through (t_0,ψ) .*

Theorem 2.4 *[30] (Existence and uniqueness) if Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and $f(t, \psi)$ is a Lipschitz mapping with respect to ψ on any compact subset of Ω, so for all $(t_0, \psi) \in \Omega$, there is a unique solution for the equation (2.7) passing through (t_0, ψ).*

2.5 Solving delay differential equations

Delay differential equations can be solved by using many methods depending on the form of the equation itself and its complexity when dealing with it. For instance, we can use the step method, the Runge-Kutta method, the Laplace method, or by using a Software such as Matlab, Maple, Mathematica,...

2.5.1 Step method

The step method (also known as "step by step method", "step method" or "successive integration method") makes it possible to numerically solve DDEs and NDEs and at the same time makes it possible to establish the existence and uniqueness of the solution. It was presented in 1965, by R. Bellman for constant delays. Others like El'sgol'ts and Norkin (1973) have shown that it also remains valid for variable delays, provided that the delay never vanishes. To fix the ideas, we consider a particular case of the following "Frisch-Holme" delay linear functional differential equation:

$$\begin{cases} x'(t) = a_1 x(t) + a_2 x(t-\tau) & \text{for all } t \in [0, \tau] \\ x(t) = \varphi(t) & \text{for all } t \in [-\tau, 0], \end{cases} \tag{2.8}$$

where a_1 and a_2 are two real constants.

Example 2.1 Consider the following equation of the Frisch-Holme type:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x(t-\tau), & For\ t \in [0, 2\tau] \\ x(t) = \varphi(t) = 1 & -\tau \leq t \leq 0. \end{cases} \tag{2.9}$$

<u>1^{st} **step:**</u> In the interval $[-\tau, 0]$

$$x(t) = 1.$$

<u>2^{st} **step:**</u> In the interval $[0, \tau]$

The integration of the two sides of the DDE from 0 to t, gives

$$x(t) = \alpha \int_0^t x(s-\tau)ds + x(0). \tag{2.10}$$

Since $0 \leq s < \tau$, so $-\tau \leq s - \tau < 0$. Since $x(t) = 1$ For $t \in [-\tau, 0[$, then

$$x(s-\tau) = 1,$$

for $s \in [0, \tau]$, which leads to

$$x(t) = \alpha t + 1,$$

in the interval $[0, \tau]$.

<u>3^{st} **step:**</u> In the interval $[\tau, 2\tau]$

The integration of the two sides from τ to t, gives

$$x(t) = \alpha \int_\tau^t x(s-\tau)ds + x(\tau).$$

Since $\tau \leq s < 2\tau$, then $0 \leq s - \tau < \tau$. Since $x(t) = \alpha t + 1$ for $t \in [0, \tau[$, then

$$x(s-\tau) = \alpha(s-\tau) + 1,$$

for $s \in [\tau, 2\tau]$, which leads to

$$x(t) = \alpha^2 \frac{t^2}{2} + \left(\alpha - \alpha^2\tau\right) t - \alpha^2 \frac{\tau^2}{2} + \alpha^2\tau^2 - \alpha\tau,$$

in the interval $[\tau, 2\tau]$.

Finally, we obtained

t	**ODE with initial condition**
$[0, \tau]$	$\begin{cases} x'(t) = \alpha \\ x(0) = 1 \end{cases}$
$[\tau, 2\tau]$	$\begin{cases} x'(t) = \alpha^2(t-\tau) + \alpha \\ x(\tau) = \alpha\tau + 1, \end{cases}$

and the solution is given by

$$x(t) = \begin{cases} \alpha t + 1 & 0 \leq t \leq \tau \\ \alpha^2 \dfrac{t^2}{2} + (\alpha - \alpha^2 \tau)\, t - \alpha^2 \dfrac{\tau^2}{2} + \alpha^2\tau^2 - \alpha\tau & \tau \leq t \leq 2\tau. \end{cases} \tag{2.11}$$

If we take $\alpha = -1$, $\tau = 1$ with the same initial condition $\varphi(t) = 1$, we will have

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ \dfrac{t^2}{2} - 2t + \dfrac{3}{2} & 1 \leq t \leq 2. \end{cases}$$

CHAPTER 3

Existence results for a fourth-order neutral delay differential equation

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3.4 Existence of positive periodic solutions 30

In this chapter, we use the Krasnoselskii fixed-point theorem combined with the Green's functions method to provide some sufficient conditions that guarantee the existence of positive periodic solutions of a class of fourth order nonlinear neutral differential equations.

3.1 Introduction

Fourth-order delay differential equations with periodic coefficients have taken great interest by many scholars due to their crucial role in providing a more accurate and realistic description of many real phenomena in different fields, ranging from life sciences to physics, technology, and engineering. For instance, they appear in viscoelastic and inelastic flows [46, 49, 51], deformation of structures [47, 50], soil settlement [25] and electric circuits. A lot of attention has been paid to this topic. We cite as recent contributions the following papers:

By virtue of the Krasnoselskii fixed point theorem, Yan Sun and Cun Zhu [67] proved the existence of positive solutions of the following fourth-order three point boundary value problem:

$$y^{(4)}(t) + f(t, y(t)) = 0, \ t \in [0,1],$$
$$y(0) = y'(0) = y''(0) = 0, \ y''(1) - \alpha y''(\eta) = \lambda,$$

where $\eta \in (0,1)$, $\alpha \in \left[0, \frac{1}{\eta}\right)$ and the parameter $\lambda \in [0, \infty)$.

Ertürk [26] investigated the following fourth-order three-point boundary value problem:

$$y^{(4)}(t) + g(t, y) = 0, \ t \in [c, d],$$
$$y(c) = y'(c) = y''(c) = 0, \ y(d) = \lambda y(\xi), \ \xi \in [c, d], \ \lambda \in \mathbb{R}.$$

He used the Banach fixed point theorem to establish the existence and uniqueness results.

In [2], Balamuralitharan employed the coincidence degree continuation theorem for studying the existence of positive periodic solutions for the following fourth-order differential equation with time-varying delay:

$$x^{(4)}(t) + ax^{(3)}(t) + \lambda f(x''(t-\tau(t))) + \lambda g(x'(t-\tau(t))) + h(x(t-\tau(t))) = \lambda p(t).$$

Tunç [68] used the Liapunov functional approach to study the asymptotic stability of zero solution of the following class of fourth-order non-linear differential equations with constant delay:

$$x^{(4)}(t) + \varphi(x''(t))\, x^{(3)}(t) + h(x'(t))\, x''(t) + \phi(x'(t-r)) + f(x(t-r)) = 0.$$

By means of Krasnoselskii's fixed point theorem, the authors of [48] established the existence of positive periodic solutions for the following class of fourth-order nonlinear neutral equations:

$$\frac{d^4}{dt^4}(x(t) - c(t)\, x(t-\tau(t))) = a(t)\, x(t) - f(t, x(t-\tau(t))).$$

In this chapter, we consider the following fourth-order nonlinear neutral differential equation:

$$\frac{d^4}{dt^4}(x(t) - g(t, x(t-\tau(t)))) = -a(t)\, x(t) + f(t, x(t-\tau(t))), \qquad (3.1)$$

where $a, \tau \in C(\mathbb{R}, (0,\infty))$, $g \in C(\mathbb{R} \times [0,\infty), \mathbb{R})$, $f \in C(\mathbb{R} \times [0,\infty), [0,\infty))$, $a, \tau, g(t,x),\ f(t,x)$ are ω periodic in t, ω is a positive constant and the function $g(t,x)$ is Lipschitz continuous in x. That is to say, there exists a positive constant k such that

$$\|g(t,x) - g(t,y)\| \leq k\,\|x - y\|, \forall t \in [0,\omega], x, y \in P_\omega. \qquad (3.2)$$

For a fixed $\omega > 0$, we consider the space

$$P_\omega = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t)\},$$

of continuous and $\omega-$periodic functions with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,w]} |x(t)|, \qquad (3.3)$$

is a Banach space. Moreover, for some positive constants $L_1, L_2 > 0$ we define the set

$$\mathbb{D} = \{\varphi \in P_\omega : L_1 \leq \varphi \leq L_2\}, \tag{3.4}$$

which is bounded and convex subset of P_ω.

We put

$$\begin{aligned} P_\omega^+ &:= \{x \in P_\omega : x > 0\}, \quad & m &:= \inf_{t\in[0,w]} a(t), \\ M &:= \sup_{t\in[0,w]} a(t), & \beta &:= \sqrt[4]{M}. \end{aligned} \tag{3.5}$$

3.2 Green function

Lemma 3.1 *The equation*

$$\frac{d^4}{dt^4}x(t) + Mx(t) = h(t), \; h(t) \in P_\omega^+, \tag{3.6}$$

has a unique ω-periodic solution

$$x(t) = \int_t^{t+\omega} G(t,s)\, h(s) ds, \tag{3.7}$$

where

$$\begin{aligned} G(t,s) &= \frac{1}{4\gamma^3}\left(A\left(\sinh\left(s-t-\frac{\omega}{2}\right)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right)\right) \\ &\quad +B\left(\cosh\gamma\left(s-t-\frac{\omega}{2}\right)\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right), \end{aligned} \tag{3.8}$$

for all $s \in [t, t+\omega]$.

$$A := \frac{\sin\left(\gamma\frac{\omega}{2}\right)\cosh\left(\gamma\frac{\omega}{2}\right) - \cos\left(\gamma\frac{\omega}{2}\right)\sinh\left(\gamma\frac{\omega}{2}\right)}{\cosh\gamma\omega - \cos\gamma\omega}, \tag{3.9}$$

and

$$B := \frac{\cos\left(\gamma\frac{\omega}{2}\right)\sinh\left(\gamma\frac{\omega}{2}\right) - \sin\left(\gamma\frac{\omega}{2}\right)\cosh\left(\gamma\frac{\omega}{2}\right)}{\cosh\gamma\omega - \cos\gamma\omega}. \tag{3.10}$$

Proof. The associated homogeneous equation of equation (3.6) is

$$\frac{d^4}{dt^4}x(t) + Mx(t) = 0,$$

where its characteristic equation is

$$\lambda^4 + M = 0,$$

and the roots of this last characteristic equation are

$$\lambda_1 = \gamma\left(1+i\right),\ \lambda_2 = \gamma\left(-1+i\right),\ \lambda_3 = \gamma\left(-1-i\right),\ \lambda_4 = \gamma\left(1-i\right),$$

where

$$\gamma = \frac{\sqrt{2}}{2}\beta.$$

So, the solution of the homogeneous equation is

$$x(t) = C_1 \exp^{\gamma(1+i)t} + C_2 \exp^{\gamma(-1+i)t} + C_3 \exp^{\gamma(-1+i)t} + C_4 \exp^{\gamma(1+i)t}.$$

We use the method of variation of parameters, to arrive at

$$\begin{aligned} C_1'(t) &= -h(t)\frac{\exp^{-t\gamma(1+i)}}{8\left(1-i\right)\gamma^3}, \\ C_2'(t) &= h(t)\frac{\exp^{t\gamma(1-i)}}{8\left(1+i\right)\gamma^3}, \\ C_3'(t) &= h(t)\frac{\exp^{t\gamma(1+i)}}{8\left(1-i\right)\gamma^3}, \\ C_4'(t) &= -h(t)\frac{\exp^{-t\gamma(1-i)}}{8\left(1+i\right)\gamma^3}. \end{aligned} \tag{3.11}$$

Since $x(t)$, $x'(t), x''(t)$, and $x'''(t)$ are periodic functions, we obtain

$$\begin{aligned} C_1(t) &= t^{t+\omega} - \int_t^{t+w} \frac{\exp^{\gamma(1+i)(\omega+s)}}{8\left(1-i\right)\gamma^3\left[1-\exp^{\gamma(1+i)\omega}\right]}h(s)ds, \\ C_2(t) &= t^{t+\omega} - \int_t^{t+w} \frac{\exp^{\gamma(1-i)s}}{8\left(1+i\right)\gamma^3\left[1-\exp^{\gamma(1-i)\omega}\right]}h(s)ds, \\ C_3(t) &= t^{t+\omega} - \int_t^{t+w} \frac{\exp^{\gamma(1+i)s}}{8\left(1-i\right)\gamma^3\left[1-\exp^{\gamma(1+i)\omega}\right]}h(s)ds, \\ C_4(t) &= t^{t+\omega} - \int_t^{t+w} \frac{\exp^{\gamma(1-i)(\omega-s)}}{8\left(1+i\right)\gamma^3\left[1-\exp^{\gamma(1-i)\omega}\right]}h(s)ds. \end{aligned} \tag{3.12}$$

Hence,

$$\begin{aligned} x(t) &= C_1(t)\exp^{\gamma(1+i)t} + C_2(t)\exp^{\gamma(-1+i)t} + C_3\exp^{\gamma(-1-i)t} + C_4\exp^{\gamma(1-i)t} \\ &= \int_t^{t+\omega} G\,(t,s)\,h\,(s)\,ds, \end{aligned}$$

where $G\,(t,s)$ is identified by (3.8). ■

Lemma 3.2 *Function $G\,(t,s)$ satisfies*

$$\int_t^{t+\omega} G\,(t,s)\,h\,(s)\,ds = \frac{1}{M},$$

and if

$$\max\{a(t) : t \in [0,\omega]\} < 4\left(\frac{\pi}{\omega}\right)^4, \tag{3.13}$$

then

$$0 < \alpha_1 < G\,(t,s) < \alpha_2,\;\; \forall t \in [0,\omega]\,, s \in [t, t+\omega]\,,$$

where

$$\alpha_1 = \frac{1}{4\gamma 3}\frac{\cos(\frac{\gamma\omega}{2})\sinh(\frac{\gamma\omega)}{2}) + \sin(\frac{\gamma\omega}{2}) - \cosh(\frac{\gamma\omega}{2})}{\cosh(\gamma\omega) - \cos\,(\gamma\omega)},$$

and

$$\alpha_2 = \frac{1}{4\gamma^3}\frac{\cos(\frac{\gamma\omega}{2})\sinh(\frac{\gamma\omega)}{2}) + \sin(\frac{\gamma\omega}{2}) - \cosh(\frac{\gamma\omega}{2})}{\cosh(\frac{\gamma\omega}{2}) - \cos\left(\frac{\gamma\omega}{2}\right)}.$$

Proof. We have

$$\begin{aligned}
\int_t^{t+\omega} G(t,s)\,ds &= \frac{1}{4\gamma^4}\left[\int_t^{t+\omega} A\left(\sinh\gamma\left(s-t-\frac{\omega}{2}\right)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right)\right] \\
&\quad + \frac{1}{4\gamma^4}\left[\int_t^{t+\omega} B\left(\cosh\gamma\left(s-t-\frac{\omega}{2}\right)\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right)ds\right] \\
&= \frac{A}{16\gamma^4}\exp^{\gamma\left(s-t-\frac{\omega}{2}\right)}\left[(\exp^{\gamma\left(s-t-\frac{\omega}{2}\right)+1}\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right. \\
&\quad \left.+\ (\exp^{\gamma(2s-2t+\omega)}+1)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right]\Big|_t^{t+w} \\
&\quad + \frac{B}{16\gamma^4}\exp^{\gamma\left(s-t-\frac{\omega}{2}\right)}\left[(\exp^{\gamma\left(2s-2t-\frac{\omega}{2}\right)}+1)\sin\gamma\left(s-t-\frac{\omega}{2}\right)\right. \\
&\quad \left.-\ (\exp^{\gamma\left(s-t-\frac{\omega}{2}\right)+1}\cos\gamma\left(s-t-\frac{\omega}{2}\right)\right]\Big|_t^{t+w} \\
&= \frac{1}{4\gamma^4}\left[2\frac{(\cosh(\frac{\gamma\omega}{2})\sin(\frac{\gamma\omega)}{2}))^2+(\sinh(\frac{\gamma\omega}{2})-\cos(\frac{\gamma\omega}{2}))^2}{\cosh\gamma\omega-\cos\gamma\omega}\right] \\
&= \frac{1}{4\gamma^4}\left[2\frac{(1/2)\cosh\gamma\omega-(1/2)\cos\gamma\omega}{\cosh\gamma\omega-\cos\gamma\omega}\right] = \frac{1}{4\gamma^4} = \frac{1}{\beta^4} = \frac{1}{M}.
\end{aligned}$$

Since

$$\left(\frac{d}{ds}\right)G(t,s) = 0,$$

if and only if $s = t + \frac{\omega}{2}$, then

$$\begin{aligned}
G(t,t) &= G(t,t+\omega) \\
&= \frac{1}{4\gamma^3}\frac{\cos(\frac{\gamma\omega}{2})\sinh(\frac{\gamma\omega)}{2})+\sin(\frac{\gamma\omega}{2})-\cosh(\frac{\gamma\omega}{2})}{\cosh(\frac{\gamma\omega}{2})-\cos\left(\frac{\gamma\omega}{2}\right)} \\
&= \alpha_2,
\end{aligned}$$

and

$$\begin{aligned}
G\left(t,t+\frac{\omega}{2}\right) &= \frac{1}{4\gamma 3}\frac{\cos(\frac{\gamma\omega}{2})\sinh(\frac{\gamma\omega)}{2})+\sin(\frac{\gamma\omega}{2})-\cosh(\frac{\gamma\omega}{2})}{\cosh(\gamma\omega)-\cos(\gamma\omega)} \\
&= \alpha_1.
\end{aligned}$$

Since

$$\max\{a(t) : t\in[0,\omega]\} < 4\left(\frac{\pi}{\omega}\right)^4,$$

we get

$$0 < \frac{\gamma\omega}{2} < \frac{\pi}{2}. \tag{3.14}$$

So,

$$\sin\frac{\gamma\omega}{2} > 0,\ 1 > \cos\frac{\gamma\omega}{2} > 0,\ \ \sinh\frac{\gamma\omega}{2} > 0. \tag{3.15}$$

Consequently

$$0 < \alpha_1 < G(t,s) < \alpha_2,\ \forall t \in [0,\omega], s \in [t, t+\omega].$$

This completes the proof. ■

3.3 Conversion of equation (3.1) into an integral equation

Lemma 3.3 *If* $f(t,x) > 0$*, and*

$$\max\{a(t) : t \in [0,\omega]\} < 4\left(\frac{\pi}{\omega}\right)^4, \tag{3.16}$$

then $x \in P_\omega$ *is a solution of equation (3.1) if and only if*

$$\begin{aligned} x(t) &= g(t, x(t-\tau(t))) + \int_t^{t+\omega} G(t,s)\left[(M - a(s))x(s)\right. \\ &\quad \left.+ f(s, x(s-\tau(s))) - Mg(t, x(s-\tau(s)))\right] ds. \end{aligned} \tag{3.17}$$

Proof. Let $x \in P_\omega$ be a solution of equation (3.1). We rewrite equation (3.1) as follows:

$$\begin{aligned} &\frac{d^4}{dt^4}(x(t) - g(t, x(t-\tau(t)))) + M(x(t) - g(t, x(t-\tau(t)))) \\ &= -a(t)x(t) + f(t, x(t-\tau(t))) + M(x(t) - g(t, x(t-\tau(t)))) \\ &= (M - a(t))x(t) + f(t, x(t-\tau(t))) - Mg(t, x(t-\tau(t))). \end{aligned} \tag{3.18}$$

According to Lemma 3.1, we obtain

$$\begin{aligned}
& x(t)-g(t,x(t-\tau(t))) \\
&= \int_t^{t+\omega} G(t,s)\left((M-a(s))\,x(s)+f(s,x(s-\tau(s)))\right) ds \\
&- \int_t^{t+\omega} G(t,s)\left(Mg(t,x(s-\tau(s)))\right) ds\,,
\end{aligned}$$

which implies that

$$\begin{aligned}
x(t) &= g(t,x(t-\tau(t)))+\int_t^{t+\omega} G(t,s)\left[(M-a(s))\,x(s)\right. \\
&\quad \left. +f(s,x(s-\tau(s)))-Mg(t,x(s-\tau(s)))\right] ds.
\end{aligned}$$

This completes the proof. ■

3.4 Existence of positive periodic solutions

In order to prove the existence of positive periodic solutions, we will use the Krasnoselskii fixed point theorem with help of some properties of the obtained Green's function. Let us define the two operators $\mathcal{F}_1, \mathcal{F}_2 : P_\omega \to P_\omega$ as follows:

$$\mathcal{F}_1(\varphi)(t) := g(t,\varphi(t-\tau(t)))\,, \tag{3.19}$$

and

$$\begin{aligned}
\mathcal{F}_2(\varphi)(t) &: =\varphi(t)=\int_t^{t+\omega} G(t,s)\left[(M-a(s))\,\varphi(s)+f(s,\varphi(s-\tau(s)))\right. \\
&\quad \left. -Mg(t,\varphi(s-\tau(s)))\right] ds.
\end{aligned} \tag{3.20}$$

So, equation (3.16) can be written as follows:

$$\varphi(t)=(\mathcal{F}_1\varphi)(t)+(\mathcal{F}_2\varphi)(t)\,. \tag{3.21}$$

Therefore, any solution of equation (3.21) is a solution of equation (3.1) and vice versa.

Lemma 3.4 *If*

$$k < 1, \tag{3.22}$$

then, $\mathcal{F}_1$ is a contraction.

Proof. It is evident that

$$(\mathcal{F}_1\varphi) + (t+\omega) = (\mathcal{F}_1\varphi)(t). \tag{3.23}$$

Now, for all φ, $\psi \in P_\omega$, we have

$$\begin{aligned} |(\mathcal{F}_1\varphi)(t) - (\mathcal{F}_1\psi)(t)| &= |g(t,\varphi(t-\tau(t))) - g(t,\psi(t-\tau(t)))| \\ &\leq \sup_{t\in[0,\omega]} |g(t,\varphi(t-\tau(t))) - g(t,\psi(t-\tau(t)))| \\ &\leq k\|\varphi-\psi\|. \end{aligned}$$

Thus,

$$\|(\mathcal{F}_1\varphi)(t) - (\mathcal{F}_1\psi)(t)\| \leq k\|\varphi-\psi\|. \tag{3.24}$$

Consequently, it follows from (3.22) that $\mathcal{F}_1 : P_\omega \to P_\omega$ is a contraction. ∎

Lemma 3.5 *If*

$$M < 4(\pi/\omega)^4,$$

and

$$0 < f(t,x) \leq C.$$

Then, $\mathcal{F}_2$ is completely continuous.

Proof. The proof will be made in three steps.

Step 1. We show that $\mathcal{F}_2$ is continuous. Let $\{y_n\}$ be a sequence of P_ω such that $y_n \to y$ in P_ω. We have

$$\begin{aligned} |\mathcal{F}_2(y_n)(t) - \mathcal{F}_2(y)(t)| &\leq \int_t^{t+\omega} G(t,s)\left((M - a(s))\right)|y_n(s) - y(s)| \\ &+ |f(s, y_n(s-\tau(s))) - f(s, y(s-\tau(s)))| \\ &+ M\left|g(t, y_n(s-\tau(s))) + Mg(t, y(s-\tau(s)))\right| ds. \end{aligned}$$

It follows from the continuity of f and g that

$$\|\mathcal{F}_2(y_n) - \mathcal{F}_2(y)\| \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Thus,$\mathcal{F}_2$ is continuous.

Step 2. We prove that $\mathcal{F}_2$ maps bounded sets into bounded sets in $(P_\omega, \|.\|)$. If $r > 0$, let

$$B_r = \{x \in P_\omega, \|x\| < r\}$$

be a bounded ball in $(P_\omega, \|.\|)$.

For $x \in B_r$ we have

$$\begin{aligned} |\mathcal{F}_2(x)(t)| &= \left|\int_t^{t+\omega} G(t,s)\left((M - a(s))x(s) + f(s, x(s-\tau(s)))\right)\right. \\ &\quad -Mg(t, x(s-\tau(s)))|\,ds \\ &\leq \int_t^{t+\omega} G(t,s)\left((M - a(s))x(s) + f(s, x(s-\tau(s)))\right) \\ &\quad -Mg(t, x(s-\tau(s)))|\,ds. \end{aligned} \tag{3.26}$$

From Lemma 3.2 and the fact that $f(t,x) \leq C$, we get

$$\begin{aligned} |\mathcal{F}_2(x)(t)| &\leq \alpha_2 \int_t^{t+\omega} (M - m)r + f(s, x(s-\iota(s))) - Mg(t, y_n(s-\iota(s)))\,ds \\ &\leq \alpha_2\omega\left((M - m)r + C\right). \end{aligned}$$

The estimation of $\|\mathcal{F}_2(x)\|$ implies

$$\|\mathcal{F}_2(x)\| \leq \alpha_2\omega\left((M - m)r + C\right).$$

This shows that $\mathcal{F}_2$ is bounded.

Step 3. We prove that $\mathcal{F}_2$ sends bounded sets into equicontinuous sets. Let $t_1, t_2 \in [0, \omega]$, $t_1 < t_2$, and B_r be a bounded set of P_ω. For all $i \in \{1, 2\}$, we have

$$\begin{aligned} |\mathcal{F}_2(x)(t_i)| &= \int_t^{t+\omega} G(t_i, s)((M - a(s))x(s) + f(s, x(s - \tau(s))) \\ &\quad - Mg(t, x(s - \iota(s))))) \, ds. \end{aligned}$$

We set

$$\mathcal{F}_3 = |\mathcal{F}_2(x)(t_2) - \mathcal{F}_2(x)(t_1)|.$$

So, we obtain

$$\begin{aligned} \mathcal{F}_3 &= \left| \int_{t_2}^{t_2+\omega} G(t_2, s)((M - a(s))x(s) + f(s, x(s - \tau(s)))) \right. \\ &\quad - Mg(t_2, x(s - \tau(s))))ds \\ &\quad - \int_{t_1}^{t_1+\omega} G(t_1, s)((M - a(s))x(s) + f(s, x(s - \tau(s)))) \\ &\quad \left. - Mg(t_1, x(s - \tau(s)))) \right| ds \\ &= \left| \int_{t_2}^{t_1+\omega} G(t_2, s)((M - a(s))x(s) + f(s, x(s - \tau(s)))) \right. \\ &\quad - Mg(t_2, x(s - \tau(s))))ds \\ &\quad + \int_{t_1+\omega}^{t_2+\omega} G(t_2, s)((M - a(s))x(s) + f(s, x(s - \tau(s)))) \\ &\quad - Mg(t_2, x(s - \tau(s))))ds \\ &\quad - \int_{t_1}^{t_2} G(t_1, s)((M - a(s))x(s) + f(s, x(s - \tau(s)))) \\ &\quad - Mg(t_1, x(s - \tau(s))))ds \\ &\quad - \int_{t_2}^{t_1+\omega} G(t_1, s)((M - a(s))x(s) + f(s, x(s - \tau(s)))) \\ &\quad \left. - Mg(t_1, x(s - \tau(s))))ds \right|. \end{aligned}$$

Which implies that

$$\begin{aligned}
\mathcal{F}_3 \leq & \left| \int_{t_2}^{t_1+\omega} G(t_2,s)\left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)ds \right. \\
& + \int_{t_1+\omega}^{t_2+\omega} G(t_2,s)\left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)ds \\
& - \int_{t_1}^{t_2} G(t_1,s)\left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)ds \\
& - \int_{t_2}^{t_1+\omega} G(t_1,s)\left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)ds \\
\leq & \int_{t_2}^{t_1+\omega} \left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)|G(t_2,s)-G(t_1,s)|\,ds \\
& + \int_{t_1+\omega}^{t_2+\omega} |G(t_2,s)|\left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)ds \\
& + \int_{t_1}^{t_2} |G(t_1,s)|\left((M-a(s))x(s)+f(s,x(s-\tau(s)))\right)ds \\
\leq & (M-m)\,r\left(\int_{t_2}^{t_1+\omega} |(G(t_2,s)-G(t_1,s))|\,ds \right. \\
& \left. + \int_{t_1+\omega}^{t_2+\omega} |G(t_2,s)|\,ds + \int_{t_1}^{t_2} |G(t_1,s)|\,ds \right).
\end{aligned}$$

As $t_2 \longrightarrow t_1$, the right-hand side of the above inequality tends to zero. By the Arzela-Ascoli theorem, we conclude that $\mathcal{F}_2$ is a completely continuous operator. This completes the proof. ■

Now, we will prove the existence of positive periodic solutions to equation (3.1) by using the Krasnoselskii fixed-point theorem.

The case where $g(t,x) < 0$

We assume that there exist nonpositive constants $K_1$and K_2 such that

$$K_1 < g(t,x) < K_2, \ \forall t \in [0,\omega], x \in \mathbb{D}. \tag{3.27}$$

Theorem 3.1 *Assume that*

$$M < 4(\pi/\omega)^4,$$

and the function f satisfies

$$\frac{L_1 - K_1}{\alpha_1\omega} \leq f(t, x(t - \tau(t))) \leq \frac{L_2}{\alpha_2\omega} - (M - m) L_2 + MK_1. \tag{3.28}$$

Then, equation (3.1) has a positive ω-periodic solution x in the subset $\mathbb{D}$.

Proof. Let us start by proving that

$$\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) \in \mathbb{D}, \ \forall \varphi, \phi \in \mathbb{D}. \tag{3.29}$$

In fact,

$$\begin{aligned}
\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) &= g(t, \varphi(t - \tau(t))) + \int_t^{t+\omega} G(t,s)((M - a(s))\phi(s) + f(s, \phi(s - \tau(s)))) \\
&\quad - Mg(t, \phi(s - \tau(s))))ds \\
&\leq \alpha_2\omega((M - m) L_2 + MK_1) + \alpha_2 \int_t^{t+\omega} f(s, \phi(s - \tau(s)))\, ds \\
&\leq \alpha_2\omega((M - m) L_2 - MK_1) + \alpha_2\omega\left(\frac{L_2}{\alpha_2\omega} - (M - m) L_2 + MK_1\right) = L_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) &= g(t, \varphi(t - \tau(t))) + \int_t^{t+\omega} G(t,s)((M - a(s))\phi(s) + f(s, \phi(s - \tau(s)))) \\
&\quad - Mg(t, \phi(s - \tau(s))))ds \\
&\geq K_1 + \alpha_1\omega \int_t^{t+\omega} f(s, \phi(s - \tau(s)))\, ds \\
&\geq K_1 + \alpha_1\omega\left(\frac{L_1 - K_1}{\alpha_1\omega}\right) = L_1,
\end{aligned}$$

which leads to

$$\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) \in \mathbb{D}, \ \forall \varphi, \phi \in \mathbb{D}.$$

From Lemma 3.4, $\mathcal{F}_1$ is a contraction. Also, from Lemma 3.5, the operator $\mathcal{F}_2$ is completely continuous. Then the Krasnoselskii fixed-point theorem ensures that $\mathcal{F}_1 + \mathcal{F}_2$ has at least a fixed point $\varphi \in \mathbb{D}$ which is a solution to equation (3.21), so equation (3.1) has at least a positive ω -periodic solution x in the subset $\mathbb{D}$. ■

The case where $g(t,x) = 0$

Theorem 3.2 *Assume that*

$$M < 4(\pi/\omega)^4,$$

and

$$\frac{L_2}{\alpha_1 \omega} \leq f(s, x(s-\tau(s))) \leq \frac{L_1}{\alpha_1 \omega} - (M-m) L_1, \ \forall t \in [0,\omega], \ x \in \mathbb{D}. \tag{3.30}$$

Then, equation (3.1) has a positive ω -periodic solution x in the subset $\mathbb{D}$.

Proof. We have $\mathcal{F}_1 = 0$. Similarly to the proof of Theorem 3.1, we show that equation (3.1) has a positive ω-periodic solution $x \in \mathbb{D}$. Since $F(t,x) > 0$, it is easy to see that $x(t) > 0$; i.e., equation (3.1) has a positive ω-periodic solution $x \in \mathbb{D}$. ■

The case where $g(t,x) > 0$

We assume that there exist nonnegative constants K_3 and K_4 such that

$$K_3 \leq g(t,x) \leq K_4, \ \forall t \in [0.\omega], \ x \in \mathbb{D}. \tag{3.31}$$

Theorem 3.3 *Assume that*

$$M < 4(\pi/\omega)^4,$$

and the function f satisfies

$$\frac{L_1 - K_1}{\alpha_1 \omega} \leq f(t, x(t-\tau(t))) \leq \frac{L_2 - K_4}{\alpha_2 \omega} - (M-m) L_2 + MK_3. \tag{3.32}$$

Then, equation (3.1) has a positive -periodic ω solution x in the subset $\mathbb{D}$.

Proof. According to Lemma 3.5, it follows that the operator $\mathcal{F}_1$ is a contraction, and from Lemma 3.5, the operator $\mathcal{F}_2$ is completely continuous. Now, we prove that

$$\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) \in \mathbb{D}, \ \forall \varphi, \phi \in \mathbb{D}.$$

We have

$$\begin{aligned}
\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) &= g(t, \varphi(t - \tau(t))) \\
&\quad + \int_t^{t+\omega} G(t,s)\left((M - a(s))\phi(s) + f(s, \phi(s - \tau(s)))\right) \\
&\quad - Mg(t, \phi(s - \tau(s)))\, ds \\
&\leq K_4 + \alpha_2\omega\left((M - m) L_2 + MK_3\right) \\
&\quad + \alpha_2 \int_t^{t+\omega} f(s, \varphi(s - \tau(s)))\, ds \\
&\leq K_4 + \alpha_2\omega\left((M - m) L_2 - MK_3\right) \\
&\quad + \alpha_2\omega\left(\frac{L_2 - K_4}{\alpha_2\omega} - (M - m) L_2 + MK_3\right) \\
&= L_2.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) &= g(t, \varphi(t - \tau(t))) \\
&\quad + \int_t^{t+\omega} G(t,s)\left((M - a(s))\phi(s) + f(s, \phi(s - \tau(s)))\right) \\
&\quad - Mg(t, \phi(s - \tau(s)))\, ds \\
&\geq K_3 + \alpha_1\omega \int_t^{t+\omega} f(s, \varphi(s - \tau(s)))\, ds \\
&\geq K_3 + \alpha_1\omega\left(\frac{L_1 - K_3}{\alpha_1\omega}\right) = L_1.
\end{aligned}$$

Thus,

$$\mathcal{F}_1(\varphi) + \mathcal{F}_2(\phi) \in \mathbb{D}, \ \forall \varphi, \phi \in \mathbb{D}.$$

By the Krasnoselskii theorem, we deduce that $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ has a fixed point which is a solution to equation (3.1), so equation (3.1) has a positive ω -periodic solution x in the subset $\mathbb{D}$. ■

Conclusion

Fourth-order neutral differential equations play crucial roles in every facet of real life since they are a powerful tool that can be used effectively for modelling a growing number of natural phenomena accurately.

Via a hybrid technique that combines the fixed point theory with the Green's functions method, we have obtained the existence results. More precisely, we have followed three main steps: first of all, we have defined an appropriate Banach space and a suitable subset of it to apply the Krasnoselskii fixed point theorem and to satisfy some mathematical requirements; we have next converted the problem at hand into an integral equation whose kernel is a Green's function and finally, by employing the Krasnoselskii fixed point theorem with the help of Arzelá-Ascoli theorem and some properties of the obtained kernel, we have proved the existence of at least one positive periodic solution.

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