الجمهورية الجزائرية الديمقراطية الشعبية

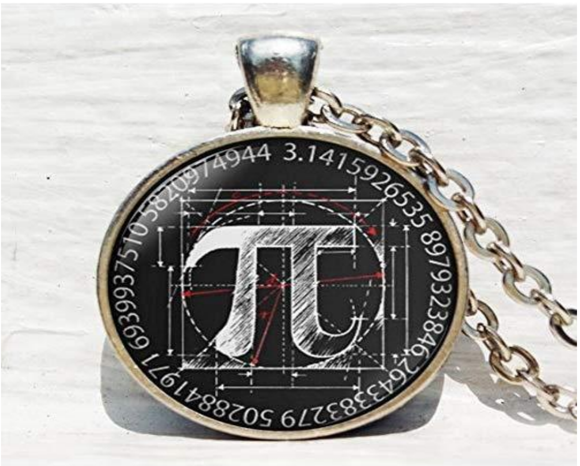
PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINIETRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

UNIVERSITY 20 AUGUST 1955-SKIKDA

FACULTY OF SCIENCES

DEPARTMENT OF MATHEMATICS



Course of:

MATHEMATICS 3

Submitted by:

Dr BENDIB EL OUAHMA

ABSTRACT

This course is intended to provide second-year science and technology students.

Academic year: 2023/2024

Semester: 3

UE: UEF 2.1.1

Material 1: Mathematics 3 **(**WHV : 67h30, Course : 3h00, TD : 1h30)

Objectives of education :

At the end of this course, the student should be able to know the different types of series and its convergence conditions thus the different types of convergence.

Recommended background knowledge

Mathematics 1 et Mathematics 2

Content of the material:

Chapter 1: Multiple integrals **3 weeks**

1.1 Recalls on the Riemann integral and on the calculation of primitives.

1.2 Double and triple integrals.

1.3 Application to the calculation of plane areas and volumes …

Chapter 2: Improper integrals **2 weeks**

2.1 Integration over a bounded interval.

2.2 Improper integrals over an infinite interval.

Chapter 3: Differential equations **3 weeks**

3.1 Recalls on ordinary differential equations.

3.2 Partial differential equations.

3.3 Special functions.

Chapter 4: Series **2 weeks**

4.1 Numerical series.

4.2 Sequences and series of functions.

4.3 Power series, Fourier series.

Chapter 5: Fourier transform **3 weeks**

5.1 Definitions and properties.

5.2 Application to the resolution of differential equations.

Chapter 6: Laplace transform **2 weeks**

6.1 Definitions and properties.

6.2 Application to the resolution of differential equations.

Evaluation method: Continuous control: 40 % ; Final examination: 60 %.

Bibliographical references:

(Depending on the availability of documentation at the establishment level, Websites...etc.)

Contents

Preface

The aim of this course is to provide general training in numerical series and integral calculus for second-year science and technology students, and to know a new mathematical tools. This course provides the fundamental concept of numerical series, multiple integrals, improper integrals, differential equation and how to use Fourier transform and Laplace transform for solving some differential equations.

1 Multiple integrals

This chapter shows how to integrate functions of two or three variables. All the basic concepts related to the methods to compute such double integrals and triple integrals are discussed.

1.1 Double integrals

1.1.1 Double integrals on rectangular domains

Theorem 1.1. *(Fubini's Theorem on rectangular domains)*
If $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous in* $D = [a, b] \times [c, d]$*, then*

$$\begin{aligned} \int\int_D f(x,y)\,dxdy &= \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx \\ &= \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy. \end{aligned}$$

Geometrical interpretation of double integral

Let $f(x,y) > 0$ for every $(x,y) \in D$ and f be continuous. We can consider the value $\int\int_D f(x,y)dxdy$ as the volume of the solid S bounded by the planes $x = a,\ x = b,\ y = c,\ y = d$ and the surface $z = f(x,y)$. (See figure(1.1)).

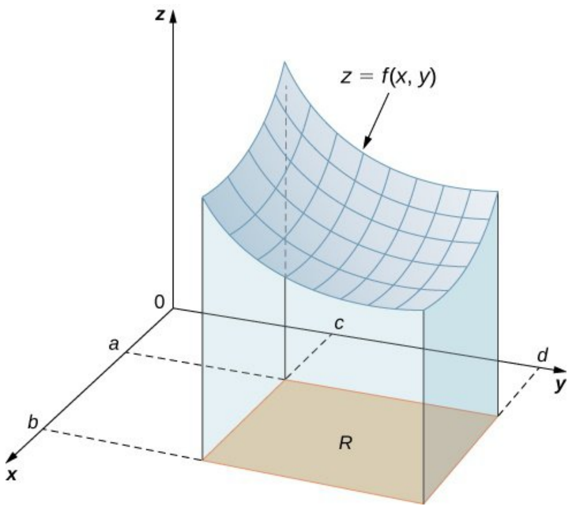


Figure 1.1: Double integral

Example 1.1. *Evaluate* $I = \int\int_D xe^{xy}dxdy$, *on the domain* $D = [0,1] \times [1,2] \subset \mathbb{R}^2$.

Solution:

$$\begin{aligned} I = \int\int_D xe^{xy}dxdy &= \int_0^1 \left[\int_1^2 xe^{xy}dy\right] dx \\ &= \int_0^1 [e^{xy}]_1^2 \, dx = \int_0^1 \left[e^{2x} - e^x\right] dx \\ &= \left[\frac{1}{2}e^{2x} - e^x\right]_0^1 = \frac{1}{2}e^2 - e^1 + \frac{1}{2} \ . \end{aligned}$$

Corollary 1.1. *Let* $D = [a,b] \times [c,d]$. *If* $f(x,y) = g(x)h(y)$ *for some functions* $g : [a,b] \longrightarrow \mathbb{R}$ *and* $h : [c,d] \longrightarrow \mathbb{R}$, *then*

$$\int\int_D g(x)h(y)dxdy = \left(\int_a^b g(x)\, dx\right) \cdot \left(\int_c^d h(y)\, dy\right).$$

Example 1.2. *Evaluate* $I = \int\int_D xy^2dxdy$, *on the domain* $D = [0,1] \times [0,2] \subset \mathbb{R}^2$.

Solution:

$$\begin{aligned} I = \int\int_D xy^2dxdy &= \left(\int_0^1 xdx\right) \cdot \left(\int_0^2 y^2dy\right) \\ &= \left(\left[\frac{x^2}{2}\right]_0^1\right) \cdot \left(\left[\frac{y^3}{3}\right]_0^2\right) = \frac{4}{3} \ . \end{aligned}$$

1.1.2 Properties of double integrals

The properties of the double integral need and deserve attention.

Property 1.1.

1. ***Linearity:*** $\int\int_D (f+g)\, dxdy = \int\int_D f dxdy + \int\int_D g dxdy$.
2. ***Constant comes outside:*** $\int\int_D \alpha f(x,y)\, dxdy = \alpha \int\int_D f(x,y)\, dxdy$.
3. *D splits into D_1 and D_2 (not overlapping): If $D = D_1 \cup D_2$ et $D_1 \cap D_2 = 0$, then*

$$\int\int_D f(x,y)\, dxdy = \int\int_{D_1} f(x,y)\, dxdy + \int\int_{D_2} f(x,y)\, dxdy.$$

1.1.3 Double integrals on non-rectangular domains (stronger form)

Theorem 1.2.
If $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous in D. If $D = \{(x,y) \in \mathbb{R}^2 /\ a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$, with g_1 and g_2 continuous functions on $[a,b]$, then

$$\int\int_D f(x,y)\, dxdy = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\right] dx.$$

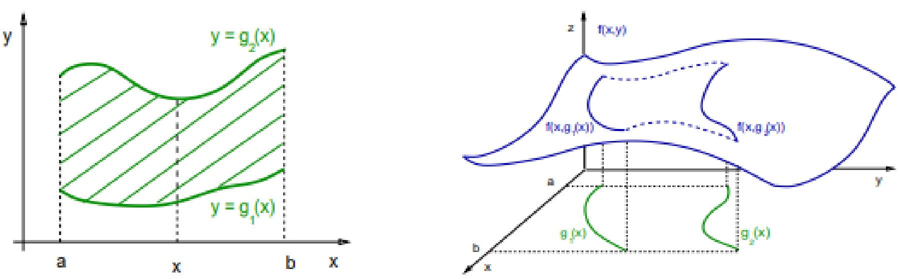


Figure 1.2

Theorem 1.3.
If $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous in D. If $D = \{(x,y) \in \mathbb{R}^2 /\ h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$, with h_1 and h_2 continuous functions on $[c,d]$, then

$$\int\int_D f(x,y)\, dxdy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\right] dy.$$

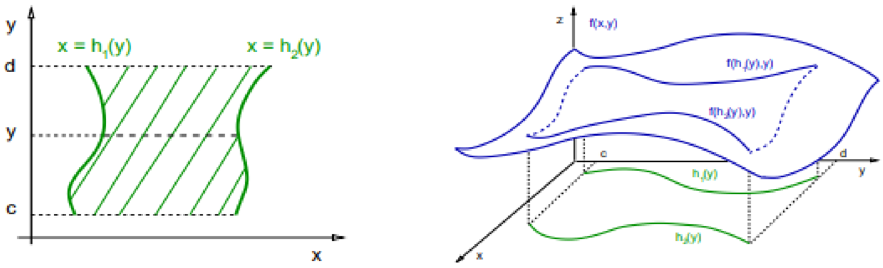


Figure 1.3

Example 1.3. *Find the integral of $f(x,y) = x^2 + y^2$, on the domain*

$$D = \left\{(x,y) \in \mathbb{R}^2 /\ 0 \leq x \leq 1,\ \ x^2 \leq y \leq x\right\}.$$

Solution

The region D is presented by

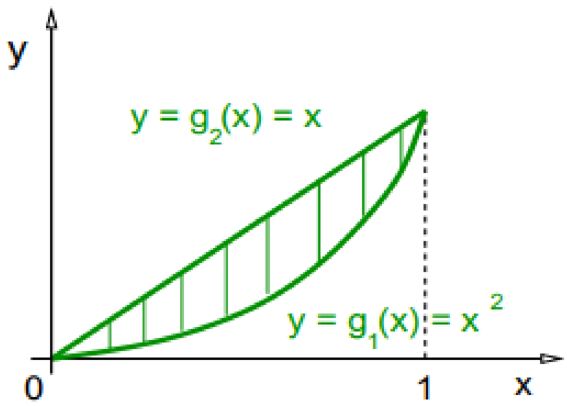


Figure 1.4

$$\begin{aligned}
\int_0^1 \left[\int_{x^2}^{x} \left(x^2 + y^2\right) dy\right] dx &= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{x} dx = \int_0^1 \left[\frac{4}{3}x^3 - \frac{x^6}{3} - x^4\right] dx \\
&= \left[\frac{x^4}{3} - \frac{x^5}{5} - \frac{x^7}{21}\right]_0^1 = \frac{3}{35}.
\end{aligned}$$

1.1.4 Plane Area and volume using double integral

Plane Area: If $f(x, y) = 1$, we get the area of the region D.

Example 1.4. *Find the area bounded by the the parabola* $y = x^2$ *and the straight line* $y = x + 2$.

Solution:

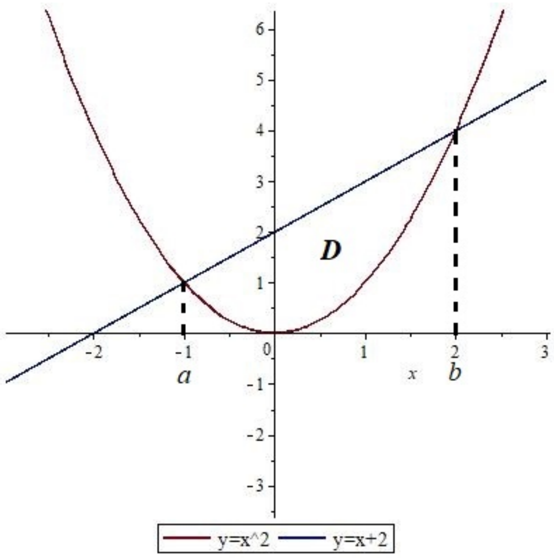


Figure 1.5

The area of the region D in figure (1.5) becomes $A = \int_a^b \int_{x^2}^{x+2} dxdy$.
We must find a and b, for that reason we put $x^2 = x + 2$ and we calculate the roots of the equation $x^2 - x - 2 = 0$. We obtain $a = -1$ and $b = 2$. Then

$$A = \int_{-1}^{2} \left(\int_{x^2}^{x+2} dy \right) dx = \int_{-1}^{2} \left(x + 2 - x^2 \right) dx = \frac{9}{2}$$

Volume: If f is continuous and $f(x, y) > 0$ on the region D, we can consider the double integral $V = \int \int_D f(x, y)dxdy$ as the volume of the solid under the surface $z = f(x, y)$ above the region D.

Example 1.5. *Find the volume bounded by the planes* $x = 0,\ \ y = 0,\ \ z = 0$ *and* $x+y+z = 1$.
Solution:

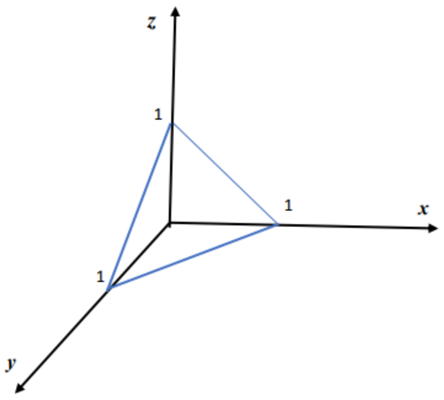


Figure 1.6

The solid goes from $z = 0$ up to the plane $x + y + z = 1$. On that plane $z = 1 - x - y$ is the height function $f(x, y)$ to be integrated. Figure (1.6) shows the domain D. To find its sides, set $z = 0$. The sides of D are the lines $x = 0$, $x = 1$, $y = 0$ and $y = 1 - x$.

$$D = \left\{(x, y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\right\}.$$

Then,

$$V = \int_0^1 \left(\int_0^{1-x} (1 - x - y)\, dy\right) dx = \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right) dx = \frac{1}{6}.$$

1.1.5 Double integration in polar coordinates

Some Cartesian double integral cannot be integrated unless it is transformed to another region of integration. Let f be a continuous function in a region $D \subset \mathbb{R}^2$. If a region D' in the θr-plane is transformed into the region D in the xy-plane by differentiable equation of the form $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r \geq 0$. Where r is the radial coordinate and θ is the polar angle. Then, the double integral of $f(x, y)$ over D is defined by the equation :

$$\int\int_D f(x, y)\, dxdy = \int\int_{D'} f(r\cos(\theta), r\sin(\theta))\, |J|\, drd\theta, \tag{1.1}$$

where J is the determinate of partial derivative and it is called Jacobian of the coordinate transformation.

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r.$$

Hence equation (1.1) becomes

$$\int\int_D f(x,y)\,dxdy = \int\int_{D'} f(r\cos(\theta), r\sin(\theta))\, r dr d\theta.$$

Example 1.6. *Compute* $\int\int_D \sqrt{x^2+y^2}dxdy$, *where*

$$D = \left\{(x,y) \in \mathbb{R}^2 : \;\; 0 \le x \le 1, \;\; 0 \le y \le 1, \;\; x^2+y^2 \le 4\right\}.$$

Solution:

Polar coordinates have $x = r\cos(\theta)$, $y = r\sin(\theta)$ *where* $r > 0$.

We have that: $x^2+y^2 \le 4 \Longrightarrow r^2\left(\cos(\theta)^2 + \cos(\theta)^2\right) \le 4 \Longrightarrow 0 \le r \le 2$.

$$\left.\begin{array}{l} 0 \le x \le 1 \Longrightarrow 0 \le r\cos(\theta) \le 1 \Longrightarrow 0 \le \cos(\theta) \le 1 \\ 0 \le y \le 1 \Longrightarrow 0 \le r\sin(\theta) \le 1 \Longrightarrow 0 \le \sin(\theta) \le 1 \end{array}\right\}\theta \in \left[0, \frac{\pi}{2}\right].$$

Then

$$\int\int_D \sqrt{x^2+y^2}dxdy = \int_0^{\frac{\pi}{2}}\int_0^2 r^2 dr d\theta = \left[\frac{r^3}{3}\right]_0^2 [\theta]_0^{\frac{\pi}{2}} = \frac{4}{3}\pi.$$

1.2 Triple integrals

1.2.1 Triple integrals on parallelepiped domain

Theorem 1.4. *(Fubini's Theorem on parallelepiped domain)*
If a function on three variables $f(x,y,z) : V \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is continuous on an domain* $V = [a_1,a_2] \times [b_1,b_2] \times [c_1,c_2]$, *then*

$$\begin{aligned} \int\int\int_V f(x,y,z)\,dxdydz &= \int_{a_1}^{a_2}\left[\int_{b_1}^{b_2}\left[\int_{c_1}^{c_2} f(x,y,z)\,dz\right]dy\right]dx \\ &= \int_{b_1}^{b_2}\left[\int_{a_1}^{a_2}\left[\int_{c_1}^{c_2} f(x,y,z)\,dz\right]dx\right]dy \\ &= \int_{c_1}^{c_2}\left[\int_{b_1}^{b_2}\left[\int_{a_1}^{a_2} f(x,y,z)\,dx\right]dy\right]dz = ... \end{aligned}$$

Geometrical interpretation of Triple integral

Let $f(x,y,z) > 0$ for every $(x,y,z) \in V \subset \mathbb{R}^3$ and f be continuous. We can show the geometrical interpretation of the triple integral $\int\int\int_V f(x,y,z)dxdydz$ by

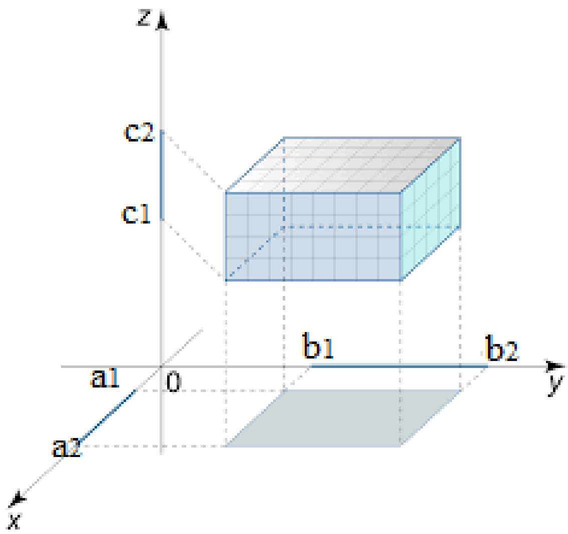


Figure 1.7: Triple integral

Example 1.7. *Evaluate* $\int\int\int_V x^2ze^y \ dxdydz$, *where*

$$V=\left\{(x,y,z)\in\mathbb{R}^3:\ \ 1\leq x\leq 2,\ \ 0\leq y\leq 1,\ \ -2\leq z\leq 1\right\}.$$

Solution:

We shall evaluate the integral in the order $dydzdx$.

$$\begin{aligned}\int\int\int_V x^2ze^y dxdydz &= \int_1^2\int_{-2}^1\int_0^1 x^2ze^y dydzdx\\ &= \int_1^2\int_{-2}^1 x^2z\left[e^y\right]_0^1 dzdx\\ &= (e-1)\int_1^2 x^2\left[\frac{z^2}{2}\right]_{-2}^1 dx\\ &= \frac{-3}{2}(e-1)\left[\frac{x^3}{3}\right]_1^2=-\frac{7}{2}(e-1).\end{aligned}$$

Corollary 1.2. *Let* $V=[a_1,a_2]\times[b_1,b_2]\times[c_1,c_2]$. *If* $f(x,y,z)=g_1(x)g_2(y)g_3(z)$ *for some functions* $g_1:[a_1,a_2]\longrightarrow\mathbb{R}$, $g_2:[b_1,b_2]\longrightarrow\mathbb{R}$ *and* $g_3:[c_1,c_2]\longrightarrow\mathbb{R}$, *then*

$$\int\int\int_V f(x,y,z)\,dxdydz=\left(\int_{a_1}^{a_2}g_1(x)\,dx\right)\cdot\left(\int_{b_1}^{b_2}g_2(y)\,dy\right)\cdot\left(\int_{c_1}^{c_2}g_3(z)\,dz\right).$$

Example 1.8.

In this example, we are going to evaluate the same integral in example 1.7. Then

$$\begin{aligned}\int\int\int_V x^2 z e^y dxdydz &= \left(\int_1^2 x^2 dx\right)\cdot\left(\int_0^1 e^y dy\right)\cdot\left(\int_{-2}^1 zdz\right)\\ &= \left(\left[\frac{x^3}{3}\right]_1^2\right)\cdot\left([e^y]_0^1\right)\cdot\left(\left[\frac{z^2}{2}\right]_{-2}^1\right)\\ &= -\frac{7}{2}(e-1).\end{aligned}$$

1.2.2 Basic properties of triple integrals

Property 1.2.

1. ***Linearity:*** *If functions f_1 and f_2 on an interval V and $c_1,\ c_2 \in \mathbb{R}$, then*

$$\begin{aligned}\int\int\int_V (c_1 f(x,y,z) + c_2 f(x,y,z))\,dxdydz &= c_1\int\int\int_V f(x,y,z)\,dxdydz\\ &+ c_2\int\int\int_V f(x,y,z)\,dxdydz.\end{aligned}$$

2. ***Additivity:*** *If a function f is integrable on an interval V and intervals V_1 and V_2 form a division of the interval V $(V = V_1 \cup V_2,\ V_1 \cap V_2 = 0)$, then*

$$\int\int\int_V f(x,y,z)\,dxdydz = \int\int\int_{V_1} f(x,y,z)\,dxdydz + \int\int\int_{V_2} f(x,y,z)\,dxdydz.$$

1.2.3 Triple integrals over regular regions

Theorem 1.5. *If a function of three variables $f(x,y,z)$ is continuous on a regular region of the type*

$$V = \left\{(x,y,z)\in\mathbb{R}^3 :\ a\le x\le b,\ g_1(x)\le y\le g_2(x),\ h_1(x,y)\le z\le h_2(x,y)\right\},$$

then

$$\int\int\int_V f(x,y,z)\,dxdydz = \int_a^b\left(\int_{g_1(x)}^{g_2(x)}\left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right)dy\right)dx.$$

Analogical propositions are valid also for the other types of regular regions.

Example 1.9. *Evaluate* $\int\int\int_V z dxdydz$*, where*

$$V = \left\{(x, y, z) \in \mathbb{R}^3 : \ 0 \leq x \leq 1, \ 0 \leq y \leq x, \ 0 \leq z \leq \sqrt{1-y^2}\right\}.$$

Solution:

$$\begin{aligned}
\int\int\int_V z \ dxdydz &= \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} z \ dzdydx = \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dydx \\
&= \int_0^1 \int_0^x \frac{1-y^2}{2} \ dydx = \int_0^1 \left[\frac{y}{2} - \frac{y^3}{6}\right]_0^x dx \\
&= \int_0^1 \left[\frac{x}{2} - \frac{x^3}{6}\right] dx = \frac{5}{24}.
\end{aligned}$$

1.2.4 Volume and mass using triple integral

Volume: If $f(x, y, z) = 1$, we get the volume of the region V.

Mass: If f is continuous and $f(x, y, z) > 0$ on the region V, we can consider the triple integral $V = \int\int\int_V f(x, y, z)dxdydz$ as the mass of the solid of V.

Example 1.10. *Find the volume of the tetrahedron bounded by the coordinate planes and* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution:

Volume$= \int\int\int_V \ dxdydz$*, where*

$$0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right), \ 0 \leq y \leq b\left(1 - \frac{x}{a}\right), \ 0 \leq x \leq a.$$

$$\begin{aligned}
V = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx &= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \ dydx \\
&= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \ dydx \\
&= \int_0^a c\left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} \ dx
\end{aligned}$$

$$= cb\int_0^a \left(\left(1-\frac{x}{a}\right) - \frac{x\left(1-\frac{x}{a}\right)}{a} - \frac{\left(1-\frac{x}{a}\right)^2}{2} \right) dx$$
$$= \frac{cb}{2}\int_0^a \left(\frac{1}{2}\left(1-\frac{x}{a}\right)^2 \right) dx = \frac{abc}{6}.$$

Here, if we take $a = 2$, $b = 1$ and $c = 4$. We obtain that $V = \frac{4}{3}$.

1.2.5 Using change of variable to calculate triple integrals

Cylindrical coordinates:

Cylindrical coordinates are convenient for representing cylindrical surfaces and surfaces for which the z-axis is the axis of symmetry. We convert a triple integral from rectangular to cylindrical coordinates (r, θ, z) by writing

$$x = r\cos(\theta), \quad y = r\sin(\theta), z = z, \quad \texttt{where } r > 0, \quad 0 \leq \theta \leq 2\pi,$$

where

- (r, θ) are a polar coordinates of the projection of $M(x, y, z) \in \mathbb{R}^3$ on the plane $z = 0$.
- z is the distance between M and the plane $z = 0$.

Then

$$\int\int\int_V f(x,y,z)\, dxdydz = \int\int\int_{V'} f(r\cos(\theta), r\sin(\theta), z)\, |J|\, drd\theta dz. \tag{1.2}$$

Where J is the determinate of partial derivative and it is called Jacobian of the coordinate transformation.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Hence equation (1.2) becomes

$$\int\int\int_V f(x,y,z)\, dxdydz = \int\int\int_{V'} f(r\cos(\theta), r\sin(\theta), z)\, r\ \ drd\theta dz,$$

where $V^{'}$ is a region in the θrz-plane.

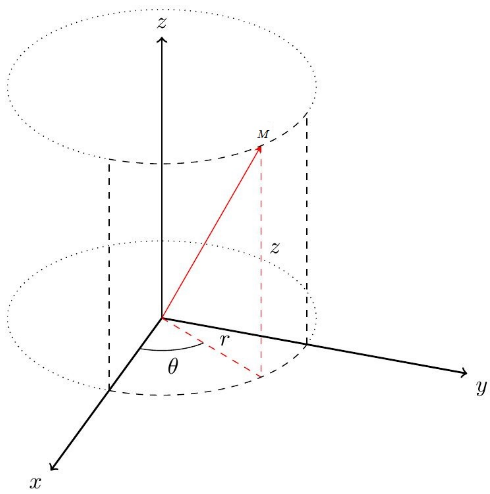


Figure 1.8: Cylindrical coordinates

Example 1.11. *Evaluate* $\int\int\int_V \frac{1}{2}\left(x^2+y^2\right)+z\ \ dxdydz$, *where*

$$V=\left\{(x,y,z)\in\mathbb{R}^3/\ x^2+y^2\leq 4,\ \ 0\leq z\leq 1\right\}.$$

Solution:

Cylindrical coordinates

$$\begin{aligned} x &= r\cos(\theta),\\ y &= r\sin(\theta),\\ z &= z. \end{aligned}$$

We have that $x^2+y^2\leq 4\Rightarrow 0\leq r\leq 2$ *and* $0\leq\theta\leq 2\pi$. *Then*

$$\begin{aligned} \int\int\int_V \frac{1}{2}\left(x^2+y^2\right)+z\ \ dxdydz &= \int_0^{2\pi}\int_0^1\int_0^2\left(\frac{r^2}{2}+z\right)\ \ drdzd\theta\\ &= \int_0^{2\pi}\int_0^1\left[\frac{r^3}{6}+rz\right]_0^2 dzd\theta\\ &= \int_0^{2\pi}\left(\frac{4}{3}+2z\right)dzd\theta=\int_0^{2\pi}\left[\frac{4z}{3}+z^2\right]_0^1 d\theta\\ &= \int_0^{2\pi}\frac{7}{3}d\theta=\frac{14\pi}{3}. \end{aligned}$$

Spherical coordinates:

Spherical coordinates represent a point $M\in\mathbb{R}^3$ in space by ordered triples (r,θ,ρ) in which

- r is the distance from M to the origin.
- θ is the angle from cylindrical coordinates $(0 \leq \theta \leq 2\pi)$.
- ρ is the angle OM makes with the positive z-axis $(0 \leq \rho \leq \pi)$.

Since

$$\begin{cases} x = r\sin(\rho)\cos(\theta), \\ y = r\sin(\rho)\sin(\theta), \\ z = r\cos(\rho). \end{cases}$$

Then

$$\int\int\int_V f(x,y,z)\,dxdydz = \int\int\int_{V'} f(r\sin(\rho)\cos(\theta), r\sin(\rho)\sin(\theta), r\cos(\rho))\, r^2\sin(\rho) \quad drd\theta d\rho,$$

where V' is a region in the $\theta r\rho$-plane.

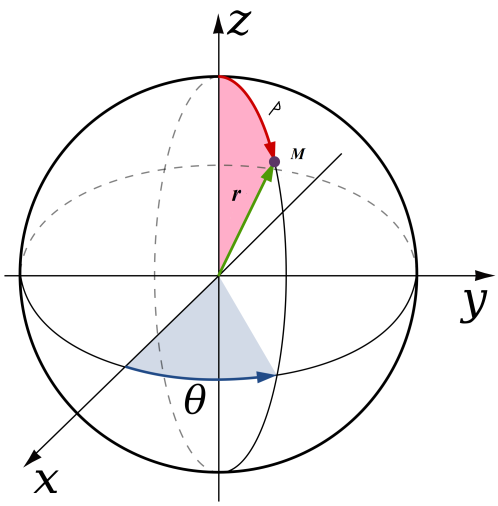


Figure 1.9: Spherical Coordinates

Example 1.12. *Evaluate* $\int\int\int_V \frac{1}{\sqrt{x^2+y^2+z^2}}\ dxdydz$, *where*

$$V = \left\{(x,y,z) \in \mathbb{R}^3 /\ x^2+y^2+z^2 \leq 1,\ \ x \geq 0,\ \ z \geq 0\right\}.$$

Solution: *Let us transform this integral in spherical coordinates by taking*

$$\begin{cases} x = r\sin(\rho)\cos(\theta), \\ y = r\sin(\rho)\sin(\theta), \\ z = r\cos(\rho). \end{cases}$$

Hence $x^2+y^2+z^2 \leq 1 \Longrightarrow 0 \leq r \leq 1$ *and* $x \geq 0 \Longrightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, *and from* $z \geq 0$ *we obtain* $0 \leq \rho \leq \frac{\pi}{2}$. *Then*

$$\begin{aligned} \int\int\int_V \frac{1}{\sqrt{x^2+y^2+z^2}}\ dxdydz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \left(\frac{1}{\sqrt{r^2}}\right) r^2\sin(\rho)\ \ drd\rho d\theta \\ &= \left(\int_0^1 r\ \ dr\right)\cdot\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\right)\cdot\left(\int_0^{\frac{\pi}{2}}\sin(\rho)\ \ d\rho\right) \\ &= \left[\frac{r^2}{2}\right]_0^1\cdot\Big[\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\Big[-\cos(\rho)\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}. \end{aligned}$$

1.3 Supplementary exercises

Exercise 01.

Evaluate the following integrals:

1. $\int\int_D x\cos(xy)\,dxdy$, where $D = \left\{\ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{\pi}{2},\ \ 0 \leq y \leq 1\right\}$.
2. $\int\int_D (x+y)\,dxdy$, where $D = \left\{\ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ \ 0 \leq y \leq 2x\right\}$.
3. $\int\int\int_V z^2ye^x dxdydz$, where $V = \left\{\ (x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ \ 1 \leq y \leq 2,\ \ -1 \leq z \leq 1\right\}$.
4. $\int\int\int_V x^2 dxdydz$, where $V = \left\{\ (x,y,z) \in \mathbb{R}^3 : -3 \leq x \leq 3,\ -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2},\ \ 0 \leq z \leq 9-x^2-y^2\right\}$.

Exercise 02.

- Find the volume of the solid bounded by the graphs $z = 4-x^2-y^2$ and $z = x^2+y^2$.

- Find the volume of the region bounded above by $z = 4 - x^2 - y^2$, below by $z = 0$ and laterally by $x^2 + y^2 \leq 1$.

Exercise 03.

Evaluate $\int\int_D (x^2 + y^2 + 1)\, dxy$ where D is the region inside the circle $x^2 + y^2 = 4$.

Exercise 04.

Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\ dzdydx.$$

2

Improper integrals

In this chapter, we will extend the concept of definite integrals where one or both of the boundaries is at infinity, and integrals with infinite discontinuity. This kind of integrals are called *improper integrals*.

Definition 2.1. *(Improper integral)*

The integral $\int_a^b f(x)dx$ *is called an* ***improper integral*** *if*

1. *One or both integration limits is infinite, i.e,* $a = -\infty$ *or* $b = \infty$ *or both.*
2. $f(x)$ *is unbounded at one or more points of* $a \leq x \leq b$.

2.1 Integration over a bounded interval

2.1.1 Half-open intervals

A half-open interval is an interval has the form $[a, b[$ or $\left]a, b\right]$, where a and b are a real numbers.

Definition 2.2.

1. *Let* $f(x) : [a, b[\longrightarrow \mathbb{R}$ *be continuous and integrable on* $[a, b[$. *Then*

$$\int_a^b f(x)dx = \lim_{x \to b^-} \int_a^x f(t)dt.$$

2. *Let* $f(x) : \left]a, b\right] \longrightarrow \mathbb{R}$ *be continuous and integrable on* $\left]a, b\right]$. *Then*

$$\int_a^b f(x)dx = \lim_{x \to a^+} \int_x^b f(t)dt.$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

Example 2.1. *Let* $f : \left]0, 1\right] \longrightarrow \mathbb{R}$ *defined by* $f(x) = \frac{1}{\sqrt{x}}$. *Rewrite* $\int_0^1 \frac{1}{\sqrt{x}}dx$ *as a limit, we obtain*

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{x \to 0^+} \int_x^1 \frac{1}{\sqrt{t}}dt \\ &= \lim_{x \to 0^+} \left[2\sqrt{t}\right]_x^1 = 2. \end{aligned}$$

The improper integral converges to 2.

Proposition 2.1. *(Riemann integration at* 0*)*
Let $f : \left]0, 1\right] \longrightarrow \mathbb{R}$ *be a function defined by* $f(x) = \frac{1}{x^\alpha}$. *The integral* $\int_0^1 \frac{1}{x^\alpha}dx$, *where* α *is a constant, converges if* $\alpha < 1$ *and diverges if* $\alpha > 0$.

2.1.2 Open intervals

An open interval is an interval has the form $\left]a, b\right[$, where a and b are a real numbers.

Definition 2.3.
Let $f : \left]a, b\right[\longrightarrow \mathbb{R}$ *be continuous and integrable on* $\left]a, b\right[$. *If for some (and then for any)* $c \in \left]a, b\right[$, f *is improperly integrable on semi-open intervals* $\left]a, c\right]$ *and* $\left[c, b\right[$. *i.e* $\int_a^c f(x)dx$ *and* $\int_c^b f(x)dx$ *are exist. The integral of* f *is defined by*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2.2 Improper integrals over an infinite interval

2.2.1 Half-open intervals

Definition 2.4.

1. *Let* $f : [a, +\infty[\longrightarrow \mathbb{R}$ *be continuous and integrable on* $[a, +\infty[$. *Then*

$$\int_a^{\infty} f(x)dx = \lim_{x \to +\infty} \int_a^x f(t)dt.$$

provided this limit exists.

2. *Let* $f : \Big] - \infty, b\Big] \longrightarrow \mathbb{R}$ *be continuous and integrable on* $\Big] - \infty, b\Big]$. *Then*

$$\int_{-\infty}^{b} f(x)dx = \lim_{x \to +\infty} \int_x^b f(t)dt.$$

provided this limit exists.

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

Example 2.2. *Determine whether the following integral converges or diverges* $\int_0^{+\infty} e^{-t}dt$.
Solution: $\forall x > 0$, *we have*

$$\int_0^{+\infty} e^{-t}dt = \lim_{x \to +\infty} \int_0^x e^{-t}dt = \lim_{x \to +\infty} \Big(1 - e^{-x}\Big) = 1.$$

Therefore the improper integral $\int_0^{+\infty} e^{-t}dt$ *converges.*

Proposition 2.2. *(Riemann integration) Let* $f : [1, +\infty[$ *a function defined by* $f(x) = \frac{1}{x^{\alpha}}$. *The integral* $\int_1^{+\infty} \frac{1}{x^{\alpha}}dx$, *where* α *is a constant, converges if* $\alpha > 1$ *and diverges if* $\alpha \leq 1$.

2.2.2 Open intervals

Definition 2.5.
Let $f : \Big]a, +\infty\Big[\longrightarrow \mathbb{R}$ *be continuous and integrable on* $\Big]a, +\infty\Big[$. *If for some (and then for any)* $c \in \Big]a, +\infty\Big[$, f *is improperly integrable on semi-open intervals* $\Big]a, c\Big]$ *and* $\Big[c, +\infty\Big[$. *i.e*

$\int_a^c f(x)dx$ and $\int_c^{+\infty} f(x)dx$ are exist. The integral of f is defined by

$$\int_a^{+\infty} f(x)dx = \int_a^c f(x)dx + \int_c^{+\infty} f(x)dx.$$

Definition 2.6.
Let $f(x)$ be continuous over $\big]-\infty,+\infty\big[$. Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{+\infty} f(x)dx,$$

provided that $\int_{-\infty}^0 f(x)dx$ and $\int_0^{+\infty} f(x)dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges.

Remark 2.1. *Let $f(x)$ be continuous over $\big]-\infty,+\infty\big[$. Then, it can be shown that, in fact,*

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx,$$

for any value of a.

Example 2.3. *The integral of the function $f(x) = \frac{1}{1+x^2}$ on $\big]-\infty,+\infty\big[$ converges and*

$$\int_{-\infty}^{+\infty} \frac{1}{1+t^2}dt = \pi.$$

2.3 Properties of the improper integral

Let $[a, b\,[$, with $-\infty < a < b \leq +\infty$.

1. **Linearity**. If f and g are integrable in every interval $[a, x] \subseteq [a, b]$, and their improper integrals are convergent in $[a, b\,[$, then $\int_a^b (\alpha f(x) + \beta g(x))\,dx =$ converges for all $\alpha, \beta \in \mathbb{R}$, and
$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

2. **Integration by parts**. Let $u, v : [a, b\,[\longrightarrow \mathbb{R}$ be two functions of class C^1 such that two of the following three integrals are convergent. Then the other integral will be convergent as well, and
$$\int_a^b u(x)v'(x)dx = \lim_{x\to b^-} (u(x)v(x)) - \int_a^b u'(x)v(x)dx.$$

3. **Change of variable**. Let $f : [a, b[\longrightarrow \mathbb{R}$ be continuous and let $\phi : [\alpha, \beta[\longrightarrow \mathbb{R}$ be of class C^1, where $-\infty < \alpha < \beta \leq +\infty$, $\phi(\alpha) = a$, $\lim\limits_{t \to \beta^-} \phi(t) = b$ and the image $\phi([\alpha, \beta[) = [a, b[$ (the image of the interval $[\alpha, \beta[$ by ϕ is $[a, b[$). Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt.$$

4. **Absolute convergence**.

   - Let $f : [a, b[\longrightarrow \mathbb{R}$ be a function that is integrable in every interval $[a, x] \subseteq [a, b]$. We say that the improper integral $\int_a^b f(x)dx$ converges absolutely if and only if the improper integral $\int_a^b |f(x)|\, dx$ converges.

   - If the improper integral $\int_a^b f(x)dx$ is absolutely convergent, then it is convergent and
$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx.$$
     The reciprocal of this proposition is not true.

5. **Quotient**. Integrals with non-negative integrands

   - If $f(x) \geq 0$ and $g(x) \geq 0$ for $a < x \leq b$, and if $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ either both converge or both diverge.
   - If $A = 0$, then $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.
   - If $A = \infty$, then $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

6. Let $f, g : [a, b[\longrightarrow \mathbb{R}$ integrable in every $[a, x] \subseteq [a, b]$. Suppose that $0 \leq f(x) \leq g(x)$ for all $x \in [a, b[$. Then

   - $\int_a^b g(x)dx$ converges $\Longrightarrow$ $\int_a^b f(x)dx$ converges, and we have that
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$
   - $\int_a^b f(x)dx$ diverges $\Longrightarrow$ $\int_a^b g(x)dx$ diverges.

2.4 Supplementary exercises

Exercise 01.

Evaluate:

1. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$.
2. $\int_0^4 \frac{dx}{\sqrt{(x-1)^2}}$.
3. $\int_0^{+\infty} \frac{dx}{(x+1)(x+2)(x+3)}$.
4. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^n}$.

Exercise 02.

Test for convergence:

1. $\int_{-2}^3 \frac{dx}{x}$.
2. $\int_{-1}^1 \frac{dx}{\sqrt{(1-x^2)}}$.
3. $\int_a^b \frac{dt}{\sqrt{(b-t)(t-a)}} \quad (a < b)$.
4. $\int_{2\pi}^{+\infty} \ln\cos\left(\frac{1}{x}\right) dx$.

Exercise 03.

Prove that $\int_0^{+\infty} e^{-\alpha x} \cos(rx)\, dx = \frac{\alpha}{\alpha^2+r^2}$ for $\alpha > 0$ and any real value of r.

3 Differential equations

In this chapter, we will describe the main ideas to solve certain differential equations. A differential equation is an equation involving an unknown function and its derivatives.
The differential equation is an equation has the form

$$F(x, y(x), y(x)^{'}, y(x)^{"}, y^{(3)} \cdots, y(x)^{(n)}) = 0,$$

where $y(x)^{'}, y(x)^{"}, y(x)^{(3)} \cdots, y(x)^{(n)}$ represent $dy/dx, dy^2/dx^2, dy^2/dx^2, \cdots, dy^n/dx^n$ respectively. A solution of a differential equation in the unknown function y and the independent variable x.

3.1 First-order differential equations

Definition 3.1. *The first order differential equation is written as*

$$F(x, y, y^{'}) = 0, \quad \texttt{or} \ y^{'} = f(x, y),$$

where $f(x, y)$ can be any function.

Definition 3.2. *(Solution)*
We denote by $y = \phi(x, c)$ a general solution of the differential equation $y^{'} = f(x, y)$, where c is a constant. $c = c_0$ can be obtained by using the initial condition $y(x_0) = y_0$, in this case $y = \phi(x, c_0)$ is called a particular solution of the differential equation.
We conclude that any differential equation has a general solution which is the set of all solutions, and a particular solution which is any one solution obtained when we use a given initial condition.

Example 3.1. *Solve the initial value problem*

$$y^{'} = \sin(x), \quad y(0) = 2.$$

***Solution**:*

Integrating both sides of this equation, we find

$y^{'} = \sin(x) \Longrightarrow \frac{dy}{dx} = \sin(x) \Longrightarrow \int dy = \int \sin(x)\, dx \Longrightarrow y(x) = -\cos(x) + c.$

Using the initial condition $y(0) = 2$ *(*$y = 2$ *when* $x = 0$*), we obtain* $y(0) = -\cos(0) + c = 2$ $\Longrightarrow c = 3.$

Then, the general solution of the initial-value problem is $y(x) = -\cos(x) + c$*, and the particular solution is* $y(x) = -\cos(x) + 3.$

3.1.1 Separable equations

A separable equation is an equation can be solved by integrating on both sides of the differential equation.

Definition 3.3. *(Differential equation with separated variables)*

A first-order differential equation is called an equation with separated variables if it can be written in the form

$$M(x)dx + N(y)dy = 0,$$

where $M(x)$ *and* $N(y)$ *are given functions.*

The solution can be obtained by integrating on both sides

$$\int M(x)dx + \int N(y)dy = c.$$

where c *is an arbitrary constant.*

Remark 3.1. *A differential equation with separated variables can also be written as*

$$y^{'} = f(x) \cdot g(y).$$

Suppose that $g(y) \neq 0$*, then, in this case the solution is defined by*

$$\frac{dy}{g(y)} = f(x)dx \Longleftrightarrow M(x)dx + N(y)dy = 0.$$

Example 3.2. *Solve* $y' = x^2\sqrt{1+y^2}$.

Solution: *This differential equation is an equation with separated variables. So*

$$\frac{dy}{dx} = x^2\sqrt{1+y^2} \Longleftrightarrow \frac{dy}{\sqrt{1+y^2}} = x^2dx \Longleftrightarrow x^2dx - \frac{1}{\sqrt{1+y^2}}dy = 0$$

To find the solution of this equation, we integrate the equation on both sides

$$\int x^2dx - \int \frac{1}{\sqrt{1+y^2}}dy = c.$$

We have $\int \frac{1}{\sqrt{a^2+y^2}}dy = \ln\left|y + \sqrt{y^2+a^2}\right|$. *Then, the general integral is given by*

$$\frac{1}{3}x^3 - \ln\left|y + \sqrt{1+y^2}\right| = c$$

Definition 3.4. *(Separable differential equation)*

A first-order differential equation is separable if it can be written in the form

$$M_1(x)N_1(y)dx + M_2(x)N_2(y)dy = 0,$$

where $M_{1,2}(x)$ *and* $N_{1,2}(y)$ *are given functions.*

We shall transform this equation to an equation with separated variables. Dividing it by $N_1(y)M_2(x)$ *we obtain*

$$\frac{M_1(x)}{M_2(x)}dx + \frac{N_2(x)}{N_1(x)}dy = 0,$$

which is an equation with separated variables.

Example 3.3. *Solve* $\frac{x^2}{\sqrt{y}}dx + x\sqrt{y}dy = 0$.

Solution:

Dividing the equation by $x\frac{1}{\sqrt{y}}$, *we get*

$$\frac{x^2\left(\sqrt{y}\right)^{-1}}{x\left(\sqrt{y}\right)^{-1}}dx + \frac{x\sqrt{y}}{x\left(\sqrt{y}\right)^{-1}}dy = 0,$$

$$xdx + ydy = 0,$$

which is an equation with separated variables. Its solution is

$$\int xdx + \int ydy = c \Longleftrightarrow x^2 + y^2 = 2c.$$

We see that $x^2 + y^2 > 0$, so we also have that $2c > 0$ and we can rewrite this equation in the form $x^2 + y^2 = k^2$, where $k^2 = 2c$.

This is an equation of the family concentric circles centered at the origin of radius $k > 0$.

3.1.2 Linear first order equations

Definition 3.5. *A linear first-order differential equation is an equation has the form*

$$y^{'} + p(x)y = q(x), \tag{3.1}$$

where $p(x)$ and $q(x)$ tow continuous functions.

Method of solution

Let

$$y(x) = u(x)v(x), \tag{3.2}$$

be a solution of (3.1). We derive (3.2), we obtain

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3.3}$$

Substituting (3.2) and (3.3) in (3.1), we get

$$u\frac{dv}{dx} + v\frac{du}{dx} + p(x)uv = q(x) \Longrightarrow u\left(\frac{dv}{dx} + p(x)v\right) + v\frac{du}{dx} = q(x).$$

We chose v such that $\frac{dv}{dx} + p(x)v(x) = 0$. Therefore, by integrating on both sides we get

$$\begin{cases} \frac{dv}{dx} + p(x)v(x) = 0 & \Longrightarrow \quad v(x) = e^{-\int p(x)dx}, \\ v(x)\frac{du}{dx} = q(x) & \Longrightarrow \quad u(x) = \int \frac{q(x)}{v(x)}dx + c. \end{cases}$$

Then the solution of (3.1) becomes

$$y(x) = v(x)\int \frac{q(x)}{v(x)}dx + cv(x).$$

Example 3.4. *Solve the differential equation* $xy' - y = 3x^3$.

Solution:

We have

$$xy' - y = 3x^3 \Longrightarrow y' - \frac{1}{x}y = 3x^2.$$

Substituting $y(x) = u(x)v(x)$ *and* $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ *in the differential equation above, we get*

$$u\left(\frac{dv}{dx} - \frac{1}{x}v\right) + v\frac{du}{dx} = 3x^2.$$

So

$$\frac{dv}{dx} - \frac{1}{x}v(x) = 0 \Longrightarrow \int \frac{dv}{v} = \int \frac{1}{x}dx \Longrightarrow \ln v = \ln x \Longrightarrow v(x) = x.$$

Substituting $v(x) = x$ *in* $v\frac{du}{dx} = 3x^2$, *we get*

$$x\frac{du}{dx} = 3x^2 \Longrightarrow du = 3x \ \ dx.$$

Integrating on both sides, we get $u(x) = \frac{3}{2}x^2 + c$. *Then, the solution is giving by*

$$y(x) = x\left(\frac{3}{2}x^2 + c\right).$$

3.1.3 Bernoulli equations

Definition 3.6. *The Bernoulli equation is an equation has the following form*

$$y' + p(x)y = q(x)y^n, \tag{3.4}$$

where p, q *are given continuous functions and* $n \in \mathbb{R}$ $(n \neq 0, 1)$.

Method of solution

The idea here is to transform the Bernoulli equation into a linear differential equation. Divide both sides of the Bernoulli equation (3.4) by y^n, we get

$$y^{-n}\frac{dy}{dx} + p(x)y^{-n+1} = q(x).$$

Introduce $z(x) = y^{-n+1}$ and its derivative

$$\frac{dz}{dx} = (-n+1)y^{-n}\frac{dy}{dx} \Longrightarrow y^{-n}\frac{dy}{dx} = \frac{1}{-n+1}\frac{dz}{dx}$$

into the differential equation above, we get the following differential equation

$$\frac{dz}{dx} + (-n+1)p(x)z(x) = (-n+1)q(x), \tag{3.5}$$

which is linear in z.

Example 3.5. *Solve* $y' + \frac{1}{x+1}y = xy^3$.

Solution:

We have $n = 3$. *Divide the differential equation above by* y^3, *we get*

$$y^{-3}y' + \frac{1}{x+1}y^{-2} = x.$$

Let $z = y^{-2}$ *and* $z' = -2y^{-3}y' \Longrightarrow y^{-3}y' = -\frac{1}{2}z'$, *then our equation becomes*

$$-\frac{1}{2}z' + \frac{1}{x+1}z = x \Longrightarrow z' - \frac{2}{x+1}z = -2x,$$

which is linear in z. *Introduce* $z = u(x)v(x)$ *and* $\frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ *in the equation above, we obtain*

$$u\left(\frac{dv}{dx} - \frac{2}{x+1}v\right) + v\frac{du}{dx} = -2x.$$

By integrating on both sides, we get

$$\begin{cases} \frac{dv}{dx} - \frac{2}{x+1}v = 0 & \Longrightarrow \quad v(x) = (x+1)^2, \\ (x+1)^2\frac{du}{dx} = -2x & \Longrightarrow \quad u(x) = -\frac{2}{x+1} - 2\ln(x+1) + c. \end{cases}$$

Thus,

$$z(x) = (x+1)^2\left[-2\left(\frac{1}{x+1} + \ln(x+1)\right) + c\right].$$

Then, the general solution becomes

$$z(x) = y^{-2} \Longrightarrow y(x) = \frac{1}{\sqrt{(x+1)^2\left[-2\left(\frac{1}{x+1} + \ln(x+1)\right) + c\right]}}.$$

3.1.4 Homogeneous equations

Definition 3.7. *(Homogeneous function)*

$f(x, y)$ is called homogeneous function of degree n if for all λ we have $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Example 3.6. *The following are homogeneous functions*

1. $f(x, y) = y\sqrt{(x^2 + y^2)}$ *homogeneous function of degree 2.*
2. $f(x, y) = \frac{\sqrt{x^2}+y}{y}$ *homogeneous function of degree 0.*
3. $f(x, y) = x^2 - 2x^3 y$ *not homogeneous.*

Definition 3.8. *(Homogeneous equation)*

The first-order differential equation

$$\frac{dy}{dx} = f(x, y), \tag{3.6}$$

is called homogeneous equation if $f(x, y)$ is a homogeneous function of degree 0.

Method of solution

To solve equation (3.6) we transform it into a separable equation. So, from the definition above we have $f(x, y) = f(\lambda x, \lambda y)$. Introduce $\lambda = \frac{1}{x}$ into the differential equation (3.6) we get

$$\frac{dy}{dx} = f\left(1, \frac{y}{x}\right), \tag{3.7}$$

Substituting $t = \frac{y}{x} \Longrightarrow y = tx$ and $\frac{dy}{dx} = t + \frac{dt}{dx} \cdot x$ in (3.7), we obtain

$$t + \frac{dt}{dx} \cdot x = f(1, t) \Longrightarrow [t - f(1, t)]\, dx + x\ \ dt = 0,$$

which is a separable equation.

Example 3.7. *Solve $xy' - y = x\cos\left(\frac{y}{x}\right)$.*

Solution:

Writing the differential equation in standard form, we obtain

$$y' = \frac{x\cos\left(\frac{y}{x}\right) + y}{x} = \frac{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda y}{\lambda x}.$$

Let $\lambda = \frac{1}{x}$. Then our equation becomes

$$y^{'} = \cos\left(\frac{y}{x}\right) + \frac{y}{x}. \tag{3.8}$$

Substituting $t = \frac{y}{x} \Longrightarrow y = tx$ and $\frac{dy}{dx} = t + \frac{dt}{dx} \cdot x$ in (3.8), we get

$$t + \frac{dt}{dx} \cdot x = \cos(t) + t. \tag{3.9}$$

Now, we integrate both sides of the equation, we get

$$\int \frac{dt}{\cos(t)} = \int \frac{dx}{x} \Longrightarrow \int \frac{\cos(t)}{\cos^2(t)} dt = \int \frac{dx}{x} \Longrightarrow \int \frac{\cos(t)}{1 - \sin^2(t)} dt = \int \frac{dx}{x}.$$

Suppose $\sin(t) = u \Longrightarrow \cos(t)dt = du$, *the integral becomes*

$$\int \frac{du}{1 - u^2} = \int \frac{dx}{x} \Longrightarrow \frac{1}{2} \ln\left(\left|\frac{1+u}{1-u}\right|\right) + c = \ln|x| \Longrightarrow \frac{1}{2} \ln\left(\left|\frac{1+\sin(t)}{1-\sin(t)}\right|\right) + c = \ln|x|.$$

The solution is given by

$$x = c\left(\frac{1 + \sin\left(\frac{y}{x}\right)}{1 - \sin\left(\frac{y}{x}\right)}\right)^{1/2}.$$

3.1.5 Nonhomogeneous first-order differential equations

Definition 3.9. *An equation is called nonhomogeneous first-order differential equation if it has the form*

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}, \tag{3.10}$$

where $a_{1,2}, b_{1,2}$ and $c_{1,2}$ are arbitrary constants.

Method of solution

If $c_1 = c_2 = 0$, the equation (3.10) becomes a homogeneous differential equation.

If $c_1 \neq 0$ or $c_2 \neq 0$, for solve the equation (3.10), we calculate $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. We consider tow cases for equation (3.10).

- **Case 1**: If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. Substituting

$$\begin{cases} x &= x_1 + h \\ y &= y_1 + k \end{cases}, \quad \text{and} \quad \frac{dy}{dx} = \frac{dy_1}{dx_1}$$

in (3.10), we obtian

$$\frac{dy_1}{dx_1} = \frac{a_1x_1 + b_1y_1 + a_1h + b_1k + c_1}{a_2x_1 + b_2y_1 + a_2h + b_2k + c_2}.$$

We chose h and k such that

$$\begin{cases} a_1h + b_1k + c_1 &= 0, \\ a_2h + b_2k + c_2 &= 0. \end{cases}$$

Then, equation (3.10) becomes

$$\frac{dy_1}{dx_1} = \frac{a_1x_1 + b_1y_1 + c_1}{a_2x_1 + b_2y_1 + c_2},$$

which is a homogeneous differential equation.

Example 3.8. *Solve* $\frac{dy}{dx} = \frac{x-y+2}{x+y+4}$.

***Solution**:*
We have $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$. *Suppose*

$$\begin{cases} x &= x_1 + h \\ y &= y_1 + k \end{cases}, \quad \textit{and} \quad \frac{dy}{dx} = \frac{dy_1}{dx_1}.$$

Then, the equation can be rewritten as

$$\frac{dy_1}{dx_1} = \frac{x_1 - y_1 + h - k + 2}{x_1 + y_1 + h + k + 4}.$$

Choosing h and k such that

$$\begin{cases} h - k + 2 &= 0 \\ h + k + 4 &= 0 \end{cases} \Longrightarrow \begin{cases} h &= -3, \\ k &= -1. \end{cases}$$

Therefore
$$\frac{dy_1}{dx_1} = \frac{x_1 - y_1}{x_1 + y_1} = \frac{\lambda x_1 - \lambda y_1}{\lambda x_1 + \lambda y_1}.$$
Introduce $\lambda = \frac{1}{x_1}$ *into the equation above we get*
$$\frac{dy_1}{dx_1} = \frac{1 - \frac{y_1}{x_1}}{1 + \frac{y_1}{x_1}}.$$
Substituting $t = \frac{y_1}{x_1} \Longrightarrow y_1 = tx_1$ *and* $\frac{dy_1}{dx_1} = t + \frac{dt}{dx_1} \cdot x_1$, *we get*
$$t + \frac{dt}{dx_1} \cdot x_1 = \frac{1-t}{1+t} \Longrightarrow \int \frac{1+t}{1-2t-t^2} dt = \int \frac{dx_1}{x_1}.$$
Then
$$\frac{-1}{2} \ln\left(1 - 2t - t^2\right) = \ln|x_1| + \ln|c| \Longrightarrow \left(1 - 2t - t^2\right)^{-\frac{1}{2}} = cx_1.$$
Since $t = \frac{y_1}{x_1} \Longrightarrow \left(1 - 2\left(\frac{y_1}{x_1}\right) - \left(\frac{y_1}{x_1}\right)^2\right)^{-\frac{1}{2}} = cx_1.$

Since $\begin{cases} x_1 &= x+3 \\ y_1 &= y+1 \end{cases}.$

Then the general integral is given by
$$\boxed{\left(1 - 2\left(\frac{y+1}{x+3}\right) - \left(\frac{y+1}{x+3}\right)^2\right)^{-\frac{1}{2}} = c\,(x+3)\,.}$$

- **Case 2**: If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. Suppose
$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \lambda \Longrightarrow a_2 = \lambda a_1, b_2 = \lambda b_1.$$
Therefore, equation (3.10) can be rewritten as
$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{\lambda\,(a_1 x + b_1 y) + c_2}.$$
Substituting $z = a_1 x + b_1 y$ and $\frac{dz}{dx} = a_1 + b_1 \frac{dy}{dx} \Longrightarrow \frac{dy}{dx} = \frac{1}{b_1}\frac{dz}{dx} - \frac{a_1}{b_1}$ in the equation above, we get
$$\frac{dz}{dx} = b_1 \left(\frac{z + c_1}{\lambda z + c_2} + \frac{a_1}{b_1}\right),$$

which is an equation with separated variables.

Example 3.9. *Solve* $\frac{dy}{dx} = \frac{3x-y}{6x-2y+5}$.

Solution:
We have $\begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} = 0$. *Then, the equation can be rewritten as*

$$\frac{dy}{dx} = \frac{3x-y}{2(3x-y)+5}.$$

Suppose $z = 3x - y$ *and* $\frac{dz}{dx} = 3 - \frac{dy}{dx} \Longrightarrow \frac{dy}{dx} = 3 - \frac{dz}{dx}$ *in the equation above, we get*

$$3 - \frac{dz}{dx} = \frac{z}{2z+5} \Longrightarrow \int \frac{2z+5}{5z+15} dz = \int dx \Longrightarrow \int \frac{2z}{5z+15} dz + \int \frac{5}{5z+15} dz = \int dx.$$

We have
$\int \frac{5}{5z+15} dz = \ln |5z+15|$,
$\int \frac{2z}{5z+15} dz = \frac{5}{2} \int \frac{2z}{5z+15} dz = \frac{2}{5} \int \frac{5z+15-15}{5z+15} dz = \frac{2}{5} \int \left(1 - \frac{15}{5z+15}\right) dz = \frac{2}{5} z - \frac{6}{5} \ln |5z+15|$.
Therefore

$$\int \frac{2z}{5z+15} dz + \int \frac{5}{5z+15} dz = \int dx \Longrightarrow \frac{2}{5} z - \frac{1}{5} \ln |5z+15| = x + c.$$

Since $z = 3x - y$, *the general integral is given by*

$$\frac{1}{5} x - \frac{2}{5} y - \frac{1}{5} \ln |15(x+1) - 5y| = c.$$

3.2 Second order linear homogeneous differential equations with constant coefficients

A second order linear differential equation with constant coefficients is an equation of the form:

$$y^{"} + ay^{'} + by = 0, \tag{3.11}$$

where a and b are a constant coefficients.

To find the general solution of equation (3.11), we suppose that $y = e^{\lambda x}$ $(\lambda \in \mathbb{R})$ is a particular solution of (3.11). Then, after replacing this solution in equation (3.11), we obtain

$$e^{\lambda x}\left(\lambda^2 + a\lambda + b\right) = 0.$$

We have $e^{\lambda x} \neq 0$. Then, we obtain the following characteristic equation

$$\lambda^2 + a\lambda + b = 0. \tag{3.12}$$

To find the roots of the characteristic equation (3.12), we calculate $\Delta = a^2 - 4b$. We consider three cases for equation (3.12)

- **Case 01**: If $\Delta > 0$, we obtain tow real and distinct roots λ_1 and λ_2. Two particular solutions are $y_1 = e^{\lambda_1 x}$ and $y_1 = e^{\lambda_2 x}$. The general solution is

  $$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

  Example: Solve $y^{"} - 5y^{'} + 4y = 0$.

  Solution:

  The characteristic equation is $\lambda^2 - 5\lambda + 4 = 0$. $\Delta = 9 > 0$, since there exist tow real and distinct roots $\lambda_1 = 1$ and $\lambda_2 = 4$. Then, the general solution is

  $$y = c_1 e^{x} + c_2 e^{4x}.$$

- **Case 02**: $\Delta = 0$, we obtain one real root λ. Tow particular solution are $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$. The general solution is

  $$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = e^{\lambda x}\left(c_1 + c_2 x\right).$$

  Example: Solve $y^{"} - 2y^{'} + y = 0$.

  Solution:

  The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$. $\Delta = 0$, since there exist one real root $\lambda = 1$. Then, the general solution is

  $$y = e^{x}\left(c_1 + c_2 x\right).$$

- **Case 03**: $\Delta < 0$, we obtain tow complex and distinct roots $\lambda_{1,2} = \alpha \pm i\beta$. Two particular solutions are $y_1 = e^{(\alpha+i\beta)x}$ and $y_2 = e^{(\alpha-i\beta)x}$. The general complex solution is

$$y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right).$$

Example: Solve $y^{"} + 2y^{'} + 5y = 0$, with given conditions $y(0) = 0,\ \ y^{'}(0) = 4$.

Solution:

The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. $\Delta = -16 < 0$, since there exist tow complex and distinct roots $\lambda_{1,2} = -1 \pm 2i$. Then, the general solution is

$$y = e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right).$$

We have $y(0) = 0 \Longrightarrow c_1 \cos(0) + c_2 \sin(0) = 0 \Longrightarrow c_1 = 0$,

and $y^{'}(0) = 4 \Longrightarrow -2c_1 \sin(0) + 2c_2 \cos(0) = 4 \Longrightarrow c_2 = 2$.

Then, $y = 2\sin(2x)$ is a particular solution.

3.3 Partial differential equations

3.3.1 General facts about PDE

Definition 3.10. *(Partial differential equations)*

Let u be a function dependent on n independent variables $x_1,\ \ x_2...x_n$. Partial differential equation is an identity that relates the independent variables $x_1,\ \ x_2...x_{n},$, the dependent variable u, and the partial derivatives of u. An n-th order partial differential equation (PDE) is an equation has the form

$$F(x_1,\ \ x_2...x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial^n u}{\partial x_1^n}, ..., \frac{\partial^n u}{\partial x_1^{m_1}, \partial x_2^{m_2}, ..., \partial x_k^{m_k}}) = g(x_1,\ \ x_2...x_n, u), \qquad (3.13)$$

where $1 \leq k \leq n$, $m_1 + m_2 + ... + m_k = n$. We denote the partial derivatives by $\frac{\partial u}{\partial x} = u_x$.

Definition 3.11. *(Order partial differential equation)*

The order of an equation is the highest derivative that appears.

Definition 3.12. *(Homogeneous partial differential equation)*

Equation (3.13) is called a homogeneous equation if $g = 0$.

Definition 3.13. *(Solution)*

A solution of a PDE is a function $u(x_1, x_2, ...)$ that satisfies the equation identically, at least in some region of the $x_1, x_2, ...$ variables.

Example 3.10. *Find the partial derivatives in order 2 of the following functions:*

$$u(x, y) = x^2(x + y), \quad u(x, y) = e^{xy}.$$

Solution:

1. $u(x, y) = x^2(x + y)$, *we have*

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy, \quad \frac{\partial u}{\partial y} = x^2.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 2y, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 2x.$$

2. $u(x, y) = e^{xy}$, *we have*

$$\frac{\partial u}{\partial x} = ye^{xy}, \quad \frac{\partial u}{\partial y} = xe^{xy}.$$

$$\frac{\partial^2 u}{\partial x^2} = y^2e^{xy}, \quad \frac{\partial^2 u}{\partial y^2} = y^2e^{xy}, \quad \frac{\partial^2 u}{\partial x \partial y} = e^{xy} + xye^{xy}.$$

3.3.2 First order partial differential equations

Definition 3.14.

The homogeneous first-order PDE in two independent variables can be written as

$$F(x, y, u(x, y), \partial u_x(x, y), \partial u_y(x, y)) = 0.$$

3.3.2.1 Some types of first order PDE

- Transport equation: $u_x + u_y = 0$.
- Shock wave equation: $u_x + uu_y = 0$.

3.3.2.2 The constant coefficient equation

Let

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0,$$

where a and b are constants not both zero.

Coordinate Method:

To solve the PDE, we change variables

$$x_1 = ax + by, \;\; y_1 = bx - ay.$$

Replace all x and y derivatives by x_1 and y_1 derivatives, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial y} = a\frac{\partial u}{\partial x_1} + b\frac{\partial u}{\partial y_1},$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial y} + \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y} = b\frac{\partial u}{\partial x_1} - a\frac{\partial u}{\partial y_1}.$$

Hence

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a\left(a\frac{\partial u}{\partial x_1} + b\frac{\partial u}{\partial y_1}\right) + b\left(b\frac{\partial u}{\partial x_1} - a\frac{\partial u}{\partial y_1}\right) = \left(a^2+b^2\right)\frac{\partial u}{\partial x_1}.$$

So, since $(a^2+b^2) \neq 0$, the equation takes the form $\frac{\partial u}{\partial x_1} = 0$ in the new variables. Thus the solution is defined by

$$u\left(x,y\right) = f\left(y_1\right) = f\left(bx - ay\right).$$

Example 3.11. *Solve the following PDE*

$$2\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial y} = 0, \;\; u(0,y) = y^2.$$

Solution: *The general solution of the PDE is $u(x,y) = f(-5x-2y)$.*
Setting $x = 0$ yields the equation $y^2 = f(-2y)$. Letting $v = -2y$, yields $f(v) = -\frac{v^2}{4}$.
Therefore,

$$u\left(x,y\right) = \frac{1}{4}\left(5x+2y\right)^2.$$

3.3.2.3 The variable coefficient equation

Consider the following partial differential equation in two independent variables,

$$a\left(x,y,u\right)\frac{\partial u}{\partial x} + b\left(x,y,u\right)\frac{\partial u}{\partial y} = g\left(x,y,u\right).$$

The method of characteristics:

Let $u = u(x, y)$ be a solution of this equation. It reduces the solution of the PDE to the solution of the ODE. The curves in the xy plane as tangent vectors. Their equations are

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{du}{dt} = c.$$

This relation can be summarized in the form

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

Thus the integral surface formed by this characteristics of adequacy is the solution sought. Every solution of the PDE is constant on the solution curves of the ODE.

Example 3.12. *Find solutions of* $3u_x - 2u_y + u = x$.

Solution*:*

The standard form of the equation is $3u_x 2u_y = x - u$. *The characteristic equations defined by*

$$\frac{dx}{3} = \frac{dy}{-2} = \frac{du}{x-u}.$$

These equations imply that $2dx = 3dy$. *This implies that the characteristic curves (lines) are* $2x + 3y = c_1$.

and $\frac{du}{dx} = \frac{1}{3}(x - u)$. *This is a linear first order differential equation,* $\frac{du}{dx} + \frac{1}{3}u = \frac{1}{3}x$. *It can be solved using the integrating factor*

$$\begin{aligned}
\mu(x) &= \exp(\frac{1}{3}\int dx) = e^{x/3} \\
\frac{d}{dx}\left(ue^{x/3}\right) &= \frac{1}{3}xe^{x/3} \\
ue^{x/3} &= \frac{1}{3}\int xe^{x/3}dx + c_2 = (x-3)\, e^{x/3} + c_2 \\
u\,(x, y) &= (x-3) + c_2 e^{-x/3}.
\end{aligned}$$

As before, we write c_2 *as an arbitrary function of* $c_1 = 2x + 3y$. *This gives the general solution*

$$u\,(x, y) = x - 3 + G(2x + 3y)e^{-x/3}.$$

3.3.3 Second order partial differential equations

Definition 3.15.

The general form of a second order PDE in two independents written as

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+p(x,y)\frac{\partial u}{\partial x}+q(x,y)\frac{\partial u}{\partial y}+g(x,y)u=g(x,y).$$

3.3.3.1 Classification of second order PDE

The type of second order PDE depends on $\Delta = b^2 - 4ac$. We consider three types for the equation

- The second order PDE is called hyperbolic if $\Delta > 0$.

  Example 3.13. *Consider the equation* $2u_{xx} - x^2y^4u_{yy} = 0$. *Here* $a = 2,\ b = 0$ *and* $c = -x^2y^4$, *and therefore* $\Delta = 8x^2y^4 > 0$. *Thus, the given equation is hyperbolic.*

- The second order PDE is called parabolic if $\Delta = 0$.

  Example 3.14. *We consider the equation* $u_{xx} + 2u_{xy} + u_{yy} = 0$. *We get* $a = 1,\ b = 2$ *and* $c = 1$. *Therefore* $\Delta = 0$. *Then, the given equation is parabolic.*

- The second order PDE is called elliptic if $\Delta < 0$.

  Example 3.15. *Let the equation* $u_{xx} + y^2u_{yy} = 0$. *We have* $a = 1,\ b = 0$ *and* $c = y^2$, *and therefore* $\Delta = -y^2 < 0$. *Thus, the given equation is elliptic.*

3.3.3.2 Main equations of physics

1. **Laplace's equation:** $u_{xx} + c^2u_{yy} = 0$. The equation is elliptic for $c = \pm 1$.

2. **Poisson's equation:** $\nabla^2 u = \delta^2(x,y)$. The equation is elliptic.

3. **Heat equation:** $u_{xx} = \frac{1}{a}u_t$. The equation is parabolic.

4. **Wave equation:** $u_{xx} = \frac{1}{a^2}u_{tt}$. The equation is parabolic.

3.3.3.3 The method of separation of variables

Consider the following second order EDP with constant coefficient

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u = 0. \tag{3.14}$$

We will look for a solution has the following form

$$u(x, y) = f(x)g(y), \tag{3.15}$$

Substituting (3.15) into the equation (3.14), we obtain

$$af^{''}(x)g(y) + bf(x)g^{''}(y) + \alpha f^{'}(x)g(y) + \beta f(x)g^{'}(y) + \gamma f(x)g(y) = 0.$$

Suppose that $f(x)g(y) \neq 0$, we get

$$a\frac{f^{''}(x)}{f(x)} + b\frac{g^{''}(y)}{g(y)} + \alpha\frac{f^{'}(x)}{f(x)} + \beta\frac{g^{'}(y)}{g(y)} + \gamma = 0.$$

Therefore, we get two EDO to solve

$$\begin{aligned} a\frac{f^{''}(x)}{f(x)} + \alpha\frac{f^{'}(x)}{f(x)} + c &= 0, \\ b\frac{g^{''}(y)}{g(y)} + \beta\frac{g^{'}(y)}{g(y)} + \gamma - c &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} af^{''}(x) + \alpha f^{'}(x) + cf(x) &= 0, \\ bg^{''}(y) + \beta g^{'}(y) + (\gamma - c)g(y) &= 0, \end{aligned}$$

where c is a constant. After that we solve these tow equations for obtain the tow solutions $f(x)$ and $g(y)$.

Example 3.16. *Solve the following equation*

$$-4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{3.16}$$

Solution:

Let $u(x, y) = f(x)g(y)$ *be a solution of the EDP* (3.16)*. Substituting this solution into the equation* (3.16)*, we obtain*

$$-4f''(x)g(y) + f(x)g''(y) = 0 \Longleftrightarrow \frac{f''(x)}{f(x)} = \frac{g''(y)}{4g(y)} = -c^2.$$

Therefore, we get the two following EDO

$$\begin{aligned} f''(x) + c^2 f(x) &= 0, \\ g''(y) + 4c^2 g(y) &= 0. \end{aligned}$$

These tow EDO have the following tow solutions

$$\begin{aligned} f(x) &= c_1 \cos(cx) + c_2 \sin(cx), \\ g(y) &= c_3 \cos(2cy) + c_4 \sin(2cy). \end{aligned}$$

Then, the general solution of the EDP (3.16) *is given by*

$$u(x, y) = (c_1 \cos(cx) + c_2 \sin(cx)) (c_3 \cos(2cy) + c_4 \sin(2cy)).$$

3.4 Special functions

3.4.1 The Gamma function

Definition 3.16. *(Gamma function)*
The Gamma function Γ *is defined as*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt,$$

where z *a complex number such that* $Re(z) > 0$*.*

Definition 3.17. *The Gamma function is the generalization of the factorial to complex numbers is*

$$\Gamma(n + 1) = n!.$$

Remark 3.2. *Integrating by parts, we get*

$$\Gamma(z+1) = z\Gamma(z).$$

3.4.2 Bessel functions

3.4.2.1 Bessel functions of the first kind

Definition 3.18. *The Bessel function is a solution of the following linear second-order differential equation*

$$x^2 y^{''} + xy^{'} + (x^2 - \lambda^2)y = 0.$$

where $\lambda \in \mathbb{C} - \{k/2 | k \in \mathbb{Z}\}$.

Definition 3.19. *(Bessel function) The Bessel function of the first kind of order* λ *is a function of the form:*

$$J_\lambda(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin(\theta)-\lambda\theta)} d\theta = \frac{1}{\pi}\int_0^{\pi} \cos\left(x\sin(\theta) - \lambda\theta\right).$$

Definition 3.20. *Let* λ *be a complex constant. Then, the series defining Bessel functions of the first kind of order* λ *and* $-\lambda$ *are*

$$J_\lambda(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\lambda+n+1\right)} \left(\frac{x}{2}\right)^{(\lambda+2n)},$$

and

$$J_{-\lambda}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(-\lambda+n+1\right)} \left(\frac{x}{2}\right)^{(-\lambda+2n)}.$$

Example 3.17. *The Bessel functions* $J_0(x)$ *and* $J_0(x)$ *of the first kind of orders 0 and 1 are*

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(n+1\right)} \left(\frac{x}{2}\right)^{(2n)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{(2n)}, \\ J_1(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(n+2\right)} \left(\frac{x}{2}\right)^{(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2\left(n+1\right)} \left(\frac{x}{2}\right)^{(2n+1)}. \end{aligned}$$

Thus, for small values of x, $J_0(x) \sim 1$ *and* $J_1(x) \sim \frac{x}{2}$. *Moreover, we have that for real values* *of* x

$$\begin{aligned}\lim_{x\longrightarrow 0} J_0(x) &= 1,\\ \lim_{x\longrightarrow 0} J_1(x) &= 0.\end{aligned}$$

3.4.3 Properties of the Bessel function

1. $J_{-\lambda}(x) = (-1)^{\lambda} J_{-\lambda}(x)$.
2. $J_{\lambda+1}(x) + J_{\lambda-1}(x) = \frac{2\lambda}{x} J_{\lambda}(x)$.
3. $J_{\lambda+1}(x) - J_{\lambda-1}(x) = -2J'_{\lambda}(x)$.
4. $J_{\lambda+1}(x) - \frac{\lambda}{x} J(x) = -J'_{\lambda}(x)$.
5. $\frac{d}{dx} x^{\lambda} J_{\lambda}(x) = x^{\lambda} \frac{d}{dx} J_{\lambda}(x) + \lambda x^{\lambda-1} J_{\lambda}(x) = x^{\lambda} J_{\lambda-1}(x)$.
6. $\frac{d}{dx} x^{-\lambda} J_{\lambda}(x) = x^{-\lambda} \frac{d}{dx} J_{\lambda}(x) - \lambda x^{-\lambda-1} J_{\lambda}(x) = -x^{-\lambda} J_{\lambda+1}(x)$.

Example 3.18. *Prove the following inequality, for all* $x \in \mathbb{C}$.

$$|J_{\lambda}(x)| \leq \frac{1}{\lambda!} \left(\frac{|x|}{2}\right)^{\lambda} e^{(|x|/2)^2}, \quad \forall \lambda \in \mathbb{Z}.$$

Solution:

$$\begin{aligned}|J_{\lambda}(x)| &= \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(\lambda+n)!} \left(\frac{x}{2}\right)^{(\lambda+2n)} \right| \\ &\leq \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n!\,(\lambda+n)!} \left(\frac{x}{2}\right)^{(\lambda+2n)} \right| \\ &= \sum_{n=0}^{\infty} \frac{1}{n!\,(\lambda+n)!} \left(\frac{|x|}{2}\right)^{(\lambda+2n)} \\ &= \left(\frac{|x|}{2}\right)^{\lambda} \sum_{n=0}^{\infty} \frac{1}{n!\,(\lambda+n)!} \left(\frac{|x|}{2}\right)^{2n} \\ &\leq \left(\frac{|x|}{2}\right)^{\lambda} \sum_{n=0}^{\infty} \frac{1}{n!\lambda!} \left(\frac{|x|}{2}\right)^{2n}.\end{aligned}$$

Since $e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, *then*

$$|J_{\lambda}(x)| \leq \frac{1}{\lambda!} \left(\frac{|x|}{2}\right)^{\lambda} e^{(|x|/2)^2}.$$

3.5 Supplementary exercises

Exercise 01.

Solve

- $y' = \frac{x+1}{y^4+1}$.
- $dy = 2x\left(y^2+9\right)dx$.
- $\frac{dy}{dx} = y^2 - 2x + 1$.
- $e^x dx - y dy = 0, \;\; y(0) = 1$.

Exercise 02.

Solve

- $y' - xy = -x, \;\; y(0) = -4$.
- $e^{-y}dy + dx + 2xdy = 0$.
- $y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}}$.
- $y' = y + 2y^5$.
- $y' = \frac{2y^4+x^4}{xy^3}$.
- $y' = \frac{2xy}{x^2-y^2}$.
- $y' = \frac{x^2+y^2}{xy}, \;\; y(1) = -2$.
- $y' = \frac{x+y+6}{2x-y}$.
- $y" - 2y' = 0$.
- $y" - 6y' + 25y = 0, \;\; y(0) = 1, y'(0) = 2$.

Exercise 03.

State whether the following statements are **True** or **False**.

- Order of the differential equation representing the family of ellipses having center at origin and force on x-axis is two.
- Degree of the differential equation $\sqrt{1 + \frac{d^2y}{dx^2}} = x + \frac{dy}{dx}$.

- $\frac{dy}{dx} + y = 5$ is a differential equation of the type $\frac{dy}{dx} + py = q$ but it can be solved using variable separable method also.
- $f(x, y) = \frac{x^2+y^2}{x-y}$ is a homogeneous function of degree 1.
- The general solution of the differential equation $x(1 + y^2)dx + y(1 + x^2)dy = 0$ is $(1 + x^2)(1 + y^2) = k$.

Exercise 04.

- Find the partial derivatives in order 2 of the following functions:
  1. $u(x, y) = e^x \cos(y)$.
  2. $u(x, y) = (x^2 + y^2) \cos(xy)$.
  3. $u(x, y) = \sqrt{1 + x^2y^2}$.
- Solve the first-order equations:
  1. $2u_x + 3u_y = 0$, with the auxiliary condition $u(0, y) = \sin(y)$.
  2. $(1 + x^2)u_x + u_y = 0.$.
  3. $xu_x + yu_y = 0$.
  4. $\sqrt{1x}2u_x + u_y = 0$, with the condition $u(0, y) = y$.
- Solve the second-order equations:
  1. $u_{xx} + 4u_{xy} + u_{yy} = 0$.
  2. $3u_y + u_{xy} = 0$.

Exercise 05.

Prove the following relation

$$\frac{d}{dx}\frac{J_{-\lambda}(x)}{J_\lambda(x)} = -\frac{2\sin(\lambda\pi)}{x\pi\left(J_\lambda(x)\right)^2}.$$

Deduce that J_λ and $J_{-\lambda}$ are linearly dependent if and only if $\lambda \in \mathbb{Z}$.

4 Series

A series is the sum of the terms in a sequence. In this chapter, we will study a different types of series and its convergence conditions.

4.1 Numerical series

Let $(U)_{n\geq 0}$ be a sequence of real or complex numbers, we define the sequence (S_n) of ***partial sums*** by

$S_0 = u_0,$

$S_1 = u_0 + u_1,$

$S_2 = u_0 + u_1 + u_2,$

$\dots$

$S_n = u_0 + u_1 + u_2 + \cdots + u_n = \sum_{k=0}^{n} u_k.$

The series for the general term u_n is denoted by

$$\sum_{n=1}^{\infty} u_n, \ \sum_{n\geq 1} u_n \ \textit{or just} \ \sum u_n.$$

Definition 4.1.

- *If* $\lim\limits_{n\to+\infty} S_n = S$, *the series* $\sum u_n$ *converges and* $\sum_{n=1}^{\infty} u_n = S$. *The limit* S *is called the sum of the series.*
- *If* $\lim\limits_{n\to+\infty} S_n$ *does not have a finite limit, the series* $\sum_{n=1}^{\infty} u_n$ *diverges.*

Example 4.1. *(Geometric series)*

Let $u_n = r^n$. *The partial sums*

$$S_n = \begin{cases} \frac{1-r^{n+1}}{1-r} & \textbf{\textit{if}} \quad r \neq 0, \\ n+1 & \textbf{\textit{if}} \quad r = 0. \end{cases}$$

$\sum_{n=0}^{+\infty} u_n$ *converges if* $|r| < 1$, *and diverges if* $|r| \geq 1$.

If the geometric series converges, then the sum of the series $S = \frac{1}{1-r}$.

Example 4.2. *(Harmonic series)*

The general term of the harmonic series is defined by $u_n = \frac{1}{n}$ *where* $n \in \mathbb{N}^*$. *The series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *converges, and*

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

4.1.1 Necessary condition for convergence

Proposition 4.1. *If* $\sum u_n$ *is convergent, then* $\lim\limits_{n\to+\infty} u_n = 0$.

Remark 4.1.

- *The exchange of this condition is false.*
- *The condition* $\lim\limits_{n\to+\infty} u_n = 0$ *is not sufficient.*
- *In the practice, we use this condition in its contraposed form:*

  If $\lim\limits_{n\to+\infty} u_n \neq 0$, *then* $\sum u_n$ *is divergent.*

Example 4.3.

- *The series* $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ *is divergent because* $\lim\limits_{n\to+\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0$.
- *The series* $\sum_{n=1}^{+\infty} e^{\sqrt{n}}$ *is divergent because* $\lim\limits_{n\to+\infty} e^{\sqrt{n}} = +\infty \neq 0$.

Property 4.1.

- *If* $\sum_{n=0}^{+\infty} u_n = U$ *and* $\sum_{n=0}^{+\infty} v_n = V$ *are convergent, then* $\sum_{n=0}^{+\infty} (\alpha u_n + \beta v_n), \alpha, \beta \in \mathbb{R}$ *converge, and* $\sum_{n=0}^{+\infty} (\alpha u_n + \beta v_n) = \alpha U + \beta V$.
- *If* $\sum_{n=0}^{+\infty} u_n$ *is convergent and* $\sum_{n=0}^{+\infty} v_n$ *is divergent, then* $\sum_{n=0}^{+\infty} (\alpha u_n + \beta v_n), \alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ *is divergent.*

Example 4.4. *The series* $\sum_{n=1}^{+\infty} \left(\frac{3}{2^n} + \frac{2}{n(n+1)}\right)$ *is convergent, because* $\sum_{n=1}^{+\infty} \left(\frac{3}{2^n}\right)$ *converge, and* $\sum_{n=1}^{+\infty} \left(\frac{2}{n(n+1)}\right)$ *converge. And, we have*

$$\sum_{n=1}^{+\infty} \left(\frac{3}{2^n} + \frac{2}{n(n+1)}\right) = 3\sum_{n=1}^{+\infty} \left(\frac{1}{2^n}\right) + 2\sum_{n=1}^{+\infty} \left(\frac{1}{n(n+1)}\right) = 5.$$

4.1.2 Positive term series

The series $\sum u_n$ is called positive term series if $u_n \geq 0$ for all $n \geq N_0,\ \ N_0 \in \mathbb{N}$.

Proposition 4.2. *Suppose $\forall n > 0$, $u_n \geq 0$, we have*

$$\sum u_n \ \texttt{converges} \Longleftrightarrow (S_n(u)) \ \ \texttt{bounded}.$$

Theorem 4.1. *(Comparison Tests)*
Let (u_n) and (v_n) be two sequences. Suppose $0 < u_n < v_n$ for all $n \in \mathbb{N}$.

- *$\sum v_n$ converges $\Longrightarrow \sum u_n$ converges.*
- *$\sum u_n$ diverges $\Longrightarrow \sum v_n$ diverges.*

Example 4.5. *Consider $\sum_{n=0}^{+\infty} \sin\left(\frac{1}{2^n}\right)$ and $\sum_{n=0}^{+\infty}\left(\frac{1}{2^n}\right)$.*
We have $0 \leq \sin\left(\frac{1}{2^n}\right) \leq \frac{1}{2^n}$ and $\sum_{n=0}^{+\infty}\left(\frac{1}{2^n}\right)$ is a geometric series convergent, then the series $\sum_{n=0}^{+\infty} \sin\left(\frac{1}{2^n}\right)$ is convergent.

Theorem 4.2. *Let (u_n) and (v^n) be two sequences with positive terms are equivalent to $+\infty$.*

$$u_n \approx_{\infty} v_n \Longleftrightarrow \lim_{n \to +\infty} \frac{u_n}{v_n} = 1.$$

Then $\sum u_n$ and $\sum v_n$ have the same behavior.

Example 4.6.
The general term of the series $\sum \frac{n^2+3n+1}{n^3+2n^2+4}$ is equivalent to $\frac{1}{n}$, and we have that $\sum \frac{1}{n}$ is divergent. Then $\sum \frac{n^2+3n+1}{n^3+2n^2+4}$ is divergent.

Theorem 4.3.

- *If the sequence $(n^{\alpha} u_n)_{n \in \mathbb{N}^*}$ converges to 0, and if $\alpha > 1$, then $\sum_{n \geq 0} u_n$ is convergent.*
- *If the sequence $(n^{\alpha} u_n)_{n \in \mathbb{N}^*}$ tends to $+\infty$, and if $\alpha < 1$, then $\sum_{n \geq 0} u_n$ is divergent.*

4.1.2.1 Riemann and Bertrand series

Definition 4.2. *(Riemann series)*

The series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ *where* $n \geq 1,\ \alpha \in \mathbb{R}$ *is called Riemann series. We have*

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{n^\alpha} &= 0, \ \ \boldsymbol{if}\ \alpha > 0.\\ &= 1, \ \ \boldsymbol{if}\ \alpha = 0.\\ &= +\infty, \ \ \boldsymbol{if}\ \alpha < 0.\end{aligned}$$

If $\alpha = 1$*, we obtain an harmonic series.*

Proposition 4.3. *The Riemann series* $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ *is*

- *Convergent if* $\alpha > 1$.
- *Divergent if* $\alpha \leq 1$.

Example 4.7.

- *The Riemann series* $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$ *is divergent, because* $\alpha = \frac{1}{2} < 1$.
- *The series* $\sum \left[(-1)^n \sin\left(\frac{\pi}{n}\right)\right]^2 = \sin^2\left(\frac{\pi}{n}\right) \approx_\infty \frac{\pi^2}{n^2}$ *converges and so by the comparison Test with the Riemann series* $\sum_{n\geq 1} \frac{1}{n^2}$, $\alpha = 2 > 1$ *which is convergent.*

Definition 4.3. *(Bertrand series)*

The Bertrand series have the form

$$\sum_{n=2}^{+\infty} \frac{1}{n^\alpha (\ln(n))^\beta} \ \ n \geq 2, \ \ and \ \ \alpha, \beta \in \mathbb{R}.$$

Proposition 4.4. *The Bertrand series* $\sum_{n=2}^{+\infty} \frac{1}{n^\alpha (\ln(n))^\beta}$ *is*

- *convergent if* $\alpha > 1, \forall \beta$.
- *divergent if* $\alpha < 1, \forall \beta$.
- *If* $\alpha = 1$*, it is*
  - *convergent if* $\beta > 1$.
  - *divergent if* $\beta \leq 1$.

Example 4.8. *Determine if the following series is convergent or divergent.*

$$\sum n^{\frac{1}{n^2}} - 1.$$

Solution:

We have $u_n = n^{\frac{1}{n^2}} - 1 = e^{\ln n^{\frac{1}{n^2}}} - 1 = e^{\frac{\ln n}{n^2}} - 1$, *by the comparison test* $e^{u_n} - 1 \approx_\infty u_n$, *we obtain*

$$e^{\frac{\ln n}{n^2}} - 1 \approx_\infty \frac{\ln n}{n^2} = \frac{1}{n^2 (\ln n)^{-1}}.$$

Then, the series $\sum n^{\frac{1}{n^2}} - 1$ *is convergent by the comparison test with the Bertrand series* $\frac{1}{n^2(\ln n)^{-1}}$ *which is convergent because* $\alpha = 2 > 1$.

4.1.2.2 Ratio tests

Theorem 4.4. *(Cauchy criterion)*

Let $\sum u_n$ *be a positive term series. Let's suppose* $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = \lim\limits_{n\to+\infty} \sqrt[n]{u_n} = l$. *Then*

- *If* $l < 1 \Longrightarrow \sum u_n$ *converges.*
- *If* $l > 1 \Longrightarrow \sum u_n$ *diverges.*
- *If* $l = 1$, *nothing can be concluded.*

Example 4.9. *Study the nature of the following series*

$$\sum \left(\frac{n^2 - 5n + 1}{n^2 - 4n + 2} \right)^{n^2}.$$

Solution:

We have $\sqrt[n]{u_n} = \left(\frac{n^2-5n+1}{n^2-4n+2}\right)^{n^2} = \left(1 - \frac{n+1}{n^2-4n+2}\right)^n$, *and* $\ln \sqrt[n]{u_n} = n \ln\left(1 - \frac{n+1}{n^2-4n+2}\right) \approx \left(\frac{-n(n+1)}{n^2-4n+2}\right) \longrightarrow -1$. *Then*

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} e^{\ln \sqrt[n]{u_n}} = e^{-1} < 1 \Longrightarrow \sum u_n \text{ converges.}$$

Theorem 4.5. *(D'Alembert rule)*

Let $\sum u_n$ *be a positive term series. Let's suppose* $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = l$. *Then*

- *If* $l < 1 \Longrightarrow \sum u_n$ *converges.*
- *If* $l > 1 \Longrightarrow \sum u_n$ *diverges.*
- *If* $l = 1$, *nothing can be concluded.*

Example 4.10. *We will use D'Alembert rule to study the nature of some series* $\sum u_n$.

1. $u_n = \frac{n!}{n^n}$, *we have*

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = (n+1) \frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+\frac{1}{n}}\right).$$

As we also have $\lim\limits_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e$. *So* $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = \frac{1}{e} < 1$. *Then,* $\sum u_n$ *is convergent.*

2. $u_n = \frac{2n}{n+2^n}$, *we have*

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{n+1}{n} \frac{n+2^n}{n+1+2^{n+1}} = \frac{1}{2} < 1.$$

Then, $\sum u_n$ *is convergent.*

3. $u_n = \frac{3n-1}{n^4+1}$, *we have*

$$\frac{u_{n+1}}{u_n} = \frac{3n+2}{3n-1} \frac{n^4+1}{(n+1)^4+1} \longrightarrow 1.$$

In this case, we can't apply D'Alembert rule to study the nature of the series. We will use Riemman rule. Therefore, $u_n \approx_\infty \frac{3}{n^3}$ *which is convergent because* $3 > 1$. *Then* $\sum u_n$ *is convergent too.*

Proposition 4.5. 1. *If* $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = l_1 \neq 0$ *and* $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = l_2 \neq 0$, *then* $l_1 = l_2$.

2. *If* $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = l$, *then* $\lim\limits_{n\to+\infty} \sqrt[n]{u_n} = l$. *(The converse is not true).*

4.1.3 Integral test

Theorem 4.6. *Suppose that* $f(x)$ *is a continuous, positive and decreasing function on the interval* $[a, \infty[$ *and that* $f(n) = u_n$ *then*

1. *If* $\int_a^{+\infty} f(x)dx$ *is convergent so is* $\sum_{n=a}^{+\infty} u_n$.

2. *If* $\int_a^{+\infty} f(x)dx$ *is divergent so is* $\sum_{n=a}^{+\infty} u_n$.

4.1.4 Alternating series

Definition 4.4. *(Alternating series)*
An alternating series is any series $\sum u_n$ *for which the series terms can be written in one of the following two forms.*

$$u_n = (-1)^n v_n \text{ or } u_n = (-1)^{n+1} v_n, \text{ where } v_n \geq 0.$$

Theorem 4.7. *(Leibniz's theorem)*
Suppose that we have an alternating series $\sum (-1)^n v_n$. If

1. *$\{v_n\}$ is a decreasing sequence. i.e ($v_{n+1} \leq v_n$)*
2. *$\lim\limits_{n\to+\infty} v_n = 0$.*

Then, the series is convergent.

Example 4.11. *Let $u_n = \sin\left(\frac{\pi n^2}{n+1}\right)$. We have*

$$u_n = \sin\left[(n-1)\pi + \frac{\pi}{n+1}\right] = (-1)^n \sin\left(\frac{\pi}{n+1}\right).$$

The series $\sum_{n>0} \sin\left(\frac{\pi n^2}{n+1}\right)$ an alternating series, where $v_n = \sin\left(\frac{\pi}{n+1}\right)$ a decreasing sequence which tends to zero. Then, the series $\sum u_n$ is convergent.

4.1.5 Absolute convergence

Definition 4.5. *(Absolute Convergence)*
We call the series $\sum u_n$ absolutely convergent if $\sum |u_n|$ is convergent.

Definition 4.6. *(Conditionally convergent)*
The series $\sum u_n$ is said conditionally convergent, if it is convergent and $\sum |u_n|$ is divergent.

Remark 4.2. *If $\sum u_n$ is absolutely convergent then it is also convergent. (The converse is not true).*

Example 4.12.

1. *The series $\sum_{n=1}^{+\infty} \left|\frac{\cos(n)}{n^2}\right| \leq \frac{1}{n^2}$ and the Riemann series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ are convergent. Then, $\sum_{n=1}^{+\infty} \frac{\cos(n)}{n^2}$ is convergent.*
2. *The series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is convergent, but the series $\sum_{n=1}^{+\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent.*

4.2 Sequences and series of functions

4.2.1 Sequence of functions

Definition 4.7. *(Sequence of functions)*
Let I be a set in $\mathbb{R}$. If there is a function $f_n : A \longrightarrow \mathbb{R}$ for each $n \in \mathbb{N}$, we call $\{f_n\}$ a sequence of functions.

4.2.1.1 Pointwise convergence

Definition 4.8.

The sequence $\{f_n\}$ converges pointwise to a function f on I, if for each $x \in I$ we have

$$\lim_{n\to+\infty} f_n(x) = f(x).$$

In other words, for each $x_0 \in I$ the sequence of numbers $f_n(x)$ converges to the function $f(x)$.

Example 4.13. *1. Let $\{f_n\}$ defined by $f_n(x) = \frac{nx^2}{1+nx^2}$ converges pointwise to the function*

$$\begin{aligned} f(x) &= 0, \ \ \textbf{\textit{if}} \ x = 0. \\ &= 1, \ \ \textbf{\textit{if}} \ x \neq 0. \end{aligned}$$

2. Let $f_n : [0, \pi] \longrightarrow \mathbb{R}$ be defined by $f_n(x) = \frac{\sin(nx)}{n}$, $n \in \mathbb{N}$ converges pointwise to 0 on $[0, \pi]$.

4.2.1.2 Uniform convergence

Definition 4.9.

Suppose that $\{f_n\}$ is a sequence of functions $f_n : I \longrightarrow \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$. Then, f_n converges to uniformly to f on I if

$$\lim_{n\to+\infty} \left(\sup_{x\in I} |f_n(x) - f(x)| \right) = 0.$$

Example 4.14.

The sequence $f_n(x) = \frac{nx^2}{1+nx}$ converges pointwise to $f(x) = x$ on $[0, +\infty[$.

We have

$$\sup_{x\in[0,+\infty[} |f_n(x) - f(x)| = \sup_{x\in[0,+\infty[} \left| \frac{nx^2}{1+nx} - x \right| = \sup_{x\in[0,+\infty[} \left| \frac{x}{1+nx} \right| = \frac{1}{n},$$

and

$$\lim_{n\to+\infty} \left(\sup_{x\in[0,+\infty[} |f_n(x) - f(x)| \right) = 0.$$

Then, the sequence $f_n(x) = \frac{nx^2}{1+nx}$ converges uniformly to x on $[0, +\infty[$.

Remark 4.3. *If the sequence $\{f_n\}$ converges uniformly, then it converges pointwise. (The converse is not true).*

4.2.1.3 Properties of uniform convergence

- **Continuity**

Theorem 4.8. *If a sequence $\{f_n\}$ of continuous functions $f_n : I \longrightarrow \mathbb{R}$ converges uniformly on $I \subset \mathbb{R}$ to f, then f is continuous on I.*

- **Differentiability**

Theorem 4.9. *Suppose that $\{f_n\}$ is a sequence of differentiable functions $f_n : [a, b] \to \mathbb{R}$ such that f_n converges pointwise to f and f_n' converges uniformly to g for some $f, g : [a, b] \to \mathbb{R}$. Then f is differentiable on $[a, b]$ and $f' = g$, and we write*

$$\left[\lim_{n\longrightarrow+\infty} f_n(x)\right]' = \lim_{n\longrightarrow+\infty} \left[f_n'(x)\right].$$

- **Integration**

Theorem 4.10. *Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ and converges uniformly to f, then we have*

$$\lim_{n\longrightarrow+\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\longrightarrow+\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Example 4.15. *The sequence $f_n(x) = \frac{ne^{-x}+x^2}{n+x}$ pointwise convergence to $f(x) = e^{-x}$ on $[0, 1]$. We have*

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\frac{ne^{-x} + x^2}{n + x} - e^{-x}\right| = \sup_{x\in[0,1]} \left|\frac{x^2 - xe^x}{n + x}\right| \leq \frac{2}{n},$$

and $\lim_{n\longrightarrow+\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$, then $\{f_n\}$ converges uniformly to $f(x) = e^{-x}$. Therefore by the previous theorem, we have

$$\lim_{n\longrightarrow+\infty} \int_0^1 \frac{ne^{-x} + x^2}{n + x}dx = \int_0^1 \lim_{n\longrightarrow+\infty} \frac{ne^{-x} + x^2}{n + x}dx = \int_0^1 e^{-x}dx = 1 - e^{-1}.$$

4.2.2 Series of functions

Definition 4.10. *(Series of functions)*
Let $f_n : I \subset \mathbb{R} \longrightarrow \mathbb{R}$. *Let* $\{f\}_n$ *be a sequence of real-valued functions defined on a subset* I. *The series*

$$\sum_{n=0}^{+\infty} f_n(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots$$

is called series of functions.

Definition 4.11. *(Pointwise convergence)*
Let $f_n : I \subset \mathbb{R} \longrightarrow \mathbb{R}$. *The series* $\sum f_n(x)$ *converges pointwise on* I *if the sequence of partial sums* $\{S_n(x)\}$ *converges pointwise on* I.

Definition 4.12. *(Absolute convergence)*
Let $f_n : I \subset \mathbb{R} \longrightarrow \mathbb{R}$. *The series* $\sum f_n(x)$ *absolutely convergent on* I *if the series* $\sum |f_n(x)|$ *converges pointwise on* I.

Definition 4.13. *(Domain of convergence)*

- *We say that the series* $\sum f_n(x)$ *convergence on* x_0, *if* $\sum f_n(x_0)$ *is convergent.*
- *The series* $\sum f_n$ *converges pointwise to* f *on* (a, b) *if* $\sum f_n$ *converges for all* $x \in (a, b)$.
- *The domain of convergence of the series* $\sum f_n$ *is defined by*

$$D = \left\{x \in (a, b) : \sum f_n \ \texttt{converges}\right\}.$$

Example 4.16. *Find the domain of convergence of the series* $\sum_{n=1}^{+\infty} f_n(x)$, *where* $f_n(x) = \frac{x^n}{n}$.
Solution:
We note that $|f_n(x)| = \left|\frac{x^n}{n}\right| \leq |x|^n$.
For $x \in]-1, 1[$, *the series* $\sum_{n=1}^{+\infty} |x|^n$ *is convergent, then* $\sum_{n=1}^{+\infty} \frac{x^n}{n}$ *is convergent.*
For $x = -1$, *the series* $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ *is an alternating series which is convergent.*
For $x = 1$, *the series* $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is an harmonic series which is divergent.*
For $|x| > 0$, *we have* $\lim_{n\to+\infty} \frac{x^n}{n} = \infty$, *then* $\sum_{n=1}^{+\infty} \frac{x^n}{n}$ *is divergent for* $|x| > 0$.
Then, the domain of convergence of the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is* $D = [-1, 1\,[$.

Definition 4.14. *(Uniform convergence)*
Let $f_n : I \subset \mathbb{R} \longrightarrow \mathbb{R}$. *Let* $\{f_n\}$ *be a sequence of real-valued functions defined on a subset*

I of $\mathbb{R}$. We say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I if for each $x \in I$, the sequence of partial sums $\{S_n(x)\}$ converges uniformly on I, where $S_n(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_n(x) = \sum_{k=0}^{n} f_k(x)$.

Definition 4.15. *(Normal convergence)*
Let $f_n : I \subset \mathbb{R} \longrightarrow \mathbb{R}$. The series $\sum f_n(x)$ is called normally convergent if the series $\sum \|f_n(x)\|_\infty$ is convergent, where $\|f_n(x)\|_\infty = \sup_{x\in I} |f_n(x)|_\infty$.

Remark 4.4. *The relationship between the different kind of convergence.*

- *Normal convergence $\Longrightarrow$ Uniform convergence.*
- *Normal convergence $\Longrightarrow$ Absolute convergence.*
- *Absolute convergence $\Longrightarrow$ Pointwise convergence.*
- *Uniform convergence $\Longrightarrow$ Pointwise convergence.*

Example 4.17. *Consider the following series $\sum_{n=1}^{+\infty} \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$.*
We have $\left|\frac{1}{n^2+x^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is convergent. Then, the series $\sum_{n=1}^{+\infty} \frac{1}{n^2+x^2}$ converges normally on $\mathbb{R}$.

4.3 Power series

Definition 4.16. *A power series (centered at 0) is a series of the form*

$$\sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots ,$$

where the $\{a_n\}$ is a sequence of real numbers and $x \in \mathbb{R}$.

Remark 4.5. *The series*

$$\sum_{n=0}^{+\infty} a_n (x-c)^n ,$$

is a power series centered at c.

Definition 4.17. *(Domain of convergence)*
Let the power series $\sum_{n=0}^{+\infty} a_n x^n$, we define his domain of convergence by

$$D = \left\{ x \in \mathbb{R} \text{ such that } \sum_{n=0}^{+\infty} a_n x^n \text{ converges} \right\}.$$

Example 4.18. *Consider* $\sum_{n=1}^{+\infty} \frac{x^n}{n^2}$.

Using d'Alembert rule, we obtain

$$\lim_{n\to+\infty}\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \lim_{n\to+\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}}\right| = \lim_{n\to+\infty}\left|\left(\frac{n}{n+1}\right)^2 x\right| = |x|\,.$$

If $|x| < 1$, *the series is absolutely convergent, and if* $|x| > 1$ *is divergent.*

If $|x| = 1$, *we have* $|f_n(x)| = \frac{|x|^n}{n^2} = \frac{1}{n}$ *which is a Riemann series convergent.*

Therefore the series $\sum_{n=1}^{+\infty} \frac{x^n}{n^2}$ *is absolutely convergent. Then the domain of convergence is* $D = [-1, 1]$.

Theorem 4.11. *(Abel's Theorem)*

Let $\sum a_n x^n$ *a power series. Suppose that for all* $x_0 \in \mathbb{R}$ *the sequence* $(a_n x_0^n)_4$ *bounded.*

- *The series* $\sum a_n x^n$ *is absolutely convergent for* $|x| < |x_0|$.
- *The series* $\sum a_n x^n$ *is normally convergent for* $|x| < r$, *where* $0 < r < |x_0|$.

4.3.1 Radius of convergence

Theorem 4.12. *Let* $\sum a_n x^n$ *be a power series, There is an* $0 \leq R \leq \infty$ *such that*

- $\sum a_n x^n$ *is absolutely convergent in* $]-R, R[$.
- $\sum a_n x^n$ *is divergent if* $|x| > R$.

Definition 4.18. *The number* $R = \sup\{r \in \mathbb{R}^+ : \sum a_n x^n \text{ converges}\} \in \mathbb{R}^+ \cup \{+\infty\}$ *is called the radius of convergence of the series* $\sum a_n x^n$.

Remark 4.6. *The radius of convergence of the series* $\sum a_n x^n$ *is characterized by*

1. *If* $|x| < R \Longrightarrow \sum a_n x^n$ *is absolutely convergent.*
2. *If* $|x| > R \Longrightarrow \sum a_n x^n$ *is divergent.*
3. *If* $|x| = R$, *nothing can be concluded.*
4. *If* $|x| \leq r < R(r > 0) \Longrightarrow \sum a_n x^n$ *is normally convergent .*

Theorem 4.13. *(Hadamard's theorem)*

The radius of convergence R *of the power series* $\sum a_n x^n$ *is given by*

$$\frac{1}{R} = \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\sqrt[n]{|a_n|}.$$

Example 4.19. *We will compute the radius of convergence of the following power series.*

1. *Let the series* $\sum_{n\geq 1} \frac{n^{2n}}{(2n)!} x^n$. *We have* $a_n = \frac{n^{2n}}{(2n)!}$, *so* R *can be determined by the ratio test*

$$\begin{aligned} \frac{1}{R} &= \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)^{2n+2}}{(2n+2)!} \frac{(2n)!}{n^{2n}}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)}{2(2n+1)} \left(\frac{n+1}{n}\right)^{2n}\right| \\ &= \lim_{n\to+\infty} \frac{(n+1)}{2(2n+1)} \left|\left(1+\frac{1}{n}\right)^{2}\right|^{2} = \frac{e^2}{4}. \end{aligned}$$

Then, the radius of convergence is $R = \frac{4}{e^2}$

2. *Consider* $\sum_{n=1}^{+\infty} \frac{x^n}{2^n}$.

We have $a_n = \frac{1}{2^n}$, *so*

$$\frac{1}{R} = \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \left(\frac{1}{2^n}\right)^{\frac{1}{n}} = \frac{1}{2}.$$

Therefore the radius of convergence is $R = 2$.

In this case, the series $\sum_{n=1}^{+\infty} \frac{x^n}{2^n}$ *converges absolutely for for every real number* x *such that* $|x| < 2$, *and diverges for* $|x| \geq 2$.

4.3.2 Properties of power series

- **Differentiation**

Property 4.2. *Suppose that the power series* $\sum a_n x^n$ *has radius of convergence* R. *Let* $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ *be a function defined on* $]-R, R]$. *Then* f *is differentiable and* $f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.

Remark 4.7. *The power series and the differentiated power series have the same radius of convergence.*

Example 4.20. *Consider* $f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} x^n$ *has a radius of convergence* $R = 1$ *wich is defined and continuous on* $]-1, 1[$. *For all* $x \in]-1, 1[$ *we have*

$$f'(x) = \sum_{n=1}^{+\infty} n \frac{(-1)^n}{n} x^{n-1} = \sum_{n=1}^{+\infty} -(-x^{n-1}) = -\sum_{n=0}^{+\infty} (-x)^n = \frac{-1}{x+1}.$$

Then, for all $x \in]-1, 1[$ *we have* $f(x) = f(0) + \int_0^x \frac{-1}{1+t} dt = -\ln(1+x)$.

Therefore $\forall x \in]-1, 1]$, $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} x^n = \ln(1+x)$.

- **Integration**

Property 4.3. *Let* $\sum a_n x^n$ *be a power series has a radius of convergence* R*, and let* $f :]-1,1[\longrightarrow \mathbb{R}$ *defined by* $f(x) = \sum_{n=0}^{+\infty} a_n x^n$*. Then for all* $[0,x] \subset]-1,1[$ *we have*

$$\int_0^x \sum_{n=0}^{+\infty} a_n t^n dt = \sum_{n=0}^{+\infty} a_n \int_0^x t^n dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}.$$

The function $f :]-1,1[\longrightarrow \mathbb{R}$ *defined by* $F(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}$ *has a radius of convergence* R*, which is primitive of* f *, and* $F^{'}(x) = f(x),\ \forall x \in]-R,R[$.

4.3.3 Power series expansion

Definition 4.19. *Let* f *be a real function. We say that* f *is a power series expansion at* x_0*, if there exist a sequence* a_n *and* $R > 0$ *such that*

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n,\ \ x \in]x_0 - R, x_0 + R[.$$

Example 4.21. *Let* f *defined on* $]-1,1[$ *by* $f(x) = \frac{1}{1-x}$.
The power series expansion of f *on* $]-1,1[$ *is:* $\sum_{n=0}^{+\infty} x^n$.

4.3.4 Taylor's series

Definition 4.20. *Suppose that* $f \in C^\infty$ *(infinitely differentiable) on a neighborhood of* 0*, then we can define its Taylor coefficients* $a_n = \frac{f^n(0)}{n!}$ *at* 0 *for every* $n \in N$*, and the Taylor series of* $f :]-R,R[\longrightarrow R$ *is given by* $\sum_{n=0}^{+\infty} \frac{f^n(0)}{n!} x^n$.

Proposition 4.6. *Let* $f :]-R,R[\longrightarrow R$ *a* C^∞ *function on a neighborhood of* 0*. If there exist* $M > 0$ *such that* $\forall n \in N$ *and* $\forall x \in]-R,R[$*, we have* $|f^n(x)| \leq M$*. Then, the Taylor series* $\sum_{n=0}^{+\infty} \frac{f^n(0)}{n!} x^n$ *of* f *is pointwise convergent on* $]-R,R[$ *and*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^n(0)}{n!} x^n,\ \ \forall x \in]-R,R[.$$

Example 4.22. *Consider* $f(x) = \ln\left(\frac{1+x}{2-x}\right)$.
We will develop the function on power series. So, we have
$f(x) = \ln\left(\frac{1+x}{2-x}\right) = \ln(1+x) - \ln(2-x)$.
$\ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n,\ R_1 = 1$.

$\ln(2-x) = \ln\left(2\left(1-\frac{x}{2}\right)\right) = \ln 2 + \ln\left(1-\frac{x}{2}\right)$ *and* $\ln\left(1-\frac{x}{2}\right) = -\sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{x}{2}\right)^n$, *then* $\ln(2-x) = \ln 2 - \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{x}{2}\right)^n$, *and* $R_2 = 2$. *Therefore*

$$f(x) = \ln\left(\frac{1+x}{2-x}\right) = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{x^n}{n} - \ln 2 + \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{x}{2}\right)^n.$$

Then, the power series expansion of f *is*

$$f(x) = -\ln 2 + \sum_{n=1}^{+\infty}\left[(-1)^{n+1} + \frac{1}{2^n n}\right]x^n,$$

and his radius of convergence is $R_f = \min(R_1, R_2) = \min(1,2) = 1$.

4.4 Fourier series

Definition 4.21. *(Periodic function)*
A function $f(x)$ *is said to be a periodic function of period* T *if there exists a positive real number* T *such that* $f(x+T) = f(x)$ *for all* $x \in \mathbb{R}$.

Example 4.23.

1. $\cos\theta = \cos(\theta + 2\pi)$, $T = 2\pi$.
2. $\sin w\theta = \sin(w\theta + 2\pi)$, $T = \frac{2\pi}{|w|}$.

Definition 4.22. *(Trigonometric polynomial)*
A trigonometric polynomial of period $T > 0$ *is an expression of the form:*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n\cos\left(\frac{2\pi}{T}nt\right) + b_n\cos\left(\frac{2\pi}{T}nt\right)\right],$$

where $t \in \mathbb{R}$ *and* $a_n, b_n \in C$. *Such a polynomial is said to be of degree* n *if either of the coefficients* a_n *or* b_n *is different from* 0.

Definition 4.23. *(Fourier series)*
Let $f : [-L, L] \longrightarrow \mathbb{R}$ *be a* $2L-$*periodic function* $[f(x+2L) = f(x)]$. *The Fourier series of* $f(x)$ *is of the form*

$$SF(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1,2,3,\cdots, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1,2,3,\cdots. \end{aligned}$$

Definition 4.24. *(Fourier series of $2\pi-$periodic function)*

Let f be a $2\pi-$periodic function. The Fourier series of $f(x)$ is of the form

$$SF(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad n = 1,2,3,\cdots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n = 1,2,3,\cdots. \end{aligned}$$

Example 4.24. *Find the Fourier series of $f(x) = \frac{4-x^2}{2}$ on $[-2,2]$.*

Solution:

f is a $4-$periodic function. The Fourier series is defined by

$$SF(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\left(\frac{n\pi x}{2}\right)\right),$$

where

$$a_0 = \int_0^2 \frac{4-x^2}{2}dx = \frac{8}{3}, \tag{4.1}$$

$$a_n = \int_0^2 \frac{4-x^2}{2}\cos\left(\frac{n\pi x}{2}\right)dx = \frac{8(-1)^{n+1}}{\pi^2 n^2}, \quad n \geq 1, \tag{4.2}$$

$$b_n = 0. \tag{4.3}$$

Then,

$$SF(f)(x) = \frac{4}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{2}\right), \quad \forall x \in [-2,2].$$

Definition 4.25. *(Fourier Cosine series)*

The coefficients of the Fourier cosine series of $f : [0, L] \longrightarrow \mathbb{R}$ is given by the coefficients of the full Fourier series of the even extension of f:

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx, \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \\ b_n &= 0. \end{aligned}$$

Definition 4.26. *(Fourier Sine series)*
The coefficients of the Fourier sine series of $f : [0, L] \longrightarrow \mathbb{R}$ is given by the coefficients of the full Fourier series of the odd extension of f:

$$\begin{aligned} a_0 &= a_n = 0, \\ b_n &= \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Example 4.25. *Decompose $f(x) = x$ into its Fourier series on the interval $[-1, 1]$.*
Solution.
We find the Fourier coefficients
a_n: Since $f(x) = x$ is odd, the a_n coefficients are zero.
$b_n = \int_{-1}^{1} x\sin(n\pi x) = -\frac{2(-1)^n}{\pi n}$.
The corresponding Fourier series of f(x)=x is given by

$$x = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n}\sin(n\pi x).$$

4.4.1 Convergence of Fourier series

4.4.2 Dirichlet's theorem

Theorem 4.14. *(Dirichlet)*
Let $f \in \mathbb{R}$ be a $2L-$periodic function and $x_0 \in \mathbb{R}$ such that:

- *Lateral limits $f(x_0^-), f(x_0^+)$ exist and are real.*
- *Lateral derivatives $f'(x_0^-), f'(x_0^+)$ exist and are real.*

Then, the Fourier series of f converges to $Sf(x_0) = f(x_0)$ if f is continuous at x_0 , and it converges to

$$Sf(x_0) = \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right],$$

if f is discontinuous at x_0.

Example 4.26. *Let $f(t) = x$ be a $2\pi-$periodic function on $]-\pi, \pi]$.*

1. *Find the Fourier series of f. Does the series converge?*

Solution:

- *We have*

$$SF(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right).$$

  f is an odd function, so $a_0 = a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x sin(nx)dx = -2\frac{(-1)^n}{n}.$$

  Then,

$$SF(f)(x) = 2\sum_{n=1}^{+\infty}\frac{(-1)^{n-1} sin(nx)}{n}.$$

- *The function f is not continuous over $]-\pi, \pi]$ because $f(\pi) = \pi$ but $\lim_{x\to-\pi^+} f(x) = -\pi$. Therefore, there is a discontinuity at $x = \pm\pi$. Then, Dirichlet's theorem affirms that the series will converge to x over $]-\pi, \pi[$, and it will converge to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ at $x = \pm\pi$.*

4.5 Supplementary exercises

Exercise 01.

Study the nature of the following series

1. $\sum_{n=1}^{+\infty} \frac{3n}{5n+1}$.
2. $\sum_{n\geq 1} (-1)^n cos\frac{1}{n}$.
3. $\sum \frac{2^n+n^3}{3^n+n^2}$.

Exercise 02. (Cauchy criterion, D'Alembert rule and alternating series)

Study the nature of the following series

1. $\sum \left(\frac{n^2-5n+1}{n^2-4n+2}\right)^{n^2}$.

2. $\sum \frac{(3n)!n!5^{4n}}{(4n-1)!3^{2n}2^{3n}}$.

3. $\sum (-1)^n \frac{lnn}{\sqrt{n}}$.

Exercise 03.

Study the pointwise and uniform convergence of the following sequence functions on I

1. $f_n(x) = \frac{x}{x+n}$, $I = [0, 1]$ and $I = [1, +\infty[$.

2. $f_n(x) = \frac{nx}{1+n^2x^2}$, $I = \mathbb{R}$.

Exercise 04.

- Prove that $\sum_{n\geq 1} \frac{cos(nx)}{n(n+1)}$ is normally convergent.

- Study the pointwise, uniform, absolute and normal convergence of the following series

  - $\sum_{n\geq 1} \sqrt{n}xe^{-n^2x}$, $x \in \mathbb{R}^+$.
  - $\sum_{n\geq 1} x^n (lnx)^2$, $x \in [0, 1]$.

Exercise 05.

Compute the radius of convergence of the following power series.

1. $\sum_{n\geq 1} \frac{n!}{2^{2n}}\sqrt{(2n)!}xn$.

2. $\sum_{n\geq 1} (ln(n!))\, xn$.

Exercise 06.

1. Develop the following functions on power series

   - $f(x) = \frac{1}{6x^2-5x+1}$.
   - $f(x) = ln(x^2 - 5x + 6)$.
   - $f(x) = \int_0^x cos(t^2)dt$.

2. Find the some of the following series:

   - $\sum_{n\geq 1} \frac{n^2+3n+2}{n!}x$.
   - $\sum_{n\geq 1} n^2x^{2n}$.

3. Prove that: $\int_0^{1/2}\left(\sum_{+\infty}^{n=0} x^n\right) dx = \sum_{n=0}^{+\infty} \frac{1}{2^{n+1}(n+1)}$.

Exercise 07.

Find the Fourier series of the following functions, and study its convergence:

1. $f(x) = \pi - |x|$, $x \in]-\pi, \pi]$.

2. $f(x) = x^2$, $[0, 2\pi[$.

3. $f(x) = e^x$, $]-\pi, \pi[$.

4. $f(x) = \begin{cases} 1 & \texttt{if} \quad x \in]0, \pi[\\ 0 & \texttt{if} \quad r = \pi \end{cases}$.

5 Fourier transform

The Fourier transform is a useful tool for solving differential equations because it can reformulate them as problems which are easier to solve.

5.1 Mathematical operator of Fourier transform

Definition 5.1. *$L^1(\mathbb{R})$ is said to be a set of piecewise continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^{+\infty} |f(t)|\, dt$ exists.*

Example 5.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(t) = \frac{1}{1+t^2}$ belongs to $L^1(\mathbb{R})$ because $\int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt = 2 \lim\limits_{x \to +\infty} \arctan(x) = \pi$.*

Definition 5.2. *(Fourier transform)*
Let $f \in L^1(\mathbb{R})$. The Fourier transform $F : \mathbb{R} \longrightarrow \mathbb{C}$ of the function $f(t)$ is defined as

$$TF(f)(t) = F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt,$$

where $w \in \mathbb{R}$.

Definition 5.3. *(Specific cases)*

- *If f is an even function. The Fourier transform of f is defined as*

$$TF(f)(t) = F(\omega) = 2 \int_{0}^{+\infty} f(t) \cos(2\pi\omega t)\, dt.$$

- *If f is an odd function. The Fourier transform of f is defined as*

$$TF(f)(t) = F(\omega) = -2i\int_0^{+\infty} f(t)\sin(2\pi\omega t)\,dt.$$

Example 5.2.

Let $a > 0$. Calculate the Fourier transform of the function $f(t) = e^{-a|t|}$.

Solution:

For all $x, y \in \mathbb{R}$, we have $\frac{d}{dt}e^{(x+iy)t} = (x+iy)e^{(x+iy)t}$. We conclude that

$$F(\omega) = \int_{-\infty}^{0} e^{at}e^{-i\omega t}dt + \int_0^{+\infty} e^{-at}e^{-i\omega t}dt = \left[\frac{e^{(a-i\omega)t}}{a-i\omega}\right]_{t\longrightarrow-\infty}^{t=0} - \left[\frac{e^{-(a+i\omega)t}}{a+i\omega}\right]_{t=0}^{t\longrightarrow+\infty}.$$

So $\lim\limits_{R\to+\infty} e^{-(a+i\omega)R} = \lim\limits_{R\to+\infty} e^{-aR}e^{-i\omega R} = 0$. Therefore $\left|e^{-i\omega R}\right| = 1$ and $\lim\limits_{R\to+\infty} e^{-aR} = 0$ for $a > 0$. $\lim\limits_{R\to-\infty} e^{(a-i\omega)R} = 0$. Then

$$F(\omega) = \frac{1}{a+i\omega} + \frac{1}{a-i\omega} = \frac{2a}{a^2+\omega^2}.$$

Definition 5.4. *(Inverse Fourier transform)*

Let $f : \mathbb{C} \longrightarrow \mathbb{R}$. The inverse Fourier transform is defined as

$$f(t) = TF^{-1}(F)(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{-i\omega t}d\omega.$$

5.2 Fourier transform table

We will present the Fourier transform table of some usual functions

$f(t)$	$TF(f)(t) = F(\omega)$	Conditions
$f(-t)$	$2\pi F(\omega)$	
$f(\alpha t)$	$\frac{1}{\|\alpha\|}F(\frac{\omega}{\alpha})$	$\alpha \in \mathbb{C}$
$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\|\omega\|}$	$a > 0$
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a > 0$
$e^{-at}u(t)$	$\frac{1}{a+i\omega}$	$a > 0$
$f^{'}(t)$	$i\omega F(\omega)$	
$f^{''}(t)$	$(i\omega)^2 F(\omega)$	
$tf(t)$	$i\frac{d}{d\omega}F(\omega)$	
$t^2 f(t)$	$i^2\frac{d^2}{d\omega^2}F(\omega)$	
$f(\frac{t-t_0}{k})$	$ke^{-i\omega t_0}F(k\omega)$	
$(f * g)(t)$	$F(\omega)G(\omega)$	

Example 5.3. *Prove that:* $TF(e^{-at^2}) = \sqrt{\frac{\pi}{a}}e^{-\omega^2/(4a)}, \ a > 0.$

Solution:

We have:

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{+\infty} e^{-i\omega t}e^{-at^2}dt \\
&= \frac{1}{-i\omega}e^{-i\omega t}e^{-at^2}\Big|_{-\infty}^{+\infty} + \frac{1}{i\omega}\int_{-\infty}^{+\infty} e^{-i\omega t}(-2at)e^{-at^2}dt \\
&= -\frac{2a}{i\omega}\int_{-\infty}^{\infty} e^{-i\omega t}(-it)e^{-at^2}dt \\
&= -\frac{2a}{i\omega}F^{'}(\omega).
\end{aligned}
$$

We obtain the differential equation with separated variables

$$-2aF^{'}(\omega) = \omega F(\omega) \Longrightarrow \frac{dF(\omega)}{F} = -\frac{1}{2a}\omega \Longrightarrow \ln F(\omega) = -\frac{1}{4a}\omega^2 + c_1 \Longrightarrow F(\omega) = ke^{-\omega^2/(4a)}.$$

We suppose $k = F(0) = \int_{-\infty}^{+\infty} e^{-i0t}e^{-at^2}dt = \sqrt{\frac{\pi}{a}}$. *Then*

$$TF(e^{-at^2}) = \sqrt{\frac{\pi}{a}}e^{-\omega^2/(4a)}.$$

5.3 Properties of Fourier transform

Let $f(t), g(t) \in L^1(\mathbb{R})$, TF the Fourier transform and TF^{-1}. The Fourier transform possesses the following properties:

- **Linearity**. For any number λ and μ (real or complex), we have

$$TF(\lambda f + \mu g)(t) = \lambda F(\omega) + \mu G(\omega).$$

- **Linearity of $T^{-1}F$**.

$$T^{-1}F(\lambda F + \mu G)(\omega) = \lambda f(t) + \mu g(t).$$

- **Differentiation in time**.

$$TF\left(\frac{d^n f}{dt^n}\right)(t) = (i\omega)^n F(\omega).$$

- **Integration**. Let f be a continuous function. Suppose that $\lim_{R\to+\infty} f(t)dt = 0$. Then for $\omega \neq 0$,

$$\int_{-\infty}^{t} f(s)ds = \frac{F(\omega)}{i\omega}.$$

- **Conjugation**. Let f accepted a Fourier transform. Then $\overline{f} : t \longrightarrow \overline{f(t)}$ accepted a Fourier transform and

$$TF(\overline{f})(t) = \overline{F(\omega)}.$$

- **Time shifting**.

$$TF(f)(t - t_0) = F(\omega)e^{-i\omega t_0}.$$

- **Modulation**.

$$e^{i\omega_0 t} f(t) = F(\omega - \omega_0).$$

- **Parseval's Theorem**

Theorem 5.1. *Let $f(t) \in L^2$, and $F(\omega)$ the Fourier transform of $f(t)$. Then we have*

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega.$$

- **Plancherel's Theorem**

Theorem 5.2. *Let $f, g \in S$ be a tow functions, we have*

$$\int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt = \int_{-\infty}^{+\infty} F(\omega)\overline{G(\omega)}d\omega.$$

5.4 Solving differential equations using Fourier transform

5.4.1 Second order differential equation

Solving differential equations was Fourier's original motivation for Fourier series and the use of the Fourier transform to this end has continued to exercise a strong influence on the theory and the applications. We will consider the following second order differential equation :

$$a_2 y^{"}(t) + a_1 y^{'}(t) + a_0 y(t) = f(t).$$

The solution procedure is presented below:
First, we take the Fourier transform of both sides

$$\begin{aligned} TF\{f(t)\} &= F(\omega), \\ TF\{y(t)\} &= Y(\omega), \\ TF\left\{y^{'}(t)\right\} &= i\omega Y(\omega), \\ TF\left\{y^{"}(t)\right\} &= (i\omega)^2 Y(\omega), \end{aligned}$$

we obtain

$$\left(a_2 (i\omega)^2 + a_1 i\omega + a_0\right) Y(\omega) = F(\omega),$$

$$\Longrightarrow Y(\omega) = \left(a_2 (i\omega)^2 + a_1 i\omega + a_0\right)^{-1} F(\omega).$$

Let $G(\omega) = \left(a_2 (i\omega)^2 + a_1 i\omega + a_0\right)^{-1}$, we then have a product of two Fourier transforms

$$Y(\omega) = G(\omega) \times F(\omega).$$

The inverse transform is

$$TF^{-1}(Y)(\omega) = TF^{-1}(G * F)(\omega) \Longrightarrow y(t) = g(t) * f(t).$$

Example 5.4. *Solve by the Fourier transform the following equation*

$$-y''(t) + y(t) = e^{-t^2}, \ \ t \in \mathbb{R}.$$

***Solution**:*

Apply Fourier transform on every term in the differential equation, we obtain

$$\begin{aligned}
TF\left\{-y''(t) + y(t)\right\} &= TF\left\{e^{-t^2}\right\} \\
-(i\omega)^2 Y(\omega) + Y(\omega) &= TF\left\{e^{-t^2}\right\} \\
\left(\omega^2 + 1\right) Y(\omega) &= TF\left\{e^{-t^2}\right\} \\
Y(\omega) &= \frac{1}{2}\frac{2}{(\omega^2+1)} TF\left\{e^{-t^2}\right\} \\
&= \frac{1}{2} TF\left\{e^{-|t|}\right\} TF\left\{e^{-t^2}\right\}
\end{aligned}$$

Then, the solution is

$$y(t) = \frac{1}{2} e^{-|t|} * e^{-t^2}.$$

5.5 Supplementary exercises

Exercise 01.

1. Let $a > 0$, and f a function defined on $\mathbb{R}$ by $f(t) = e^{-a|t|}$.

   - Compute the Fourier transform of f.
   - With a particular value of a, deduct the value of the integral

$$\int_0^{+\infty} \frac{cos(\omega t)}{1+\omega^2} d\omega.$$

2. Compute the Fourier transform of $f(t) = \frac{sin(a\omega)}{\omega}$, $a > 0$.

3. Find the Fourier transform of $f(t) = te^{-\alpha t}$, $\alpha > 0$.

Exercise 02.

Prove the following Fourier transform

- $TF\left\{\frac{1}{a^2+t^2}\right\} = \frac{\pi}{|a|}e^{-|a\omega|}, a \in \mathbb{R}.$
- $TF\left\{e^{-a\left|t^2\right|}\right\} = \sqrt{\frac{\pi}{a}}e^{-\frac{t^2}{4a}}.$
- If $f(t) = \begin{cases} 1 & \texttt{if } |t| \leq L \\ 0 & \texttt{if } |t| > L \end{cases}$, we have $TF\{f\} = 2\frac{sin(Lt)}{t}$.

Exercise 03.

Solve the following differential equations

- $y'' + 2y'(t) + y(t) = e^{-t}$.
- $-\frac{1}{\omega}y''(t) + y(t) = f(t)$, $\omega \in \mathbb{R}^*$.

6

Laplace transform

The Laplace transform is used in engineering and physics to simplify the computations needed to solve some problems. In this chapter, we'll use Laplace transform to solve differential equations.

6.1 Mathematical operator of Laplace transform

Definition 6.1. *(Laplace transform)*
Let $f : [0, +\infty[\longrightarrow \mathbb{R}$ be a function defined for $t \geq 0$. The Laplace transform of the function $f(t)$ is defined as

$$\mathscr{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)dt = F(s).$$

where $s \in \mathbb{R}$.

Example 6.1. *Let's calculate the Laplace transform of the function $f(t) = t^2$.*

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \mathscr{L}\left\{t^2\right\} = \int_0^{+\infty} e^{-st}t^2 dt \\
&= \left[-\frac{1}{s}t^2 e^{-st}\right]_0^{+\infty} + \int_0^{+\infty} \frac{2}{s} t e^{-st} dt \\
&= \left[-\frac{1}{s}t^2 e^{-st}\right]_0^{+\infty} - \left[\frac{2}{s^2} t e^{-st}\right]_0^{+\infty} - \int_0^{+\infty} \frac{2}{s^2} e^{-st} dt = \frac{2}{s^3}.
\end{aligned}$$

6.1.1 Table of Laplace transform

Here, we will present the Laplace transform table of some usual functions.

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	Conditions
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{kt}	$\frac{1}{s-k}$	$s > k$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$s > \|k\|$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$s > \|k\|$

Example 6.2. *Calculate the Laplace transform of $f(t) = \sin(3t)$.*
Using the table of Laplace transform, we obtain: $\mathscr{L}\{f(t)\} = \frac{3}{s^2+9}$.

6.2 Laplace transform properties

In the following section, we will present a few important properties of the Laplace transform.

- **Linearity.** Let f and g two functions for which the Laplace transform is defined, for any constants $\alpha, \beta \in \mathbb{R}$ we have

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}.$$

proof.

$$\begin{aligned} \mathscr{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^{+\infty} e^{-st}(\alpha f(t) + \beta g(t)) \\ &= \alpha \int_0^{+\infty} e^{-st} f(t)dt + \beta e^{-st} \int_0^{+\infty} g(t)dt \\ &= \alpha \mathscr{L}\{f(t)\} + \mathscr{L}\{g(t)\}. \end{aligned}$$

Example 6.3. *Find Laplace transform of the function: $f(t) = 2\cos(t) - 5t^3$.*
Solution.

$$\begin{aligned} \mathscr{L}\left\{2\cos(t) - 5t^3\right\} &= 2\mathscr{L}\{\cos(t)\} - 5\mathscr{L}\left\{t^3\right\} \\ &= \frac{2s}{s^2+1} - \frac{30}{s^4}. \end{aligned}$$

- **Scaling.** If $\mathscr{L}\{f(t)\} = F(s)$, then for any $a > 0$ we have $\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$.

  Proof. We have $\mathscr{L}\{f(at)\} = \int_0^{+\infty} e^{-st} f(at)dt$. Using the change of variables $at = u$, we obtain

$$\begin{aligned}
\mathscr{L}\{f(at)\} &= \int_0^{+\infty} e^{-s\frac{u}{a}} f(u) d\left(\frac{u}{a}\right), \\
&= \frac{1}{a}\int_0^{+\infty} e^{-\frac{s}{a}u} f(u) du \\
&= \frac{1}{a}F\left(\frac{s}{a}\right).
\end{aligned}$$

  Example 6.4. *Perform the Laplace transform of the function: $f(t) = \sin(3t)$.*

  Solution. *From the table of Laplace transform, we have $\mathscr{L}\{\sin(t)\} = \frac{1}{s^2+1}$. Then*

$$\mathscr{L}\{\sin(3t)\} = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2 + 1} = \frac{3}{s^2+9}.$$

- **Shifting property.**

  1. If $\mathscr{L}\{f(t)\} = F(s)$, then for any $a \in \mathbb{R}$, we have $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$.

     Proof.

$$\mathscr{L}\left\{e^{at} f(t)\right\} = \int_0^{+\infty} e^{at} e^{-st} f(t)dt = \int_0^{+\infty} e^{-(s-a)t} f(t)dt = F(s-a).$$

     Example 6.5. *Find $\mathscr{L}\{e^{-4t}t^3\}$.*

     Solution. *We have $\mathscr{L}\{t^3\} = \frac{6}{s^4} = F(s)$, then*

$$\mathscr{L}\left\{e^{-4t}t^3\right\} = F(s+4) = \frac{6}{(s+4)^4}.$$

  2. Let $\mathscr{L}\{f(t)\} = F(s)$. If for any $a > 0$,

$$g(t) = \begin{cases} f(t-a) & t \geq a, \\ 0 & t < a. \end{cases}$$

     Then, $\mathscr{L}\{g(t)\} = e^{-as}F(s)$.

- **Differentiation**. If $f(t)$ is continuous, and $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f\left(0^+\right).$$

A recurrence relation for Laplace transform of higher order (n) derivatives of function $f(t)$ may be expressed as:

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - \cdots - f^{(n-1)}(0^+).$$

- **Laplace integral**. If $\mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\left\{\int_0^t f(x)dx\right\} = \frac{F(s)}{s}.$$

Example 6.6. *Let calculate* $\mathscr{L}\left\{\int_0^t e^{-6x}dx\right\}$.
We have $\mathscr{L}\{e^{-6t}\} = F(s) = \frac{1}{s+6}$, *so*

$$\mathscr{L}\left\{\int_0^t e^{-6x}dx\right\} = \frac{F(s)}{s} = \frac{1}{s}\frac{1}{s+6} = \frac{1}{s^2+6s}.$$

- **t-multiplication** . If $\mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}.$$

- **t-division**. If $\mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(u)du.$$

Example 6.7. *Prove that* $\int_0^{+\infty} \frac{\sin(t)}{t}dt = \frac{\pi}{2}$.
Solution.
From the definition of the Laplace transform, we have $\mathscr{L}\left\{\frac{\sin(t)}{t}\right\} = \int_0^{+\infty} e^{-st}\frac{\sin(t)}{t}dt$.
In the other side, we have $\mathscr{L}\left\{\frac{\sin(t)}{t}\right\} = \int_s^{+\infty} \frac{1}{u^2+1}du = \arctan(u)\Big]_s^{+\infty} = \frac{\pi}{2} - \arctan(s)$.
From this equality, we get

$$\int_0^{+\infty} e^{-st}\frac{\sin(t)}{t}dt = \frac{\pi}{2} - \arctan(s).$$

Then, for $s = 0$, we obtain

$$\int_0^{+\infty} \frac{\sin(t)}{t}dt = \frac{\pi}{2}.$$

- **Inverse Laplace transform**

1. **Definition**. Let $F(s)$ be the Laplace transform of the function $f(t)$. The inverse Laplace transform defined by:

$$\mathscr{L}^{-1}\{F(s)\} = f(t).$$

Example 6.8. *Find the inverse Laplace transform of the following functions:*

$$F(s) = \frac{1}{s^2}, \ \ F(s) = \frac{s}{s^2+4}.$$

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t).$

2. **Property**.

   - **Linear**. $\mathscr{L}^{-1}\{\alpha F + \beta G\} = \alpha\mathscr{L}^{-1}\{F\} + \beta\mathscr{L}^{-1}\{G\}.$
   - If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then for any $a > 0$, we have $\mathscr{L}^{-1}\{F(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right).$
   - If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then $\mathscr{L}^{-1}\{F(s+a)\} = e^{-at}f(t).$
   - If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then $\mathscr{L}^{-1}\left\{F^{'}(s)\right\} = -tf(t).$
   - If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then $\mathscr{L}^{-1}\left\{\int_s^{+\infty} F(u)du\right\} = \frac{f(t)}{t}.$

6.3 Solving differential equations using Laplace transform

6.3.1 Second order differential equation

Laplace transform method is found to be effective in solving differential equations. Let the following second order differential equation with the given conditions:

$$a_2 y^{"}(t) + a_1 y^{'}(t) + a_0 y(t) = f(t), \ \ y(0) = \alpha, \ \ y^{'}(0) = \beta.$$

The solution procedure is presented below:

- Apply Laplace transform on every term in the differential equation, we obtain

$$\begin{aligned} \mathscr{L}\{f(t)\} &= F(s), \\ \mathscr{L}\{y(t)\} &= Y(s), \\ \mathscr{L}\left\{y^{'}(t)\right\} &= sY(s) - y(0), \\ \mathscr{L}\left\{y^{"}(t)\right\} &= s^2Y(s) - sy(0) - y^{'}(0). \end{aligned}$$

- After apply the given values of the given conditions, we can obtain the expression:

$$\left(a_2s^2 + a_1s + a_0\right)Y(s) + \phi(s) = F(s) \Rightarrow Y(s) = \frac{F(s) - \phi(s)}{\left(a_2s^2 + a_1s + a_0\right)},$$

where $\phi(s) = -[(s+1)\alpha + \beta]$.

- To find the solution of the differential equation we use the inverse Laplace transform of $Y(s)$, i.e. $y(t) = \mathscr{L}^{-1}(Y(s))$.

Example 6.9. *Solve the following differential equation with given conditions:*

$$y^{"} - 2y^{'} + y = te^t, \qquad y(0) = 1, \;\; y^{'}(0) = 0.$$

Solution.
Apply Laplace transform to every term in the differential equation, we obtain

$$\begin{aligned} \mathscr{L}\left\{y^{"} - 2y^{'} + y\right\} &= \mathscr{L}\left\{te^t\right\} \\ \mathscr{L}\left\{y^{"}\right\} - 2\mathscr{L}\left\{y^{'}\right\} + \mathscr{L}\{y\} &= \mathscr{L}\left\{te^t\right\}. \end{aligned}$$

Apply the given conditions, we get

$$\begin{aligned} \left(s^2Y(s) - sy(0) - y^{'}(0)\right) - 2\left(sY(s) - y(0)\right) + Y(s) &= \frac{1}{(s-1)^2} \\ \left(s^2Y(s) - s\right) - 2\left(sY(s) - 1\right) + Y(s) &= \frac{1}{(s-1)^2} \\ \left(s^2 - 2s + 1\right)Y(s) = -s + 2 &= \frac{1}{(s-1)^2}. \end{aligned}$$

Then, we obtain the expression:

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^2}\Big[\frac{1}{(s-1)^2}+s-2\Big], \\ Y(s) &= \frac{1}{(s-1)^4}-\frac{1}{(s-1)^2}+\frac{1}{(s-1)}. \end{aligned}$$

The solution of this differential equation will be obtained by the inverse Laplace transform of $Y(s)$. So, the inversion of $Y(s)$ in the above form is:

$$y(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s-1)}\right\}.$$

The solution of the differential equation is obtained by:

$$y(t) = e^t\Big[\frac{t^3}{6} - t + 1\Big].$$

6.4 Supplementary exercises

Exercise 01.

Find the Laplace transforms of the following functions:

1. $f(t) = t^2 + t - e^{-3t}$.

2. $f(t) = (t^2 + t + 1)\, e^{-2t}$.

3. $f(t) = (\cos(2t) - \sin(t))\, e^{-3t}$.

Exercise 02.

Evaluate the following inverse Laplace transforms:

1. $\mathscr{L}^{-1}\left\{\frac{s+2}{(s+3)(s+4)}\right\}$.

2. $\mathscr{L}^{-1}\left\{\frac{3}{(s+5)^2}\right\}$.

3. $\mathscr{L}^{-1}\left\{\frac{3}{s+2} - \frac{1}{s^3}\right\}$.

Exercise 03.

Use the Laplace transform method to solve the following differential equations with given conditions:

1. $y^{"} + 2y^{'} + 2y = 0,\ \ y(0) = 1,\ \ y^{'}(0) = 1.$

2. $y^{"} + 5y^{'} + 4y = e^{-2t},\ \ y(0) = 1,\ \ y^{'}(0) = 0.$

3. $y^{"} - 4y = 3e^{-t} - t^{2},\ \ y(0) = 0,\ \ y^{'}(0) = 1.$

4. $y^{"} + 2y^{'} + 5y = e^{-t}\sin(t),\ \ y(0) = 0,\ \ y^{'}(0) = 1.$

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