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Thesis

A view to obtaining the diploma of

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Option: ***Numerical analysis, partial differential equations and applications***

Existence and asymptotic behavior of the solution of some evolution problems

Presented by:

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Dedication

I dedicate this humble work to:

My dear parents whose love for me is boundless, and who taught me the value of hard work. I thank you not only for being my parents, but also for being my friends, teachers and mentors.

To my husband for his motivation and moral encouragement.

To my daughter, my inspiration.

To my beautiful sister and dear brothers, who supported and helped me a lot in completing this work.

To my wonderful teachers who have made me an ambitious person.

To my extended family.

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My warm thanks go to my thesis co-supervisor ***Pr. GUESMIA Amar*** for his guidance, toleration and encouragement.

I wish to express my gratitude to ***Dr. KHEMIS Rabeh****,* ***Pr. Salah ZITOUNI, Dr. Fares YAZID*** *and* ***Dr. KHENNICHE Ghania*** *because they accepted to be the rapporteurs of this thesis.*

I would also like to sincerely thank my mother and father because they encouraged me a lot during the completion of this thesis. Finally, I would like to express my gratitude to the people who contributed directly or indirectly to the development of this work.

Abstract

In this thesis, we study the existence, uniqueness and the stability of the solutions of certain evolution problems where the dissipation is induced by the presence of delay. In this regard, under some adequate assumptions, we examine three problems and arrive at an exponential decay result. The first problem is concerned with the thermoelastic system of swelling porous-elastic soils with a second sound and delay term, the second one is related to the Von kármán beam with delay time and microtemperature effect, while the last one is about the thermoelastic Bresse system with distributed neutral delay and a second sound.

Using the semigroups theory, we establish the existence and uniqueness of solutions of our problems. Then, we show its exponential stability by the energy method, which depends on the construction of a suitable Lyapunov functional.

Keywords: Evolution problems, swelling porous system, Von kármán system, Bresse system, delay term, semigroups theory, Lyapunov functional, exponential stability.

Résumé

Dans cette thèse, nous étudions l'existence, l'unicité et la stabilité des solutions de certains problèmes d'évolution où la dissipation est induite par la présence d'un retard. À cet égard, sous certaines hypothèses adéquates, nous examinons trois problèmes et arrivons à un résultat de décroissance exponentielle. Le premier problème concerne le système thermoélastique des sols poreux-élastiques gonflants avec un second son et un terme de retard, le deuxième est lié à la poutre de Von Kármán avec un terme de retard et un effet de microtempérature, tandis que le dernier concerne le système thermoélastique de Bresse avec un retard neutre distribué et un second son.

En utilisant la théorie des semi-groupes, nous établissons l'existence et l'unicité des solutions de nos problèmes. Ensuite, nous montrons sa stabilité exponentielle par la méthode de l'énergie, qui dépend de la construction d'une fonctionnelle de Lyapunov appropriée.

Mots-clés: Problèmes d'évolution, système de poreux gonflé, système de Von kármán, système de Bresse, terme de retard, théorie des semigroupes, fonctionnelle de Lyapunov, stabilité exponentielle.

الملخص:

ندرس في هذه الأطروحة الوجود، الوحدانية والاستقرار لحلول بعض مسائل التطور حيث يحدث التبديد عن طريق وجود التأخير. في هذا الصدد، وفي ظل بعض الافتراضات المناسبة، فإننا ندرس ثلاث مسائل حتى نصل إلى نتيجة الاضمحلال الأسي. تتعلق المشكلة الأولى بالنظام المرن الحراري للتربة المسامية المرنة المتضخمة مع الصوت الثاني وحد التأخير الثابت، وتتعلق المشكلة الثانية بحزمة فون كارمان مع حد التأخير الثابت وتأثير درجة الحرارة الدقيقة، بينما تتعلق المشكلة الأخيرة بنظام بريس الحراري المرن مع تأخير محايد موزع والصوت الثاني.

باستخدام نظرية شبه الزمرة، نثبت وجود ووحدانية حلول مسائلنا. ثم نظهر استقرارها الأسي من خلال طريقة الطاقة، والتي تعتمد على بناء دالة ليابونوف المناسبة.

الكلمات المفتاحية:

مشاكل التطور، نظام مسامي منتفخ، نظام فون كارمان، نظام بريس، حد التأخير، نظرية شبه الزمرة، دالة ليابونوف، الاستقرار الأسي.

Contents

Introduction

Partial differential equations (PDEs) which typically rely on time (evolution problems) can be used to model many physics problems as swelling of soils, vibratory movements, etc. Whereas many works in recent years have interestes on the study of the existence and stability of development problems. In this thesis we were focused on the study of the well-posedness and the stabilization of some evolution problems.

The goal of stabilization is to reduce vibrations through feedback, as a result, it entails ensuring that a dissipation mechanism ensures that the energy of the solutions decreases toward 0 in a more or less gradual manner.

More specifically, the stabilization problem that interests us entails determining the asymptotic behavior of the energy that we note as $E(t)$ (it is the norm solution in the state space), studying its limit, if this limit is null providing an estimate of the rate decrease to zero. It is possible to study different types of stabilization, we mention among them:

▶ Strong stabilization:

$$E\left(t\right) \longrightarrow 0, \ as \ t \longrightarrow \infty.$$

▶ Exponential (uniform) stabilization (where we are interested in the fastest decrease of the energy):

$$E\left(t\right) \leq ce^{-\delta t}, \ \forall t > 0, \ \left(c, \delta > 0\right).$$

▶ Polynomial stabilization:

$$E\left(t\right) \leq ct^{-\delta}, \ \forall t > 0, \ \left(c, \delta > 0\right).$$

To determine the system's desired stability results, we use the multiplier technique which mostly depends on the construction of an appropriate Lyapunov functional, where in order to achieve the exponential decay, we should find that all energy terms are negative, but as the energy is quadratic it will be difficult to use it that's why we split the energy and we obtain the Lyapunov functional $L(t)$. Then, we prove the equivalence between them $L \sim E$, we mean

$$C_1 E\left(t\right) \leq L\left(t\right) \leq C_2 E\left(t\right), \forall t \geq 0. \tag{1}$$

Where, C_1 and C_2 are two positive constants. To prove the exponential stability, we state and prove this main result

$$E\left(t\right) \leq k_0 e^{-k_1 t}, \ \forall t \geq 0,$$

where, k_0 and k_1 are two positive constants. Then, we show that L satisfies

$$L^{'}\left(t\right) \leq \alpha L\left(t\right) \quad , \ \forall t \geq 0, \tag{2}$$

where, $\alpha \geq 0$. A simple integration of (2) over $(0, t)$ yields the desired exponential stability result.

The asymptotically stable system may be almost destabilized under the effect of time delay. As its effects appear in a wide range of processes both natural and manmade including biology, chemistry, medicine, physics, economics, engineering, etc. (See [1]), as a result the issue of good status and stability outcomes of delay systems is of practical and theoretical importance to many physical systems.

We can take the phenomenon of lightning and thunder as an example about the time delay, since they both occur at the same time but we hear lightning then we see the thunder. Reforestation is another example in nature where a cut forest, after replanting, will take at least 20 years before reaching any kind of maturity, it would be much longer for certain species of trees as redwoods. Therefore, it is clear that time delays must be included in any mathematical model of forest harvesting and regeneration. This is why the most realistic model should include some of the system's past history, where generally Models incorporating past history include delay differential equations (DDEs), in which the derivatives of some unknown functions at present time depend on the values of the functions at previous times.

The control of partial differential equations with time delay effects has become an active area of research see [21, 68, 69], where there are several types of delays. In this thesis, we have focused on three of them, which are the constant delay, the distributed delay and the neutral delay that occur in the second (upper) derivative. Many researchers have looked at the impact of the delay term on the asymptotic behavior of solutions, that may lead to a wild-behaved system instead of a well one as Datko et al. [22] and Xu et al. [67] whom demonstrated that the system uniformaly asymptotically stable in the lack of delay may become unstable with an arbitrarily small delay.

In 2006 [58] Nicaise and Pignotti examined the wave equation with a delay term in the boundary condition and considered the wave equation with a delayed velocity term and mixed Dirichlet–Neumann boundary condition in both cases, they demonstrated that the energy is exponentially stable in the presence of delay ($\mu_2 > 0$), under the assumption $\mu_2 < \mu_1$. In contrast, in the opposite case ($\mu_2 \geq \mu_1$) the solution is unstable. Then, in 2008 (see [59]) they obtained an exponential stability result of the same system by replacing the constant delay with a distributed delay.

Guesmia [33] in 2013, demonstrated that even in the presence of a small delay the memory term's unique dissipation is strong enough to exponentially stabilize the system which is the second-order abstract linear equation with infinite memory as a dissipation and constant delay. Also in [42] Khochemane et al., motivated by Apalara [5] examined the porous elastic system with past history and nonlinear damping term and showed the general decay of the problem in both equations of the system for the case of equal wave propagation speed.

On the other hand, concerning the asymptotic behavior with neutral delay Tatar in 2017 [66] considered a damped wave equation with neutral delay and he proved that the solution decays exponentially under some conditions on the kernel of distributed neutral delay.

The objective of this thesis is to investigate the well-posedness and the stabilization of certain hyperbolic systems (evolution problems) where the dissipation is introduced by the presence of time delays and arrive at an exponential decay result for the one-dimensional case.

The swelling porous-elastic system

It is well-known that swelling of soils, wood, fiber drying, plants, paper and other issues are problems in relation to the theory of porous media. Several recent articles, such as Payne et al. and various references therein have introduced continuum theories for fluids entering elastic porous surfaces. In 1994 Eringen [24] proposed his theory in which it is imposed using the second law of thermodynamics a continuous theory for viscous fluids, elastic materials and gases. By underlining the balancing criteria for each component of the combination, he was able to find the field equations for a heat-conducting mixture, check [6] for further information about a complete history of the general theory of mixes and an evaluation of its current state.

Heaving may occur if the soil pressure is greater than the main structure [46], where, the ability of soil to inflate due to capillary action induced by absorption of subsurface water or shrink due to dryness caused by changes in the weather is proportional to its initial dry density. As shown in figure 1

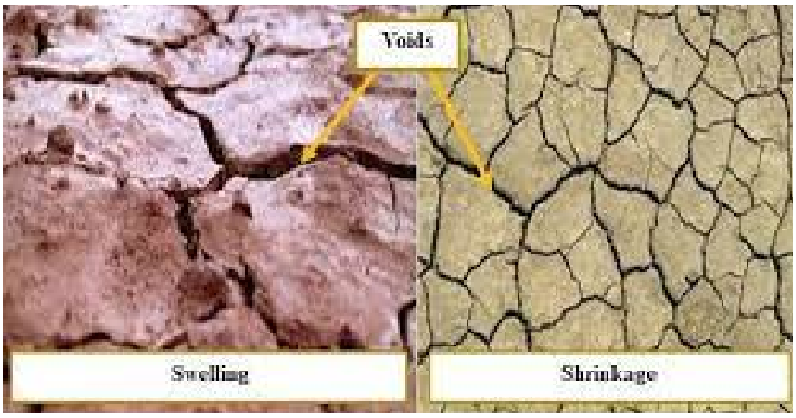


figure1: Swelling and shrinkage of soil

It is important to note that swelling soils contain clay minerals that attract and absorb water potentially increasing pressure (see [35]), which can lead to major engineering issues like cracking of buried pipes and buckling of pavements, particularly when it exceeds 10% in most expansive clays. To clarify this, we show this figure that represents, the phenomenon of swelling and expasion of the soil when they absorb water

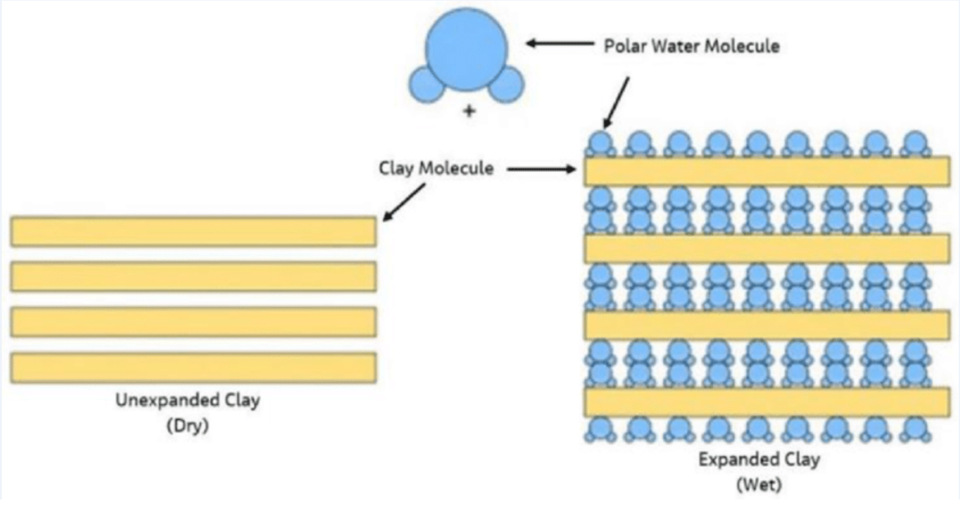


Figure2: the soil swelling process

So to deal with this problem, it is necessary to evaluate the potential of swelling and then propose several techniques to avoid such damage, such as the use of strong enough structures, for more information about swelling soil the reader is directed to [38].

Karalis [39], looked at the asymptotic behavior of swelling (also known as expansive) soils which had previously been investigated using porous media theory, for more detail about the relation between the elasticity theory and the porous media theory we cite the works of Cowin, S. C. and Nunziato, J. W. [20] in 1983 and Cowin, S. C. [19] in 1985.

The original field equations for theory of swelling porous elastic soils is given mathematically by the following two basic evolution equations (see [48]):

$$\begin{aligned} \rho_1\varphi_{tt} &= P_{1x} - G_1 + H_1, \\ \rho_2\psi_{tt} &= P_{2x} + G_2 + H_2. \end{aligned} \tag{3}$$

The functions (P_1, P_2) represent the partial tensions, (G_1, G_2) the internal body forces and (H_1, H_2) the external forces, the constituents φ and ψ represent the displacement of the fluid and the elastic solid material respectively. The duo positive constant coefficients ρ_1 and ρ_2 are the densitie of each constituent, where the partial tensions (P_1, P_2) are given by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \mathcal{M} \begin{pmatrix} \varphi_x \\ \psi_x \end{pmatrix},$$

where $\mathcal{M}$ is the positive definite matrix $\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$, i.e., $a_2^2 < a_1a_3$.

Quintanilla [62] investigated (3) by taking

$$G_1 = G_2 = \xi(z_t - u_t), \quad H_1 = a_3 z_{xxt}, \quad H_2 = 0,$$

where ξ is a positive coefficient, with initial and homogeneous Dirichlet boundary conditions and obtained an exponential stability result.

The physical dilemma of the infinite speed of heat propagation is predicted by the model using the classic Fourier's law of heat conduction (which states that the heat flux is proportional to the temperature gradient). To put it another way, any thermal disruption in one part of the body has an immediate influence on the rest of the body. Non-classical theories devoid of this flaw have been developed over the previous three decades, these new theories employ modified versions of the classical Fourier's law of heat conduction, resulting in hyperbolic-type heat transport equations that accept finite speeds for thermal signals. Heat transmission is considered as a wave phenomena rather than a diffusion process, according to these ideas. Second sound refers to a wave like thermal disturbance where the first sound being the ordinary sound and nonclassical theories that anticipate the existence of such disturbances are known as theories with finite wave speeds or theories with second sound. This theory proposes replacing the classic Fourier's law

$$\beta q + \theta_x = 0,$$

by a modified law of heat conduction called Cattaneo's law

$$\gamma q_t + \beta q + \theta_x = 0.$$

As a result, instead of the typical diffusion, heat is transmitted by a wave propagation process, resolving the physical dilemma of infinite heat propagation speed, for more on Cattaneo's law and thermoelasticity with second sound see [54, 4] and the references therein.

The Von kármán beam system

The phenomenon of vibrations arises basically in all mechanical structures in the field of engineering. However, certain types of these vibrations are undesired because they have a negative effect on these structures, as they can cause fractures or even destroy them, furthermore they may pose a threat to the user himself. The external environment, atmosphere and water as well as a shock with other structures they are all sources of dynamic excitations that causes these vibrations.

Many constructions in various sectors of engineering are made up of one or more beams. Depending on the nature and type of vibrations, these beams have varied models. In contrast to other base models (such as the Timoshenko model or the Euler-Bernoulli), the Von kármán model is more suitable because it considers both transverse and longitudinal displacements when vibrating a narrow body with a significant deflection. As a result, numerous strategies are employed to eliminate or reduce the consequences of these vibrations, such as the Von kármán beam stability problem which has attracted the interest of many researchers, with a substantial range of literature addressing the issues of existence, uniqueness and asymptotic behavior in time when damping effects are taken into account, as well as other relevant aspects (see refs. [12, 47] and the references therein for more information).

The controllability and stabilization of the Von kármán system were investigated by Horn and Lasiecka in 1995 (see [37]). In 1990, Lagnese [44] investigated a one-dimensional Von kármán beam with internal damping using nonlinear boundary feedback, they were able to achieve model uniform stabilization. Furthermore, in 1998 Benabdallah and Teniou stabilized the system by linking it to two heat equations: one for the longitudinal component and the other for the transversal component (see [8]), in this regard we refer also to the work of Benabdallah and Lasiecka [7].

Green and Naghdi suggested three models that allow heat to be transmitted in the form of thermal waves at constrained speeds, this happens when the beam is at a low temperature, the reader is directed to [29, 30].

Many articles have looked into the stabilization of systems using boundary damping (often in conjunction to internal damping), see Favini et al. [27], Lagnese and Leugering [45], Puel and Tucsnak [61] and the references therein.

In 2013, Djebabla and Tatar [23] achieved an exponential stabilization result of the one-dimensional full von Kármán beam using the dissipative effect through heat conduction acting on the longitudinal component and the frictional damping acting on the transversal component of the beam.

The stability of the wave equation with delay has lately become a hot topic of research, where several authors demonstrating that delays can destabilize a system that is asymptotically stable without them (see [68, 59] for more details).

Various sorts of dissipative mechanisms were examined by numerous writers in order to get stability results, temperature and microtemperature elements were included into the theory by Lesan [47], and Lesan and Quintanilla [49], we can refer to some references in this topic [61, 40]. Furthermore, in 2005 another work [16] demonstrated that mixing temperature and microtemperature cause exponential stability.

Most phenomena are naturally dependent on their current state as well as previous events, which is why time delay occurs in many applications. Introducing distributed delay, varying delay or constant delay has been a important research topic in (PDEs) for decades and has gotten a lot of attention (see for example [36, 50, 43, 41, 11]). We recall that the delay term became a source of instability when it was demonstrated that a slight delay in a boundary control may change a well-behaved hyperbolic system into a wild one, resulting in instability.

The thermoelastic Bresse system

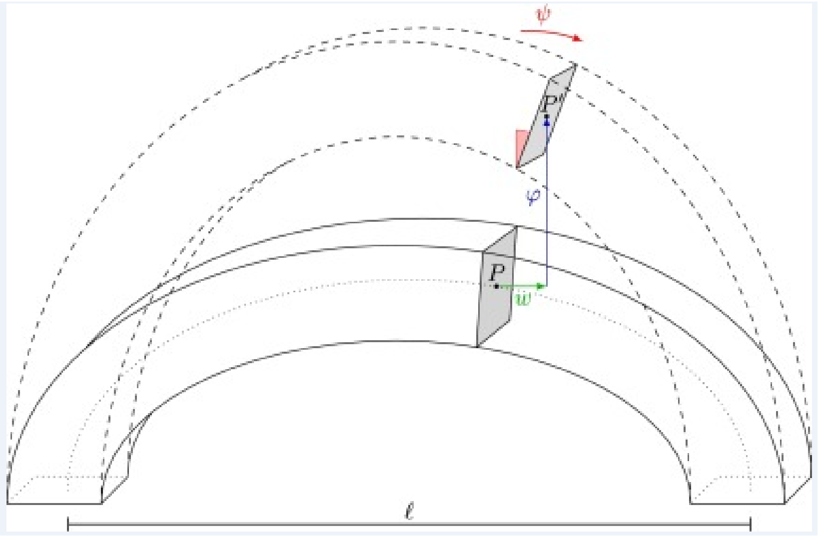
In 1859, Bresse [14] developed the circular arc problem which consists of three wave equations where the main variables ϕ, ψ and w represent the vertical, longitudinal and shear angle displacements respectively. The following is the structure of this system:

$$\begin{cases} \rho_1\phi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases} \tag{4}$$

where F_i, $i = 1, 2, 3$ determine the external forces exerted on the object and the coefficients N, Q and M are used to signify respectively the axial force, the shear force and the bending moment, which defined as follows

$$N = k_0 \left(w_x - l\phi\right), \ Q = k\left(\phi_x + lw + \psi\right), \ M = b\psi_x.$$

The following figure represent the displacement of a particle in the axis of the beam due to deformation.



Then, in 2014 Soriano et al [65] considered (4) when $F_1 = a(x)\, g_1(\phi_t)$, $F_2 = g_2(\psi_t)$, $F_3 = \gamma(x)\, g_3(w_t)$, where $a, \gamma \in L^\infty(0, L)$ and the functions g_1, g_2 and g_3 are continuous and monotone, on the same topic Alabau Boussouira et al [3] demonstrated that the system is exponentially stable when

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ and } k = k_0, \tag{5}$$

other than that, the system's exponential stability is lacking. Later, in [25] Fatori and Monteiro developed the result.

The Bresse system (4) reduces to the well-known Timoshenko system where the longitudinal displacement w is not considered $(l = 0)$, on the other hand, the energy associated with this system (4) remains constant when the time t evolves, because it is an undamped system. That's why, different types of dampings should be added to the equations or at the limit, in order to stabilize this system that has been studied by many authors (see [34, 9, 53]).

In 2009 Liu and Rao [51] considered the thermoelastic Bresse system which describes the motion of a linear planar, shearable thermoelastic beam, under initial and specific boundary conditions, they showed that the system is exponentially stable in the case of equal wave speeds ([5]). Otherwise, they proved that the energy of the system decays polynomially.

Later, in 2010 Fatori and Muñoz Rivera [26] found the same result as Liu and Rao in [51] when they considered the same system without θ_1 and the last equation.

Many research has looked at the impact of the delay term on the asymptotic behavior of solutions that lead to instability in systems that are uniformly stable in the absence of delay (for more discussions see [21]).

There are other types of delays as the neutral delay where in 2020 [64] Seghour et al. studied the thermoelastic laminated system with distributed delay of neutral type and they proved the exponentially stable of the system in the case of equal wave speeds, other than that with an additional assumption on the kernel they demonstrated a polynomial stability.

Objective of the thesis

In this thesis, we examine three problems. The first problem is a one-dimensional swelling porous-elastic system with second sound and internal delay term, in the case of equal wave speeds, we establish the well-posedness and the stability results of the system via the semigroup theory and the energy method respectively and we find that the combination of the frictional damping and the impact of the heat flux is strong enough to cause an exponential decay of the system even if the delay is a source of instability.

The second is a one-dimensional Von kármán beam with delay term coupled to a microtemperature equation. Under suitable assumptions on the weight of delay and a microtemperature effect we prove the existence and uniqueness of the solution by using the semigroup theory, then we study the exponential stability by using the energy method.

The last problem is about a one-dimensional thermoelastic Bresse system, with distributed neutral delay and a second sound. We prove that the system stabilize exponentially although the presence of the delay under suitable assumptions on the kernel of neutral delay term.

This thesis follows the following structure.

Chapter 1:

In this chapter, we recall some preliminaries notions, theorems and results.

Chapter 2:

In this chapter, we study the well posedness and energy decay of swelling porous elastic system with a second sound and delay term. We first give the well-posedness of the system by using the semigroup method. Then, we show that the system is exponentially stable under the assumption of equal wave speeds by using the energy method.

Chapter 3:

Chapter 3 focused on studying the one-dimensional Von kármán beam with delay term coupled to a microtemperature equation. In the beginning we examine the well posedness of the system by using the semigroup method, then we prove that the solution of the system decay to zero exponentially. It is divided into three sections: preliminaries, well posedness of the problem and the result of stability of this system.

Chapter 4:

In this last chapter, we establish an energy decay of one-dimensional thermoelastic Bresse system with distributed neutral delay and a second sound. We show that the system is well-posed by using the Faedo-Galerkin method then we prove that the system is exponentially stable by introducing a suitable Lyaponov functional.

Publications

[1] Abdelli Manel and Bouzettouta Lamine. "Energy decay of one-dimensional thermoelastic Bresse system with distributed neutral delay and a second sound." Studies in Engineering and Exact Sciences 5.1 (2024): 2932-2956.

Chapter 1

Preliminaries

In this chapter, we introduce some fundamental definitions, theorems and properties in functional analysis to be used throughout this thesis.

1.1 Fondamental spaces

1.1.1 Hilbert spaces

Hilbert spaces arise naturally and frequently in mathematics and physics, typically as function spaces.

Definition 1.1 [15] Let $\mathcal{H}$ be a vectorial space. A scalar product (u, v) is a bilinear form on $\mathcal{H} \times \mathcal{H}$ with values in $\mathbb{R}$ such that

$$\begin{array}{lll} (u,v) = (v,u) & \forall u, v \in \mathcal{H} & (symmetry), \\ (u,u) \geq 0 & \forall u \in \mathcal{H} & (positive), \\ (u,u) \neq 0 & \forall u \neq 0 & (definite). \end{array}$$

Let us keep in mind that a scalar product achieve the Cauchy–Schwarz inequality

$$|(u,v)| \ \leq (u,u)^{1/2}(v,v)^{1/2} \quad \forall u, v \in \mathcal{H}.$$

Definition 1.2 [15] A Hilbert space is a vector space $\mathcal{H}$ provided with a scalar product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm that allows $\mathcal{H}$ to be complete.

1.1.2 Sobolev Spaces

Sobolev spaces are particularly functional spaces in mathematical analysis that can be used to solve problems involving partial differential equations, it consists of the functions of $L^p(\Omega)$.

We start with the Lebesgue spaces $L^p(\Omega)$.

The $L^p(\Omega)$ spaces:

Definition 1.3 [15] Let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ and $1 \leq p < \infty$, we define the standard Lebesgue space $L^p(\Omega)$ as follows

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R};\ f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

If $p = \infty$, we obtain

$$\begin{aligned} L^\infty(\Omega) = \{ & f : \Omega \to \mathbb{R};\ f \text{ is measurable and there exists a constant } C \\ & \text{such that, } \ |f(x)| \leq C \text{ a.e in } \Omega \}. \end{aligned}$$

Notation 1.1 *Spaces $L^p(\Omega)$ supplied with the following norms:*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

and

$$\|f\|_\infty = \inf \{ C, \ |f(x)| \leq C \ a.e \ in \ \Omega \}.$$

Remark 1.1 for $p = 2$, $L^2(\Omega)$ is a Hilbert space equipped with the following inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx.$$

The Sobolev spaces $\mathcal{H}^m$:

Definition 1.4 [15] The Sobolev space $\mathcal{H}^m(\Omega)$ $(m \in \mathbb{N})$ is defined as

$$\begin{aligned} \mathcal{H}^m(\Omega) = \Bigg\{ & u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad for\ all\ \alpha \in \mathbb{N}^n \ \text{ with } |\alpha| \in \sum_{j=1}^n \alpha_j \leq m, \\ & \text{where the derivatives } D^\alpha u \text{ are taken in the weak sense} \Big\}, \end{aligned}$$

that makes $\mathcal{H}^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{\mathcal{H}^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx,$$

with the norm

$$\|u\|_{\mathcal{H}^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \left(\|\partial^\alpha u\|_{L^2} \right)^2 \right)^{\frac{1}{2}}.$$

Remark 1.2 This is how $\mathcal{H}_0^m(\Omega)$ is described

$$\mathcal{H}_0^m(\Omega) = \left\{ u \in \mathcal{H}^m(\Omega),\ u = u' = ... = u^{(m-1)} = 0 \ on\ \partial\Omega \right\}.$$

It is important to understand the difference between

$$\mathcal{H}_0^2(\Omega) = \left\{ u \in \mathcal{H}^2(\Omega),\ u = u' = 0 \ on\ \partial\Omega \right\}.$$

and

$$\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega) = \left\{ u \in \mathcal{H}^2(\Omega),\ u = 0 \ on\ \partial\Omega \right\}.$$

1.2 Some beneficial inequalities

In this part, we recall certain inequalities that will be used in the following chapters.

Theorem 1.1 *[15] **(Young's Inequality)** For all $a, b \in \mathbb{R}^+$, we have*

$$a, b \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{1.1}$$

where ϵ is any positive constant.

Theorem 1.2 *[15] **(Poincaré's inequality)** Suppose Ω is a bounded interval. Then there exists a positive constant C (depending on $|\Omega| < \infty$) such that*

$$\|u\|_{\mathcal{H}^1(\Omega)} \leq C \left\| u' \right\|_{L^2(\Omega)}, \quad for\ all\ u \in \mathcal{H}_0^1(\Omega). \tag{1.2}$$

Theorem 1.3 *[15] **(Hölder's Inequality)** Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$, assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg|\, dx \leq \|f\|_p \|g\|_q. \tag{1.3}$$

Lemma 1.1 *[15] **(Cauchy-Schwarz Inequality)** Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|. \tag{1.4}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Remark 1.3 The Cauchy-Schwarz inequality is a special case of the Hölder inequality in the case $p = 2$, $q = 2$.

Lemma 1.2 *[28] There exists a positive constant c such that the following inequality holds for every* $(\varphi, \psi, \omega) \in \left[H_0^1(0, L)\right]^3$

$$\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx \leq c \int_0^L \left[b\psi_x^2 + k(\varphi_x + \psi + l\omega)^2 + k_0(\omega_x - l\varphi)^2\right] dx. \tag{1.5}$$

1.3 Existence and uniqueness theorems

1.3.1 Semi-groups theorem

$\mathcal{H}$ denotes a Hilbert space, throughout this section.

The semi-groups is a theory used to establish the existence and uniqueness of solutions. Typically, the first step in studying the solution's well-posedness is to convert the evolution system of partial differential equations into a Cauchy problem on a suitable Hilbert space $\mathcal{H}$, also known as the energy space.

$$\begin{cases} \phi'(t) + \mathcal{A}\phi(t) = 0, \\ \phi(0) = \phi_0. \end{cases}$$

Where $\mathcal{A}$ is an unbounded operator on $\mathcal{H}$. Then, in order to conclude the existence of a solution in a specific Hilbert space, we demonstrate that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$. So, to begin, we first introduce some fundamental concepts about semigroups.

Definition 1.5 [15] Let $\mathcal{H}$ be a Hilbert space with scalar product $(.,.)$ and norm associated $\|.\|$ and let $\mathcal{A} : D(A) \subset \mathcal{H} \to \mathcal{H}$ be an unbounded linear operator. We say that $\mathcal{A}$ is monotone if

$$\langle \mathcal{A}\phi, \phi \rangle \geq 0, \ \forall \phi \in D(\mathcal{A}),$$

it is called maximal monotone if, as well

$$R(\mathcal{I}d + \mathcal{A}) = \mathcal{H}, \text{ i.e.,}$$

$$\forall f \in \mathcal{H}, \exists \phi \in D(\mathcal{A}) \text{ such that } \phi + A\phi = f.$$

Proposition 1.1 *[15] Let* $\mathcal{A}$ *be a maximal dissipative operator. Then* $D(\mathcal{A})$ *is dense in* $\mathcal{H}$.

Definition 1.6 [60] Let X be a Banach space. A one parameter family $\{S(t)\}_{t\geq 0}$ of bounded linear operators defined from X into X is a strongly continuous semigroup of bounded linear operators on X if:

- $S(0)=I$ (I identity operator on X).
- $S(t+s)=S(t)S(s)$ for every $t,s\geq 0$.
- $S(t)x\rightarrow x$, as $t\rightarrow 0$, $\forall x\in X$.

Such a semigroup is called a C_0-semigroup.

Definition 1.7 [60] We call infinitesimal generator of the C_0-semigroup $\{S(t)\}_{t\geq 0}$ any operator $\mathcal{A}$ defined on the set

$$D(\mathcal{A})=\left\{x\in X:\lim_{t\rightarrow 0}\frac{S(t)x-x}{t}\text{ exists}\right\},$$

by

$$\mathcal{A}x=\lim_{t\rightarrow 0}\frac{S(t)x-x}{t},\ x\in D(\mathcal{A}).$$

Definition 1.8 A C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$ is said to be of contractions if

$$\|S(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}\leq 1,\forall t\geq 0.$$

Now, we mention here Hille–Yosida Theorem: Lumer-Phillips form, which is a powerful and fundamental tool linking energy dissipation properties of an unbounded operator $\mathcal{A}:D(A)\subset\mathcal{H}\rightarrow\mathcal{H}$ by the existence, uniqueness and regularity of the solutions of a differential equation

Theorem 1.4 *[15] **(Hille–Yosida)** Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $\phi_0\in D(\mathcal{A})$ there exists a unique function*

$$\phi\in C([0,\infty[,D(\mathcal{A}))\cap C^1([0,\infty[,\mathcal{H}),$$

satisfying

$$\begin{cases}\phi'+\mathcal{A}(t)\phi=0 & \text{on } [0,\infty[,\\ \phi(0)=\phi_0.\end{cases}$$

Moreover,

$$|\phi(t)|\leq|\phi_0|,\ \forall t\geq 0 \text{ and } |\frac{d\phi}{dt}(t)|=|A\phi(t)|\leq|A\phi_0|,\ \forall t>0.$$

Theorem 1.5 *[52] **(Lumer-Phillips)** Let $\mathcal{A}$:$D(\mathcal{A})\subseteq\mathcal{H}\rightarrow\mathcal{H}$ a linear operator and $D(\mathcal{A})$ is dense in $\mathcal{H}$. Then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if*

i) $\mathcal{A}$ is dissipative,

ii) There exists $\lambda>0$ such that $Im(\lambda I-\mathcal{A})=\mathcal{H}$ ($\mathcal{A}$ is maximal).

1.3.2 Lax-Milgram theorem

Ordinary differential equations and linear partial differential equations can both be solved using the Lax-Milgram theorem, which is a simple and effective method leading to a unique solution to the weak formulation of the problem.

Theorem 1.6 *[15] **(Lax-Milgram theorem)** Let $a(\cdot,\cdot)$ be a bilinear form on a Hilbert space $\mathcal{H}$ equipped with norm $\|\cdot\|_{\mathcal{H}}$ and the following properties:*

i) $a(\cdot,\cdot)$ is continuous, if

$$\exists \beta_1 > 0 \text{ such that } |a(u,v)| \leq \beta_1 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \ \forall u,v \in \mathcal{H}.$$

ii) $a(\cdot,\cdot)$ coercive (or $\mathcal{H}$-elliptic), if

$$\exists \alpha > 0 \text{ such that } a(v,v) \geq \alpha \|v\|_{\mathcal{H}}^2, \ \forall v \in \mathcal{H}.$$

iii) L is a linear mapping on $\mathcal{H}$ (thus L is continuous), if

$$\exists \beta_2 > 0 \text{ such that } |L(v)| \leq \beta_2 \|v\|_{\mathcal{H}}, \ \forall v \in \mathcal{H}.$$

Then there exists a unique $u \in \mathcal{H}$ such that

$$a(u,v) = L(v), \ \forall v \in \mathcal{H}.$$

Chapter 2

Well-posedness and energy decay of swelling porous elastic soils with a second sound and delay term

2.1 Presentation of the problem

In the present chapter, we consider the swelling problem in porous elastic soils with an internal delay term acting on the transverse displacement and a linear frictional damping, where the heat flux is given by Cattaneo's law, it has the form

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0 & \text{in } (0,1)\times(0,\infty), \\ \gamma q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \theta(x,0) = \theta_0(x) & x\in(0,1), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ q(x,0) = q_0(x) & x\in(0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0 \text{ in } (0,\infty). \end{cases} \tag{2.1}$$

Here, the dependent variables $\varphi = \varphi(x,t)$, $\psi = \psi(x,t)$, $\theta = \theta(x,t)$ and $q = q(x,t)$ represent the displacement of the fluid with density ρ_1, elastic solid material with density ρ_2, the difference temperature and the heat flux, respectively. The coefficients ρ_1, ρ_2, ρ_3, a_1, a_2, a_3, β, δ, μ_1, τ are positive constants and $a_2 \in \mathbb{R}^*$ is satisfying the relation $a_2^2 < a_1a_3$, the term $\mu_1\varphi_t$ represents a frictional damping, $\mu_2\varphi_t(x,t-\tau)$ represents the time delay and $\tau > 0$ represents the respective retardation time, $\mu_2 \in \mathbb{R}^*$ is a real number such that,

$$\mu_2 < \mu_1. \tag{2.2}$$

Finally, φ_0, φ_1, ψ_0, ψ_1, θ_0 and q_0 are initial data and f_0 is history function belong to an appropriate functional spaces.

This system (2.1) is a result of the two basic evolution equations for the theory of swelling of one-dimensional porous elastic soils (3), where $G_1 = G_2 = 0$,

$$H_1 = -\mu_1\varphi_t - \mu_2\varphi_t(x, t-\tau) \text{ and } H_2 = \delta\theta_x,$$

with the second sound equations.

Our goal in this chapter is to study how the damping term and the time delay affect the system's (2.1) ability to stabilize over time. We investigate the well-posedness result via the semigroup method and the asymptotic behavior of the solution of (2.1) in the case of equal wave speeds

$$\frac{a_1}{\rho_1} = \frac{a_3}{\rho_2}.$$

2.2 Well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (2.5) using the semigroup theory [60] by introducing the following new dependent variable to deal with the delay feedback term, as in [59]

$$z(x, \rho, t) = \varphi_t(x, t-\tau\rho) \text{ in } (0,1) \times (0,1) \times (0,\infty), \tag{2.3}$$

then, the above variable z satisfies

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \text{ in } (0,1) \times (0,1) \times (0,\infty). \tag{2.4}$$

Therefore, the problem (2.1) is equivalent to

$$\begin{cases} \rho_1\varphi_{tt}-a_1\varphi_{xx}-a_2\psi_{xx}+\mu_1\varphi_t+\mu_2 z(x,1,t)=0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt}-a_3\psi_{xx}-a_2\varphi_{xx}+\delta\theta_x=0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t+q_x+\delta\psi_{tx}=0 & \text{in } (0,1)\times(0,\infty), \\ \gamma q_t+\beta q+\theta_x=0 & \text{in } (0,1)\times(0,\infty), \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } (0,1)\times(0,1)\times(0,\infty), \\ z(x,0,t)=\varphi_t(x,t) & \text{in } (0,1)\times(0,\infty), \\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \theta(x,0)=\theta_0(x) & x\in(0,1), \\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\ q(x,0)=q_0(x) & x\in(0,1), \\ \varphi(0,t)=\varphi(1,t)=\psi_x(0,t)=\psi_x(1,t)=\theta(0,t)=\theta(1,t)=0 & \text{in } (0,\infty), \\ z(x,\rho,0)=f_0(x,-\tau\rho) & \text{in } (0,1)\times(0,1). \end{cases} \tag{2.5}$$

For any regular solution of (2.5) the energy $E(t)$ defined by

$$E(t)=\frac{1}{2}\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_3\theta^2+\gamma q^2+\left(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x\right)^2 +\left(a_1-\frac{a_2^2}{a_3}\right)\varphi_x^2+\mu_2\tau\int_0^1 z^2(x,\rho,t)d\rho\right\}dx. \tag{2.6}$$

Now, we introduce the vector function

$$\phi=(\varphi,u,\psi,v,\theta,q,z)^T,$$

where,

$$u=\varphi_t,\ \ v=\psi_t,$$

then, the system (2.5) can be written as,

$$\begin{cases} \dfrac{\partial\phi}{\partial t}+\mathcal{A}\phi(t)=0, \quad t>0, \\ \phi(0)=\phi_0=(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,q_0,f_0)^T, \end{cases} \tag{2.7}$$

where, ϕ is the solution of (2.7), ϕ_0 is the initial condition and the operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is the linear operator defined by

$$\mathcal{A} = \begin{pmatrix} 0 & -I & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{a_1}{\rho_1}\partial_x^2 & \dfrac{\mu_1}{\rho_1}I & -\dfrac{a_2}{\rho_1}\partial_x^2 & 0 & 0 & 0 & \dfrac{\mu_2}{\rho_1}I \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ -\dfrac{a_2}{\rho_2}\partial_x^2 & 0 & -\dfrac{a_3}{\rho_2}\partial_x^2 & 0 & \dfrac{\delta}{\rho_2}\partial_x & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\delta}{\rho_3}\partial_x & 0 & \dfrac{1}{\rho_3}\partial_x & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{\gamma}\partial_x & \dfrac{\beta}{\gamma}I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\tau}\partial_\rho \end{pmatrix},$$

where, I denotes the identity operator.
So,

$$\mathcal{A}\phi = \begin{pmatrix} -u \\ -\dfrac{a_1}{\rho_1}\varphi_{xx} - \dfrac{a_2}{\rho_1}\psi_{xx} + \dfrac{\mu_1}{\rho_1}u + \dfrac{\mu_2}{\rho_1}z(x,1,t) \\ -v \\ -\dfrac{a_3}{\rho_2}\psi_{xx} - \dfrac{a_2}{\rho_2}\varphi_{xx} + \dfrac{\delta}{\rho_2}\theta_x \\ \dfrac{1}{\rho_3}q_x + \dfrac{\delta}{\rho_3}v_x \\ \dfrac{\beta}{\gamma}q + \dfrac{1}{\gamma}\theta_x \\ \dfrac{1}{\tau}z_\rho(x,\rho,t) \end{pmatrix},$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L_*^2(0,1) \times L^2((0,1)\times(0,1)),$$

such that,

$$\begin{aligned} H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\ L_*^2(0,1) &= \left\{ \psi \in L^2(0,1) : \int_0^1 \psi(x)\,dx = 0 \right\}, \\ H_*^2(0,1) &= \{\psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0\}. \end{aligned}$$

2.2. Well-posedness of the problem

Let $\phi = (\varphi, u, \psi, v, \theta, q, z)^T \in \mathcal{H}$, $\tilde{\phi} = \left(\tilde{\varphi}, \tilde{u}, \tilde{\psi}, \tilde{v}, \tilde{\theta}, \tilde{q}, \tilde{z}\right)^T \in \mathcal{H}$, we equip the Hilbert space $\mathcal{H}$ with the inner product through

$$\begin{aligned}\langle \phi, \widetilde{\phi} \rangle_{\mathcal{H}} = &\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)(\frac{a_2}{\sqrt{a_3}}\widetilde{\varphi}_x + \sqrt{a_3}\widetilde{\psi}_x)dx + (a_1 - \frac{a_2^2}{a_3})\int_0^1 \varphi_x \widetilde{\varphi}_x dx + \rho_1 \int_0^1 u\tilde{u}dx \\ &+ \rho_2 \int_0^1 v\tilde{v}dx + \rho_3 \int_0^1 \theta\widetilde{\theta}dx + \gamma \int_0^1 q\tilde{q}dx + \mu_2\tau \int_0^1\int_0^1 z(x,\rho,t)\widetilde{z}(x,\rho,t)d\rho dx.\end{aligned} \tag{2.8}$$

The domain of $\mathcal{A}$ is given by,

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \phi \in \mathcal{H} : \ \varphi \in H^2(0,1) \cap H_0^1(0,1), \ \psi \in H_*^2(0,1) \cap H_*^1(0,1), \\ u, \theta \in H_0^1(0,1), \ v, q \in H_*^1(0,1), \\ z, z_\rho \in L^2((0,1),(0,1)), \ z(x,0,t) = u. \end{array} \right\}.$$

$D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have now the following existence and uniqueness result.

Theorem 2.1 *Let $\phi_0 \in \mathcal{H}$, then there exists a unique solution $\phi \in C(\mathbb{R}_+, \mathcal{H})$ of problem (2.7). Moreover, if $\phi_0 \in D(\mathcal{A})$, then $\phi \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})$.*

Proof. To prove that the operator $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$. First, we show that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. For any $\phi = (\varphi, u, \psi, v, \theta, q, z)^T \in D(\mathcal{A})$, we get

$$\langle \mathcal{A}\phi, \phi \rangle = \left\langle \begin{pmatrix} -u \\ -\frac{a_1}{\rho_1}\varphi_{xx} - \frac{a_2}{\rho_1}\psi_{xx} + \frac{\mu_1}{\rho_1}u + \frac{\mu_2}{\rho_1}z(x,1,t) \\ -v \\ -\frac{a_3}{\rho_2}\psi_{xx} - \frac{a_2}{\rho_2}\varphi_{xx} + \frac{\delta}{\rho_2}\theta_x \\ \frac{1}{\rho_3}q_x + \frac{\delta}{\rho_3}v_x \\ \frac{\beta}{\gamma}q + \frac{1}{\gamma}\theta_x \\ \frac{1}{\tau}z_\rho(x,\rho,t) \end{pmatrix}, \begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ \theta \\ q \\ z \end{pmatrix} \right\rangle.$$

By using the inner product (2.8), we obtain

$$\begin{aligned}\langle \mathcal{A}\phi, \phi \rangle = \mu_1 \int_0^1 u^2 dx + \beta \int_0^1 q^2 dx + +\mu_2 \int_0^1 uz(x,1,t)dx \\ + \mu_2 \int_0^1\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx.\end{aligned} \tag{2.9}$$

For the last term of (2.9), we have

$$\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho = \frac{1}{2}\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho$$
$$= \frac{1}{2}\left[z^2(x,1,t) - z^2(x,0,t)\right],$$

then,

$$\mu_2\int_0^1\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx = \frac{\mu_2}{2}\int_0^1 \left[z^2(x,1,t) - z^2(x,0,t)\right]dx. \tag{2.10}$$

The use of Young's inequality, gives

$$-\mu_2\int_0^1 u(x)\,z(x,1,t)dx \le \frac{\mu_2}{2}\int_0^1 u^2(x)\,dx + \frac{\mu_2}{2}\int_0^1 z^2(x,1,t)dx, \tag{2.11}$$

substituting (2.11) and (2.10) into (2.9) and using the fact (2.3), we obtain

$$\langle \mathcal{A}\phi,\phi\rangle_{\mathcal{H}} \ge \beta\int_0^1 q^2dx + (\mu_1-\mu_2)\int_0^1 u^2dx.$$

From (2.2), we conclude that $\langle \mathcal{A}\phi,\phi\rangle_{\mathcal{H}} \ge 0$, which implies that $\mathcal{A}$ is monotone.

Now, we prove that the operator $I+\mathcal{A}$ is surjective.

For any $G=(g_1,g_2,g_3,g_4,g_5,g_6,g_7)^T \in \mathcal{H}$, we prove that there exists $\phi\in D(\mathcal{A})$, such that

$$(\mathcal{A}+I)\,\phi = G. \tag{2.12}$$

That is,

$$\begin{cases} \varphi - u = g_1, \\ (\rho_1+\mu_1)\,u - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_2 z(x,1,t) = \rho_1 g_2, \\ \psi - v = g_3, \\ \rho_2 v - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x = \rho_2 g_4, \\ \rho_3\theta + q_x + \delta v_x = \rho_3 g_5, \\ (\gamma+\beta)\,q + \theta_x = \gamma g_6, \\ \tau z(x,\rho,t) + z_\rho(x,\rho,t) = \tau g_7. \end{cases} \tag{2.13}$$

We note that the last equation in (2.13) with $z(x,0,t)=u$ has a unique solution

$$z(x,\rho,t) = e^{-\tau\rho}u + \tau e^{-\tau\rho}\int_0^\rho e^{\tau k}g_7(x,k)\,dk. \tag{2.14}$$

From the sixth equation in $(2.13)_6$, we deduce

$$\theta_x = \gamma g_6 - (\gamma+\beta)\,q, \tag{2.15}$$

$$\theta = \gamma \int_0^x g_6 dx - (\gamma + \beta) \int_0^x q dx, \tag{2.16}$$

then, $\theta(0) = \theta(1) = 0$. Inserting $u = \varphi - g_1$ and substituting (2.14) into $(2.13)_2$, we get

$$\rho_1\varphi - \rho_1 g_1 + \mu_1\varphi - \mu_1 g_1 - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_2 e^{-\tau}\varphi - \mu_2 e^{-\tau} g_1 + \mu_2\tau e^{-\tau}\int_0^1 e^{\tau k} g_7(x,k)\,dk = \rho_1 g_2,$$

then,

$$\left(\rho_1 + \mu_1 + \mu_2 e^{-\tau}\right)\varphi - a_1\varphi_{xx} - a_2\psi_{xx} = \rho_1 g_2 + \left(\rho_1 + g_1 + \mu_2 e^{-\tau}\right) g_1 - \mu_2\tau e^{-\tau}\int_0^1 e^{\tau k} g_7(x,k)\,dk.$$

Now, by inserting $v = \psi - g_3$ and substituting (2.15) into $(2.13)_4$, we have

$$\rho_2\psi - \rho_2 g_3 - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\gamma g_6 - \delta(\gamma+\beta) q = \rho_2 g_4,$$

so,

$$\rho_2\psi - a_3\psi_{xx} - a_2\varphi_{xx} - \delta(\gamma+\beta) q = \rho_2(g_4 + g_3) - \delta\gamma g_6.$$

In the same way, by substituting (2.16) in $(2.13)_5$, we obtain

$$-q_x + \delta g_{3x} + \rho_3(\gamma+\beta)\int_0^x q dx - \delta\psi_x - \rho_3\gamma\int_0^x g_6 dx = -\rho_3 g_5,$$

then,

$$-q_x - \delta\psi_x + \rho_3(\gamma+\beta)\int_0^x q dx = \rho_3\left(\gamma\int_0^x g_6 dx - g_5\right) - \delta g_{3x}.$$

Consequently, we get

$$\begin{cases} -a_1\varphi_{xx} - a_2\psi_{xx} + \mu\varphi = h_1 \in L^2(0,1), \\ \rho_2\psi - a_3\psi_{xx} - a_2\varphi_{xx} - \delta(\gamma+\beta) q = h_2 \in L^2_*(0,1), \\ -q_x - \delta\psi_x + \rho_3(\gamma+\beta)\int_0^x q dx = h_3 \in L^2(0,1), \end{cases} \tag{2.17}$$

where,

$$\begin{aligned} &\mu = \mu_1 + \rho_1 + \mu_2 e^{-\tau}, \\ &h_1 = \rho_1 g_2 + \mu g_1 - \mu_2\tau e^{-\tau}\textstyle\int_0^1 g_7(x,k)\, e^{\tau k} dk, \\ &h_2 = \rho_2(g_4 + g_3) - \delta\gamma g_6, \\ &h_3 = -\delta g_{3x} - \rho_3\left(g_5 - \gamma\textstyle\int_0^x g_6(y)\,dy\right). \end{aligned} \tag{2.18}$$

To solve (2.17) we consider this variational formulation

$$B\left((\varphi,\psi,q), \left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right) = F\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right), \tag{2.19}$$

where, $B : \left[H_0^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1)\right]^2 \rightarrow \mathbb{R}$ is the bilinear form

$$\begin{aligned}
B\left((\varphi,\psi,q),\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right) &= a_2 \int_0^1 \left(\psi_x\tilde{\varphi}_x + \varphi_x\tilde{\psi}_x\right)dx + \mu\int_0^1 \varphi\tilde{\varphi}dx + \rho_2\int_0^1 \psi\tilde{\psi}dx \\
&+ a_1\int_0^1 \varphi_x\tilde{\varphi}_x dx + a_3\int_0^1 \psi_x\tilde{\psi}_x dx - \delta(\gamma+\beta)\int_0^1 q\tilde{\psi}dx \\
&+ (\gamma+\beta)\int_0^1 q\tilde{q}dx + \delta(\gamma+\beta)\int_0^1 \psi\tilde{q}dx \\
&+ \rho_3(\gamma+\beta)^2\int_0^1\left(\int_0^x q(y)\,dy\int_0^x \tilde{q}(y)\,dy\right)dx,
\end{aligned}$$

and, $F : \left[H_0^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1)\right] \rightarrow \mathbb{R}$ is the linear form

$$F\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right) = \int_0^1 h_1\tilde{\varphi}dx + \int_0^1 h_2\tilde{\psi}dx + \int_0^1 h_3\int_0^x \tilde{q}(y)\,dydx.$$

Now, for $V = H_0^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1)$ equipped with the norm

$$\|(\varphi,\psi,q)\|_V^2 = \|\varphi_x\|_2^2 + \|\psi_x\|_2^2 + \|\varphi\|_2^2 + \|\psi\|_2^2 + \|q\|_2^2,$$

we can see that B and F are bounded. Furthermore, using integration by parts, we obtain

$$\begin{aligned}
B\left((\varphi,\psi,q),\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right)_V &= 2a_2\int_0^1 \psi_x\varphi_x dx + \mu\int_0^1 \varphi^2 dx + \rho_2\int_0^1 \psi^2 dx + a_1\int_0^1 \varphi_x^2 dx \\
&+ a_3\int_0^1 \psi_x^2 dx + (\gamma+\beta)\int_0^1 q^2 dx + \rho_3(\gamma+\beta)^2\int_0^1\left(\int_0^x q(y)\,dy\right)^2 dx \\
&\geq c\,\|(\varphi,\psi,q)\|_V^2,
\end{aligned}$$

thus B is coercive. On the other hand, using Cauchy-Schwarz and Poincare's inequalities, we obtain

$$B\left((\varphi,\psi,q),\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right)_V \leq n_1\|(\varphi,\psi,q)\|_V\left\|\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right\|_V.$$

Similarly,

$$F\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right) \leq n_2\left\|\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right\|_V.$$

Consequently, by Lax-Milgram theorem, system (2.17) has a unique solution

$$\varphi \in H_0^1(0,1),\ \psi \in H_*^1(0,1),\ q \in L_*^2(0,1),$$

substituting φ, ψ and q into $(2.13)_1$, $(2.13)_3$ and $(2.13)_6$ respectively, we obtain

$$u \in H_0^1(0,1),\ v \in H_*^1(0,1),\ \theta \in H_0^1(0,1),$$

similarly, inserting u into (2.14) and bearing in mind $(2.13)_7$, we obtain

$$z, z_\rho \in L^2((0,1),(0,1)).$$

Now, if $(\tilde{\varphi}, \tilde{q}) = (0,0) \in H_0^1(0,1) \times L_*^2(0,1)$, then, (2.19) gives

$$a_2 \int_0^1 \varphi_x \tilde{\psi}_x dx + a_3 \int_0^1 \psi_x \tilde{\psi}_x dx + \rho_2 \int_0^1 \psi \tilde{\psi} dx - \delta(\gamma+\beta) \int_0^1 q \tilde{\psi} dx = \int_0^1 h_2 \tilde{\psi} dx, \ \forall \tilde{\psi} \in H_*^1(0,1), \tag{2.20}$$

which implies,

$$\rho_2 \psi = a_2 \varphi_{xx} + \delta(\gamma+\beta) q + h_2 \in L^2(0,1), \tag{2.21}$$

consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\psi \in H^2(0,1) \cap H_*^1(0,1).$$

Moreover, (2.20) is also true for any $\phi \in C^1([0,1]) \subset H_*^1(0,1)$. Hence, we have

$$\rho_2 \int_0^1 \psi \phi dx + \int_0^1 (-a_2 \varphi_{xx} - \delta(\gamma+\beta) q - h_2) \phi dx = 0,$$

for all $\phi \in C^1([0,1])$. Thus, using integration by parts and bearing in mind (2.21), we obtain

$$\psi_x(1)\phi(1) - \psi_x(0)\phi(0) = 0, \ \forall \phi \in C^1([0,1]),$$

therefore, $\psi_x(0) = \psi_x(1) = 0$. So, we obtain

$$-a_1 \varphi_{xx} = a_2 \psi_{xx} - \mu\varphi + h_1 \in L^2(0,1),$$

$$-q_x = \delta \psi_x - \rho_3(\gamma+\beta) \int_0^x q dx + h_3 \in L^2(0,1),$$

thus, we have

$$\varphi \in H^2(0,1) \cap H_0^1(0,1), \ q \in H_*^1(0,1).$$

Finally, by using the regularity theory to solve linear elliptic equations ensures the presence of unique solution $\phi \in D(\mathcal{A})$ such that (2.12) is satisfied. Consequently, $\mathcal{A}$ is a maximal operator. Hence, the result of Theorem 2.1 follows from the Lumer-Phillips theorem. ■

2.3 Stability results

In this section, we use the energy method to prove our stability result for the energy of the solution of the system (2.5). First, we state and prove the following lemma.

Lemma 2.1 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.5) and assume (2.2) holds. Then, the energy functional $E(t)$, defined by (2.6) satisfies*

$$E'(t) \leq -\beta \int_0^1 q^2 dx - m_0 \int_0^1 \varphi_t^2 dx, \quad \forall t \geq 0, \tag{2.22}$$

where $m_0 = (\mu_1 - \mu_2)$.

Proof. Multiplying (2.5)$_1$, (2.5)$_2$, (2.5)$_3$ and (2.5)$_4$ by φ_t, ψ_t, θ and q respectively, and integrating over $(0,1)$ and using integration by parts and the boundary conditions, we obtain

$$\begin{cases} \rho_1 \int_0^1 \varphi_{tt}\varphi_t dx - a_1 \int_0^1 \varphi_{xx}\varphi_t dx - a_2 \int_0^1 \psi_{xx}\varphi_t dx + \mu_1 \int_0^1 \varphi_t^2 dx + \mu_2 \int_0^1 \varphi_t z(x,1,t) dx = 0, \\ \rho_2 \int_0^1 \psi_{tt}\psi_t dx - a_3 \int_0^1 \psi_{xx}\psi_t dx - a_2 \int_0^1 \varphi_{xx}\psi_t dx + \delta \int_0^1 \theta_x \psi_t dx = 0, \\ \rho_3 \int_0^1 \theta_t \theta dx + \int_0^1 q_x \theta dx + \delta \int_0^1 \psi_{tx}\theta dx = 0, \\ \gamma \int_0^1 q_t q dx + \beta \int_0^1 q q dx + \int_0^1 \theta_x q dx = 0, \end{cases}$$

the integration par parts, gives

$$\begin{cases} \frac{\rho_1}{2}\frac{d}{dt} \int_0^1 \varphi_t^2 dx + a_1 \int_0^1 \varphi_x \varphi_{tx} dx + a_2 \int_0^1 \psi_x \varphi_{tx} dx + \mu_1 \int_0^1 \varphi_t^2 dx + \mu_2 \int_0^1 \varphi_t z(x,1,t) dx = 0, \\ \frac{\rho_2}{2}\frac{d}{dt} \int_0^1 \psi_t^2 dx + a_3 \int_0^1 \psi_x \psi_{tx} dx + a_2 \int_0^1 \varphi_x \psi_{tx} dx - \delta \int_0^1 \theta \psi_{tx} dx = 0, \\ \frac{\rho_3}{2}\frac{d}{dt} \int_0^1 \theta^2 dx + \int_0^1 q_x \theta dx + \delta \int_0^1 \psi_{tx}\theta dx = 0, \\ \frac{\gamma}{2}\frac{d}{dt} \int_0^1 q^2 dx + \beta \int_0^1 q^2 dx + \int_0^1 \theta_x q dx = 0. \end{cases} \tag{2.23}$$

Summing up the equations (2.23)$_1$, (2.23)$_2$, (2.23)$_3$ and (2.23)$_4$ member to member, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\theta^2 + \gamma q^2\right) dx + a_1 \int_0^1 \varphi_x\varphi_{tx} dx + a_2 \int_0^1 \left(\psi_x\varphi_{tx} + \varphi_x\psi_{tx}\right) dx,$$

$$+a_3 \int_0^1 \psi_x\psi_{tx} dx + \int_0^1 \left(q_x\theta + \theta_x q\right) dx + \mu_1 \int_0^1 \varphi_t^2 dx + \beta \int_0^1 q^2 dx + \mu_2 \int_0^1 \varphi_t z(x,1,t) dx = 0,$$

noticing that,

$$a_1 \int_0^1 \varphi_x \varphi_{tx} dx = \frac{a_1}{2}\frac{d}{dt}\int_0^1 \varphi_x^2 dx,$$

$$a_3 \int_0^1 \psi_x \psi_{tx} dx = \frac{a_3}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx,$$

$$a_2 \int_0^1 \left(\psi_x\varphi_{tx} + \varphi_x\psi_{tx}\right) dx = a_2 \frac{d}{dt}\int_0^1 \varphi_x\psi_x dx,$$

$$\int_0^1 \left(q_x\theta + \theta_x q\right) dx = \int_0^1 \frac{d}{dx}\left(q\theta\right) dx = q\left(1,t\right)\theta\left(1,t\right) - q\left(0,t\right)\theta\left(0,t\right) = 0.$$

We obtain,

$$\frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + a_1\varphi_x^2 + \rho_2\psi_t^2 + a_3\psi_x^2 + \rho_3\theta^2 + \gamma q^2 + 2a_2\varphi_x\psi_x\right\} dx$$
$$= -\mu_1 \int_0^1 \varphi_t^2 dx - \beta \int_0^1 q^2 dx - \mu_2 \int_0^1 \varphi_t z(x,1,t) dx,$$

we have,

$$a_1\varphi_x^2 + a_3\psi_x^2 + 2a_2\varphi_x\psi_x = \left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 + \left(a_1 - \frac{a_2^2}{a_3}\right)\varphi_x^2.$$

Consequently, we obtain

$$\frac{1}{2}\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_3\theta^2+\gamma q^2+(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2+(a_1-\frac{a_2^2}{a_3})\varphi_x^2\right\}dx \tag{2.24}$$
$$=-\mu_1\int_0^1\varphi_t^2dx-\beta\int_0^1 q^2dx-\mu_2\int_0^1\varphi_t z(x,1,t)dx.$$

Multiplying (2.5)$_5$ by $\mu_2 z$, integrating the product over $(0,1)\times(0,1)$ and recalling that

$$z(x,0,t)=\varphi_t(x,t), \tag{2.25}$$

gives,

$$\tau\mu_2 z_t(x,\rho,t)z(x,\rho,t)=-\mu_2 z(x,\rho,t)z_\rho(x,\rho,t),$$

integrating over $(0,1)\times(0,1)$, gives

$$\begin{aligned}\tau\mu_2\frac{d}{2dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx &= -\mu_2\int_0^1\left(\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)d\rho\right)dx\\ &= -\mu_2\int_0^1\left(\frac{1}{2}z^2(x,\rho,t)\Big|_{\rho=0}^{\rho=1}\right)dx\\ &= -\frac{\mu_2}{2}\int_0^1 z^2(x,1,t)dx+\frac{\mu_2}{2}\int_0^1 z^2(x,0,t)dx,\end{aligned}$$

and recalling (2.25), we obtain

$$\tau\mu_2\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx=-\frac{\mu_2}{2}\int_0^1 z^2(x,1,t)dx+\frac{\mu_2}{2}\int_0^1\varphi_t^2dx. \tag{2.26}$$

A combination of (2.24) and (2.26), gives

$$\begin{aligned}E^{'}(t) \;&=\; -\left(\mu_1-\frac{\mu_2}{2}\right)\int_0^1\varphi_t^2dx-\beta\int_0^1 q^2dx-\mu_2\int_0^1\varphi_t z(x,1,t)dx \\ &\quad -\frac{\mu_2}{2}\int_0^1 z^2(x,1,t)dx.\end{aligned} \tag{2.27}$$

Meanwhile, using Young's inequality, gives

$$-\mu_2\int_0^1\varphi_t z(x,1,t)dx\le\frac{\mu_2}{2}\int_0^1\varphi_t^2dx+\frac{\mu_2}{2}\int_0^1 z^2(x,1,t)dx. \tag{2.28}$$

Inserting (2.28) into (2.27) and using (2.2), we obtain (2.6) and (2.22).

The proof is complete. ■

Lemma 2.2 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (2.5). Then, the functional*

$$F_1(t) = \rho_1 \int_0^1 \varphi_t \varphi dx - \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi dx,$$

satisfies, the estimate

$$F_1^{'}(t) \leq -\frac{1}{2}(a_1 - \frac{a_2^2}{a_3}) \int_0^1 \varphi_x^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx + \epsilon_0 \int_0^1 \psi_t^2 dx + C_{\epsilon_0} \int_0^1 \varphi_t^2 dx \tag{2.29}$$
$$+ C \int_0^1 \theta^2 dx + C_0 \int_0^1 z^2(x,1,t) dx.$$

Proof. Differentiating $F_1(t)$ using $(2.5)_1$ and $(2.5)_2$, gives

$$\begin{aligned} F_1^{'}(t) &= \rho_1 \int_0^1 \varphi_t^2 dx + \int_0^1 \varphi[a_1 \varphi_{xx} + a_2 \psi_{xx} - \mu_1 \varphi_t - \mu_2 z(x,1,t) dx] dx \\ &- \frac{a_2}{a_3} \int_0^1 [a_3 \psi_{xx} + a_2 \varphi_{xx} - \delta \theta_x] \varphi dx - \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx \\ &= \rho_1 \int_0^1 \varphi_t^2 dx - a_1 \int_0^1 \varphi_x^2 dx - \frac{\mu_1}{2} \int_0^1 \varphi^2 dx - \mu_2 \int_0^1 \varphi z(x,1,t) dx \\ &- \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx + \frac{a_2^2}{a_3} \int_0^1 \varphi_x^2 dx - \frac{\delta a_2}{a_3} \int_0^1 \varphi_x \theta dx, \end{aligned}$$

then,

$$\begin{aligned} F_1^{'}(t) &= -(a_1 - \frac{a_2^2}{a_3}) \int_0^1 \varphi_x^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx - \mu_2 \int_0^1 \varphi z(x,1,t) dx \\ &- \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx - \frac{\delta a_2}{a_3} \int_0^1 \varphi_x \theta dx, \end{aligned} \tag{2.30}$$

using Young's inequality in the terms in the right hand side of (2.30) gives:

$$-\frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx \leq \epsilon_0 \int_0^1 \psi_t^2 dx + C_{\epsilon_0} \int_0^1 \varphi_t^2 dx, \tag{2.31}$$

$$-\frac{\delta a_2}{a_3} \int_0^1 \varphi_x \theta dx \leq \frac{1}{4}(a_1 - \frac{a_2^2}{a_3}) \int_0^1 \varphi_x^2 dx + C \int_0^1 \theta^2 dx, \tag{2.32}$$

by applying Cauchy-Schwarz's, Young's and Poincaré inequalities, we get

$$-\mu_2 \int_0^1 \varphi z(x,1,t) dx \leq \frac{1}{4}(a_1 - \frac{a_2^2}{a_3}) \int_0^1 \varphi_x^2 dx + C_0 \int_0^1 z^2(x,1,t) dx. \tag{2.33}$$

Finally, by substuting (2.31), (2.32) and (2.33) into (2.30) we get the estimate (2.29). ■

Lemma 2.3 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.5). Then, the functional*

$$
\begin{aligned}
F_2\left(t\right) = {} & \rho_1 a_2 \int_0^1 \varphi_t(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx - \frac{a_2^2}{a_3}\rho_2 \int_0^1 \psi_t(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx \\
& + \frac{\mu_1 a_2^2}{2\sqrt{a_3}} \int_0^1 \varphi^2 dx + \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi\varphi dx,
\end{aligned}
$$

satisfies, the estimate

$$
\begin{aligned}
F_2^{'}\left(t\right) \leq {} & -\frac{a_2^2 \rho_2 \sqrt{a_3}}{4a_3} \int_0^1 \psi_t^2 dx + 2C_{\epsilon_1} \int_0^1 \varphi_x^2 dx + C \int_0^1 \varphi_t^2 dx \\
& + C_{\epsilon_1} \int_0^1 z(x,1,t)dx + 2\epsilon_1 \int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)^2 dx.
\end{aligned}
\tag{2.34}
$$

Proof. Differentiating $F_2\left(t\right)$ using $(2.5)_1$ and $(2.5)_2$, gives

$$
\begin{aligned}
F_2^{'}\left(t\right) = {} & a_2 \int_0^1 [a_1\varphi_{xx} + a_2\psi_{xx} - \mu_1\varphi_t - \mu_2 z(x,1,t)](\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx \\
& + \rho_1 a_2 \int_0^1 \varphi_t(\frac{a_2}{\sqrt{a_3}}\varphi_t + \sqrt{a_3}\psi_t)dx \\
& - \frac{a_2^2}{a_3} \int_0^1 [a_3\psi_{xx} + a_2\varphi_{xx} - \delta\theta_x](\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx \\
& - \frac{a_2^2}{a_3}\rho_2 \int_0^1 \psi_t(\frac{a_2}{\sqrt{a_3}}\varphi_t + \sqrt{a_3}\psi_t)dx + \frac{\mu_1 a_2^2}{\sqrt{a_3}} \int_0^1 \varphi\varphi_t dx \\
& + \mu_1 a_2 \sqrt{a_3} \left(\int_0^1 \psi_t\varphi dx + \int_0^1 \psi\varphi_t dx \right) \\
= {} & -a_1 a_2 \int_0^1 \varphi_x(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)dx \\
& - a_2^2 \int_0^1 \psi_x(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)dx \\
& - \frac{\mu_1 a_2^2}{\sqrt{a_3}} \int_0^1 \varphi\varphi_t dx - \mu_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t\psi dx \\
& - a_2\mu_2 \int_0^1 z(x,1,t)(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx + \rho_1 \frac{a_2^2}{\sqrt{a_3}} \int_0^1 \varphi_t^2 dx \\
& + \rho_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t\psi_t dx + a_2^2 \int_0^1 \psi_x(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)dx \\
& + \frac{a_2^3}{a_3} \int_0^1 \varphi_x(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)dx - \frac{a_2^3\rho_2}{a_3\sqrt{a_3}} \int_0^1 \varphi_t\psi_t dx \\
& - \frac{a_2^2\rho_2\sqrt{a_3}}{a_3} \int_0^1 \psi_t^2 dx + \frac{\mu_1 a_2^2}{\sqrt{a_3}} \int_0^1 \varphi\varphi_t dx + \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi_t\varphi dx \\
& + \mu_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t\psi dx,
\end{aligned}
$$

next,

$$\begin{aligned} F_2^{'}(t) &= -a_2(a_1 - \frac{a_2^2}{a_3}) \int_0^1 \varphi_x(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)dx \\ &- a_2\mu_2 \int_0^1 z(x,1,t)(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx + \rho_1 \frac{a_2^2}{\sqrt{a_3}} \int_0^1 \varphi_t^2 dx \\ &+ \rho_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t \psi_t dx - \frac{a_2^3\rho_2}{a_3\sqrt{a_3}} \int_0^1 \psi_t \varphi_t dx - \frac{a_2^2\rho_2\sqrt{a_3}}{a_3} \int_0^1 \psi_t^2 dx \\ &+ \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi_t \varphi dx. \end{aligned} \tag{2.35}$$

By using Young's inequality in (2.35), we obtain

$$\begin{aligned} -a_2(a_1 - \frac{a_2^3}{a_3}) \int_0^1 \varphi_x(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)dx &\leq \epsilon_1 \int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)^2 dx \\ &+ C_{\epsilon_1} \int_0^1 \varphi_x^2 dx, \end{aligned} \tag{2.36}$$

$$-\frac{a_2^3\rho_2}{a_3\sqrt{a_3}} \int_0^1 \psi_t \varphi_t dx \leq \frac{a_2^2\rho_2\sqrt{a_3}}{4a_3} \int_0^1 \psi_t^2 dx + C_1 \int_0^1 \varphi_t^2 dx, \tag{2.37}$$

$$\rho_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t \psi_t dx \leq \frac{a_2^2\rho_2\sqrt{a_3}}{4a_3} \int_0^1 \psi_t^2 dx + C_1 \int_0^1 \varphi_t^2 dx, \tag{2.38}$$

using Young's and Poincaré inequalities, yields

$$\mu_1 a_2 \sqrt{a_3} \int_0^1 \psi_t \varphi dx \leq \frac{a_2^2\rho_2\sqrt{a_3}}{4a_3} \int_0^1 \psi_t^2 dx + C_{\epsilon_1} \int_0^1 \varphi_x^2 dx, \tag{2.39}$$

applying Cauchy-Schwarz's, Young's and Poincaré inequalities, gives

$$\begin{aligned} - a_2\mu_2 \int_0^1 z(x,1,t)(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)dx \quad &\leq \quad \epsilon_1 \int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)^2 dx \\ &+ C_{\epsilon_1} \int_0^1 z^2(x,1,t)dx. \end{aligned} \tag{2.40}$$

Finally, by substuting (2.36), (2.37), (2.38), (2.39) and (2.40) into (2.35) we obtain the estimate (2.34). ■

Lemma 2.4 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.5). Then, the functional*

$$F_3(t) = \rho_2 \int_0^1 \psi\psi_t dx + \rho_1 \int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx,$$

satisfies, the estimate

$$
\begin{aligned}
F_3^{'}(t) \leq & -\frac{1}{2}(a_1-\frac{a_2^2}{a_3})\int_0^1 \varphi_x^2 dx-\frac{1}{2}\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2 dx \\
& +C_2\int_0^1 \theta^2 dx \\
& +\mu_1\int_0^1 z^2(x,1,t)dx+\rho_1\int_0^1 \varphi_t^2 dx+\rho_2\int_0^1 \psi_t^2 dx.
\end{aligned} \tag{2.41}
$$

Proof. Taking the derivative of $F_3(t)$, using $(2.5)_1$ and $(2.5)_2$ and integration by parts, we obtain

$$
\begin{aligned}
F_3^{'}(t) &= \rho_2\int_0^1 \psi_t^2 dx+\int_0^1 [a_3\psi_{xx}+a_2\varphi_{xx}-\delta\theta_x]\psi dx+\rho_1\int_0^1 \varphi_t^2 dx \\
&+\int_0^1 \varphi[a_1\varphi_{xx}+a_2\psi_{xx}-\mu_1\varphi_t-\mu_2 z(x,1,t)]dx+\frac{\mu_1}{2}\int_0^1 \varphi^2 dx \\
&= \rho_2\int_0^1 \psi_t^2 dx-a_3\int_0^1 \psi_x^2 dx-a_2\int_0^1 \varphi_x\psi_x dx+\delta\int_0^1 \theta\psi_x dx+\rho_1\int_0^1 \varphi_t^2 dx \\
&- a_1\int_0^1 \varphi_x^2 dx-a_2\int_0^1 \varphi_x\psi_x dx-\frac{\mu_1}{2}\int_0^1 \varphi^2 dx-\mu_2\int_0^1 \varphi z(x,1,t)dx \\
&+\frac{\mu_1}{2}\int_0^1 \varphi^2 dx.
\end{aligned}
$$

Using Young's and Poincaré inequalities, establish the estimate (2.41). ■

Lemma 2.5 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (2.5). Then, the functional*

$$
F_4(t)=\gamma\rho_3\int_0^1 \theta\left(\int_0^x q(y)\,dy\right)dx,
$$

satisfies the estimate

$$
F_4^{'}(t)\leq -\frac{\rho_3}{2}\int_0^1 \theta^2 dx+\epsilon_2\int_0^1 \psi_t^2 dx+c\left(1+\frac{1}{\epsilon_2}\right)\int_0^1 q^2 dx. \tag{2.42}
$$

Proof. Taking the derivative of $F_4(t)$, using the third and the fourth equations in (2.5), gives

$$
\begin{aligned}
F_4^{'}(t) &= \gamma\rho_3\int_0^1 \theta_t\left(\int_0^x q(y)\,dy\right)dx+\gamma\rho_3\int_0^1 \theta\frac{d}{dt}\left(\int_0^x q(y)\,dy\right)dx \\
&= \gamma\int_0^1 (-q_x-\delta\psi_{tx})\left(\int_0^x q(y)\,dy\right)dx+\rho_3\int_0^1 \theta\int_0^x (-\beta q-\theta_x)\,dydx \\
&= \gamma\int_0^1 q^2 dx+\delta\gamma\int_0^1 \psi_t q dx-\beta\rho_3\int_0^1 \theta\int_0^x q dydx-\rho_3\int_0^1 \theta\int_0^x \theta_x dydx.
\end{aligned}
$$

Now, we use Young's and Cauchy-Schwarz's inequalities to obtain (2.42). ■

2.3. Stability results

Lemma 2.6 *Let* $(\varphi,\psi,\theta,q,z)$ *be the solution of* (2.5)*. Then, the functional*

$$F_5(t)=\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx,$$

satisfies the estimate,

$$F_5^{'}(t)\leq -m_1\int_0^1 z^2(x,1,t)dx-m_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx+\int_0^1\varphi_t^2dx. \tag{2.43}$$

Proof. Differentiating $F_5(t)$ and using $(2.5)_5$, gives

$$\begin{aligned}
F_5^{'}(t)&=2\tau\int_0^1\int_0^1 e^{-\tau\rho}z(x,\rho,t)z_t(x,\rho,t)d\rho dx\\
&=2\tau\int_0^1\int_0^1 e^{-\tau\rho}z\left(-\frac{1}{\tau}z_\rho(x,\rho,t)\right)d\rho dx\\
&=-2\int_0^1\int_0^1 e^{-\tau\rho}z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx\\
&=-\int_0^1\int_0^1\frac{d}{d\rho}\left[e^{-\tau\rho}z^2(x,\rho,t)\right]d\rho dx\\
&-\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx\\
&=-\int_0^1 e^{-\tau}z^2(x,1,t)dx+\int_0^1 z^2(x,0,t)dx\\
&-\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx,
\end{aligned}$$

integration by parts and using the fact that $z(x,0,t)=\varphi_t(x,t)$ and $e^{-\tau}\leq e^{-\tau\rho}\leq 1$, for all $\rho\in[0,1]$, we get

$$\begin{aligned}
F_5^{'}(t)\ &\leq\ -m_1\int_0^1 z^2(x,1,t)dx-m_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx\\
&+\int_0^1\varphi_t^2dx.
\end{aligned}$$

Since $-e^{-\tau}$ is an increasing function, setting $m_1=e^{-\tau}$, we obtain (2.43). ■

Now, we define the Lyapunov functional $L\left(t\right)$ by

$$L(t) = NE(t) + \sum_{i=1}^{5} N_i F_i(t), \tag{2.44}$$

where N_i, $i = 1, 2, ..., 5$ are positive constants to be properly chosen later.

Lemma 2.7 *there exists two positive constants C_1 and C_2 such that the Lyapunov functional $L\left(t\right)$ satisfies*

$$C_1 E\left(t\right) \leq L\left(t\right) \leq C_2 E\left(t\right), \ \ \forall t \geq 0. \tag{2.45}$$

In other words, the functions E and L are equivalent.

Proof. Let

$$L\left(t\right) = NE\left(t\right) + N_1 F_1 + N_2 F_2 + 2F_3 + N_3 F_4 + N_4 F_5,$$

$$\begin{aligned}
\left|L\left(t\right) - NE\left(t\right)\right| \leq{} & N_1 \rho_1 \int_0^1 \left|\varphi_t \varphi\right| dx - N_1 \frac{a_2}{a_3} \rho_2 \int_0^1 \left|\psi_t \varphi\right| dx \\
& + N_2 \rho_1 a_2 \int_0^1 \left|\varphi_t (\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi)\right| dx \\
& - N_2 \frac{a_2^2}{a_3} \rho_2 \int_0^1 \left|\psi_t (\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi)\right| dx \\
& + N_2 \frac{\mu_1 a_2^2}{2\sqrt{a_3}} \int_0^1 \varphi^2 dx + N_2 \mu_1 a_2 \sqrt{a_3} \int_0^1 \left|\psi \varphi\right| dx \\
& + 2\rho_2 \int_0^1 \left|\psi \psi_t\right| dx + 2\rho_1 \int_0^1 \left|\varphi \varphi_t\right| dx + \mu_1 \int_0^1 \varphi^2 dx \\
& + N_3 \gamma \rho_3 \int_0^1 \left|\theta \left(\int_0^x q\left(y\right) dy\right)\right| dx \\
& + N_4 \tau \int_0^1 \int_0^1 e^{-\tau \rho} z^2(x, \rho, t) d\rho dx.
\end{aligned}$$

Exploiting Young inequality, we get

$$\begin{aligned}
|L(t) - NE(t)| \leq {} & \frac{N_1\rho_1}{2}\int_0^1 \varphi_t^2 dx + \frac{N_1\rho_1}{2}\int_0^1 \varphi^2 dx \\
& + \frac{N_1 a_2}{2a_3}\rho_2 \int_0^1 \psi_t^2 dx + \frac{N_1 a_2}{2a_3}\rho_2\int_0^1 \varphi^2 dx \\
& + \frac{N_2\rho_1 a_2}{2}\int_0^1 \varphi_t^2 dx \\
& + \frac{N_2\rho_1 a_2}{2}\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)^2 dx \\
& + \frac{N_2 a_2^2}{2a_3}\rho_2\int_0^1 \psi_t^2 dx \\
& + \frac{N_2 a_2^2}{2a_3}\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)^2 dx \\
& + N_2\frac{\mu_1 a_2^2}{2\sqrt{a_3}}\int_0^1 \varphi^2 dx + \frac{N_2\mu_1 a_2\sqrt{a_3}}{2}\int_0^1 \psi^2 dx \\
& + \frac{N_2\mu_1 a_2\sqrt{a_3}}{2}\int_0^1 \varphi^2 dx + \rho_2\int_0^1 \psi^2 dx \\
& + 2\rho_2\int_0^1 \psi_t^2 dx + \rho_1\int_0^1 \varphi^2 dx + \rho_1\int_0^1 \varphi_t^2 dx \\
& + \mu_1\int_0^1 \varphi^2 dx + \frac{N_3\gamma\rho_3}{2}\int_0^1 \theta^2 dx \\
& + \frac{N_3\gamma\rho_3}{2}\int_0^1 \left(\int_0^x q(y)\,dy\right)^2 dx \\
& + N_4\tau\int_0^1\int_0^1 e^{-\tau\rho} z^2(x,\rho,t)\,d\rho\, dx.
\end{aligned}$$

With the fact that $e^{-\tau\rho} \leq 1$, for all $\rho \in [0,1]$ and Poincaré's inequality, we see that

$$\int_0^1 \varphi^2 dx \leq \int_0^1 \varphi_x^2 dx,$$

$$\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi)^2 dx \leq \int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)^2 dx,$$

$$\int_0^1 \left(\int_0^x q(y)\,dy\right)^2 dx \leq \int_0^1 q^2 dx,$$

$$\int_0^1 \psi^2 dx \leq \int_0^1 \psi_x^2 dx.$$

So,

$$\begin{aligned}
|L(t)-NE(t)| \leq{}& \frac{\rho_1}{2}\left(N_1+N_2a_2+2\right)\int_0^1 \varphi_t^2 dx \\
&+\frac{\rho_2}{2}\left(\frac{N_1a_2}{a_3}+\frac{N_2a_2^2}{a_3}+2\right)\int_0^1 \psi_t^2 dx \\
&+\frac{1}{2}\left(N_1\rho_1+\frac{N_1a_2}{a_3}\rho_2+N_2\frac{\mu_1a_2^2}{\sqrt{a_3}}\right. \\
&\left.+2\mu_1+2\rho_1+N_2\mu_1a_2\sqrt{a_3}\right)\int_0^1 \varphi_x^2 dx \\
&+\frac{1}{2}\left(N_2\rho_1a_2+\frac{N_2a_2^2}{a_3}\right)\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2 dx \\
&+\frac{1}{2}\left(N_2\mu_1a_2\sqrt{a_3}+2\rho_2\right)\int_0^1 \psi_x^2 dx \\
&+\frac{N_3\gamma\rho_3}{2}\int_0^1 \theta^2 dx+\frac{N_3\gamma\rho_3}{2}\int_0^1 q^2 dx \\
&+N_4\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)\,d\rho dx.
\end{aligned}$$

So,

$$\begin{aligned}
|L(t)-NE(t)| \leq{}& C\int_0^1 \left(\varphi_t^2+\psi_t^2+\theta^2+q^2\right. \\
&\left.+(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2+\varphi_x^2\right) dx \\
&+C\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx \\
\leq{}& CE(t),
\end{aligned}$$

where,

$$\begin{aligned}
C \;=\; \max&\left\{\frac{\rho_1}{2}\left(N_1+N_2a_2+2\right),\; \frac{\rho_2}{2}\left(\frac{N_1a_2}{a_3}+\frac{N_2a_2^2}{a_3}+2\right),\right. \\
&\frac{1}{2}\left(N_1\rho_1+\frac{N_1a_2}{a_3}\rho_2+N_2\frac{\mu_1a_2^2}{\sqrt{a_3}}+2\mu_1+2\rho_1+N_2\mu_1a_2\sqrt{a_3}\right), \\
&\frac{1}{2}\left(N_2\rho_1a_2+\frac{N_2a_2^2}{a_3}\right),\; \frac{1}{2}\left(N_2\mu_1a_2\sqrt{a_3}+2\rho_2\right), \\
&\left.\frac{N_3\gamma\rho_3}{2},\; \frac{N_3\gamma\rho_3}{2},\; \frac{N_4}{2}\tau\right\}.
\end{aligned}$$

Then,

$$|L(t)-NE(t)| \leq CE(t),$$

which yields,

$$(N-C)\,E\,(t)\leq L\,(t)\leq (N+C)\,E\,(t)\,.$$

Consequently, By choosing N large enough, we obtain the estimate (2.45). ■

Now, we are ready to state and prove the main result of this section.

Theorem 2.2 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (2.5). Then, the energy functional (2.6) satisfies,*

$$E\,(t)\leq k_0e^{-k_1t},\ \forall t\geq 0, \tag{2.46}$$

where, k_0 and k_1 are two positive constants.

Proof. by differentiating (2.44) and recalling (2.22), (2.29), (2.34), (2.41), (2.42) and (2.43), we obtain

$$\begin{aligned}
\frac{dL\,(t)}{dt} &\leq N\left[-\beta\int_0^1 q^2dx-(\mu_1-|\mu_2|)\int_0^1\varphi_t^2dx\right]\\
&+N_1\left[-\frac{1}{2}(a_1-\frac{a_2^2}{a_3})\int_0^1\varphi_x^2dx+\rho_1\int_0^1\varphi_t^2dx+\epsilon_0\int_0^1\psi_t^2dx\right.\\
&\left.+C_{\epsilon_0}\int_0^1\varphi_t^2dx+C\int_0^1\theta^2dx+C_0\int_0^1 z^2(x,1,t)dx\right]\\
&+N_2\left[-\frac{a_2^2\rho_2\sqrt{a_3}}{4a_3}\int_0^1\psi_t^2dx+2C_{\epsilon_1}\int_0^1\varphi_x^2dx+C\int_0^1\varphi_t^2dx\right.\\
&\left.+C_{\epsilon_1}\int_0^1 z(x,1,t)dx+2\epsilon_1\int_0^1(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2dx\right]\\
&+2\left[-\frac{1}{2}(a_1-\frac{a_2^2}{a_3})\int_0^1\varphi_x^2dx-\frac{1}{2}\int_0^1(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2dx\right.\\
&\left.+C_2\int_0^1\theta^2dx+\mu_1\int_0^1 z^2(x,1,t)dx+\rho_1\int_0^1\varphi_t^2dx+\rho_2\int_0^1\psi_t^2dx\right]\\
&+N_3\left[-\frac{\rho_3}{2}\int_0^1\theta^2dx+\epsilon_2\int_0^1\psi_t^2dx+c\left(1+\frac{1}{\epsilon_2}\right)\int_0^1 q^2dx\right]\\
&+N_4\left[-m_1\int_0^1 z^2(x,1,t)dx-\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx+\int_0^1\varphi_t^2dx\right].
\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
L^{'}(t) \leq & -\left[N\left(\mu_1-|\mu_2|\right)-N_1\left(\rho_1+C_{\epsilon_0}\right)-CN_2-2\rho_1-N_4\right] \\
& \int_0^1 \varphi_t^2 dx \\
& -\left[\frac{a_2^2\rho_2\sqrt{a_3}}{4a_3}N_2-\epsilon_0 N_1-2\rho_2-\epsilon_2 N_3\right]\int_0^1 \psi_t^2 dx \\
& -\left[\frac{1}{2}(a_1-\frac{a_2^2}{a_3})N_1-2N_2C_{\epsilon_1}+\left(a_1-\frac{a_2^2}{a_3}\right)\right]\int_0^1 \varphi_x^2 dx \\
& -\left[1-2\epsilon_1 N_2\right]\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2 dx \\
& -\left[\beta N-c\left(1+\frac{1}{\epsilon_2}\right)N_3\right]\int_0^1 q^2 dx \\
& -\left[\frac{\rho_3}{2}N_3-CN_1-2C_2\right]\int_0^1 \theta^2 dx \\
& -\left[m_1N_4-N_2C_{\epsilon_1}-2\mu_1-C_0N_1\right]\int_0^1 z^2(x,1,t)dx \\
& -\tau N_4\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx.
\end{aligned}
$$

By setting,

$$\epsilon_0=\frac{1}{N_1},\quad \epsilon_2=\frac{1}{N_3},\quad \epsilon_1=\frac{1}{4N_2},$$

we obtain,

$$
\begin{aligned}
L^{'}(t) \leq & -\left[N\left(\mu_1-|\mu_2|\right)-N_1\left(\rho_1+C_{\epsilon_0}\right)-C_1N_2-2\rho_1-N_4\right] \\
& \int_0^1 \varphi_t^2dx \\
& -\left[\frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2-2\rho_2-2\right]\int_0^1 \psi_t^2dx \\
& -\left[\frac{1}{2}(a_1-\frac{a_2^2}{a_3})N_1-N_2C_{\epsilon_1}+\left(a_1-\frac{a_2^2}{a_3}\right)\right]\int_0^1 \varphi_x^2dx \\
& -\frac{1}{2}\int_0^1 (\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x)^2dx \\
& -\left[\beta N-c\left(1+N_3\right)N_3\right]\int_0^1 q^2dx \\
& -\left[\frac{\rho_3}{2}N_3-CN_1-2C_2\right]\int_0^1 \theta^2dx \\
& -\left[m_1N_4-N_2C_{\epsilon_1}-2\mu_1-C_0N_1\right]\int_0^1 z^2(x,1,t)dx \\
& -\tau N_4\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx.
\end{aligned}
$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose N_2 large enough such that

$$
\alpha_1=\frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2-2\rho_2-2>0,
$$

then, we choose N_1 large enough such that,

$$
\alpha_2=\frac{N_1}{2}(a_1-\frac{a_2^2}{a_3})-N_2C_{\varepsilon_1}+\left(a_1-\frac{a_2^2}{a_3}\right)>0,
$$

once N_1 is fixed, we choose N_3 large enough such that,

$$
\alpha_3=\frac{\rho_3}{2}N_3-CN_1-2C_2>0,
$$

for any N_1, N_2 and N_3 choosing N_4 large enough that,

$$
m_1N_4-N_2C_{\varepsilon_1}-2\mu_1-C_0N_1>0.
$$

Finally, we choose N large enough (even larger so that (2.45) remains valid) so that,

$$
\alpha_4=N\left(\mu_1-|\mu_2|\right)-N_1\left(\rho_1+C_{\varepsilon_0}\right)-C_1N_2-2\rho_1-N_4>0,
$$

$$
\alpha_5=\beta N-c\left(1+N_3\right)N_3>0,
$$

where, $\alpha_0 = \tau N_4$, we obtain

$$L^{'}(t) \leq -\int_0^1 \left(\alpha_4 \varphi_t^2 + \alpha_1 \psi_t^2 + \alpha_3 \theta^2 + \alpha_5 q^2 + \frac{1}{2}\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 + \alpha_2 \varphi_x^2 \right) - \alpha_0 \int_0^1 \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx.$$

By (2.6), we obtain

$$L^{'}(t) \leq -\sigma_0 E(t) \quad , \forall t \geq 0, \tag{2.47}$$

for some $\sigma_0 > 0$. A combination of (2.45) and (2.47) gives

$$L^{'}(t) \leq -k_1 L(t) \quad , \forall t \geq 0, \tag{2.48}$$

where $k_1 = \alpha_0 / C_2$. A simple integration of (2.48) over $(0,t)$ yields

$$L(t) \leq L(0) e^{-k_1 t}, \ \forall t \geq 0. \tag{2.49}$$

Finally, by combining (2.45) and (2.49) we obtain, (2.46) with $k_0 = \frac{C_2 E(0)}{C_1}$, which completes the proof. ■

Chapter 3

Exponential stability of the Von Kármán beam with delay-time and microtemperature effect

3.1 Presentation of the problem

In this chapter, we look at a one-dimensional Von kármán beam with internal delay acting on the first equation coupled to a microtemperature equation

$$\begin{cases} \phi_{tt} - d_1 \left[\left(u_x + \frac{1}{2} \left(\phi_x \right)^2 \right) \phi_x \right]_x + d_2 \phi_{xxxx} + \mu_1 \phi_t + \mu_2 \phi_t(x, t - \tau) = 0, \\ u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} \left(\phi_x \right)^2 \right) \right]_x + d w_x = 0, \\ c w_t - k_1 w_{xx} + d u_{tx} + k_2 w = 0, \end{cases} \tag{3.1}$$

in $\Omega \times (0, \infty)$, where $\Omega = [0, L]$ and $d_1, d_2, d, \mu_1, c, k_1$ and k_2 are positive constants. The quantities $u(x,t)$ represent the longitudinal displacement, $\phi(x,t)$ the transversal displacement and $w(x,t)$ the microtemperatures. We complement system (3.1) with boundary conditions

$$\begin{cases} u_x(0,t) = u_x(L,t) = \phi(0,t) = \phi(L,t) = 0, & t > 0, \\ \phi_x(0,t) = \phi_x(L,t) = 0, & t > 0, \end{cases} \tag{3.2}$$

and the initial data

$$\begin{cases} u(.,0) = u_0, \quad u_t(.,0) = u_1, & x \in (0, L), \\ \phi(.,0) = \phi_0, \quad \phi_t(.,0) = \phi_1, \quad w(.,0) = w_0, & x \in (0, L), \\ \phi_t(x,t) = f_0(x,t), & x \in (0, L). \end{cases} \tag{3.3}$$

Here, we will prove the stability results for problems (3.1)–(3.3) under the assumption

$$\mu_2 \leq \mu_1. \tag{3.4}$$

The purpose of this chapter is to prove that the microtemperature effect is powerful enough to uniformly stabilize the system (3.1) even when time delay is present.

3.2 Well-posedness of the problem

In this section, we prove the existence and uniqueness of solution of problem (3.1)–(3.3) using the semigroup theory by introducing the following new variable, as in [59]:

$$z(x,\rho,t) = \phi_t(x, t-\tau\rho),\ x \in (0,L)\,,\ \rho \in (0,1)\,,\ t \in (0,\infty)\,,$$

then, the above variable z satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ \textit{in}\ (0,L)\times(0,1)\times(0,\infty).$$

Therefore, problem (3.1) takes the form

$$\begin{cases} \phi_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(\phi_x\right)^2\right)\phi_x\right]_x + d_2\phi_{xxxx} + \mu_1\phi_t + \mu_2 z(x,1,t) = 0, \\ u_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(\phi_x\right)^2\right)\right]_x + dw_x = 0, \\ cw_t - k_1 w_{xx} + du_{tx} + k_2 w = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \end{cases} \tag{3.5}$$

and the initial conditions

$$\begin{cases} u(.,0) = u_0, \quad u_t(.,0) = u_1, & x \in (0,L), \\ \phi(.,0) = \phi_0, \quad \phi_t(.,0) = \phi_1, \quad w(.,0) = w_0, & x \in (0,L), \\ \phi_t(x,t) = f_0(x,t), & x \in (0,L), \\ z(x,\rho,0) = f_0(x,\rho,t) \text{ in } (0,L)\times(0,1)\,, & \\ z(x,0,t) = \phi_t(x,t) \text{ in } (0,L)\times(0,\infty)\,. & \end{cases} \tag{3.6}$$

For $U = (\phi, \phi_t, u, u_t, w, z)^T$, then $U_t = (\phi_t, \phi_{tt}, u_t, u_{tt}, w_t, z_t)^T$, and we introduce two new dependent variables

$$\varphi = \phi_t, \psi = u_t,$$

then, the system (3.5)–(3.6) can be written as

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathcal{A}U + \mathcal{F}(U), \\ U(0) = (\phi_0, \phi_1, u_0, u_1, w_0, f_0)^T, \end{cases} \tag{3.7}$$

with the linear problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathcal{A}U, \\ U(0) = (\phi_0, \phi_1, u_0, u_1, w_0, f_0)^T. \end{cases} \tag{3.8}$$

Where, the linear operator $\mathcal{A}$ is defined as follow:

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ -d_2\partial_x^4 & -\mu_1 I & 0 & 0 & 0 & -\mu_2 I \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & d_1\partial_x^2 & 0 & -d\partial_x & 0 \\ 0 & 0 & 0 & -d\partial_x & k_1\partial_x^2 - k_2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\tau}\partial_\rho \end{pmatrix},$$

so,

$$\mathcal{A}U = \begin{pmatrix} \varphi \\ -d_2\phi_{xxxx} - \mu_1\varphi - \mu_2 z(x,1,t) \\ \psi \\ d_1 u_{xx} - d w_x \\ k_1 w_{xx} - d\psi_x - k_2 w \\ -\dfrac{1}{\tau} z_\rho(x,\rho,t) \end{pmatrix}, \tag{3.9}$$

and,

$$\mathcal{F}(U) = \begin{pmatrix} 0 \\ d_1\left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\phi_x\right]_x \\ 0 \\ \dfrac{d_1}{2}(\phi_x)_x^2 \\ 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

3.2. Well-posedness of the problem

The domain of $\mathcal{A}$ is given by

$$\begin{aligned}D(\mathcal{A}) = \Big\{&(\phi,\varphi,u,\psi,w,z)^T \in \left[H^4(0,L)\cap H_0^2(0,L)\right]\\&\times H_0^1(0,L)\times\left[H^2(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\\&\times L^2(0,L)\times L^2\left((0,L)\times H_0^1(0,L)\right),\\&\varphi = z(x,0)\ \ in\ \ (0,L)\Big\}.\end{aligned}$$

Denote by H the Hilbert space

$$\begin{aligned}H = \{&\left[H^4(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\\&\times\left[H^2(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\times L^2(0,L)\\&\times L^2\left((0,L)\times H_0^1(0,L)\right)\}.\end{aligned}$$

$D(\mathcal{A})$ is dense in H.

We will show that $\mathcal{A}$ generates a C_0 semigroup on H. For $U=(\phi,\varphi,u,\psi,w,z)^T$, $\tilde{U}=\left(\tilde{\phi},\tilde{\varphi},\tilde{u},\tilde{\psi},\tilde{w},\tilde{z}\right)^T$, the Hilbert space H, equipped with the following inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle &= \int_0^L \varphi\tilde{\varphi}dx + \int_0^L \psi\tilde{\psi}dx + c\int_0^L w\widetilde{w}dx + d_2\int_0^L \phi_{xx}\tilde{\phi}_{xx}dx\\&+ d_1\int_0^L u_x\tilde{u}_x dx + \mu_2\tau\int_0^L\int_0^1 z(x,\rho,t)\widetilde{z}(x,\rho,t)d\rho dx.\end{aligned}$$

We have the following existence and uniqueness result.

Theorem 3.1 *([12])Let $(\phi,\varphi,u,\psi,w,z)^T \in H$, For any initial datum $U_0\in H$, there exists a unique solution $U\in C(\mathbb{R}^+,H)$ of problem (3.8). Moreover, if $U_0\in D(\mathcal{A})$, then, $U\in C(\mathbb{R}^+,D(\mathcal{A}))\cap C^1(\mathbb{R}^+,H)$.*

Proof. To show that the operator $\mathcal{A}$ generates a C_0-semigroup on H, firstly we prove that the operator $\mathcal{A}$ is dissipative.

Let $U=(\phi,\varphi,u,\psi,w,z)^T$, then

$$\begin{aligned}\langle\mathcal{A}U,U\rangle &= \left\langle\begin{pmatrix}\varphi\\ -d_2\phi_{xxxx}-\mu_1\varphi-\mu_2 z(x,1,t)\\ \psi\\ d_1u_{xx}-dw_x\\ k_1w_{xx}-d\psi_x-k_2w\\ -\frac{1}{\tau}z_\rho(x,\rho,t)\end{pmatrix},\begin{pmatrix}\phi\\ \varphi\\ u\\ \psi\\ w\\ z\end{pmatrix}\right\rangle\\
&= -\mu_1\int_0^L\varphi^2dx-d_2\int_0^L\phi_{xxxx}\varphi dx-\mu_2\int_0^L z(x,1,t)\varphi dx\\
&+d_1\int_0^L u_{xx}\psi dx-d\int_0^L w_x\psi dx+k_1\int_0^L w_{xx}wdx\\
&-d\int_0^L\psi_x wdx-k_2\int_0^L w^2dx-\mu_2\int_0^L\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx\\
&+d_2\int_0^L\phi_{xx}\varphi_{xx}dx+d_1\int_0^L u_x\psi_x dx,\end{aligned}$$

then, we obtain

$$\begin{aligned}\langle\mathcal{A}U,U\rangle = -\mu_1\int_0^L\varphi^2dx-\mu_2\int_0^L\varphi z(x,1,t)dx-k_1\int_0^L w_x^2dx\\
-k_2\int_0^L w^2dx-\mu_2\int_0^L\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx,\end{aligned}\tag{3.11}$$

by integrating by parts in ρ, we get

$$\begin{aligned}\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho &= \frac{1}{2}\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho\\
&= \frac{1}{2}\left[z^2(x,1,t)-z^2(x,0,t)\right],\end{aligned}$$

then,

$$\mu_2\int_0^L\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx=\frac{\mu_2}{2}\int_0^L\left[z^2(x,1,t)-z^2(x,0,t)\right]dx,\tag{3.12}$$

as result from (3.11) and (3.12), we obtain

$$\begin{aligned}\langle\mathcal{A}U,U\rangle = -\mu_1\int_0^L\varphi^2(x)\,dx-k_1\int_0^L w_x^2dx-k_2\int_0^L w^2(x)\,dx\\
-\frac{\mu_2}{2}\int_0^L z^2(x,\rho,t)dx+\frac{\mu_2}{2}\int_0^L\varphi^2(x)\,dx-\mu_2\int_0^L\varphi(x)\,z(x,1,t)dx,\end{aligned}\tag{3.13}$$

using Young's inequality, the last term in (3.13), gives

$$-\mu_2\int_0^L\varphi(x)\,z(x,1,t)dx\le\frac{\mu_2}{2}\int_0^L\varphi^2(x)\,dx+\frac{\mu_2}{2}\int_0^L z^2(x,1,t)dx,\tag{3.14}$$

substituting (3.14) into (3.13), yields

$$\langle \mathcal{A}U, U\rangle \leq -(\mu_1-\mu_2)\int_0^L \varphi^2(x)\,dx - k_1\int_0^L w_x^2 dx - k_2\int_0^L w^2(x)\,dx.$$

It follows that $\langle \mathcal{A}U, U\rangle \leq 0$, which implies that $\mathcal{A}$ is dissipative.

Now, we prove that the operator $\lambda I - \mathcal{A}$ is surjective. For any $G = (g_1, g_2, g_3, g_4, g_5, g_6)^T \in H$, we prove that there exists $U = (\phi, \varphi, u, \psi, w, z)^T \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})\,U = G, \tag{3.15}$$

where (3.15) is equivalent to the following system

$$\begin{cases} \lambda\phi - \varphi = g_1, \\ \lambda\varphi + d_2\phi_{xxxx} + \mu_1\varphi + \mu_2 z(x,1,t) = g_2, \\ \lambda u - \psi = g_3, \\ \lambda\psi - d_1 u_{xx} + d w_x = g_4, \\ \lambda w - k_1 w_{xx} + d\psi_x + k_2 w = g_5, \\ \lambda z(x,\rho,t) + \dfrac{1}{\tau} z_\rho(x,\rho,t) = g_6, \end{cases} \tag{3.16}$$

from the first and third equations in (3.16), we get

$$\begin{cases} \varphi = \lambda\phi - g_1, \\ \psi = \lambda u - g_3, \end{cases} \tag{3.17}$$

then, $\varphi \in H_0^1(0,L)$ and $\psi \in H_0^1(0,L)$, also, we have $z(x,0,t) = \varphi_t(x,t)$, then, by using the last equation in (3.16),

$$\lambda z(x,\rho,t) + \frac{1}{\tau} z_\rho(x,\rho,t) = g_6, \;\; x \in (0,L),$$

we obtain,

$$\frac{dz}{d\rho} = -\lambda\tau z,$$

then,

$$\frac{dz}{z} = -\lambda\tau d\rho,$$

therefore,

$$\begin{aligned} \ln|z| &= -\lambda\tau\rho + c \\ z &= c_1(x,\rho)\exp(-\lambda\tau\rho), \end{aligned}$$

so, we have

$$\begin{aligned} z_\rho &= c_{1\rho}\exp(-\lambda\tau\rho) - \lambda\tau c_1\exp(-\lambda\tau\rho) \\ &= c_{1\rho}\exp(-\lambda\tau\rho) - \lambda\tau z, \end{aligned}$$

furthermore,

$$\begin{aligned} z_\rho(x,\rho,t) + \lambda\tau z(x,\rho,t) &= c_{1\rho}\exp\left(-\lambda\tau\rho\right) \\ &= \tau g_6, \end{aligned}$$

then, we get

$$c_1 = \tau\int_0^\rho e^{\lambda\tau\rho} g_6\left(x,\rho,t\right) d\rho,$$

as a result, we note that the last equation in (3.16) has a unique solution,

$$z(x,\rho,t) = e^{-\lambda\tau\rho}\varphi(x,t) + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau\rho} g_6\left(x,\rho,t\right) d\rho. \tag{3.18}$$

Substituting φ given by (3.17), we get

$$z(x,\rho,t) = \lambda\phi(x)e^{-\lambda\tau\rho} - g_1 e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau\rho} g_6\left(x,\rho,t\right) d\rho, \tag{3.19}$$

using (3.4) and (3.16), the functions ϕ, u satisfy the following system:

$$\begin{cases} \lambda^2\phi + d_2\phi_{xxxx} + \mu_1\varphi + \mu_2 z(x,1,t) = \lambda g_1 + g_2 \\ \lambda^2 u - d_1 u_{xx} + dw_x = \lambda g_3 + g_4, \end{cases}$$

by solving (3.16), we find $(\phi,u) \in [H^4(0,L)\cap H_0^2(0,L)] \times [H^2(0,L)\cap H_0^2(0,L)]$, where

$$\begin{cases} \int_0^L \left(\lambda^2\phi\eta + d_2\phi_{xx}\eta_{xx} + \mu_1\varphi\eta + \eta\mu_2 z(x,1,t)\right)dx = \int_0^L\left(\lambda g_1 + g_2\right)\eta dx, \\ \int_0^L \left(\lambda^2 u\zeta - d_1 u_x\zeta_x + dw\zeta_x\right)dx = \int_0^L \left(\lambda g_3 + g_4\right)\zeta dx, \end{cases} \tag{3.20}$$

for all $(\eta,\zeta) \in H_0^1(0,L)\times H_0^1(0,L)$, from (3.19), we obtain

$$z(x,1,t) = \lambda\phi(x)e^{-\lambda\tau} - g_1(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^\rho e^{\lambda\tau\rho} g_6\left(x,\rho,t\right) d\rho.$$

To solve (3.20) we consider this equivalent variational formulation

$$B((w,u),(\eta,\zeta)) = A(\eta,\zeta), \tag{3.21}$$

where, $B : [H_0^2(0,L)\times H_0^1(0,L)]^2 \to \mathbb{R}$ is the bilinear form

$$\begin{aligned} B((w,u),(\eta,\zeta)) = &\int_0^L ((\lambda^2\phi + \mu_1\varphi)\eta + d_2\phi_{xx}\eta_{xx} + \eta\lambda\mu_2\phi(x)e^{-\lambda\tau})dx \\ &+ \int_0^L \left(\lambda^2 u\zeta + \zeta_x(dw - d_1 u_x)\right)dx, \end{aligned}$$

and, $A : H_0^2(0,L)\times H_0^1(0,L) \to \mathbb{R}$ is the linear form

$$A(\eta,\zeta) = \int_0^L (\lambda g_1 + g_2)\eta dx + \int_0^L \left(\lambda g_3 + g_4\right)\zeta dx + \eta\int_0^L z_0(x,t)dx,$$

so, B is continuous and coercive and A is continuous, by the Lax-Milgram theorem, we obtain that problem (3.21) admits a unique solution $(\phi,u) \in H^2(0,L)\times H_0^1(0,L)$ for all $(\eta,\zeta) \in H_0^1(0,L)\times H_0^1(0,L)$. So by using the classical elliptic regularity it follows from (3.20) that $(w,u) \in H^4(0,L)\times H_0^2(0,L)$, as a result, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Hence, the result of 3.1 follows from the Lumer Phillips theorem. ■

3.3 Stability results

In this section, we state and prove our stability result for the energy of the solution of system (3.5) by using the energy method. To achieve our goal, we need the following lemma.

Lemma 3.1 *Let* (ϕ, u, w, z) *be the solution of* (3.5)*, and assume* (3.4) *holds. then, the energy functional* $E(t)$*, define by*

$$E(t) = \frac{1}{2}\int_0^L \left\{ \phi_t^2 + u_t^2 + cw^2 + d_2\phi_{xx}^2 + d_1\left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)^2\right] \right. \qquad (3.22)$$
$$\left. + \frac{\mu_2}{2}\tau \int_0^1 z^2(x,\rho,t)d\rho \right\} dx,$$

and satisfies

$$E^{'}(t) \leq -k_1\int_0^L w_x^2 dx - k_2\int_0^L w^2 dx - m_0\int_0^L \phi_t^2 dx, \quad \forall t \geq 0, \qquad (3.23)$$

where, $m_0 = (\mu_1 - \mu_2)$.

Proof. Multiplying equations in (3.5) by ϕ_t, u_t and w respectively, integrating over $\Omega = [0, L]$ and using integration by parts and the boundary conditions, get

$$\begin{cases} \int_\Omega \phi_{tt}\phi_t dx - d_1\int_0^L \left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\phi_x\right]_x \phi_t dx \\ +d_2\int_0^L \phi_{xxxx}\phi_t dx \\ +\mu_1\int_0^L \phi_t\phi_t dx + \mu_2\int_0^L z(x,1,t)\phi_t dx = 0, \\ \int_0^L u_{tt}u_t dx - d_1\int_0^L \left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\right]_x u_t dx \\ +d\int_0^L w_x u_t dx = 0, \\ c\int_0^L w_t w dx - k_1\int_0^L w_{xx}w dx + d\int_0^L u_{tx}w dx \\ +k_2\int_0^L wwdx = 0, \end{cases}$$

integrating par parts, gives

$$\begin{cases} \frac{d}{2dt}\int_0^L \phi_t^2 dx + d_1\int_0^L \left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\phi_x\right]\phi_{tx} dx \\ +d_2\int_0^L \phi_{xx}\phi_{txx} dx \\ +\mu_1\int_0^L \phi_t^2 dx + \mu_2\int_0^L z(x,1,t)\phi_t dx = 0, \\ \frac{d}{2dt}\int_0^L u_t^2 dx + d_1\int_0^L \left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\right] u_{tx} dx \\ -d\int_0^L w u_{tx} dx = 0, \\ \frac{c}{2}\frac{d}{dt}\int_0^L w^2 dx + k_1\int_0^L w_x^2 dx + d\int_0^L u_{tx}w dx \\ +k_2\int_0^L w^2 dx = 0, \end{cases} \qquad (3.24)$$

Summing up the equations (3.24)$_1$,(3.24)$_2$ and (3.24)$_3$ member to member, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L\left(\phi_t^2+u_t^2+w^2\right)dx+d_2\int_0^L\phi_{xx}\phi_{txx}dx+\mu_1\int_0^L\varphi_t^2dx\\&+d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\phi_x\right]\phi_{tx}dx\end{aligned}$$

$$+d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\right]u_{tx}dx+k_1\int_0^L w_x^2dx+k_2\int_0^L w^2dx+\mu_2\int_0^L z(x,1,t)\phi_tdx=0,$$

noticing that,

$$d_2\int_0^L\phi_{xx}\phi_{txx}dx=\frac{d_2}{2}\frac{d}{dt}\int_0^L\phi_{xx}^2dx,$$

then,

$$\begin{aligned}&d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\phi_x\right]\phi_{tx}dx+d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\right]u_{tx}dx\\=\;&d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\right]\left[\phi_x\phi_{tx}+u_{tx}\right]dx\\=\;&d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\right]\left[\frac{1}{2}\frac{d}{dt}\left(\phi_x\right)^2+u_{tx}\right]dx\\=\;&d_1\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\right]\left[\frac{1}{2}\left(\phi_x\right)^2+u_x\right]_t dx\\=\;&\frac{1}{2}\frac{d}{dt}\int_0^L\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\right]^2dx.\end{aligned}$$

So, we get

$$\begin{aligned}&\frac{1}{2}\int_0^L\left\{\phi_t^2+u_t^2+w^2+d_2\phi_{xx}^2+d_1\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)^2\right]\right\}dx\\&=-\mu_1\int_0^L\phi_t^2dx-k_1\int_0^L w_x^2dx-k_2\int_0^L w^2dx-\mu_2\int_0^L\phi_tz(x,1,t)dx.\end{aligned}\tag{3.25}$$

Now, multiplying (3.5)$_5$ by $|\mu_2|\,z$, integrating the product over $(0,L)\times(0,1)$ and recalling that

$$z(x,0,t)=\phi_t(x,t),\tag{3.26}$$

gives,

$$\tau\left|\mu_2\right|z_t(x,\rho,t)z(x,\rho,t)=-\left|\mu_2\right|z(x,\rho,t)z_\rho(x,\rho,t),$$

integrating over $(0,L)\times(0,1)$, gives

$$\begin{aligned}\tau\left|\mu_{2}\right|\frac{d}{2dt}\int_{0}^{L}\int_{0}^{1}z^{2}(x,\rho,t)d\rho dx &= -\left|\mu_{2}\right|\int_{0}^{L}\left(\int_{0}^{1}z(x,\rho,t)z_{\rho}(x,\rho,t)d\rho\right)dx\\ &= -\left|\mu_{2}\right|\int_{0}^{L}\left(\frac{1}{2}z^{2}(x,\rho,t)\Big|_{\rho=0}^{\rho=1}\right)dx\\ &= -\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}z^{2}(x,1,t)dx\\ &+\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}z^{2}(x,0,t)dx,\end{aligned}$$

and recalling (3.26), we obtain

$$\begin{aligned}\tau\left|\mu_{2}\right|\frac{1}{2}\frac{d}{dt}\int_{0}^{L}\int_{0}^{1}z^{2}(x,\rho,t)d\rho dx &= -\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}z^{2}(x,1,t)dx\\ &+\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}\phi_{t}^{2}dx.\end{aligned}\tag{3.27}$$

A combination of (3.25) and (3.27) gives

$$\begin{aligned}E^{'}(t) &= -\left(\mu_{1}-\frac{\left|\mu_{2}\right|}{2}\right)\int_{0}^{L}\phi_{t}^{2}dx-\beta\int_{0}^{L}q^{2}dx-\mu_{2}\int_{0}^{L}\phi_{t}z(x,1,t)dx\\ &-\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}z^{2}(x,1,t)dx.\end{aligned}\tag{3.28}$$

Meanwhile, by using Young's inequality, we get

$$-\mu_{2}\int_{0}^{L}\phi_{t}z(x,1,t)dx\leq\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}\phi_{t}^{2}dx+\frac{\left|\mu_{2}\right|}{2}\int_{0}^{L}z^{2}(x,1,t)dx.\tag{3.29}$$

Inserting (3.29) into (3.28) and using (3.4), we get (3.22) and (3.23).

The proof is complete. ■

Lemma 3.2 *Let (ϕ,u,w,z) be the solution of (3.5). Then, the functional*

$$F_{1}(t)=\int_{0}^{L}\left(u_{t}u+\frac{1}{2}\phi_{t}\phi+\frac{k_{1}}{4}\phi^{2}\right)dx,$$

satisfies, the estimate

$$\begin{aligned}F_{1}^{'}(t)\leq & -d_{1}\int_{0}^{L}\left(u_{x}+\frac{1}{2}(\phi_{x})^{2}\right)^{2}dx-\frac{d_{2}}{2}\int_{0}^{L}(\phi_{xx})^{2}dx\\ &+\int_{0}^{L}\left(u_{t}^{2}+\frac{1}{2}\phi_{t}^{2}\right)dx+\varepsilon_{1}\int_{0}^{L}u_{x}^{2}dx\\ &+\frac{d}{4\varepsilon_{1}}\int_{0}^{L}w^{2}dx+C_{1}\int_{0}^{L}\phi_{x}^{2}dx+\frac{C_{2}}{4\varepsilon_{1}}\int_{0}^{L}z^{2}(x,1,t)dx.\end{aligned}\tag{3.30}$$

Proof. Differentiating $F_1(t)$ using $(3.5)_1$ and $(3.5)_2$, gives

$$
\begin{aligned}
F_1^{'}(t) &= \int_0^L u_t^2 dx + \int_0^L u_{tt} u dx + \frac{1}{2}\int_0^L \phi_t^2 dx + \frac{1}{2}\int_0^L \phi_{tt}\phi dx \\
&+ \frac{k_1}{4}\int_0^L \phi\phi_t dx, \\
&= \int_0^L u_t^2 dx + \int_0^L \left(d_1 \left[\left(u_x + \frac{1}{2}(\phi_x)^2 \right) \right]_x - d w_x \right) u dx \\
&+ \frac{1}{2}\int_0^L \phi_t^2 dx \\
&+ \frac{1}{2}\int_0^L \left(d_1 \left[\left(u_x + \frac{1}{2}(\phi_x)^2 \right) \phi_x \right]_x - d_2\phi_{xxxx} - \mu_1\phi_t - \mu_2 z(x,1,t) \right) \phi dx \\
&+ \frac{k_1}{4}\int_0^L \phi\phi_t dx,
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \int_0^L u_t^2 dx - d_1 \int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right) u_x dx \\
& + d\int_0^L w u_x dx + \frac{1}{2}\int_0^L \phi_t^2 dx \\
& - \frac{d_1}{2}\int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right) \phi_x^2 dx - \frac{d_2}{2}\int_0^L \phi_{xx}^2 dx \\
& - \frac{\mu_2}{2}\int_0^L z(x,1,t)\phi dx,
\end{aligned} \tag{3.31}
$$

by using Young's inequality, we find

$$
d\int_0^L w u_x dx \le \varepsilon_1 \int_0^L u_x^2 dx + \frac{d}{4\varepsilon_1}\int_0^L w^2 dx, \tag{3.32}
$$

by applying Young's and Poincaré inequalities, we get

$$
-\frac{\mu_2}{2}\int_0^L z(x,1,t)\phi dx \le C_1 \int_0^L \phi_x^2 dx + \frac{C_2}{4\varepsilon_1}\int_0^L z^2(x,1,t) dx, \tag{3.33}
$$

we have, as well

$$
\begin{aligned}
& -d_1\int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right) u_x dx - \frac{d_1}{2}\int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right) \phi_x^2 dx \qquad (3.34) \\
= \quad & -d_1\int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right)\left(u_x + \frac{1}{2}(\phi_x)^2 \right) dx \\
= \quad & -d_1\int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right)^2 dx.
\end{aligned}
$$

Now, by substuting (3.32), (3.33) and (3.34) into (3.31) we get the estimate (3.30). ■

3.3. Stability results

Lemma 3.3 *Let* (ϕ, u, w, z) *be the solution of* (3.5). *Then, the functional*

$$F_2(t) = \int_0^L \left(\int_0^x w(t,y)\,dy \right) u_t dx,$$

satisfies, the estimate

$$\begin{aligned} F_2^{'}(t) \quad &\le \quad -\left(\frac{d}{c} - \frac{k_1}{2c} - \frac{k_2\varepsilon_2}{c} \right) \int_0^L u_t^2 dx + d_1\varepsilon_2 \int_0^L \left(u_x + \frac{1}{2}(\phi_x)^2 \right)^2 dx \\ &+ \frac{k_1}{2} \int_0^L w_x^2 dx + L(\varepsilon_2) \int_0^L w^2 dx, \end{aligned} \tag{3.35}$$

where, $L(\varepsilon_2) = \left(\frac{d_1}{4\varepsilon_2} + d + \frac{k_2}{4\varepsilon_2} \right)$

Proof. Differentiating $F_2(t)$ using $(3.5)_2$ and $(3.5)_3$, gives

$$F_2^{'}(t) = \int_0^L \left(\int_0^x \frac{d}{dt} w(t,y)\,dy \right) u_t dx + \int_0^L u_{tt} \left(\int_0^x w(t,y)\,dy \right) dx,$$

$$\begin{aligned} \int_0^L \left(\int_0^x \frac{d}{dt} w(t,y)\,dy \right) u_t dx &= \int_0^L \left(\int_0^x w_t(t,y)\,dy \right) u_t dx \\ &= \int_0^L \left(\int_0^x \left(\frac{k_1}{c} w_{xx} - \frac{d}{c} u_{tx} - \frac{k_2}{c} w \right) dy \right) u_t dx \\ &= \frac{k_1}{c} \int_0^L u_t w_x dx - \frac{d}{c} \int_0^L u_t^2 dx \\ &- \frac{k_2}{c} \int_0^L u_t \int_0^x w dy dx, \end{aligned} \tag{3.36}$$

applying Young's inequality, we obtain,

$$\frac{k_1}{c} \int_0^L u_t w_x dx \le \frac{k_1}{2c} \int_0^L u_t^2 + \frac{k_1}{2} \int_0^L w_x^2 dx, \tag{3.37}$$

$$-\frac{k_2}{c} \int_0^L u_t \int_0^x w dy dx \le \frac{k_2\varepsilon_2}{c} \int_0^L u_t^2 dx + \frac{k_2}{4\varepsilon_2} \left(\int_0^x w dy \right)^2 dx, \tag{3.38}$$

by substuting (3.37) and (3.38) into (3.36), we get

$$\begin{aligned} \int_0^L \left(\int_0^x \frac{d}{dt} w(t,y)\,dy \right) u_t dx \quad &\le \quad \frac{k_1}{2c} \int_0^L u_t^2 + \frac{k_1}{2} \int_0^L w_x^2 dx - \frac{d}{c} \int_0^L u_t^2 dx \\ &+ \frac{k_2\varepsilon_2}{c} \int_0^L u_t^2 dx + \frac{k_2}{4\varepsilon_2} \left(\int_0^x w dy \right)^2 dx, \end{aligned} \tag{3.39}$$

by recalling Cauchy–Schwarz inequality for the last term (3.39), we find

$$\begin{aligned}\int_0^x w(t,y)\,dy &\leq \left(\int_0^x 1dy\right)\left(\int_0^x w^2(t,y)\,dy\right) \\ &\leq x\left(\int_0^x w^2(t,y)\,dy\right) \\ &\leq \left(\int_0^1 w^2dx\right),\end{aligned} \tag{3.40}$$

by substituting (3.40) into (3.39), we obtain

$$\begin{aligned}\int_0^L\left(\int_0^x \frac{d}{dt}w(t,y)\,dy\right)u_tdx \quad &\leq \quad \left(-\frac{d}{c}+\frac{k_1}{2c}+\frac{k_2\varepsilon_2}{c}\right)\int_0^L u_t^2 \\ &+\frac{k_1}{2}\int_0^L w_x^2dx+\frac{k_2}{4\varepsilon_2}\int_0^L w^2dx,\end{aligned} \tag{3.41}$$

$$\begin{aligned}\int_0^L u_{tt}\left(\int_0^x w(t,y)\,dy\right)dx &= \int_0^L\left(d_1\left[\left(u_x+\frac{1}{2}(\phi_x)^2\right)\right]_x - dw_x\right)\left(\int_0^x w(t,y)\,dy\right)dx \\ &= -d_1\int_0^L\left[\left(u_x+\frac{1}{2}(\phi_x)^2\right)\right]wdx \\ &+d\int_0^L w^2dx,\end{aligned}$$

by using Young's inequality, we get

$$\begin{aligned}-d_1\int_0^L\left[\left(u_x+\frac{1}{2}(\phi_x)^2\right)\right]wdx \quad &\leq \quad d_1\varepsilon_2\int_0^L\left(u_x+\frac{1}{2}(\phi_x)^2\right)^2dx \\ &+\frac{d_1}{4\varepsilon_2}\int_0^L w^2dx,\end{aligned}$$

then, we obtain

$$\begin{aligned}\int_0^L u_{tt}\left(\int_0^x w(t,y)\,dy\right)dx \quad &\leq \quad d_1\varepsilon_2\int_0^L\left(u_x+\frac{1}{2}(\phi_x)^2\right)^2dx \\ &+\left(\frac{d_1}{4\varepsilon_2}+d\right)\int_0^L w^2dx,\end{aligned} \tag{3.42}$$

combining (3.41) and (3.42), gives the estimate (3.35). ■

Lemma 3.4 *Let (ϕ,u,w,z) be the solution of (3.5). Then, the functional*

$$F_3(t)=\tau\int_0^L\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx,$$

satisfies, the estimate

$$F_3^{'}(t) \leq -m_1 \int_0^L z^2(x,1,t)dx - m_1\tau \int_0^L \int_0^1 z^2(x,\rho,t)d\rho dx + \int_0^L \phi_t^2 dx. \tag{3.43}$$

Proof. Differentiating $F_3(t)$, gives

$$\begin{aligned}
F_3^{'}(t) &= 2\tau \int_0^L \int_0^1 e^{-\tau\rho} z(x,\rho,t) z_t(x,\rho,t) d\rho dx \\
&= 2\tau \int_0^L \int_0^1 e^{-\tau\rho} z(x,\rho,t) \left(-\frac{1}{\tau} z_\rho(x,\rho,t) \right) d\rho dx \\
&= -2 \int_0^L \int_0^1 e^{-\tau\rho} z(x,\rho,t) z_\rho(x,\rho,t) d\rho dx \\
&= - \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-\tau\rho} z^2(x,\rho,t) \right) d\rho dx \\
&- \tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx \\
&= - \int_0^L e^{-\tau} z^2(x,1,t)dx + \int_0^L z^2(x,0,t)dx \\
&- \tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx,
\end{aligned}$$

integration by parts and using the fact that $z(x,0,t) = \phi_t(x,t)$ and $e^{-\tau} \leq e^{-\tau\rho} \leq 1$, we get for all $\rho \in [0,1]$

$$F_3^{'}(t) \leq - \int_0^L e^{-\tau} z^2(x,1,t)dx - \tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx + \int_0^L \phi_t^2 dx.$$

Since, $-e^{-\tau}$ is an increasing function, setting $m_1 = e^{-\tau}$, we obtain (3.43). ■

Now, we define the Lyapunov functional $L(t)$ by

$$L(t) = NE(t) + N_1F_1 + N_2F_2 + N_3F_3, \tag{3.44}$$

where, N,N_1,N_2 and N_3 are positive constants.

Lemma 3.5 *there exists two positive constants C_1 and C_2 such that the Lyapunov functional $L(t)$ satisfies,*

$$C_1E(t) \leq L(t) \leq C_2E(t), \ \forall t \geq 0. \tag{3.45}$$

In other words, the functions E and L are equivalent.

Proof. Let

$$L(t) = NE(t) + N_1F_1 + N_2F_2 + N_3F_3,$$

$$\begin{aligned}|L(t) - NE(t)| \leq{} & N_1 \int_0^L |u_t u|\, dx + \frac{N_1}{2}\int_0^L |\phi_t\phi|\, dx + \frac{N_1 k_1}{4}\int_0^L \phi^2 dx \\ & + N_2 \int_0^L \left| \left(\int_0^x w(t,y)\, dy \right) u_t \right| dx \\ & + N_3\tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx.\end{aligned}$$

Exploiting Young inequality, we get

$$\begin{aligned}|L(t) - NE(t)| \leq{} & \frac{N_1}{2}\int_0^L u_t^2 dx + \frac{N_1}{2}\int_0^L u^2 dx + \frac{N_1}{2}\int_0^L \phi_t^2 dx \\ & + \frac{N_1}{2}\int_0^L \phi^2 dx \\ & + \frac{N_1 k_1}{4}\int_0^L \phi^2 dx + \frac{N_2}{2}\int_0^L \left(\int_0^x w(t,y)\, dy \right)^2 dx \\ & + \frac{N_2}{2}\int_0^L u_t^2 dx \\ & + N_3\tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx.\end{aligned}$$

By exploiting Poincaré inequality, and the fact that $e^{-\tau\rho} \leq 1$, for all $\rho \in [0,1]$ we see that,

$$\int_0^L \phi^2 dx \leq \int_0^L \phi_{xx}^2 dx,$$

$$\frac{N_1}{2}\int_0^L u^2 dx + \frac{N_1 k_1}{4}\int_0^L \phi^2 dx \leq \frac{N_1}{2}\int_0^L \left[\left(u_x + \frac{1}{2}(\phi_x)^2 \right)^2 \right] dx,$$

$$\int_0^L \left(\int_0^x w(t,y)\, dy \right)^2 dx \leq \int_0^L w^2 dx,$$

So,

$$\begin{aligned}|L(t) - NE(t)| \leq{} & \left(\frac{N_1}{2} + \frac{N_2}{2} \right) \int_0^L u_t^2 dx + \frac{N_1}{2}\int_0^L \phi_t^2 dx \\ & + \frac{N_1}{2}\int_0^L \phi_{xx}^2 dx \\ & + \frac{N_2}{2}\int_0^L w^2 dx + \frac{N_1}{2}\int_0^L \left[\left(u_x + \frac{1}{2}(\phi_x)^2 \right)^2 \right] dx \\ & + N_4\tau \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx.\end{aligned}$$

So,

$$\begin{aligned} L\left(t\right) \leq C\int_{0}^{L}\left(\phi_{t}^{2}+u_{t}^{2}+cw^{2}+d_{2}\phi_{xx}^{2}+d_{1}\left[\left(u_{x}+\frac{1}{2}\left(\phi_{x}\right)^{2}\right)^{2}\right]\right)dx \\ +C\int_{0}^{L}\int_{0}^{1}z^{2}(x,\rho,t)d\rho dx, \\ \leq CE\left(t\right), \end{aligned}$$

where,

$$C=\max\left\{\left(\frac{N_{1}}{2}+\frac{N_{2}}{2}\right),\ \frac{N_{1}}{2},\ \frac{N_{2}}{2},\ \frac{N_{4}}{2}\tau\right\},$$

then,

$$\left|L\left(t\right)-NE\left(t\right)\right|\leq CE\left(t\right),$$

which yields,

$$\left(N-C\right)E\left(t\right)\leq L\left(t\right)\leq\left(N+C\right)E\left(t\right).$$

Consequently, by choosing N large enough we obtain estimate (3.45). ■

Now, it's time to state and demonstrate the section's major result.

Theorem 3.2 *Let (ϕ,u,w,z) be the solution of (3.5). Then, the energy functional (3.22) decays exponentially, i.e., there exist two positive constants k_0 and k_1 independent of the initial data such that,*

$$E\left(t\right)\leq k_{0}e^{-k_{1}t},\ \forall t\geq 0, \tag{3.46}$$

where, k_0 and k_1 are two positive constants.

Proof. by differentiating (3.44) and recalling (3.23), (3.30), (3.35) and (3.43), we obtain

$$\begin{aligned}\frac{dL(t)}{dt} \leq & -\frac{N_1 d_2}{2}\int_0^L (\phi_{xx})^2\,dx \\ & -\left(Nm_0-\frac{N_1}{2}-N_3\right)\int_0^L \phi_t^2 dx \\ & -\left(\frac{N_2 d}{c}-\frac{N_2 k_1}{2c}-\frac{N_2 k_2 \varepsilon_2}{c}-N_1\right)\int_0^L u_t^2 dx \\ & -\left(Nk_2-\frac{N_1 d}{4\varepsilon_1}-N_2\left(\frac{d_1}{4\varepsilon_2}+d+\frac{k_2}{4\varepsilon_2}\right)\right)\int_0^L w^2 dx \\ & -\left(N_1 d_1-N_2 d_1 \varepsilon_2\right)\int_0^L \left(u_x+\frac{1}{2}(\phi_x)^2\right)^2 dx \\ & -\left(m_1 N_3-\frac{N_1 C_2}{4\varepsilon_1}\right)\int_0^L z^2(x,1,t)dx. \\ & -N_3\tau m_1\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx. \\ & +\frac{N_2 k_1}{2}\int_0^L w_x^2 dx.\end{aligned}$$

By setting,

$$\varepsilon_1=\frac{d}{4N_1},\quad \varepsilon_2=\frac{c}{N_2},$$

we obtain,

$$\begin{aligned}\frac{dL(t)}{dt} \leq & -\frac{N_1 d_2}{2}\int_0^L (\phi_{xx})^2\,dx \\ & -\left(Nm_0-\frac{N_1}{2}-N_3\right)\int_0^L \phi_t^2 dx \\ & -\left(\frac{N_2 d}{c}-\frac{N_2 k_1}{2c}-k_2-N_1\right)\int_0^L u_t^2 dx \\ & -\left(Nk_2-1-\frac{d_1}{4c}-N_2 d-\frac{k_2}{4c}\right)\int_0^L w^2 dx \\ & -\left(N_1 d_1-c d_1\right)\int_0^L \left(u_x+\frac{1}{2}(\phi_x)^2\right)^2 dx \\ & -\left(m_1 N_3-\frac{C_2}{d}\right)\int_0^L z^2(x,1,t)dx \\ & -N_3\tau m_1\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx.\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_3 large enough, such that

$$m_1 N_3-\frac{C_2}{d}>0,$$

then we choose N_1 large enough, such that

$$\alpha_1 = N_1 d_1 - c d_1 > 0,$$

Once N_1 is fixed, we then choose N_2 large enough, such that

$$\alpha_2 = \frac{N_2 d}{c} - \frac{N_2 k_1}{2c} - k_2 - N_1 > 0.$$

Finally, we choose N large enough (even larger so that (3.45) remains valid), so that

$$\alpha_3 = N m_0 - \frac{N_1}{2} - N_3 > 0,$$

$$\alpha_4 = N k_2 - 1 - \frac{d_1}{4c} - N_2 d - \frac{k_2}{4c},$$

where, $\alpha_0 = N_3 \tau m_1$ and $\alpha_5 = \dfrac{N_1 d_2}{2}$,we obtain

$$\begin{aligned} L^{'}(t) \leq &- \int_0^L \left(\alpha_5 \left(\phi_{xx}\right)^2 dx + \alpha_3 \phi_t^2 + \alpha_2 u_t^2 + \alpha_4 w^2 + \alpha_1 \left(u_x + \frac{1}{2} \left(\phi_x\right)^2 \right)^2 \right) dx \\ &- \alpha_0 \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx. \end{aligned}$$

By (3.22), we obtain

$$L^{'}(t) \leq -\sigma_0 E(t) \quad , \forall t \geq 0, \tag{3.47}$$

for some $\sigma_0 > 0$. A combination of (3.45) and (3.47) gives

$$L^{'}(t) \leq -k_1 L(t) \quad , \forall t \geq 0, \tag{3.48}$$

where $k_1 = \alpha_0 / C_2$. A simple integration of (3.48) over $(0,t)$, yields

$$L(t) \leq L(0) e^{-k_1 t}, \ \forall t \geq 0. \tag{3.49}$$

Finally, by combining (3.45) and (3.49) we obtain, (3.46) with $k_0 = \dfrac{C_2 E(0)}{C_1}$, which completes the proof. ■

Chapter 4

Energy decay of one-dimensional thermoelastic Bresse system with distributed neutral delay and a second sound

4.1 Presentation of the problem

In this chapter, we consider the following Bresse system with distributed neutral delay [2]

$$\begin{cases} \rho_1 \phi_{tt} - k\left(\phi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\phi\right) = 0, \\ \rho_2 \left(\psi_t + \int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)^{'} - b\psi_{xx} + k\left(\phi_x + lw + \psi\right) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\phi\right)_x + kl\left(\phi_x + lw + \psi\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0, \end{cases} \tag{4.1}$$

where, $(x,t) \in (0,1) \times \mathbb{R}_+$, with the following initial and boundary conditions:

$$\begin{cases} \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ \theta(x,0) = \theta_0(x) & in\ \left(0,\infty\right), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ q(x,0) = q_0(x), & in\ \left(0,\infty\right), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & in\ \left(0,\infty\right), \\ \phi(0,t) = \phi(1,t) = w(0,t) = w(1,t) = 0, & \forall t \geq 0, \\ \psi(0,t) = \psi(1,t) = 0, & \forall t \geq 0, \end{cases} \tag{4.2}$$

where, $\rho_1, \rho_2, \rho_3, k, l, k_0, b, \alpha, \gamma, \beta$ are positive constants and the initial data $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, q_0)$ belongs to a suitable Sobolev space, and the integral represents the neutral delay term where, k is the relaxation function already specified in the preliminaries.

In this section, we present our assumptions on both kernels and we are in need to announce this Lemma, which will be used in the next sections, in order to make the computations easier.

(H1) The kernel k is a non negative continuously differentiable and summable function satisfying

$$k^{'}(t) \leq 0,\ \forall t \geq 0,\ \bar{k} = \int_0^{\infty} k(s)\,ds < 1.$$

(H2) $\exp(\varsigma t)\,k(t) \in L^1(\mathbb{R}_+)$ for some $\varsigma > 0$.

Note that if $\int_0^{+\infty} e^{\varsigma s} k(s)\,ds < \infty$ and $\lim\limits_{t\to\infty} \exp(\varsigma t)\,k(t) < \infty$, then

$$\begin{aligned} \int_0^{+\infty} e^{\varsigma s}\left|k^{'}(s)\right| ds &= -\int_0^{+\infty} e^{\varsigma s} k^{'}(s)\,ds = -e^{\varsigma s}k(s)\big|_0^{\infty} \\ +\varsigma \int_0^{+\infty} e^{\varsigma s} k(s)\,ds &< \infty. \end{aligned}$$

Lemma 4.1 **(1)** *For any function $\psi \in C^1([0,\infty); L^2(0,1))$, and any $k \in C^1([0,\infty))$, we have the following identity*

$$\begin{aligned} \int_0^1 \psi(t)\left(\int_0^t k(t-s)\,\psi_t(s)\,ds\right) dx = {}& -\frac{1}{2}\left(k^{'}\Box\psi\right)(t) \\ & + \frac{1}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k(t-s)\,\psi^2(s)\,ds\right) dx \\ & + \frac{k(t)}{2}\int_0^1 \psi^2 dx - k(t)\int_0^1 \psi(0)\,\psi(t)\,dx, \end{aligned}$$

where,

$$(k\Box\psi) = \int_0^t k(t-s) \parallel \psi(t) - \psi(s) \parallel^2 ds,\ t \geq 0.$$

We also need the following parameters to establish an exponential stability results

$$k = k_0,$$

and,

$$\xi = \left(1 - \frac{k\rho_3\alpha}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\gamma^2\alpha}{b} = 0.$$

4.1. Presentation of the problem

Lemma 4.2 *There exists a positive constant c such that the following inequality holds for every $(\phi, \psi, w) \in \left[H_0^1(0, L)\right]^3$,*

$$\int_0^L \left(\phi_x^2 + \psi_x^2 + w_x^2\right) dx \leq c \int_0^1 \left[\psi_x^2 + (\phi_x + lw + \psi)^2 + (w_x - l\phi)^2\right] dx.$$

For completeness, we present the following existence and uniqueness result without proof, which can be demonstrated using the Faedo-Galerkin method.

Theorem 4.1 *Let $\left[(\phi_0, \phi_1), (\psi_0, \psi_1), (w_0, w_1), (\theta_0, \theta_1), (q_0, q_1)\right]^5 \in H_0^1(0,1) \times L^2(0,1)$.*

Assume that ***(H1)*** *and* ***(H2)*** *are satisfied. Then, the problem (4.1)–(4.2) has a unique solution $(\phi, \psi, w, \theta, q)$, such that,*

$$(\phi, \psi, w, \theta, q) \in C(\mathbb{R}^+, H_0^1(0,1)) \cap C^1\left(\mathbb{R}^+, L^2(0,1)\right).$$

Our purpose in this paper is to establish an exponential stability results although there is a dissipation caused by the neutral delay when $\xi = 0$, $k = k_0$ and l is small enough.

4.2 Stability result

In this section, we state and prove our stability results for the energy of the solution of system (4.1)–(4.2), using the Lyapunov functional which is equivalent to the energy functional. The following technical lemmas are required to fulfill our objective.

First, we introduce the energy associated to the system (4.1)–(4.2). using the multiplier technique.

Lemma 4.3 *Let $(\phi, \psi, w, \theta, q)$ be the solution of (4.1)–(4.2).Then, the energy functional, define by*

$$\begin{aligned} E(t) = \frac{1}{2} \int_0^1 & \left[\rho_1 \phi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 + b\psi_x^2 + \rho_3 \theta^2 + \alpha q^2 \right. \\ & \left. + k(\phi_x + lw + \psi)^2 + k_0 (w_x - l\phi)^2 + \rho_2 \int_0^t k(t-s)\, \psi_t^2(s)\, ds \right] dx, \end{aligned} \tag{4.3}$$

satisfies,

$$E'(t) \leq -\frac{\rho_2 k(t)}{2} \int_0^1 \psi_t^2 dx + \frac{\rho_2}{2} \left(k' \Box \psi_t\right)(t)\, dx - \beta \int_0^1 q^2 dx, \tag{4.4}$$

where,

$$(k \Box \psi) = \int_0^t k(t-s) \parallel \psi(t) - \psi(s) \parallel^2 ds, \ t \geq 0.$$

Proof. Multiplying (4.1)$_1$, (4.1)$_2$, (4.1)$_3$, (4.1)$_4$ and (4.1)$_5$ by ϕ_t, ψ_t, w_t, θ, and q respectively and integrating over $(0,1)$,

$$\begin{cases} \rho_1 \int_0^1 \phi_{tt}\phi_t dx - k\int_0^1 \phi_t\left(\phi_x + lw + \psi\right)_x dx - k_0 l \int_0^1 \phi_t\left(w_x - l\phi\right) dx = 0, \\ \rho_2 \int_0^1 \psi_{tt}\psi_t dx + \rho_2 \int_0^1 \left[\psi_t \left(\int_0^t k\left(t-s\right)\psi_t\left(s\right) ds\right)'\right] dx - b\int_0^1 \psi_t \psi_{xx} dx \\ +k\int_0^1 \psi_t\left(\phi_x + lw + \psi\right) dx + \gamma\int_0^1 \psi_t\theta_x dx = 0, \\ \rho_1 \int_0^1 w_{tt}w_t dx - k_0\int_0^1 w_t\left(w_x - l\phi\right)_x dx + kl\int_0^1 \psi_t\left(\phi_x + lw + \psi\right) dx = 0, \\ \rho_3 \int_0^1 \theta\theta_t dx + \int_0^1 \theta q_x dx + \gamma\int_0^1 \theta\psi_{tx} dx = 0, \\ \alpha\int_0^1 qq_t dx + \beta\int_0^1 qq dx + \int_0^1 q\theta_x dx = 0, \end{cases}$$

using integration by parts and the boundary conditions, we obtain

$$\begin{cases} \frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \phi_t^2 dx + k\int_0^1 \phi_{tx}\left(\phi_x + lw + \psi\right) dx - k_0 l\int_0^1 \phi_t\left(w_x - l\phi\right) dx = 0, \\ \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx + \rho_2\int_0^1 \left[\psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right) ds\right)'\right] dx + b\int_0^1 \psi_x^2 dx \\ +k\int_0^1 \psi_t\left(\phi_x + lw + \psi\right) dx + \gamma\int_0^1 \psi_t\theta_x dx = 0, \\ \frac{\rho_1}{2}\frac{d}{dt}\int_0^1 w_t^2 dx + k_0\int_0^1 w_{tx}\left(w_x - l\phi\right) dx + kl\int_0^1 w_t\left(\phi_x + lw + \psi\right) dx = 0, \\ \frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \int_0^1 \theta q_x dx - \gamma\int_0^1 \theta_x\psi_t dx = 0, \\ \frac{\alpha}{2}\frac{d}{dt}\int_0^1 q^2 dx + \beta\int_0^1 q^2 dx - \int_0^1 q_x\theta dx = 0, \end{cases}$$

we have also,

$$\begin{aligned} & k\int_0^1 \phi_{tx}\left(\phi_x + lw + \psi\right) dx + k\int_0^1 \psi_t\left(\phi_x + lw + \psi\right) dx + lk\int_0^1 w_t\left(\phi_x + lw + \psi\right) dx \\ & = k\int_0^1 \left(\phi_x + lw + \psi\right)\left(\phi_{xt} + lw_t + \psi_t\right) dx \\ & = k\int_0^1 \left(\phi_x + lw + \psi\right)\left(\phi_x + lw + \psi\right)_t dx \\ & = \frac{k}{2}\frac{d}{dt}\int_0^1 \left(\phi_x + lw + \psi\right)^2 dx, \end{aligned}$$

and,

$$\begin{aligned} & -lk_0\int_0^1 \phi_t\left(w_x - l\phi\right) dx + k_0\int_0^1 w_{xt}\left(w_x - l\phi\right) dx \\ & = k_0\int_0^1 \left(w_x - l\phi\right)\left(w_x - l\phi\right)_t dx \\ & = \frac{k_0}{2}\frac{d}{dt}\int_0^1 \left(w_x - l\phi\right)^2 dx. \end{aligned}$$

Now, by summing them up, we get

$$\frac{d}{2dt}\int_0^1\left(\rho_1\phi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+b\psi_x^2+\rho_3\theta^2+\alpha q^2+k\left(\phi_x+lw+\psi\right)^2+k_0\left(w_x-l\phi\right)^2\right)dx$$
$$+\rho_2\int_0^1\left[\psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)'\right]dx=0. \tag{4.5}$$

A simple integration, gives

$$\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)'=k\left(t\right)\psi_t\left(0\right)+\int_0^t k\left(t-s\right)\psi_{tt}\left(s\right)ds.$$

Then,

$$\begin{aligned}&\rho_2\int_0^1\left[\psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)'\right]dx\\&=\rho_2\int_0^1\psi_t\left(k\left(t\right)\psi_t\left(0\right)+\int_0^t k\left(t-s\right)\psi_{tt}\left(s\right)ds\right)dx\\&=\rho_2 k\left(t\right)\int_0^1\psi_t\left(0\right)\psi_t dx+\rho_2\int_0^1\psi_t\left(\int_0^t k\left(t-s\right)\psi_{tt}\left(s\right)ds\right)dx.\end{aligned}$$

By applying Lemma's (1) result, we obtain

$$\begin{aligned}&\rho_2\int_0^1\left[\psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)'\right]dx\\&=\rho_2 k\left(t\right)\int_0^1\psi_t\left(0\right)\psi_t dx-\frac{\rho_2}{2}\left(k'\Box\psi_t\right)\left(t\right)\\&+\frac{\rho_2}{2}\frac{d}{dt}\int_0^1\left(\int_0^t k\left(t-s\right)\psi_t^2\left(s\right)ds\right)dx\\&+\frac{\rho_2 k\left(t\right)}{2}\int_0^1\psi_t^2 dx-\rho_2 k\left(t\right)\int_0^1\psi_t\left(0\right)\psi_t dx.\end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.5), gives

$$\begin{aligned}&\frac{d}{2dt}\int_0^1\left[\rho_1\phi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+b\psi_x^2+\rho_3\theta^2+\alpha q^2\right.\\&\left.+k\left(\phi_x+lw+\psi\right)^2+k_0\left(w_x-l\phi\right)+\rho_2\int_0^t k\left(t-s\right)\psi_t^2\left(s\right)ds\right]dx\\&=-\frac{\rho_2 k\left(t\right)}{2}\int_0^1\psi_t^2 dx+\frac{\rho_2}{2}\left(k'\Box\psi_t\right)\left(t\right)-\beta\int_0^1 q^2 dx,\end{aligned}$$

which complete the proof. ■

4.2. Stability result

Lemma 4.4 *Let (ϕ,ψ,w,θ,q) be the solution of (4.1)-(4.2). Then, the functional*

$$F_1(t)=\alpha\rho_3\int_0^1\theta\left(\int_0^x q(y)\,dy\right)dx, \tag{4.7}$$

satisfies, the estimate

$$F_1'(t)\le-\frac{\rho_3}{2}\int_0^1\theta^2dx+\varepsilon_1\int_0^1\psi_t^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 q^2dx. \tag{4.8}$$

where, $\varepsilon_1>0$.

Proof. Taking the derivative of (4.7) and using the fourth and the fifth equations into (4.1), gives

$$\begin{aligned}F_1'(t)&=\alpha\rho_3\int_0^1\theta_t\left(\int_0^x q(y)\,dy\right)dx+\alpha\rho_3\int_0^1\theta\frac{d}{dt}\left(\int_0^x q(y)\,dy\right)dx\\&=\alpha\int_0^1(-q_x-\gamma\psi_{tx})\left(\int_0^x q(y)dy\right)dx\\&+\rho_3\int_0^1\theta\int_0^x(-\beta q-\theta_x)\,dydx\\&=\alpha\int_0^1 q^2dx+\gamma\alpha\int_0^1\psi_tq dx-\beta\rho_3\int_0^1\theta\int_0^x qdydx\\&-\rho_3\int_0^1\theta\int_0^x\theta_xdydx.\end{aligned}$$

Now, we use Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}F_1'(t)\le{}&\alpha\int_0^1 q^2dx+\gamma\alpha\varepsilon_1\int_0^1\psi_t^2dx+\frac{\gamma\alpha}{\varepsilon_1}\int_0^1q^2dx-\rho_3\int_0^1\theta^2dx\\&+\frac{\beta\rho_3}{2}\int_0^1\theta^2dx+\frac{\beta\rho_3}{2}\int_0^1\left(\int_0^x qdy\right)^2dx,\end{aligned}$$

then,

$$\begin{aligned}F_1'(t)\ &\le\ \left(\alpha+\frac{\gamma\alpha}{\varepsilon_1}+\frac{\beta\rho_3}{2}\right)\int_0^1q^2dx+\left(\frac{\beta\rho_3}{2}-\rho_3\right)\int_0^1\theta^2dx\\&\quad+\gamma\alpha\varepsilon_1\int_0^1\psi_t^2dx.\end{aligned}$$

Thus, we get the estimate (4.8). ■

Lemma 4.5 *Let* $(\phi, \psi, w, \theta, q)$ *be the solution of (4.1)–(4.2). Then, the functional*

$$F_2(t) = -\rho_1 \int_0^1 (\phi\phi_t + ww_t)\, dx, \tag{4.9}$$

satisfies the estimate,

$$\begin{aligned} F_2^{'}(t) \leq & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_1 \int_0^1 w_t^2 dx + c \int_0^1 \psi_x^2 \\ & + c \int_0^1 (w_x - l\phi)^2\, dx \\ & + c \int_0^1 (\phi_x + lw + \psi)^2\, dx, \end{aligned} \tag{4.10}$$

Proof. By differentiating (4.9) and using $(4.1)_1$ and $(4.1)_3$, we obtain

$$\begin{aligned} F_2^{'}(t) = & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_1 \int_0^1 \phi\phi_{tt} dx - \rho_1 \int_0^1 w_t^2 dx \\ & - \rho_1 \int_0^1 ww_{tt} dx \\ = & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_1 \int_0^1 w_t^2 dx \\ & - \int_0^1 \phi \left(k (\phi_x + lw + \psi)_x + lk_0 (w_x - l\phi)\right) dx \\ & - \int_0^1 w \left(k_0 (w_x - l\phi)_x - kl (\phi_x + lw + \psi)\right) dx \\ = & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_1 \int_0^1 w_t^2 dx + k \int_0^1 \phi_x (\phi_x + lw + \psi)\, dx \\ & - lk_0 \int_0^1 \phi (w_x - l\phi)\, dx \\ & + k_0 \int_0^1 w_x (w_x - l\phi)\, dx + kl \int_0^1 (\phi_x + lw + \psi)\, w dx \\ & + k \int_0^1 \psi (\phi_x + lw + \psi)\, dx - k \int_0^1 \psi (\phi_x + lw + \psi)\, dx, \end{aligned}$$

then,

$$\begin{aligned} F_2^{'}(t) = & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_1 \int_0^1 w_t^2 dx \\ & + k \int_0^1 (\phi_x + lw + \psi)^2\, dx \\ & + k_0 \int_0^1 (w_x - l\phi)^2\, dx - k \int_0^1 \psi (\phi_x + lw + \psi)\, dx, \end{aligned} \tag{4.11}$$

using Young's and Poincaré's inequalities, to estimate the last term, gives

$$-k\int_0^1 \psi\left(\phi_x+lw+\psi\right)dx \leq c_pk\int_0^1 \psi_x^2dx + \frac{k}{2}\int_0^1 \left(\phi_x+lw+\psi\right)^2dx. \tag{4.12}$$

Now, by using the result of Lemma 2, we obtain

$$c_pk\int_0^1 \psi_x^2dx \leq c\int_0^1 \left[\psi_x^2+\left(\phi_x+lw+\psi\right)^2+\left(w_x-l\phi\right)^2\right]dx. \tag{4.13}$$

By substituting (4.12) and (4.13) into (4.11), we obtain the result. ■

Lemma 4.6 *Let (ϕ,ψ,w,θ,q) be the solution of (4.1)–(4.2). Then, the functional*

$$F_3\left(t\right)=\rho_2\int_0^1 \psi\left(\psi_t+\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)dx, \tag{4.14}$$

satisfies, for any $\varepsilon_3>0$ the estimate

$$\begin{aligned}F_3^{'}\left(t\right) \leq& -\left(b-\frac{c_pk}{4\varepsilon_3}-\frac{\gamma}{2}\right)\int_0^1 \psi_x^2dx+\frac{3\rho_2}{2}\int_0^1 \psi_t^2dx\\&+\frac{\gamma}{2}\int_0^1 \theta^2dx+\varepsilon_3k\int_0^1 \left(\phi_x+lw+\psi\right)^2dx\\&+\frac{\rho_2\bar{k}}{2}\int_0^1\left(\int_0^t k\left(t-s\right)\psi_t^2\left(s\right)ds\right)dx.\end{aligned}$$

Proof. Taking the derivative of (4.7), and using the second equation in (4.1), gives

$$\begin{aligned}F_3^{'}\left(t\right) =& \rho_2\int_0^1 \psi_t^2dx+\rho_2\int_0^1 \psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)dx\\&+\rho_2\int_0^1 \psi\left(\psi_t+\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)^{'}dx\\=&\rho_2\int_0^1 \psi_t^2dx+\rho_2\int_0^1 \psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)dx\\&+\rho_2\int_0^1 \psi\psi_{tt}dx+\int_0^1 \psi\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)^{'}dx.\end{aligned}$$

Integration by parts, yields

$$\begin{aligned}F_3^{'}\left(t\right) =& \rho_2\int_0^1 \psi_t^2dx+\rho_2\int_0^1 \psi_t\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)dx\\&-\int_0^1 \psi\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)^{'}dx\\&-b\int_0^1 \psi_x^2dx-k\int_0^1 \psi\left(\phi_x+lw+\psi\right)dx+\gamma\int_0^1 \psi_x\theta dx\\&+\int_0^1 \psi\left(\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right)^{'}dx.\end{aligned} \tag{4.15}$$

Using Young's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} &\rho_2 \int_0^1 \psi_t \left(\int_0^t k(t-s)\, \psi_t(s)\, ds \right) dx \\ \leq\ & \frac{\rho_2}{2} \int_0^1 \psi_t^2 dx + \frac{\rho_2}{2} \int_0^1 \left(\int_0^t k(t-s)\, \psi_t(s)\, ds \right)^2 dx, \end{aligned}$$

then,

$$\begin{aligned} \int_0^1 \left(\int_0^t k(t-s)\, \psi_t(s)\, ds \right)^2 dx &= \int_0^1 \left(\int_0^t \sqrt{k(t-s)} \sqrt{k(t-s)} \psi_t(s)\, ds \right)^2 dx \\ &\leq \int_0^1 \left[\left(\int_0^t \left(\sqrt{k(t-s)} \right)^2 ds \right)^{\frac{1}{2}} \right. \\ &\left. \left(\int_0^t \left(\sqrt{k(t-s)} \psi_t(s) \right)^2 ds \right)^{\frac{1}{2}} \right]^2 dx \\ &= \int_0^1 \left[\left(\int_0^t k(t-s)\, ds \right) \left(\int_0^t k(t-s)\, \psi_t^2(s)\, ds \right) \right] dx \\ &= \int_0^t k(t-s)\, ds \int_0^1 \left(\int_0^t k(t-s)\, \psi_t^2(s)\, ds \right) dx. \end{aligned}$$

As,

$$\int_0^t k(t-s)\, ds \leq \int_0^\infty k(t-s)\, ds = \bar{k},$$

we obtain,

$$\int_0^1 \left(\int_0^t k(t-s)\, \psi_t(s)\, ds \right)^2 dx \leq \bar{k} \int_0^1 \left(\int_0^t k(t-s)\, \psi_t^2(s)\, ds \right) dx.$$

So,

$$\begin{aligned} \rho_2 \int_0^1 \psi_t \left(\int_0^t k(t-s)\, \psi_t(s)\, ds \right) dx \ \leq\ & \frac{\rho_2}{2} \int_0^1 \psi_t^2 dx \\ & + \frac{\rho_2 \bar{k}}{2} \int_0^1 \left(\int_0^t k(t-s)\, \psi_t^2(s)\, ds \right) dx, \end{aligned}$$

Also, by using Young's and Poincaré's inequalities, we get

$$-k \int_0^1 \psi \left(\phi_x + lw + \psi \right) dx \leq \frac{c_p k}{4\varepsilon_3} \int_0^1 \psi_x^2 dx + \varepsilon_3 k \int_0^1 \left(\phi_x + lw + \psi \right)^2 dx,$$

$$\gamma \int_0^1 \psi_x \theta dx \leq \frac{\gamma}{2} \int_0^1 \psi_x^2 dx + \frac{\gamma}{2} \int_0^1 \theta^2 dx,$$

substituting into (4.15) leads to the estimate (**??**). ■

4.2. Stability result

Lemma 4.7 *Let $(\phi, \psi, w, \theta, q)$ be the solution of (4.1)–(4.2). Then, the functional*

$$F_4(t) = -\rho_1 \int_0^1 \left[\phi_t (w_x - l\phi) + w_t (\phi_x + lw + \psi)\right] dx, \tag{4.16}$$

satisfies the estimate,

$$\begin{aligned} F_4'(t) \leq & -k_0 l \int_0^1 (w_x - l\phi)^2 dx - \frac{l\rho_1}{2} \int_0^1 w_t^2 dx \\ & + l\rho_1 \int_0^1 \phi_t^2 dx + kl \int_0^1 (\phi_x + lw + \psi)^2 dx \\ & + \frac{\rho_1}{2l} \int_0^1 \psi_t^2 dx, \end{aligned}$$

Proof. By differentiating (4.16) and using $(4.1)_1$ and $(4.1)_3$, we arrive at

$$\begin{aligned} F_4'(t) = & -\rho_1 \int_0^1 \phi_{tt} (w_x - l\phi) dx - \rho_1 \int_0^1 \phi_t (w_x - l\phi)_t dx \\ & - \rho_1 \int_0^1 w_{tt} (\phi_x + lw + \psi) dx - \rho_1 \int_0^1 w_t (\phi_x + lw + \psi)_t dx \\ = & -k \int_0^1 (\phi_x + lw + \psi)_x (w_x - l\phi) dx - k_0 l \int_0^1 (w_x - l\phi)^2 dx \\ & - \rho_1 \int_0^1 \phi_t (w_x - l\phi)_t dx - \rho_1 \int_0^1 w_t (\phi_x + lw + \psi)_t dx \\ & - k_0 \int_0^1 (w_x - l\phi)_x (\phi_x + lw + \psi) dx + kl \int_0^1 (\phi_x + lw + \psi)^2 dx, \end{aligned}$$

by exploiting the condition $k = k_0$, we obtain

$$\begin{aligned} F_4'(t) = & -k_0 l \int_0^1 (w_x - l\phi)^2 dx + kl \int_0^1 (\phi_x + lw + \psi)^2 dx \\ & - \rho_1 \int_0^1 \phi_t (w_x - l\phi)_t dx - \rho_1 \int_0^1 w_t (\phi_x + lw + \psi)_t dx, \end{aligned}$$

which implies,

$$\begin{aligned} F_4'(t) = & -k_0 l \int_0^1 (w_x - l\phi)^2 dx + kl \int_0^1 (\phi_x + lw + \psi)^2 dx - l\rho_1 \int_0^1 w_t^2 dx \\ & + \rho_1 \int_0^1 \phi_{tx} w_t dx + l\rho_1 \int_0^1 \phi_t^2 dx - \rho_1 \int_0^1 w_t \phi_{xt} dx - \rho_1 \int_0^1 w_t \psi_t dx, \end{aligned}$$

noticing that $\rho_1 \int_0^1 \phi_{tx} w_t dx = -\rho_1 \int_0^1 w_t \phi_{xt} dx$, gives

$$\begin{aligned} F_4'(t) = & -k_0 l \int_0^1 (w_x - l\phi)^2 dx - l\rho_1 \int_0^1 w_t^2 dx + l\rho_1 \int_0^1 \phi_t^2 dx \\ & - \rho_1 \int_0^1 w_t \psi_t dx + kl \int_0^1 (\phi_x + lw + \psi)^2 dx, \end{aligned}$$

using Young's inequalitie,

$$-\rho_1 \int_0^1 w_t \psi_t dx \leq \frac{\rho_1}{2} \int_0^1 w_t^2 dx + \frac{\rho_1}{2} \int_0^1 \psi_t^2 dx,$$

yields, estimate (**??**). ■

Lemma 4.8 *Let $(\phi, \psi, w, \theta, q)$ be the solution of (4.1)–(4.2). Then, under the condition $k = k_0$ the functional*

$$\begin{aligned} F_5(t) &= \rho_2 \int_0^1 \psi_t (\phi_x + lw + \psi) dx + \frac{b\rho_1}{k} \int_0^1 \psi_x \phi_t dx \\ &\quad - \int_0^1 \left(\int_0^t k(t-s) \psi_t^2(s) ds \right) dx, \end{aligned} \tag{4.17}$$

satisfies, for any $\varepsilon_5 > 0$ the estimate

$$\begin{aligned} F_5'(t) &\leq -\frac{k}{2} \int_0^1 (\phi_x + lw + \psi)^2 dx + \frac{b}{\gamma\alpha} \xi \int_0^1 \theta_x (\phi_x + lw + \psi) dx \\ &+ \frac{lb}{2} \int_0^1 \psi_x^2 dx + \varepsilon_5 \int_0^1 w_t^2 dx + \varepsilon_5 \int_0^1 (w_x - l\phi)^2 dx \\ &+ \left(\rho_2 + \frac{\rho_2 l}{2} + \frac{b}{2\gamma} \chi \right) \int_0^1 \psi_t^2 dx \\ &+ \left(\frac{k}{2} + \frac{b}{2\gamma} \chi + \frac{lb}{4\varepsilon_5} \chi \right) \int_0^1 q^2 dx \\ &+ \left(\frac{b\rho_3}{\gamma} \chi + \frac{(\rho_3 k_0 lb\chi)^2}{4(\gamma\rho_1)^2 \varepsilon_5} \right) \int_0^1 \theta^2 dx. \end{aligned} \tag{4.18}$$

Proof. By a simple differentiation of (4.17) and using $(4.1)_1$ and $(4.1)_2$, we obtain

$$\begin{aligned} F_5'(t) &= \rho_2 \int_0^1 \psi_{tt} (\phi_x + lw + \psi) dx + \rho_2 \int_0^1 \psi_t (\phi_x + lw + \psi)_t dx \\ &+ \frac{b\rho_1}{k} \int_0^1 \psi_{xt} \phi_t dx + \frac{b\rho_1}{k} \int_0^1 \psi_x \phi_{tt} dx \\ &- \int_0^1 \left(\int_0^t k(t-s) \psi_t^2(s) ds \right)' dx, \end{aligned}$$

then,

$$\begin{aligned}
F_5'(t) = &-\int_0^1 \left[\left(\int_0^t k(t-s)\psi_t(s)\,ds\right)'(\phi_x + lw + \psi)\right]dx \\
&+ \rho_2 \int_0^1 \psi_t (\phi_x + lw + \psi)_t\,dx \\
&+ b\int_0^1 \psi_{xx}(\phi_x + lw + \psi)\,dx - k\int_0^1 (\phi_x + lw + \psi)^2\,dx \\
&- \gamma \int_0^1 \theta_x (\phi_x + lw + \psi)\,dx \\
&+ \frac{b\rho_1}{k}\int_0^1 \psi_{xt}\phi_t dx - \int_0^1 \left(\int_0^t k(t-s)\psi_t^2(s)\,ds\right)' dx \\
&+ b\int_0^1 \psi_x (\phi_x + lw + \psi)_x\,dx + lb\int_0^1 \psi_x (w_x - l\phi)\,dx,
\end{aligned}$$

using integration by parts, gives

$$\begin{aligned}
F_5'(t) = &-k\int_0^1 (\phi_x + lw + \psi)^2\,dx - \int_0^1 \left(\int_0^t k(t-s)\psi_t(s)\,ds\right)'(\phi_x + lw + \psi)\,dx \\
&+ \rho_2 \int_0^1 \psi_t (lw + \psi)_t\,dx - \gamma\int_0^1 \theta_x (\phi_x + lw + \psi)\,dx \\
&+ b\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 \psi_{xt}\phi_t dx - \int_0^1 \left(\int_0^t k(t-s)\psi_t^2(s)\,ds\right)' dx \\
&+ lb\int_0^1 \psi_x (w_x - l\phi)\,dx, \qquad (4.19)
\end{aligned}$$

from (4.1)$_4$, we obtain

$$\begin{aligned}
\int_0^1 \psi_{xt}\phi_t dx &= \frac{1}{\gamma}\int_0^1 (-\rho_3\theta_t - q_x)\,\phi_t dx \\
&= -\frac{\rho_3}{\gamma}\int_0^1 \theta_t\phi_t dx - \frac{1}{\gamma}\int_0^1 q_x\phi_t dx \\
&= -\frac{\rho_3}{\gamma}\int_0^1 \theta_t\phi_t dx + \frac{1}{\gamma}\int_0^1 q\phi_{tx} dx,
\end{aligned}$$

so,

$$\begin{aligned}
\int_0^1 \psi_{xt}\phi_t dx = &-\frac{\rho_3}{\gamma}\frac{d}{dt}\int_0^1 \theta\phi_t dx + \frac{\rho_3}{\gamma}\int_0^1 \theta\phi_{tt} dx \\
&+ \frac{1}{\gamma}\int_0^1 q(\phi_x + lw + \psi)_t\,dx - \frac{1}{\gamma}\int_0^1 q(lw + \psi)_t\,dx, \qquad (4.20)
\end{aligned}$$

then,

$$\int_0^1 \theta\phi_{tt} dx = -\frac{k}{\rho_1}\int_0^1 \theta_x(\phi_x + lw + \psi)\,dx + \frac{k_0 l}{\rho_1}\int_0^1 \theta(w_x - l\phi)\,dx, \qquad (4.21)$$

by using (4.1)$_5$, we obtain

$$\begin{aligned}
\frac{1}{\gamma}\int_0^1 q\left(\phi_x+lw+\psi\right)_t dx &= \frac{1}{\gamma}\frac{d}{dt}\int_0^1 q\left(\phi_x+lw+\psi\right)dx \\
&- \frac{1}{\gamma}\int_0^1 q_t\left(\phi_x+lw+\psi\right)dx \\
&= \frac{1}{\gamma}\frac{d}{dt}\int_0^1 q\left(\phi_x+lw+\psi\right)dx \\
&- \frac{1}{\gamma\alpha}\int_0^1 \left(-\beta q-\theta_x\right)\left(\phi_x+lw+\psi\right)dx \\
&= \frac{1}{\gamma}\frac{d}{dt}\int_0^1 q\left(\phi_x+lw+\psi\right)dx \\
&+ \frac{\beta}{\gamma\alpha}\int_0^1 q\left(\phi_x+lw+\psi\right)dx \\
&+ \frac{1}{\gamma\alpha}\int_0^1 \theta_x\left(\phi_x+lw+\psi\right)dx. \qquad (4.22)
\end{aligned}$$

substituting (4.21) and (4.22) into (4.20), we obtain

$$\begin{aligned}
\int_0^1 \psi_{xt}\phi_t dx = &-\frac{\rho_3}{\gamma}\frac{d}{dt}\int_0^1 \theta\phi_t dx \\
&+ \left(\frac{1}{\gamma\alpha}-\frac{\rho_3 k}{\gamma\rho_1}\right)\int_0^1 \theta_x\left(\phi_x+lw+\psi\right)dx \\
&+ \frac{\rho_3 k_0 l}{\gamma\rho_1}\int_0^1 \theta\left(w_x-l\phi\right)dx \\
&- \frac{1}{\gamma}\int_0^1 q\left(lw+\psi\right)_t dx + \frac{1}{\gamma}\frac{d}{dt}\int_0^1 q\left(\phi_x+lw+\psi\right)dx \\
&+ \frac{\beta}{\gamma\alpha}\int_0^1 q\left(\phi_x+lw+\psi\right)dx, \qquad (4.23)
\end{aligned}$$

substituting (4.23) into (4.19), leads to

$$\begin{aligned}
F_5'(t) &= -\frac{b\rho_3}{\gamma}\chi\frac{d}{dt}\int_0^1 \theta\phi_t dx - k\int_0^1 (\phi_x + lw + \psi)^2 dx \\
&+ \frac{b}{\gamma}\chi\frac{d}{dt}\int_0^1 q(\phi_x + lw + \psi)\,dx + \frac{b\beta}{\gamma\alpha}\chi\int_0^1 q(\phi_x + lw + \psi)\,dx \\
&+ \rho_2\int_0^1 \psi_t (lw+\psi)_t\,dx - \gamma\int_0^1 \theta_x(\phi_x + lw + \psi)\,dx \\
&- \int_0^1 \left(\int_0^t k(t-s)\,\psi_t^2(s)\,ds\right)' dx + lb\int_0^1 \psi_x (w_x - l\phi)\,dx \\
&+ \frac{\rho_3 k_0 lb}{\gamma\rho_1}\chi\int_0^1 \theta(w_x - l\phi)\,dx - \frac{b}{\gamma}\chi\int_0^1 q(lw+\psi)_t\,dx \\
&+ \frac{b}{\gamma\alpha}\left[\left(1 - \frac{\alpha\rho_3 k}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\gamma^2\alpha}{b}\right]\int_0^1 \theta_x(\phi_x + lw + \psi)\,dx \\
&- \int_0^1 \left(\int_0^t k(t-s)\,\psi_t(s)\,ds\right)' (\phi_x + lw + \psi)\,dx.
\end{aligned}$$

By simplifying the last inequality, using Young's and Cauchy-Schwartz inequalities and the fact that $k = k_0$, we get the desired result. ■

Lemma 4.9 *Let* $(\phi, \psi, w, \theta, q)$ *be the solution of (4.1)–(4.2). Then, the functional*

$$F_6(t) = e^{-\varsigma t}\int_0^1 \left(\int_0^t e^{\varsigma s}\tilde{H}_1(t-s)\,\psi_t^2(s)\,ds\right) dx,$$

satisfy, $\forall t \geq 0$,

$$F_6'(t) = -\varsigma F_6(t) + \tilde{H}_1(0)\int_0^1 \psi_t^2 dx - \int_0^1 \left(\int_0^t k(t-s)\,\psi_t^2(s)\,ds\right) dx, \tag{4.24}$$

where, $\tilde{H}_1(t) = \int_t^\infty e^{\varsigma s}|k(s)|\,ds$.

Proof. By a simple integration, we obtain

$$F_6'(t) = -\varsigma F_6(t) + e^{-\varsigma t}\int_0^1 \left(\int_0^t e^{\varsigma t}\tilde{H}_1(0)\,\psi_t^2 + \int_0^t e^{\varsigma s}e^{\varsigma(t-s)}k(t-s)\,\psi_t^2(s)\,ds\right) dx,$$

which gives the estimate (4.24). ■

Now, we define the Lyapunov functional $L(t)$ by

$$L(t) = NE(t) + N_1F_1 + N_2F_2 + N_3F_3 + \frac{1}{l}F_4 + N_5F_5 + N_6F_6, \tag{4.25}$$

where, N, N_1, N_2, N_5, and N_6 are positive constants.

Lemma 4.10 *Let* $(\phi, \psi, w, \theta, q)$ *be the solution of (4.1)–(4.2), there exists two positive constants* C_1 *and* C_2 *such that the Lyapunov functional* $L(t)$, *satisfies*

$$C_1\left(E\left(t\right)+F_6\left(t\right)\right)\leq L\left(t\right)\leq C_2\left(E\left(t\right)+F_6\left(t\right)\right),\ \ \forall t\geq 0. \tag{4.26}$$

Proof. From (4.25), we have

$$\begin{aligned}
&\left|L\left(t\right)-NE\left(t\right)-N_6F_6\left(t\right)\right| \\
&\leq N_1\alpha\rho_3\int_0^1\left|\theta\right|\left(\int_0^x\left|q\left(y\right)\right|dy\right)dx \\
&+N_2\rho_1\int_0^1\left|\left(\phi\phi_t+ww_t\right)\right|dx \\
&+2\rho_2\int_0^1\left|\psi\right|.\left|\psi_t+\int_0^t k\left(t-s\right)\psi_t\left(s\right)ds\right|dx \\
&+\frac{\rho_1}{l}\int_0^1\left|\phi_t\left(w_x-l\phi\right)+w_t\left(\phi_x+lw+\psi\right)\right|dx \\
&+\rho_2\int_0^1\left|\psi_t\right|\left|\left(\phi_x+lw+\psi\right)\right|dx+\frac{b\rho_1}{k}\int_0^1\left|\psi_x\right|\left|\phi_t\right|dx \\
&+\int_0^1\left|\int_0^t k\left(t-s\right)\psi_t^2\left(s\right)ds\right|dx.
\end{aligned}$$

Exploiting Cauchy-Schwarz's, Young's and Poincaré inequalities, gives

$$\begin{aligned}
&\left|L\left(t\right)-NE\left(t\right)-N_6F_6\left(t\right)\right| \\
&\leq C\int_0^1\left[\rho_1\phi_t^2+\rho_2\psi_t^2+\rho_1w_t^2+b\psi_x^2+\rho_3\theta^2+\alpha q^2\right. \\
&\left.+k\left(\phi_x+lw+\psi\right)^2+k_0\left(w_x-l\phi\right)^2+\rho_2\int_0^t k\left(t-s\right)\psi_t^2\left(s\right)ds\right]dx \\
&\leq CE\left(t\right),
\end{aligned}$$

which yields,

$$\left(N-C\right)E\left(t\right)+N_6F_6\left(t\right)\leq L\left(t\right)\leq\left(N+C\right)E\left(t\right)+N_6F_6\left(t\right).$$

Consequently, by choosing N large enough, we obtain the estimate (4.26), with

$$\begin{aligned}
C_1&=\min\left\{N-C,\ N_6\right\}, \\
C_2&=\max\left\{N+C,\ N_6\right\}.
\end{aligned}$$

■

Now, we are ready to state and prove the main result of this section.

Theorem 4.2 *Let* $(\phi, \psi, w, \theta, q)$ *be the solution of* (4.1)–(4.2) *and assume that* ***(H1)-(H2)*** *hold,* $k = k_0$*, and* $\xi = 0$*. Then, the energy functional* (4.3) *satisfies,*

$$E(t) \leq k_0 e^{-k_1 t}, \ \forall t \geq 0, \tag{4.27}$$

where, k_0 and k_1 are two positive constants.

Proof. by differentiating (4.25) and recalling (4.8), (4.10), (**??**), (**??**), (4.18) and (4.24), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -(k_0 - cN_2 - N_5\varepsilon_5) \int_0^1 (w_x - l\phi)^2 \, dx \\
& - \left(\frac{N_5 k}{2} - cN_2 - N_3\varepsilon_3 k - k \right) \int_0^1 (\phi_x + lw + \psi)^2 \, dx \\
& - \left(\frac{N\rho_2 k(t)}{2} - \tilde{H}_1(0) N_6 - N_5 \left(\rho_2 + \frac{\rho_2 l}{2} + \frac{b}{2\gamma}\chi \right) \right. \\
& \left. - -\frac{\rho_1}{2l^2} - \frac{3N_3\rho_2}{2} - N_1\varepsilon_1 \right) \int\limits_0^1 \psi_t^2 dx \\
& - (N_2\rho_1 - \rho_1) \int_0^1 \phi_t^2 dx \\
& - \left(N_2\rho_1 + \frac{\rho_1}{2} - N_5\varepsilon_5 \right) \int_0^1 w_t^2 dx \\
& - \left(N_3 \left(b - \frac{c_p k}{4\varepsilon_3} - \frac{\gamma}{2} \right) - cN_2 - \frac{N_5 l b}{2} \right) \int_0^1 \psi_x^2 dx \\
& - \left(\frac{N_1\rho_3}{2} - \frac{N_3\gamma}{2} - N_5 \left(\frac{b\rho_3}{\gamma}\chi + \frac{(\rho_3 k_0 l b \chi)^2}{4(\gamma\rho_1)^2 \varepsilon_5} \right) \right) \int\limits_0^1 \theta^2 dx \\
& - \left(\beta N - cN_1 \left(1 + \frac{1}{\varepsilon_1} \right) \right. \\
& \left. - N_5 \left(\frac{k}{2} + \frac{b}{2\gamma}\chi + \frac{lb}{4\varepsilon_5}\chi \right) \right) \int_0^1 q^2 dx \\
& - \left(N_6 - \frac{N_3\rho_2\bar{k}}{2} \right) \int_0^1 \left(\int_0^t k(t-s) \psi_t^2(s) \, ds \right) dx \\
& + \frac{N\rho_2}{2} \left(k' \square \psi_t \right)(t) \, dx - \varsigma N_6 F_6(t) .
\end{aligned}$$

By setting,

$$\varepsilon_1 = \frac{1}{N_1}, \quad \varepsilon_5 = \frac{1}{N_5}, \quad \varepsilon_3 = \frac{1}{N_3},$$

we obtain,

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(k_0 - cN_2 - 1\right)\int_0^1 \left(w_x - l\phi\right)^2 dx \\
& -\left(\frac{N_5 k}{2} - cN_2 - 2k\right)\int_0^1 \left(\phi_x + lw + \psi\right)^2 dx \\
& -\left(\frac{N\rho_2 k(t)}{2} - \tilde{H}_1(0)N_6 - N_5\left(\rho_2 + \frac{\rho_2 l}{2} + \frac{b}{2\gamma}\chi\right)\right. \\
& \left. -\left(-\frac{\rho_1}{2l^2} - \frac{3N_3\rho_2}{2} - 1\right)\right)\int\limits_0^1 \psi_t^2 dx \\
& -\left(N_2\rho_1 - \rho_1\right)\int_0^1 \phi_t^2 dx \\
& -\left(N_2\rho_1 + \frac{\rho_1}{2} - 1\right)\int_0^1 w_t^2 dx \\
& -\left(N_3\left(b - \frac{N_3 c_p k}{4} - \frac{\gamma}{2}\right) - cN_2 - \frac{N_5 lb}{2}\right)\int_0^1 \psi_x^2 dx \\
& -\left(\frac{N_1\rho_3}{2} - \frac{N_3\gamma}{2} - N_5\left(\frac{b\rho_3}{\gamma}\chi + \frac{N_5\left(\rho_3 k_0 lb\chi\right)^2}{4\left(\gamma\rho_1\right)^2}\right)\right)\int\limits_0^1 \theta^2 dx \\
& -\left(\beta N - cN_1\left(1 + N_1\right) - N_5\left(\frac{k}{2} + \frac{b}{2\gamma}\chi + \frac{N_5 lb}{4}\chi\right)\right)\int_0^1 q^2 dx \\
& -\left(N_6 - \frac{N_3\rho_2\bar{k}}{2}\right)\int_0^1\left(\int_0^t k(t-s)\,\psi_t^2(s)\,ds\right)dx \\
& + \frac{N\rho_2}{2}\left(k'\square\psi_t\right)(t)\,dx - \varsigma N_6 F_6(t).
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_2 large enough, such that

$$\alpha_0 = k_0 - cN_2 - 1 > 0, \qquad \alpha_1 = N_2\rho_1 + \frac{\rho_1}{2} - 1 > 0,$$

$$\alpha_2 = N_2\rho_1 - \rho_1 > 0,$$

then, we choose N_5 large enough, such that

$$\alpha_3 = \frac{N_5 k}{2} - cN_2 - 2k > 0,$$

once N_2 and N_5 are fixed, then we choose N_3 large enough, such that

$$\alpha_4 = N_3\left(b - \frac{N_3 c_p k}{4} - \frac{\gamma}{2}\right) - cN_2 - \frac{N_5 lb}{2} > 0,$$

also, we choose N_6 large enough, so that

$$\alpha_5 = N_6 - \frac{N_3 \rho_2 \bar{k}}{2} > 0,$$

next, we choose N_1 large enough, such that

$$\alpha_6 = \frac{N_1 \rho_3}{2} - \frac{N_3 \gamma}{2} - N_5 \left(\frac{b \rho_3}{\gamma} \chi + \frac{N_5 \left(\rho_3 k_0 l b \chi \right)^2}{4 \left(\gamma \rho_1 \right)^2} \right) > 0,$$

finally, we choose N large enough (even larger so that (4.26) remains valid), so that

$$\alpha_7 = \frac{N \rho_2 k\left(t\right)}{2} - \tilde{H}_1\left(0\right) N_6 - N_5 \left(\rho_2 + \frac{\rho_2 l}{2} + \frac{b}{2\gamma} \chi \right) - \frac{\rho_1}{2 l^2} - \frac{3 N_3 \rho_2}{2} - 1 > 0,$$

$$\alpha_8 = \beta N - c N_1 \left(1 + N_1\right) - N_5 \left(\frac{k}{2} + \frac{b}{2\gamma} \chi + \frac{N_5 l b}{4} \chi \right).$$

All these choices, lead to

$$\begin{aligned}
\mathcal{L}'\left(t\right) \leq & -\int_0^1 [\alpha_2 \phi_t^2 + \alpha_7 \psi_t^2 + \alpha_1 w_t^2 + \alpha_4 \psi_x^2 + \alpha_6 \theta^2 + \alpha_8 q^2 \\
& + \alpha_3 \left(\phi_x + l w + \psi \right)^2 + \alpha_0 \left(w_x - l \phi \right)^2 + \alpha_5 \int_0^t k\left(t-s\right) \psi_t^2\left(s\right) ds \Big] dx \\
& - \varsigma N_6 F_6\left(t\right).
\end{aligned}$$

By (4.3), we obtain

$$L'\left(t\right) \leq -\sigma_0 E\left(t\right) + F_6\left(t\right) \quad , \; \forall t \geq 0, \tag{4.28}$$

for some $\sigma_0 > 0$. A combination of the right side of (4.26) and (4.28), gives

$$L'\left(t\right) \leq -k_1 L\left(t\right) \quad , \; \forall t \geq 0, \tag{4.29}$$

where $k_1 = \frac{\sigma_0}{C_2}$. A simple integration of (4.29) over $\left(0, t\right)$, yields

$$L\left(t\right) \leq L\left(0\right) e^{-k_1 t}, \; \forall t \geq 0. \tag{4.30}$$

Finally, using the fact that $F_6\left(t\right)$ is positive and combining (4.26) and (4.30), we obtain (4.27) which completes the proof. ■

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