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Mémoire

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Geometric mean of accretive matrices

Option : Commande Optimale et Système dynamique

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Dedication

No pleasure can equal that of sharing your happiness with the
People you love.

Arrived at the end of my studies, I have the great honor to dedicate
This modest work:

To my dear mother to whom I owe who I am, she was always there for me and who never stopped paying for my happiness.

To my dear father, for all the advice he gave me, the support he showed me and the sacrifices he made to see me succeed.

To my dear brothers.

To my dear sisters.

To all my big family.

To all my friends .

To all those I love and who love me.

Selma …

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A dissertation is not an end in itself, but a special moment in a student's journey. We have always waited for the writing of this page which would mark the end of it.

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KERBOUA Selma.......

ABSTRACT

The theory of geomtric mean is well established for positive matrices,with more then 40 years old research heritage.In this thesis,we investigate this theory for accretive matrices,extending the notion of geomtric mean from the context of positive matrices to that of accretive matrices.By switching from positive to accretive matrices,it is possible to get new forms that are interest.The main outcomes of this thesis include,but are not limited to,extening the concept of geomtric mean to the context of accretive matrices,extending a number of well-known inequalities in a similar manner,and investigating the numerical radius of accretive matrices.The latter topic is particularly intriguing because it repreents a fresh approach to the discipline and is straightforward for positive matrices.

Keywords: Geomtric mean, accretive matrices, numerical raduis, positive matrices, inequalities.

الملخص

يعود تاريخ نظرية الوسط الهندسي للمصفوفات لاكثر من ٤٠ سنة في مجال البحث وتم من خلالها التطرق للمصفوفات الموجبة. وتهدف هذه الأطروحة الى دراسة المصفوفات ذات الجزء الحقيقي الموجب حيث نقوم بتوسيع مفهوم الوسط الهندسي بالانتقال من المصفوفات الموجبة الى المصفوفات ذات الجزء الحقيقي الموجب. ويسمح هذا الانتقال باعطاء تعميم لنتائج معروفة واشكال جديدة والتي لم تكن تشكل اهمية بالنسبة للمصفوفات الموجبة. وتتركز النتائج الرئيسية لهذه الاطروحة، على سبيل المثال لا الحصر، على توسيع مفهوم الوسط الهندسي للمصفوفات ذات الجزء الحقيقي الموجب، وتوسيع العديد من المتباينات الشائعة وكذا دراسة نصف القطر العددي للمصفوفات ذات الجزء الحقيقي الموجب. ونذكر ان الموضوع ذو اهمية كبيرة لكونه موضوعا سطحيا في دراسة المصفوفات الموجبة.

الكلمات المفتاحية: المتوسط الهندسي، المصفوفات ذات الجزء الحقيقي الموجب، نصف القطر العددي، المصفوفات الموجبة، المتباينات.

RÉSUMÉ

La théorie de la moyenne géométrique des matrices remonte à plus de 40 ans dans le domaine de la recherche, où les matrices positives ont été abordées. Cette thèse vise à étudier les matrices ayant une partie réelle positive, en élargissant le concept de la moyenne géométrique en passant des matrices positives aux matrices ayant une partie réelle positive. Ce passage permet de généraliser des résultats connus et de nouvelles formes qui n'étaient pas importantes pour les matrices positives. Les résultats principaux de cette thèse se concentrent, entre autres, sur l'élargissement du concept de la moyenne géométrique des matrices aux matrices ayant une partie réelle positive, l'élargissement de nombreuses inégalités courantes ainsi que l'étude du demi-rayon numérique des matrices ayant une partie réelle positive. Il convient de noter que ce sujet revêt une grande importance car il s'agit d'un sujet superficiel dans l'étude des matrices positives.

Mots clés : Moyenne géométrique, matrices ayant une partie réelle positive, demi-rayon numérique, matrices positives, inégalités.

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Introduction

Let $\mathcal{M}_n$ stand for the algebra of all $n \times n$ complex matrices. Positive matrices offer numerous desirable characteristics that are regarded extensions of the corresponding characteristics of positive numbers. A matrix $A \in \mathcal{M}_n$ is said to be positive if $\langle Ax, x \rangle > 0$ for all non-zero vectors $x \in \mathbb{C}^n$.

In this literature, numerous academics have expressed an interest in studying inequalities among positive matrices. Due to its significance in the area of matrix theory and its siblings, inequalities among geometric means in particular have drawn the attention of many scholars.

A geometric means is a binary operation with a number of desirable qualities, as we shall see later in this thesis. It is defined by

$$A\sharp_\lambda B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\lambda A^{\frac{1}{2}},$$

where $A, B \in \mathcal{M}_n$ are positive.

When discussing the geometric mean, we have to mention the arithmetic mean and the harmonic mean, which are defined as follows:

$$A\nabla_\lambda B = (1-\lambda)A + \lambda B, A!_\lambda B = ((1-\lambda)A^{-1} + \lambda B^{-1})^{-1},$$

where $0 \leq \lambda \leq 1$.

These three means are related via the relation

$$A!_r B \leq A\sharp_r B \leq A\nabla_r B; 0 \leq r \leq 1.$$

This inequality, also known as the arithmetic-geometric-harmonic mean inequality, is one of the most fundamental in the theory of matrix means. The extension of the geometric mean theory from the context of positive matrices to that of accretive matrices is the focus of this thesis. Those matrices are the ones with positive real parts. That is, $A \in \mathcal{M}_n$ is said to be accretive if $\Re A$ is positive. The first difficulty in this transition is that even if A and B are accretive, then $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is not strictly accretive and is not positive. Because the definition of geometric mean for positive matrices depends on the fact that $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is positive when both A, B are positive. this problem is extremely difficult.

The integral representation of a geometric mean is used to get around this problem. In a manner similar to positive matrices, this detour enables the extension of the geometric mean definition to the context of accretive matrices and enables the discovery of other intriguing inequalities.

It is also important to point out that this thesis helps as a survey of results involving positive matrices, since these results follow as special cases of the new results involving accretive matrices. In the sequel, we shall remind the reader frequently of this fact: where positive versions are revealed from the accretive ones.

There are three chapters in this thesis. We introduce the necessary definitions and attributes in the first chapter so that the reader may understand the thesis with ease. The second chapter, which is where the body of the thesis is found, defines the geometric mean of accretive matrices and covers a wide range of related subjects and inequalities. This will involve the expansion of numerous well-known positive matrices results. The third chapter will discuss inequalities for the numerical radius of accretive matrices will be covered in the third chapter as a novel approach to this area of study. This method is particularly interesting because it studies such inequalities for positive matrices is straightforward because the numerical radius of positive matrices is the same as their operator norm

CHAPTER 1

Basics on matrices

Throughout this chapter we introduce some basic concepts and results. That will be used frequently in this thesis. We refer the reader to (Bhatia, 1997) and (Simon, 1979).

1.1 Basic definitions in matrix theory

In this section, we give some basic properties of matrices.

Definition 1.1.1. *If $A \in \mathcal{M}_n$, then the matrix $A^* : \mathcal{M}_n \to \mathcal{M}_n$ that satisfies*

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \text{ for all } x, y \in \mathbb{C}^n$$

is called the adjoint Hermitian conjugate of A.

Definition 1.1.2. *Let $A \in \mathcal{M}_n$.*

(a) If $A^* = A$, then A is called Hermitian matrix.

(b) If $A^*A = AA^* = I$, then A is called a unitary matrix.

(c) If $\langle Ax, x\rangle \geq 0$ for all $x \in H$, then A is called a positive semi−definite matrix.

(d) If A is positive semi−definite matrix and is invertible, it is called a positive matrix.

(e) If $A^*A = AA^*$, then A is called a normal matrix.

Throughout this thesis, the notation $A \geq 0$ means that the matrix is positive semi−definite. For two Hermition matrices A, B the notation $B \geq A$ or $A \leq B$ means $B - A \geq 0$. On the other hand, the notation $A > 0$ is used to mean that A is positive. The notation $\mathcal{M}_n^+$ is used to denote the class of all positive matrices.

Theorem 1.1.3. *Let* $A \in \mathcal{M}_n$*. If* $A \geq 0$*, then there exists a unique matrix* $B \in \mathcal{M}_n$ *such that* $B \geq 0$ *and* $B^2 = A$*. Here* B *is called the positive square root of* A *and is denoted by* $A^{1/2}(or\ \sqrt{A})$*.*

The following is a basic property.

Theorem 1.1.4. *If* $A \in \mathcal{M}_n$*, then* A^*A *and* AA^* *are positive semi−definite matrices.*

Definition 1.1.5. *If* $A \in \mathcal{M}_n$*, then the absolute value of* A *is the positive square root of the positive matrix* A^*A*, i.e.,* $|A| = (A^*A)^{1/2}$*.*

Definition 1.1.6. *The Cartesian decomposition of matrix* $A \in \mathcal{M}_n$*, is given by*

$$A = \Re A + i\Im A, \ \textit{with}\ \Re A = \frac{A + A^*}{2} \ \textit{and}\ \Im A = \frac{A - A^*}{2i},$$

where $\Re A$ *is the real part of* A *and* $\Im A$ *is the imaginary part of* A*.*

Definition 1.1.7. *The spectrum of a matrix* $A \in \mathcal{M}_n$ *is the set*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \ \textit{is not invertible in}\ \mathcal{M}_n\},$$

where I *is the identity matrix in* $\mathcal{M}_n$*.*

Theorem 1.1.8. (***Spectral Mapping Theorem***) *Let* $A \in \mathcal{M}_n$ *and let* p *be any polynomial. Then*

$$\sigma(p(A)) = p(\sigma(A)).$$

Remark 1.1.9. *Let* $A \in \mathcal{M}_n$*. Then*

$$\sigma(A^*) = \{\bar{\lambda} : \lambda \in \sigma(A)\}.$$

where $\bar{\lambda}$ *is the complex conjugate of* λ*.*

Moreover, if A *is invertible, then*

$$\sigma(A^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(A)\}.$$

Definition 1.1.10. *Let* $\lambda_1, \lambda_2, ..., \lambda_n$ *be the eigenvalues of matrix* A*. Then the trace of* A *is given by*

$$trA = \sum_{i=1}^{n} \lambda_i.$$

Definition 1.1.11. *The singular values of a matrix* $A \in \mathcal{M}_n$*, denoted by* $s_1(A), s_2(A), \cdots, s_n(A)$*, are the eigenvalues of* $|A|$ *arranged in decreasing order, and repeated according to multiplicity. That is, the singular values of* A *are the square roots of the eigenvalues of* A^*A*.*

1.2 Unitarily invariant norms

In this section, we study some classes of norms on $\mathcal{M}_n$ that have the unitary invariance property. These classes proved to be useful in different areas of science such as geometry, physics, analysis, and applied mathematics.

Definition 1.2.1. *A real valued function $N : X \to \mathbb{R}$, where X is real (or complex) vector space is said to be a norm if for every scalar α, and $x, y \in X$ we have:*

(a) $N(x) \geq 0$.

(b) $N(x) = 0$ *if and only if* $x = 0$.

(c) $N(\alpha x) = |\alpha|\, N(x)$.

(d) $N(x + y) \leq N(x) + N(y)$.

Definition 1.2.2. *A norm $|||\cdot|||$ on $\mathcal{M}_n$ is said to be unitarily invariant if*

$$|||UAV||| = |||A|||,$$

for all unitary matrices $U, V \in \mathcal{M}_n$ and all $A \in \mathcal{M}_n$.

Example 1.2.3. *The usual operator (or the spectral) norm, which is denoted by $||\cdot||$ is defined for any $A \in \mathcal{M}_n$ by*

$$||A|| = \sup_{||x||=1} ||Ax|| = \sup_{||x||=||y||=1} |\langle Ax, y\rangle| = s_1(A).$$

When A is Hermition, this norm is given by

$$||A|| = \sup_{||x||=1} |\langle Ax, x\rangle| \,.$$

Example 1.2.4. *The Schatten p-norm of matrix A , denoted by* $||\cdot||_p$, *where* $1 \le p < \infty$ *is defined by*

$$||A||_p = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p} .$$

It is clear that $||A||_p = (\text{tr } |A|^p)^{1/p} = (\text{tr } |A^*|^p)^{1/p}$.

Proposition 1.2.5. *Let* $A \in \mathcal{M}_n$. *Then* $\|A\| = \lim_{p\longrightarrow\infty} \|A\|_p = s_1(A)$.

Here $||A||_2$ is called the Hilbert-Schmidt norm and $||A||_1$ is called the trace norm.

Proposition 1.2.6. *Let* $A \in \mathcal{M}_n$. *Then*

$$||A||_2 = \left(tr\ |A|^2\right)^{1/2} \ \ and \ \ ||A||_1 = tr\ |A| \,.$$

Example 1.2.7. *The Ky Fan k-norm, denoted by* $||\cdot||_{(k)}$, *are defined for any matrix* $A \in \mathcal{M}_n$, *by*

$$||A||_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k = 1, 2, \cdots, n.$$

The following is an important property of unitarily invariant norm

Proposition 1.2.8. *Let* $A \in \mathcal{M}_n$. *We can easily prove the following properties for unitarily invariant norms:*

(a) $|||A||| = |||A^*|||$.

(b) $|||A||| = \left|\left|\left|(A^*A)^{1/2}\right|\right|\right| = |||\ |A|\ |||\,.$

(c) $|||A^*A||| = |||AA^*|||\,.$

(d) $|||AB^*||| = |||\ |A|\,|B|\ |||\,.$

Theorem 1.2.9. *Let $A \in \mathcal{M}_n$ be invertible . Then*

$$\|A\|^{-1} \leq \|A^{-1}\|. \tag{1.2.1}$$

Theorem 1.2.10. *Let $A, B \in \mathcal{M}_n^+$. Then*

$$\parallel AB \parallel \leq \frac{1}{4} \parallel (A+B)^2 \parallel . \tag{1.2.2}$$

For further results on unitarily invariant norms, we refer to (Bhatia, 1997) and (Simon, 1979).

1.3 Matrix monotone and matrix convex functions

In this part, we examine matrix monotone functions, a unique and significant class of functions with a number of unique characteristics. These characteristics have a strong connection to those of the matrix convex functions. We'll study both courses.

First, we need to remind the reader of the meaning of $f(A)$, when A is any matrix.

Let $f : \mathcal{D} \to \mathbb{C}$ be an analytic complex function on a complex domain $\mathcal{D}$. The Cauchy integral formula assures that for $a \in \mathcal{D}$,

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz,$$

where Γ is a simple closed curve in $\mathcal{D}$ that winds once around a. Extending this definition to matrices is possible using the Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z\mathcal{I} - A)^{-1} dz, \tag{1.3.1}$$

where Γ is a simple closed curve in the definition of A that winds once around each eigenvalue of A. Of course, Γ must lie in $\mathbb{D}$.

For example, letting $f : \mathbb{C} \backslash (-\infty, 0] \to \mathbb{C}$ be $f(z) = z^{\lambda}, 0 < \lambda < 1$, we define

$$A^{\lambda} = \frac{1}{2\pi i} \int_{\Gamma} z^{\lambda}(z\mathcal{I} - A)^{-1} dz, \tag{1.3.2}$$

where Γ is any closed curve avoiding $(-\infty, 0]$ in the resolvent of A, so that Γ winds once around each eigenvalue of A.

So, fractional powers are not only defined for positive matrices. They can be defined for any matrix whose eigenvalues are not in $(-\infty, 0]$.

Definition 1.3.1. *Let $f : J \to \mathbb{R}$ be a function where $J \subseteq \mathbb{R}$ is an interval. Then f is called matrix monotone on J if for any two Hermition matrices A and B with spectra contained in J, $A \leq B$ implies that $f(A) \leq f(B)$.*

The following function is an example of a function that is not matrix monotone, but it is monotone as a real function.

Example 1.3.2. *Consider the function $f(x) = x^2$, where $x \geq 0$. Let A and B be the 2×2 matrices defined as $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. It is clear that $B - A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ that is $B \geq A$. But $B^2 - A^2 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ is not positive semi-definite, since it has a negative* *eigenvalue, so* $B^2 \not\geq A^2$.

Definition 1.3.3. *Let* $f : J \to \mathbb{R}$ *be a function where* $J \subseteq \mathbb{R}$ *is an interval. Then* f *is called matrix convex if for any Hermition matrices* A *and* B *with spectra contained in* J, *we have*

$$f(\alpha A + (1-\alpha)B) \leq \alpha f(A) + (1-\alpha)f(B)$$

for all $\alpha \in [0, 1]$.

The following function is an example of a function that is not matrix convex, but it is convex as a real function.

Example 1.3.4. *Consider the function* $f(x) = x^3, x \geq 0$. *Let* A *and* B *be the* 2×2 *matrices defined as* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. *Then* A *and* B *are positive and*

$$\frac{A^3 + B^3}{2} - \left(\frac{A+B}{2}\right)^3 = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$$

which is not positive since it has a negative eigenvalue. So $\frac{A^3+B^3}{2} \not\leq \left(\frac{A+B}{2}\right)^3$, *that is* f *is not matrix convex, on* $[0, \infty)$.

Example 1.3.5. *The function* $f(x) = ax+b$ *where* $a, b \in \mathbb{R}$ *and* $a \geq 0$ *is matrix monotone function. It is matrix convex for every* $a, b \in \mathbb{R}$.

Definition 1.3.6. *Let* $f : J \to \mathbb{R}$ *be a function where* $J \subseteq \mathbb{R}$ *is an interval. Then* f *is called matrix concave if for any Hermition matrices* A *and* B *with spectra contained in* J, *we have*

$$f(\alpha A + (1-\alpha)B) \geq \alpha f(A) + (1-\alpha)f(B), \tag{1.3.3}$$

for all $\alpha \in [0,1]$.

Remark 1.3.7. *f is called matrix concave if and only if $-f$ is matrix convex.*

Although convex(concave) functions have no relations in generale with monotone functions, the following proposition gives an interesting relation, when talking about matrix convexity (concavity). See (Uchiyama, 2010) and (Ando and Hiai, 2011) for details.

Proposition 1.3.8. *Let $f : (0,\infty) \to [0,\infty)$ be continuous. Then*

(i) f is matrix monotone decreasing if and only if f is matrix convex and $f(\infty) < \infty$.

(ii) f is matrix monotone increasing if and only if f is matrix concave.

1.4 Numerical range and numerical radius

Definition 1.4.1. *The numerical range of matrix $A \in \mathcal{M}_n$ is the subset of the complex plane $\mathbb{C}$, given by*

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C},\ \|x\| = 1\}.$$

Notice that when $A \in \mathcal{M}_n^+$, then $W(A) \subseteq (0,\infty)$.

The following are basic properties of the numerical range.

Theorem 1.4.2. *Let $A, B \in \mathcal{M}_n$, and let $\alpha, \beta \in \mathbb{C}$. Then*

(a) $W(\alpha I + \beta A) = \alpha + \beta W(A)$.

(b) $W(A + B) \subseteq W(A) + W(B)$.

(c) $W(A^*) = \{\bar{\lambda} : \lambda \in W(A)\}$.

(d) $W(UAU^*) = W(A)$ *for all unitary matrices* $U \in \mathcal{M}_n$.

(e) The numerical range of matrix $A \in \mathcal{M}_n$ is a compact set.

Theorem 1.4.3. (***Toeplitz-Hausdorff***) *The numerical range of matrix $A \in \mathcal{M}_n$ is a convex subset of $\mathbb{C}$.*

Theorem 1.4.4. (***Spectral Inclusion***) *The spectrum of matrix $A \in \mathcal{M}_n$ is contained in its numerical range. That is,*

$$\sigma(A) \subseteq W(A).$$

It follows from the previous theorem that **conv** $\sigma(A) \subseteq W(A)$ holds for any matrix $A \in \mathcal{M}_n$, where **conv** $\sigma(A)$ is the convex hull of the set $\sigma(A)$. The next theorem asserts that this containment becomes an equality when A is normal.

Theorem 1.4.5. *If $A \in \mathcal{M}_n$ is normal, then*

$$\textbf{\textit{conv}}\ \sigma(A) = W(A).$$

Among the important scalar quantities related to matrices, is the so called numerical raduis, defined as follows.

Definition 1.4.6. *The numerical radius of matrix $A \in \mathcal{M}_n$ is given by*

$$w(A) = \sup_{\lambda \in W(A)} |\lambda| = \sup_{x \in \mathbb{C}^n,\ \|x\|=1} |\langle Ax, x\rangle| \,.$$

It is well known that $w(\cdot)$ defines a norm on $\mathcal{M}_n$ that is equivalent to the operator norm, via the relation see (Halmos, 1982).

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|, A \in \mathcal{M}_n. \tag{1.4.1}$$

It is easy to verify that $w(\cdot)$ defines a norm on $\mathcal{M}_n$. This norm is weakly unitarily invariant ($i.e., w(A) = w(UAU^*)$ for all matrices A and all unitary matrices U in $\mathcal{M}_n(\mathbb{C})$ and satisfies $w(A) = w(A^*)$ for all $A \in \mathcal{M}_n$.

The following useful theorem provides an alternative way to compute the numerical radius of matrix.

Theorem 1.4.7. *Let $A \in \mathcal{M}_n$. Then*

$$\begin{aligned} w(A) &= \sup_{\theta \in \mathbb{R}} \left\| \Re(e^{i\theta} A) \right\| \\ &= \sup_{\theta \in \mathbb{R}} \left\| \Im(e^{i\theta} A) \right\| \end{aligned}$$

Now we give some properties for the numerical radius.

Theorem 1.4.8. *Let $A, B \in \mathcal{M}_n$. Then*

(a) $w(A + B) \leq w(A) + w(B)$.

(b) $w(AB) \leq 4w(A)w(B)$.

(c) If $AB = BA$, then $w(AB) \leq 2w(A)w(B)$.

(d) If A is normal, then $w(A) = \|A\|$.

(e) If A, B are normal, then $w(AB) \leq w(A)w(B)$.

(f) $w(A) = w(UAU^*)$ for all matrice A and all unitary matrice U in $\mathcal{M}_n$.

(g) $w(A) = w(A^*)$ for all $A \in \mathcal{M}_n$.

Proposition 1.4.9. *Let $A, B \in \mathcal{M}_n$ be positive semi− definite. Then*

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{2}\|A+B\|. \tag{1.4.2}$$

Proposition 1.4.10. *Let $A \in \mathcal{M}_n$. Then*

$$w(\Re A) \leq w(A). \tag{1.4.3}$$

Proof :

$$w(\Re A) = w(\frac{A+A^*}{2}) \leq \frac{w(A)+w(A^*)}{2} = w(A).$$

□

1.5 Positive linear maps

In this section, we state some fundamental properties of possitive linear maps.

The definition of unital positive linear maps is as follows:

Definition 1.5.1. *A map $\Phi : \mathcal{M}_n \to \mathcal{M}_r$ is linear if it is additive and homogeneous, i.e. $\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$ for any $\lambda, \mu \in \mathbb{C}$ and for any $A, B \in \mathcal{M}_n$.*

A linear map $\Phi : \mathcal{M}_n \to \mathcal{M}_r$ is positive if it preserves the matrix order $\geq$, i.e. $A \in \mathcal{M}_n^+$ implies $\Phi(A) \in \mathcal{M}_r^+$.

A linear map $\Phi : \mathcal{M}_n \to \mathcal{M}_r$ is called unital if it preserves the identity matrix, i.e. if $\Phi(I) = I$.

Proposition 1.5.2. *Let $\Phi : \mathcal{M}_n \to \mathcal{M}_r$ be a positive linear map. Then*

$A \le B$ *implies* $\Phi(A) \le \Phi(B)$.

$\Phi(A^*) = \Phi^*(A)$.

The following two propositions are given in (Furuta et al, 2005), (Bhatia, 2007), respectively.

Proposition 1.5.3. *Let Φ be a positive linear map. Then for any matrices $A, B \in \mathcal{M}_n^+$,*

$$\Phi(B)\Phi(A)^{-1}\Phi(B) \le \Phi(BA^{-1}B). \tag{1.5.1}$$

Proposition 1.5.4. *Let $A \in \mathcal{M}_n^+$ and Φ be a positive linear map. Then*

$$\Phi(A^{-1}) \ge \ \Phi^{-1}(A). \tag{1.5.2}$$

1.6 Geometric mean of positive matrices

In 1980, Kubo and Ando introduced the notion of a matrix means between two positive matrices (kubo and Ando, 1980). A matrix mean σ on $\mathcal{M}_n^+$ is a binary operation $A\sigma B$ satisfying the following requirements:

- $A \le C$ and $B \le D$ imply $A\sigma B \le C\sigma D$; for any $A, B, C, D \in \mathcal{M}_n^+$.
- $C^*(A\sigma B)C = (C^*AC)\sigma(C^*BC)$; for any $A, B \in \mathcal{M}_n^+$ and any invertible $C \in \mathcal{M}_n$.
- $A_k \downarrow_k A$ and $B_k \downarrow_k B$ imply $(A_k\sigma B_k) \downarrow_k (A\sigma B)$; for any $A_k, B_k, A, B \in \mathcal{M}_n^+$.
- $I\sigma I = I$.

Standard examples of matrix means are given by (Pusz and Woronowicz, 1975).

- The weighted arithmetic mean $A\nabla_\lambda B = (1-\lambda)A + \lambda B$,

- The weighted Harmonic mean $A!_\lambda B = ((1-\lambda)A^{-1} + \lambda B^{-1})^{-1}$,
- The weighted geometric mean $A\sharp_\lambda B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\lambda}A^{\frac{1}{2}}$,

where $A, B \in \mathcal{M}_n^+$ and $0 \le \lambda \le 1$. When $\lambda = \frac{1}{2}$, we drop λ from the above notations, and we simply write $\nabla, !$ and $\sharp$.

In particular, we have $!_\lambda \le \sharp_\lambda \le \nabla_\lambda$, see (Ando, 1978). That is, if $A, B \in \mathcal{M}_n^+$, then

$$A!_\lambda B \le A\sharp_\lambda B \le A\nabla_\lambda B, \;\; 0 \le \lambda \le 1.$$

Proposition 1.6.1. *Let $A, B \in \mathcal{M}_n$ be positive, then*

$$|||A\sharp_\lambda B||| \le |||A|||\sharp_\lambda|||B|||. \tag{1.6.1}$$

Proposition 1.6.2. *Let $A, B \in \mathcal{M}_n^+$, then for any unit vector x,*

$$\langle (A\sharp_\lambda B)x, x\rangle \le \langle Ax, x\rangle \sharp_\lambda \langle Bx, x\rangle . \tag{1.6.2}$$

Proposition 1.6.3. *(**Ando's inequality**) Let $A, B \in \mathcal{M}_n^+$ and Φ be a positive unital linear map. Then*

$$\Phi(A\sharp_\lambda B) \le \Phi(A)\sharp_\lambda\Phi(B). \tag{1.6.3}$$

Proposition 1.6.4. *Let $A, B \in \mathcal{M}_n^+$. Then for any positive unital linear map Φ, it holds*

$$\Phi(A\sharp_\lambda B) \ge \Phi(A)\sharp_\lambda\Phi(B), \;\lambda \in (-1, 0), \tag{1.6.4}$$

providing a reversed version of (1.6.3).

Proposition 1.6.5. *Let $A, B \in \mathcal{M}_n$ be such that $0 < mI \leq A, B \leq MI$, for some positive scalars m, M, and let $f, g \in \mathfrak{m}$. Then for every unital positive linear map Φ,*

$$\Phi^2(\frac{A+B}{2}) \leq K(h)^2\Phi^2(A\sharp_\lambda B), \tag{1.6.5}$$

where $h = \frac{M}{m}$ and $k(h) = \frac{(h+1)^2}{4h}$ is the well known Kantorovich constant.

Proposition 1.6.6. *Let $A, B \in \mathcal{M}_n^+$. Then for $\lambda \in (-1, 0)$,*

$$(1-\lambda)A + \lambda B \leq A\sharp_\lambda B. \tag{1.6.6}$$

The theory of geometric mean for positive matrices has been well developed and studied in the literature. We refer the reader to (Ando, 1978), (kubo and Ando, 1980), (Nishio and Ando, 1976), (Mathias, 1992), (Ando, 1979), (Furuta, 2002) and (Furuta and Yanagide, 1998) as a sample of articles treating this topic.

CHAPTER 2

Some new inequalities for geometric mean

2.1 Accretive matrices

Our goal in this section is about introducing means of a special class of matrices, that is wider than that of positive ones. namely, accretive matrices.

Definition 2.1.1. *A matrix $A \in \mathcal{M}_n$ is said to be accretive if its real part is positive (i.e., $\Re A > 0$.)*

When studying properties of accretive matrices, it is necessary to recall the definition of sectorial matrices.

Definition 2.1.2. *For $0 \leq \alpha < \frac{\pi}{2}$, we define the sector*

$$S_\alpha = \{z \in \mathbb{C} : \Re(z) > 0, |\Im(z)| \leq \tan(\alpha)\Re(z)\}.$$

A matrix A whose numerical range is a subset of a sector S_α, is called a sectorial matrix. It is clear that a sectorial matrix is necessarily accretive, in this thesis, it will be implicitly understood that the notions of S_α is defined only when $0 \leq \alpha < \frac{\pi}{2}$.

Remark 2.1.3. *The numerical range $W(A)$ of an accretive matrix A satisfies*

$$W(A) := \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\} \subset \textit{the right half complex plane.}$$

In the following, we list some properties of accretive (sectorial) matrices.

Proposition 2.1.4. *The class of accretive matrices is*

(1) a convex cone of $\mathcal{M}_n$.

(2) invariant under inversion.

It is well known that for any matrix $A \in \mathcal{M}_n$, $|||\Re A||| \leq |||A|||$, for any unitarily invariant norm $|||\cdot|||$ on $\mathcal{M}_n$. The following lemma presents a reversed version of this inequality for sectorial matrices see (Zhang, 2015).

Proposition 2.1.5. *Let $A \in \mathcal{M}_n$ be accretive matrix such that $W(A) \subset S_\alpha$ and let $|||\cdot|||$ be any unitarily invariant norm on $\mathcal{M}_n$. Then*

$$\cos\alpha\, |||A||| \leq \ |||\Re(A)||| \leq |||A|||.$$

Other intersting needed properties of accretive matrices are given in (Lin, 2015), (Drury, 2014), (Choi et al, 2019) and (Drury 2015) as follows.

Proposition 2.1.6. *If $A \in \mathcal{M}_n$ is accretive, then*

$$\Re(A^{-1}) \leq (\Re A)^{-1}$$

This inquality can be reversed as follows.

Lemma 2.1.7. *If $A \in \mathcal{M}_n$ is accretive with $W(A) \subset S_\alpha$, then*

$$\sec^2(\alpha)\ \ \Re(A^{-1}) \geq (\Re A)^{-1}$$

Lemma 2.1.8. *Let $A \in \mathcal{M}_n$ is accretive with $W(A) \subset S_\alpha$ and $t \in (0,1)$. Then $W(A^t) \subset S_{t\alpha}$.*

Also note that $W(A^{-t}) \subset S_{t\alpha}$. This follows because $W(A^{-1}) \subset S_\alpha$ when $W(A) \subset S_\alpha$.

2.2 Properties of geometric mean

In this section,we explore more properties of the geometric mean of accretive matrices.

Definition 2.2.1. *let A,$B \in \mathcal{M}_n$ be accretive matrices, then*

$$A\sharp_\lambda B = \int_0^1 A!_t B \; dv_\lambda(t) \ \ ,0 < \lambda < 1$$

Where v_λ is a probability measure on $[0,1]$ given by $dv_\lambda(t) = \dfrac{\sin(\lambda\pi)}{\pi}\dfrac{t^{\lambda-1}}{(1-t)^\lambda}dt$

Corollary 2.2.2. *Let $A, B \in \mathcal{M}_n$ be accretive matrices. Then for $\lambda \in (0,1)$,*

$$(A\sharp_\lambda B)^{-1} = A^{-1}\sharp_\lambda B^{-1}.$$

Theorem 2.2.3. *Let $A, B \in \mathcal{M}_n$ be accretive matrices and $0 < \lambda < 1$, then*

$$\Re(A!_\lambda B) \geq (\Re A)!_\lambda(\Re B)$$

Proof : Consider the matrix convex function $f(A) = (\Re(A^{-1}))^{-1}$, then

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)f(A) + \lambda f(B)$$

$$(\Re((1-\lambda)A + \lambda B)^{-1})^{-1} \leq (1-\lambda)(\Re A^{-1})^{-1} + \lambda(\Re B^{-1})^{-1}$$

$$(\Re((1-\lambda)A^{-1} + \lambda B^{-1})^{-1})^{-1} \leq (1-\lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1}$$

$$\Re((1-\lambda)A^{-1} + \lambda B^{-1})^{-1} \geq ((1-\lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1})^{-1}$$

$$\Re(A!_\lambda B) \geq (\Re A)!_\lambda(\Re B),$$

this completes the proof. □

Theorem 2.2.4. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$. Then for $0 < \lambda < 1$*

$$\Re(A!_\lambda B) \leq \sec^2(\alpha)(\Re A)!_\lambda(\Re B) \tag{2.2.1}$$

Proof : Let A, B be sectorial matrices, we have

$$\begin{aligned}\Re(A!_\lambda B) &= \Re((1-\lambda)A^{-1} + \lambda B^{-1})^{-1} \\ &\leq ((1-\lambda)\Re A^{-1} + \lambda \Re B^{-1})^{-1} \\ &\leq (\cos^2(\alpha)(1-\lambda)(\Re A)^{-1} + \cos^2(\alpha)\lambda(\Re B)^{-1})^{-1} \\ &\leq \sec^2(\alpha)\ ((1-\lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1})^{-1} \\ &= \sec^2(\alpha)\ (\Re A)!_\lambda(\Re B),\end{aligned}$$

which complete the proof □

Theorem 2.2.5. *Let $A, B \in \mathcal{M}_n$ be accrative matrices and let $0 < \lambda < 1$,then*

$$\Re(A\sharp_\lambda B) \geq (\Re A)\sharp_\lambda(\Re B)$$

Proof : By Definition 2.2.1, we have

$$A\sharp_\lambda B = \int_0^1 A!_\lambda B\ dv(t)$$
$$\Re(A\sharp_\lambda B) = \int_0^1 \Re(A!_\lambda B)\ dv(t) \geq \int_0^1 (\Re A)!_\lambda(\Re B)\ dv(t) \geq (\Re A)\sharp_\lambda(\Re B)$$

which complete the proof □

Theorem 2.2.6. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$.then for $0 < \lambda < 1$*

$$\Re(A\sharp_\lambda B) \leq \sec^2(\alpha)(\Re A)\sharp_\lambda(\Re B)$$

Proof : we have

$$\Re(A\sharp_\lambda B) = \int_0^1 \Re(A!_t B)dv(t) \leq \int_0^1 \sec^2(\alpha)(\Re A)!_t(\Re B)dv(t) = \sec^2(\alpha)(\Re A)\sharp_\lambda(\Re B).$$

This completes the proof. □

Theorem 2.2.7. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$, then for $0 < \lambda < 1$*

$$|||A\sharp_\lambda B||| \leq \sec^2(\alpha)\ |||A|||\sharp_\lambda|||B|||$$

Proof : Let $A, B \in \mathcal{M}_n$ be sectorial matrices, then

$$|||A\sharp_\lambda B||| \leq \sec(\alpha)\ |||\Re(A\sharp_\lambda B)||| \leq \sec^3(\alpha)\ |||(\Re A)\sharp_\lambda(\Re B)|||$$
$$\leq \sec^3(\alpha)\ |||\Re A|||\sharp_\lambda|||\Re B||| \leq \sec^3(\alpha)\ |||A|||\sharp_\lambda|||B|||.$$

Finally we have,

$$|||A\sharp_\lambda B||| \leq \sec^3(\alpha)\ |||A|||\sharp_\lambda|||B|||$$

□

Theorem 2.2.8. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$. Then for $0 < \lambda < 1$*

$$\cos^2(\alpha)\Re(A!_\lambda B) \leq \Re(A\sharp_\lambda B) \leq \sec^2(\alpha)\Re(A\nabla_\lambda B). \tag{2.2.2}$$

Proof : First,

$$\Re(A\sharp_\lambda B) \leq \sec^2(\alpha)(\Re A\sharp_\lambda \Re B) \leq \sec^2(\alpha)\Re(A)\nabla_\lambda \Re(B) = \sec^2(\alpha)\Re(A\nabla_\lambda B).$$

Thus, we have shown the first inequality. To show the second inequality, we have

$$\Re(A\sharp_\lambda B) \geq \Re(A)\sharp_\lambda \Re(B) \geq \Re(A)!_\lambda \Re(B) \geq \cos^2(\alpha)\ \Re(A!_\lambda B).$$

This shows the second desired inequality, and the proof is complete. □

Lemma 2.2.9. *Let $A, B \in \mathcal{M}_n$ be accretive matrices. Then for $\lambda \in (0,1)$,*

$$\cos^3\alpha\ |||A\sharp B||| \leq |||\mathcal{H}_\lambda(A,B)||| \leq \frac{\sec^3\alpha}{2}\ |||A+B|||, \tag{2.2.3}$$

for any unitarily invariant norm on $\mathcal{M}_n$.

2.3 Geometric mean with negative power

In this part, we discuss geometric mean, for $\lambda \in (-1,0)$.

Definition 2.3.1. *Let $A, B \in \mathcal{M}_n$ be accretive and let $\lambda \in (-1,0)$. Then*

$$\begin{aligned} A\sharp_\lambda B &= \int_0^1 \left[\frac{A}{s} - \frac{1-s}{s}(A!_s B)\right] d\nu(s) \\ &= \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\nu(s), \end{aligned}$$

for some probability mesure $\nu(s)$ on $[0,1]$

Now we have the reversed version of Theorem 2.2.5.

Theorem 2.3.2. *Let $A, B \in \mathcal{M}_n$ be accretive and let $\lambda \in (-1, 0)$. Then*

$$\Re(A\sharp_\lambda B) \leq \Re A\sharp_\lambda \Re B. \tag{2.3.1}$$

Proof : We have

$$\begin{aligned}
\Re(A\sharp_\lambda B) &= \Re\left(\int_0^1 \left[\frac{A}{s} - \frac{1-s}{s}(A!_s B)\right] d\nu(s)\right) && \text{(by Definition 2.3.1)} \\
&= \int_0^1 \left[\frac{\Re A}{s} - \frac{1-s}{s}\Re(A!_s B)\right] d\nu(s) \\
&\leq \int_0^1 \left[\frac{\Re A}{s} - \frac{1-s}{s}(\Re A!_s \Re B)\right] d\nu(s) && \text{(by Theorem 2.2.3)} \\
&= \Re A\sharp_\lambda \Re B, && \text{(by Definiition 2.3.1)}
\end{aligned}$$

completing the proof. □

Proposition 2.3.3. *Let $A, B \in \mathcal{M}_n$ be accretive matrices. Then for $-1 < \lambda < 0$*

$$A\sharp_\lambda B = A(A^{-1}\sharp_{-\lambda} B^{-1})A \tag{2.3.2}$$

Theorem 2.3.4. *Let $B \in \mathcal{M}_n$ be accretive matrix such that $W(B) \subset S_\alpha$ and $A \in \mathcal{M}_n^+$. Then for $-1 < \lambda < 0$*

$$\cos^{2\lambda}\alpha\,(\Re A\sharp_\lambda \Re B) \leq \Re(A\sharp_\lambda B). \tag{2.3.3}$$

Proof : For $\lambda \in (-1,0)$, we have for any vector $x \in \mathbb{C}^n$,

$$\begin{aligned}
\langle \Re(A\sharp_\lambda B)x, x\rangle &= \Re\left\langle A(A^{-1}\sharp_{-\lambda}B^{-1})Ax, x\right\rangle \\
&= \left\langle \Re(A^{-1}\sharp_{-\lambda}B^{-1})Ax, Ax\right\rangle \\
&\geq \left\langle (\Re(A^{-1})\sharp_{-\lambda}\Re(B^{-1}))Ax, Ax\right\rangle \qquad \text{(by Theorem 2.2.5)} \\
&\geq \left\langle ((\Re A)^{-1}\sharp_{-\lambda}\cos^2\alpha\ (\Re B)^{-1})Ax, Ax\right\rangle \qquad \text{(by Proposition 2.1.6)} \\
&= \cos^{2\lambda}\alpha\ \left\langle (\Re A((\Re A)^{-1}\sharp_{-\lambda}(\Re B)^{-1})\Re A)x, x\right\rangle \qquad \text{(since } A = \Re A) \\
&= \cos^{2\lambda}\alpha\ \langle (\Re A\sharp_\lambda \Re B)x, x\rangle .
\end{aligned}$$

This completes the proof. □

Next, we present an accretive verision of (1.6.6).

Theorem 2.3.5. *Let $B \in \mathcal{M}_n$ be accretive matrix such that $W(B) \subset S_\alpha$ and $A \in \mathcal{M}_n^+$. Then for $-1 < \lambda < 0$*

$$\cos^{2\lambda}\alpha\ \Re((1-\lambda)A + \lambda B) \leq \Re(A\sharp_\lambda B). \tag{2.3.4}$$

Proof : By (1.6.6) and Theorem 2.3.4, we have

$$\cos^{2\lambda}\alpha\ \Re((1-\lambda)A+\lambda B) = \cos^{2\lambda}\alpha\ ((1-\lambda)\Re A + \lambda\Re B) \leq \cos^{2\lambda}\alpha\ (\Re A\sharp_\lambda\Re B) \leq \Re(A\sharp_\lambda B),$$

completing the proof. □

2.4 geometric mean and positive linear maps

In this section, we present some inequalities related to the geometric mean and positive linear maps of accretive matrices. For this purpose, we notice that if Φ is a unital positive linear map and A is any matrix, then

$$\Re\Phi(A) = \Phi(\Re A). \tag{2.4.1}$$

Now we state the sectorial version of (1.6.3).

Theorem 2.4.1. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$. and let Φ be a unital positive linear map. Then for $0 < \lambda < 1$*

$$\Re\Phi(A\sharp_\lambda B) \leq \sec^2\alpha\, \Re\left(\Phi(A)\sharp_\lambda \Phi(B)\right). \tag{2.4.2}$$

Proof : Using the definition of $A\sharp_\lambda B$, we get

$$\begin{aligned}
\cos^2\alpha\, \Re\Phi(A\sharp_\lambda B) &= \Phi(\cos^2\alpha\, \Re(A\sharp_\lambda B)) \quad (\text{by } (2.4.1)) \\
&\leq \Phi(\Re A\sharp_\lambda \Re B) \qquad (\text{by Theorem } 2.2.6) \\
&\leq \Phi(\Re A)\sharp_\lambda \Phi(\Re B) \quad (\text{by } (1.6.3)) \\
&= \Re\Phi(A)\sharp_\lambda \Re\Phi(B) \quad (\text{by } (2.4.1)) \\
&\leq \Re(\Phi(A)\sharp_\lambda \Phi(B)) \quad (\text{by Theorem } 2.2.5),
\end{aligned}$$

which completes the proof. □

As an application of Theorem 2.4.1, we present the following accretive version of Proposition 1.6.2.

Corollary 2.4.2. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$, then for any unit vector x, we have*

$$\Re\left\langle (A\sharp_\lambda B)x, x\right\rangle \leq \sec^2\alpha\, \Re\left(\langle Ax, x\rangle \sharp_\lambda \langle Bx, x\rangle\right). \tag{2.4.3}$$

Proof : Letting $\Phi(A) = \langle Ax, x\rangle$ in Theorem 2.4.1. Then Φ is a unital positive linear map and

$$\cos^2\alpha\, \Re\Phi(A\sharp_\lambda B) \leq \Re\left(\Phi(A)\sharp_\lambda\Phi(B)\right) \Rightarrow \cos^2\alpha\, \Re\left\langle (A\sharp_\lambda B)x, x\right\rangle \leq \Re\left(\langle Ax, x\rangle \sharp_\lambda \langle Bx, x\rangle\right),$$

which completes the proof □

Theorem 2.4.3. *Let $A, B \in \mathcal{M}_n$ be accretive matrices with $W(A), W(B) \subset S_\alpha$ such that $0 < mI \leq \Re A, \Re B \leq MI$. Then for every unital positive linear map Φ,*

$$\| \Phi(\Re(A\nabla_\lambda B)\Phi^{-1}(\Re(A\sharp_\lambda B)) \| \leq \sec^4\alpha\, K(h), \tag{2.4.4}$$

where $K(h) = \dfrac{(M+m)^2}{4Mm}$ and $\|\cdot\|$ denote the usual operator norm.

Proof : Since $0 < mI \leq \ \Re A \leq MI$,

we have

$$(MI - \Re A)(mI - \Re A)(\Re A)^{-1} \leq 0,$$

which is equivalent to

$$\Re A + Mm(\Re A)^{-1} \leq (M+m)I,$$

since $(\Re A)^{-1} \geq \Re A^{-1}$, we have

$$(1-\lambda)\Re A + (1-\lambda)Mm\Re A^{-1} \leq (1-\lambda)(M+m)I. \tag{2.4.5}$$

Similarly

$$\lambda\Re B + \lambda Mm\Re B^{-1} \leq \lambda(M+m)I. \tag{2.4.6}$$

Adding (2.4.5) and (2.4.6), we get

$$\Re(A\nabla_\lambda B) + Mm\Re(A!_\lambda B)^{-1} \leq \ (M+m)I. \tag{2.4.7}$$

By computation, we have

$$\begin{aligned}
& \| \sec^4\alpha\ Mm\Phi(\Re(A\nabla_\lambda B))\Phi^{-1}(\Re(A\sharp_\lambda B)) \| \\
\leq\ & \frac{1}{4} \| \sec^4\alpha\ \Phi(\Re(A\nabla_\lambda B)) + Mm\Phi^{-1}(\Re(A\sharp_\lambda B)) \|^2 && \text{(by Theorem 1.2.10)} \\
\leq\ & \frac{1}{4} \| \sec^4\alpha\ \Phi(\Re(A\nabla_\lambda B)) + Mm\Phi\left((\Re(A\sharp_\lambda B))^{-1}\right) \|^2 && \text{(by (1.5.2))} \\
\leq\ & \frac{1}{4} \| \sec^4\alpha\ \Phi(\Re(A\nabla_\lambda B)) + \sec^2(\alpha)Mm\Phi((\Re(A!_\lambda B))^{-1}) \|^2 && \text{(by (2.2.2))} \\
\leq\ & \frac{1}{4} \| \sec^4\alpha\ \Phi(\Re(A\nabla_\lambda B)) + \sec^4\alpha\ Mm\Phi(\Re(A!_\lambda B)^{-1}) \|^2 \\
\leq\ & \frac{1}{4} \| \sec^4\alpha\ \Phi(\Re(A\nabla_\lambda B) + Mm\Re(A!_\lambda B)^{-1}) \|^2 \\
\leq\ & \frac{1}{4}\sec^8\alpha\ (M+m)^2. && \text{(by(2.4.7))}
\end{aligned}$$

That is

$$\| \Phi(\Re(A\nabla_\lambda B))\Phi^{-1}(\Re(A\sharp_\lambda B)) \| \leq \sec^4\alpha\ K(h),$$

which completes the proof.

□

Now we can present the accretive version of (1.6.4).

Theorem 2.4.4. *Let B be accretive matrices such that $W(B) \subset S_\alpha$ and $A > 0$ and let Φ be a positive unital linear map. Then for $\lambda \in (-1, 0)$,*

$$\cos\alpha\ \Re(\Phi(A)\sharp_\lambda\Phi(B)) \leq \Re\Phi(A\sharp_\lambda B). \tag{2.4.8}$$

Proof : By Theorem 2.3.4 and then (1.6.4), we have

$$\Re\Phi(A\sharp_\lambda B) = \Phi(\Re(A\sharp_\lambda B)) \geq \cos^{2\lambda}\alpha\ \Phi(\Re A\sharp_\lambda\Re B) \geq \cos^{2\lambda}\alpha\Phi(\Re A)\sharp_\lambda\Phi(\Re B).$$

This, with Theorem 2.3.2, yield

$$\Re\Phi(A\sharp_\lambda B) \geq \cos^{2\lambda}\alpha\Re(\Phi(A)\sharp_\lambda\Phi(B)),$$

completing the proof.

When $B \in \mathcal{M}_n^+$, then α can be taken as $\alpha = 0$, which then retrieves (1.6.4) as a special case of Theorem 2.4.4. □

CHAPTER 3

The numerical radii of accretive matrices

In this chapter, we present many new inequalities for the numerical radius of accretive matrices. The importance of this study is the presence of a new approach that treats a specific class of matrices, namely the accretive ones. While some of these inequalities can be considered as refinements of other existing ones, others present new insights to some known results for positive matrices.

3.1 Accretive versions of some known numerical radius inequalities

First, we have the simple sectorial version of (1.4.1).

Proposition 3.1.1. *Let A be accretive matrices such that $W(A) \subset S_\alpha$. Then*

$$\cos\alpha \,\|A\| \le w(A) \le \|A\| \tag{3.1.1}$$

Proof : Noting that $w(\Re A) = \|\Re A\|$, since $\Re A > 0$, Proposition 2.1.5 implies

$$\cos\alpha \,\|A\| \le \|\Re A\| = w(\Re A) \le w(A) \le \|A\|.$$

This completes the proof. □

Remark 3.1.2. *Notice that when $0 < \alpha < \frac{\pi}{3}$, $\cos\alpha > \frac{1}{2}$. This means that, for such α,*

$$\frac{1}{2}\|A\| < \cos\alpha \,\|A\| \le w(A) \le \|A\|,$$

providing a considerable refinement of the left inequality in (1.4.1).

While $w(\Re A) \le w(A)$ for any matrix A, a reversed version can be found via sectorial matrices, as follows.

Corollary 3.1.3. *Let A be accretive matrices such that $W(A) \subset S_\alpha$. Then*

$$w(A) \leq \sec\alpha \, w(\Re A). \tag{3.1.2}$$

Proof : Let A be accretive matrices such that $W(A) \subset S_\alpha$. Then $w(\Re A) = \|\Re A\|$, since $\Re A > 0$. Proposition 3.1.1 implies

$$w(A) \leq \|A\| \leq \sec\alpha \, \|\Re A\| = \sec\alpha \, w(\Re A).$$

□

Notice that when A is positive, then $\alpha = 0$ and the above inequality becomes $w(A) \leq w(\Re A)$, which then implies $w(A) = w(\Re A)$, as well known for positive matrices.

In the following result, we present a new submultiplicative inequality for the numerical radius. Recall that for general $A, B \in \mathcal{M}_n$, one has $w(AB) \leq 4w(A)w(B)$. When A and B commute, the factor 4 can be reduced to 2, while it can be reduced to 1 when A and B are normal, see (Halmos, 1982). The following result presents a bound when A and B are sectorial with $0 < \alpha < \frac{\pi}{3}$.

Theorem 3.1.4. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$. Then*

$$w(AB) \leq \sec^2\alpha \, w(A)w(B). \tag{3.1.3}$$

Proof : We have

$$
\begin{aligned}
w(AB) &\leq \|AB\| && \text{(by (1.4.1))} \\
&\leq \|A\|\|B\| \\
&\leq \sec^2\alpha\ \|\Re A\|\|\Re B\| && \text{(by Proposition 2.1.5)} \\
&= \sec^2\alpha\ w(\Re A)w(\Re B) \\
&\leq \sec^2\alpha\ w(A)w(B) && \text{(by Proposition 1.4.10)},
\end{aligned}
$$

which completes the proof.

When A, B are positive, then $\alpha = 0$, and we obtain the well known inequality $w(AB) \leq w(A)w(B)$. □

Proposition 3.1.5. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$. Then $W(A\sharp_\lambda B) \subset \mathcal{S}_\alpha$.*

Proof : Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$. and notice that Definition 2.2.1

$$A\sharp_\lambda B = \int_0^1 (A!_s B) d\nu_\lambda(s)$$

for some positive measure $\nu_\lambda(s)$ on $[0,1]$. Then for any unit vector $x \in \mathbb{C}$, we have

$$
\begin{aligned}
\langle A\sharp_\lambda Bx, x\rangle &= \int_0^1 \langle (A!_s B)x, x\rangle\, d\nu_\lambda(s) \\
&= \int_0^1 h(s) d\nu_\lambda(s) \text{ (where } h(s) = \langle (A!_s B)x, x\rangle) \\
&= c + id,
\end{aligned}
$$

where

$$c = \Re \int_0^1 h(s) d\nu_\lambda(s), d = \Im \int_0^1 h(s) d\nu_\lambda(s).$$

We notice that for each $s \in [0,1]$, $W(h(s)) \subset S_\alpha$ since $W(A), W(B) \subset S_\alpha$. This is due to the fact that S_α is invariant under inversion and addition. To show that $W(A\sharp_\lambda B) \subset S_\alpha$, we need to show that $\langle (A\sharp_\lambda B)x, x\rangle \subset S_\alpha$, or $|d| \leq \tan(\alpha)c$. In fact, we have

$$\begin{aligned} |d| &= \left| \Im \int_0^1 h(s) d\nu_\lambda(s) \right| \\ &\leq \int_0^1 |\Im h(s)| \, d\nu_\lambda(s) \\ &\leq \int_0^1 \tan(\alpha) \Re h(s) d\nu_\lambda(s) \quad (\text{since } W(h(s)) \subset S_\alpha) \\ &= \tan(\alpha)c. \end{aligned}$$

This shows that $A\sharp_\lambda B$ is sectorial and completes the proof. □

3.2 The numerical radius and geometric means

Now we present another new type of numerical radius inequalities, where the numerical radius of geometric mean is discussed. When $A, B \in \mathcal{M}_n$ are positive, then

$$A\sharp B > 0 \Rightarrow w(A\sharp B) = \|A\sharp B\|.$$

This makes the study of numerical radius inequalities of means of positive matrices trivial.

Theorem 3.2.1. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$.*

Then for $0 < \lambda < 1$

$$w(A\sharp_\lambda B) \leq \sec^3\alpha\, w(A)^{1-\lambda} w(B)^\lambda. \tag{3.2.1}$$

Proof : Noting Proposition 3.1.5, we have

$$\begin{aligned}
w(A\sharp_\lambda B) &\leq \|A\sharp_\lambda B\| \\
&\leq \sec\alpha\, \|\Re(A\sharp_\lambda B)\| \qquad \text{(by Proposition 2.1.5)} \\
&\leq \sec^3\alpha\, \|\Re(A)\sharp_\lambda \Re(B)\| \qquad \text{(by Theorem 2.2.6)} \\
&\leq \sec^3\alpha\, (\|\Re(A)\|\ \sharp_\lambda\ \|\Re(B)\|) \qquad \text{(by Proposition 1.6.1)} \\
&= \sec^3\alpha\, (w(\Re A)\ \sharp_\lambda\ w(\Re B)) \\
&\leq \sec^3\alpha\, (w(A)\ \sharp_\lambda\ w(B)) \qquad \text{(by Proposition 1.4.10)} \\
&= \sec^3\alpha\, w(A)^{1-\lambda} w(B)^\lambda,
\end{aligned}$$

which completes the proof. □

Theorem 3.2.2. *Let* $B \in \mathcal{M}_n$ *be accretive matrix such that* $W(B) \subset S_\alpha$ *and* $A \in \mathcal{M}_n^+$*. Then for* $-1 < \lambda < 0$

$$\cos^6\alpha\, w^{-1}(A^{-2}) w^{-(\lambda+1)}(A) w^\lambda(B) \leq w(A\sharp_\lambda B). \tag{3.2.2}$$

Proof : Let $x \in \mathbb{C}^n$ be a unit vector. Then

$$\begin{aligned}
w^{-1}(A\sharp_\lambda B) &\le \sec^3\alpha\, w((A\sharp_\lambda B)^{-1}) = \sec^3\alpha\, w(A^{-1}\sharp_\lambda B^{-1}) \\
&= \sec^3\alpha \max_{\|x\|=1} \left|\left\langle (A^{-1}\sharp_\lambda B^{-1})x, x\right\rangle\right| \\
&= \sec^3\alpha \max_{\|x\|=1} \left|\left\langle A^{-1}(A^{-1}\sharp_{-\lambda} B^{-1})^{-1}A^{-1}x, x\right\rangle\right| \\
&= \sec^3\alpha \max_{\|x\|=1} \left|\left\langle A^{-1}(A\sharp_{-\lambda} B)A^{-1}x, x\right\rangle\right| \\
&= \sec^3\alpha \max_{\|x\|=1} \left\{ \|A^{-1}x\|^2 \left|\left\langle (A\sharp_{-\lambda} B)\frac{A^{-1}x}{\|A^{-1}x\|}, \frac{A^{-1}x}{\|A^{-1}x\|}\right\rangle\right| \right\} \\
&\le \sec^3\alpha \max_{\|x\|=1} \left\langle A^{-2}x, x\right\rangle \max_{\|x\|=1} \left|\left\langle (A\sharp_{-\lambda} B)\frac{A^{-1}x}{\|A^{-1}x\|}, \frac{A^{-1}x}{\|A^{-1}x\|}\right\rangle\right| \\
&= \sec^3\alpha\, w(A^{-2})w(A\sharp_{-\lambda} B) \\
&\le \sec^6\alpha\, w(A^{-2})w^{1+\lambda}(A)w^{-\lambda}(B),
\end{aligned}$$

which completes the proof. □

In a similar way, the Heinz mean satisfies similar property, as follows.

First, we define the Heinz mean of accretive matrices A, B by

$$\mathcal{H}_\lambda(A, B) = \frac{A\sharp_\lambda B + A\sharp_{1-\lambda} B}{2} \qquad 0 < \lambda < 1 \tag{3.2.3}$$

Theorem 3.2.3. *Let $A, B \in \mathcal{M}_n$ be accretive matrices. Then for $\lambda \in (0, 1)$,*

$$w(\mathcal{H}_\lambda(A, B)) \le \sec^3\alpha\, \mathcal{H}_\lambda(w(A), w(B)). \tag{3.2.4}$$

Proof : Compute

$$\begin{aligned}
w(\mathcal{H}_\lambda(A,B)) &= w\left(\frac{A\sharp_\lambda B + A\sharp_{1-\lambda}B}{2}\right) && \text{(by (3.2.3))}\\
&\le \frac{w(A\sharp_\lambda B) + w(A\sharp_{1-\lambda}B)}{2}\\
&\le \frac{\sec^3\alpha}{2}\left(w^{1-t}(A)w^\lambda(B) + w^\lambda(A)w^{1-\lambda}(B)\right) && \text{(by (3.2.1))}\\
&= \sec^3\alpha\, \mathcal{H}_\lambda(w(A), w(B)).
\end{aligned}$$

The proof is complete.

□

A Heinz-type inequality for the numerical radii of accretive matrices may be stated as follows.

Theorem 3.2.4. *Let $A, B \in \mathcal{M}_n$ be accretive matrices. Then for $\lambda \in (0,1)$,*

$$\cos^4\alpha\, w(A\sharp B) \le w(\mathcal{H}_\lambda(A,B)) \le \frac{\sec^4\alpha}{2}\, w(A+B). \tag{3.2.5}$$

Proof : We prove the first inequality.

$$\begin{aligned}
w(A\sharp B) &\le \|A\sharp B\| && \text{(by Proposition 3.1.1)}\\
&\le \sec^3\alpha\, \|\mathcal{H}_\lambda(A,B)\| && \text{(by Lemma 2.2.9)}\\
&\le \sec^4\alpha\, w(\mathcal{H}_\lambda(A,B)). && \text{(by Proposition 3.1.1)}
\end{aligned}$$

We now prove the second inequality.

$$\begin{aligned} w(\mathcal{H}_\lambda(A,B)) &\leq \|\mathcal{H}_\lambda(A,B)\| && \text{(by Proposition 3.1.1)} \\ &\leq \sec^3\alpha \, \|\frac{A+B}{2}\| && \text{(by Lemma 2.2.9)} \\ &\leq \frac{\sec^4\alpha}{2}\, w(A+B). && \text{(by Proposition 3.1.1)} \end{aligned}$$

□

The proof is complete.

Corollary 3.2.5. *Let $A, B \in \mathcal{M}_n$ be accretive matrices. Then for $\lambda \in (0,1)$,*

$$\cos\alpha \, w^{\frac{1}{2}}(AB) \leq \mathcal{H}_\lambda(w(A), w(B)). \tag{3.2.6}$$

Proof : By Theorem 3.1.4, we get

$$\cos\alpha \, w^{\frac{1}{2}}(AB) \leq \sqrt{w(A)w(B)} \leq \mathcal{H}_t(w(A), w(B)).$$

□

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