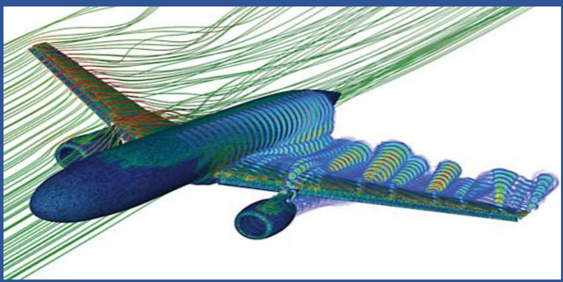
People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
20 August 1955-Skikda University
Faculty of Technology
Department of Technology

Momentum Transfer

Course and Exercises



Pedagogical Handout

Third year Petrochemical Engineering

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Preface

Momentum Transfer plays an important role in various applications such as the automotive sector, airplanes, spacecraft propulsion, biomedical engineering, petroleum, chemical industries, etc.

This pedagogical handout entitled "Momentun Transfer: course and exercises", has been produced in accordance with the new science and technology pedagogical program.

This pedagogical handout is aimed at students of the 3^{rd} year Petrochemical Engineering.

This course has been developed over several years of teaching Momentum Transfer to petrochemical engineering students.

This course consists of four chapters.

The first chapter presents a reminder where we define the properties of fluids and fluid statics.

The second chapter deals with the balances of matter, quantity of motion and energy.

The third chapter focuses on fluid dynamics. The equation of motion of real fluids, flow regimes and head losses are explained.

The fourth chapter deals with pumps and pumping. Pumps and network calculations are described in detail.

Each chapter concludes with exercises.

Dr. Aicha BOUHEZZA

May, 2025

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Chapter I
Reminders

I.1 INTRODUCTION

Fluid is a material substance in which its particles can move easily relative to each other. The fluid is therefore an isotropic medium, without rigidity, deformable, continuous and which can flow.

Based on viscosity and compressibility, fluids can be classified into the following categories: Ideal fluid, Real fluid, Incompressible fluid and Compressible fluid.

I.2 PROPERTIES OF FLUIDS

I.2.1 PHYSICAL QUANTITIES OF A FLUID

I.2.1.1 Temperature

- ✓ **Relative temperature scale**

Celsius scale (°C): In the international system (SI), temperature is divided into 100 graduations.

- ✓ **Absolute temperature scale**

Kelvin scale (K): In the international system (SI): ***T (K) = T (°C) + 273,15***

I.2.1.2 Density

The density (ρ) of a substance is the ratio between the mass (m) and the volume (V) occupied.

$$\boldsymbol{\rho = \frac{m}{V}} \qquad \text{(I.1)}$$

ρ is expressed in *kg/m³*

V: volume in m^3,

m: mass in *kg*.

For gases, the density depends on the temperature and the pressure. For an ideal gas, the density can be calculated from its equation of state.

$$\boldsymbol{P\,V = m\,R\,T \;\Rightarrow\; P = \rho\,R\,T \;\Rightarrow\; \rho = \frac{P}{R\,T}} \qquad \text{(I.2)}$$

Where :

P: absolute pressure (*Pa*),

T : absolute temperature (*K*),

R : gas constant (*J/kg.K*).

$$\boldsymbol{R = \frac{r}{M_{gaz}}}$$

r : universal gas constant, (*r = 8,314 kJ/kmol.K*).

M_{gaz} : molar mass of the gas (*kg/mol*).

For the air, *R = 287 J /kg.K.*

I.2.1.3 Specific volume

Specific volume ($\boldsymbol{\nu}$) is the reciprocal of density.

$$\boldsymbol{\nu = \frac{1}{\rho} = \frac{V}{m}} \qquad \text{(I.3)}$$

ν is expressed (*m³/kg*).

I.2.1.4 Specific weight

The specific weight or weight density ($\boldsymbol{\gamma_s}$) is the weight of a unit volume.

$$\boldsymbol{\gamma_s = \rho\, g = \frac{m\, g}{V}} \qquad \text{(I.4)}$$

$\boldsymbol{\gamma_s}$ is expressed (*N/m³*).

m : mass (*kg*),

g : gravitational acceleration (*m/s²*),

V : volume (*m³*).

I.2.1.5 Specific gravity

The specific gravity (SG) of a fluid is the ratio between the density of the fluid considered and the density of the standard fluid taken under the same temperature and pressure conditions.

$$\mathbf{SG} = \frac{\boldsymbol{fluid\ density}}{\boldsymbol{density\ of\ a\ standard\ fluid}} = \frac{\boldsymbol{\rho}}{\boldsymbol{\rho_{ref}}} \quad \text{(dimensionless)} \qquad \text{(I.5)}$$

Generally, the standard fluid is :

- ✓ Water for liquids ($SG = \frac{\rho_{liquid}}{\rho_{water}}$),
- ✓ Air for gases ($SG = \frac{\rho_{gas}}{\rho_{air}}$).

I.2.2 VISCOSITY OF A FLUID

Some fluids flow more easily than others. This means that each fluid has a property that governs its flow. This property is actually called viscosity.

Viscosity is a property of a liquid due to cohesion and interaction between molecules, which exhibits resistance to deformation. This property appears when the fluid is in motion.

I.2.2.1 Dynamic viscosity μ

To obtain a relation for viscosity, we consider two coaxial circular cylinders, of slightly different radius and whose intermediate space is filled with a fluid.

If the outer cylinder is driven with a motor at constant angular speed (**ω**), we see that the inner cylinder tends to rotate in the same direction. To keep it stationary, we must therefore apply a torque (**T**) in the opposite direction.

The distance (**h**) between the two cylinders being small compared to their radius (**r**), we can schematize the experiment by considering a mobile plane moving parallel to a fixed plane parallel to (ox) of area **A = 2 π r L**, at the distance (h) and with the speed **U = ω r.**

The fluid in contact with the upper plate adheres to the surface of the plate and moves with it at the same speed *U*, and shear stress acts on the fluid layer while the fluid in contact with the fixed plate will have zero velocity (no-slip condition).

The shear stress is:

$$\boldsymbol{\tau} = \frac{\boldsymbol{F}}{\boldsymbol{A}} \tag{I.6}$$

Or :

τ : The shear stress (*N/m²*).

A : the area of the plate (m^2).

When the fluid moves in parallel layers, the flow is called laminar, therefore the fluid speed changes linearly between **zero** and *U*, and thus the velocity profile and the velocity gradient are:

$$\boldsymbol{u(y)} = \frac{\boldsymbol{y}}{\boldsymbol{h}}\boldsymbol{U} \quad \text{and} \quad \frac{\boldsymbol{du}}{\boldsymbol{dy}} = \frac{\boldsymbol{U}}{\boldsymbol{h}} \tag{I.7}$$

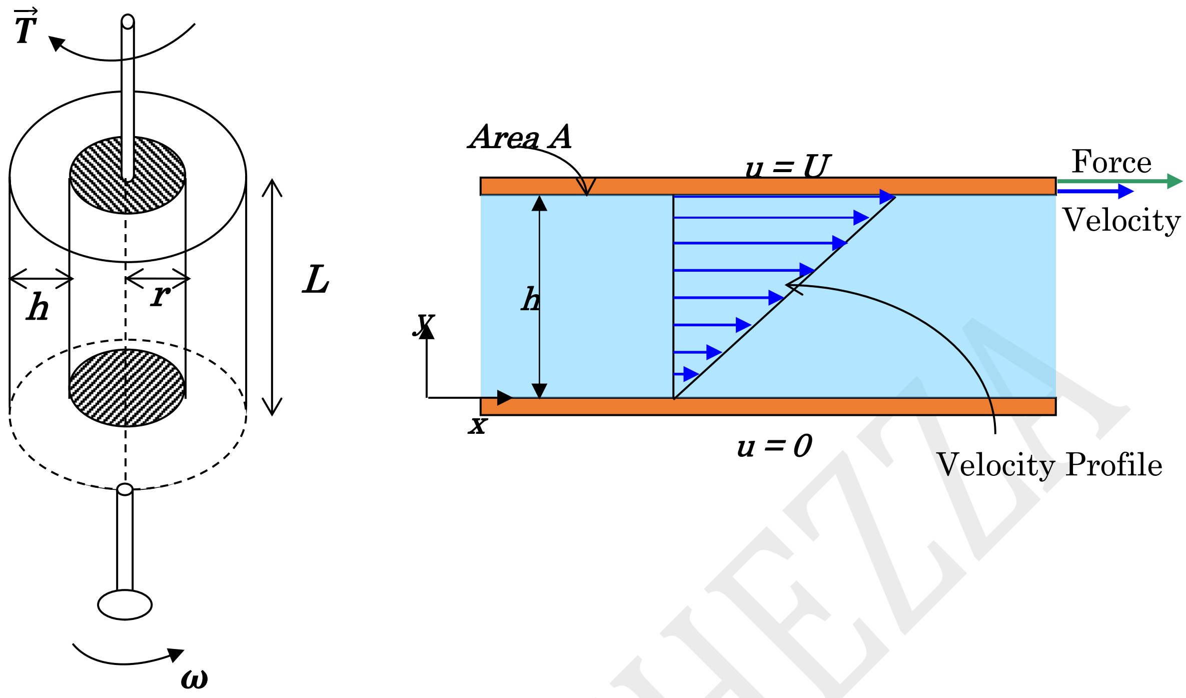


Figure I.1 : *Behavior of a fluid in laminar flow between two parallel plates.*

The force $\boldsymbol{F}$ depends on the area $\boldsymbol{A}$ and the velocity gradient $\frac{du}{dy}$.

Newton's law establishes that its norm is proportional to these two quantities:

$$F = \tau A = \mu\ A\ \frac{du}{dy} \tag{I.8}$$

$$\tau = \frac{F}{A} = \mu\ \frac{du}{dy} \tag{I.9}$$

μ is called the dynamic (or absolute) viscosity of the fluid.

In the SI units, the unit of dynamic viscosity is ***kg/(m.s)*** or Pascal.second (**Pa.s**).

Or equivalently: *1 kg/m.s =1 Pa.s = 1 N.s/m²*

Other units:

1 kg/(m.s) = 10 poise

I.2.2.2 Viscosity variation

Viscosity depends on :

- *The nature of the fluid*
- *The temperature* : Viscosity varies with temperature : $\mu = f(T)$

For liquids, viscosity decreases as temperature increases.

For gases, viscosity increases as temperature increases.

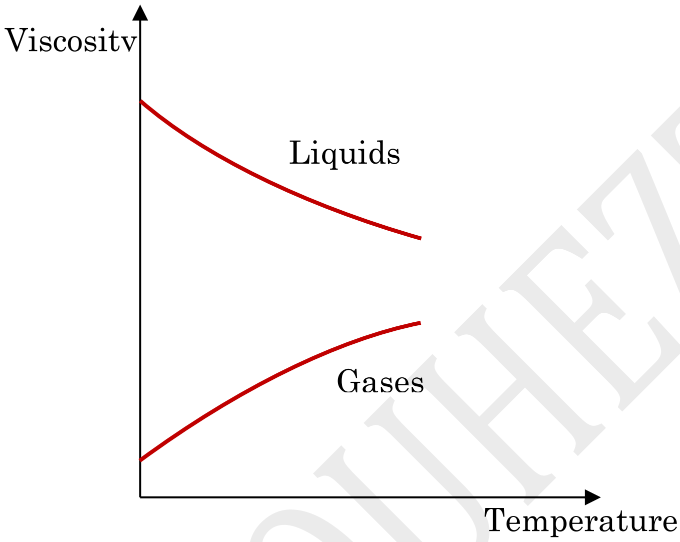


Figure I.2 : *Variation of viscosity as a function of temperature for gases and liquids*

- *The pressure* : For liquids, dynamic viscosity varies with pressure according to the law : $\frac{\mu}{\mu_0} = a^{\left(\frac{P}{P_0} - 1\right)}$, where P_0 is the atmospheric pressure, and a depends on the nature of the fluid (a = 1,003 for mineral oils).
- *The deformation* : Depending on the relationship between shear stress and velocity gradient, different types of fluids are distinguished :

✓ **Newtonian fluids** : a fluid is said to be Newtonian when the viscous stress tensor is a linear function of the strain rate tensor. The proportionality factor is called viscosity $\boldsymbol{\mu}$, it is constant and independent of the shear rate. The equation describing Newtonian behavior is :

$$\boldsymbol{\tau = \mu \frac{du}{dy}}$$

This is the case for gases, vapors, pure liquids of low molar mass, etc.

✓ **Non-Newtonian fluids** : a fluid is said to be non-Newtonian when the viscous stress tensor is not a linear function of the stress rate tensor. There are several types of non-Newtonian fluid (Figure I.3) :

- **Rheofluidifying or pseudoplastic fluid** (fats, mayonnaises, etc.).
- **Rheothickening or dilatant fluid** (Solutions with high powder concentration (water-sand), ...).
- **Bingham fluid** (Chocolate, paints, soups, soap, ...).

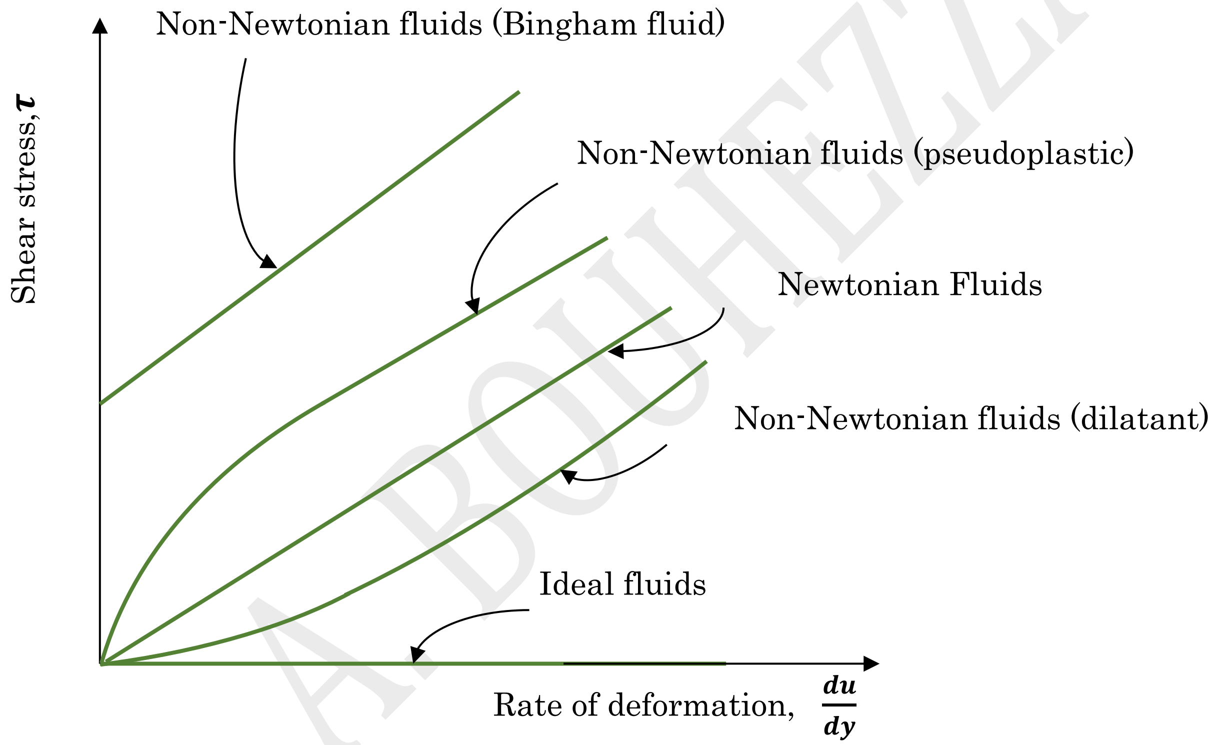


Figure I.3 : *Variation of shear stress as a function of shear rate for Newtonian and non-Newtonian fluids*

I.2.2.3 Kinematic viscosity υ

Kinematic viscosity represents the ratio between dynamic viscosity and density of a fluid.

$$\boldsymbol{\nu} = \frac{\boldsymbol{\mu}}{\boldsymbol{\rho}} \qquad \text{(I.10)}$$

In the SI units, the unit of kinematic viscosity is (***m²/ s***).

Other units:

1 m²/s = 10⁴ stoke.

I.3 FLUID STATICS

I.3.1 FUNDAMENTAL EQUATION OF FLUID STATICS

We consider a small fluid element of parallelepiped shape (Fig. I.4), dimensions (***dx***, ***dy***, ***dz***) and the element mass ***dm***. This fluid element is in equilibrium in the gravitational field, so the sum of the forces exerted on it is zero ($\sum \overrightarrow{F_{Ext}} = \vec{0}$).

The forces acting on this fluid element are as follows:

- Body force : the fluid weight is given by $dm\,\vec{g} = \rho\, dx\, dy\, dz\vec{g}$.
- Surface forces : the resultant of the pressure forces $d\vec{F}$ with three components (dF_x, dF_y , dF_z) in the three directions (***x, y, z***).

$$d\vec{F} = -\left(\frac{\partial\, P(x,y,z)}{\partial x}\vec{\imath} + \frac{\partial\, P(x,y,z)}{\partial y}\vec{\jmath} + \frac{\partial\, P(x,y,z)}{\partial z}\vec{k}\right)\, dx\, dy\, dz$$

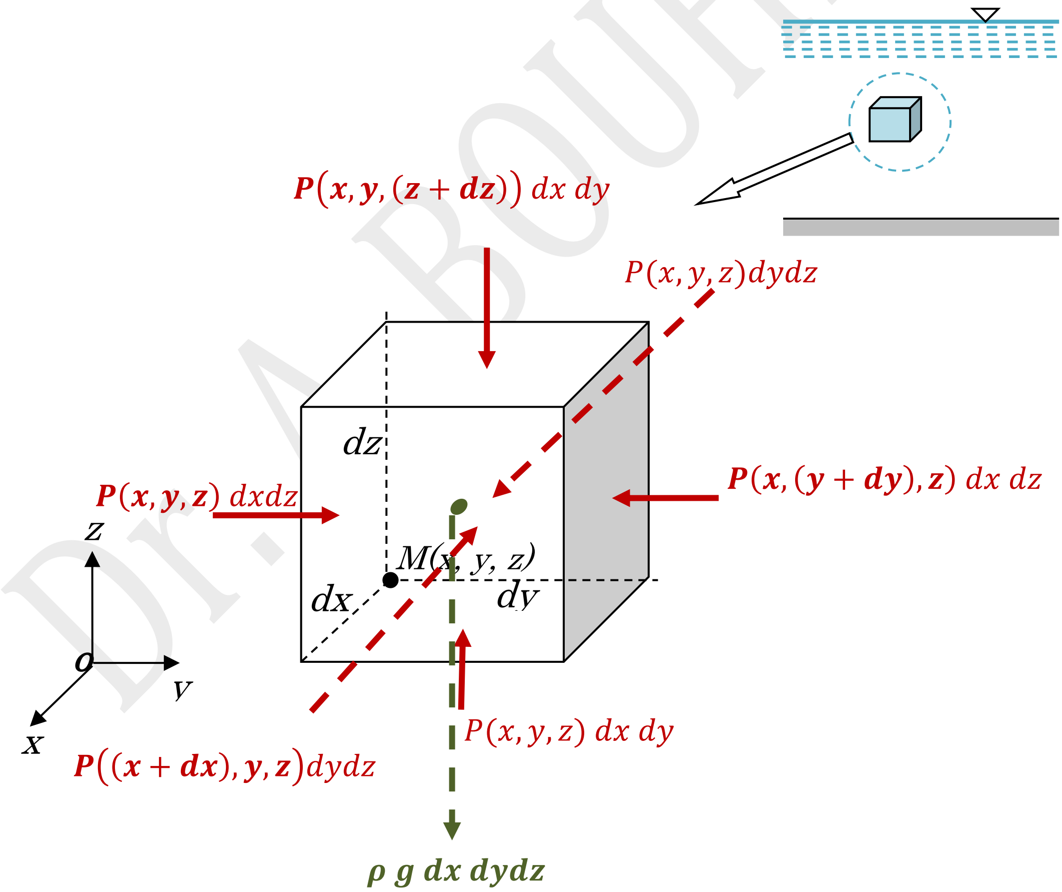


Figure I.4 : *Forces acting on a small fluid element in equilibrium in the gravitational field*

Therefore, the fundamental differential equation of fluid statics in the gravitational field is given by the following relation:

$$\frac{dP(z)}{dz} = -\rho g \qquad \text{or} \qquad dP(z) = -\rho g \, dz \qquad \text{(I.11)}$$

Equation (I.11) is the fundamental equation for fluids at rest in the gravitational field.

I.3.1.1 Incompressible Fluid: Hydrostatics Equation

The density is constant (ρ = constant) (incompressible fluid). Furthermore, for most engineering applications, the variation of the acceleration of gravity is negligible (small variation with altitude), so $\boldsymbol{g = constant}$.

We have the fundamental equation for fluids at rest:

$$\frac{dP(z)}{dz} = -\rho g \qquad \text{(I.12)}$$

And as $\boldsymbol{\rho =}$ constant and $\boldsymbol{g = constant}$

By integrating equation (I.12), we get:

$$P(z) = \int dP(z) = -\int \rho \, g \, dz = -\rho \, g \, z + constant \qquad \text{(I.13)}$$

Or

$$P(z) + \rho \, g \, z = constant \qquad \text{(I.14)}$$

This relationship (I.14) *is called the hydrostatic equation.*

I.3.2 HYDROSTATIC FORCE ON A PLANE SURFACE

We consider a flat plate inclined at an angle α with respect to the horizontal, with area $\boldsymbol{A}$ and center of gravity of plane surface (***G)***. In most engineering applications, the plate is on one side in contact with a fluid of density $\boldsymbol{\rho}$ and on the other side is open to the atmosphere. Therefore, the plate is subjected to two pressure forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ normal to the surface and in opposite directions. The force $\overrightarrow{F_1}$ is due to the fluid pressure and the force $\overrightarrow{F_2}$ is due to the atmospheric pressure. The resultant $\vec{F}$ of these pressure forces is called the hydrostatic force.

This force $\vec{F}$ is always normal to the surface. The point of application of the hydrostatic force $\vec{F}$ is the point C_P, which is called the center of pressure.

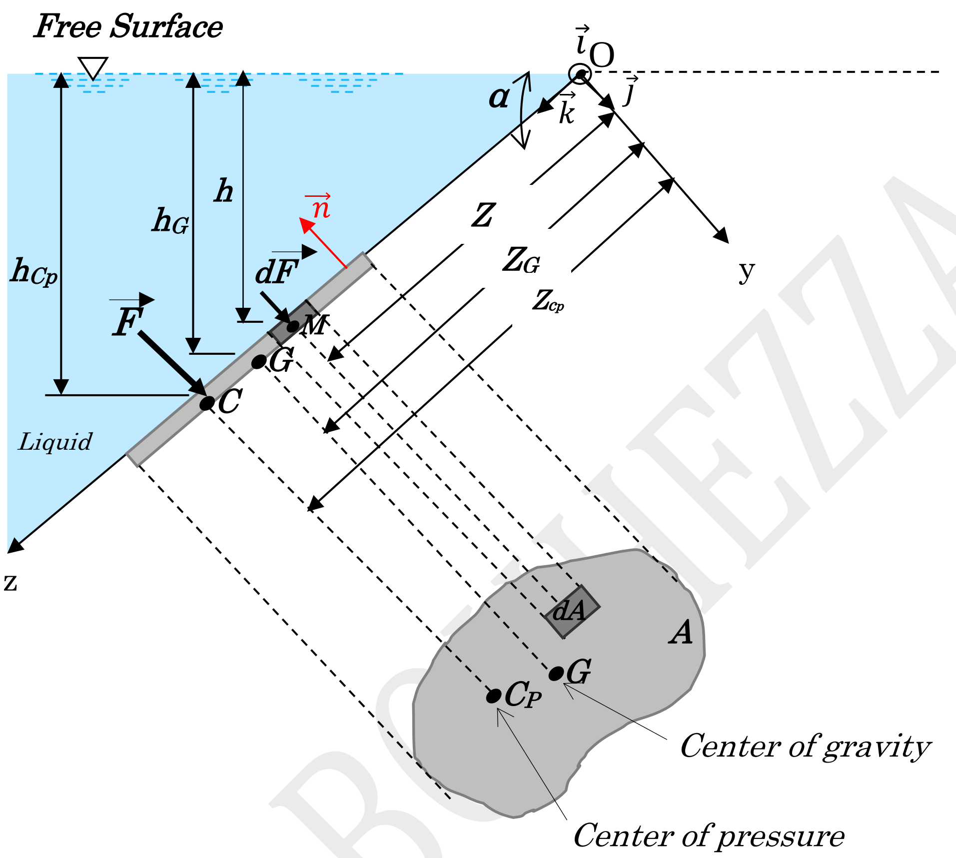


Figure I.5 : *Hydrostatic force and center of pressure on an inclined plane surface in contact with a liquid at rest*

II.3.2.1 Magnitude of the hydrostatic force $\vec{F}$

The magnitude of the hydrostatic force is given by:

$$F = \int dF = \int_A \rho g h dA = \rho g \int_A h dA$$

$$F = \rho g h_G A \tag{I.15}$$

Where :

A : the area of the submerged surface,

Z_G : the y coordinate of the center of gravity of area $\boldsymbol{A}$ measured from the x axis which passes through O,

h_G : the vertical distance from the free surface to the centroid of the area $\boldsymbol{A}$,

ρ : density of the fluid,

g : the acceleration of gravity.

II.3.2.2 Center of pressure (Point of application of the hydrostatic force $\vec{F}$)

The point of application of the hydrostatic force $\vec{\boldsymbol{F}}$ or center of pressure is given by:

$$\overrightarrow{\boldsymbol{OC_P}} \wedge \vec{\boldsymbol{F}} = \int \overrightarrow{\boldsymbol{OM}} \wedge \boldsymbol{d}\vec{\boldsymbol{F}} \tag{I.16}$$

$$\boldsymbol{Z_{C_P}\, F = \int Z\, dF = \int_A\ Z\, \rho\, g\, h\, dA} \tag{I.17}$$

$$\boldsymbol{Z_{C_P} = Z_G + \frac{I_{(xx,G)}}{A\, Z_G}} \tag{I.18}$$

And as $\boldsymbol{h_G = Z_G \sin\alpha}$ and $\boldsymbol{h_{C_P} = Z_C \sin\alpha}$, the previous formula becomes:

$$\boldsymbol{h_{C_P} = h_G + \frac{I_{(xx,G)}(\sin\alpha)^2}{S h_G}} \tag{I.19}$$

Where :

A : the area of the submerged surface,

Z_G : the Z coordinate of the center of gravity $\boldsymbol{G}$ of area $\boldsymbol{A}$ measured from the $\boldsymbol{x}$ axis which passes through O,

Z_{CP} : the Z coordinate of the center of pressure C_P of area $\boldsymbol{A}$ measured from the $\boldsymbol{x}$ axis which passes through O,

h_G : the vertical distance from the free surface to the centroid of the area $\boldsymbol{A}$,

h_{Cp} : the vertical distance from the free surface to the center of pressure C_P,

ρ : density of the fluid,

g : the acceleration of gravity.

I.3.3 BUOYANCY AND STABILITY

Consider a cubic solid body of volume (L^3) and density ρ_{solid} immersed in an incompressible fluid of density ρ_{fluid}.

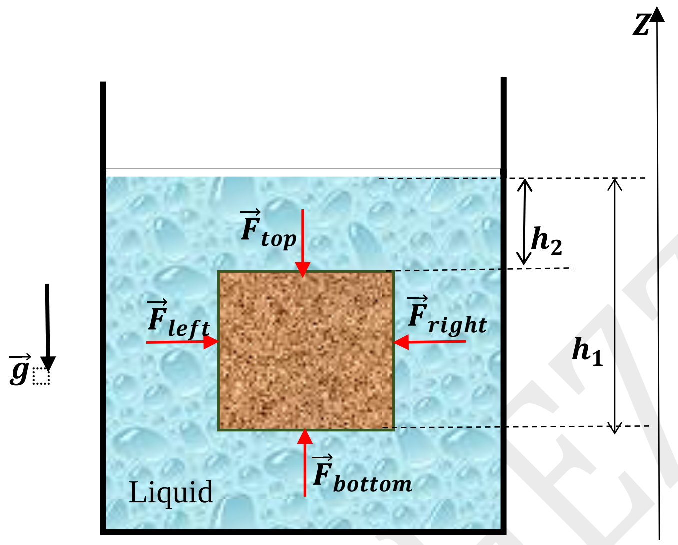


Figure I.6 : *The hydrostatic pressure forces acting in a body*

The hydrostatic pressure forces acting on this body are:

- Hydrostatic pressure forces on vertical walls $\vec{F}_{bottom}$ and $\vec{F}_{top}$
- Hydrostatic pressure forces on the horizontal walls $\vec{F}_{right}$ and $\vec{F}_{left}$ (zero resultant on the horizontal walls).

  The resultant of the pressure forces is therefore:

$$\vec{F} = \vec{F}_{bottom} + \vec{F}_{top} \quad \text{(I.20)}$$

$$\vec{F} = -P_2\, L^2\, \vec{k} + P_1\, L^2\, \vec{k} = -\rho_{fluid}\, g\, h_2\, L^2\, \vec{k} + \rho_{fluid}\, g\, h_1\, L^2\, \vec{k} \quad \text{(I.21)}$$

$$\vec{F} = \rho_{fluid}\, g\, L^2\, (h_1 - h_2)\, \vec{k} = \rho_{fluid}\, g\, L^2\, L\, \vec{k} = \rho_{fluid}\, g\, L^3\, \vec{k} \quad \text{(I.22)}$$

$$\vec{F} = \rho_{fluid}\, g\, \mathrm{V}_{body\ immersed}\, \vec{k} = -\rho_{fluid}\, \mathrm{V}_{body\ immersed}\, \vec{g} \quad \text{(I.23)}$$

The resultant $\vec{F}$ is ascending (directed from bottom to top).

This force $\vec{F}$ is opposed to gravity and is called buoyant force and is denoted by F_B.

$$\overrightarrow{F_B} = -\rho_{fluid}\ V_{body\ immersed}\ \vec{g} = -\rho_{fluid}\ V_{fluid\ displaced}\ \vec{g} \quad (I.24)$$

$$\overrightarrow{F_B} = \rho_{fluid}\ V_{body\ immersed}\ g = \rho_{fluid}\ V_{fluid\ displaced}\ g \quad (I.25)$$

$$V_{body\ immersed} = V_{fluid\ displaced} \quad (I.26)$$

Where:

$V_{body\ immersed}$: volume of the immersed solid.

$V_{fluid\ displaced}$: volume of the displaced fluid.

ρ_{fluid} : density of the fluid.

$\overrightarrow{F_B}$: buoyant force.

Buoyant force is vertical, and directed from bottom to top. The point of application of buoyant force is at the center of gravity of the volume of fluid displaced.

Archimedes' principle

The buoyant force acting on a body of uniform density immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.

- **Equilibrium of immersed or floating bodies**

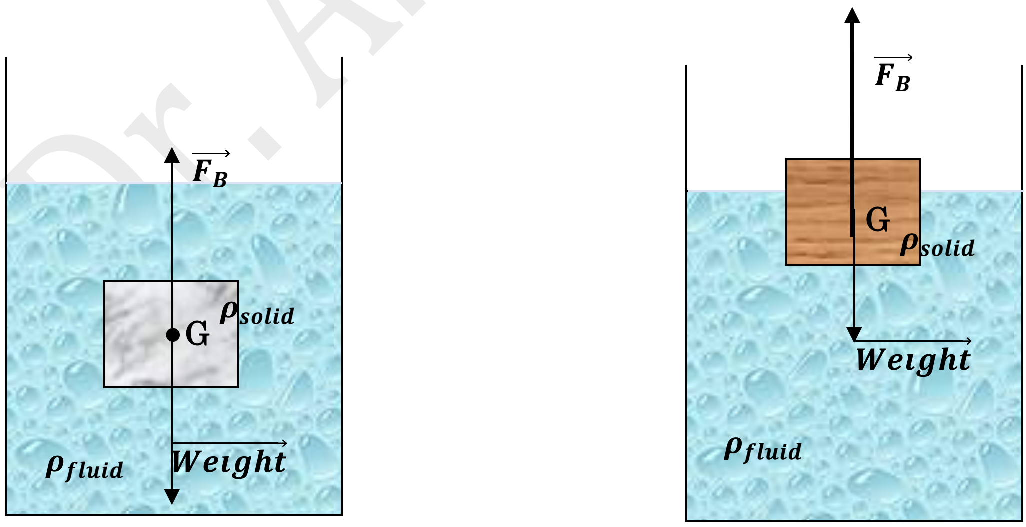


Figure I.13 : *(a) Immersed body , (b) Floating body*

The body (immersed or floating) is subjected to two forces $\overrightarrow{\boldsymbol{Weight}}$ et $\overrightarrow{\boldsymbol{F_B}}$:

The body is in equilibrium $\Longrightarrow$ $\overrightarrow{\boldsymbol{Weight}} + \overrightarrow{\boldsymbol{F_B}} = \vec{\mathbf{0}}$ (I.27)

By projecting (oz), we get:

$$\boldsymbol{\rho_{solid}\, g}\, \mathsf{V}_{\boldsymbol{solid}} = \boldsymbol{\rho_{fluid}\, g}\, \mathsf{V}_{\boldsymbol{body\ immersed}} = -\boldsymbol{\rho_{fluide}\ g}\, \mathsf{V}_{\boldsymbol{fluid\ displaced}} \tag{I.28}$$

$$\frac{V_{\boldsymbol{body\ immersed}}}{\mathsf{V}_{\boldsymbol{solid}}} = \frac{\boldsymbol{\rho_{solid}}}{\mathsf{V}_{\boldsymbol{fluid\ displaced}}} \tag{I.29}$$

(a) $\mathsf{V}_{\boldsymbol{body\ immersed}} = \mathsf{V}_{\boldsymbol{solid}}$ (immersed body and remains between two fluid layers),

Therefore, : $\boldsymbol{\rho_{solid}} = \boldsymbol{\rho_{fluid}}$.

(b) $\mathsf{V}_{\boldsymbol{body\ immersed}} < \mathsf{V}_{\boldsymbol{solid}}$ (floating body), therefore : $\boldsymbol{\rho_{solid}} < \boldsymbol{\rho_{fluid}}$

- **The body sinks to the bottom**

In static equilibrium $\sum \vec{\boldsymbol{F}} = \vec{\mathbf{0}}$

$$\overrightarrow{\boldsymbol{Weight}} + \overrightarrow{\boldsymbol{F_B}} + \vec{\boldsymbol{R}} = \vec{\mathbf{0}} \tag{I.30}$$

By projecting (oz), we get:

$$\boldsymbol{\rho_{solid}\, g}\, \mathsf{V}_{\boldsymbol{solid}} = \boldsymbol{\rho_{fluid}\, g}\, \mathsf{V}_{\boldsymbol{body\ immersed}} + \boldsymbol{R} \tag{I.31}$$

$\mathsf{V}_{\boldsymbol{solid}} = \mathsf{V}_{\boldsymbol{body\ immersed}} \Longrightarrow \boldsymbol{\rho_{solid}} > \boldsymbol{\rho_{fluid}}$

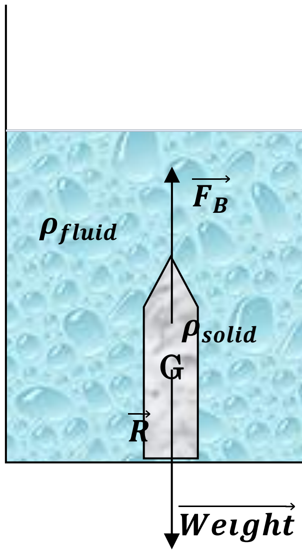


Figure I.14 : *The body sinks to the bottom*

EXERCISES

EXERCISE No. 1

The specific gravity of petrol is 0.7. Determine its density, specific weight and specific volume.

EXERCISE No. 2

Calculate the specific weight (γ_s), density (ρ) and specific gravity (SG) of one liter of a liquid which weight 7 N. ρ_{water} = 1000 kg/m^3 , g = 9.81 N/kg.

EXERCISE No. 3

A glycerin tank has a mass of 1200 kg and a volume of 952 *L*.

Determine the weight (W), density (ρ), specific weight (γ_s) and specific gravity (SG) of the glycerin in the tank. ρ_{water} = 1000 kg/m^3 , g = 9.81 N/kg.

EXERCISE No. 4

A Newtonian fluid with a dynamic viscosity of 19 poise flows through a circular duct in steady state (See Fig. Ex.2). The velocity distribution is given by this equation. $\boldsymbol{u}\,(\boldsymbol{y}) = \frac{U_{max}}{R^2}\,(\boldsymbol{R}^2 - (\boldsymbol{r})^2)$. Where $U_{max} = 0.9\,m/s$, R =0.51 cm.

Determine the shear stress on the wall and at the centerline.

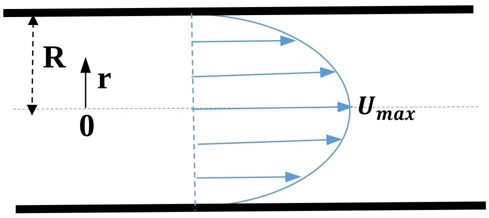


Fig. Ex.4

EXERCISE No. 5

A Newtonian fluid flows on a flat plate at steady state, has a specific gravity SG = 0.92 and a kinematic viscosity υ = 4 stokes.

The velocity profile along a flow section (Fig. Ex.5) is given by:

$$\boldsymbol{v} = \boldsymbol{v}_{max}\sqrt{\left(\frac{y}{h}\right)} \quad , \quad v_{max} = 3\,m/s$$

Determine the shear stress on the plate.

h =1 m , $\rho_{water} = 10^3$ kg/m³.

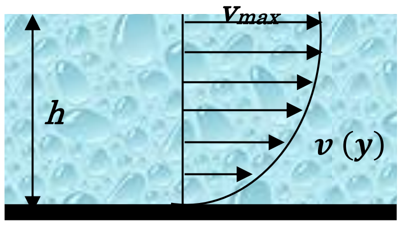


Fig.Ex.5

EXERCISE No. 6

Calculate the dynamic viscosity of an oil, which is used for lubrication between a square plate of size 0.8 m x 0.8 m and an inclined plane with angle of inclination 30°. The weight of the square plate is 300 N and it slides down the inclined plane with a uniform velocity of 0.3 m/s. The thickness of oil film is 1.5 mm.

EXERCISE No. 7

A moving plate lies between two fixed plates, as shown in Fig.Ex.7. Two Newtonian fluids with different viscosities (μ_1 and μ_2) are on each side of the moving plate.

The velocity distribution between the walls on either side of the moving plate is assumed to be linear.

Determine the shear stresses acting on the fixed plates when the moving plate moves at a speed of 4 m/s.

$h_1 = 6\ mm\ ;\ h_1 = 3\ mm\ ;\ \mu_1 = 0.2\ poise\ ;\ \mu_2 = 0.1\ poise.$

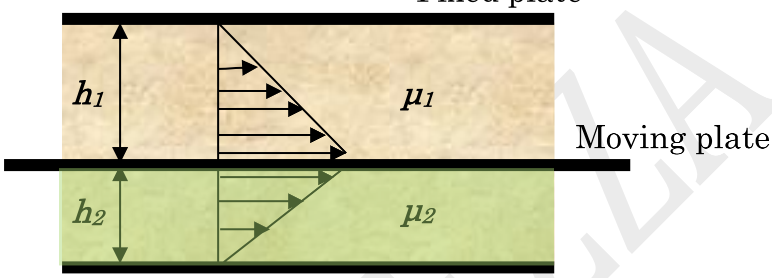


Fig.Ex.7

<u>**EXERCISE No. 8**</u>

Consider a cylindrical tube 3 km long, 10 cm in diameter, through which a liquid of dynamic viscosity μ = 0.4 poise flows. Assume that the velocity distribution in the cross section of the tube is given by the parabolic equation:

$\boldsymbol{u = 10(y - y^2)}$

1) The viscous friction force per unit area on the wall?

2) The viscous friction force per unit area at 4 cm from the wall?

3) The total frictional force exerted on the tube?

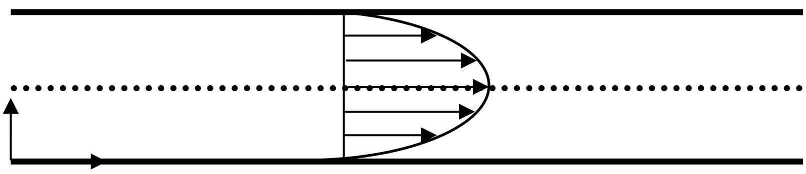


Fig.Ex.8

EXERCISE No. 9

A Newtonian fluid with a specific gravity ***SG = 0.92*** and a kinematic viscosity υ = ***4 stoke*** flows on a flat plate in steady state.

The velocity profile is shown in Fig. Ex.9 and follows the law $\boldsymbol{u}\,(\boldsymbol{y}) = \boldsymbol{U}\ \mathbf{sin}\left(\frac{\pi\, y}{2\, h}\right)$. Determine the shear stress on the plate as a function of $\boldsymbol{U}$ and $\boldsymbol{h}$.

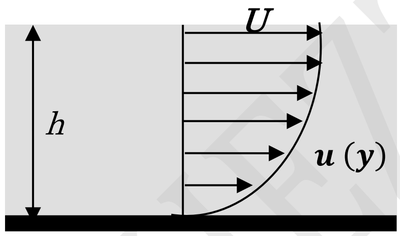


Fig.Ex.9

EXERCISE No. 10

Two oil tanks are connected to each other through a manometer. Determine the pressure difference between the two tanks ($P_A - P_B$). The specific gravities of oil and mercury are 0.92 and 13.6, respectively.

$\rho_{water} = 10^3$ kg/m^3 , g = 9.81 m/s^2 .

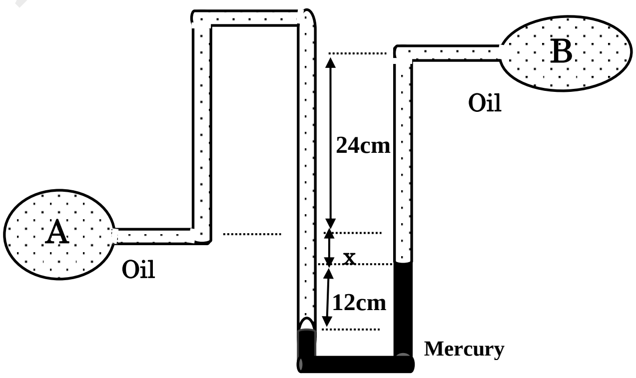


Fig. Ex.10

EXERCISE No. 11

Calculate h and θ, for the installation shown in Fig. Ex.11.

$P_B - P_A = 20$ kPa,

$x = 26.8$ cm ; $\rho_{water} = 10^3$ kg/m^3 ;

$SG_{mercury} = 13.6$; g = 9.81 m/s^2

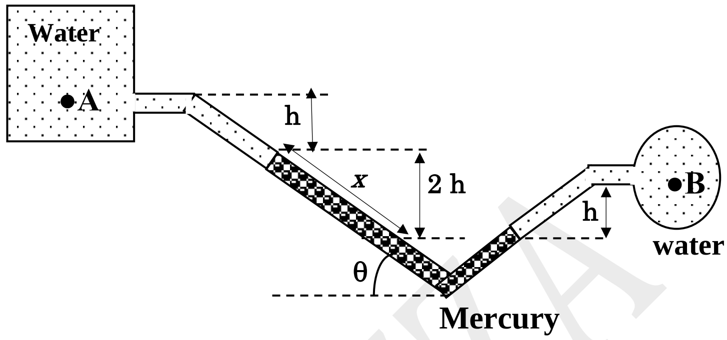


Fig. Ex.11

EXERCISE No. 12

A differential manometer is connected at the two points A and B as show in Fig. Ex. 12. At B air pressure is 9.81 N/ cm^2, find the pressure at A.

$\rho_{water} = 10^3$ kg/m^3 ; $SG_{mercury} = 13.6$; $S_{Goil} = 0.9$; g = 9.81 m/s^2

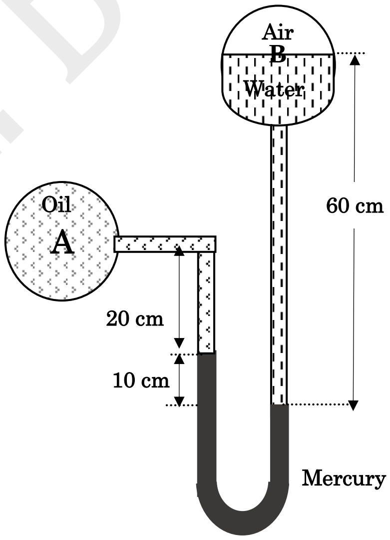


Fig. Ex.12

EXERCISE No. 13

What is the pressure at A if the manometer M indicates a pressure of 30 kPa ?

$\rho_{water} = 10^3$ kg/m^3 , g = 9.81 m/s^2, SG_{liquid} = 2.4 , h_1 = 8 cm ; h_2 = 50 cm ; L_1 = 6 cm ; L_2 = 6 cm .

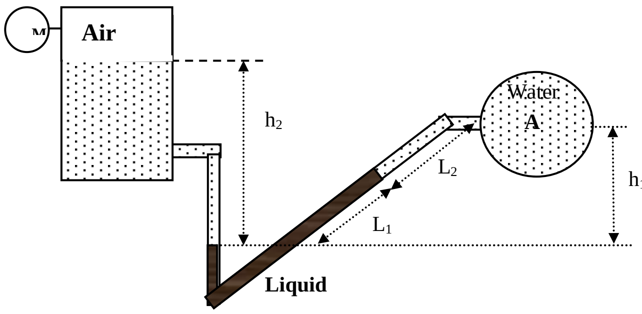


Fig. Ex.13

EXERCISE No. 14

Consider a tank open to the air and filled with water up to a height H.

At the bottom of the tank is a vertical rectangular valve (OB) (L width, and $\overline{(OB)}$ height) which allows the water to be emptied (see Fig.Ex.14).

1) Calculate the height H.

2) Represent and determine the hydrostatic force exerted by the water on the valve.

3) Find the position of the center of pressure.

$\rho_{water} = 10^3$ kg/m^3 ; $SG_{mercury}$ = 13.6 ; $\overline{(OB)} = 2h_1$; g = 9.81 m/s^2 ; h_1 = = 0.30 m ; h_2=0.40 m ; L = 0.1 m

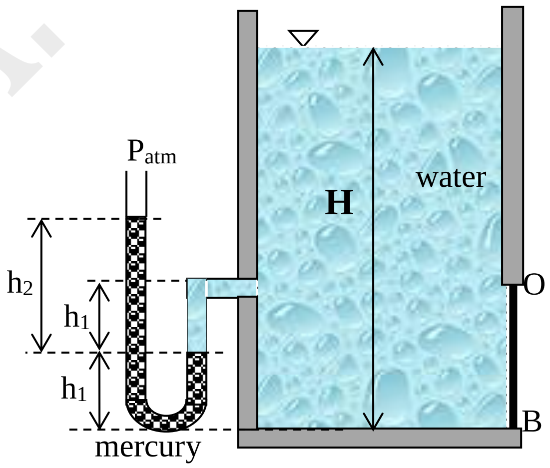


Fig. Ex.14

EXERCISE No. 15

A vertical sluice gate is used to cover an opening in a dam. The opening is 2 m wide and 1.2 m high. On one side of the gate, there is a liquid with a specific density of 1.45, up to a height of 1.5 m above the top of the gate, while water is available on the other side up to a height that touches the top of the gate (Fig.Ex.15).

1) Find the hydrostatic pressure forces acting on the gate, and positions of the pressure centers of these forces.

2) Find also the force acting horizontally at the top of the gate which is capable of opening it. Assume that the gate is hinged at the bottom (O).

$\rho_{water} = 10^3$ kg/m^3 ; $SG_{liquid} = 1.45$; $g = 9.81$ m/s^2 ; $H = 1.5$ m , $h_1 = = 1.20\ m$; $L = 2m$

$$I_{GG(Rectangle)} = \frac{a \times b^3}{12}$$

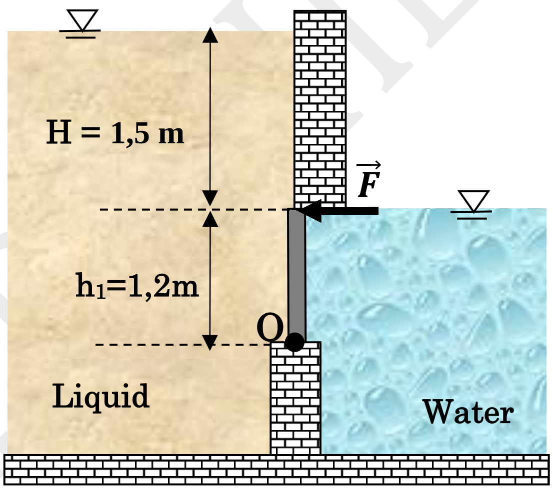


Fig. Ex.15

EXERCISE No. 16

An inclined rectangular gate is used to close an opening in a dam. The gate has length (AB), width L and mass (M). This gate can be pivoted around a horizontal axis located at (A), as shown in Fig. Ex.16.

1) Express and calculate the hydrostatic pressure force exerted by the water on the gate and its center of application.

2) What is the mass (M) of the gate to ensure closure?

$\rho_{water} = 10^3$ kg/m^3 ; $g = 9.81$ m/s^2 ; $H = 2$ m ; $AB = 3$ m ; $L = 1$m, $\alpha = 45°$

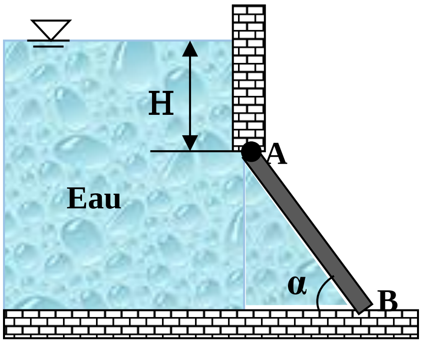


Fig.Ex.16

EXERCISE No. 17

The weight of an object in water is 1962 N. Find its weight in air and its specific gravity. Its dimensions are 1.5 m × 1.0 m × 2.0 m.

$\rho_{water} = 10^3$ kg/m^3 ; $g = 9.81$ m/s^2

EXERCISE No. 18

Find the density of a metallic body which floats at the interface of water and mercury such that 50% of its volume is submerged in water and 50% in mercury.

$\rho_{water} = 10^3$ kg/m^3 ; $g = 9.81$ m/s^2 ; $SG_{mercury} = 13.6$.

Fig.Ex.18

EXERCISE No. 19

At the bottom of a basin, containing water there is a triangular gate in a vertical position (the base of the triangle L is at the bottom) which allows the water to be emptied. This gate can pivot around a horizontal axis located at (B). To ensure the closing of this gate, a mechanical force $\vec{\mathbf{F}}$ is applied to the outside at point (O) (Fig.Ex.19).

1) Express and calculate the force exerted by the water on the gate (F_{water}).

2) Find the location of the pressure center.

3) What force must be applied at point (O) to ensure the gate closes?

$\rho_{water} = 10^3$ kg/m^3 ; g = 9.81 m/s^2, BO =3m ; H = 2m ; L = 1m.

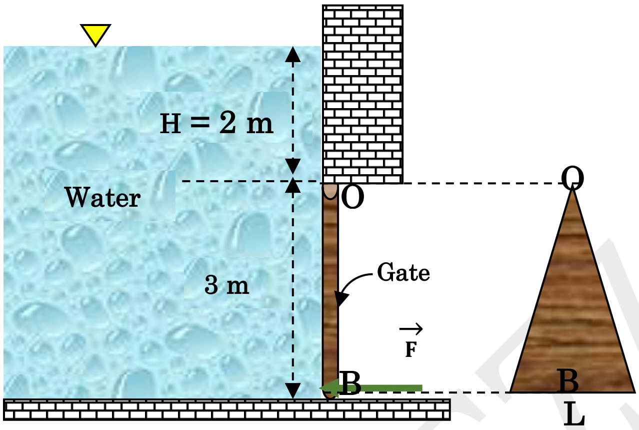


Fig.Ex.19

Chapter II
Mass, Momentum and Energy Balances

II.1 INTRODUCTION

In this chapter we will study the fundamental principles of physics (conservation laws), and how they can be applied to fluid mechanics problems.

II.2 DEFINITIONS

II.2.1 Description of fluid movement

The movement of fluids is governed by principles known as conservation laws or the balance of certain physically measurable quantities (vector or scalar) of an isolated system which is invariant over time. These laws govern several physical quantities such as the conservation of mass, the conservation of momentum and the conservation of energy. These conservation balances can be formulated mathematically by a Lagrangian description (Material volume) or Eulerian description (control volume).

II.2.1 Lagrangian and Eulerian Descriptions

There are two distinct ways to describe movement:

The first method is the Lagrangian method, where describing Lagrangian requires tracking individual fluid particles as they move, and determining how the fluid properties associated with these particles change as a function of time.

This means that the fluid particles are identified, and their properties are determined as they move.

Here, it is the evolution of the position of the particles which allows the description of the flow.

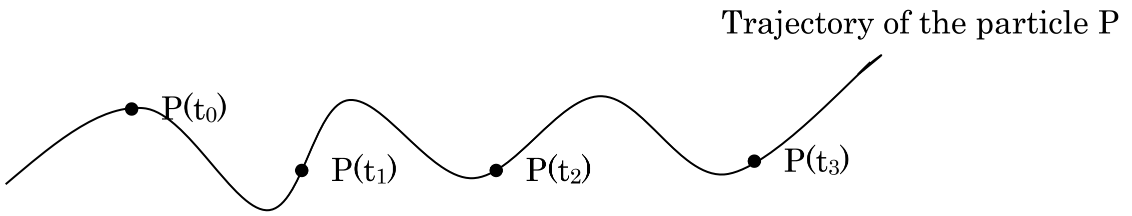


Figure II. 1: Trajectory of a fluid particle

At a time t_0 the initial position of a fluid particle is $P(t_0)$ identified by the vector $\vec{R}\ (t_0)$ the coordinates (x_0, y_0, z_0).
The unknowns of Lagrange are the coordinates, at time t, of the position defined by the position vector x:

$$\vec{R} = \vec{R}\ (R_0, t) = \begin{cases} x\ (x_0, y_0, z_0, t\ ) \\ y\ (x_0, y_0, z_0, t\ ) \\ z\ (x_0, y_0, z_0, t\ ) \end{cases}$$

Velocity vector $\quad \vec{v}\ (\ t) = \frac{d\vec{R}\ (R_0, t)}{dt}$ (II.1)

Acceleration vector $\quad \vec{a}\ (\ t) = \frac{d^2\ \vec{R}\ (x_0, t)}{dt^2}$ (II.2)

The second method is the Eulerian method, where in the Eulerian description of fluid flow, a finite volume called a flow domain or control volume is defined, through which fluid flows in and out. Instead of tracking individual fluid particles, we define field variables, functions of space and time, within the control volume.
The field variable at a particular location at a particular time is the value of the variable for whichever fluid particle happens to occupy that location at that time.
For example, the pressure field is a scalar field variable; for general unsteady three-dimensional fluid flow in Cartesian coordinates,
Pressure field: $P = P(x, y, z, t)$
We define the velocity field as a vector field variable in similar fashion,
Velocity field: $\vec{V} = \vec{V}\ (x, y, z, t)$
Likewise, the acceleration field is also a vector field variable,
Acceleration field: $\vec{a} = \vec{a}\ (x, y, z, t)$
This description of the flow consists of establishing at a given time t all the speeds associated with each point in the space occupied by the fluid.
At each instant t, the fluid flow is described by means of a velocity vector field.

$$\vec{V}\ (u, v, w) = u\ (x, y, z, t)\ \vec{\imath} + \ v\ (x, y, z, t)\ \vec{\jmath} + w\ (x, y, z, t)\ \vec{k} \qquad \text{(II.3)}$$

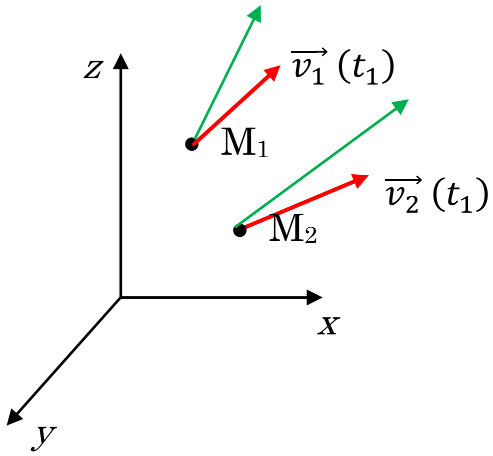


Figure II. 2: Flow of a fluid

In this description by Euler, the curve which, at each of its points, is tangent to the velocity vectors is called a ***streamline***.

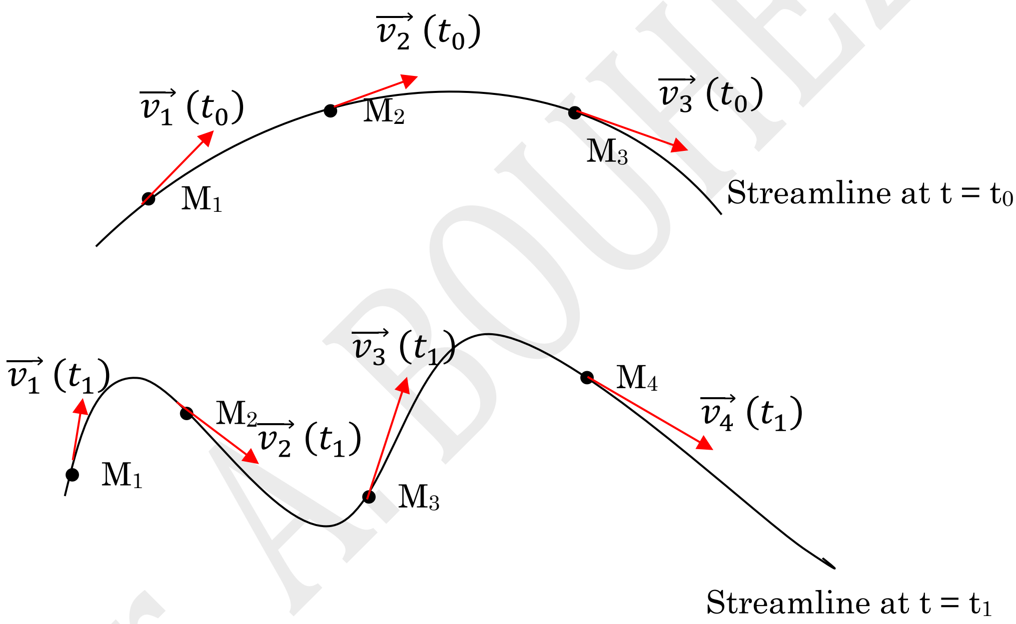


Figure II. 3 : Streamline

Stream tube: The streamlines that lie on a closed curve and form a tube are called stream tubes.

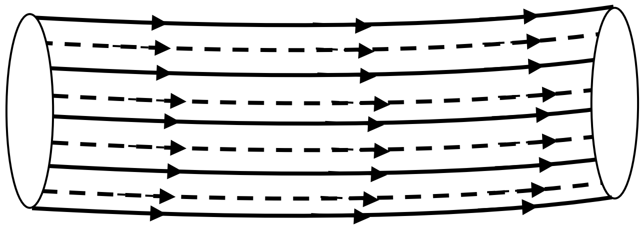


Figure II. 4 : Stream tube

II.2.3 Steady Flow

A flow is said to be steady (or stationary) when all the characteristic quantities of the flow and the properties of the fluid such as (pressure, temperature, density, speed, etc.) at each point remain constant over time.

$\left(\frac{\partial v}{\partial t} = 0\right)$, $\left(\frac{\partial P}{\partial t} = 0\right)$, ...

In this case:

- The streamlines are fixed in space;
- The trajectories coincide with the streamlines.

II.3 MATERIAL DERIVATIVE (OR TOTAL DERIVATIVE)

II.3.1 Scalar quantity Φ

Consider a scalar physical quantity Φ linked to the moving fluid particle (density, pressure, temperature, etc.).

If this quantity is expressed as a function of the Euler variables ***Φ(x, y, z, t)***.

During the time ***dt***, the particle moves from the point M(x, y, z) to the point $\grave{M}$(x+dx, y+dy, z+dz) with $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$ and $v_z = \frac{dz}{dt}$, $d\vec{M} = \vec{v}\,dl$

At (t) M(x, y, z) $\longrightarrow$ $\grave{M}$(x+dx, y+dy, z+dz) at (t+dt)

The quantity Φ varies from (DΦ) such that:

$$D\Phi = \frac{\partial \Phi}{\partial x}\,dx + \frac{\partial \Phi}{\partial y}\,dy + \frac{\partial \Phi}{\partial z}\,dz + \frac{\partial \Phi}{\partial t}\,dt \qquad \text{(II.4)}$$

$$\frac{D\Phi}{\mathrm{Dt}} = \frac{\partial\Phi}{\partial x}\frac{dx}{dt} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt} + \frac{\partial\Phi}{\partial z}\frac{dz}{dt} + \frac{\partial\Phi}{\partial t} \tag{II.5}$$

$$\frac{D\Phi}{\mathrm{Dt}} = \frac{\partial\Phi}{\partial t} + v_x\frac{\partial\Phi}{\partial x} + v_y\frac{\partial\Phi}{\partial y} + v_z\frac{\partial\Phi}{\partial z} \tag{II.6}$$

$$\boldsymbol{\frac{D\Phi}{\mathrm{Dt}} = \frac{\partial\Phi}{\partial t} + \left(\vec{v}\,.\,\overrightarrow{grad}\right)\Phi} \quad \text{Material Derivative of } \Phi \tag{II.7}$$

II.3.2 Vector quantity $\vec{A}$

In the case of any vector quantity $\vec{A}\,(M,t)$. The vector quantity is expressed as a function of the Euler variables $\vec{A}\,(x,y,z,t)$.

The particle derivative of $\vec{A}$ is written:

$$\boldsymbol{\frac{D\vec{A}}{\mathrm{Dt}} = \frac{\partial\vec{A}}{\partial t} + \left(\vec{v}\,.\,\overrightarrow{grad}\right)\vec{A}} \tag{II.8}$$

$\frac{\partial\vec{A}}{\partial t}$: the local derivative which indicates a non-permanent character of $\vec{A}$.

$\left(\vec{v}\,.\,\overrightarrow{grad}\right)$: the convective derivative which indicates a non-uniform character of $\vec{A}$.

II.4 ACCELERATION OF A FLUID PARTICLE

- **Euler approach**

The acceleration of a fluid particle is the material derivative of the velocity.

$$\boldsymbol{\vec{a}\,(\vec{r}\,,t) = \frac{D\vec{v}}{\mathrm{Dt}} = \frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\,.\,\overrightarrow{grad}\right)\vec{v}} \tag{II.9}$$

The term $\boldsymbol{\frac{\partial\vec{v}}{\partial t}}$ represents the local or temporal acceleration.

The term $\boldsymbol{\left(\vec{v}\,.\,\overrightarrow{grad}\right)\vec{v}}$ represents the convective acceleration.

$$\boldsymbol{\vec{a}\,(\vec{r}\,,t) = \frac{D\vec{v}}{\mathrm{Dt}} = \frac{\partial\vec{v}}{\partial t} + \overrightarrow{grad}\left(\frac{v^2}{2}\right) + \overrightarrow{rot}(\vec{v})\wedge\vec{v}} \tag{II.10}$$

$$\boldsymbol{\vec{a}\,(\vec{r}\,,t) = \frac{D\vec{v}}{\mathrm{Dt}} = \frac{\partial\vec{v}}{\partial t} + \overrightarrow{grad}\left(\frac{v^2}{2}\right) + 2\,\vec{\Omega}\wedge\vec{v} \quad \text{with} \quad \vec{\Omega} = \frac{1}{2}\,\overrightarrow{rot}(\vec{v})} \tag{II.11}$$

II.5 MASS CONSERVATION

II.5.1 Mass flow rate and volume flow rate

- **Mass flow rate $\dot{m}$**

The amount of mass passing through a cross-section per unit time is called the mass flow rate $\dot{m}$.

$$\delta\dot{m} = \rho\, V_n\, dA \qquad \text{(II.12)}$$

$$\dot{m} = \int \delta\dot{m} \;=\; \int_A \rho\, \left(\vec{V}\,.\vec{n}\right)\, dA = \int_A \rho\, V_n\, dA \qquad \text{(II.13)}$$

$\dot{m}$ is expressed in (kg/s).

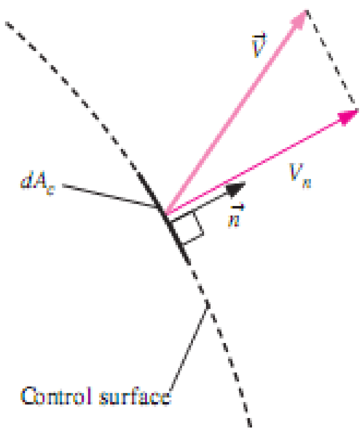


Figure II. 5 : Calculation of flow rate through a section

- **Average Velocity**

$$V_{avg} \;=\; \frac{1}{A}\int_A \; V_n\, dA \qquad \text{(II.14)}$$

- **Incompressible fluid**

$$\dot{m} = \rho\, V_{avg}\, A \qquad \text{(II.15)}$$

- **Volume flow rate Q**

The volume of fluid passing through a cross-section per unit time is called the volume flow rate Q.

$$\boldsymbol{Q = \int_A V_n \, dA = V_{avg} \, A} \tag{II.16}$$

$\boldsymbol{Q}$ is expressed in (m³/s).

- **Relationship between the two flow rates**

$$\boldsymbol{\dot{m} = \rho \, Q} \tag{II.17}$$

II.4.2 Mass conservation equation (Continuity equation)

The continuity equation expresses the fundamental principle of mass conservation. *The net mass transfer to or from a control volume during a time interval (dt) is equal to the net change (increase or decrease) of the total mass within the control volume during (dt).*

We consider a fluid volume element (parallelepiped) $\boldsymbol{d\forall = dx\, dy\, dz}$

Let $\boldsymbol{\vec{V}\,(u, v, w, t)}$ be the velocit of this fluid, a function of the Euler variables.

The velocity field is expanded in Cartesian coordinates (x, y, z) as

$$\boldsymbol{\vec{V}\,(u, v, w, t) = u(x, y, z, t)\,\vec{i} + v(x, y, z, t)\,\vec{j} + w(x, y, z, t)\,\vec{k}}$$

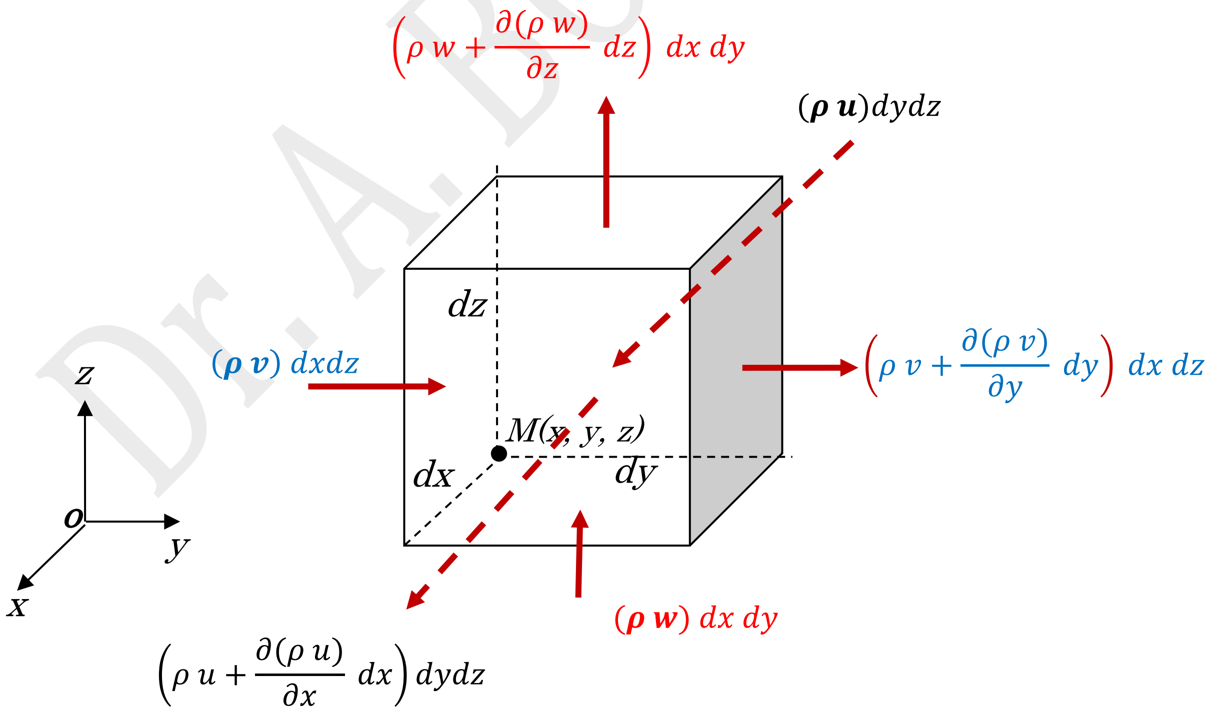


Figure II.6: Principle of mass conservation

The variation of this mass during (dt) is written:

$$\boldsymbol{dm = \frac{\partial(\rho\, dV)}{\partial t}\, dt = \frac{\partial\rho}{\partial t}\, dx\, dy\, dz\, dt} \tag{II.18}$$

This variation must then be equal to the sum of the mass of fluid, which enters and exits through the six faces of the fluid element.

Total mass entering the fluid element during time (dt) is expressed by:

$$\rho\, u\, dy\, dz\, dt + \rho\, v\, dx\, dz\, dt + \rho\, w\, dx\, dy\, dt = \rho\,(u\, dy\, dz + v\, dx\, dz + w\, dx\, dy\,)dt$$

Total mass exiting in the fluid element during time (dt) is expressed by:

$$(\rho\, u)_{x+dx}\; dy\, dz\, dt + (\rho\, v)_{y+dy}\, dx\, dz\, dt + (\rho\, w)_{z+dz} dx\, dy\, dt$$

A first-order development allows us to write:

$$(\rho\, u)_{x+dx} = \rho\, u + \frac{\partial(\rho\, u)}{\partial x}\, dx \tag{II.19}$$

$$(\rho\, v)_{y+dy} = \rho\, v + \frac{\partial(\rho\, v)}{\partial y}\, dy \tag{II.20}$$

$$(\rho\, u)_{z+dz} = \rho\, w + \frac{\partial(\rho\, w)}{\partial z}\, dz \tag{II.21}$$

So the total mass coming out is:

$$\left(\rho\, u + \frac{\partial(\rho\, u)}{\partial x}\, dx\right)\, dy\, dz\, dt + \left(\rho\, v + \frac{\partial(\rho\, v)}{\partial y}\, dy\right)\, dx\, dz\, dt + \left(\rho\, w + \frac{\partial(\rho\, w)}{\partial z}\, dz\right) dx\, dy\, dt$$

We calculate the difference dm:

$$dm = m_{in} - m_{out}$$

$$dm = -\left[\frac{\partial(\rho\, u)}{\partial x} + \frac{\partial(\rho\, v)}{\partial y} + \frac{\partial(\rho\, w)}{\partial z}\right] dx\;\, dy\, dz\, dt \tag{II.22}$$

By equalizing the two values of dm (equation II.15 and equation II.19), we obtain the continuity equation:

$$\boldsymbol{\frac{\partial\rho}{\partial t}\, dx\, dy\, dz\, dt = -\left[\frac{\partial(\rho\, u)}{\partial x} + \frac{\partial(\rho\, v)}{\partial y} + \frac{\partial(\rho\, w)}{\partial z}\right] dx\;\, dy\, dz\, dt} \tag{II.23}$$

$$\boldsymbol{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\, u)}{\partial x} + \frac{\partial(\rho\, v)}{\partial y} + \frac{\partial(\rho\, w)}{\partial z} = 0} \tag{II.24}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\,\vec{v}) = 0 \qquad \text{(II.25)}$$

This is the continuity equation.

Continuity equation in cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho\,u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho\,u_\theta)}{\partial \theta} + \frac{\partial(\rho\,u_z)}{\partial z} = 0 \qquad \text{(II.26)}$$

- ***Special cases***

➢ *Steady compressible flow*

$$\vec{\nabla}.(\rho\,\vec{v}) = 0 \qquad \text{(II.27)}$$

Steady continuity equation

➢ *Steady incompressible flow*

$$\vec{\nabla}.(\,\vec{v}) = 0 \qquad \text{(II.28)}$$

Incompressible continuity equation

➢ *For a one-dimensional steady flow, the* continuity equation *is written:*

$$\sum \dot{m}_{in} = \sum \dot{m}_{out} \qquad \text{(II.29)}$$

$$\dot{m}_1 + \dot{m}_2 = \dot{m}_3 \qquad \text{(II.30)}$$

For an incompressible fluid:

$$\sum Q_{in} = \sum Q_{out} \qquad \text{(II.31)}$$

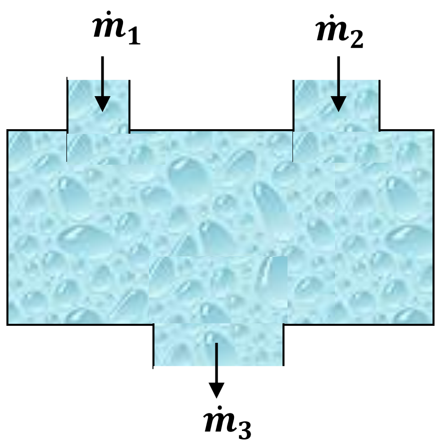


Figure II.7 : Principle of conservation of mass for a one-dimensional steady flow

II.6 MOMENTUM CONSERVATION EQUATION

II.6.1 MOMENTUM BALANCE

Consider a control volume CV(t) bounded by a closed surface A(t) in a moving fluid.

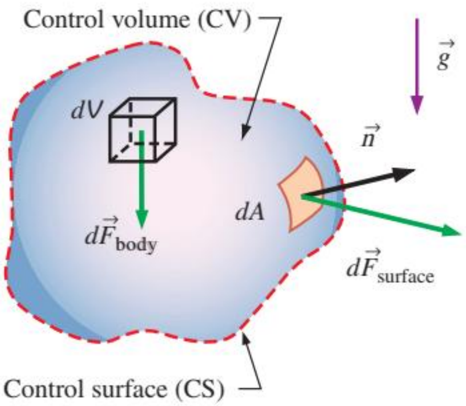


Figure II. 8 : Principle of momentum conservation

The external forces acting on the control volume are:

- The body forces (such as gravity, electric, and magnetic forces):

$$\sum \vec{F}_{body} = \iiint_{CV(t)} \rho\, \vec{F}\, dV$$

- Surface forces (such as pressure and viscous forces and reaction forces (other) at points of contact): $\sum \vec{F}_{pressure} = -\iint_{CS(t)} P\, \vec{n}\, dA$,

$$\sum \vec{F}_{viscous} = \iint_{CS(t)} \vec{T}\, dA \quad \text{and} \sum \vec{F}_{other}$$

Total force : $\sum \vec{F}_{Total} = \sum \vec{F}_{body} + \sum \vec{F}_{surface} = \sum \vec{F}_{body} + \sum \vec{F}_{viscous} + \sum \vec{F}_{pressure} + \sum \vec{F}_{other}$

The momentum balance allows us to write:

$$\frac{D}{Dt}\left[\iiint_{D(t)} \rho\, \vec{v}\, dV\right] = \sum \vec{F}_{Total} \qquad \text{(II.32)}$$

$$\frac{D}{Dt}\left[\iiint_{D(t)} \rho\,\vec{v}\ dV\right] = \sum \vec{F}_{body} + \sum \vec{F}_{viscous} + \sum \vec{F}_{pressure} + \sum \vec{F}_{other} \quad \text{(II.33)}$$

II.6.2 THE LINEAR MOMENTUM EQUATION

The fundamental principle of dynamics applied to a particle is written:
Newton's second law for a system of mass m subjected to net force ΣF is expressed as

$$\sum \overrightarrow{F_{ext}} = m\,\vec{\gamma} = \frac{D(m\,\vec{V})}{Dt} \quad \text{(II.34)}$$

Where $m\,\vec{V}$ is the linear momentum of the system
Newton's second law can be stated as the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.

$$\delta m = \rho\, d\forall$$

$$m = \int_{sys} \delta m = \int_{sys} \rho\, d\forall \quad \text{(II.35)}$$

$$\sum \overrightarrow{F_{ext}} = \frac{D}{Dt} \int_{sys} \rho\ \vec{V}\, d\forall \quad \text{(II.36)}$$

$\rho\ \vec{V}\, d\forall$ is the momentum of a differential element.
The Reynolds transport theorem is expressed for linear momentum as

$$\frac{D(m\,\vec{V})_{sys}}{Dt} = \frac{D}{Dt}\int_{CV} \rho\ \vec{V}\, d\forall + \int_{CS} \rho\ \vec{V}\,(\vec{V}_r\,.\vec{n})dA \quad \text{(II.37)}$$

General expression:

$$\sum \overrightarrow{F_{ext}} = \frac{D}{Dt}\int_{CV} \rho\ \vec{V}\, d\forall + \int_{CS} \rho\ \vec{V}\,(\vec{V}_r\,.\vec{n})dA \quad \text{(II.38)}$$

Where

$\vec{V}_r = (\vec{V} - \vec{V}_{CS})$ is the fluid velocity relative to the control surface.

$\vec{V}$ is the fluid velocity as viewed from an inertial reference frame.

For a fixed control volume (no motion or deformation of the control volume), $\vec{V}_r = \vec{V}$, and the linear momentum equation becomes

$$\sum \overrightarrow{F_{ext}} = \frac{D}{Dt}\int_{CV} \rho\ \vec{V}\, d\forall\ +\ \int_{CS} \rho\ \vec{V}\left(\vec{V}.\vec{n}\right)dA \tag{II.39}$$

- **Special Cases**

➢ *Steady flow:*

$$\sum \overrightarrow{F_{ext}} = \int_{CS} \rho\ \vec{V}\left(\vec{V}_r.\vec{n}\right)dA \tag{II.40}$$

➢ *Steady flow and fixed control volume*

$$\sum \overrightarrow{F_{ext}} = \int_{CS} \rho\ \vec{V}\left(\vec{V}.\vec{n}\right)dA \tag{II.41}$$

The mass flow rate $\dot{m}$ into or out of the control volume across an inlet or outlet at which ρ is nearly constant is

Mass flow rate: $\quad \dot{m} = \int_A \rho\left(\vec{V}.\vec{n}\right) dA = \rho\, V_{avg}\, A$ (II.42)

If $\vec{V}$ were uniform ($\vec{V} = \overrightarrow{V_{avg}}$) across the inlet or outlet.

Momentum flow rate across a uniform inlet or outlet:

$$\int_A \rho\,\vec{V}\left(\vec{V}.\vec{n}\right) dA = \rho\, V_{avg}\, A\, \overrightarrow{V_{avg}} = \dot{m}\, \overrightarrow{V_{avg}} \tag{II.43}$$

- **Momentum-Flux Correction Factor, β**

Unfortunately, the velocity across most inlets and outlets of practical engineering interest is not uniform. But a dimensionless correction factor β, called the momentum-flux correction factor, is required, as first shown by the French scientist Joseph.

➢ *Fixed control volume*

➢ *Momentum flow rate across inlet or outlet:*

$$\int_A \rho\,\vec{V}\left(\vec{V}.\vec{n}\right) dA = \beta\, \dot{m}\, \overrightarrow{V_{avg}} \tag{II.44}$$

$$\sum \overrightarrow{F_{ext}} = \frac{D}{Dt}\int_{CV} \rho\ \vec{V}\ d\mathrm{V}\ +\ \sum_{out} \beta\ \dot{m}\ \overrightarrow{V_{avg}}\ - \sum_{in} \beta\ \dot{m}\ \overrightarrow{V_{avg}} \qquad \text{(II.45)}$$

Where:

$\beta = 1$, for the case of uniform flow over an inlet or outlet.

$\beta = \dfrac{\int_A \rho\, \vec{V}(\vec{V}.\vec{n})\, dA}{\dot{m}\ \overrightarrow{V_{avg}}}$, for the general case.

- **Steady Flow**

If the flow is also steady, the time derivative term in Eq. (II.45) vanishes and we are left with:

$$\sum \overrightarrow{F_{ext}} = \sum_{out} \beta\ \dot{m}\ \overrightarrow{V_{avg}}\ - \sum_{in} \beta\ \dot{m}\ \overrightarrow{V_{avg}} \qquad \text{(II.47)}$$

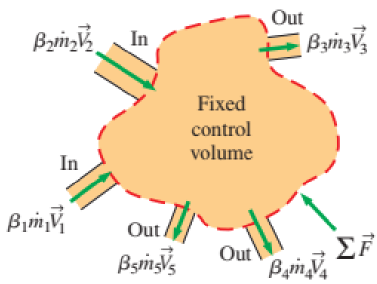


Figure II.9: The control volume contain multiple inlets and outlets

➤ **Steady Flow with One Inlet and One Outlet**

$$\sum \overrightarrow{F_{ext}} = \beta\ \dot{m}\ \overrightarrow{V_2} - \beta\ \dot{m}\ \overrightarrow{V_1} \qquad \text{(II.48)}$$

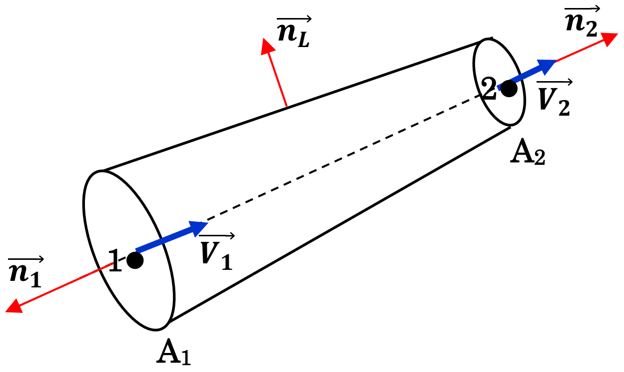


Figure II.10: The control volume contain one inlet and one outlet

- **Flow with no external Forces**

No external forces, $\sum \overrightarrow{\boldsymbol{F_{ext}}} = \mathbf{0}$

$$\mathbf{0} = \frac{D}{Dt}\int_{CV} \rho\ \vec{V}\ d\forall\ +\ \sum_{out} \beta\ \dot{m}\ \overrightarrow{V_{avg}}\ - \sum_{in} \beta\ \dot{m}\ \overrightarrow{V_{avg}} \qquad \text{(II.49)}$$

When the mass m of the control volume remains constant, we find:

$$\frac{D}{Dt}\int_{CV} \rho\ \vec{V}\ d\forall\ = m_{CV}\ \vec{a} \qquad \text{(II.50)}$$

In the case of a solid body, the thrust force is:

$$\sum \overrightarrow{F_{thrust}} = m_{body}\ \vec{a} = \sum_{out} \beta\ \dot{m}\ \overrightarrow{V_{avg}}\ - \sum_{in} \beta\ \dot{m}\ \overrightarrow{V_{avg}} \qquad \text{(II.51)}$$

II.7 CONSERVATION OF ENERGY EQUATION

The energy equation can be obtained by applying the first law of thermodynamics to an element of fluid in motion.

This law states that energy is neither created nor destroyed during a process, only changes forms. Therefore, every bit of energy must be considered during a process.

The conservation of energy principle for any system can be expressed simply as

$$\Delta E_{system} = E_{in} - E_{out} \qquad \text{(II.52)}$$

Consider a control volume CV(t) of a moving fluid (Fig.II.11).

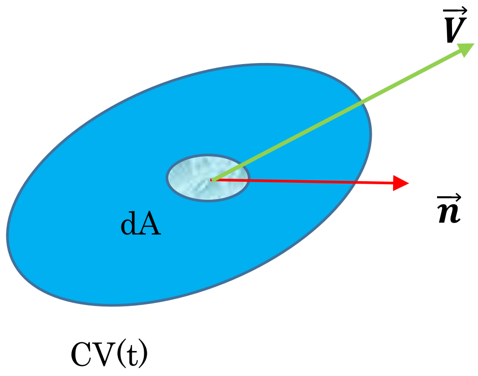


Figure II.11: Control volume of a fluid in motion

Consider a control volume CV(t) of a moving fluid (Fig.II.1). The total energy of the system at time t, equal to the sum of the internal and kinetic energies, is written as.

$$\boldsymbol{E_{system}} = \int_{system} \boldsymbol{\rho\, e\, d}\mathrm{V} \qquad \text{(II.53)}$$

Where :

$\boldsymbol{e}$ is the total energy per unit mass, as a function of Euler variables.

The conservation of energy of a closed system over time is given by.

$$\boldsymbol{dE_{system} = dQ - dW} \qquad \text{(II.54)}$$

Where :

$\boldsymbol{dE_{system}}$ is the energy change of the moving system.
$\boldsymbol{dQ}$ is the heat received by the moving system.

The conservation of energy for a fixed quantity of mass can be expressed in rate form as

$$\boldsymbol{\frac{dE_{system}}{dt} = \frac{dQ}{dt} - \frac{dW}{dt} = \dot{Q} - \dot{W}} \qquad \text{(II.55)}$$

$$\boldsymbol{\frac{dE_{system}}{dt} = \frac{d}{dt}\left(\int_{system} \rho\, e\, d\mathrm{V}\right) = \dot{Q} - \dot{W}} \qquad \text{(II.56)}$$

$$\frac{dE_{system}}{dt} = \frac{d}{dt}\left(\int_{system} \boldsymbol{\rho\, e\, d}\forall\right) = \int_{CV}\left[\frac{\partial}{\partial t}(\boldsymbol{\rho\, e}) + \boldsymbol{\nabla}(\boldsymbol{\rho\, e.V})\right] \boldsymbol{d}\forall \qquad \text{(II.57)}$$

The work rate of external forces is:

$$\dot{\boldsymbol{W}} = -\int_{CV}(\boldsymbol{F_{volume}} .\boldsymbol{V})\boldsymbol{d}\forall - \int_{CS}(\boldsymbol{F_{surface}} .\boldsymbol{V})\boldsymbol{d}\forall \qquad \text{(II.58)}$$

$$\dot{\boldsymbol{W}} = -\int_{CV}(\boldsymbol{F_v} .\boldsymbol{V})\boldsymbol{d}\forall - \int_{CV}\left(-\boldsymbol{\nabla}.(\boldsymbol{P.V}) + \boldsymbol{\nabla}.(\boldsymbol{\tau.V})\right)\boldsymbol{d}\forall \qquad \text{(II.59)}$$

The rate of heat transfer to the system is:

$$\dot{\boldsymbol{Q}} = \int_{CV}(\dot{\boldsymbol{q}}_{volume})\boldsymbol{d}\forall - \int_{CS}(\dot{\boldsymbol{q}}_{surface} .\boldsymbol{n})\boldsymbol{dA} \qquad \text{(II.60)}$$

$$\dot{\boldsymbol{Q}} = \int_{CV}(\dot{\boldsymbol{q}}_v)\boldsymbol{d}\forall - \int_{CV}(\boldsymbol{\nabla}.\dot{\boldsymbol{q}}_s)\boldsymbol{d}\forall \qquad \text{(II.61)}$$

Finally, we obtain the energy conservation equation.

$$\frac{dE_{system}}{dt} = \int_{CV}\left[\frac{\partial}{\partial t}(\boldsymbol{\rho\, e}) + \boldsymbol{\nabla}(\boldsymbol{\rho\, e.V})\right] \boldsymbol{d}\forall = \int_{CV}(\boldsymbol{F_v} .\boldsymbol{V})\boldsymbol{d}\forall + \int_{CV}\left(-\boldsymbol{\nabla}.(\boldsymbol{P.V}) + \boldsymbol{\nabla}.(\boldsymbol{\tau.V})\right)\boldsymbol{d}\forall + \int_{CV}(\dot{\boldsymbol{q}}_v)\boldsymbol{d}\forall - \int_{CV}(\boldsymbol{\nabla}.\dot{\boldsymbol{q}}_s)\boldsymbol{d}\forall \qquad \text{(II.62)}$$

$$\int_{CV}\left[\frac{\partial}{\partial t}(\boldsymbol{\rho\, e}) + \boldsymbol{\nabla}(\boldsymbol{\rho\, e.V}) - (\boldsymbol{F_v} .\boldsymbol{V}) - \left(-\boldsymbol{\nabla}.(\boldsymbol{P.V}) + \boldsymbol{\nabla}.(\boldsymbol{\tau.V})\right) - (\dot{\boldsymbol{q}}_v) + (\boldsymbol{\nabla}.\dot{\boldsymbol{q}}_s)\right] \boldsymbol{d}\forall = \boldsymbol{0} \qquad \text{(II.63)}$$

$$\frac{\partial}{\partial t}(\boldsymbol{\rho\, e}) + \boldsymbol{\nabla}(\boldsymbol{\rho\, e.V}) = +(\boldsymbol{F_v} .\boldsymbol{V}) + \left(-\boldsymbol{\nabla}.(\boldsymbol{P.V}) + \boldsymbol{\nabla}.(\boldsymbol{\tau.V})\right) + (\dot{\boldsymbol{q}}_v) - (\boldsymbol{\nabla}.\dot{\boldsymbol{q}}_s) \qquad \text{(II.64)}$$

- **The Energy Equation in Temperature Terms**

The energy equation for a Newtonian fluid where electromagnetic energy transfer and radiation are neglected is written as follows.

$$\rho C_p \frac{DT}{Dt} = \nabla . k\, \nabla T + \beta T \frac{Dp}{Dt} + \mu \Phi \qquad \text{(II.65)}$$

C_p is the specific heat at constant pressure.

k is the thermal conductivity.

p is the pressure.

β is the coefficient of thermal expansion.

Φ is the viscous dissipation function.

μ is the dynamic viscosity.

The coefficient of thermal expansion β is a material property defined as:

$$\beta = -\frac{1}{\rho} \left[\frac{\partial \rho}{\partial T}\right]_p \qquad \text{(II.66)}$$

In Cartesian coordinates Φ is given by :

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \qquad \text{(II.67)}$$

Special Cases

- **Incompressible Fluid:** $\beta = 0$ and $C_p = C_v = C$

$$\rho C_p \frac{DT}{Dt} = \nabla . k\, \nabla T + \mu \Phi \qquad \text{(II.68)}$$

- **Incompressible fluid with constant thermal conductivity :**

$$\rho C_p \frac{DT}{Dt} = k\, \nabla^2 T + \mu \Phi \qquad \text{(II.69)}$$

EXERCICES

EXERCICE N°1

The water flows through a rectangular channel 2 m wide, as shown in Fig.Ex.1. The free surface is at h =1 m from the bottom of the channel.
The velocity distribution along a flow section is given by:

$$\boldsymbol{v = v_{max}\left(\frac{y}{h}\right)^{1/2} \qquad , \quad v_{max} = 3\ m/s}$$

Determine volume flow rate, mass flow rate and average velocity.

$\rho_{water} = 10^3\ kg/m^3$

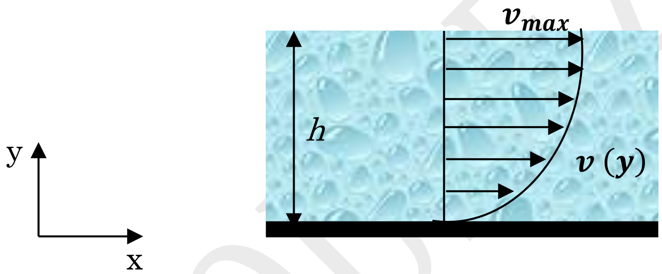


Fig. Ex.1

EXERCICE N°2

A 60 cm diameter pipe is subdivided into two branches, 40 cm and 30 cm in diameter. If the water flow rate in the main pipe is 1.5 m³/s and the average velocity in the 30 cm diameter pipe is 7.5 m/s.
Determine the volume flow rate, mass flow rate and average velocity in the 40 cm diameter pipe.

$\rho_{water} = 10^3\ kg/m^3$

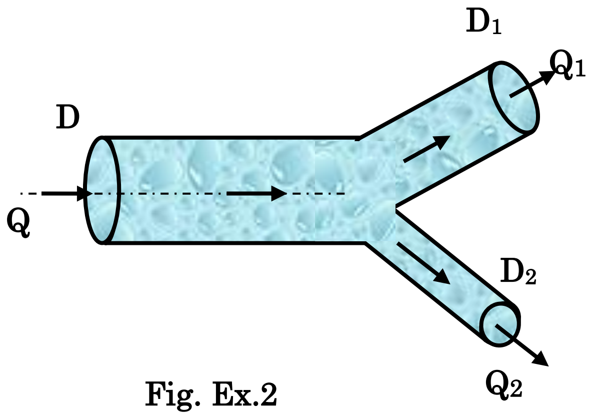


Fig. Ex.2

EXERCICE N°3

The flow of an incompressible fluid is described by: $\vec{V} = 3y\,\vec{i} + 2x\,\vec{j}$

Does this flow satisfy the continuity equation?

EXERCICE N°4

Consider the following velocity field:

$u = -2x \quad , v = 2y + 2$

Show that the continuity equation is satisfied.

EXERCICE N°5

Consider steady, incompressible flows defined by the following velocity fields:

1) $u = -x/(x^2 + y^2) \quad ; v = -y/(x^2 + y^2)$.

2) $u = 8/(x^2 + y^2) \quad ; v = 8xy$

EXERCICE N°6

Water flows through the pipe shown in Fig. Ex.6. Water is considered an ideal fluid. The figure is in a horizontal position.

Determine the forces exerted by the pipe wall on the water.

$\rho_{water} = 10^3$ kg/m³ ; g = 10 m/s² ; $D_1 = 40$ cm ; $D_2 = 20$ cm ; $V_1 = 5$ m/s ; $\theta = 60°$; $P_1 = 3 \times 10^5$ Pa.

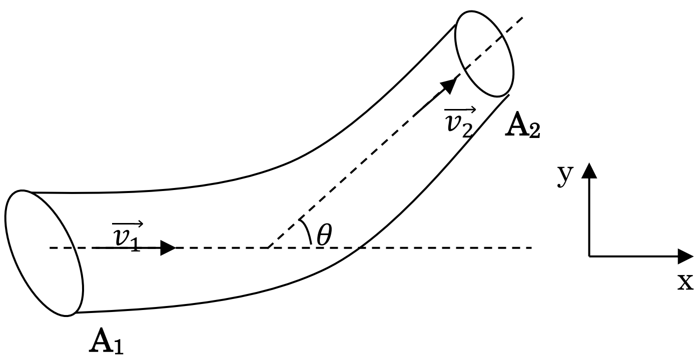


Fig. Ex. 6

EXERCICE N°7

A reducing elbow in a horizontal pipe is used to deflect water flow by an angle θ=45° from the flow direction while accelerating it. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 150 cm^2 at the inlet and 25 cm^2 at the exit. The elevation difference between the centers of the exit and the inlet is 40 cm. The mass of the elbow and the water in it is 50 kg.

Determine the anchoring force needed to hold the elbow in place.

Take the momentum flux correction factor to be 1.03 at both the inlet and outlet.

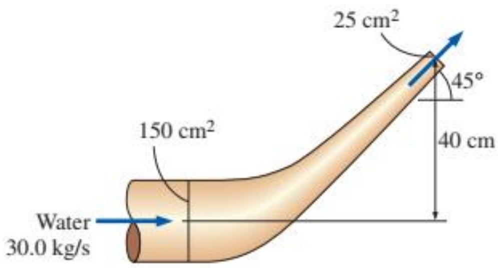


Fig. Ex.7

EXERCICE N°8

An oil with specific gravity SG = 0.920, flows through a horizontal curved pipe as shown in Fig. Ex.8. The flow rate is 250 *l/s*. All head losses are negligible.

Find the magnitude and direction of the force exerted by the oil on the bent pipe.

$\rho_{eau} = 10^3\ kg/m^3$; $g = 9.81\ m/s^2$; $D = 300\ mm$; $P_1 = P_2 = 39.24 \times 10^4\ Pa$.

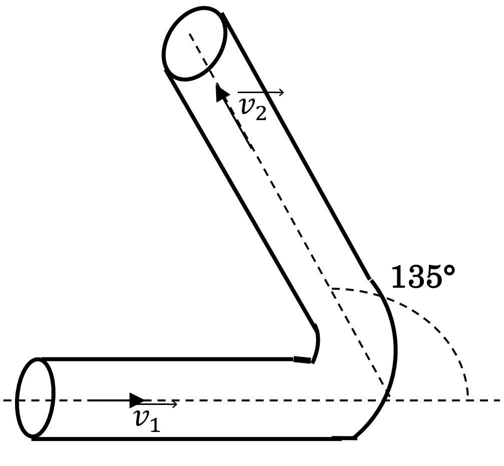


Fig. Ex. 8

EXERCICE N°9

A nozzle of diameter 2 cm is fitted to a duct of diameter 4 cm. Determine the force exerted by the nozzle on the water which is flowing through the pipe at the rate of 0.02 m³/s .

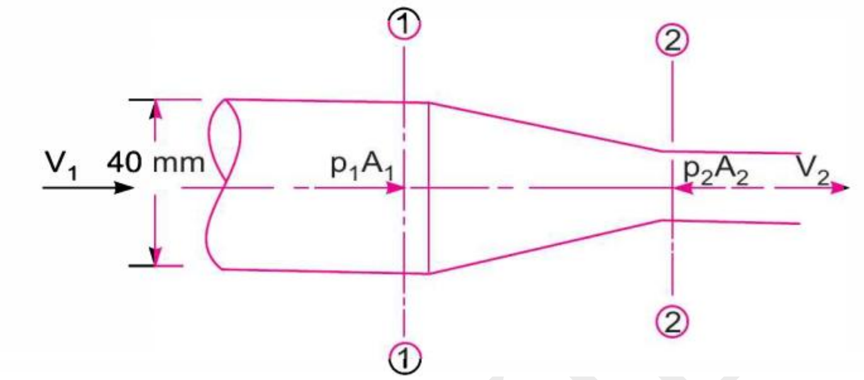


Fig.Ex.9

EXERCICE N°10

An incompressible fluid of density ρ and viscosity μ flows through a horizontal curved duct that rotates the flow by 180°. The cross-sectional area of the conduit remains constant, as in Fig.Ex.10.

Determine the horizontal force of the fluid force exerted on the walls of the duct.

$A_1 = A_2 = 0.025$ m², ρ = 998.2 kg/m³, $\beta_1 = 1.01$, $\mu = 1.003 \times 10^{-3}$, $\beta_2 = 1.03$, $V_1 = 10$ m/s, $P_{1,gage} = 78.47$ kPa, and $P_{2,gage} = 65.23$ kPa

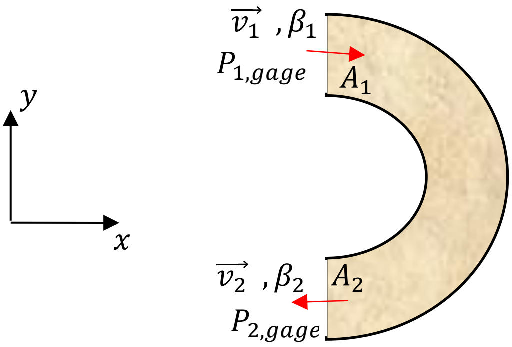


Fig. Ex.10

EXERCICE N°11

Water flows through a horizontal curved duct as shown in Fig.Ex.11.

All head losses are neglected.

Calculate and represent the force exerted by the water on the curved duct.

$\rho_{water} = 10^3$ kg/m^3, $g = 9.81$ m/s^2, $D_1 = 250$ mm, $D_2 = 150$ mm, $P_1 = 300$ kPa, $V_1 = 8$ m/s

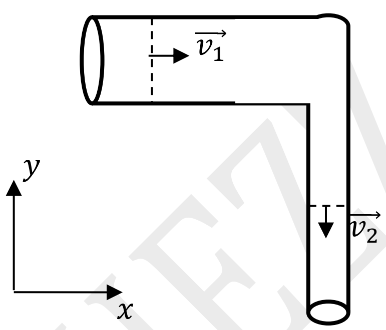


Fig.Ex.11

EXERCICE N°12

A duct of 30 cm diameter conveying 0.30 m^3/s of water has a right angled bend in a horizontal plane. Calculate and represent the force exerted on the bend.

$\rho_{water} = 10^3$ kg/m^3, $g = 9.81$ m/s^2, $P_1 = 24.525$ N/ cm^2, $P_2 = 23.544$ N/ cm^2

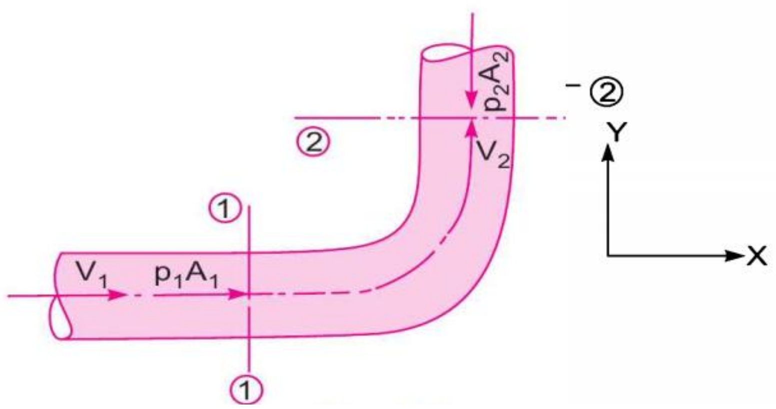


Fig.Ex.12

EXERCICE N°13

A 90° elbow in a horizontal pipe is used to direct water flow upward at a rate of 40 kg/s. The diameter of the entire elbow is 10 cm. The elbow discharges water into the atmosphere, and thus the pressure at the exit is the local atmospheric pressure. The elevation difference between the centers of the exit and the inlet of the elbow is 50 cm. The weight of the elbow and the water in it is considered to be negligible.

1) Determine the gage pressure at the center of the inlet of the elbow.

2) Determine the anchoring force needed to hold the elbow in place.

Take the momentum-flux correction factor to be 1.03 at both the inlet and the outlet.

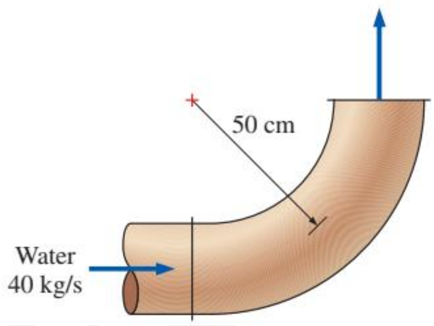
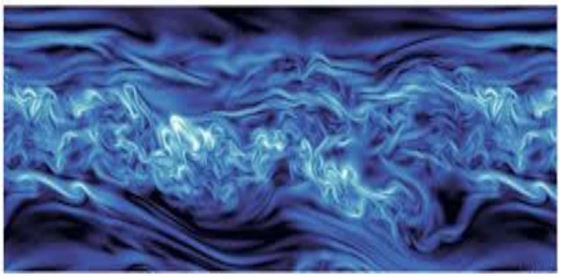
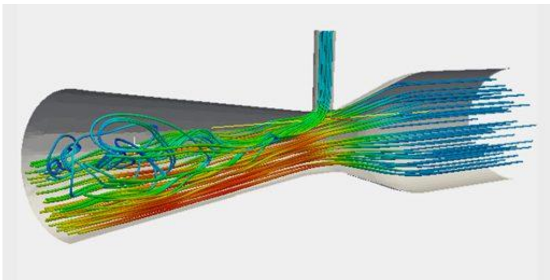


Fig.Ex.13

Chapter III
Fluid Dynamics





III.1 INTRODUCTION

In this chapter, we will study the flows of real fluids; we must also take into account the effect of friction forces due to the viscosity of the fluid.

III.2 STRESSES AND STRAINS IN CONTINUOUS MEDIA

For a real (viscous) fluid in motion, the surface forces are no longer only normal to the surface. There are tangential stresses due to viscosity (friction).
At a point (M) of a surface $\boldsymbol{dA}$, the surface force is expressed as:

$$\boldsymbol{d\vec{F}} = \boldsymbol{\overrightarrow{T_n}\, dA} \tag{III.1}$$

Where $\boldsymbol{\overrightarrow{T_n}}$ is the stress vector acting on the surface with normal $\boldsymbol{\vec{n}}$**.**

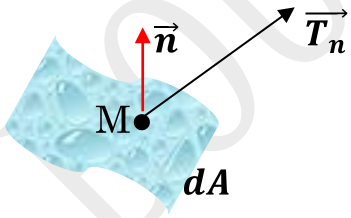


Figure III. 1: Stress vector acting on the surface dA

Consider a surface perpendicular to the (ox) axis.

The normal to this surface is : $\boldsymbol{\vec{n}} = \boldsymbol{\overrightarrow{e_x}}$

The stress exerted on this surface is then noted $\boldsymbol{\overrightarrow{T_x}}$ and can be decomposed as:

$$\boldsymbol{\overrightarrow{T_x}} = \boldsymbol{\sigma_{xx}\,\overrightarrow{e_x} + \sigma_{yx}\,\overrightarrow{e_y} + \sigma_{zx}\,\overrightarrow{e_z}} \tag{III.2}$$

We note that the components $\boldsymbol{\sigma_{yx}}$ and $\boldsymbol{\sigma_{zx}}$ are tangential components.

We will rather denote them $\boldsymbol{\tau_{yx}}$ and $\boldsymbol{\tau_{zx}}$ to distinguish them from the normal component $\boldsymbol{\sigma_{xx}}$.

We can similarly consider the surface perpendicular to the axis (oy).

We therefore have the stress:

$$\overrightarrow{T_y} = \sigma_{xy}\,\overrightarrow{e_x} + \sigma_{yy}\,\overrightarrow{e_y} + \sigma_{zy}\,\overrightarrow{e_z} \qquad \text{(III.3)}$$

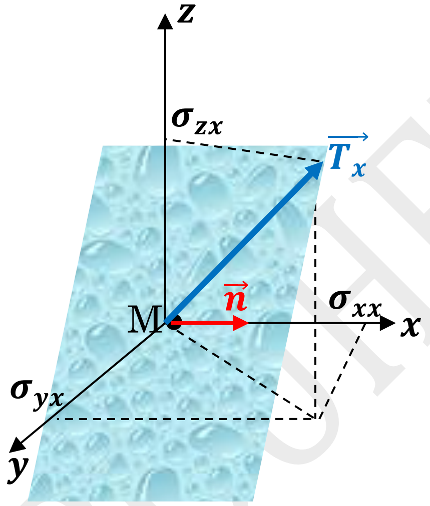


Figure III. 2 : Components of the stress $\overrightarrow{T_x}$

And for the surface perpendicular to the axis (oz), the stress is expressed by:

$$\overrightarrow{T_z} = \sigma_{xz}\,\overrightarrow{e_x} + \sigma_{yz}\,\overrightarrow{e_y} + \sigma_{zz}\,\overrightarrow{e_z} \qquad \text{(III.4)}$$

Now consider a surface of any orientation. In the Cartesian coordinates, its normal can be decomposed into:

$$\vec{n} = n_x\,\overrightarrow{e_x} + n_y\,\overrightarrow{e_y} + n_z\,\overrightarrow{e_z} \qquad \text{(III.5)}$$

In this case, the stress acting on this surface is expressed as:

$$\overrightarrow{T_n} = n_x\,\overrightarrow{T_x} + n_y\,\overrightarrow{T_y} + n_z\,\overrightarrow{T_z} \qquad \text{(III.6)}$$

$$\overrightarrow{\boldsymbol{T_n}} = \boldsymbol{n_x}\left(\boldsymbol{\sigma_{xx}}\,\overrightarrow{\boldsymbol{e_x}} + \boldsymbol{\tau_{yx}}\,\overrightarrow{\boldsymbol{e_y}} + \boldsymbol{\tau_{zx}}\,\overrightarrow{\boldsymbol{e_z}}\right) + \boldsymbol{n_y}\left(\boldsymbol{\tau_{xy}}\,\overrightarrow{\boldsymbol{e_x}} + \boldsymbol{\sigma_{yy}}\,\overrightarrow{\boldsymbol{e_y}} + \boldsymbol{\tau_{zy}}\,\overrightarrow{\boldsymbol{e_z}}\right) + \boldsymbol{n_z}\left(\boldsymbol{\tau_{xz}}\,\overrightarrow{\boldsymbol{e_x}} + \boldsymbol{\tau_{yz}}\,\overrightarrow{\boldsymbol{e_y}} + \boldsymbol{\sigma_{zz}}\,\overrightarrow{\boldsymbol{e_z}}\right) \tag{III.7}$$

So finally:

$$\overrightarrow{\boldsymbol{T_n}} = \begin{bmatrix} \boldsymbol{\sigma_{xx}} & \boldsymbol{\tau_{xy}} & \boldsymbol{\tau_{xz}} \\ \boldsymbol{\tau_{yx}} & \boldsymbol{\sigma_{yy}} & \boldsymbol{\tau_{yz}} \\ \boldsymbol{\tau_{zx}} & \boldsymbol{\tau_{zy}} & \boldsymbol{\sigma_{zz}} \end{bmatrix} \begin{bmatrix} \boldsymbol{n_x} \\ \boldsymbol{n_y} \\ \boldsymbol{n_z} \end{bmatrix} \Longrightarrow \overrightarrow{\boldsymbol{T_n}} = \overline{\overline{\boldsymbol{T}}}\,\vec{\boldsymbol{n}} \tag{III.8}$$

Where:

$\overline{\overline{\boldsymbol{T}}}$: Stress tensor

This stress tensor can then be decomposed into the sum of a spherical tensor and a zero trace tensor:

$$\overline{\overline{\boldsymbol{T}}} = \boldsymbol{\alpha}\,\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\check{\mathrm{T}}}} \tag{III.9}$$

$$\boldsymbol{\alpha}\,\overline{\overline{\boldsymbol{I}}} = \boldsymbol{\alpha} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\overline{\overline{\check{\mathrm{T}}}} = \begin{bmatrix} \boldsymbol{\sigma_{xx}} & \boldsymbol{\tau_{xy}} & \boldsymbol{\tau_{xz}} \\ \boldsymbol{\tau_{yx}} & \boldsymbol{\sigma_{yy}} & \boldsymbol{\tau_{yz}} \\ \boldsymbol{\tau_{zx}} & \boldsymbol{\tau_{zy}} & \boldsymbol{\sigma_{zz}} \end{bmatrix}$$: Viscosity stress tensor

Hence:

$$\begin{cases} \boldsymbol{\sigma_{xx}} = \boldsymbol{\alpha} + \grave{\boldsymbol{\sigma}}_{xx} \\ \boldsymbol{\sigma_{yy}} = \boldsymbol{\alpha} + \grave{\boldsymbol{\sigma}}_{yy} \\ \boldsymbol{\sigma_{zz}} = \boldsymbol{\alpha} + \grave{\boldsymbol{\sigma}}_{zz} \end{cases} \text{ and } \grave{\boldsymbol{\sigma}}_{xx} + \grave{\boldsymbol{\sigma}}_{yy} + \grave{\boldsymbol{\sigma}}_{zz} = \mathbf{0} = -\mathbf{3}\boldsymbol{\alpha} + \boldsymbol{\sigma_{xx}} + \boldsymbol{\sigma_{yy}} + \boldsymbol{\sigma_{zz}} \Longrightarrow$$

$$\boldsymbol{\alpha} = \frac{\mathbf{1}}{\mathbf{3}}\left(\boldsymbol{\sigma_{xx}} + \boldsymbol{\sigma_{yy}} + \boldsymbol{\sigma_{zz}}\right) \tag{III.10}$$

$$\overrightarrow{\boldsymbol{T_n}} = \overline{\overline{\boldsymbol{T}}}\,\vec{\boldsymbol{n}} = \boldsymbol{\alpha}\,\overline{\overline{\boldsymbol{I}}}\,\vec{\boldsymbol{n}} + \overline{\overline{\check{\mathrm{T}}}}\,\vec{\boldsymbol{n}} = \boldsymbol{\alpha}\,\vec{\boldsymbol{n}} + \overline{\overline{\check{\mathrm{T}}}}\,\vec{\boldsymbol{n}} \tag{III.11}$$

$$\boldsymbol{d}\vec{\boldsymbol{F}} = \overrightarrow{\boldsymbol{T_n}}\,\boldsymbol{dA} = \boldsymbol{\alpha}\,\vec{\boldsymbol{n}}\,\boldsymbol{dA} + \overline{\overline{\check{\mathrm{T}}}}\,\vec{\boldsymbol{n}}\,\boldsymbol{dA} \tag{III.12}$$

$\boldsymbol{\alpha}\,\vec{\boldsymbol{n}}\,\boldsymbol{dA}$: Force normal to the surface, it is therefore the hydrostatic pressure force.

$$\alpha = -P \Longrightarrow P = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) \tag{III.13}$$

$$d\vec{F} = -P\ \vec{n}\, dA + \overline{\overline{\mathrm{T}}}\,\vec{n}\, dA \tag{III.14}$$

III.3 EQUATION OF REAL FLUIDS MOTION

We take the differential fluid element as a material element instead of a control volume $(d\forall = dx\, dy\, dz)$. In other words, we think of the fluid within the differential element as a tiny system of fixed identity, moving with the flow. The acceleration of this fluid element is $\vec{a} = \frac{D\vec{V}}{Dt}$ by definition of the material acceleration. By Newton's second law applied to a material element of fluid,

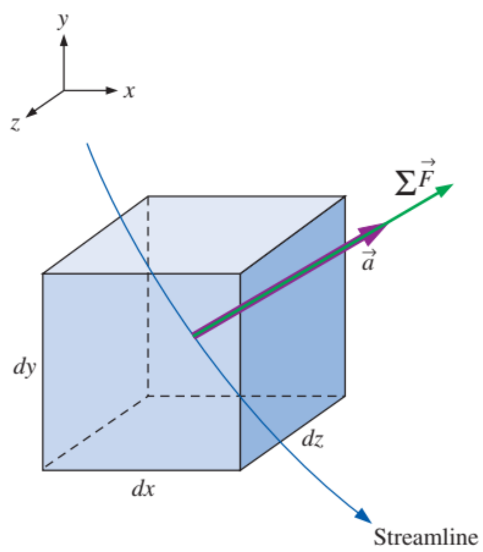


Figure III. 3 : Fluid element in motion

$$\sum \overrightarrow{F_{ext}} = m\,\vec{a} = \rho\, d\forall\, \frac{D\vec{V}}{Dt} \tag{III.15}$$

The forces acting on the fluid element are as follows:

- body forces: the weight of the fluid given by : $dm\,\vec{g} = \rho\, d\forall\,\vec{g}$

- Surface forces : $\boldsymbol{d\vec{F} = \overrightarrow{T_n}\, dA = \bar{\bar{T}}\,\vec{n}\, dA} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \vec{n}\, dA$

We can decompose $\boldsymbol{d\vec{F}}$ into three components : $\boldsymbol{d\vec{F} = dF_x\vec{\imath} + dF_y\vec{\jmath} + dF_z\vec{k}}$

The following component (y) is written:

$$dF_y = \left[(\tau_{yx})_{x+dx} - (\tau_{yx})_x\right] dy\, dz + \left[(\sigma_{yy})_{y+dy} - (\sigma_{yy})_y\right] dx\, dz + \left[(\tau_{yz})_{z+dz} - (\tau_{yz})_z\right] dx\, dy \quad \text{(III.16)}$$

By a first-order development, we have:

$$(\tau_{yx})_{x+dx} = (\tau_{yx})_x + \frac{\partial \tau_{yx}}{\partial x} dx \quad \text{(III.17)}$$

$$(\sigma_{yy})_{y+dy} = (\sigma_{yy})_y + \frac{\partial \sigma_{yy}}{\partial y} dy \quad \text{(III.18)}$$

$$(\tau_{yz})_{z+dz} = (\tau_{yz})_z + \frac{\partial \tau_{yz}}{\partial z} dz \quad \text{(III.19)}$$

Therefore,

$$dF_y = \frac{\partial \tau_{yx}}{\partial x} dx\, dy\, dz + \frac{\partial \sigma_{yy}}{\partial y} dx\, dy\, dz + \frac{\partial \tau_{yz}}{\partial z} dx\, dy\, dz \quad \text{(III.20)}$$

$$dF_y = \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) dx\, dy\, dz = \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) dV \quad \text{(III.21)}$$

And by analogy, according to the other axes:

$$dF_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) dx\, dy\, dz = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) dV \quad \text{(III.22)}$$

$$dF_z = \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) dx\, dy\, dz = \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) dV \quad \text{(III.23)}$$

Which can be written as:

$$d\vec{F} = \vec{\nabla}\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} dV = \vec{\nabla}\,\overline{\overline{T}}\, d\vee \qquad \text{(III.24)}$$

Therefore :

$$d\vec{F} + \rho\, d\vee\, \vec{g} = \rho\, d\vee\, \frac{D\vec{V}}{Dt} \qquad \text{(III.25)}$$

$$\vec{\nabla}\,\overline{\overline{T}}\, d\vee + \rho\, d\vee\, \vec{g} = \rho\, dV\, \frac{D\vec{V}}{Dt} \qquad \text{(III.26)}$$

$$\vec{\nabla}\,\overline{\overline{T}} + \rho\, \vec{g} = \rho\, \frac{D\vec{V}}{Dt} \qquad \text{(III.27)}$$

We can then show the two parts of the stress tensors:

$$\vec{\nabla}\,\overline{\overline{T}} = -\vec{\nabla}P + \vec{\nabla}\,\overline{\overline{\tilde{T}}} \qquad \text{(III.28)}$$

So finally:

$$\rho\, \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \vec{\nabla}\,\overline{\overline{\tilde{T}}} + \rho\, \vec{g} \qquad \text{(III.29)}$$

This is the fundamental equation of real fluid dynamics.

III.3 .1 THE NAVIER–STOKES EQUATIONS

The Navier–Stokes equations are differential equations for:

- Newtonian fluids, defined as fluids for which the shear stress is linearly proportional to the shear strain rate (viscosity, $\mu = constant$).
- Incompressible fluids, $\rho = constant$.

Therefore, for Newtonian and incompressible fluids, equation (III.28) becomes:

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\,.\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\ \vec{g} + \mu\,\nabla^2\vec{V} \qquad \text{(III.30)}$$

This equation (III.30) is called the Navier–Stokes equation for incompressible flow with constant viscosity.

➢ Navier–Stokes Equations in Cartesian Coordinates

The Navier–Stokes equation (III.30) is expanded in Cartesian coordinates (***x, y, z***) and $\vec{V}(u, v, w)$:

x – component:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho\,g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{(III.31)}$$

y – component:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho\,g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(III.32)}$$

z – component:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho\,g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \text{(III.33)}$$

➢ Navier–Stokes Equations in Cylindrical Coordinates

The Navier–Stokes equation (III.30) is expanded in Cylindrical Coordinates (r, θ, z) and $\vec{V}(u_r, u_\theta, u_z)$:

r – component:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho\,g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right] \qquad \text{(III.34)}$$

θ – component:

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r\,u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho\,g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right] \qquad \text{(III.35)}$$

z – component:

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho\, g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \quad \text{(III.36)}$$

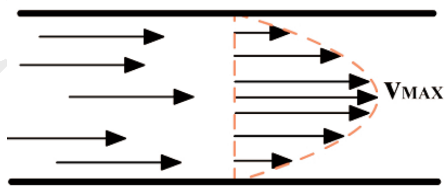
III.4 FLOW REGIMES

III.4.1 Reynolds experience

The experiment carried out by REYNOLDS (Osborne Reynolds (British engineer (1842 – 1912)) is schematized in figure (III.4). The liquid flows through a straight cylindrical pipe at a velocity regulated by a valve. Near the inlet section is a dye source, whose density is close to that of the liquid in the tank, which allows a colored net to be observed. At low flow velocities, the dye stream flows without mixing with other layers of the liquid (laminar flow). For average flow velocities, the colored net is animated by pulsations (transient phase). At high speeds, increased pulsations cause the colored net to break up. A disordered movement of fluid particles is observed in any section of the pipe (turbulent regime).

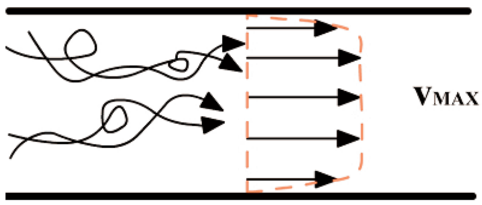
- **Laminar flow**

At low velocities, the fluid flow is laminar, characterized by straight and smooth streamlines and highly ordered motion. The distribution of velocities is parabolic.



- **Turbulent flow**

When the velocity increases above a critical value, the fluid flow becomes turbulent, characterized by velocity fluctuations and highly turbulent motion. The streamlines are not parallel. The velocity profile is flattened.



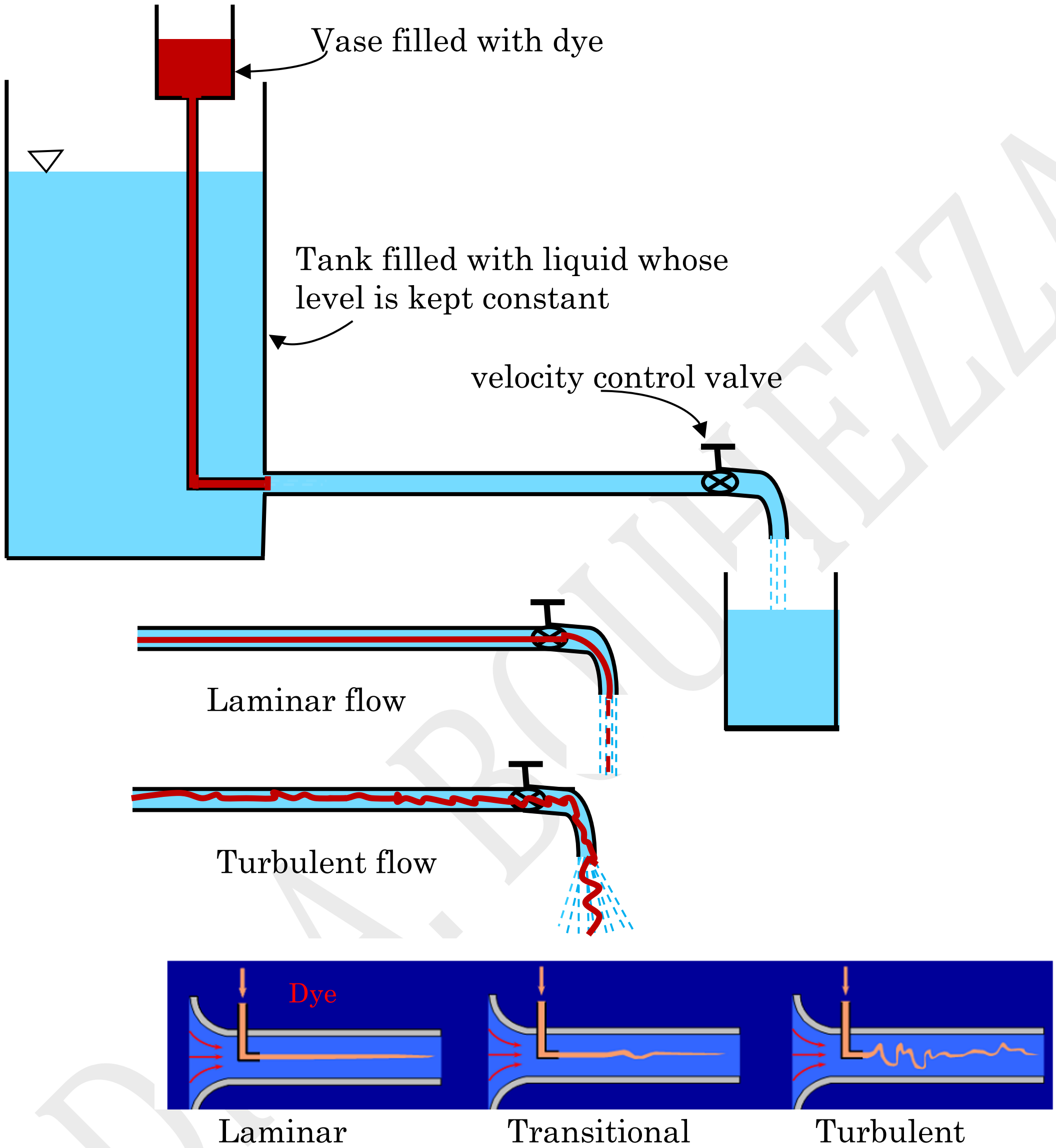


Figure III.4 : Reynolds experience

III.4.2 Reynolds Number

The transition from laminar to turbulent flow depends on the flow velocity, geometry, surface roughness, surface temperature, fluid type, and other factors. After extensive experiments in the 1880s, Osborne Reynolds discovered that the flow regime depends primarily on the ratio of inertial forces to viscous forces in the fluid. This ratio is called the Reynolds number "Re." For internal flow in a circular tube, this number is given by the following expression:

$$\boldsymbol{Re} = \frac{\boldsymbol{inertial\ forces}}{\boldsymbol{viscous\ forces}} = \frac{\boldsymbol{\rho\, V_{avg}\, D}}{\boldsymbol{\mu}} = \frac{\boldsymbol{V_{avg}\, D}}{\boldsymbol{\nu}} \qquad \text{(III.37)}$$

Where:

Re : Reynolds number (dimensionless)

D : Pipe diameter (m)

V_{avg} : Average fluid flow velocity (m/s)

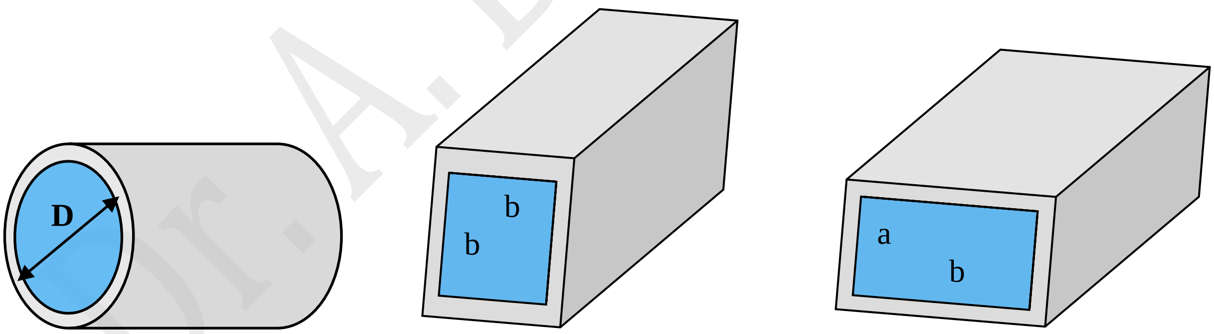
ρ : Fluid density (kg/ m^3)

μ : Dynamic fluid viscosity (Pa. s)

ν : Kinematic fluid viscosity (m^2/s)

In the case of fluid flow in a non-circular pipe, **D** is replaced by the ***hydraulic diameter D_h***:

$$\boldsymbol{D_h = 4 \times \frac{cross\text{−}sectional\ area}{wetted\ perimeter} = 4 \times \frac{A_c}{Per}} \qquad \text{(III.38)}$$



$$\boldsymbol{D_h = 4\,\frac{(\frac{\pi D^2}{4})}{\pi D} = D} \quad ; \quad \boldsymbol{D_h = 4\,\frac{b^2}{4b} = b} \quad ; \quad \boldsymbol{D_h = 4\,\frac{ab}{2(a+b)} = \frac{2\,ab}{(a+b)}}$$

Figure III.5: Hydraulic diameter of some ducts

The transition point from laminar to turbulent flow corresponds to a critical Reynolds number "Re_{cr}". The value of the critical Reynolds number is different for different geometries and flow conditions. For internal flow in a circular tube, the generally accepted critical value of the Reynolds number is $Re_{cr} = 2300$.

- If $Re \lesssim 2300$ laminar flow
- If $2300 \lesssim Re \lesssim 4000$ transitional zone
- If $Re \gtrsim 4000$ turbulent flow

III.5 PRESSURE DROP AND HEAD LOSS

For a real fluid flow, the head is not constant along a streamline, or as they are often called, the pressure loss. This pressure loss depend on the shape, dimensions and roughness of the pipe, the flow velocity and the viscosity of the fluid. We can therefore distinguish two types of head losses.

III.5.1 Major head Losses

Major head losses are the energy dissipations of the fluid during its flow. They are due to the resistance to flow resulting from the viscosity of the fluid on the one hand, and to the friction of the fluid on the internal wall of the straight pipe during the flow of the fluid on the other hand.
Major head losses are calculated using the Darcy-Weisbach formula:

Major head losses: $$\boldsymbol{h_{L,\ major} = f\,\frac{L\,V_{avg}^2}{2\,g\,D_h}} \qquad \text{(m)} \qquad \text{(III.39)}$$

Pressure loss: $$\boldsymbol{\Delta P_{L,\ major} = f\,\frac{\rho\,L\,V_{avg}^2}{2\ D_h}} \qquad \text{(Pa)} \qquad \text{(III.40)}$$

Where:

h_L : head loss due to friction (m)

L : length of the pipe (m)

D_h : hydraulic diameter of the pipe (m)

V_{avg} : Average velocity of fluid (m/s)

g : gravitational acceleration (m/s^2)

f : Darcy friction factor (dimensionless), it depends on the flow regime and the roughness of the pipe, ; $f = F\left(Re, \frac{\varepsilon}{D}\right)$.

Depending on the flow regime, different cases are distinguished:

- ***Case of laminar flow (Re < 2300)***

For the laminar flow, the roughness of the pipe has no discernible effect, and the pressure loss coefficient f was determined analytically by Poiseuille:

$$\boldsymbol{f = \frac{64}{Re} = \frac{64\,\mu}{\rho\, D\, V_{avg}}} \tag{III.41}$$

Poiseuille's Law

For laminar flow, in a horizontal cylindrical pipe, of length **L** and diameter **D**, the volume flow rate of the fluid is given by:

The average velocity for laminar flow in a horizontal pipe is,

$$\mathbf{V_{avg}} = \frac{\mathbf{D^2}}{\mathbf{32\,\mu\,L}}\boldsymbol{\Delta P_L} \tag{III.42}$$

Then the volume flow rate for laminar flow through a horizontal pipe of diameter **D** and length **L** becomes

$$\boldsymbol{Q} = \mathbf{V_{avg}\,A_c} = \frac{\mathbf{\pi\,D^4}}{\mathbf{128\,\mu\,L}}\boldsymbol{\Delta P_L} \tag{III.43}$$

This equation (III.43) is known as Poiseuille's law

- ***Case of turbulent flow (Re > 4000)***

For the turbulent flow regime, the relationship between the pressure drop coefficient , the Reynolds number (Re) and the relative roughness $(\frac{\varepsilon}{D})$ is more complex. A model of this relationship is provided by the Colebrook – White equation:

$$\boldsymbol{\frac{1}{\sqrt{f}} = -2\,log\left(\frac{\varepsilon}{3{,}71\;D} + \frac{2{,}51}{Re\sqrt{f}}\right)} \tag{III.44}$$

To simplify this relationship, we can seek to know whether the flow is hydraulically smooth or rough to evaluate the predominance of the two terms.

- For smooth turbulent flow the Colebrook – White formula becomes:

$$\frac{1}{\sqrt{f}} = -2\,log\left(\frac{2{,}51}{Re\sqrt{f}}\right):$$ Prandtl equation. (III.45)

- For fully rough turbulent flow the Colebrook–White formula becomes:

$$\frac{1}{\sqrt{f}} = -2\,log\left(\frac{\varepsilon}{3{,}71\,D}\right):$$ Von Kármán equation. (III.46)

A simpler empirical formula valid for smooth turbulent flow ($2300 < Re < 10^5$), is given by Blasius : $f = 0{,}3164\ Re^{-0{,}25}$ (III.47)

III.4.2 MOODY Chart

The Moody diagram is a dimensionless graph that plots the head loss coefficient (f) as a function of the Reynolds number (Re) and the relative roughness ($\frac{\varepsilon}{D}$) of the pipe for a full-section flow in a circular pipe. The Moody diagram is plotted from the Colebrook–White equation. This dimensionless graph is used to determine the head loss coefficient (f).

- **Presentation of the diagram**

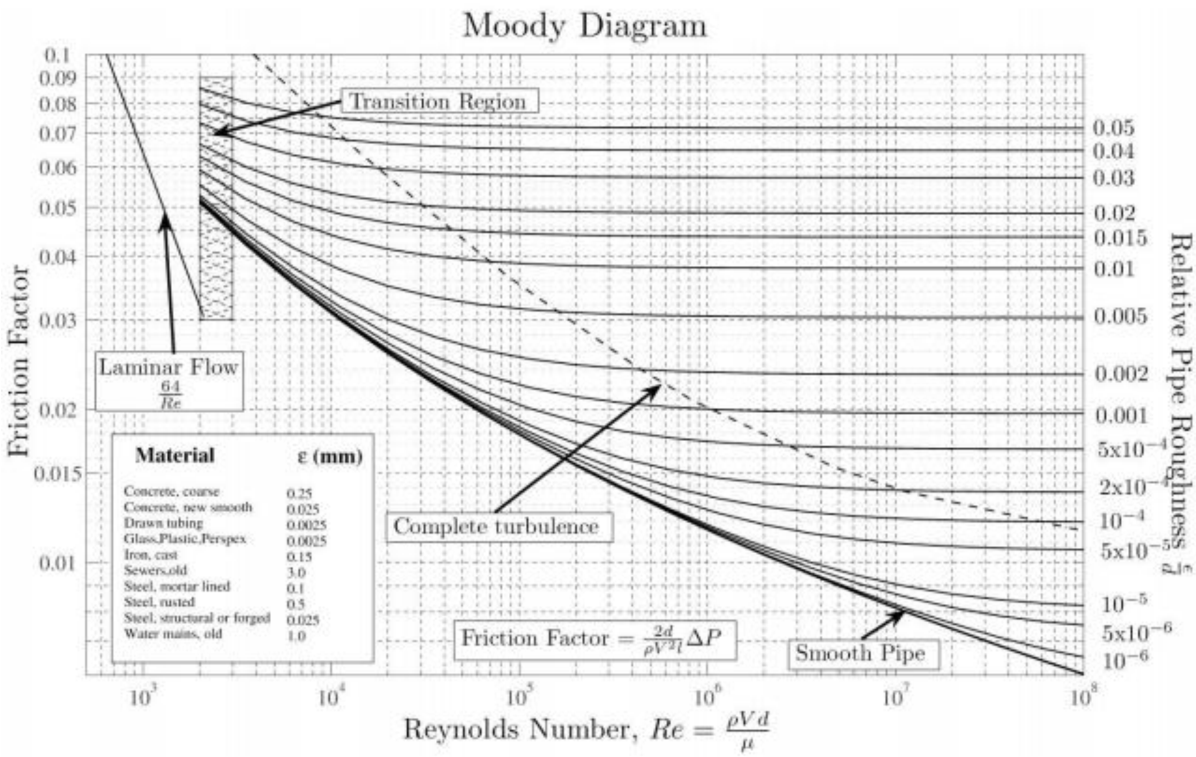


Figure III.5 : *Moody chart*

- **Use in laminar flow**

The coefficient $\boldsymbol{f}$ is read directly from the line (Laminar flow ($\frac{\boldsymbol{64}}{\boldsymbol{Re}}$)).

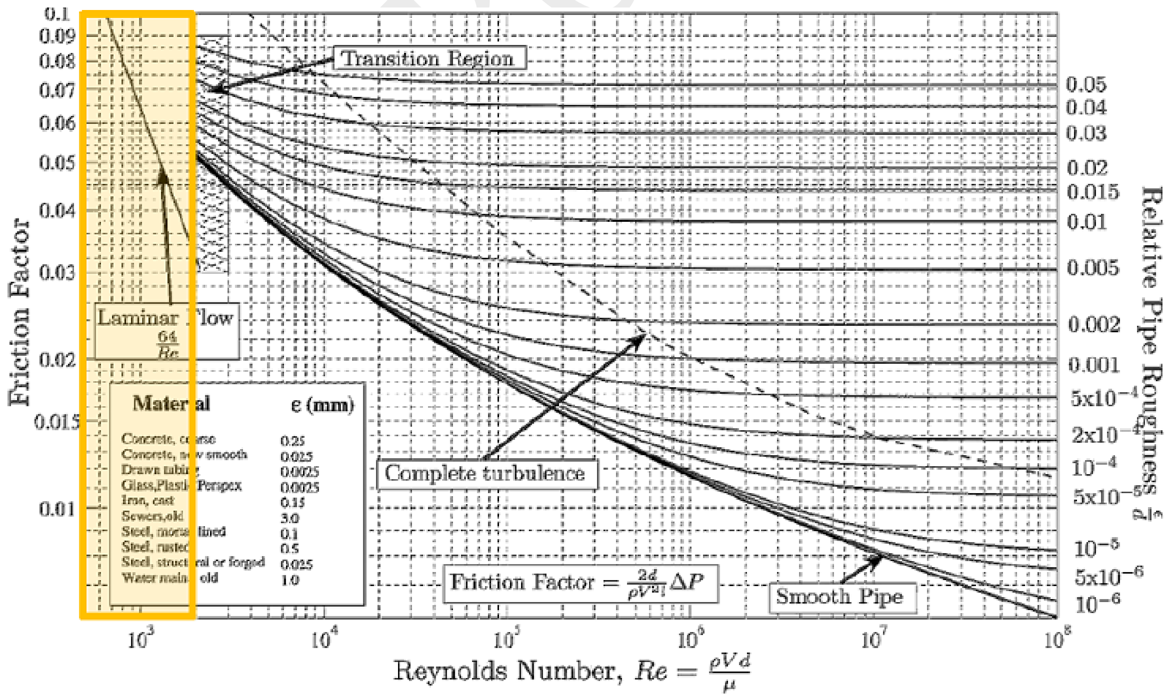


Figure III.6 : Moody chart used in laminar flow

- **Use in turbulent flow**

We calculate the relative roughness ($\frac{\varepsilon}{D}$) and select the corresponding curve.

We determine the Reynolds number and read f at the intersection of the curve and the vertical.

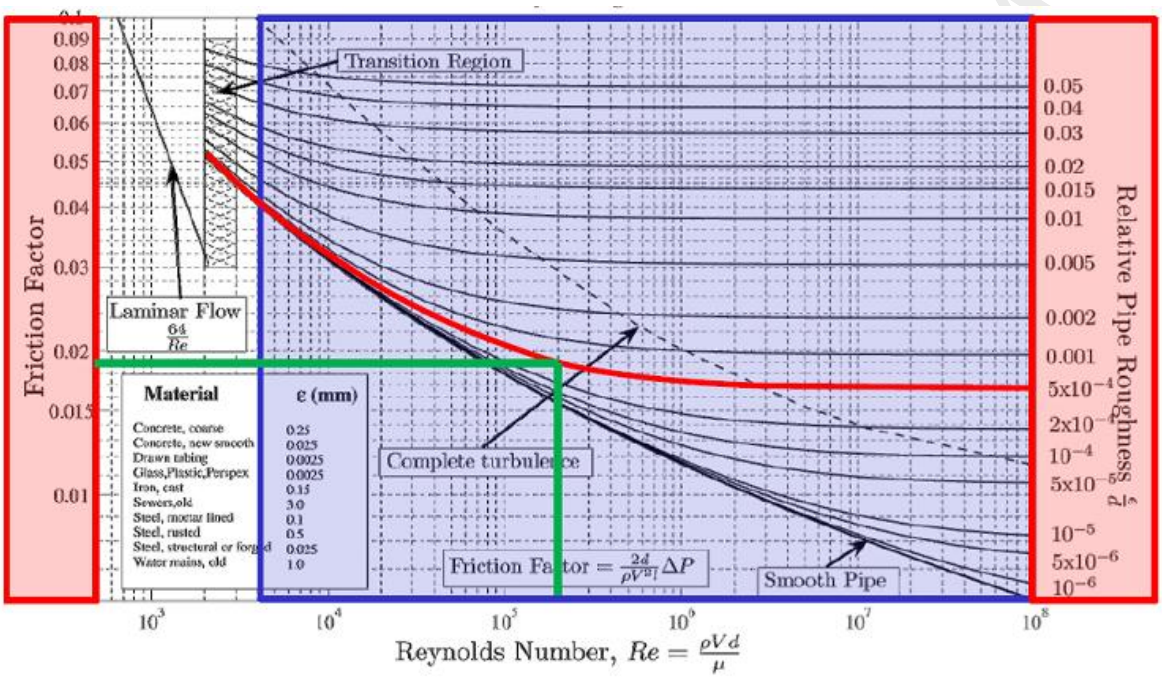


Figure III.7 : Moody chart used in turbulent flow

III.4.3 Minor head Losses

The minor head losses are due to accidents encountered in the pipes (valves, bends, elbows, tees, inlets, exits, expansions, contractions, etc.). They are calculated using the following general formula:

Minor head losses $\boldsymbol{h_{L,minor}} = \boldsymbol{K_L \frac{V^2}{2\,g}}$ (m) (III.48)

Pressure losses $\boldsymbol{\Delta P_{L,minor}} = \boldsymbol{K_L \frac{\rho\, V^2}{2}}$ (Pa) (III.49)

Where:

$\boldsymbol{K_L}$: minor loss coefficient (dimensionless), $\boldsymbol{K_L}$ depends on the nature of the deformation.

III.4.3.1 Case of Sudden Expansion

The minor head loss in a sudden expansion is given by:

$$h_{L,minor} = \frac{(V_1 - V_2)^2}{2\,g} \tag{III.50}$$

As it is a sudden expansion, we write:

$$h_{L,minor} = K_L \frac{V_1^2}{2\,g} \tag{III.51}$$

The minor loss coefficient is given by:

$$h_{L\,,minor} = \frac{\frac{(V_1 - V_2)^2}{2\,g}}{\frac{V_1^2}{2\,g}} \tag{III.52}$$

Using the continuity equation, we obtain:

$$K_L = \left(1 - \frac{A_1}{A_2}\right)^2 \tag{III.53}$$

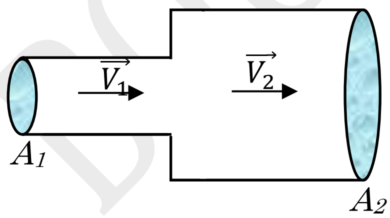


Figure III.8 : Sudden Expansion

III.4.3.2 Case of Sudden contraction

The minor head loss in a sudden contraction is given by:

$$h_{L\,,minor} = K_L \frac{V_2^2}{2\,g} \tag{III.54}$$

The minor head loss coefficient $\boldsymbol{K_L}$ is given by the following table:

$\frac{A_1}{A_2}$	0,01	0,10	0,20	0,30	0,40	0,60	0,80	1
K_L	0,5	0,45	0,41	0,36	0,30	0 ,18	0,06	0

Tableau III.1 : The minor head loss coefficient for sudden constrictions

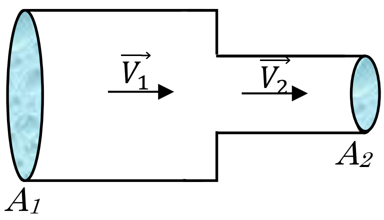


Figure III.9 : Sudden constriction

III.4.3.3 Case of Bends

The minor head loss in a bend is given by:

$$h_{L,minor} = K_L \frac{V^2}{2\,g} \tag{III.55}$$

The minor head coefficient K_L depends on the ratio of the radius of bend and the diameter of the pipe, and on the angle of the bend:

Smooth bend : $$K_L = \left[0,131 + 1,847\left(\frac{D}{2\,R}\right)^{\frac{7}{2}}\right]\frac{\theta}{90°} \tag{III.56}$$

Rough bend: $$K_L = 0,42\left(\frac{D}{R}\right)^{0,5} \tag{III.57}$$

Where:

K_L: Minor head loss coefficient in bends (dimensionless);

D: pipe diameter (m);

R: radius of bend (m);

θ: bend angle (degrees).

III.4.4 Total head losses

The total head losses in a piping system are determined from:

Total head losses (general):

$$h_{L,\ Total} = h_{L,\ major} + h_{L,\ minor} \tag{III.58}$$

$$\boldsymbol{h_{L,\ Total}} = \sum_i \boldsymbol{f_i} \frac{\boldsymbol{L_i}}{\boldsymbol{D_i}} \frac{\boldsymbol{V_i^2}}{\boldsymbol{2\,g}} + \sum_j \boldsymbol{K_{L,j}} \frac{\boldsymbol{V_j^2}}{\boldsymbol{2\,g}} \tag{III.59}$$

III.4 GENERALIZED BERNOULLI'S EQUATION

When a piping system (from point 1 to point 2) includes a pump and/or a turbine, the generalized Bernoulli equation for a fluid real in steady flow is expressed as follows:

$$\frac{\boldsymbol{P_1}}{\boldsymbol{\rho g}} + \boldsymbol{\alpha_1} \frac{\boldsymbol{V_1^2}}{\boldsymbol{2\,g}} + \boldsymbol{Z_1} + \boldsymbol{h_{pump,\ u}} = \frac{\boldsymbol{P_2}}{\boldsymbol{\rho g}} + \boldsymbol{\alpha_2} \frac{\boldsymbol{V_2^2}}{\boldsymbol{2\,g}} + \boldsymbol{Z_2} + \boldsymbol{h_{turbine,\ e}} + \boldsymbol{h_{L,\ Total}} \tag{III.60}$$

Where:

$\boldsymbol{h_{pump,\ u}} = \frac{\dot{\boldsymbol{W}}_{\boldsymbol{pump,\ u}}}{\boldsymbol{\rho g\,Q}}$ is the is the useful pump head supplied to the fluid (m).

$\dot{\boldsymbol{W}}_{\boldsymbol{pump,\ u}}$ is the useful pumping power supplied to the fluid (Watt)

$$\boldsymbol{\eta_{pump}} = \frac{\dot{\boldsymbol{W}}_{\boldsymbol{pump,\ u}}}{\dot{\boldsymbol{W}}_{\boldsymbol{pump}}}$$

$\boldsymbol{\eta_{pump}}$ is the efficiency of the pump.

$\dot{\boldsymbol{W}}_{\boldsymbol{pump}}$ is the mechanical power input (Watt).

$\boldsymbol{h_{turbine,\ e}} = \frac{\dot{\boldsymbol{W}}_{\boldsymbol{turbine,\ e}}}{\boldsymbol{\rho g\,Q}}$ is the turbine head extracted from the fluid (m).

$\dot{\boldsymbol{W}}_{\boldsymbol{turbine,\ e}}$ is the mechanical power extracted from the fluid by the turbine (Watts).

$$\boldsymbol{\eta_{turbine}} = \frac{\dot{\boldsymbol{W}}_{\boldsymbol{turbine}}}{\dot{\boldsymbol{W}}_{\boldsymbol{turbine,\ e}}}$$

$\boldsymbol{\eta_{turbine}}$ is the efficiency of the turbine.

$\dot{\boldsymbol{W}}_{\boldsymbol{turbine}}$ is the mechanical power output (Watts).

$h_{L,\ Total}$ is the total head losses (m).

α_1 and α_2 are the kinetic energy correction factors.

Fully developed turbulent flow $\alpha_1 = \alpha_2 \cong 1.05$

Fully developed laminar flow $\alpha_1 = \alpha_2 \cong 2$

EXERCISES

EXERCICE N°1

Consider a layer, of thickness h, of a viscous incompressible liquid, of viscosity μ and density ρ, flowing on an inclined plane making an angle α with the horizontal. In steady-state flow, this is unidirectional and the only non-zero component of velocity v depends only on z, i.e. $v(z)$. The free surface corresponds to $z = 0$, and the plane supporting the layer to $z = h$.

1) Under these conditions, show that the general Navier-Stokes equation:

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho[\vec{v}.\vec{\nabla}]\vec{v} = -\vec{\nabla}\,p + \rho\vec{g} + \mu\Delta\vec{v} \quad \text{is reduced to: } 0 = -\vec{\nabla}\,p + \rho\vec{g} + \mu\Delta\vec{v}$$

2) Show that in the absence of an external pressure gradient $\frac{\partial p}{\partial x} = 0$.

3) Determine the boundary conditions in terms of velocity.

4) Deduce the velocity profile of such a flow.

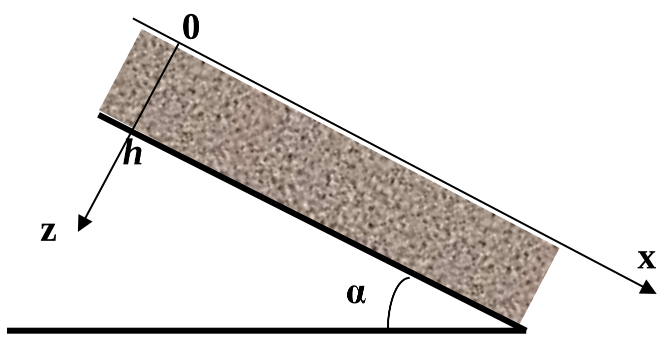


Fig. Ex. 1

EXERCICE N°2

Consider the steady, two-dimensional, incompressible velocity field of

$\vec{\boldsymbol{V}} = (\boldsymbol{a\,x} + \boldsymbol{b})\vec{\boldsymbol{\imath}} + (-\,\boldsymbol{a\,y} + \boldsymbol{c\,x})\vec{\boldsymbol{\jmath}}$, where $\boldsymbol{a}$, b, and c are constants.

The flow is steady and incompressible. The fluid has constant properties. The flow is two-dimensional in the xy –plane. Gravity does not act in either the x or y direction.

Calculate the pressure as a function of x and y.

EXERCICE N°3

Consider the following steady, two-dimensional, incompressible velocity field:

$\vec{\boldsymbol{V}} = -(\boldsymbol{a\,x^2})\vec{\boldsymbol{\imath}} + (\boldsymbol{2a\,x\,y})\vec{\boldsymbol{\jmath}}$, where $\boldsymbol{a}$ is a constant.

The fluid has constant properties. Gravity does not act in either the x or y direction.

Find the expression of the pressure as a function of x and y.

EXERCICE N°4

Consider the steady, plane flow of an incompressible, viscous fluid of constant viscosity μ between two parallel plates (Fig. Ex. 4). The plates move in their plane with constant velocity $U > 0$.

The flow is assumed to take place a with velocity parallel to the axis x, and the pressure depends only on x : $\boldsymbol{V} = \boldsymbol{u}(\boldsymbol{y})$ **and** $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{y})$

1) Demonstrate that $\frac{\partial p}{\partial x}$ is constant. Suppose is constant $\frac{\partial p}{\partial x} = -\boldsymbol{C}$

2) For this case, what are the boundary conditions for velocity?

3) Find the expression for the velocity profile.

4) Calculate the volume flow rate Q between the two plates of height 2 h and depth l =1.

Fig. Ex.4

EXERCICE N°5

If the pump's efficiency is 70%, what power does the pump need to deliver gasoline (SG = 0.850) with a volume flow of 1 m^3/s to tank B?

$\boldsymbol{\upsilon = 5 \times 10^{-5}\ m^2/s}$, the pipe is made of steel ($\boldsymbol{\varepsilon = 0.45\ mm}$), the minor head loss coefficients (($K_{L,entrance} = 0{,}5$, $K_{L,exit} = 1$, $K_{L,elbow} = 0{,}2$), total pipe length L =300 m, pipe diameter D = 60 cm.

$Z_1 = 20$ m , $Z_2 = 70$ m , $\rho_{water} = 10^3$ kg/m^3 ; g = 9.81 m/s^2 .

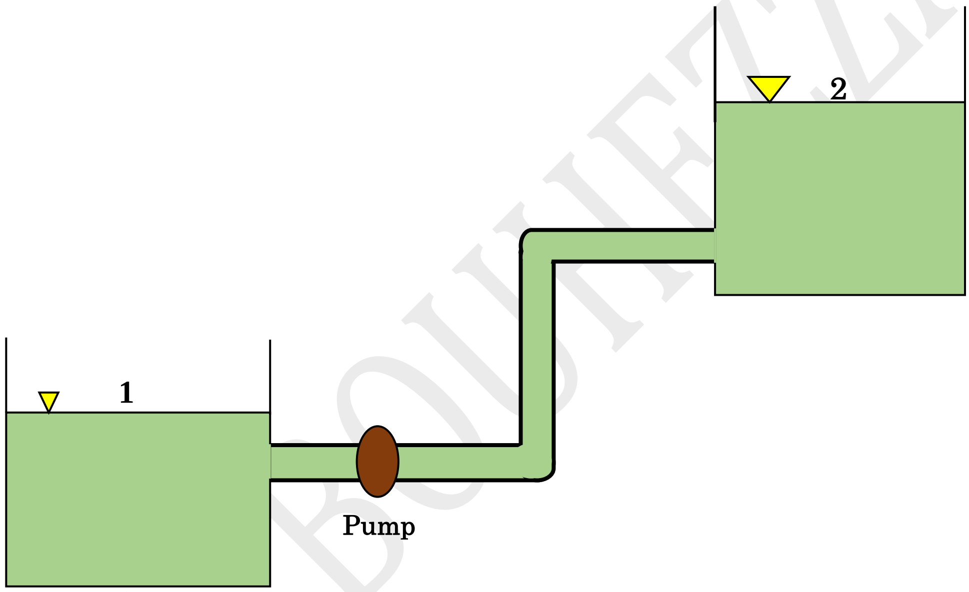


Fig. Ex.5

EXERCICE N°6

In the system shown in Fig. Ex. 5, the pump is to deliver oil with specific gravity SG = 0.762 from reservoir 1 to reservoir 2 at a flow rate of 160 l/s.

The kinematic viscosity of this oil is $\boldsymbol{\upsilon = 54.8 \times 10^{-6}\ m^2/s}$. The minor head losses are considered negligible compared to the major head losses.

Calculate the power supplied to the system.

$Z_1 = 15$ m , $Z_2 = 3$ m et $Z_4 = 60$ m, $\rho_{water} = 10^3$ kg/m^3 ; g = 9.81 m/s^2 .

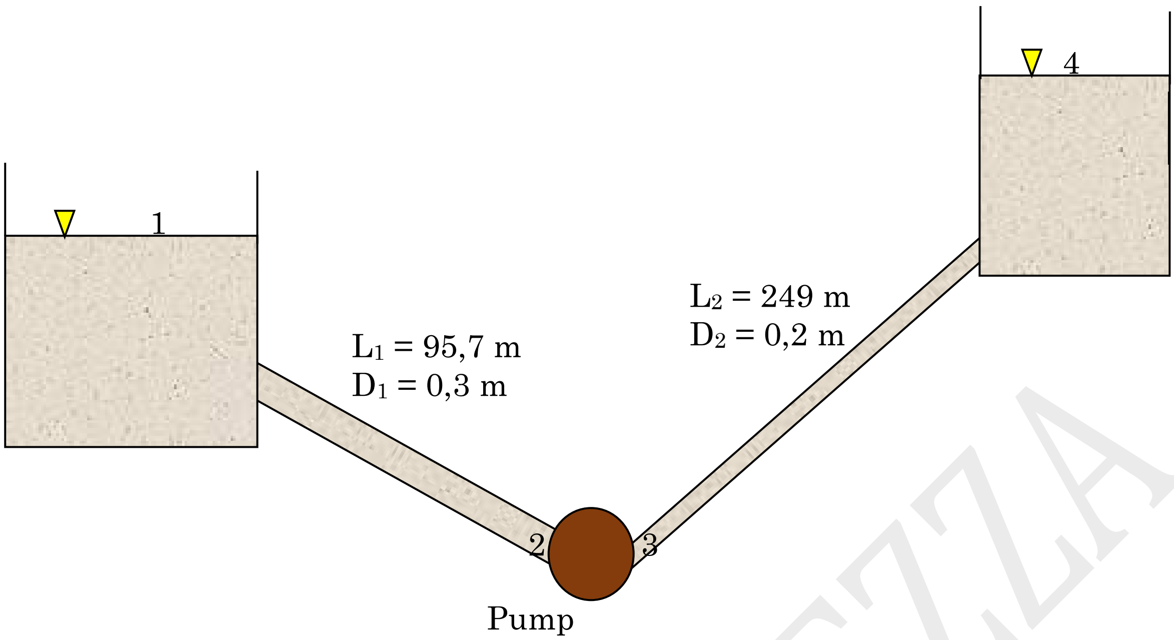


Fig. Ex. 6

EXERCICE N°7

Consider the fully developed flow of glycerin at 40 °C through a 70-m-long, 4-cm-diameter, horizontal, circular pipe. If the flow velocity at the centerline is measured to be u_{max} = 6 m/s. The flow is steady, incompressible, and fully developed. The density and dynamic viscosity of glycerin at 40 °C are 1252 kg/m^3 and 3.073 poise, respectively. The velocity profile in fully developed flow in a circular pipe is expressed as: $u(r) = u_{max}\left(1 - \frac{r^2}{R^2}\right)$

Determine the average velocity, volume flow rate and the pressure difference across this 70-m-long section of the pipe.

EXERCICE N°8

In the system shown in Fig. Ex.8, water flows through a pipe at a flow rate of 55 *l/min*. Calculate the pressure at point (1).

The pipe is made of copper ($\boldsymbol{\varepsilon = 0{,}005\ mm}$), $\mu_{water} = 10^{-3}$ kg/m.s ,

$\rho_{water} = 10^3$ kg/m^3 , $g = 9{,}81$ m/s^2.

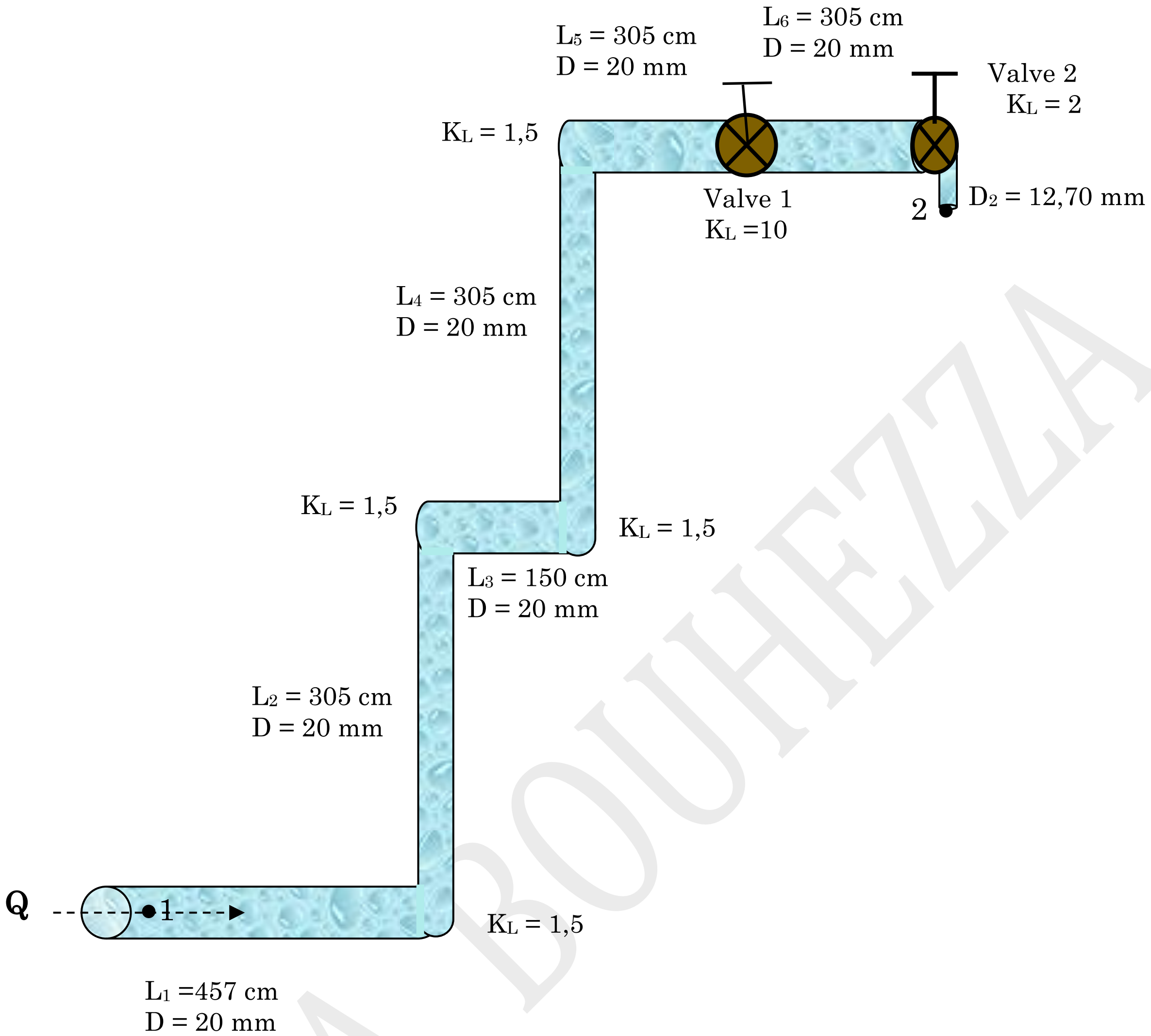


Fig. Ex. 8

EXERCICE N°9

A Venturi meter equipped with a differential pressure gage is used to measure the flow rate of water through a 5-cm-diameter horizontal pipe. The diameter of the Venturi neck is 3 cm, and the measured pressure drop is 5 kPa. Taking the discharge coefficient to be 0.98, determine the volume flow rate of water and the average velocity through the pipe.

$\boldsymbol{\rho = 999.1}$ **kg/m³**

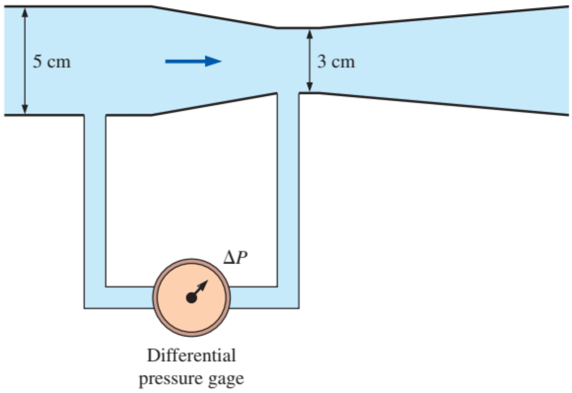


Fig. Ex. 9

EXERCICE N°10

Fuel oil of specific gravity 0.95 and dynamic viscosity μ flows from point 1 to point 2 in a horizontal cylindrical pipe of length L = 2 km and diameter D = 25 cm, with a volume flow rate Q = 40 l/s. Pressures at point 1 and point 2 are 3 bar and 1.82 bar respectively. The flow regime is assumed to be laminar, for which the flow rate is given by :

$$\boldsymbol{Q} = \frac{[(P_1 - P_2)\pi D^4]}{128\,\mu\,L}$$

1) Calculate the average flow velocity of the fuel oil.

2) Calculate the dynamic viscosity and kinematic viscosity of the fuel oil.

3) Evaluate the Reynolds number of this flow and justify its laminar nature.

4) Calculate the velocity at the center of the pipe $V_{Avg} = \frac{1}{2} V_{max}$.

5) Calculate the head losses.

$\rho_{water} = 10^3$ kg/m^3 , g = 9.81 m/s^2.

Fig. Ex. 10

<u>EXERCICE N°11</u>

A large tank "1" supplies a large tank "2" with water at a flow rate of 50 l/min through two cylindrical pipes AB and CD. Water is a real fluid with a kinematic viscosity of 10^{-6} m²/s. The first pipe AB has a length L_1 =10 m and a diameter D_1 = 30 mm. The second pipe CD has a length L_2 = 20 m and a diameter D_2 = 20 mm (Fig. Ex. 11).

The minor head loss coefficients at the pipe inlet K_L = 0.5, at the pipe outlet K_L = 1 and at the elbows K_L = 0.75 are given.

Calculate the hydraulic power supplied by the pump to ensure this flow rate.

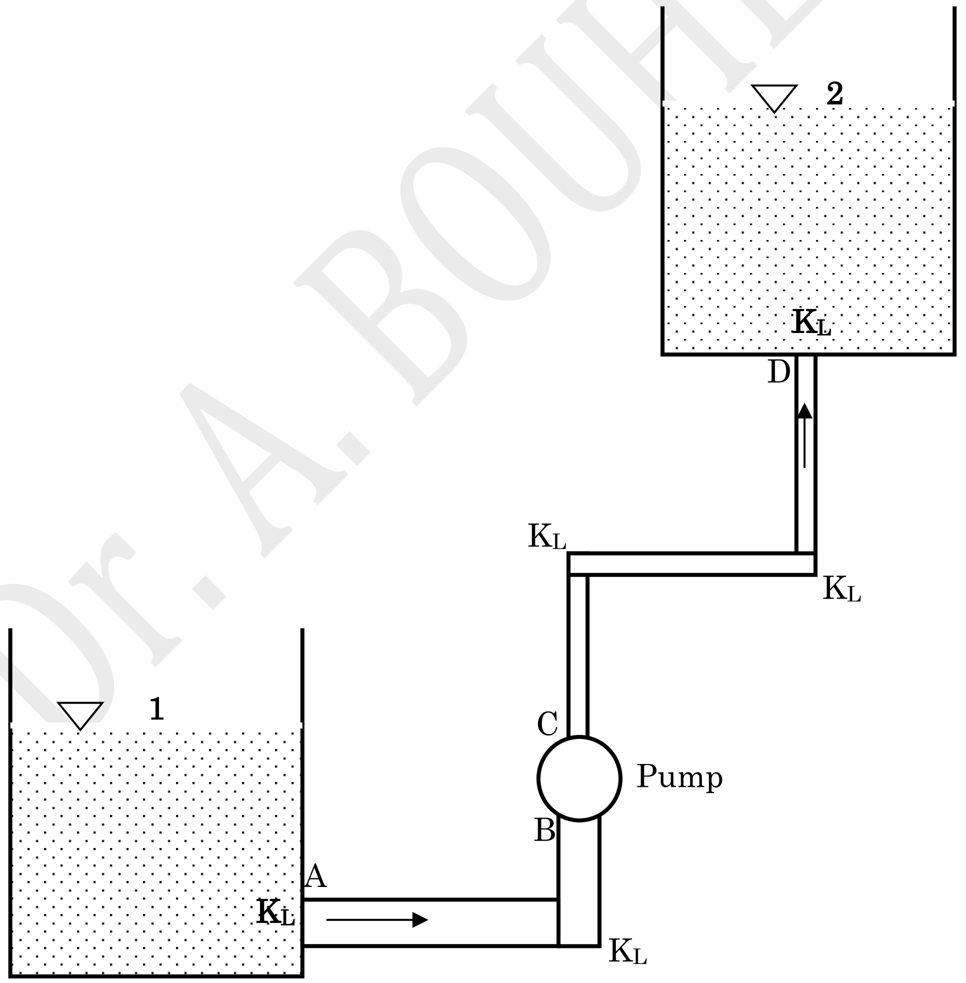


Fig. Ex.11

EXERCICE N°12

A flow nozzle equipped with a differential pressure gage is used to measure the flow rate of water through a 3-cm-diameter horizontal pipe. The nozzle outlet diameter is 1.5 cm, and the measured pressure drop is 3 kPa. Determine the volume flow rate of water, the average velocity through the pipe, and the head loss.

$\boldsymbol{\mu = 1.307 \times 10^{-3}}$ **kg/m.s**

$\boldsymbol{\rho = 999.7}$ **kg/m^3**

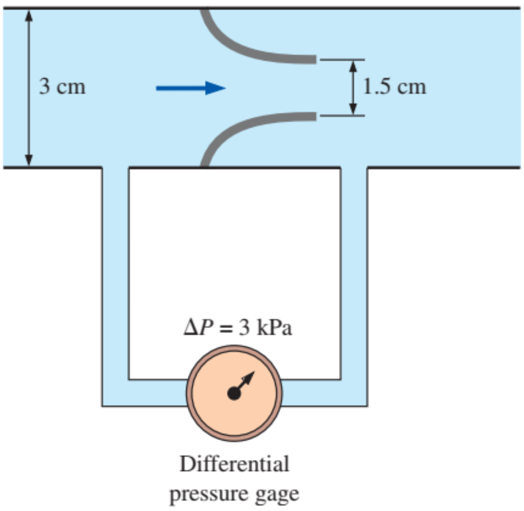


Fig. Ex.12

Chapter IV
Pumps and Pumping




IV.1 GENERAL ABOUT PUMPS

IV.1.1 Pump Definition

Pumps are devices that transfer energy between a fluid and a mechanical device, drawing fluid from an area of low pressure to a region of higher pressure. Machines communicate with fluid by converting the mechanical energy of their motor into hydraulic energy.

The pump is represented by the following schematic:

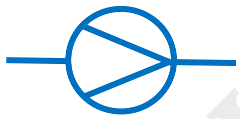


Figure IV. 1: Pump schematic

IV.1.2 Role of a Pump

The role of the pump is to distribute fluids in pipes, so the use of pumps is essential to:

- Ensure consistent flow rates,
- Ensure high flow rates,
- Ensure fluids are transported to high altitudes,
- Overcome load loss due to pipeline length and various accidents (valves, expansion, contradiction, elbows, etc.).

IV.1.3 Pump installation types

There are two types of pump assembly:

- Charge Pump assembly
- Suction Pump

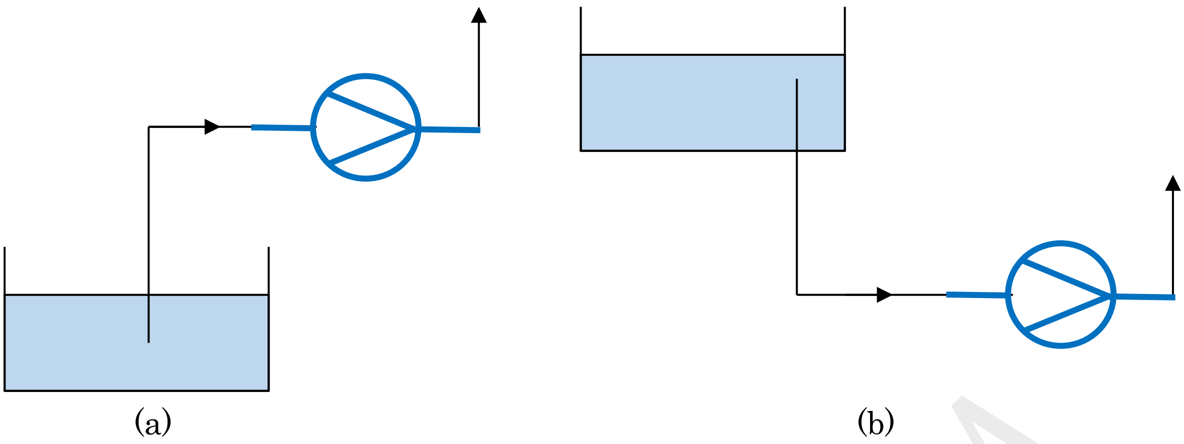


Figure IV. 2: (a) Charge Pump (b) Suction Pump

IV.1.4 Pump Classification

There are two main types of pumps:

- Volumetric pumps,
- Hydrodynamic pumps or turbo-pumps.

IV.1.4.1 Volumetric pumps: A volumetric pump is a mechanical device used to transfer a fluid from point A to point B by the movement of a pumping chamber. These pumps are capable of delivering very high loads at a low flow rate. There are different types of volumetric pumps (Fig. 3):

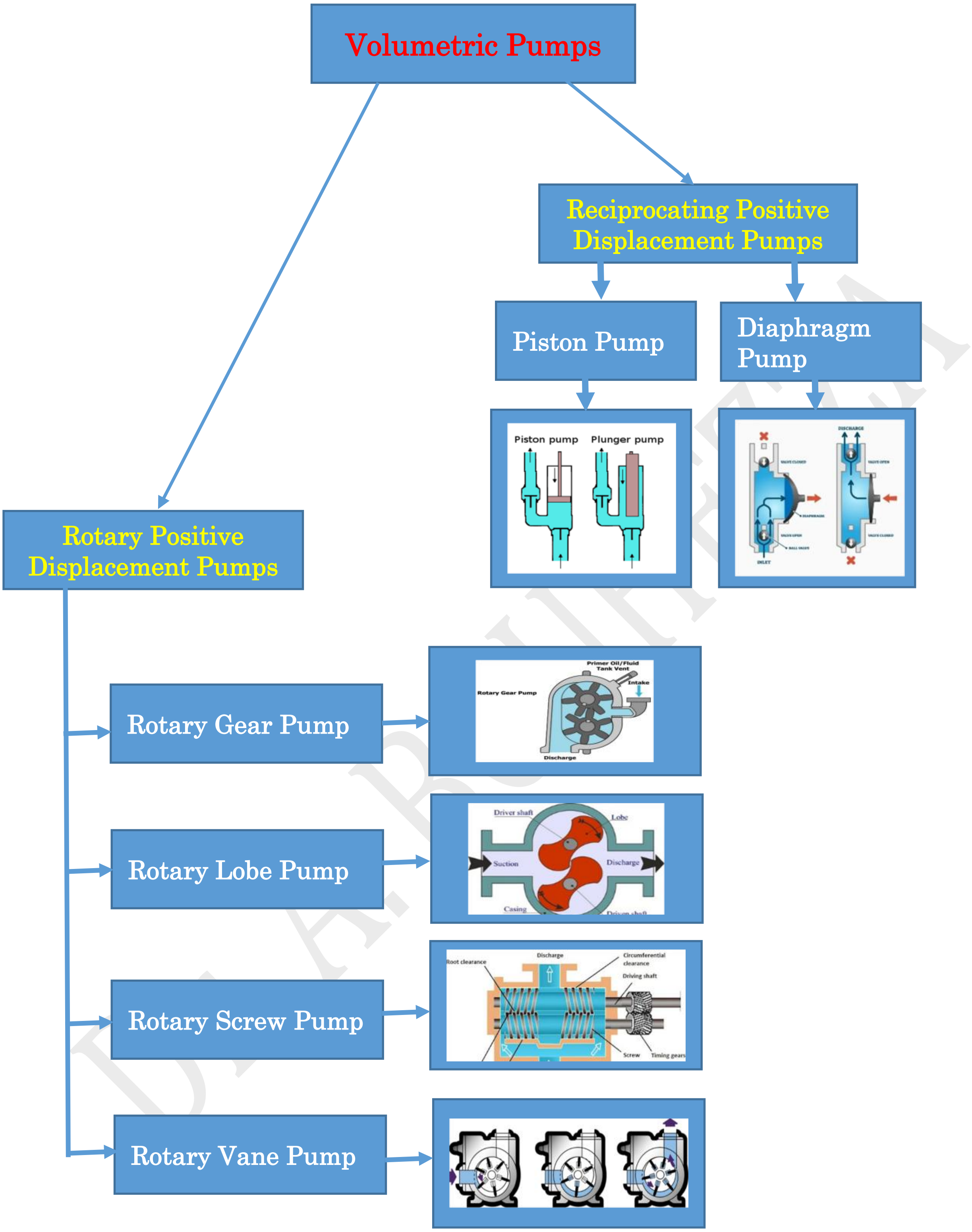


Figure IV. 3 : Different types of volumetric pumps

IV.1.4.2 Turbo-Pumps or Hydrodynamic Pumps

In turbo-pumps, a wheel with rotating blades or vanes provides kinetic energy to the fluid, some of which is converted into pressure by reducing the speed in a component called a recuperator. These pumps are load-limited but can operate at very high flow rates. There are different types of turbo-pumps (Fig. 5):

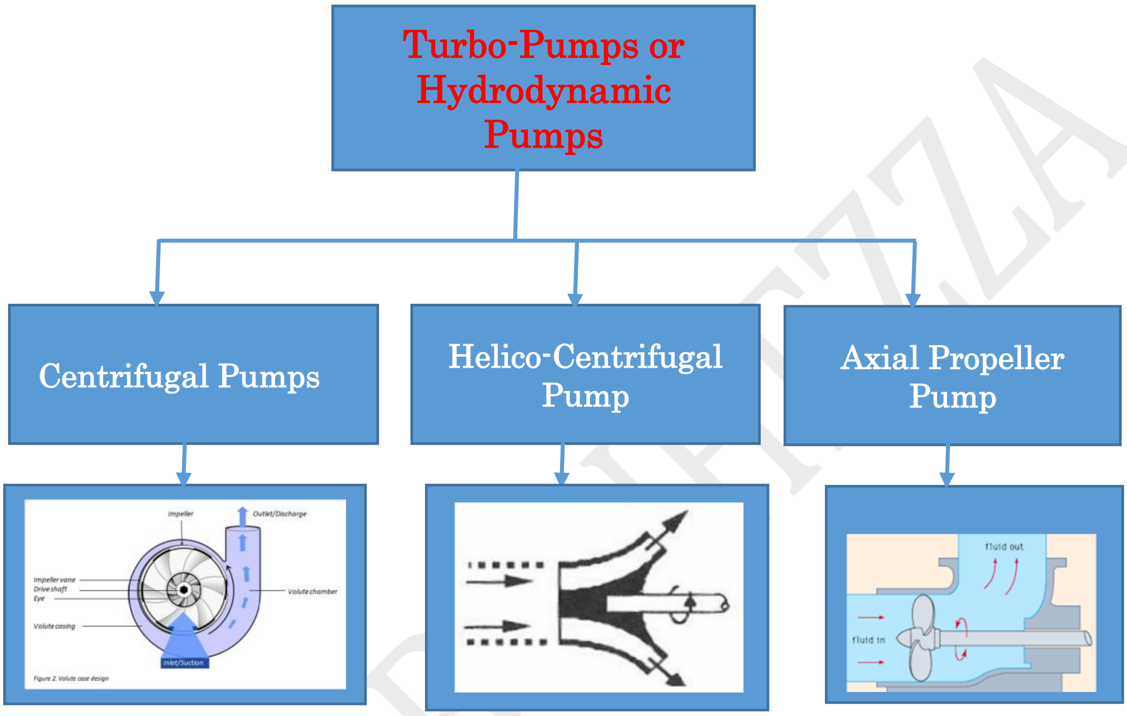


Figure IV. 4 : Turbo-Pumps or Hydrodynamic Pumps

IV.1.5 Applications for volumetric and hydrodynamic pumps

The areas of use of these two groups of pumps are:

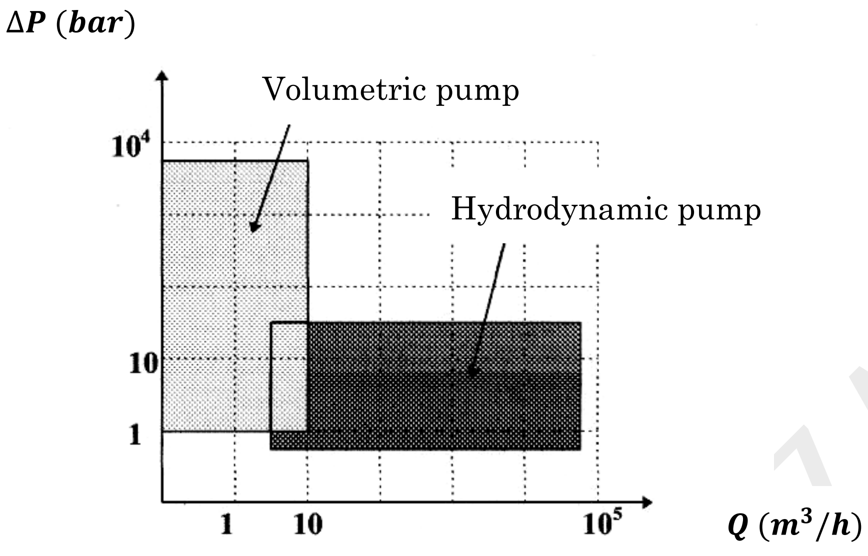


Figure IV. 5: Areas of use of volumetric and hydrodynamic pumps

IV.1.6 Pump Characteristics

The most important pump characteristics are as follows:

IV.1.6.1 Rotation velocity

Rotational velocity is the number of revolutions the pump makes per unit of time "N" in [rpm]. It can be converted to angular velocity:

$$\boldsymbol{\omega = \frac{2\pi N}{60}} \quad \text{(rad/s)} \tag{IV.1}$$

IV.1.6.2 Pump flow rate

Some basic parameters are used to analyze pump performance. The mass flow rate $\dot{\boldsymbol{m}}$ of fluid through the pump is the basic pump performance parameter. For incompressible flow, it is common to use volume flow rate $\boldsymbol{Q}$ instead of mass flow rate. In the turbomachinery industry, the volume flow rate is called capacity.

$$\boldsymbol{Q = \frac{\dot{m}}{\rho}} \quad \text{(m}^3\text{/s)} \tag{IV.2}$$

In the case of volumetric pumps, the notion of displacement (D) is much more widely used: this is the volume flow rate delivered by a pump for one revolution.

$$\boldsymbol{D} = \frac{\boldsymbol{Q}}{\boldsymbol{N}} \quad (m^3/rev) \qquad (IV.3)$$

IV.1.6.3 Manometric head

The Manometric head is defined as the head against which a pump has to work. Manometric head is denoted Hm. The Manometric head is expressed in meters. There are two different operating modes.

IV.1.6.3.1 Discharge Installation

The discharge installation (figure 4.3) is characterized by a suction tank (or lower tank) level higher than the pump shaft level.

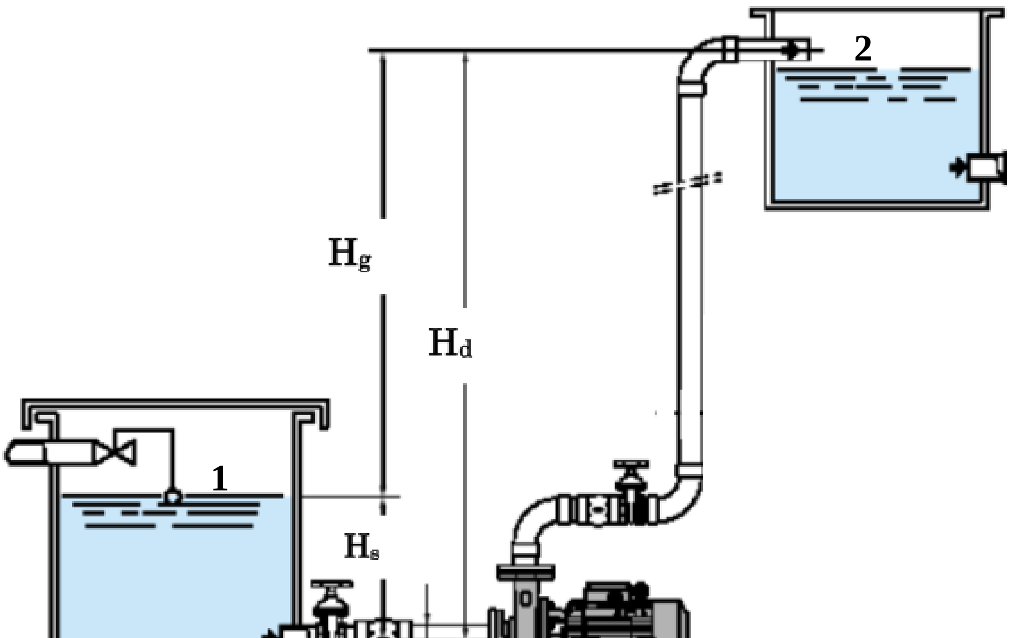


Figure IV. 6 : Discharge Installation

- **Determination of the Manometric Head**

To obtain the manometric head the pump must provide, we apply Bernoulli's equation between points 1 and 2, we get.

$$Hm = \frac{P_2}{\rho g} - \frac{P_1}{\rho g} + \frac{V_2^2}{2\,g} - \frac{V_1^2}{2\,g} + Z_2 - Z_1 + h_{L,\ Total} \qquad \text{(IV.4)}$$

Or $Z_2 - Z_1 = H_d - H_s = H_g$

$$Hm = \frac{P_2}{\rho g} - \frac{P_1}{\rho g} + \frac{V_2^2}{2\,g} - \frac{V_1^2}{2\,g} + H_g + h_{L,\ Total} \qquad \text{(IV.5)}$$

IV.1.6.3.1 Suction Installation

The suction installation is characterized by a suction tank level (or lower tank level) below the level of the pump shaft.

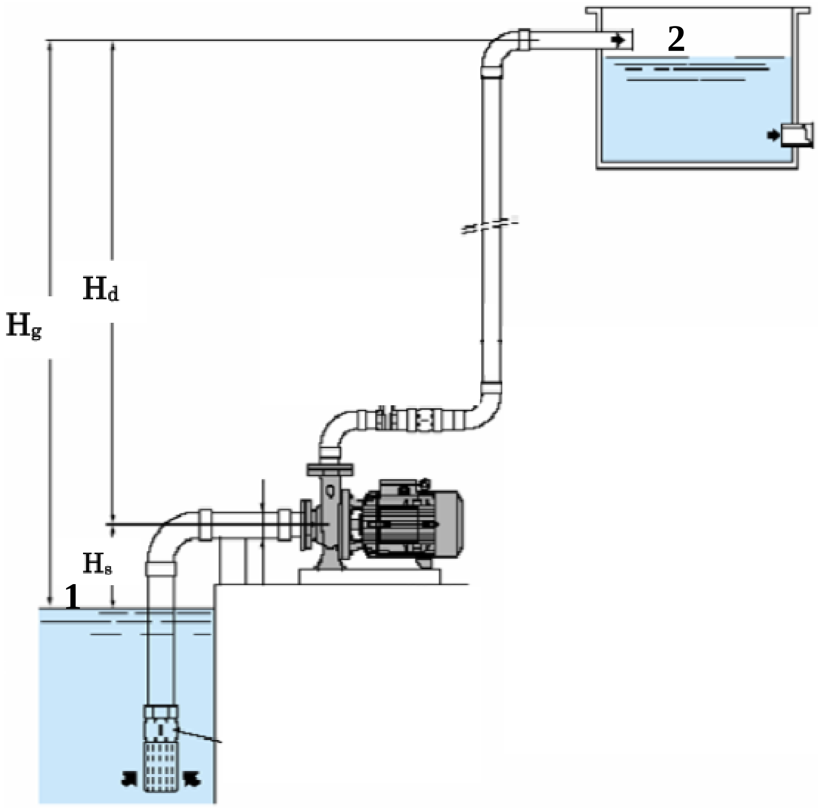


Figure IV. 6 : Suction Installation

- **Determination of the Manometric Head**

Applying Bernoulli's theorem between 1 and 2 gives us.

$$\boldsymbol{Hm} = \frac{P_2}{\rho g} - \frac{P_1}{\rho g} + \frac{V_2^2}{2\,g} - \frac{V_1^2}{2\,g} + Z_2 - Z_1 + h_{L,\ Total} \qquad \text{(IV.6)}$$

Or $\quad Z_2 - Z_1 = H_d + H_s = H_g$

$$\boldsymbol{Hm} = \frac{P_2}{\rho g} - \frac{P_1}{\rho g} + \frac{V_2^2}{2\,g} - \frac{V_1^2}{2\,g} + H_g + h_{L,\ Total} \qquad \text{(IV.7)}$$

IV.1.6.4 Pump Specific Velocity

Another useful parameter is called the specific speed of the pump ($\boldsymbol{N_{sp}}$), which is a dimensionless parameter that is constant no matter what the pump speed is:

$$N_{sp} = \frac{\omega\, Q^{\frac{1}{2}}}{(g\ \ Hm)^{3/4}} \qquad \text{(IV.8)}$$

Where

$\boldsymbol{N_{sp}}$: Pump Specific Speed (dimensionless)

$\boldsymbol{Q}$: volume flow rate (m^3/s)

$\boldsymbol{g}$: gravitational acceleration (m/s^2)

$\boldsymbol{Hm}$: Manometric head (m)

$\boldsymbol{\omega}$: rotational velocity (red/s)

The specific speed of the pump can be used to determine the operating characteristics of the pump at its optimum conditions (best efficiency point) and select the pump where:

$\boldsymbol{N_{sp}}$ LOW: Centrifugal Pumps $\Longrightarrow$ Low flow rates for high Hm.

$\boldsymbol{N_{sp}}$ FORT: Helical Pumps $\Longrightarrow$ High flow rates for low Hm.

IV.1.7 Efficiency and Power of a Pump

IV.1.7.1 Pump Shaft Power

Pump shaft power or absorbed power is the power supplied to the pump motor (consumed power), which is also the input power to the pump.

Shaft power is defined as

$$\boldsymbol{\eta_{motor} = \frac{\dot{W}_{shaft}}{\dot{W}_{elec}}} \qquad \text{(IV.9)}$$

$$\boldsymbol{\dot{W}_{shaft} = \eta_{motor}\, \dot{W}_{elec}} \qquad \text{(IV.10)}$$

$$\boldsymbol{\dot{W}_{shaft} = T_{shaft} \times \omega} \qquad \text{(IV.11)}$$

Where

$\boldsymbol{\dot{W}_{elec}}$: Electric power (watts)

$\boldsymbol{\eta_{motor}}$: motor efficiency

$\boldsymbol{T_{shaft}}$: torque supplied to the shaft (N.m)

$\boldsymbol{\omega}$: rotational velocity (red/s)

IV.1.7.2 Pump Hydraulic Power (useful)

Hydraulic power (useful) is the power delivered by pump in the form of flow rate and head.

Hydraulic power is defined by

$$\boldsymbol{\dot{W}_{pump,u} = \rho \times Q \times g \times Hm} \qquad \text{(IV.12)}$$

Where

$\boldsymbol{\dot{W}_{pump,u}}$: useful pumping power (watts)

$\boldsymbol{\rho}$: density of fluid (kg/m^3)

$\boldsymbol{g}$: gravitational acceleration (m/s^2)

Hm: Manometric head (m)

IV.1.7.3 Pump Efficiency

The pump efficiency is the ratio between the two hydraulic powers (useful) and the pump shaft power (absorbed):

$$\eta_{pump} = \frac{\dot{W}_{pump,u}}{\dot{W}_{shaft}} = \frac{\rho \times Q \times g \times Hm}{T_{shaft} \times \omega} \quad \text{(IV.13)}$$

IV.1.8 Pumps in Series and Parallel

By combining pumps in series or in parallel, you can achieve specific operating points that would be impossible with a single pump.

IV.1.8.1 Pumps in Series

When the required load exceeds what can be achieved with one pump, two or more pumps can be installed in series (**Fig. IV. 8**).

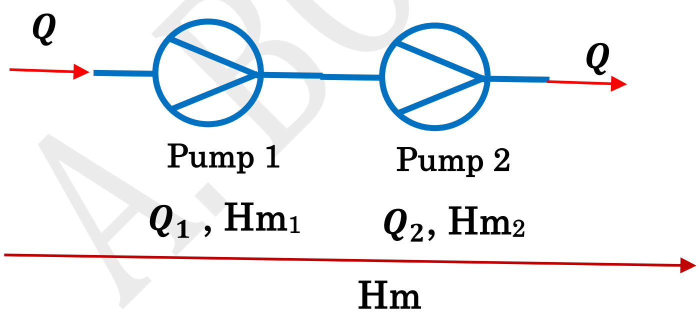


Figure IV. 8 : Pumps in series

Relationship between volume flow rates :

$$Q = Q_1 = Q_2 \quad \text{(IV.14)}$$

Relationship between manometric heads :

$$Hm = Hm_1 + Hm_2 \quad \text{(IV.15)}$$

- **Graphical result**

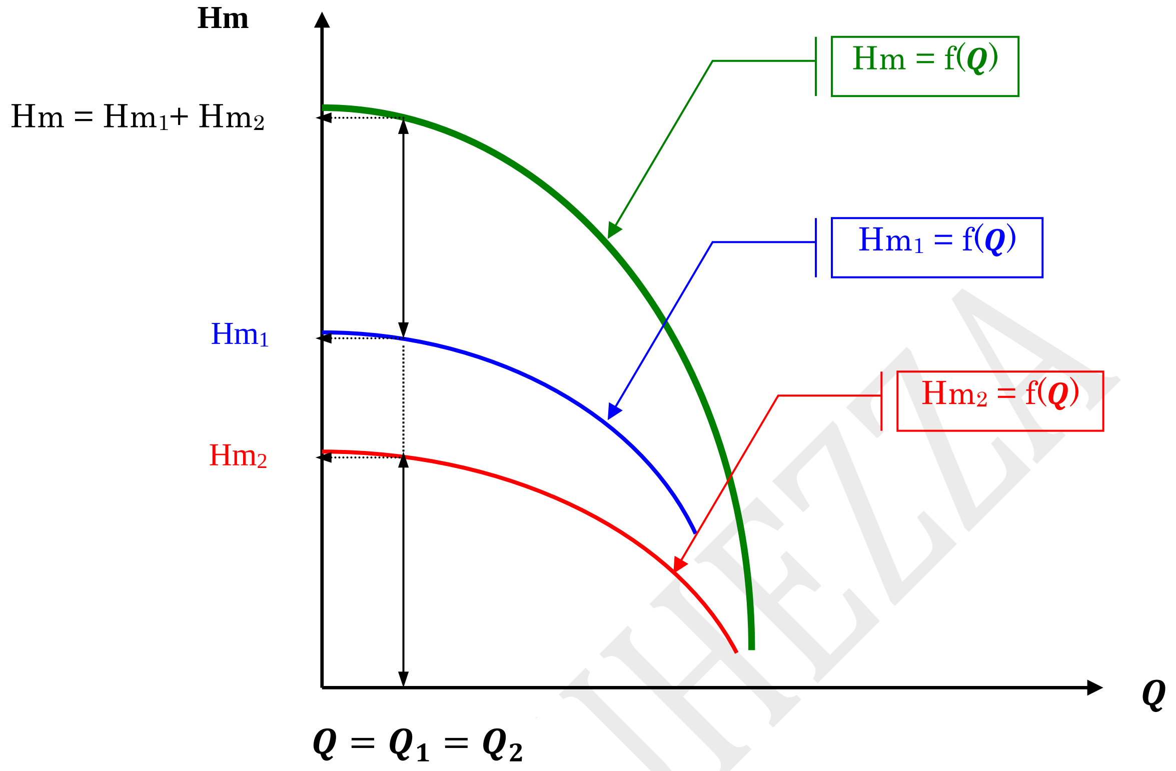


Figure IV. 9 : Resultant curve of two pumps in series

- **Case of n** Pumps in series

Flow rates : $\boldsymbol{Q = Q_1 = Q_2 = \cdots = Q_n}$ (IV.16)

Manometric heads : $\boldsymbol{Hm = \sum_i^n Hm_i}$ (IV.17)

IV.1.8.2 Pumps in Parallel

When the flow rate of a single pump is lower than required, two or more pumps can be installed in parallel (**Fig. IV. 10**).

Relationship between volume flow rates :

$$\boldsymbol{Q = Q_1 + Q_2} \quad \text{(IV.18)}$$

Relationship between manometric heads :

$$\boldsymbol{Hm = Hm_1 = Hm_2} \quad \text{(IV.19)}$$

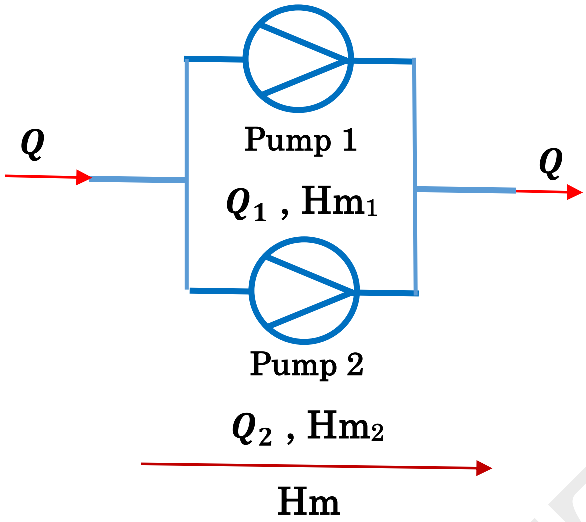


Figure IV. 10 : Pumps in Parallel

- **Graphical result**

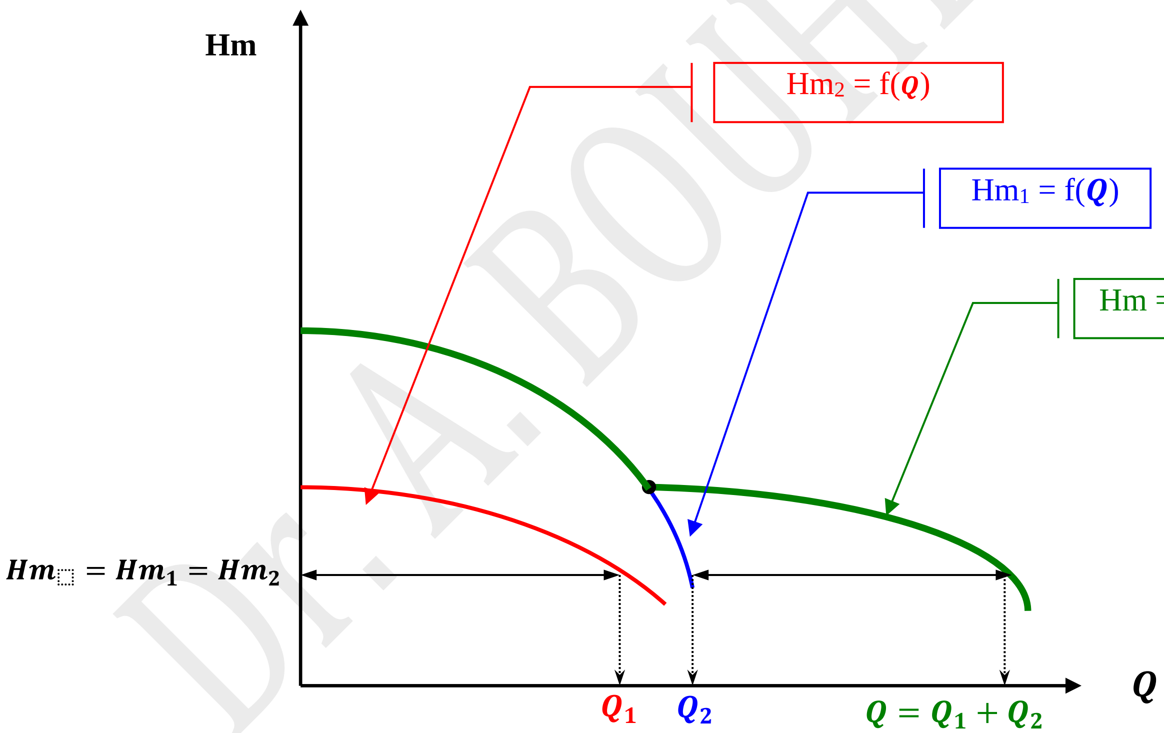


Figure IV. 11 : Resultant curve of two pumps in parallel

- **Case of n** Pumps in parallel

Flow rates : $Q = \sum_i^n Q_i$ (IV.20)

Manometric heads : $Hm = Hm_1 = Hm_2 = \cdots = Hm_n$ (IV.21)

IV.2 CALCULATION OF HYDRAULIC NETWORKS

IV.2.1 HYDRAULIC NETWORKS

Hydraulic networks or distribution networks are the entire hydraulic circuit. These hydraulic networks may include: tanks, pipes, special parts (valves, elbows, …), connections, …

There are different types of network construction:

- **Ramified network**

Ramified networks are simple, economical and low safety.

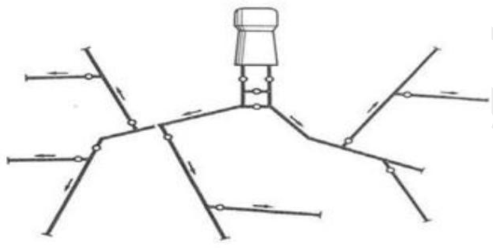


Figure IV. 12 : Ramified network

- **Mesh network**

Mesh networks are more secure, more difficult to calculate and more expensive, but better than Ramified networks.

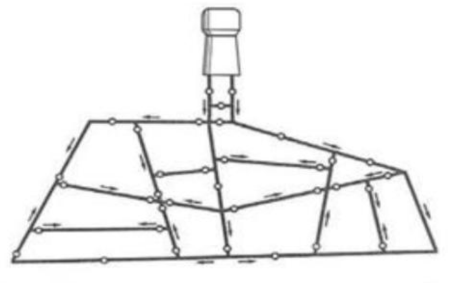


Figure IV. 13 : Mesh network

IV.2.1 HYDRAULIC NETWORK CALCULATIONS

To calculate the network, we can apply Kirchhoff's two laws.

First law: law of knots

At any node in the network, the algebraic sum of the flow rates is zero.

$$\sum_i^n \boldsymbol{Q_i} = \mathbf{0} \tag{IV.22}$$

Second law: law of loops

The algebraic sum of total head losses is zero along a loop.

$$\sum \boldsymbol{h_L} = \mathbf{0} \tag{IV.23}$$

- **Hardy Cross-Method**

Head loss in loop

$$\sum \boldsymbol{h_L} = \mathbf{0} \tag{IV.24}$$

Since this equality is not verified in the first approximation, the flow rate must be corrected using the following formula.

$$\Delta \boldsymbol{Q} = \frac{-\sum \boldsymbol{h_L}}{\mathbf{2}\sum \frac{\boldsymbol{h_L}}{\boldsymbol{Q}}} \tag{IV.25}$$

Corrected $\boldsymbol{Q}$ for single pipe in one loop

$$\boldsymbol{Q_{new}} = \boldsymbol{Q_{old}} + \Delta \boldsymbol{Q} \tag{IV.26}$$

If the second law has not yet been verified, the flow rates will have to be corrected by a new $\Delta \boldsymbol{Q}$ value, calculated as before.

EXERCISES

EXERCICE N°1

Consider the following water network:

1) Write the Bernoulli equation between 1 and 2.

2) Express the Manometric head.

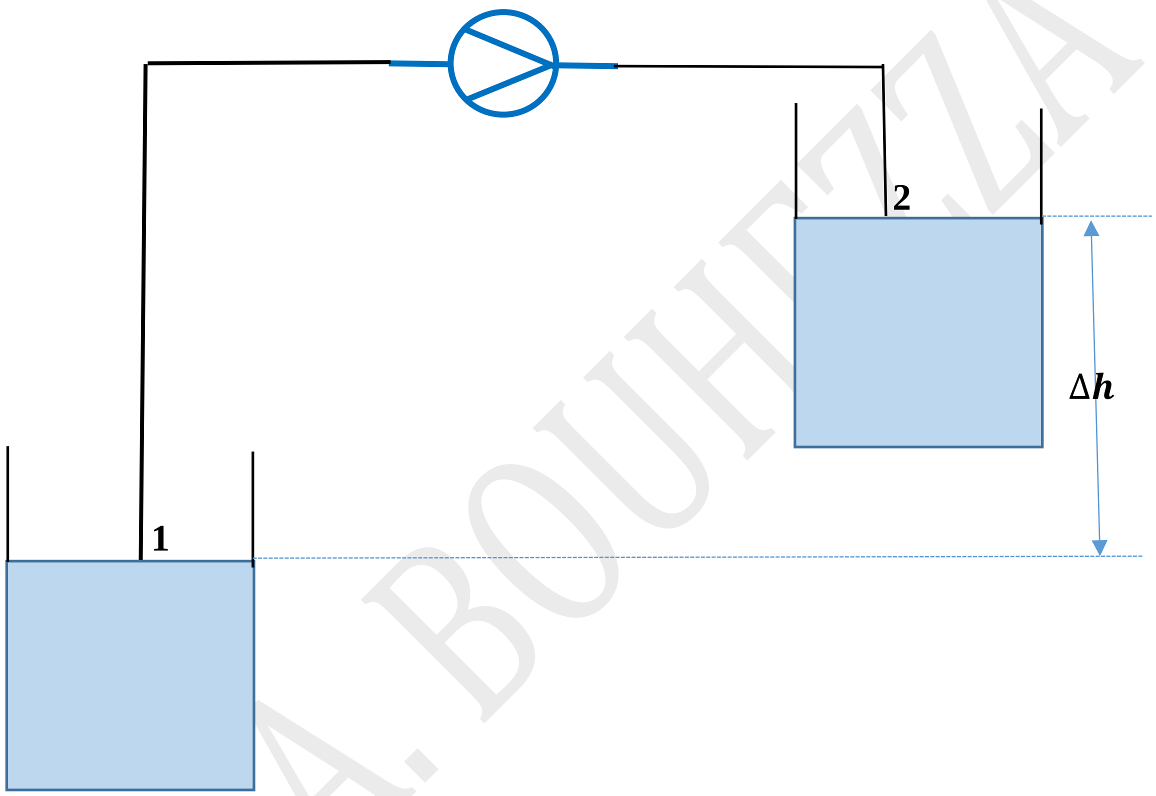


Fig. Ex. 1

EXERCICE N°2

Consider the following water network:

Give the expressions for Manometric head Hm and hydraulic power as a function of H.

Given that:

Pipe length: L = 3 H,

Pipe diameter: D = 600 mm,

$\boldsymbol{f}$ = 0.023 (major head loss coefficient),

Water velocity in the pipe: V = 3 m/s,

$\sum \boldsymbol{K_L}$ (in the pipe) = 20,

g = 10 m/s^2,

$\rho_{water} = 10^3$ kg/m^3

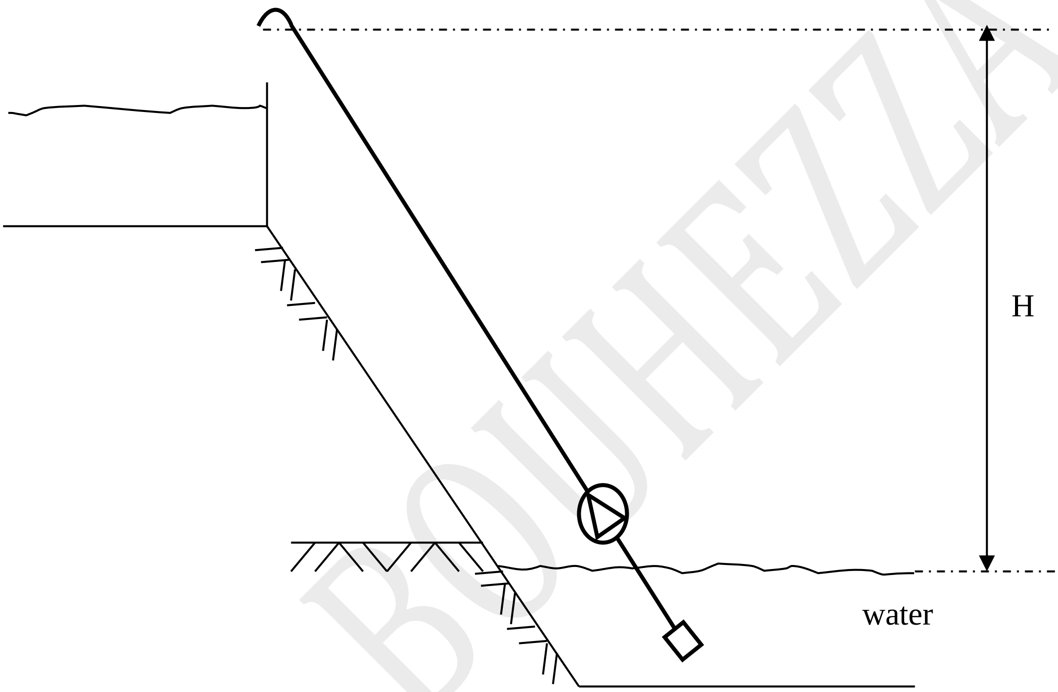


Fig. Ex. 2

EXERCICE N°3

A pump delivers water from a large reservoir at a flow rate of 15 *l/s* through a cylindrical pipe of length L_1=2.5 m and diameter D = 40 cm, to another cylindrical pipe of length L_2 = 4.5 m and diameter d = 20 cm (see Fig. EX.3). Water is a real fluid with dynamic viscosity $\mu = 10^{-2}$ poise. The outlet pressure is 478 kPa. All minor head losses are negligible.

$\rho_{water} = 10^3\ kg/m^3$; $g = 9{,}81\ m/s^2$.

Calculate Manometric head Hm and hydraulic power.

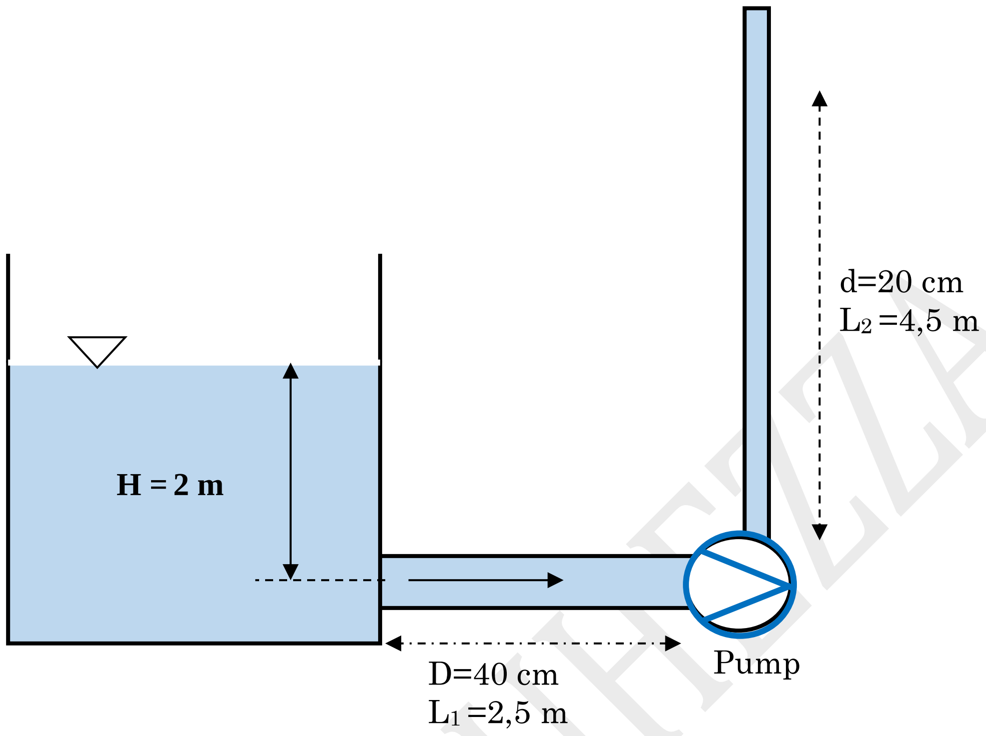


Fig. Ex. 3

EXERCICE N°4

In the system shown in Fig. 3, the pump is to deliver water at a flow rate of 90 l/s from tank A to tank B. The kinematic viscosity of the water is $\boldsymbol{\nu = 10^{-2}}$ **Stokes.** All minor head losses are negligible.

($\mathbf{(Z_B - Z_A) = 30\ m}$**,** $\boldsymbol{\rho_{water} = 10^3\ kg.m^{-3}}$ and $\mathbf{g = 9{,}81\ m/s^2}$.

1) Calculate the average flow velocities in the pipes.
2) Calculate the major head losses.
3) Calculate Manometric head Hm and hydraulic power.
4) This pump is driven by an electric motor. The unit's global efficiency is 85%. Calculate the electrical power consumed.

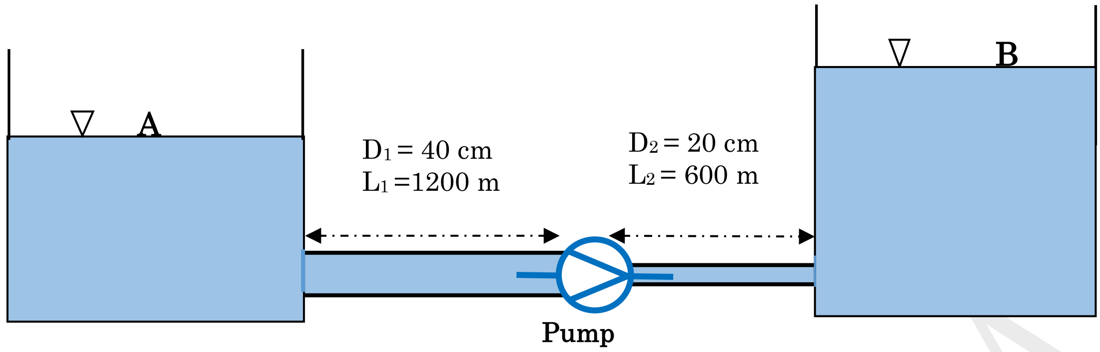


Fig. Ex. 4

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