

ALGERIAN DEMOCRATIC AND POPULAR REPUBLIC

MINISTRY OF HIGHER EDUCATION

AND SCIENTIFIC RESEARCH

20 AUGUST 1955 UNIVERSITY SKIKDA



FACULTY OF SCIENCES

DEPARTMENT OF MATHEMATICS



Probability exercises

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UNIVERSITY YEAR : 2024-2025

Probability exercises with solutions

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20 august 1955 university Skikda

University year : 2024-2025

Contents

Preface	iv
1 Basic Concepts of probability	1
1.1 Summary of concepts	1
1.2 Exercises	3
1.3 Solutions	5
2 Random variables	14
2.1 Reminder	14
2.1.1 Discrete random variables	14
2.1.2 Continuous random variables	16
2.1.3 Probability inequalities	18
2.2 Exercises	18
2.3 Solution	26
3 Probability distribution	72
3.1 Reminder	73
3.1.1 Summary of discrete probability distributions	73
3.1.2 Summary of continuous probability distributions	75
3.1.3 Approximations	78
3.2 Exercises	78
3.3 Solutions	90
Appendix	124

CONTENTS

BIBLIOGRAPHY

140

List of Figures

2.1	Area under the curve of $f(x)$ and over the interval $[a, b]$	16
2.2	The probability mass function of the random variable X	29
2.3	The cumulative distribution function of the random variable X	30

List of Tables

3.1	The continuity correction between normal and binomial distributions. . . .	78
3.2	Distribution of blood types in Austria	80
3.3	Sensory Rates of 15 Subjects	88
3.4	Blood Type Probabilities	94
3.5	Table of cumulative normal distribution for negative z-values:	124
3.6	Table of cumulative normal distribution for positive z-values:	125
3.7	Table of Poisson distribution for a given value of λ , entry indicates the probability of a specified value of X	126
3.8	Table of Binomial distribution for all values ≤ 0.0005 have been left out: . .	128
3.9	Table of Chi-square distribution for different values of k :	134
3.10	Table of Student distribution for different values of t :	135
3.11	Table of Fisher distribution for different values of α :	136

Preface

This manuscript is part of a collection of solved exercises designed as a supplement to the "Probability" course. It was developed specifically for second-year mathematics students at the University of August 20, 1955, Skikda.

The contents are divided into three chapters. Each chapter begins with a concise review of key probability concepts, followed by a series of exercises with detailed solutions. Some exercises are designed to be of direct application as homework or test questions, while others are intended to clarify concepts or to introduce more extensively theorems and techniques not covered in the text.

For any feedback, suggestions, or corrections related to this document, you are kindly invited to contact me so that appropriate revisions and improvements can be made.

Basic Concepts of probability

Contents

1.1 Summary of concepts	1
1.2 Exercises	3
1.3 Solutions	5

1.1 Summary of concepts

Let Ω be the sample space and A be an event associated with a random experiment. Then the probability of event A occurring is denoted as $P(A)$, is denoted by

$$P(A) = \frac{\text{card}(A)}{\text{card}(\Omega)},$$

satisfying the following axioms:

1. $0 \leq P(A) \leq 1$.
2. $P(\Omega) = 1$.
3. If A and B are disjoint events,

$$P(A \cup B) = P(A) + P(B).$$

4. More generally, if $A_1, A_2, \dots, A_n, \dots$ are disjoint events,

$$P(A_1 \cup A_2 \cup \dots \cup A_n \dots) = P(A_1) + P(A_2) + \dots + P(A_n) \dots$$

1.1 Summary of concepts

and properties:

1. If $B \subset A$, $P(B) \leq P(A)$.
2. If \bar{A} is the complementary event of A , $P(\bar{A}) = 1 - P(A)$.
3. In addition, If A and B are any events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

Conditional probability:

The conditional probability of the event B under that the event A has happened, is denoted by $P(B|A)$ and defined as

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, \quad \text{provided } P(A) \neq 0.$$

and satisfying :

- i) $P(\Omega|A) = P(\Omega/B) = 1$.
- ii) $P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$.
- iii) $P(\bar{A}|B) = 1 - P(A|B)$.

Product Theorem on probability:

If A_1, \dots, A_n a family of n events such that: $P(A_1 \cap A_2 \cap \dots \cap A_n) \neq 0$, then

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

Theorem of total probability:

Let $\{B_1, B_2, \dots, B_n\}$ be a partition of the sample Ω . Let A be any event associated with Ω , then

$$P(A) = \sum_{k=1}^n P(A \cap B_k) = \sum_{k=1}^n P(A|B_k) P(B_k).$$

Bayes Theorem:

$$P(B_k|A) = \frac{P(B_k) \cdot P(A|B_k)}{\sum_{k=1}^n P(B_k) P(A|B_k)}.$$

1.2 Exercises

Independent events

If the two events A and B are independent, the product theorem takes the form

$$P(A \cap B) = P(A)P(B).$$

Thus,

i) $P(A|B) = P(A)$, provided $P(B) \neq 0$

ii) $P(B|A) = P(B)$, provided $P(A) \neq 0$

1.2 Exercises

Exercise 1.1. 1. How many probabilities P can be defined on $\Omega = \{a, b, c\}$ such that:

(a) $P(\{a, b\}) = \frac{1}{4}$

(b) $P(\{a, b\}) = P(\{b, c\}) = \frac{1}{4}$

(b) $P(\{a, b\}) = P(\{b, c\}) = \frac{3}{4}$

2. Let A and B be two events in a probability space $(\Omega, \mathcal{P}(\Omega), P)$.

- Show that:

$$P(A \cap B) - P(A)P(B) = -P(A \cap B^c) + P(A)P(B^c)$$

Exercise 1.2. Let $(\Omega, \mathcal{P}(\Omega), P)$ be a probability space.

1. Let A , B , and C be three events. Determine $P(A \cup B \cup C)$.

2. Let A and B be two independent events. Prove that:

- A and \bar{B} are also independent.
- $P(B|A) = P(B|\bar{A})$.

3. Let A and B be two events. If $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cup B) = 0.65$, are the events A and B independent?

1.2 Exercises

Exercise 1.3. Two dice are thrown together. Let A be the event "getting 6 on the first die" and B be the event "getting 2 on the second die". Are the events A and B independent?

Exercise 1.4. A committee of 4 students is selected at random from a group consisting of 8 boys and 4 girls. Given that there is at least one girl on the committee, calculate the probability that there are exactly 2 girls on the committee.

Exercise 1.5. Let A and B be two candidates seeking admission in a college. We are given that the probability that A is selected is 0.7, and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

Exercise 1.6. Three machines E_1 , E_2 , and E_3 in a certain factory produce 50%, 25%, and 25%, respectively, of the total daily output of electric tubes. It is known that 4% of the tubes produced on each of machines E_1 and E_2 are defective, and that 5% of those produced on E_3 are defective.

- If one tube is picked up at random from a day's production, calculate the probability that it is defective.
- Calculate the probability that the defective tube was produced on machine E_1 .

Exercise 1.7. A car manufacturing factory has two plants, X and Y . Plant X manufactures 70% of cars and plant Y manufactures 30%. 80% of the cars at plant X and 90% of the cars at plant Y are rated of standard quality. A car is chosen at random and is found to be of standard quality. What is the probability that it has come from plant X ?

Exercise 1.8. A merchant sells spare parts of the same kind. They are supplied by four factories: u_1 , u_2 , u_3 , u_4 , with probabilities 36%, 20%, 40%, and 10% respectively. The defective parts have probabilities 10%, 5%, 4%, 7%, and 20% from u_1 , u_2 , u_3 , and u_4 respectively. A person buys a part. What is the probability that it is defective? What is the probability that the defective part purchased comes from u_3 ?

1.3 Solutions

Exercise 1.9. A survey was conducted in a city in March 2021 about the reaction to the COVID-19 vaccine. The results showed the following:

- 20% of the citizens received the Pfizer vaccine.
- 30% received the Astrazeneca vaccine.
- 50% received the Sputnik V vaccine.

We conduct a systematic examination of vaccines: for this, we apply a test that reacts to 4% of citizens who take the Sputnik V vaccine, 5% of those who received the Astrazeneca vaccine, and 7% of the citizens who received the Pfizer vaccine. When we select a citizen at random;

1. What is the probability of a randomly selected citizen having had a reaction?
2. What is the probability that a citizen had received the Sputnik V vaccine and had a reaction?
3. What is the probability that a citizen had received the Pfizer vaccine given that they had a reaction?

1.3 Solutions

Solution 1.1.

1. We define a probability P such that:

$$P(\{a\}) + P(\{b\}) + P(\{c\}) = 1.$$

(a). $P(\{a, b\}) = \frac{1}{4}$ We have

$$P(\{a\}) + P(\{b\}) = \frac{1}{4} \Rightarrow P(\{c\}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Let $P(\{a\}) = p$, then:

$$P(\{b\}) = \frac{1}{4} - p, \quad P(\{c\}) = \frac{3}{4}.$$

We require:

$$0 \leq p \leq \frac{1}{4}, \quad 0 \leq P(\{b\}) \leq \frac{1}{4}.$$

So there are an infinity of possible probabilities.

1.3 Solutions

(b). $P(\{a, b\}) = P(\{b, c\}) = \frac{1}{4}$

$$P(\{a\}) + P(\{b\}) = \frac{1}{4} \quad (1.1)$$

$$P(\{b\}) + P(\{c\}) = \frac{1}{4} \quad (1.2)$$

$$P(\{a\}) + P(\{b\}) + P(\{c\}) = 1. \quad (1.3)$$

Add (1.1) and (1.2):

$$P(\{a\}) + 2P(\{b\}) + P(\{c\}) = \frac{1}{2}. \quad (1.4)$$

Subtract (1.4) from (1.3):

$$(1.3) - (1.4) \Rightarrow -P(\{b\}) = \frac{1}{2} \Rightarrow P(\{b\}) = -\frac{1}{2}$$

which is not possible, there is no probability verifying the given conditions.

(c). $P(\{a, b\}) = P(\{b, c\}) = \frac{3}{4}$. We have

$$P(\{a\}) + P(\{b\}) = \frac{3}{4} \quad (1.5)$$

$$P(\{b\}) + P(\{c\}) = \frac{3}{4} \quad (1.6)$$

$$P(\{a\}) + P(\{b\}) + P(\{c\}) = 1. \quad (1.7)$$

Add (1.5) and (1.6):

$$P(\{a\}) + 2P(\{b\}) + P(\{c\}) = \frac{3}{2}. \quad (1.8)$$

Subtract (1.7) from (1.8):

$$(1.8) - (1.7) \Rightarrow P(\{b\}) = \frac{1}{2}.$$

Now substitute into (1.5) and (1.6):

From (1.4):

$$P(\{a\}) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

From (1.5):

$$P(\{c\}) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

Thus:

$$P(\{a\}) + P(\{b\}) + P(\{c\}) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

There is one probability and only one possible one.+

1.3 Solutions

2.

$$P(A \cap B) - P(A)P(B) = -P(A \cap B^c) + P(A)P(B^c).$$

We have:

$$P(B^c) = 1 - P(B), \quad P(A) = P(A \cap B) + P(A \cap B^c).$$

Thus:

$$\begin{aligned} P(A \cap B) - P(A)P(B) &= P(A) - P(A \cap B) - P(A)P(B) \\ &= P(A)[1 - P(B)] - P(A \cap B) \\ &= P(A)P(B^c) - P(A \cap B) \\ &= -P(A \cap B) + P(A)P(B^c). \end{aligned}$$

Solution 1.2.

1. The formula of $P(A \cup B \cup C)$:

For any three events A , B , and C , the probability of their union is given by the inclusion-exclusion formula:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

2. Properties of independent events:

Assume that A and B are independent; that is,

$$P(A \cap B) = P(A)P(B).$$

(a). We need to verify that

$$P(A \cap \bar{B}) = P(A)P(\bar{B}).$$

Note that:

$$P(A \cap \bar{B}) = P(A) - P(A \cap B).$$

Since A and B are independent, substitute $P(A \cap B) = P(A)P(B)$:

$$P(A \cap \bar{B}) = P(A) - P(A)P(B) = P(A)[1 - P(B)].$$

But $1 - P(B) = P(\bar{B})$, so:

$$P(A \cap \bar{B}) = P(A)P(\bar{B}),$$

proving that A and \bar{B} are independent.

1.3 Solutions

(b). $P(B | A) = P(B | \bar{A})$.

First, compute $P(B | A)$:

$$P(B | A) = \frac{P(A \cap B)}{P(A)}.$$

Since A and B are independent, $P(A \cap B) = P(A)P(B)$, so:

$$P(B | A) = \frac{P(A)P(B)}{P(A)} = P(B).$$

Next, compute $P(B | \bar{A})$:

$$P(B | \bar{A}) = \frac{P(B \cap \bar{A})}{P(\bar{A})}.$$

Notice that:

$$P(B) = P(B \cap A) + P(B \cap \bar{A}),$$

and using independence $P(B \cap A) = P(A)P(B)$, we have:

$$P(B \cap \bar{A}) = P(B) - P(A)P(B) = P(B)[1 - P(A)].$$

Since $P(\bar{A}) = 1 - P(A)$, it follows that:

$$P(B | \bar{A}) = \frac{P(B)[1 - P(A)]}{1 - P(A)} = P(B).$$

Therefore,

$$P(B | A) = P(B) = P(B | \bar{A}).$$

3.

Suppose that for two events A and B , we have:

$$P(A) = 0.5, \quad P(B) = 0.3, \quad \text{and} \quad P(A \cup B) = 0.65.$$

Using the formula for the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

we solve for $P(A \cap B)$:

$$0.65 = 0.5 + 0.3 - P(A \cap B) \implies P(A \cap B) = 0.5 + 0.3 - 0.65 = 0.15.$$

For A and B to be independent, we must have:

$$P(A \cap B) = P(A)P(B) = 0.5 \times 0.3 = 0.15.$$

Since both calculations yield 0.15, the events A and B are indeed independent.

1.3 Solutions

Solution 1.3. Let $A = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$ and $B = \{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)\}$. The intersection of A and B is:

$$A \cap B = \{(6, 2)\}.$$

Next, we calculate the probabilities:

$$P(A) = \frac{6}{36}, \quad P(B) = \frac{6}{36}, \quad P(A \cap B) = \frac{1}{36}.$$

Events A and B are independent if:

$$P(A \cap B) = P(A) \cdot P(B).$$

Substitute the values:

$$P(A \cap B) = \frac{1}{36}, \quad P(A) \cdot P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

we conclude that A and B are independent events.

Solution 1.4. Let A denote the event that at least one girl will be chosen, and B the event that exactly 2 girls will be chosen. We are asked to find $P(B|A)$.

Since A denotes the event that at least one girl will be chosen, A' denotes the complement event, where no girl is chosen, i.e., 4 boys are chosen. Then,

$$P(A') = \frac{\binom{8}{4}}{\binom{12}{4}} = \frac{70}{495} = \frac{14}{99}.$$

Hence,

$$P(A) = 1 - P(A') = 1 - \frac{14}{99} = \frac{85}{99}.$$

Next, we compute $P(A \cap B)$, which is the probability of selecting exactly 2 boys and 2 girls. The number of ways to choose 2 girls and 2 boys is:

$$P(A \cap B) = \frac{\binom{8}{2}\binom{4}{2}}{\binom{12}{4}} = \frac{6 \times 28}{495} = \frac{56}{165}.$$

Finally, we can find $P(B|A)$:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{56}{165} \times \frac{99}{85} = \frac{168}{425}.$$

Thus, $P(B|A) = \frac{168}{425}$.

1.3 Solutions

Solution 1.5. Let p be the probability that B is selected.

The probability that exactly one of A and B is selected is given as 0.6:

$$P(\text{Exactly one of } A, B \text{ is selected}) = 0.6$$

This can be expressed as:

$$P(A \cap B') + P(A' \cap B) = 0.6.$$

This represents the events where either A is selected and B is not, or B is selected and A is not.

Expanding these terms:

$$P(A)P(B') + P(A')P(B) = 0.6.$$

Substitute the known values:

$$(0.7)(1 - p) + (0.3)(p) = 0.6.$$

Now, solve for p :

$$0.7 - 0.7p + 0.3p = 0.6$$

$$0.7 - 0.4p = 0.6$$

$$0.7 - 0.6 = 0.4p$$

$$0.4p = 0.1$$

$$p = \frac{0.1}{0.4} = 0.25.$$

Thus, the probability that B is selected is 0.25.

Solution 1.6.

- Let A_1, A_2, A_3 be the events that the tube is produced on machines E_1, E_2, E_3 , respectively.

- Let D be the event that the picked-up tube is defective.

We are given the following probabilities:

$$P(A_1) = \frac{50}{100} = \frac{1}{2}, \quad P(A_2) = \frac{25}{100} = \frac{1}{4}, \quad P(A_3) = \frac{25}{100} = \frac{1}{4}.$$

$$P(D|A_1) = \frac{4}{100} = \frac{1}{25}, \quad P(D|A_2) = \frac{4}{100} = \frac{1}{25}, \quad P(D|A_3) = \frac{5}{100} = \frac{1}{20}.$$

The total probability of a defective tube, $P(D)$, is given by:

$$P(D) = P(A_1)P(D|A_1) + P(A_2)P(D|A_2) + P(A_3)P(D|A_3)$$

1.3 Solutions

$$= \frac{1}{2} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{20} = 0.425.$$

Thus, $P(D) = 0.253$.

Now, we need to calculate $P(A_1|D)$, the probability that the tube is produced on machine E_1 , given that the tube is defective. Using Bayes' Theorem:

$$P(A_1|D) = \frac{P(A_1)P(D|A_1)}{P(D)}.$$

Substitute the values:

$$P(A_1|D) = \frac{\frac{1}{2} \times \frac{1}{25}}{\frac{17}{400}} = \frac{8}{17} \approx 0.471.$$

Thus, the probability that the tube is produced on machine E_1 , given that the tube is defective, is approximately $P(A_1|D) \approx 0.471$.

Solution 1.7.

- Let E be the event that the car is of standard quality.

- Let B_1 and B_2 be the events that the car is manufactured in plants X and Y, respectively. Now

$$P(B_1) = \frac{70}{100} = \frac{7}{10}, \quad P(B_2) = \frac{30}{100} = \frac{3}{10}$$

$P(E | B_1)$ = Probability that a standard quality car is manufactured in plant X

$$P(E | B_1) = \frac{80}{100} = \frac{8}{10}$$

$$P(E|B_2) := \frac{90}{100} = \frac{9}{10}$$

$P(B_1|E)$:= Probability that a standard quality car has come from plant X

$$P(B_1|E) = \frac{P(B_1) \cdot P(E|B_1)}{P(B_1) \cdot P(E|B_1) + P(B_2) \cdot P(E|B_2)}$$

$$P(B_1|E) = \frac{\frac{7}{10} \times \frac{8}{10}}{\frac{7}{10} \times \frac{8}{10} + \frac{3}{10} \times \frac{9}{10}} = \frac{56}{83}.$$

Hence, the required probability is $\frac{56}{83}$.

Solution 1.8. Let the events be defined as follows:

- u_i : "a part comes from factory i " for $i = 1, 2, 3, 4$.

1.3 Solutions

- D : "the part is defective"

We are given the following probabilities:

$$P(u_1) = 0.30, \quad P(u_2) = 0.20, \quad P(u_3) = 0.40, \quad P(u_4) = 0.10$$

$$P(D | u_1) = 0.10, \quad P(D | u_2) = 0.05, \quad P(D | u_3) = 0.70, \quad P(D | u_4) = 0.20$$

1. The probability that a purchased part is defective:

Using the law of total probability:

$$P(D) = \sum_{i=1}^4 P(u_i) \cdot P(D | u_i)$$

$$P(D) = (0.30)(0.10) + (0.20)(0.05) + (0.40)(0.70) + (0.10)(0.20)$$

$$P(D) = 0.03 + 0.01 + 0.28 + 0.02 = 0.34.$$

2. The probability that the defective part came from (u_3):

Using Bayes' Theorem:

$$P(u_3 | D) = \frac{P(u_3) \cdot P(D | u_3)}{P(D)}$$

$$P(u_3 | D) = \frac{0.40 \cdot 0.70}{0.34} = \frac{0.28}{0.34} \approx 0.8235.$$

Solution 1.9. Let the events be defined as follows:

- S : "the citizen who received the Sputnik V vaccine"
- A : "the citizen who received the Astrazeneca vaccine"
- P : "the citizen who received the Pfizer vaccine"
- R : "the citizen who had a reaction to the test"

We are given:

$$P(S) = 0.5, \quad P(A) = 0.3, \quad P(P) = 0.2.$$

$$P(R|S) = 0.04, \quad P(R|A) = 0.05, \quad P(R|P) = 0.07.$$

1.3 Solutions

1. The probability that the test causes a reaction is:

$$\begin{aligned}P(R) &= P(R \cap S) + P(R \cap A) + P(R \cap P) \\&= P(R|S)P(S) + P(R|A)P(A) + P(R|P)P(P) \\&= 0.04 \times 0.5 + 0.05 \times 0.3 + 0.07 \times 0.2 \\&= 0.02 + 0.015 + 0.014 \\&= 0.049.\end{aligned}$$

2. The probability that a citizen received the Sputnik V vaccine and had a reaction is:

$$P(S \cap R) = P(R|S)P(S) = 0.04 \times 0.5 = 0.02.$$

3. The probability that a citizen received the Pfizer vaccine given that they had a reaction is:

$$P(P|R) = \frac{P(P \cap R)}{P(R)}.$$

Applying Bayes' Theorem:

$$\begin{aligned}P(P|R) &= \frac{P(R|P)P(P)}{P(R|S)P(S) + P(R|A)P(A) + P(R|P)P(P)} \\&= \frac{0.07 \times 0.2}{0.04 \times 0.5 + 0.05 \times 0.3 + 0.07 \times 0.2} \\&= \frac{0.014}{0.049} \\&\approx 0.2857.\end{aligned}$$

Random variables

Contents

2.1	Reminder	14
2.1.1	Discrete random variables	14
2.1.2	Continuous random variables	16
2.1.3	Probability inequalities	18
2.2	Exercises	18
2.3	Solution	26

2.1 Reminder

2.1.1 Discrete random variables

Let X be a discrete random variable, we call the probability function of X (i.e, pf), the function defined by

$$P : \mathbb{R} \longrightarrow [0, 1]$$

$$x_i \longmapsto P(X = x_i)$$

The probability distribution of a discrete random variable X can be represented by the table below

x_i	x_1	x_2	...	x_n
$p(x_i) = P(X = x_i)$	$p(x_1)$	$p(x_2)$...	$p(x_n)$

and has the following basic properties:

2.1 Reminder

1. $0 \leq P(X = x_i) \leq 1$, for all $x_i \in \mathbb{R}$.
2. $\sum_i P(X = x_i) = 1$.
3. $P(X = x_i) = 0$ for all X outside a discrete set.

If X is discrete random variable then

1. $P(a < X < b) = \sum_{x=a+1}^{b-1} p(x)$.
2. $P(a \leq X < b) = \sum_{x=a}^{b-1} p(x)$.
3. $P(a < X \leq b) = \sum_{x=a+1}^b p(x)$.
4. $P(a \leq X \leq b) = \sum_{x=a}^b p(x)$.

The cumulative distribution function (CDF) is the function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined by

$$F_X(x) = P(X \leq x).$$

The mean of a discrete random variable X is given by

$$E(X) = \sum_{i=1}^n x_i P(X = x_i)$$

and satisfying the following properties:

1. $E(a) = a$.
2. Linearity: $E(aX + bY) = aE(X) + bE(Y)$.
3. Monotonicity: If $X \leq Y$ (i.e., $X(\omega) \leq Y(\omega)$ for $\omega \in \Omega$), then $E(X) \leq E(Y)$.

The r -th moment of a random variable X is defined to be $E[X^r]$ and given by

$$E[X^r] = \sum_{i=1}^n x_i^r P(X = x_i).$$

The variance of X , denoted $Var(X)$ or σ^2 is defined by

$$\sigma^2 = Var(X) = E[(X - E(X))^2]$$

and the standard deviation as

$$\sigma = \sqrt{Var(X)}.$$

The important properties of the variance are:

2.1 Reminder

1. $\text{Var}(X) \geq 0$.
2. If c is a constant, $\text{Var}(c) = 0$.
3. If a and b are real numbers, $\text{Var}(aX + b) = a^2 \text{Var}(X)$.
4. $\text{Var}(X) = E(X^2) - (E(X))^2$.
5. $\text{Var}(X) \leq E(X^2)$.

2.1.2 Continuous random variables

A continuous random variable X is a real-valued function $X : \Omega \rightarrow \mathbb{R}$ for which

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

where f is a function satisfying:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{+\infty} f(x) dx = 1$.

The function f is called the probability density function (pdf).

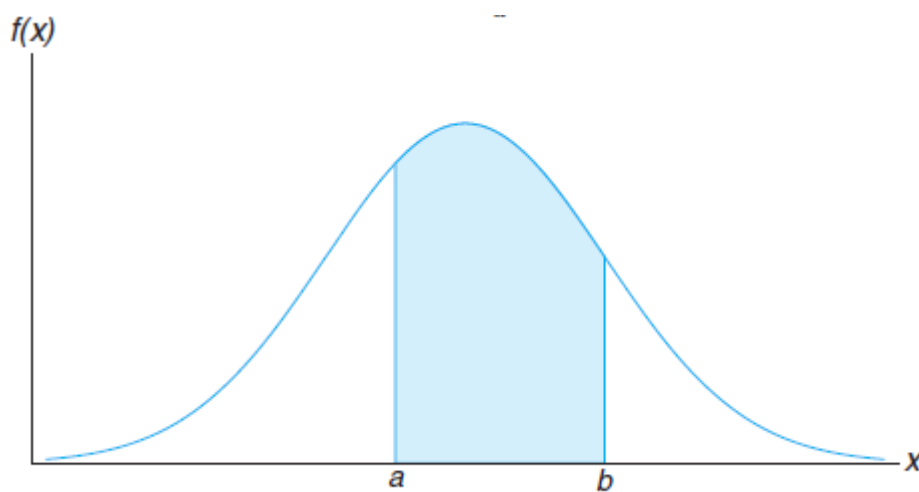


Figure 2.1: Area under the curve of $f(x)$ and over the interval $[a, b]$.

The cumulative distribution function (CDF) F_X of a continuous random variable X with probability density function $f(x)$ is given by

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt \text{ for all } x \in \mathbb{R}.$$

2.1 Reminder

and satisfying the following properties:

- $P(a \leq X \leq b) = F_X(b) - F_X(a)$.
- F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.
- $\lim_{x \rightarrow -\infty} F_X(x) = P(X \leq -\infty) = 0$.
- $\lim_{x \rightarrow +\infty} F_X(x) = P(X \leq +\infty) = 1$.

In addition, we have

$$\begin{aligned} F_X(b) - F_X(a) &= P(a < X < b) = P(a \leq X < b) \\ &= P(a < X \leq b) = P(a \leq X \leq b). \end{aligned}$$

The mean of a continuous random variable X with probability density function $f(x)$ is given by

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

and satisfying the following properties:

1. $E(a) = a$.
2. $E(aX + bY) = aE(X) + bE(Y)$.
3. If $X \leq Y$, then $E(X) \leq E(Y)$.

The r -th moment of a random variable X is defined to be $E[X^r]$ and given by

$$E[X^r] = \int_{-\infty}^{+\infty} x^r f(x) dx.$$

The variance of X , denoted $Var(X)$ or σ^2 is defined by

$$\sigma^2 = Var(X) = E[(X - E(X))^2]$$

and the standard deviation of X is

$$\sigma = \sqrt{Var(X)}.$$

Note that the variance satisfying the same properties of discrete case.

2.2 Exercises

2.1.3 Probability inequalities

2.1.3.1 Markov's inequality

$$P(X \geq \lambda) \leq \frac{E(X)}{\lambda}, \text{ for all } \lambda > 0.$$

2.1.3.2 Chebyshev's inequality

$$P(|X - E(X)| \geq \lambda) \leq \frac{\text{Var}(X)}{\lambda^2}, \text{ for all } \lambda > 0.$$

2.1.3.3 Cauchy-Schwarz inequality

$$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}.$$

2.1.3.4 Minkowski's inequality

$$E(|X + Y|^p)^{\frac{1}{p}} \leq E(|X|^p)^{\frac{1}{p}} + E(|Y|^p)^{\frac{1}{p}}.$$

2.1.3.5 Jensen's inequality

If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is convex on $[a, b]$, then

$$E(\varphi(X)) \geq \varphi(E(X)).$$

If φ is concave, then

$$E(\varphi(X)) \leq \varphi(E(X)),$$

where the convex function satisfying

$$\varphi(\alpha x + (1 - \alpha)y) \leq \alpha\varphi(x) + (1 - \alpha)\varphi(y).$$

and the concave function satisfying

$$\varphi(\alpha x + (1 - \alpha)y) \geq \alpha\varphi(x) + (1 - \alpha)\varphi(y).$$

2.2 Exercises

Exercise 2.1. Four balls are to be drawn without replacement from a box containing 8 red and 4 white balls. If X denotes the number of red balls drawn, find the probability distribution of X . Determine the mean and variance.

2.2 Exercises

Exercise 2.2. A discrete random variable X has the following probability distribution:

k	1	2	3	4	5	6	7
$P(X = k)$	C	$2C$	$2C$	$3C$	C^2	$2C^2$	$7C^2 + C$

Find the value of C . Also find the mean of the distribution.

Exercise 2.3. A random variable X takes the values $-2, -1, 0, 1$ with probabilities $\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$ respectively.

- Find and draw the probability distribution and cumulative distribution function.

Exercise 2.4. From the following probability distribution for a random variable X :

k	-3	-2	-1	0	1	2	3
$P(X = k)$	0.05	0.10	0.30	0.00	0.30	0.15	0.10

- Compute: $E(X)$, $E(2X \pm 3)$, $E(24X + 5)$, $V(X)$.

Exercise 2.5. A continuous random variable has the pdf for the random variable X as follows:

$$f(x) = \begin{cases} ke^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Determine the constant k , mean and variance.

Exercise 2.6. Is $f(x)$ represents the density of random variable has the pdf for a random variable X :

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Determine the cumulative distribution function (CDF) $F_X(x)$.

2.2 Exercises

Exercise 2.7. If $f(x)$ is a probability density function (pdf) defined as:

$$f(x) = \begin{cases} kx^3 & \text{for } 0 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

- Find the value of k and find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Exercise 2.8. The following is probability density function (pdf) for the random variable U is:

$$f_U(u) = \begin{cases} c \exp\left(-\frac{u}{4}\right) & \text{for } 0 \leq u \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- find the value of c must have, and evaluate $f_U(0.5)$.

Exercise 2.9. Given the probability density function (pdf) of the delivery time X :

$$f_X(x) = \begin{cases} \frac{x}{10} & \text{for } 0 \leq x \leq 2, \\ c & \text{for } 2 < x \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

1. What is the range Ω_X ?
2. What is the value of c that makes f_X a valid density function?
3. Find the cumulative distribution function (CDF) of X , $F_X(x)$, and make sure to define it piecewise for any real number x .
4. What is the probability that the delivery arrives between 2pm and 6pm?
5. What is the expected time that the package arrives at?

Exercise 2.10. We roll two dice and define Z as the random variable equal to the absolute difference between the numbers rolled. Determine the probability distribution of Z , its cumulative distribution function (CDF), its expectation, and its variance.

2.2 Exercises

Exercise 2.11. We consider a biased die with faces numbered from 1 to 6, and let X be the random variable representing the number on the top face. We are told that the probability of landing on a given face is proportional to the number on that face.

1. Determine the probability distribution of X , and calculating its expectation.
2. Determine the probability distribution of Y , and its expectation.

Exercise 2.12. Among the following functions defined on \mathbb{R} , determine which ones are the density of a random variable with a probability density. If applicable, calculate their cumulative distribution function (CDF) and specify if they have an expectation.

1. $f_1(x) = \cos(x)$ for $x \in [0, \pi/2)$, and 0 otherwise.

2. $f_2(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

3. $f_3(x) = \frac{e^x}{(e^x+1)^2}$ for $x \in \mathbb{R}$.

4. $f_4(x) = \begin{cases} 1+x, & \text{for } x \in [-1, 0], \\ 1-x, & \text{for } x \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$

5. $f_5(x) = \frac{1}{|x|^3}$ for $|x| > 1$, and 0 otherwise.

6. $f_6(x) = \sin(x) + 1$ for $x \in \mathbb{R}$.

Exercise 2.13. Determine if the following functions are probability density functions (pdf), and if so, determine the cumulative distribution function (CDF) of the associated random variable.

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ 4te^{-2t}, & \text{if } t \geq 0. \end{cases}$$

$$h(t) = \begin{cases} 0, & \text{if } t < 0, \\ \frac{1}{2\ln 2} e^{-t} \ln(1+e^t), & \text{if } t \geq 0. \end{cases}$$

Consider the change of variable $u = e^{-t}$ for solving the second function.

2.2 Exercises

Exercise 2.14. Calculate, if they exist, the expectation and variance of the random variable X , whose density function is:

$$g(t) = \begin{cases} 0 & \text{if } t < 0, \\ 4te^{-2t} & \text{if } t \geq 0. \end{cases}$$

and

$$h(t) = \begin{cases} 0 & \text{if } t < 1, \\ \frac{4\ln(t)}{t^3} & \text{if } t \geq 1. \end{cases}$$

Exercise 2.15. Let X be a random variable with the density function f defined on \mathbb{R} by:

$$f(x) = \begin{cases} kx(x-2) & \text{if } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

1. Determine the value of k so that f is a valid probability density function for X .
2. Determine the cumulative distribution function F of X , and compute the expectation and variance of X .

Exercise 2.16. Let f be the function defined on \mathbb{R} by:

$$f(x) = \frac{1}{2(1+|x|)^2}, \quad x \in \mathbb{R}.$$

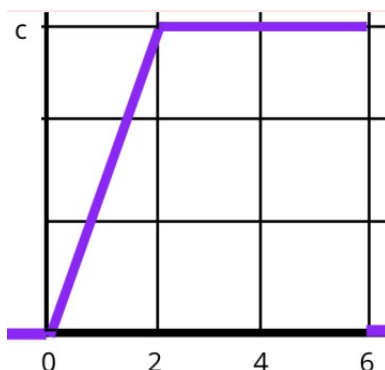
1. Prove that $f(x)$ is a probability density function.
2. Express $G(y)$ in term of $F(x)$.
3. Compute the density of Y .
4. Identify the distribution of Y .

Exercise 2.17. Suppose the number of hours that a package gets delivered past noon is modeled by the following probability density function (pdf):

$$f_X(x) = \begin{cases} \frac{x}{10}, & 0 \leq x \leq 2, \\ c, & 2 < x \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Exercises

Here is a graph of the pdf as described above:



1. What is the range Ω_X .
2. What is the value of c that makes $f_X(x)$ a valid density function.
3. Find the cumulative distribution function (CDF) of X , denoted as $F_X(x)$, ensuring that it is defined piecewise for any real number x .
4. What is the probability that the delivery arrives between 2pm and 6pm?
5. What is the expected time that the package arrives at?

Exercise 2.18. Suppose X is a continuous random variable with the probability density function (pdf):

$$f_X(x) = \begin{cases} cx^2, & 0 \leq x \leq 9, \\ 0, & \text{otherwise.} \end{cases}$$

1. Write an expression for the value of c that makes X a valid probability density function.
2. Set up expressions (integrals) for the mean $E[X]$ and variance $\text{Var}(X)$.
3. Find the cumulative distribution function (CDF) of X , denoted as $F_X(x)$.

Exercise 2.19. Suppose X is a continuous random variable with the probability density function (pdf):

$$f_X(x) = \begin{cases} \frac{c}{x^2}, & 1 \leq x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Exercises

1. Write an expression for the value of c that makes X a valid probability density function.
2. Set up expressions (integrals) for the mean $E[X]$ and variance $\text{Var}(X)$.
3. Find the cumulative distribution function (CDF) of X , denoted as $F_X(x)$.

Exercise 2.20. The probability distribution of a discrete random variable X is defined by:

$$P(X = x) = \begin{cases} \frac{c}{15}, & \text{for } x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

Find $E(X)$.

Exercise 2.21. The probability function of the discrete random variable X is shown in the table.

x	$P(X = x)$
1	0.2
2	a
3	0.25
4	b

Given that $E(X) = 2.95$, find the values of a and b .

Exercise 2.22. The probability distribution of the discrete random variable X is given by:

$$P(X = x) = \begin{cases} \frac{x}{30} & x = 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}$$

Find: $E(X)$, $E(3X - 2)$, $E(10 - X)$.

Exercise 2.23. Given the probability distribution of the random variable T :

t	$P(T = t)$
0	0.05
1	0.35
2	0.45
3	0.15

2.2 Exercises

Find the variance and standard deviation of T .

Exercise 2.24. The mean of the score of a single throw of a normal dice is 3.5 with variance $\frac{35}{12}$. In a game, the score is doubled and 3 subtracted. What are the mean and variance of the score in this game?

Exercise 2.25. The discrete random variable S is such that $E(S) = 3$ and $\text{Var}(S) = 7.5$. Find the value of:

(a) $E(S^2)$.

(b) $E[(S + 1)^2]$.

Exercise 2.26. Consider a real-valued random variable with probability density function given by

$$f(t) = \begin{cases} at + b & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

1. For which value(s) of a and b is the function f a valid probability density function?
2. Let $b = 2$. Determine the cumulative distribution function (CDF) F of the random variable X .
3. Compute $P(X \geq \frac{1}{2})$, $E(X)$, and $\text{Var}(X)$.

Exercise 2.27. Let $k > 1$. We define the function f by:

$$\forall t \in \mathbb{R}, \quad f(t) = \begin{cases} \frac{k}{t^{k+1}} & \text{if } t \in [1, +\infty[, \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that f is a probability density function of a real-valued random variable X .
2. Determine the cumulative distribution function (CDF) F of the random variable X .
3. Does X have an expectation? If so, calculate it.

2.3 Solution

4. Let a and b be two real numbers strictly greater than 1. Calculate:

$$P(X > a), \quad P(X > a + b \mid X > a).$$

Show that this conditional probability increases with a .

5. Let $Y = \ln X$. Determine the cumulative distribution function G of Y , and deduce the distribution of Y .

Exercise 2.28. Consider $X \sim B(n, p)$ and we want to use Chebyshev's inequality to find an upper bound on $P(X \geq qn)$ for $p < q < 1$.

Exercise 2.29. Suppose $X \sim \mathcal{E}(\lambda)$. Using Chebyshev's inequality, we estimate $P(|X - E[X]| > b)$ for $b > 0$.

Exercise 2.30. Suppose $X \sim \mathcal{E}(\lambda)$. Using Markov's inequality, we estimate $P(X > a)$ for $a > 0$ and compare it with the exact value of this probability.

2.3 Solution

Solution 2.1.

1. The probability distributions of X :

Since 4 balls have to be drawn, therefore, X can take the values 0, 1, 2, 3, 4. Thus $X(\Omega) = \{0, 1, 2, 3, 4\}$, we have

$$\begin{aligned} P(X = 0) &= P(\text{no red ball and 4 white balls}) \\ &= \frac{\binom{8}{0} \times \binom{4}{4}}{\binom{12}{4}} \\ &= \frac{1}{495}. \end{aligned}$$

$$\begin{aligned} P(X = 1) &= P(\text{1 red ball and 3 white balls}) \\ &= \frac{\binom{8}{1} \times \binom{4}{3}}{\binom{12}{4}} \\ &= \frac{32}{495}. \end{aligned}$$

2.3 Solution

$$\begin{aligned}P(X = 2) &= P(2 \text{ red balls and } 2 \text{ white balls}) \\&= \frac{\binom{8}{2} \times \binom{4}{2}}{\binom{12}{4}} \\&= \frac{168}{495}.\end{aligned}$$

$$\begin{aligned}P(X = 3) &= P(3 \text{ red balls and } 1 \text{ white ball}) \\&= \frac{\binom{8}{3} \times \binom{4}{1}}{\binom{12}{4}} \\&= \frac{224}{495}.\end{aligned}$$

$$\begin{aligned}P(X = 4) &= P(4 \text{ red balls and } 0 \text{ white ball}) \\&= \frac{\binom{8}{4} \times \binom{4}{0}}{\binom{12}{4}} \\&= \frac{70}{495}.\end{aligned}$$

Thus, the probability distribution of X is given by the following table:

k	0	1	2	3	4
$P(X=k)$	$\frac{1}{495}$	$\frac{32}{495}$	$\frac{168}{495}$	$\frac{224}{495}$	$\frac{70}{495}$

2. The mean $E(X)$:

The mean of a discrete random variable is given by:

$$E(X) = \sum_{k=0}^4 kP(X = k).$$

Substitute the values:

$$\begin{aligned}E(X) &= 0 \cdot \frac{1}{495} + 1 \cdot \frac{32}{495} + 2 \cdot \frac{168}{495} + 3 \cdot \frac{224}{495} + 4 \cdot \frac{70}{495} \\&= 0 + \frac{32}{495} + \frac{336}{495} + \frac{672}{495} + \frac{280}{495} \\&= \frac{32 + 336 + 672 + 280}{495} \\&= \frac{1320}{495} \\&\approx 2.67\end{aligned}$$

2.3 Solution

Thus, the mean $E(X)$ is (approximately) 2.67.

3. The variance $\text{Var}(X)$:

The variance is given by:

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

First, calculate $E(X^2)$:

$$E(X^2) = \sum_{k=0}^4 k^2 P(X = k).$$

Substitute the values:

$$\begin{aligned} E(X^2) &= 0^2 \cdot \frac{1}{495} + 1^2 \cdot \frac{32}{495} + 2^2 \cdot \frac{168}{495} + 3^2 \cdot \frac{224}{495} + 4^2 \cdot \frac{70}{495} \\ &= \frac{32}{495} + 4 \cdot \frac{168}{495} + 9 \cdot \frac{224}{495} + 16 \cdot \frac{70}{495} \\ &= \frac{32}{495} + \frac{672}{495} + \frac{2016}{495} + \frac{1120}{495} \\ &= \frac{32 + 672 + 2016 + 1120}{495} \\ &= \frac{3840}{495} \\ &\approx 7.76 \end{aligned}$$

Now, calculate the variance:

$$\begin{aligned} \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= 7.76 - (2.67)^2 \\ &= 7.76 - 7.13 \\ &\approx 0.63 \end{aligned}$$

Thus, the variance $\text{Var}(X)$ is (approximately) 0.63.

Solution 2.2.

1. **The value C :** Since the sum of all probabilities must be equal to 1, we have:

$$\begin{aligned} C + 2C + 2C + 3C + C^2 + 2C^2 + (7C^2 + C) &= 1 \\ 10C^2 + 9C &= 1 \\ 10C^2 + 9C - 1 &= 0 \\ (10C - 1)(C + 1) &= 0 \end{aligned}$$

2.3 Solution

Thus, the possible values for C are:

$$C = \frac{1}{10} \quad \text{or} \quad C = -1$$

Since probability values must be non-negative, we discard $C = -1$, thus:

$$C = \frac{1}{10}.$$

2. The mean $E(X)$:

The mean of a discrete probability distribution is given by:

$$E(X) = \sum_{k=1}^7 kP(X = k).$$

Substituting the given values:

$$\begin{aligned} E(X) &= 1 \times \frac{1}{10} + 2 \times \frac{2}{10} + 3 \times \frac{2}{10} + 4 \times \frac{3}{10} + 5 \times \left(\frac{1}{10}\right)^2 + 6 \times 2 \times \left(\frac{1}{10}\right)^2 + 7 \times \left(7 \left(\frac{1}{10}\right)^2 + \frac{1}{10}\right) \\ &= \frac{10 + 40 + 60 + 120 + 5 + 12 + 49 + 70}{100} \\ &= 3.66. \end{aligned}$$

Thus, the mean of the given probability distribution is:

$$E(X) = 3.66$$

Solution 2.3.

The probability distribution and cumulative distribution function of X :

$X = x$	-2	-1	0	1
$P(X)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$F(X) = P(X \leq x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{8}{8} = 1$

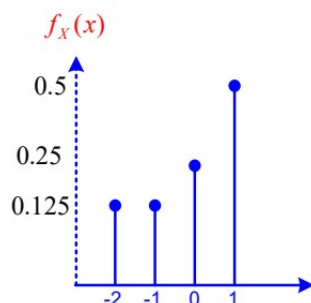


Figure 2.2: The probability mass function of the random variable X .

2.3 Solution

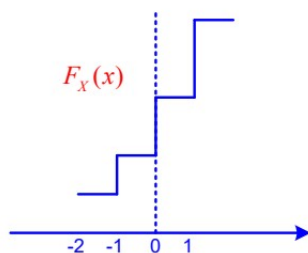


Figure 2.3: The cumulative distribution function of the random variable X .

Solution 2.4.

1. **Mean** $E(X)$: The expected value of X is given by:

$$E(X) = \sum_k kP(X = k).$$

Substituting the values from the table:

$$\begin{aligned} E(X) &= (-3 \times 0.05) + (-2 \times 0.10) + (-1 \times 0.30) + (0 \times 0.00) + (1 \times 0.30) + (2 \times 0.15) + (3 \times 0.10) \\ &= -0.15 - 0.20 - 0.30 + 0 + 0.30 + 0.30 + 0.30 \\ &= 0.25. \end{aligned}$$

2. **Expected Value** $E(2X \pm 3)$: From the linearity property, we have:

$$E(aX + b) = aE(X) + b.$$

- For $E(2X + 3)$:

$$\begin{aligned} E(2X + 3) &= 2 \times E(X) + 3 \\ &= 2 \times 0.25 + 3 \\ &= 0.50 + 3 \\ &= 3.5. \end{aligned}$$

- For $E(2X - 3)$:

$$\begin{aligned} E(2X - 3) &= 2 \times E(X) - 3 \\ &= 2 \times 0.25 - 3 \\ &= 0.50 - 3 \\ &= -2.5. \end{aligned}$$

2.3 Solution

3. **Expected Value** $E(24X + 5)$: Using the same property (linearity), we have:

$$\begin{aligned} E(24X + 5) &= 24 \times E(X) + 5 \\ &= 24 \times 0.25 + 5 \\ &= 6 + 5 \\ &= 11. \end{aligned}$$

4. **Variance** $Var(X)$: The variance of X is given by:

$$Var(X) = E(X^2) - (E(X))^2.$$

First, calculate $E(X^2)$:

$$E(X^2) = \sum_k k^2 P(X = k).$$

Substitute the values:

$$\begin{aligned} E(X^2) &= (-3^2 \times 0.05) + (-2^2 \times 0.10) + (-1^2 \times 0.30) + (0^2 \times 0.00) + (1^2 \times 0.30) + (2^2 \times 0.15) \\ &\quad + (3^2 \times 0.10) \\ &= (9 \times 0.05) + (4 \times 0.10) + (1 \times 0.30) + (0 \times 0.00) + (1 \times 0.30) + (4 \times 0.15) + (9 \times 0.10) \\ &= 0.45 + 0.40 + 0.30 + 0 + 0.30 + 0.60 + 0.90 \\ &= 2.95. \end{aligned}$$

Thus, the variance:

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= 2.95 - (0.25)^2 \\ &= 2.95 - 0.0625 \\ &= 2.8875. \end{aligned}$$

Solution 2.5.

1. **The value k** : To determine the constant k , the probability density function (pdf) must satisfy:

$$\int_{-\infty}^{+\infty} f(x) dx = 1.$$

2.3 Solution

Since $f(x) = 0$ for $x \leq 0$, the integral reduces to:

$$\begin{aligned}\int_0^{+\infty} k e^{-x} dx &= 1 \\ k \int_0^{+\infty} e^{-x} dx &= 1 \\ k[-e^{-x}]_0^{+\infty} &= 1 \\ k(0 - (-1)) &= 1.\end{aligned}$$

Thus, $k = 1$.

2. Mean $E(X)$: The expected value $E(X)$ is given by:

$$\begin{aligned}E(X) &= \int_{-\infty}^{+\infty} x f(x) dx \\ &= \int_0^{+\infty} x e^{-x} dx\end{aligned}$$

We use integration by parts. Let:

$$u = x \quad \text{and} \quad dv = e^{-x} dx.$$

Then:

$$du = dx \quad \text{and} \quad v = -e^{-x}.$$

Using the integration by parts formula:

$$\int u dv = uv - \int v du,$$

we get:

$$\begin{aligned}\int_0^{+\infty} x e^{-x} dx &= [-x e^{-x}]_0^{+\infty} + \int_0^{+\infty} e^{-x} dx \\ &= [-x e^{-x}]_0^{+\infty} + \int_0^{+\infty} e^{-x} dx \\ &= 0 + 1 \\ &= 1.\end{aligned}$$

Thus, the expected value is:

$$E(X) = 1.$$

3. Variance $Var(X)$: The variance $Var(X)$ is given by:

$$Var(X) = E(X^2) - (E(X))^2.$$

2.3 Solution

We need to calculate $E(X^2)$. So, the expected value of X^2 is:

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx = \int_0^{+\infty} x^2 e^{-x} dx.$$

Using integration by parts again. Let:

$$u = x^2 \quad \text{and} \quad dv = e^{-x} dx.$$

Then:

$$du = 2x dx \quad \text{and} \quad v = -e^{-x}.$$

We have:

$$\begin{aligned} \int_0^{+\infty} x^2 e^{-x} dx &= \left[-x^2 e^{-x} \right]_0^{+\infty} + 2 \int_0^{+\infty} x e^{-x} dx \\ &= 0 + 2E(X) \end{aligned}$$

Thus:

$$E(X^2) = 2.$$

Finally, the variance is:

$$\begin{aligned} V(X) &= E(X^2) - (E(X))^2 \\ &= 2 - 1^2 \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

Solution 2.6.

1. The probability density: To determine if $f(x)$ is a valid probability density function, we need to check if it satisfies the condition:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Since $f(x) = 0$ outside of the interval $-1 < x < 1$, we have:

$$\begin{aligned} \int_{-1}^1 \frac{1}{2}(x+1) dx &= \frac{1}{2} \int_{-1}^1 (x+1) dx \\ &= \frac{1}{2} \left(\int_{-1}^1 x dx + \int_{-1}^1 1 dx \right) \\ &= \frac{1}{2} \left(\left[\frac{x^2}{2} \right]_{-1}^1 + [x]_{-1}^1 \right) \\ &= \frac{1}{2} (0 + 2) \\ &= 1. \end{aligned}$$

2.3 Solution

Hence

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

is a density function.

2. Cumulative distribution function: The cumulative distribution function (CDF) $F_X(x)$ is defined as:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt.$$

We will define the CDF piecewise based on the given pdf.

Case 1: $x < -1$

For $x < -1$, the pdf $f(x) = 0$, so:

$$F_X(x) = 0.$$

Case 2: $-1 \leq x \leq 1$

For $-1 \leq x \leq 1$, the pdf is $f(x) = \frac{1}{2}(x+1)$. The CDF is:

$$\begin{aligned} F_X(x) &= \int_{-1}^x \frac{1}{2}(t+1) dt \\ &= \frac{1}{2} \int_{-1}^x (t+1) dt \\ &= \frac{1}{2} \left[\frac{t^2}{2} + t \right]_{-1}^x \\ &= \frac{1}{2} \left[\left(\frac{x^2}{2} + x \right) - \left(\frac{(-1)^2}{2} + (-1) \right) \right] \\ &= \frac{1}{2} \left[\frac{x^2}{2} + x - \left(\frac{1}{2} - 1 \right) \right] \\ &= \frac{1}{2} \left[\frac{x^2}{2} + x + \frac{1}{2} \right]. \end{aligned}$$

Thus, for $-1 \leq x \leq 1$:

$$F_X(x) = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4}.$$

Case 3: $x > 1$

For $x > 1$, the pdf is 0, and the total probability is 1. Therefore, the CDF is:

$$F_X(x) = 1.$$

2.3 Solution

The cumulative distribution function $F_X(x)$ is:

$$F_X(x) = \begin{cases} 0 & \text{for } x < -1, \\ \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4} & \text{for } -1 \leq x \leq 1, \\ 1 & \text{for } x > 1. \end{cases}$$

Solution 2.7.

The value k : To find the value of k , we use the property that the total probability must be equal to 1:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Since $f(x) = 0$ outside the interval $0 \leq x \leq 3$, the integral reduces to:

$$\begin{aligned} \int_0^3 kx^3 dx &= 1 \\ k \int_0^3 x^3 dx &= 1 \\ k \left[\frac{x^4}{4} \right]_0^3 &= 1. \end{aligned}$$

So, we get:

$$k \left(\frac{3^4}{4} - 0 \right) = 1 \quad \Rightarrow \quad k \times \frac{81}{4} = 1.$$

Thus, the value of k is $\frac{4}{81}$.

2. The probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$:

We have:

$$\begin{aligned} P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right) &= \int_{\frac{1}{2}}^{\frac{3}{2}} f(x) dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{4}{81} x^3 dx. \\ &= \frac{4}{81} \int_{\frac{1}{2}}^{\frac{3}{2}} x^3 dx = \frac{4}{81} \left[\frac{x^4}{4} \right]_{\frac{1}{2}}^{\frac{3}{2}} \\ &= \frac{4}{81} \left(\frac{\left(\frac{3}{2}\right)^4}{4} - \frac{\left(\frac{1}{2}\right)^4}{4} \right) = \frac{4}{81} \left(\frac{81}{16} - \frac{1}{16} \right) \\ &= \frac{4}{81} \times \frac{80}{64} = \frac{5}{81}. \end{aligned}$$

2.3 Solution

Thus, the probability between $\frac{1}{2}$ and $\frac{3}{2}$ is

$$P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right) = \frac{5}{81}.$$

Solution 2.8.

1. The value c :

To ensure that $f_U(u)$ is a valid pdf, we must satisfy the condition that the total probability is equal to 1:

$$\int_{-\infty}^{\infty} f_U(u) du = 1$$

Since $f_U(u) = 0$ outside the interval $0 \leq u \leq 1$, the integral reduces to:

$$\begin{aligned} \int_0^1 c \exp\left(-\frac{u}{4}\right) du &= 1 \\ c \int_0^1 \exp\left(-\frac{u}{4}\right) du &= 1 \\ c \left[-4 \exp\left(-\frac{u}{4}\right)\right]_0^1 &= 1 \\ c \left(-4 \left(\exp\left(-\frac{1}{4}\right) - 1\right)\right) &= 1 \\ c \times 4 \left(1 - \exp\left(-\frac{1}{4}\right)\right) &= 1. \end{aligned}$$

Thus:

$$c = \frac{1}{4 \left(1 - \exp\left(-\frac{1}{4}\right)\right)}.$$

2. $F_U(0.5)$:

The cumulative distribution function (CDF) $F_U(u)$ is:

$$F_U(u) = \int_0^u f_U(t) dt.$$

For $u = 0.5$:

$$F_U(0.5) = \int_0^{0.5} c \exp\left(-\frac{t}{4}\right) dt.$$

2.3 Solution

Using the same integral technique as above:

$$\begin{aligned}F_U(0.5) &= c \left[-4 \exp\left(-\frac{t}{4}\right) \right]_0^{0.5} \\F_U(0.5) &= c \times 4 \left(1 - \exp\left(-\frac{0.5}{4}\right) \right) \\F_U(0.5) &= \frac{1}{4 \left(1 - \exp\left(-\frac{1}{4}\right) \right)} \times 4 \left(1 - \exp\left(-\frac{0.5}{4}\right) \right) \\F_U(0.5) &= \frac{1 - \exp\left(-\frac{0.5}{4}\right)}{1 - \exp\left(-\frac{1}{4}\right)}.\end{aligned}$$

Solution 2.9.

1. The range Ω_X :

The range Ω_X is the set of all possible values that X can take. From the given pdf, we see that the function is defined for x between 0 and 6, inclusive of the interval $[0, 6]$. Hence, the range is:

$$\Omega_X = [0, 6].$$

2. The value c :

For $f_X(x)$ to be a valid probability density function, the total probability must be equal to 1, i.e.,

$$\int_{-\infty}^{\infty} f_X(x) dx = 1.$$

Since the pdf is zero outside the range $[0, 6]$, we have:

$$\begin{aligned}\int_0^2 \frac{x}{10} dx + \int_2^6 c dx &= 1 \\ \frac{1}{10} \int_0^2 x dx + c \int_2^6 dx &= 1 \\ \frac{1}{10} \left[\frac{x^2}{2} \right]_0^2 + c [x]_2^6 &= 1 \\ 0.2 + 4c &= 1.\end{aligned}$$

So, $4c = 1 - 0.2 = 0.8$, we get $c = \frac{0.8}{4} = 0.2$.

Thus, the value of c that makes $f_X(x)$ a valid density function is $c = 0.2$.

2.3 Solution

3. The cumulative distribution function (CDF) $F_X(x)$:

We have

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt.$$

Case 1: $x < 0$

For $x < 0$, $f_X(x) = 0$, so:

$$F_X(x) = 0.$$

Case 2: $0 \leq x \leq 2$

For $0 \leq x \leq 2$, $f_X(x) = \frac{x}{10}$. Thus,

$$F_X(x) = \int_0^x \frac{t}{10} dt = \frac{1}{10} \int_0^x t dt = \frac{1}{10} \left[\frac{t^2}{2} \right]_0^x = \frac{x^2}{20}.$$

So, for $0 \leq x \leq 2$:

$$F_X(x) = \frac{x^2}{20}.$$

Case 3: $2 < x \leq 6$

For $2 < x \leq 6$, $f_X(x) = 0.2$. We already have $F_X(2) = \frac{2^2}{20} = \frac{4}{20} = 0.2$. Therefore,

$$F_X(x) = F_X(2) + \int_2^x 0.2 dt = 0.2 + 0.2(x - 2).$$

Simplifying:

$$F_X(x) = 0.2 + 0.2x - 0.4 = 0.2x - 0.2.$$

So, for $2 < x \leq 6$:

$$F_X(x) = 0.2x - 0.2.$$

Case 4: $x > 6$

For $x > 6$, $f_X(x) = 0$, and we know that the total probability is 1. Therefore,

$$F_X(x) = 1.$$

The cumulative distribution function $F_X(x)$ is:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{x^2}{20} & \text{for } 0 \leq x \leq 2, \\ 0.2x - 0.2 & \text{for } 2 < x \leq 6, \\ 1 & \text{for } x > 6. \end{cases}$$

2.3 Solution

4. $P(2 \leq X \leq 6)$:

We want to find $P(2 \leq X \leq 6)$, which is:

$$P(2 \leq X \leq 6) = F_X(6) - F_X(2).$$

From the CDF:

$$F_X(6) = 0.2(6) - 0.2 = 1.2 - 0.2 = 1,$$

$$F_X(2) = \frac{2^2}{20} = \frac{4}{20} = 0.2.$$

So,

$$P(2 \leq X \leq 6) = 1 - 0.2 = 0.8.$$

5. The mean $E(X)$:

The expected value $E(X)$ is given by:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Since the pdf is zero outside the interval $[0, 6]$, we compute the integral over this interval:

$$E(X) = \int_0^2 x \cdot \frac{x}{10} dx + \int_2^6 x \cdot 0.2 dx.$$

We have

$$\int_0^2 x \cdot \frac{x}{10} dx = \frac{1}{10} \int_0^2 x^2 dx = \frac{1}{10} \left[\frac{x^3}{3} \right]_0^2 = \frac{1}{10} \times \frac{8}{3} = \frac{8}{30} = \frac{4}{15}.$$

and

$$\int_2^6 x \cdot 0.2 dx = 0.2 \int_2^6 x dx = 0.2 \left[\frac{x^2}{2} \right]_2^6 = 0.2 \left(\frac{36}{2} - \frac{4}{2} \right) = 0.2 \times 16 = 3.2.$$

Thus,

$$E(X) = \frac{4}{15} + 3.2 = \frac{4}{15} + \frac{48}{15} = \frac{52}{15}.$$

So, the expected time that the package arrives is:

$$E(X) = \frac{52}{15} \approx 3.47 \text{ hours.}$$

Solution 2.10. Let

$$Z = |X_1 - X_2|$$

where X_1 and X_2 represent the outcomes of the two dice.

1. The Probability Distribution of Z :

The possible values of Z are:

$$Z \in \{0, 1, 2, 3, 4, 5\}.$$

2.3 Solution

- $P(Z = 0)$ occurs when the two dice show the same number. There are 6 possible outcomes: (1,1), (2,2), (3,3), (4,4), (5,5), (6,6). Therefore:

$$P(Z = 0) = \frac{6}{36} = \frac{1}{6}$$

- $P(Z = 1)$ occurs when the difference between the dice is 1. There are 10 possible outcomes: (1,2), (2,1), (2,3), (3,2), (3,4), (4,3), (4,5), (5,4), (5,6), (6,5). Therefore:

$$P(Z = 1) = \frac{10}{36} = \frac{5}{18}$$

- $P(Z = 2)$ occurs when the difference between the dice is 2. There are 8 possible outcomes: (1,3), (2,4), (3,1), (3,5), (4,2), (5,3), (6,4), (4,6). Therefore:

$$P(Z = 2) = \frac{8}{36} = \frac{2}{9}$$

- $P(Z = 3)$ occurs when the difference between the dice is 3. There are 6 possible outcomes: (1,4), (2,5), (3,6), (4,1), (5,2), (6,3). Therefore:

$$P(Z = 3) = \frac{6}{36} = \frac{1}{6}$$

- $P(Z = 4)$ occurs when the difference between the dice is 4. There are 4 possible outcomes: (1,5), (2,6), (5,1), (6,2). Therefore:

$$P(Z = 4) = \frac{4}{36} = \frac{1}{9}$$

- $P(Z = 5)$ occurs when the difference between the dice is 5. There are 2 possible outcomes: (1,6), (6,1). Therefore:

$$P(Z = 5) = \frac{2}{36} = \frac{1}{18}$$

Thus, the probability distribution of Z is tabulate as:

z	0	1	2	3	4	5
$P(Z = z)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

2. Cumulative Distribution Function (CDF) of Z

The cumulative distribution function $F_Z(z)$ is the sum of the probabilities up to z . We calculate:

2.3 Solution

$$F_Z(z) = \begin{cases} 0 & \text{for } z < 0 \\ \frac{1}{6} & \text{for } 0 \leq z < 1 \\ \frac{1}{6} + \frac{5}{18} = \frac{13}{36} & \text{for } 1 \leq z < 2 \\ \frac{13}{36} + \frac{2}{9} = \frac{29}{36} & \text{for } 2 \leq z < 3 \\ \frac{29}{36} + \frac{1}{6} = \frac{35}{36} & \text{for } 3 \leq z < 4 \\ \frac{35}{36} + \frac{1}{9} = \frac{39}{36} = 1 & \text{for } 4 \leq z < 5 \\ 1 & \text{for } z \geq 5 \end{cases}$$

3. The mean $E[Z]$:

The expectation is calculated as follows:

$$\begin{aligned} E[Z] &= \sum_{z=0}^5 z \cdot P(Z = z) \\ &= 0 \cdot \frac{1}{6} + 1 \cdot \frac{5}{18} + 2 \cdot \frac{2}{9} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{9} + 5 \cdot \frac{1}{18} \\ &= \frac{5}{18} + \frac{8}{18} + \frac{9}{18} + \frac{8}{18} + \frac{5}{18} \\ &= \frac{35}{18} \\ &\approx 1.944. \end{aligned}$$

4. The variance $\text{Var}(Z)$:

To calculate the variance, we first compute $E[Z^2]$:

$$\begin{aligned} E[Z^2] &= \sum_{z=0}^5 z^2 \cdot P(Z = z) \\ &= 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{5}{18} + 2^2 \cdot \frac{2}{9} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{9} + 5^2 \cdot \frac{1}{18} \\ &= 0 + \frac{5}{18} + \frac{8}{9} + \frac{9}{6} + \frac{16}{9} + \frac{25}{18} \\ &= \frac{105}{18} \\ &= \frac{35}{6}. \end{aligned}$$

2.3 Solution

The variance is then:

$$\begin{aligned}\text{Var}(Z) &= E[Z^2] - (E[Z])^2 \\ &= \frac{35}{6} - \left(\frac{35}{18}\right)^2 \\ &= \frac{35}{6} - \frac{1225}{324} \\ &= \frac{1890}{324} - \frac{1225}{324} \\ &= \frac{665}{324} \\ &\approx 2.052.\end{aligned}$$

Solution 2.11.

1. The probability distribution of X , and its expectation:

Let the probability of landing on face i be proportional to i , where $i = 1, 2, 3, 4, 5, 6$. The total sum of the probabilities must be 1, so we first determine the constant of proportionality.

The probability $P(X = i)$ is given by:

$$P(X = i) = k \cdot i$$

where k is the constant of proportionality. To determine k , we use the fact that the total sum of the probabilities must be 1:

$$\begin{aligned}\sum_{i=1}^6 P(X = i) &= 1 \\ \sum_{i=1}^6 k \cdot i &= 1 \\ k \cdot (1 + 2 + 3 + 4 + 5 + 6) &= 1 \\ k \cdot 21 = 1 &\Rightarrow k = \frac{1}{21}.\end{aligned}$$

Thus, the probability distribution of X is:

$$P(X = i) = \frac{i}{21} \quad \text{for } i = 1, 2, 3, 4, 5, 6.$$

2.3 Solution

Now, we calculate the expectation $E[X]$:

$$\begin{aligned} E[X] &= \sum_{i=1}^6 i \cdot P(X=i) \\ &= 1 \cdot \frac{1}{21} + 2 \cdot \frac{2}{21} + 3 \cdot \frac{3}{21} + 4 \cdot \frac{4}{21} + 5 \cdot \frac{5}{21} + 6 \cdot \frac{6}{21} \\ &= \frac{1}{21} + \frac{4}{21} + \frac{9}{21} + \frac{16}{21} + \frac{25}{21} + \frac{36}{21} \\ &= \frac{91}{21} \\ &\approx 4.33 \end{aligned}$$

Thus, the expectation of X is 4.33.

2. The probability distribution of Y , and its expectation:

Let $Y = \frac{1}{X}$. We need to determine the probability distribution of Y and calculate its expectation. First, we find the possible values of Y . Since X can take values from 1 to 6, the corresponding values of $Y = \frac{1}{X}$ will be:

$$Y \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}$$

The probability distribution of Y is the same as the probability distribution of X , but with the corresponding values of Y . Thus, the probability mass function for Y is:

$$P(Y=y) = P\left(X = \frac{1}{y}\right) = \frac{1/y}{21} = \frac{1}{21y}$$

where $y \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}$.

Now, we calculate the expectation $E[Y]$:

$$\begin{aligned} E[Y] &= \sum_{y \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}} y \cdot P(Y=y) \\ &= 1 \cdot \frac{1}{21 \cdot 1} + \frac{1}{2} \cdot \frac{1}{21 \cdot \frac{1}{2}} + \frac{1}{3} \cdot \frac{1}{21 \cdot \frac{1}{3}} + \frac{1}{4} \cdot \frac{1}{21 \cdot \frac{1}{4}} + \frac{1}{5} \cdot \frac{1}{21 \cdot \frac{1}{5}} + \frac{1}{6} \cdot \frac{1}{21 \cdot \frac{1}{6}} \\ &= \frac{1}{21} + \frac{1}{21} + \frac{1}{21} + \frac{1}{21} + \frac{1}{21} + \frac{1}{21} \\ &= \frac{6}{21} \\ &= \frac{2}{7} \\ &\approx 0.286 \end{aligned}$$

Thus, the expectation of Y is 0.286.

2.3 Solution

Solution 2.12.

1. $f_1(x) = \cos(x)$ for $x \in [0, \pi/2)$, and 0 otherwise.

$$\int_0^{\pi/2} \cos(x) dx = \sin(x) \Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1.$$

Thus, $f_1(x)$ is a valid probability density function. The CDF is:

$$F_1(x) = \int_0^x \cos(t) dt = \sin(x) \quad \text{for } x \in [0, \pi/2).$$

The expectation is:

$$E[X] = \int_0^{\pi/2} x \cos(x) dx.$$

To calculate this integral, we use integration by parts. Let:

$$u = x, \quad dv = \cos(x) dx$$

Then,

$$du = dx, \quad v = \sin(x)$$

From the integration by parts formula, we get

$$\begin{aligned} E[X] &= [x \sin(x)]_0^{\pi/2} - \int_0^{\pi/2} \sin(x) dx \\ &= \left(\frac{\pi}{2} \sin(\pi/2) - 0 \right) - [-\cos(x)]_0^{\pi/2} \\ &= \frac{\pi}{2} - (-\cos(\pi/2) + \cos(0)) \\ &= \frac{\pi}{2} - (0 + 1) \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

So, the expectation is $\frac{\pi}{2} - 1$.

2. $f_2(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

This is the standard Cauchy distribution. We check normalization:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

Using the known integral:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

2.3 Solution

So, to normalize, we divide by π , giving the pdf:

$$f_2(x) = \frac{1}{\pi(1+x^2)}.$$

The CDF is:

$$F_2(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x).$$

The expectation does not exist because the integral for $E[X]$ diverges for the Cauchy distribution:

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{1+x^2} dx.$$

This integral does not converge, hence the expectation does not exist.

3. $f_3(x) = \frac{e^x}{(e^x+1)^2}$ for $x \in \mathbb{R}$.

We check normalization:

$$\int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} dx.$$

This can be verified using substitution or recognizing it as the derivative of a standard sigmoid function. The integral evaluates to 1, so $f_3(x)$ is a valid pdf.

The CDF $F_3(x)$ is:

$$F_3(x) = \int_{-\infty}^x \frac{e^t}{(e^t+1)^2} dt.$$

The result involves the sigmoid function and is difficult to express in elementary functions.

The expectation is:

$$E[X] = \int_{-\infty}^{\infty} x \frac{e^x}{(e^x+1)^2} dx.$$

This integral can be computed numerically and has a finite value.

$$4. f_4(x) = \begin{cases} 1+x, & \text{for } x \in [-1, 0], \\ 1-x, & \text{for } x \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

First, check normalization:

$$\int_{-1}^0 (1+x) dx + \int_0^1 (1-x) dx.$$

Compute each integral:

$$\int_{-1}^0 (1+x) dx = \left[x + \frac{x^2}{2} \right]_{-1}^0 = (0+0) - (-1 + \frac{1}{2}) = \frac{1}{2}.$$

$$\int_0^1 (1-x) dx = \left[x - \frac{x^2}{2} \right]_0^1 = (1 - \frac{1}{2}) - (0-0) = \frac{1}{2}.$$

2.3 Solution

So, the total integral is:

$$\frac{1}{2} + \frac{1}{2} = 1.$$

Thus, $f_4(x)$ is a valid pdf. The CDF is:

$$F_4(x) = \begin{cases} 0, & x < -1, \\ \frac{x^2}{2} + x, & -1 \leq x < 0, \\ 1 - \frac{(1-x)^2}{2}, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

The expectation is:

$$E[X] = \int_{-1}^0 x(1+x) dx + \int_0^1 x(1-x) dx.$$

Compute each integral:

$$\int_{-1}^0 x(1+x) dx = \int_{-1}^0 (x+x^2) dx = \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-1}^0 = 0 - \left(\frac{1}{2} - \frac{1}{3} \right) = -\frac{1}{6}$$

$$\int_0^1 x(1-x) dx = \int_0^1 (x-x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

So,

$$E[X] = -\frac{1}{6} + \frac{1}{6} = 0.$$

5. $f_5(x) = \frac{1}{|x|^3}$ for $|x| > 1$, and 0 otherwise.

Check normalization:

$$\int_{-\infty}^{-1} \frac{1}{|x|^3} dx + \int_1^{\infty} \frac{1}{|x|^3} dx.$$

These integrals are:

$$\int_1^{\infty} \frac{1}{x^3} dx = \frac{1}{2}.$$

So, the total integral is:

$$\frac{1}{2} + \frac{1}{2} = 1.$$

However, $f_5(x)$ is not a valid pdf because it diverges at $x = 0$ (it is not integrable over the entire range). Therefore, $f_5(x)$ is not a valid probability density.

6. $f_6(x) = \sin(x) + 1$ for $x \in \mathbb{R}$.

The function oscillates between 0 and 2 and does not approach 0 fast enough to ensure the total probability is finite. Therefore, $f_6(x)$ is not a valid pdf because it doesn't decay at infinity and doesn't integrate to 1.

2.3 Solution

Solution 2.13.

1. The function $f(t)$:

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ 4te^{-2t}, & \text{if } t \geq 0. \end{cases}$$

(i) For a function to be a probability density function (pdf), it must satisfy the following two conditions:

1. The function must be non-negative for all t , i.e., $f(t) \geq 0$.

2. $\int_{-\infty}^{\infty} f(t) dt = 1$.

- For $t \geq 0$, $f(t) = 4te^{-2t}$, which is clearly non-negative.

- For $t < 0$, $f(t) = 0$, which is also non-negative.

(ii) The function f is continuous on $] -\infty; 0]$; and as a product of continuous functions, the function

$$t \mapsto 4te^{-2t}$$

is continuous on $]0; +\infty[$. Thus, f is continuous on \mathbb{R}^* . Hence f is continuous on \mathbb{R} .

(iii) As f is null on $] -\infty; 0[$, the integral $\int_{-\infty}^0 f(t) dt$ is convergent and equals 0.

On $]0; +\infty[$, f is continuous so the integral $\int_0^{+\infty} f(t) dt$ only raises an issue at $+\infty$.

Now, let's check the normalization condition by integrating $f(t)$ over $t \in [0, \infty)$:

$$\int_0^{\infty} f(t) dt = \int_0^{\infty} 4te^{-2t} dt.$$

Let $A > 0$, We use integration by parts to solve this integral. Let:

$$u = t, \quad dv = 4e^{-2t} dt.$$

Then:

$$du = dt, \quad v = -2e^{-2t}.$$

From the integration by parts formula, we get

$$\begin{aligned} \int_0^A 4te^{-2t} dt &= \left[-2te^{-2t}\right]_0^A + \int_0^A 2e^{-2t} dt \\ &= -2Ae^{-2A} - e^{-2A} + 1. \end{aligned}$$

2.3 Solution

Now,

$$\lim_{A \rightarrow +\infty} -2Ae^{-2A} - e^{-2A} + 1 = 1.$$

So, the integral $\int_0^{+\infty} f(t) dt$ is convergent and equals 1. We conclude that $\int_{-\infty}^{+\infty} f(t) dt$ is convergent and equals 1.

Thus f is therefore a probability density function.

The cumulative distribution function (CDF) is given by:

$$F(t) = \int_{-\infty}^t f(u) du$$

- For $t < 0$, since $f(t) = 0$ for $t < 0$, the CDF is:

$$F(t) = 0 \quad \text{for } t < 0$$

- For $t \geq 0$, we have:

$$F(t) = \int_0^t 4ue^{-2u} du$$

This is the same integral we computed earlier, but with the upper limit t . From the previous calculation, we know:

$$F(t) = 1 - e^{-2t}(1 + 2t)$$

Thus, the CDF is:

$$F(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1 - e^{-2t}(1 + 2t), & \text{if } t \geq 0. \end{cases}$$

2. The function $h(t)$:

$$h(t) = \begin{cases} 0, & \text{if } t < 0, \\ \frac{1}{2\ln 2} e^{-t} \ln(1 + e^t), & \text{if } t \geq 0. \end{cases}$$

(i) First, we need to verify that $h(t)$ is non-negative and integrates to 1.

1. For $t \geq 0$, $h(t) = \frac{1}{2\ln 2} e^{-t} \ln(1 + e^t)$ is clearly non-negative, as the expression is positive for all $t \geq 0$.
2. For $t < 0$, $h(t) = 0$, which is non-negative.

(ii) The function h is continuous on $] -\infty; 0]$; and as a product of continuous functions, the function

$$t \mapsto \frac{1}{2\ln 2} e^{-t} \ln(1 + e^t)$$

is continuous on $] 0; +\infty[$. Thus, h is continuous on \mathbb{R}^* . Hence h is continuous on \mathbb{R} .

2.3 Solution

(iii) As h is null on $]-\infty; 0[$, the integral $\int_{-\infty}^0 h(t) dt$ is convergent and equals 0.

On $[0; +\infty[$, h is continuous so the integral $\int_0^{+\infty} h(t) dt$ only raises an issue at $+\infty$.

Let $A > 0$, using the change of variable $u = e^{-t}$ (which gives us $t = -\ln(u)$ and $dt = -\frac{1}{u} du$):

$$\begin{aligned} \int_0^A \frac{1}{2\ln 2} e^{-t} \ln(1 + e^t) dt &= \frac{1}{2\ln 2} \int_1^{e^{-A}} u \ln\left(1 + \frac{1}{u}\right) \left(-\frac{1}{u}\right) du \\ &= -\frac{1}{2\ln 2} \int_1^{e^{-A}} \ln\left(1 + \frac{1}{u}\right) du \\ &= -\frac{1}{2\ln 2} \int_1^{e^{-A}} (\ln(1 + u) - \ln(u)) du \\ &= -\frac{1}{2\ln 2} [(1 + u) \ln(1 + u) - (1 + u) - u \ln(u) + u]_1^{e^{-A}} \\ &= -\frac{1}{2\ln 2} \left((1 + e^{-A}) \ln(1 + e^{-A}) - 1 - e^{-A} \ln(e^{-A}) - 2\ln 2 + 1 \right) \\ &= 1 - \frac{(1 + e^{-A}) \ln(1 + e^{-A}) - Ae^{-A}}{2\ln 2}. \end{aligned}$$

Now,

$$\lim_{A \rightarrow +\infty} 1 - \frac{(1 + e^{-A}) \ln(1 + e^{-A}) - Ae^{-A}}{2\ln 2} = 1$$

So, the integral $\int_0^{+\infty} h(t) dt$ is convergent and equals 1. We conclude that $\int_{-\infty}^{+\infty} h(t) dt$ is convergent and equals 1.

Thus h is therefore a probability density function.

The CDF is given by:

$$F(t) = \int_{-\infty}^t h(u) du$$

- For $t < 0$, since $h(t) = 0$ for $t < 0$, the CDF is:

$$F(t) = 0 \quad \text{for } t < 0$$

- For $t \geq 0$, we have:

$$F(t) = \int_0^t \frac{1}{2\ln 2} e^{-u} \ln(1 + e^u) du$$

we get:

$$F(t) = 1 - \frac{(1 + e^{-t}) \ln(1 + e^{-t}) - te^{-t}}{2\ln 2}$$

2.3 Solution

Thus, the CDF is:

$$F(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1 - \frac{(1+e^{-t})\ln(1+e^{-t}) - te^{-t}}{2\ln 2}, & \text{if } t \geq 0. \end{cases}$$

Solution 2.14.

1. The density function $g(t)$:

(a). The expectation of a continuous random variable X with density $g(t)$ is defined as:

$$E[X] = \int_{-\infty}^{\infty} tg(t) dt.$$

Since $g(t) = 0$ for $t < 0$, we can restrict the integral to $t \geq 0$, where $g(t) = 4te^{-2t}$. Thus, the expectation becomes:

$$E[X] = \int_0^{\infty} t \cdot 4te^{-2t} dt = 4 \int_0^{\infty} t^2 e^{-2t} dt.$$

Now, we need to compute the integral:

$$I = \int_0^{\infty} t^2 e^{-2t} dt.$$

This is a standard integral, which can be evaluated using the Gamma function or integration by parts. The standard result for this integral is:

$$\int_0^{\infty} t^2 e^{-2t} dt = \frac{2}{(2)^3} = \frac{1}{4}.$$

Thus, the expectation is:

$$E[X] = 4 \times \frac{1}{4} = 1.$$

(b). The variance of a random variable X is given by:

$$\text{Var}(X) = E[X^2] - (E[X])^2.$$

We already know that $E[X] = 1$, so now we need to compute $E[X^2]$, which is:

$$E[X^2] = \int_0^{\infty} t^2 \cdot 4te^{-2t} dt = 4 \int_0^{\infty} t^3 e^{-2t} dt.$$

Now, we solve the integral:

$$J = \int_0^{\infty} t^3 e^{-2t} dt.$$

2.3 Solution

This is another standard integral, and its result is:

$$\int_0^{\infty} t^3 e^{-2t} dt = \frac{6}{(2)^4} = \frac{3}{8}.$$

Thus:

$$E[X^2] = 4 \times \frac{3}{8} = \frac{3}{2}.$$

Now, we can compute the variance:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{3}{2} - 1^2 = \frac{3}{2} - 1 = \frac{1}{2}.$$

2. The density function $h(t)$

(a). The expectation is given by:

$$E[Y] = \int_1^{\infty} t \cdot \frac{4 \ln(t)}{t^3} dt = \int_1^{\infty} \frac{4 \ln(t)}{t^2} dt.$$

Let $A > 1$, using integration by parts, we get

$$\begin{aligned} \int_1^A \frac{4 \ln(t)}{t^2} dt &= \left[-\frac{4 \ln(t)}{t} \right]_1^A + \int_1^A \frac{4}{t^2} dt \\ &= -\frac{4 \ln(A)}{A} - \frac{4}{A} + 4. \end{aligned}$$

Now,

$$\lim_{A \rightarrow +\infty} -\frac{4 \ln(A)}{A} - \frac{4}{A} + 4 = 4$$

So, the integral $\int_0^{+\infty} h(t) dt$ is convergent and equals 1. We conclude that $\int_{-\infty}^{+\infty} h(t) dt$ is convergent and equals 1.

Therefore, the expectation is:

$$E[Y] = 4.$$

(b). The variance is given by:

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2.$$

We already know that $E[Y] = 4$. Now, we compute $E[Y^2]$:

$$E[Y^2] = \int_1^{\infty} t^2 \cdot \frac{4 \ln(t)}{t^3} dt = \int_1^{\infty} \frac{4 \ln(t)}{t} dt.$$

This integral diverges. In fact, the integral $\int_1^{\infty} \frac{\ln(t)}{t} dt$ is divergent, which implies that $E[Y^2] = \infty$.

Therefore, the variance $\text{Var}(Y)$ does not exist.

2.3 Solution

Solution 2.15.

1. The value k

For $f(x)$ to be a valid probability density function, it must satisfy the following conditions:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$,
- $\int_{-\infty}^{\infty} f(x) dx = 1$.

Since $f(x) = 0$ outside the interval $[0, 2]$, we have

$$\begin{aligned}\int_0^2 kx(x-2) dx &= 1 \\ k \left[\int_0^2 x^2 dx - 2 \int_0^2 x dx \right] &= 1 \\ k \left[\left[\frac{x^3}{3} \right]_0^2 - 2 \left[\frac{x^2}{2} \right]_0^2 \right] &= 1 \\ k \left(\frac{8}{3} - 4 \right) &= 1\end{aligned}$$

$$k \left(\frac{8}{3} - 4 \right) = 1 \Rightarrow k \left(\frac{8}{3} - \frac{12}{3} \right) = 1 \Rightarrow k \left(\frac{-4}{3} \right) = 1 \Rightarrow k = -\frac{3}{4}.$$

Thus, the value of k is $-\frac{3}{4}$.

2. The cumulative distribution function:

The cumulative distribution function (CDF) $F(x)$ is defined as:

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f(t) dt.$$

We need to compute $F(x)$ for different values of x :

- For $x < 0$, $f(x) = 0$, so:

$$F(x) = 0 \quad \text{for } x < 0.$$

- For $0 \leq x \leq 2$, we compute the integral:

$$\begin{aligned}F(x) &= \int_0^x -\frac{3}{4}t(t-2) dt = -\frac{3}{4} \int_0^x (t^2 - 2t) dt \\ &= -\frac{3}{4} \left(\left[\frac{t^3}{3} \right]_0^x - 2 \left[\frac{t^2}{2} \right]_0^x \right) = -\frac{3}{4} \left(\frac{x^3}{3} - x^2 \right).\end{aligned}$$

Therefore, for $0 \leq x \leq 2$,

$$F(x) = -\frac{3}{4} \left(\frac{x^3}{3} - x^2 \right).$$

2.3 Solution

- For $x > 2$, $f(x) = 0$, so:

$$F(x) = 1 \quad \text{for } x > 2.$$

Thus, the cumulative distribution function is:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ -\frac{3}{4} \left(\frac{x^3}{3} - x^2 \right) & \text{if } 0 \leq x \leq 2, \\ 1 & \text{if } x > 2. \end{cases}$$

To compute the expectation and variance of X , we use the following formulas:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx,$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx,$$

and the variance is given by:

$$\text{Var}(X) = E[X^2] - (E[X])^2.$$

Thus

$$E[X] = 1, \text{Var}(X) = \frac{1}{5}$$

Solution 2.16.

1. The probability density function.

For $f(x)$ to be a probability density function, it must satisfy the following two conditions:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$,
- The integral of $f(x)$ over \mathbb{R} must be 1:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

- For first condition: $f(x) \geq 0$

Clearly, for all $x \in \mathbb{R}$, $|x| \geq 0$, so $1 + |x| \geq 1$, implying that $(1 + |x|)^2 \geq 1$. Therefore,

$$f(x) = \frac{1}{2(1 + |x|)^2} \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

- For second condition: the normalization condition

2.3 Solution

Now, we compute the integral of $f(x)$ over \mathbb{R} :

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{1}{2(1+|x|)^2} dx.$$

Due to the symmetry of $f(x)$ around $x = 0$, we can compute the integral for $x \geq 0$ and multiply by 2:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{2(1+|x|)^2} dx &= 2 \int_0^{\infty} \frac{1}{2(1+x)^2} dx \\ &= \int_0^{\infty} \frac{1}{(1+x)^2} dx \\ &= \left[-\frac{1}{1+x} \right]_0^{\infty} \\ &= 0 - (-1) \\ &= 1. \end{aligned}$$

Thus, the normalization condition holds:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Since both conditions are satisfied, $f(x)$ is a valid probability density function.

2. Express $G(y)$ in terms of $F(x)$.

Let $Y = \ln(1 + |X|)$. The cumulative distribution function of Y is:

$$G(y) = P(Y \leq y) = P(\ln(1 + |X|) \leq y).$$

Exponentiating both sides:

$$\ln(1 + |X|) \leq y \quad \Rightarrow \quad 1 + |X| \leq e^y \quad \Rightarrow \quad |X| \leq e^y - 1.$$

Thus,

$$G(y) = P(|X| \leq e^y - 1) = F(e^y - 1),$$

where $F(x)$ is the cumulative distribution function of X .

3. The density of Y .

The density of Y , denoted $g_Y(y)$, is the derivative of $G(y)$:

$$g_Y(y) = \frac{d}{dy} G(y) = \frac{d}{dy} F(e^y - 1).$$

Using the chain rule:

$$g_Y(y) = f(e^y - 1) \cdot \frac{d}{dy} (e^y - 1) = f(e^y - 1) \cdot e^y.$$

2.3 Solution

Substitute $f(x) = \frac{1}{2(1+|x|)^2}$ into the equation:

$$g_Y(y) = \frac{1}{2(1 + |e^y - 1|)^2} \cdot e^y.$$

Thus, the probability density function of Y is:

$$g_Y(y) = \frac{e^y}{2(1 + |e^y - 1|)^2}.$$

4. The distribution of Y .

The density function of Y is:

$$g_Y(y) = \frac{e^y}{2(1 + |e^y - 1|)^2}.$$

This form suggests that Y follows a logistic distribution or log-normal distribution, depending on the exact behavior of $|e^y - 1|$. More specifically, given that $Y = \ln(1 + |X|)$, the distribution of Y is related to the log-normal distribution, as it involves the logarithm of a shifted absolute value.

Solution 2.17.

1. The range Ω_X :

The random variable X takes values in the interval where $f_X(x)$ is positive. Thus, the range of X is:

$$\Omega_X = [0, 6].$$

2. The value c

To ensure that $f_X(x)$ is a valid probability density function, it must satisfy the normalization condition:

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x) dx &= 1 \\ \int_0^2 \frac{x}{10} dx + \int_2^6 c dx &= 1 \\ \frac{1}{10} \int_0^2 x dx + c \int_2^6 dx &= 1 \\ \frac{1}{10} \times \frac{x^2}{2} \Big|_0^2 + c(6-2) &= 1 \\ \frac{1}{10} \times \frac{4}{2} + 4c &= 1 \\ 0.2 + 4c &= 1. \end{aligned}$$

2.3 Solution

thus:

$$4c = 0.8 \Rightarrow c = 0.2.$$

3. CDF $F_X(x)$

The cumulative distribution function (CDF) is given by:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt.$$

- For $x < 0$:

$$F_X(x) = 0.$$

- For $0 \leq x \leq 2$:

$$F_X(x) = \int_0^x \frac{t}{10} dt = \frac{1}{10} \times \frac{t^2}{2} \Big|_0^x = \frac{x^2}{20}.$$

- For $2 < x \leq 6$:

$$\begin{aligned} F_X(x) &= F_X(2) + \int_2^x 0.2 dt \\ &= \frac{2^2}{20} + 0.2(x - 2) \\ &= 0.2 + 0.2x - 0.4 \\ &= 0.2x - 0.2. \end{aligned}$$

- For $x > 6$:

$$F_X(x) = 1.$$

Thus, the CDF is:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{20}, & 0 \leq x \leq 2, \\ 0.2x - 0.2, & 2 < x \leq 6, \\ 1, & x > 6. \end{cases}$$

4. Probability that the delivery arrives between 2 PM and 6 PM

The probability that the delivery time X falls between 2 PM and 6 PM is given by:

$$P(2 \leq X \leq 6) = \int_2^6 f_X(x) dx.$$

From the given PDF, for $2 < x \leq 6$, we have $f_X(x) = c$. From the normalization condition, we previously found $c = 0.2$. Thus,

$$P(2 \leq X \leq 6) = \int_2^6 0.2 dx.$$

2.3 Solution

Evaluating the integral:

$$P(2 \leq X \leq 6) = 0.2(6 - 2) = 0.2 \times 4 = 0.8.$$

We can compute $P(2 \leq X \leq 6)$ using different approaches:

1. Geometric Approach From the given PDF, for $2 < x \leq 6$, the density function is constant with $f_X(x) = 1/5$. The probability is simply the area of the rectangle formed by the base $(6-2) = 4$ and height $1/5$:

$$P(2 \leq X \leq 6) = \text{Base} \times \text{Height} = 4 \times \frac{1}{5} = \frac{4}{5}.$$

2. We can also use the cumulative distribution function (CDF) computed earlier:

$$P(2 \leq X \leq 6) = F_X(6) - F_X(2).$$

From the CDF:

$$F_X(6) = \frac{6}{5}, \quad F_X(2) = \frac{2}{10} = \frac{1}{5}.$$

Thus,

$$P(2 \leq X \leq 6) = \left(\frac{6}{5} - \frac{1}{5}\right) - \frac{1}{10} = 1 - \frac{1}{5} = \frac{4}{5}.$$

Both methods confirm that the probability is $\frac{4}{5} = 0.8$.

5. Expected delivery time $E[X]$

The expected value of X is given by:

$$\begin{aligned} E[X] &= \int_0^6 x f_X(x) dx \\ &= \int_0^2 x \frac{x}{10} dx + \int_2^6 x(0.2) dx \\ &= \frac{1}{10} \int_0^2 x^2 dx + 0.2 \int_2^6 x dx \\ &= \frac{1}{10} \times \frac{x^3}{3} \Big|_0^2 + 0.2 \times \frac{x^2}{2} \Big|_2^6 \\ &= \frac{1}{10} \times \frac{8}{3} + 0.2 \times \left(\frac{6^2}{2} - \frac{2^2}{2}\right) \\ &= \frac{8}{30} + 3.2 \\ &\approx 3.467. \end{aligned}$$

Thus, the expected delivery time is approximately 3.467 hours past noon, or around 3:28 PM.

2.3 Solution

Solution 2.18.

1. The value c :

To ensure that $f_X(x)$ is a valid probability density function, we require:

$$\int_{-\infty}^{\infty} f_X(y) dy = 1.$$

Since $f_X(x) = cx^2$ for $0 \leq x \leq 9$ and 0 otherwise, we have:

$$\begin{aligned}\int_0^9 cx^2 dx &= 1 \\ c \int_0^9 x^2 dx &= 1 \\ c \left[\frac{x^3}{3} \right]_0^9 &= 1 \\ c \left(\frac{9^3}{3} - \frac{0^3}{3} \right) &= 1 \\ c \times \frac{729}{3} &= 1 \\ 243c &= 1\end{aligned}$$

thus $c = \frac{1}{243}$.

2. Mean and Variance:

The expected value is given by:

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty} xf_X(x) dx = \int_0^9 x \cdot \frac{1}{243} x^2 dx \\ &= \frac{1}{243} \int_0^9 x^3 dx = \frac{1}{243} \frac{x^4}{4} \Big|_0^9 \\ &= \frac{1}{243} \times \frac{9^4}{4} = \frac{27}{4} \\ &= 6.75.\end{aligned}$$

2.3 Solution

Similarly, we compute $E[X^2]$:

$$\begin{aligned} E[X^2] &= \int_0^9 x^2 \cdot \frac{1}{243} x^2 dx \\ &= \frac{1}{243} \int_0^9 x^4 dx \\ &= \frac{1}{243} \left. \frac{x^5}{5} \right|_0^9 \\ &= \frac{1}{243} \frac{9^5}{5} \\ &= \frac{243}{5} \\ &= 48.6. \end{aligned}$$

The variance is given by:

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= 48.6 - (6.75)^2 \\ &= 48.6 - 45.5625 \\ &= 3.0375. \end{aligned}$$

3. Cumulative distribution function:

The cumulative distribution function (CDF) is defined as:

$$F_X(t) = P(X \leq t) = \int_{-\infty}^t f_X(y) dy.$$

We consider three cases:

- If $t < 0$, then $F_X(t) = 0$ since the probability of getting a negative number is zero.
- If $0 \leq t \leq 9$, we compute:

$$\begin{aligned} F_X(t) &= \int_0^t \frac{1}{243} s^2 ds \\ &= \frac{1}{243} \left[\frac{s^3}{3} \right]_0^t \\ &= \frac{1}{243} \times \frac{t^3}{3} \\ &= \frac{t^3}{729}. \end{aligned}$$

2.3 Solution

- If $t > 9$, then $F_X(t) = 1$.

Thus, the CDF is:

$$F_X(t) = \begin{cases} 0, & t < 0, \\ \frac{t^3}{729}, & 0 \leq t \leq 9, \\ 1, & t > 9. \end{cases}$$

Solution 2.19.

1. The value c :

We have

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x) dx &= 1 \\ \int_1^{\infty} \frac{c}{x^2} dx &= 1 \\ c \int_1^{\infty} x^{-2} dx &= 1 \\ c \left[\frac{-1}{x} \right]_1^{\infty} &= 1 \\ c(0 - (-1)) &= 1 \\ c &= 1. \end{aligned}$$

2. Mean and Variance:

The expected value is given by:

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty} x f_X(x) dx \\ &= \int_1^{\infty} x \cdot \frac{1}{x^2} dx \\ &= \int_1^{\infty} \frac{1}{x} dx \\ &= \ln x \Big|_1^{\infty} \\ &= \ln \infty - \ln 1 = \infty. \end{aligned}$$

Since this integral diverges, $E[X]$ is undefined.

Similarly, for $E[X^2]$:

$$E[X^2] = \int_1^{\infty} x^2 \cdot \frac{1}{x^2} dx = \int_1^{\infty} dx.$$

Since this integral also diverges, $E[X^2]$ is also undefined. Hence, the variance is not defined.

2.3 Solution

3. Cumulative distribution function:

The cumulative distribution function (CDF) is defined as:

$$F_X(t) = P(X \leq t) = \int_{-\infty}^t f_X(y) dy.$$

We consider two cases:

- If $t < 1$, then $F_X(t) = 0$.
- If $t \geq 1$, we have:

$$\begin{aligned} F_X(t) &= \int_1^t \frac{1}{x^2} dx \\ &= \left[\frac{-1}{x} \right]_1^t \\ &= \left(-\frac{1}{t} + 1 \right) \\ &= 1 - \frac{1}{t} \end{aligned}$$

Thus, the CDF is:

$$F_X(t) = \begin{cases} 0, & t < 1, \\ 1 - \frac{1}{t}, & t \geq 1. \end{cases}$$

Solution 2.20.

- The probability distribution of X is shown here.

x	$P(X = x)$
1	$1/15$
2	$2/15$
3	$3/15$
4	$4/15$
5	$5/15$

- $E(X) = \left(1 \cdot \frac{1}{15}\right) + \left(2 \cdot \frac{2}{15}\right) + \left(3 \cdot \frac{3}{15}\right) + \left(4 \cdot \frac{4}{15}\right) + \left(5 \cdot \frac{5}{15}\right) = \frac{55}{15}$.

Solution 2.21.

2.3 Solution

x	$P(X = x)$
1	0.2
2	a
3	0.25
4	b

We have the total probability must be 1, so

$$a + b = 1 - (0.2 + 0.25) = 1 - 0.45 = 0.55.$$

and $E(X) = 2.95$, so

$$(1 \cdot 0.2) + 2a + (3 \cdot 0.25) + 4b = 2.95$$

$$0.2 + 2a + 0.75 + 4b = 2.95$$

$$2a + 4b = 2.95 - 0.2 - 0.75$$

$$2a + 4b = 2$$

$$a + 2b = 1$$

Therefore, $a = 0.1$ and $b = 0.45$.

Solution 2.22.

- The probability distribution of X is:

x	$P(X = x)$
4	$4/30$
5	$5/30$
6	$6/30$
7	$7/30$
8	$8/30$

$$\begin{aligned} E(X) &= \left(4 \cdot \frac{4}{30}\right) + \left(5 \cdot \frac{5}{30}\right) + \left(6 \cdot \frac{6}{30}\right) + \left(7 \cdot \frac{7}{30}\right) + \left(8 \cdot \frac{8}{30}\right) \\ &= \frac{16 + 25 + 36 + 49 + 64}{30} \\ &= \frac{190}{30} = \frac{19}{3}. \end{aligned}$$

2.3 Solution

- Using the linearity of expectation:

$$E(3X - 2) = 3E(X) - 2 = 3\left(\frac{19}{3}\right) - 2 = 19 - 2 = 17.$$

- Using the linearity of expectation:

$$E(10 - X) = E(-X + 10) = -E(X) + 10 = -\frac{19}{3} + 10 = \frac{-19 + 30}{3} = \frac{11}{3}.$$

Solution 2.23.

- Calculate $E(T)$:

$$\begin{aligned} E(T) &= \sum t \cdot P(T = t) \\ &= (0 \cdot 0.05) + (1 \cdot 0.35) + (2 \cdot 0.45) + (3 \cdot 0.15) \\ &= 0 + 0.35 + 0.90 + 0.45 \\ &= 1.7. \end{aligned}$$

- Calculate $E(T^2)$:

$$\begin{aligned} E(T^2) &= \sum t^2 \cdot P(T = t) \\ &= (0^2 \cdot 0.05) + (1^2 \cdot 0.35) + (2^2 \cdot 0.45) + (3^2 \cdot 0.15) \\ &= (0 \cdot 0.05) + (1 \cdot 0.35) + (4 \cdot 0.45) + (9 \cdot 0.15) \\ &= 0 + 0.35 + 1.8 + 1.35 \\ &= 3.5. \end{aligned}$$

- Calculate $Var(T)$:

$$\begin{aligned} Var(T) &= E(T^2) - [E(T)]^2 \\ &= 3.5 - (1.7)^2 \\ &= 3.5 - 2.89 \\ &= 0.61. \end{aligned}$$

- Calculate the standard deviation σ :

$$\sigma = \sqrt{Var(T)} = \sqrt{0.61} \approx 0.781$$

2.3 Solution

Solution 2.24. Let X be the usual score of a single throw of a normal dice. Given:

$$E(X) = 3.5, \text{Var}(X) = \frac{35}{12}.$$

The new score is given by $Y = 2X - 3$.

- **Mean of the new score ($E(Y)$):**

Using the linearity of expectation:

$$\begin{aligned} E(Y) &= E(2X - 3) = 2E(X) - 3 \\ &= 2(3.5) - 3 = 7 - 3 \\ &= 4. \end{aligned}$$

- **Variance of the new score ($\text{Var}(Y)$):**

Using the property $\text{Var}(aX + b) = a^2 \text{Var}(X)$:

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(2X - 3) = 2^2 \text{Var}(X) \\ &= 4 \text{Var}(X) = 4 \frac{35}{12} \\ &= \frac{35}{3}. \end{aligned}$$

Solution 2.25.

(a). **The value of $E(S^2)$:**

Use the formula

$$\text{Var}(S) = E(S^2) - (E(S))^2.$$

So

$$7.5 = E(S^2) - 3^2,$$

Thus $E(S^2) = 16.5$.

(b). **The value of $E[(S + 1)^2]$:**

We have:

$$E[(S + 1)^2] = E(S^2 + 2S + 1)$$

2.3 Solution

Using 'expectation of sum = sum of expectations'

$$\begin{aligned}E[(S + 1)^2] &= E(S^2) + E(2S + 1) \\&= E(S^2) + 2E(S) + 1 \\&= 16.5 + 2(3) + 1 \\&= 16.5 + 6 + 1 \\&= 23.5;\end{aligned}$$

Solution 2.26. Consider a real-valued random variable with probability density function given by

$$f(t) = \begin{cases} at + b & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

1. **The values of a and b :**

A function f is a valid pdf if:

- $f(t) \geq 0$ for all t
- $\int_{-\infty}^{\infty} f(t) dt = 1$

Since $f(t) = 0$ outside $[0, 1]$, we compute:

$$\int_0^1 (at + b) dt = \left[\frac{a}{2}t^2 + bt \right]_0^1 = \frac{a}{2} + b = 1$$

So, the condition becomes:

$$\frac{a}{2} + b = 1 \quad \Rightarrow \quad a = 2(1 - b)$$

Also, to ensure $f(t) \geq 0$ on $[0, 1]$, we check:

$$f(0) = b \geq 0, \quad f(1) = a + b = 2(1 - b) + b = 2 - b \geq 0$$

So:

$$b \in [0, 2]$$

Therefore:

$$a = 2(1 - b), \quad b \in [0, 2]$$

2.3 Solution

2. The cumulative distribution function (CDF) :

Let $b = 2$. Then:

$$a = 2(1 - 2) = -2$$

So:

$$f(t) = -2t + 2, \quad \text{for } t \in [0, 1]$$

For the CDF:

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ \int_0^t (-2u + 2) du & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t > 1. \end{cases}$$

we have:

$$\int_0^t (-2u + 2) du = [-u^2 + 2u]_0^t = -t^2 + 2t$$

thus:

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ -t^2 + 2t & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

3. $P(X \geq \frac{1}{2})$, $E(X)$, $Var(X)$:

- $P(X \geq \frac{1}{2})$:

$$\begin{aligned} P(X \geq \frac{1}{2}) &= \int_{1/2}^1 (-2t + 2) dt \\ &= [-t^2 + 2t]_{1/2}^1 \\ &= (-1 + 2) - \left(-\frac{1}{4} + 1\right) \\ &= 1 - \left(\frac{3}{4}\right) \\ &= \frac{1}{4}. \end{aligned}$$

2.3 Solution

- *Expectation:*

$$\begin{aligned} E(X) &= \int_0^1 t(-2t+2) dt \\ &= \int_0^1 (-2t^2+2t) dt \\ &= \left[-\frac{2}{3}t^3 + t^2 \right]_0^1 \\ &= -\frac{2}{3} + 1 \\ &= \frac{1}{3}. \end{aligned}$$

- *Second moment:*

$$\begin{aligned} E(X^2) &= \int_0^1 t^2(-2t+2) dt = \int_0^1 (-2t^3+2t^2) dt \\ &= \left[-\frac{1}{2}t^4 + \frac{2}{3}t^3 \right]_0^1 = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}. \end{aligned}$$

- *Variance:*

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

Solution 2.27.

1. The density function:

(a). We have:

- For $t \in [1, \infty)$, $f(t) = \frac{k}{t^{k+1}} > 0$ since $k > 1$, $t > 0$.

- For $t < 1$, $f(t) = 0$. Thus:

$$\forall t \in \mathbb{R}, \quad f(t) \geq 0.$$

f is positive function.

(b). The function f is:

- Continuous on $[1, \infty)$ since $\frac{k}{t^{k+1}}$ is continuous for $t > 1$,
- Constant and continuous on $(-\infty, 1)$ as $f(t) = 0$,
- Right-continuous at $t = 1$:

$$\lim_{t \rightarrow 1^+} f(t) = \frac{k}{1^{k+1}} = k = f(1).$$

2.3 Solution

Thus, f is continuous on $(-\infty, 1)$ and $[1, \infty)$ (piecewise continuous on \mathbb{R}).

(c). We Have:

$$\int_{-\infty}^{\infty} f(t) dt = \int_1^{\infty} \frac{k}{t^{k+1}} dt.$$

Since $k > 1$, the integral converges:

$$\int_1^{\infty} \frac{k}{t^{k+1}} dt = k \int_1^{\infty} t^{-(k+1)} dt = k \left[\frac{t^{-k}}{-k} \right]_1^{\infty} = k \left(0 - \left(-\frac{1}{k} \right) \right) = 1.$$

The function $f(t)$ satisfies:

- $f(t) \geq 0$ for all $t \in \mathbb{R}$,
- f is piecewise continuous on \mathbb{R} ,
- $\int_{-\infty}^{\infty} f(t) dt = 1$.

Therefore, $f(t)$ is a probability density function.

2. The cumulative distribution:

- For $t < 1$, we have:

$$F(t) = \int_{-\infty}^t f(u) du = 0.$$

- For $t \geq 1$:

$$F(t) = \int_1^t \frac{k}{u^{k+1}} du = k \left[\frac{u^{-k}}{-k} \right]_1^t = 1 - \frac{1}{t^k}.$$

Thus,

$$F(t) = \begin{cases} 0 & \text{if } t < 1, \\ 1 - \frac{1}{t^k} & \text{if } t \geq 1. \end{cases}$$

3. The expectation:

$$E[X] = \int_1^{\infty} t \cdot \frac{k}{t^{k+1}} dt = k \int_1^{\infty} t^{-k} dt.$$

This integral exist and converges because $k > 1$:

$$\int_1^{\infty} t^{-k} dt = \left[\frac{t^{-k+1}}{-k+1} \right]_1^{\infty} = \frac{1}{k-1}.$$

Thus,

$$E[X] = \frac{k}{k-1}.$$

4. $P(X > a)$, $P(X > a + b | X > a)$:

2.3 Solution

Let $a > 1$ and $b > 0$. we get

$$P(X > a) = 1 - F(a) = \frac{1}{a^k},$$
$$P(X > a + b \mid X > a) = \frac{P(X > a + b)}{P(X > a)} = \frac{1/(a + b)^k}{1/a^k} = \left(\frac{a}{a + b}\right)^k.$$

To show that $\left(\frac{a}{a+b}\right)^k$ increases with a , observe that the function $\frac{a}{a+b}$ is increasing in a because its derivative is positive, and since the power function $x \mapsto x^k$ is increasing for $x > 0$, the composition is increasing.

5. The CDF G and the distribution of Y :

Let $Y = \ln X$. Thus

$$G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\ln X \leq y) = \mathbb{P}(X \leq e^y) = F(e^y).$$

So:

$$G(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 - \frac{1}{e^{ky}} & \text{if } y \geq 0. \end{cases}$$

This is the CDF of an exponential distribution with parameter k , since:

$$G(y) = 1 - e^{-ky}, \quad \text{for } y \geq 0.$$

Thus,

$$Y = \ln X \sim \mathcal{E}(k).$$

Solution 2.28.

Mean and Variance:

From the properties of the Binomial distribution:

$$E[X] = np, \quad \text{Var}(X) = np(1 - p).$$

Define the deviation:

$$b = (q - p)n > 0.$$

Rewriting the probability in terms of deviation from the mean:

$$P(X \geq qn) = P(X - np \geq (q - p)n).$$

Using Chebyshev's inequality:

$$P(|X - np| \geq (q - p)n) \geq P(X - np \geq (q - p)n).$$

2.3 Solution

Thus, we obtain:

$$\begin{aligned}P(X \geq qn) &\leq \frac{\text{Var}(X)}{(q-p)^2 n^2} \\ &\leq \frac{np(1-p)}{(q-p)^2 n^2} \\ &\leq \frac{p(1-p)}{(q-p)^2 n}.\end{aligned}$$

This provides an upper bound on $P(X > qn)$ using **Chebyshev's inequality**, which shows that the probability of X deviating significantly from its expectation decreases as n increases.

Solution 2.29.

Mean and Variance:

For an exponential distribution with rate parameter λ , the expectation and variance are given by:

$$E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

Chebyshev's inequality states that

$$P(|X - E[X]| > b) \leq \frac{\text{Var}(X)}{b^2}.$$

Substituting the expectation and variance for the exponential distribution:

$$P\left(|X - \frac{1}{\lambda}| > b\right) \leq \frac{\frac{1}{\lambda^2}}{b^2}$$

$$P\left(|X - \frac{1}{\lambda}| > b\right) \leq \frac{1}{\lambda^2 b^2}.$$

The upper bound on the probability that X deviates from its mean by more than b is:

$$P(|X - E[X]| > b) \leq \frac{1}{\lambda^2 b^2}.$$

This bound decreases as b increases, indicating that large deviations from the mean are increasingly unlikely.

2.3 Solution

Solution 2.30.

(a). *The estimate of $P(X > a)$*

For an exponential distribution with rate parameter λ , we have:

$$E[X] = \frac{1}{\lambda}.$$

Thus, applying Markov's inequality:

$$P(X > a) \leq \frac{E[X]}{a} = \frac{1}{\lambda a}.$$

(b). **Exact Probability Computation:**

For an exponential distribution, the survival function (complementary cumulative distribution function) is:

$$P(X > a) = e^{-\lambda a}.$$

(c). **Comparison:**

Since $e^{-\lambda a}$ decreases exponentially, it provides a much tighter bound than Markov's inequality, especially for large a . Markov's inequality gives a rougher estimate because it only uses the expectation, without considering the actual distribution shape.

Thus:

Markov's inequality provides an upper bound for $P(X > a)$, but it is often loose compared to the exact probability. The exponential function $e^{-\lambda a}$ decreases much faster than $\frac{1}{\lambda a}$, indicating that Markov's bound overestimates the tail probability.

Probability distribution

Contents

3.1	Reminder	73
3.1.1	Summary of discrete probability distributions	73
3.1.2	Summary of continuous probability distributions	75
3.1.3	Approximations	78
3.2	Exercises	78
3.3	Solutions	90

3.1 Reminder

3.1 Reminder

3.1.1 Summary of discrete probability distributions

Property	Bernoulli Distribution
Notation	$X \sim B(p)$
Probability Mass Function (PMF)	$P(X = x) = p^x(1 - p)^{1-x}, \quad x = 0, 1$
Cumulative Distribution Function (CDF)	$P(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$
Mean	$\mu = p$
Variance	$\sigma^2 = p(1 - p)$
Standard Deviation	$\sigma = \sqrt{p(1 - p)}$

Property	Binomial Distribution
Notation	$X \sim B(n, p)$
Probability Mass Function (PMF)	$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, 2, \dots, n$
Cumulative Distribution Function (CDF)	$P(X \leq x) = \sum_{k=0}^x \binom{n}{k} p^k (1 - p)^{n-k}$
Mean	$\mu = np$
Variance	$\sigma^2 = np(1 - p)$
Standard Deviation	$\sigma = \sqrt{np(1 - p)}$

Property	Multinomial Distribution
Notation	$\mathbf{X} = (X_1, X_2, \dots, X_k) \sim \mathcal{M}(n; p_1, p_2, \dots, p_k)$
Probability Mass Function (PMF)	$P(X_1 = x_1, \dots, X_k = x_k) = \frac{n!}{x_1! x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$ <p>where $\sum_{i=1}^k x_i = n, \sum_{i=1}^k p_i = 1$</p>
Cumulative Distribution Function (CDF)	Computed via cumulative sums of PMF.
Mean	$E[X_i] = np_i$ for each $i = 1, 2, \dots, k$
Variance	$\text{Var}(X_i) = np_i(1 - p_i)$
Covariance	$\text{Cov}(X_i, X_j) = -np_i p_j$ for $i \neq j$

3.1 Reminder

Property	Hypergeometric Distribution
Notation	$X \sim \mathcal{H}(N, K, n)$
Probability Mass Function (PMF)	$P(X = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$ for $\max(0, n - (N - K)) \leq k \leq \min(n, K)$
Cumulative Distribution Function (CDF)	Computed via cumulative sum of PMF.
Mean	$E[X] = n \cdot \frac{K}{N}$
Variance	$\text{Var}(X) = n \cdot \frac{K}{N} \cdot \left(1 - \frac{K}{N}\right) \cdot \frac{N-n}{N-1}$
Standard Deviation	$\sigma = \sqrt{\text{Var}(X)}$

Property	Polyhypergeometric Distribution
Notation	$\mathbf{X} = (X_1, X_2, \dots, X_k) \sim \mathcal{MH}(N_1, N_2, \dots, N_k; n)$
Probability Mass Function (PMF)	$P(X_1 = x_1, \dots, X_k = x_k) = \frac{\prod_{i=1}^k \binom{N_i}{x_i}}{\binom{N}{n}}$, with $\begin{cases} \sum_{i=1}^k x_i = n \\ \sum_{i=1}^k N_i = N \end{cases}$
CDF	Often tabulated or computed numerically
Mean	$E[X_i] = n \cdot \frac{N_i}{N}$
Variance	$\text{Var}(X_i) = n \cdot \frac{N_i}{N} \cdot \left(1 - \frac{N_i}{N}\right) \cdot \frac{N-n}{N-1}$
Standard Deviation	$\sigma_i = \sqrt{\text{Var}(X_i)}$

Property	Geometric Distribution
Notation	$X \sim G(p)$
Probability Mass Function (PMF)	$P(X = k) = (1 - p)^{k-1} p$, for $k = 1, 2, 3, \dots$
Cumulative Distribution Function (CDF)	$P(X \leq k) = 1 - (1 - p)^k$
Mean	$E[X] = \frac{1}{p}$
Variance	$\text{Var}(X) = \frac{1-p}{p^2}$
Standard Deviation	$\sigma = \sqrt{\frac{1-p}{p^2}}$

3.1 Reminder

Property	Poisson Distribution
Notation	$X \sim P(\lambda)$
Probability Mass Function (PMF)	$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$, for $k = 0, 1, 2, \dots$
Cumulative Distribution Function (CDF)	$P(X \leq x) = \sum_{k=0}^x \frac{\lambda^k e^{-\lambda}}{k!}$
Mean	$E[X] = \lambda$
Variance	$\text{Var}(X) = \lambda$
Standard Deviation	$\sigma = \sqrt{\lambda}$

3.1.2 Summary of continuous probability distributions

Property	Uniform Distribution: $X \sim \mathcal{U}(a, b)$
PDF	$f(x) = \frac{1}{b-a}$, $a \leq x \leq b$
CDF	$F(x) = \frac{x-a}{b-a}$, $a \leq x \leq b$
Mean	$\mu = \frac{a+b}{2}$
Variance	$\sigma^2 = \frac{(b-a)^2}{12}$
Standard Deviation	$\sigma = \frac{b-a}{\sqrt{12}}$

Property	Exponential Distribution: $X \sim \mathcal{E}(\lambda)$
PDF	$f(x) = \lambda e^{-\lambda x}$, $x \geq 0$
CDF	$F(x) = 1 - e^{-\lambda x}$, $x \geq 0$
Mean	$\mu = \frac{1}{\lambda}$
Variance	$\sigma^2 = \frac{1}{\lambda^2}$
Standard Deviation	$\sigma = \frac{1}{\lambda}$

Property	Normal Distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$
PDF	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	No closed form; denoted $\Phi\left(\frac{x-\mu}{\sigma}\right)$
Mean	μ
Variance	σ^2
Standard Deviation	σ

3.1 Reminder

Property	Log-normal Distribution: $X \sim \mathcal{LN}(\mu, \sigma^2)$
PDF	$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0$
CDF	No closed form; expressed as $\Phi\left(\frac{\ln x - \mu}{\sigma}\right)$
Mean	$E[X] = e^{\mu + \frac{\sigma^2}{2}}$
Variance	$\text{Var}(X) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$
Standard Deviation	$\sigma_X = \sqrt{\text{Var}(X)}$

Property	Gamma Distribution: $X \sim \Gamma(\alpha, \beta)$
PDF	$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0$
CDF	No closed form; often denoted as $P(\alpha, x/\beta)$
Mean	$\mu = \alpha\beta$
Variance	$\sigma^2 = \alpha\beta^2$
Standard Deviation	$\sigma = \sqrt{\alpha}\beta$

Property	Beta Distribution: $X \sim \text{Beta}(\alpha, \beta)$
PDF	$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1$
CDF	No closed form; often written as $I_x(\alpha, \beta)$
Mean	$\mu = \frac{\alpha}{\alpha + \beta}$
Variance	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$
Standard Deviation	$\sqrt{\text{Variance}}$

Property	Cauchy Distribution: $X \sim \text{Cauchy}(x_0, \gamma)$
PDF	$f(x) = \frac{1}{\pi\gamma \left[1 + \left(\frac{x-x_0}{\gamma}\right)^2\right]}$
CDF	$F(x) = \frac{1}{\pi} \arctan\left(\frac{x-x_0}{\gamma}\right) + \frac{1}{2}$
Mean	Undefined
Variance	Undefined
Standard Deviation	Undefined

3.1 Reminder

Property	Chi-square Distribution: $X \sim \chi^2(k)$
PDF	$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}, \quad x > 0$
CDF	Evaluated using regularized gamma function
Mean	$\mu = k$
Variance	$\sigma^2 = 2k$
Standard Deviation	$\sigma = \sqrt{2k}$

Property	Student's t-Distribution: $T \sim t_\nu$
PDF	$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
CDF	Evaluated numerically
Mean	$\mu = 0$ for $\nu > 1$
Variance	$\frac{\nu}{\nu-2}, \quad \nu > 2$
Standard Deviation	$\sqrt{\text{Variance}}$

Property	F Distribution: $X \sim F(d_1, d_2)$
PDF	$f(x) = \frac{\left(\frac{d_1}{d_2}\right)^{d_1/2} x^{d_1/2-1}}{B(d_1/2, d_2/2) \left(1 + \frac{d_1}{d_2}x\right)^{(d_1+d_2)/2}}$
CDF	Evaluated via beta regularization
Mean	$\mu = \frac{d_2}{d_2-2}, \quad d_2 > 2$
Variance	$\frac{2d_2^2(d_1+d_2-2)}{d_1(d_2-2)^2(d_2-4)}, \quad d_2 > 4$
Standard Deviation	$\sqrt{\text{Variance}}$

Property	Weibull Distribution: $X \sim \text{Weibull}(k, \lambda)$
PDF	$f(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, \quad x \geq 0$
CDF	$F(x) = 1 - e^{-(x/\lambda)^k}$
Mean	$\mu = \lambda \Gamma\left(1 + \frac{1}{k}\right)$
Variance	$\lambda^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right]$
Standard Deviation	$\sqrt{\text{Variance}}$

3.1.3 Approximations

Approximation of a binomial distribution by a Poisson distribution

If X follows a Binomial $B(n, p)$, with n is large ($n > 20$) and p is small ($p \leq 0.1$) and $np \leq 5$. Then for all $k \geq 0 : P(X = k) \simeq P(Y = k)$, where Y follows a Poisson distribution with parameter $\lambda = np$.

Approximation of a hypergeometric distribution by a binomial distribution The condition for approximating the hypergeometric distribution with the binomial distribution is that N should be at least 10 times larger than n ($N \geq 10n$).

Approximation of a binomial distribution by a normal distribution

When n large ($n \geq 30$), if $np \geq 5$ and $n(1-p) \geq 5$, then we see that the binomial distribution can be approximated by a normal distribution with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$.

Binomial	Normal
If $P(X = n)$	use $P(n - 0.5 < X < n + 0.5)$
If $P(X > n)$	use $P(X > n + 0.5)$
If $P(X \leq n)$	use $P(X < n + 0.5)$
If $P(X < n)$	use $P(< X < n - 0.5)$
If $P(X \geq n)$	use $P(X > n - 0.5)$

Table 3.1: The continuity correction between normal and binomial distributions.

Approximation of a Poisson distribution by a normal distribution

The Poisson distribution ($X \sim P(\lambda)$) can also be approximated by a normal distribution ($X \sim \mathcal{N}(\lambda, \sqrt{\lambda})$) under certain conditions, particularly when the mean of the Poisson distribution is sufficiently large, typically $\lambda \geq 20$.

3.2 Exercises

Exercise 3.1. A multiple-choice question (MCQ) consists of 5 statements. For each statement, you must answer true (T) if the statement is always true, false (F) if it is always false, or false (P) if you cannot conclude. An answer to a MCQ is a sequence of 5 letters from T, F, or P.

1. What is the number of possible answers for the MCQ?

3.2 Exercises

2. A candidate answers randomly, and X is the random variable giving the number of correct answers. Indicate the distribution followed by X and specify its parameters. Calculate $E(X)$

Exercise 3.2. A game of chance consists of a device randomly throwing a dart at a target having the following shape:

W	W	W	W	W	W	W	W	W	Y	Y	Y	G	G	R
R	G	G	Y	Y	Y	W	W	W	W	W	W	W	W	W

The dart always hits one square and only one. The 30 white (W), yellow (Y), green (G), or red (R) squares all have the same probability of being hit. If the dart hits a red square, the player wins 8 DA; if the dart hits a green square, the player wins 5 DA; if the dart hits a yellow square, the player wins nothing and loses nothing; if the dart hits a white square, the player loses a DA, the letter a denoting a positive real number.

1. Let X be the random variable representing the player's algebraic gain (counted negatively when they lose).
 - a) Give the probability distribution of X .
Calculate a so that the game is fair (i.e., so that $E(X)$ is zero).
2. A player is considered a winner if they have obtained a strictly positive gain.
 - a) What is the probability p that a player wins?
 - b) A player plays 5 consecutive independent games. What is the probability that they win exactly twice? Exactly 5 times? What is the average number of winning games?

Exercise 3.3. Suppose a card is drawn randomly from an ordinary deck of playing cards, and then put back in the deck. This is repeated five times. What is the probability of drawing 1 spade, 1 heart, 1 diamond, and 2 clubs?

Exercise 3.4. Suppose we have a bowl with 10 marbles - 2 red marbles, 3 green marbles, and 5 blue marbles. We randomly select 4 marbles from the bowl, with replacement. What is the probability of selecting 2 green marbles and 2 blue marbles?

3.2 Exercises

Exercise 3.5. *In a certain town, 40% of the eligible voters prefer candidate A, 10% prefer candidate B, and the remaining 50% have no preference. You randomly sample 10 eligible voters. What is the probability that 4 will prefer candidate A, 1 will prefer candidate B, and the remaining 5 will have no preference?*

Exercise 3.6. *In Austria, the distribution of blood types in the population is as follows:*

Blood Type	Percentage (%)
O+	30%
A+	33%
B+	12%
AB+	6%
O-	7%
A-	8%
B-	3%
AB-	1%

Table 3.2: Distribution of blood types in Austria

If 15 Austrian citizens are chosen at random, what is the probability that:

- 3 have blood type O⁺
- 2 have blood type A⁺
- 3 have blood type B⁺
- 2 have blood type AB⁺
- 1 has blood type O⁻
- 2 have blood type A⁻
- 1 has blood type B⁻
- 1 has blood type AB⁻

3.2 Exercises

Exercise 3.7. Three light bulbs are randomly selected simultaneously from a batch of 15, five of which are defective. Calculate the probability of the following events:

- a) At least one bulb is defective;
- b) All three bulbs are defective;
- c) Exactly one bulb is defective.

Exercise 3.8. The oral exam for a competitive exam has a total of 100 topics; candidates randomly select three topics and then choose the topic covered from among these three. A candidate presents having reviewed 60 of the 100 topics.

1. What is the probability that the candidate has reviewed:
 - a) All three topics selected;
 - b) Exactly two of the three topics;
 - c) None of the three topics.
2. Define a random variable associated with this problem and give its probability distribution and expected value.

Exercise 3.9. A population has on average one person taller than 1.90 m in 80 people. For 100 people, calculate the probability that there is at least one person taller than 1.90 m. For 300 people, calculate the probability that there is at least one person taller than 1.90 m.

Exercise 3.10. The number of miles that a particular car can run before its battery wears out is exponentially distributed with an average of 10000 miles. The owner of the car needs to take a 500-mile trip. What is the probability that he will be able to complete the trip without having to replace the car battery?

Exercise 3.11. Let X be a random variable that follows a uniform distribution on the interval $[-5, 15]$.

1. Calculate the following probabilities:

3.2 Exercises

- $P(X \leq 2)$
- $P(-1 \leq X \leq 1)$
- $P_{(X>0)}(-1 \leq X \leq 2)$ (Conditional probability given $X > 0$)

2. Let Y be a random variable defined by:

$$Y = \frac{X + 5}{10}$$

Calculate the conditional probability:

$$P_{(X \leq 10)}(Y \geq 1)$$

Exercise 3.12. Let X be a random variable that follows the exponential distribution with parameter $\lambda = 3$, (i.e.; $f(x) = \lambda e^{-\lambda x}$)

Calculate: $P(X \leq 2)$, $P(X \geq 4)$, $P(2 \leq X \leq 4)$, $P_{(X \geq 2)}(X \geq 4)$.

Exercise 3.13. Let X be a random variable following the normal distribution $\mathcal{N}(200; 15^2)$.

- Determine the real $u > 0$ such that $P(200 - 2u \leq X \leq 200 + 2u) = 0,9$.

Exercise 3.14. Let $X \sim \mathcal{N}(\mu = 120, \sigma^2)$. Determine the standard deviation σ such that:

$$P(100 \leq X \leq 140) = 0.92$$

Exercise 3.15. A survey is conducted among families of four to determine their monthly milk purchases. Across all respondents, milk consumption is a Gaussian population with a mean of 20 liters and a standard deviation of 6. For the purpose of designing an advertising campaign, we want to know the percentage of low consumers (i.e., less than 10 liters/month) and the percentage of high consumers (i.e., more than 30 liters/month).

1. Calculate these two percentages.
2. Below what number of liters purchased are 75% of consumers?

3.2 Exercises

3. How many liters at most do half of consumers consume?
4. Above what consumption level do 1/3 of the population and 2/3 of the population consume?

Exercise 3.16. A company operates aircrafts with a capacity of 320 passengers. A statistical study shows that 5 out of 100 passengers who made a reservation do not show up for boarding. Thus, we assume that the probability that a passenger with a reservation does not show up is 0.05.

1. The company accepts 327 reservations for a flight. Let X be the random variable representing the number of passengers who actually show up for boarding.
 - (a) What probability distribution does X follow?
 - (b) By which normal distribution can we approximate the distribution of X ? The parameters should be given to the nearest unit.
 - (c) Using the normal approximation, calculate $P(X \leq 320)$. Do you think the risk taken by the company in accepting 327 reservations is significant?
2. Would it be reasonable for the company to accept 330 reservations on the same flight? 335 reservations?
3. The company accepts 337 reservations on the same flight of capacity 320 passengers. If 310 people have already arrived at boarding, what is the probability that fewer than 320 passengers will show up for boarding in total?

Exercise 3.17. A horticulturist packages flower bulbs. A bulb is said to sprout if it gives rise to a flowering plant. It is considered that the horticulturist has a very large number of bulbs, and the probability that a bulb sprouts is $p = 0.83$. He randomly selects 100 bulbs successively from this stock. Let X be the random variable representing the number of bulbs that sprout.

1. What is the probability distribution of X ?
2. What is the probability that exactly 3 bulbs sprout?
3. What is the probability that at least one bulb sprouts?

3.2 Exercises

4. On average, how many bulbs will sprout in a sample of 100?
5. Can this distribution be approximated by another one? If yes, which one?
6. Calculate $P(76 \leq X \leq 90)$

Exercise 3.18. A study conducted on young children shows that first words appear on average at $\mu = 11.5$ months with a standard deviation $\sigma = 3.2$ months. Assuming that the age distribution is normal, answer the following:

1. What proportion of children spoke their first words before 10 months?
2. What proportion of children spoke their first words after 18 months?
3. What proportion of children spoke their first words between 8 and 12 months?
4. Above what age are the top one-third (33%) of children found?

Exercise 3.19. In a company, a machine produces parts that must meet very precise specifications. A sample of n randomly selected parts is examined, and X denotes the random variable representing the number of defective parts.

Part I – First Adjustment

After a first adjustment, 30% of the parts are found to be defective. For $n = 5$:

1. What is the probability distribution of X ? Compute its expected value and standard deviation.
2. What is the probability that exactly 2 parts are defective?
3. What is the probability that there is no more than 1 defective part?
4. Determine the most probable value of X and compute the associated probability.

Part II – Second Adjustment

After a second adjustment, the proportion of defective parts becomes 5%. For $n = 100$:

1. What distribution can approximate the probability law of X ? Justify.
2. Compute the probability of finding no defective parts.

3.2 Exercises

3. Compute the probability of finding exactly 2 defective parts.
4. Compute the probability that the number of defective parts is between 2 and 4 (inclusive).

Exercise 3.20. At the beginning of the school year, all 6th-grade students took a vocabulary test. The average score obtained was $\mu = 42.34$ points (out of 60), with a standard deviation of $\sigma = 18.22$. The scores are normally distributed.

1. What proportion of students scored less than 30 points?
2. What proportion of students scored more than 50 points?
3. Above what score are:
 - 80% of students?
 - 30% of students?

Exercise 3.21. The waiting time T , expressed in minutes, at each checkout of a supermarket is modeled by an exponential distribution with strictly positive parameter λ .

1. Individual Checkout Waiting Time

- a) Let $T \sim \text{Exp}(\lambda)$. Determine an exact expression for $P(T \leq t)$, and then for $P(T > t)$.

Let $\lambda = 0.1$ for the rest of the exercise.

- b) Give an exact expression for the conditional probability:

$$P(T > t + s \mid T > t).$$

- c) Given that a customer has already waited 10 minutes, determine the probability that their total waiting time does not exceed 15 minutes.

Provide:

- an exact expression
- an approximate numerical value (rounded to 0.01)

3.2 Exercises

2. Multiple Checkouts

Suppose the waiting time at each checkout is independent from the others. Currently, 6 checkouts are open. Let X be the random variable representing the number of checkouts for which the waiting time exceeds 10 minutes.

- Describe the distribution of X , including its parameters.
- The manager opens additional checkouts if at least 4 out of 6 checkouts have a waiting time exceeding 10 minutes. Determine the probability of this event to within 0.01.

Exercise 3.22. Let T be the lifetime (in hours) of a mechanical component, modeled by a Weibull distribution with parameters:

$$\text{shape } k = 2 \quad \text{and} \quad \text{scale } \lambda = 1000.$$

That is, $T \sim \text{Weibull}(k = 2, \lambda = 1000)$ with probability density function:

$$f(t) = \frac{k}{\lambda} \left(\frac{t}{\lambda} \right)^{k-1} e^{-(t/\lambda)^k}, \quad t \geq 0$$

- Find the probability that the component lasts more than 1200 hours.
- Compute the expected value $E[T]$.
- Determine the median lifetime.

Exercise 3.23.

- Find $\Gamma\left(\frac{7}{2}\right)$.
- Find the value of the following integral:

$$I = \int_0^{\infty} x^6 e^{-5x} dx.$$

Exercise 3.24. Using the properties of the Gamma function, show that the Gamma probability density function integrates to 1. That is, for $\alpha > 0$ and $\lambda > 0$, show that:

$$\int_0^{\infty} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = 1.$$

3.2 Exercises

Exercise 3.25. Engineers designing the next generation of space shuttles plan to include two fuel pumps—one active, the other in reserve. If the primary pump malfunctions, the second is automatically brought online. Suppose a typical mission is expected to require that fuel be pumped for at most 50 hours. According to the manufacturer’s specifications, pumps are expected to fail once every 100 hours.

What is the probability that such a fuel pump system would not remain functioning for the full 50 hours?

Exercise 3.26. Let X be a chi-square random variable with 10 degrees of freedom. What is the upper fifth percentile?

Exercise 3.27. What is the tenth percentile of a chi-square distribution with 10 degrees of freedom?

Exercise 3.28. What is the probability that a chi-square random variable with 10 degrees of freedom is greater than 15.99?

Exercise 3.29. A production plant produces items that have a probability X of being defective. The plant manager does not know X , but from past experience she expects this probability to be equal to 4%. Furthermore, she quantifies her uncertainty about X by attaching a standard deviation of 2% to her 4% estimate.

After consulting with an expert in statistics, the manager decides to use a Beta distribution to model her uncertainty about X .

How should she set the two parameters of the distribution in order to match her priors about the expected value and the standard deviation of X ?

Exercise 3.30. Suppose the time to failure T , in hours, of a bearing in a mechanical shaft is a Weibull random variable with shape parameter k and scale parameter λ . Determine:

- The mean time until failure.
- The probability that a bearing lasts at least 5000 hours.

3.2 Exercises

Exercise 3.31. Given that the time to failure T of a bearing follows a Weibull distribution with parameters:

$$k = \frac{1}{3}, \quad \lambda = 4000,$$

we are asked to compute:

- The mean time until failure.
- The probability that the bearing lasts at least 5000 hours.

Exercise 3.32. Suppose the lifetime of a motor has a lognormal distribution. What is the probability the lifetime exceeds 12000 hours if the mean and standard deviation of the normal random variable are 11 hours and 1.3 hours, respectively?

Exercise 3.33. Suppose you conduct a study on acupuncture to determine its effectiveness in relieving pain. Sensory rates for 15 subjects are recorded as follows:

Subject	Rate	Subject	Rate	Subject	Rate
1	8.6	6	7.3	11	10.3
2	9.4	7	9.2	12	5.4
3	7.9	8	9.6	13	8.1
4	6.8	9	8.7	14	5.5
5	8.3	10	11.4	15	6.9

Table 3.3: Sensory Rates of 15 Subjects

Construct a 95% confidence interval for the mean sensory rate, assuming the population is normally distributed.

Exercise 3.34. The Human Toxome Project (HTP) aims to understand the extent of industrial pollution in the human body. In October 2008, cord blood samples from 20 newborns in the United States were tested for 430 industrial compounds, including chemicals linked to toxicity of the brain, immune, and reproductive systems.

3.2 Exercises

79 145 147 160 116 100
159 151 156 126 137 83
156 94 121 144 123 114
139 99

By using sample data, construct a 90% confidence interval for the mean number of targeted industrial chemicals found in an infant's blood.

Exercise 3.35. A random sample of statistics students were asked to estimate the number of hours they spend watching television in an average week. The responses are shown below:

0 3 1 20 9
5 10 1 10 4
14 2 4 4 5

Exercise 3.36. Let's return to the example where the feeding habits of two species of net-casting spiders are studied. The species *Deinopis* and *Menneus* coexist in eastern Australia. The following summary statistics were obtained on the size (in millimeters) of the prey of the two species:

- Adult *Deinopis*: $n_1 = 10$, $\bar{x}_1 = 10.26$ mm, $s_1^2 = s_D^2$
- Adult *Menneus*: $n_2 = 10$, $\bar{x}_2 = 9.02$ mm, $s_2^2 = s_M^2$

Estimate, with 95% confidence, the ratio of the two population variances:

$$\frac{\sigma_1^2}{\sigma_2^2}.$$

Exercise 3.37. Let X be a chi-square random variable with 3 degrees of freedom. Compute the following probability:

$$P(1.5 < X < 5.9).$$

3.3 Solutions

Exercise 3.38. Let X be a chi-square random variable with 3 degrees of freedom. Compute the following probability:

$$P(0.35 < X < 7.81).$$

Exercise 3.39. Let X_1 and X_2 be two independent normal random variables with mean $\mu = 0$ and variance $\sigma^2 = 16$.

Compute the Probability

$$P\left(X_1^2 + X_2^2 < \frac{1}{2}\right).$$

Exercise 3.40. Let X_1 and X_2 be two independent normal random variables with mean $\mu = 0$ and variance $\sigma^2 = 16$.

Compute the following probability:

$$P(X_1^2 + X_2^2 > 8).$$

Exercise 3.41. Suppose that the random variable X has a Chi-square distribution with 5 degrees of freedom.

Define the random variable Y as follows:

$$Y = 3X + 7.$$

Compute the expected value of Y .

Exercise 3.42. Let $X \sim \chi^2(5)$ and define $Y = e^{1-X}$. Compute $E[Y]$.

3.3 Solutions

Solution 3.1.

1. Each of the 5 statements in the MCQ can be answered with one of the three options: True (T), False (F), or Cannot conclude (P). Therefore, the number of possible answers is:

$$3^5 = 243$$

243 possible answers.

3.3 Solutions

2. Let X be the random variable representing the number of correct answers when a candidate answers randomly.

Since each answer is chosen independently and randomly among the 3 possible choices, the probability of choosing the correct answer for any statement is: $p = \frac{1}{3}$.

The variable X then follows a binomial distribution with parameters $n = 5$ and $p = \frac{1}{3}$:

$$X \sim B\left(5, \frac{1}{3}\right)$$

The expected value of X is:

$$E(X) = n \cdot p = 5 \cdot \frac{1}{3} = \frac{5}{3}.$$

Solution 3.2.

1.

(a). **Probability distribution of X :**

Let X be the player's gain:

- $P(\text{Red squares}) = P(X = 8) = \frac{2}{30}$
- $P(\text{Green squares}) = P(X = 5) = \frac{4}{30}$
- $P(\text{Yellow squares}) = P(X = 0) = \frac{6}{30}$
- $P(\text{White squares}) = P(X = -a) = \frac{18}{30}$

which can be tabulate as

X	8	5	0	$-a$
$P(X)$	$\frac{2}{30}$	$\frac{4}{30}$	$\frac{6}{30}$	$\frac{18}{30}$

(b). **The value a :**

We find the value a so that the game is fair ($E(X) = 0$). We calculate the expected value:

$$\begin{aligned} E(X) &= 8 \cdot \frac{2}{30} + 5 \cdot \frac{4}{30} + 0 \cdot \frac{6}{30} + (-a) \cdot \frac{18}{30} \\ &= \frac{16}{30} + \frac{20}{30} - \frac{18a}{30} \\ &= \frac{36 - 18a}{30}. \end{aligned}$$

3.3 Solutions

For a fair game, $E(X) = 0$:

$$\frac{36 - 18a}{30} = 0 \Rightarrow 36 - 18a = 0 \Rightarrow a = 2.$$

2.

A player is considered a winner if they obtain a strictly positive gain (i.e., $X > 0$).

(a). **Value p :**

The probability p that a player wins is

$$p = P(X = 5) + P(X = 8) = \frac{4}{30} + \frac{2}{30} = \frac{6}{30} = \frac{1}{5}.$$

(b).

Let Y be the number of winning games out of 5. Y follows a binomial distribution:

$$Y \sim \mathcal{B}(n = 5, p = 1/5)$$

- Probability of exactly 2 wins:

$$P(Y = 2) = \binom{5}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 = 10 \cdot \frac{1}{25} \cdot \frac{64}{125} = \frac{640}{3125} = 0.2048.$$

- Probability of exactly 5 wins:

$$P(Y = 5) = \binom{5}{5} \left(\frac{1}{5}\right)^5 = 1 \cdot \frac{1}{3125} = 0.00032.$$

- Expected number of winning games:

$$E(Y) = n \cdot p = 5 \cdot \frac{1}{5} = 1.$$

Solution 3.3. Each card draw is independent, and the deck always has:

$$\text{Total cards} = 52, \quad \text{Cards per suit} = 13$$

So, the probability of drawing any specific suit is:

$$P(\text{suit}) = \frac{13}{52} = \frac{1}{4}.$$

We are performing 5 independent draws, and we want the following combination of suits:

$$1 \text{ Spade, } 1 \text{ Heart, } 1 \text{ Diamond, } 2 \text{ Clubs}$$

3.3 Solutions

The probability is a multinomial probability given by:

$$\begin{aligned} P &= \frac{5!}{1!1!1!2!} \cdot \left(\frac{1}{4}\right)^1 \cdot \left(\frac{1}{4}\right)^1 \cdot \left(\frac{1}{4}\right)^1 \cdot \left(\frac{1}{4}\right)^2 \\ &= \frac{120}{2} \cdot \left(\frac{1}{4}\right)^5 \\ &= 60 \cdot \frac{1}{1024} \\ &= \frac{15}{256} \\ &\approx 0.05859. \end{aligned}$$

Solution 3.4. Each draw is independent (with replacement), and the probability of drawing a marble of each color is:

$$P(\text{red}) = \frac{2}{10}, \quad P(\text{green}) = \frac{3}{10}, \quad P(\text{blue}) = \frac{5}{10} = \frac{1}{2}$$

We are interested in:

- 2 green marbles,
- 2 blue marbles,
- 0 red marbles

The probability is a multinomial probability given by:

$$\begin{aligned} P &= \frac{4!}{2!2!0!} \cdot \left(\frac{3}{10}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{2}{10}\right)^0 \\ &= 6 \cdot \frac{9}{100} \cdot \frac{1}{4} \\ &= \frac{27}{200} \\ &\approx 0.135. \end{aligned}$$

Solution 3.5. We have:

$$P_A = 0.4, \quad P_B = 0.1, \quad P_N = 0.5$$

3.3 Solutions

Let:

$$n = 10, \quad x_A = 4, \quad x_B = 1, \quad x_N = 5$$

The probability is a multinomial probability given by:

$$\begin{aligned} P &= \frac{n!}{x_A! x_B! x_N!} \cdot P_A^{x_A} \cdot P_B^{x_B} \cdot P_N^{x_N} \\ &= \frac{10!}{4! \cdot 1! \cdot 5!} \cdot (0.4)^4 \cdot (0.1)^1 \cdot (0.5)^5 \\ &= 2520 \cdot (0.0256) \cdot (0.1) \cdot (0.03125) \\ &\approx 0.2016. \end{aligned}$$

Solution 3.6. This is a classic multinomial probability problem, where we perform 15 independent trials and classify each outcome into one of 8 categories.

Let the probabilities for each blood type be:

Category	Blood Type	Probability
P_1	O ⁺	0.30
P_2	A ⁺	0.33
P_3	B ⁺	0.12
P_4	AB ⁺	0.06
P_5	O ⁻	0.07
P_6	A ⁻	0.08
P_7	B ⁻	0.03
P_8	AB ⁻	0.01

Table 3.4: Blood Type Probabilities

Let the observed counts be:

$$x_1 = 3, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 2, \quad x_5 = 1, \quad x_6 = 2, \quad x_7 = 1, \quad x_8 = 1$$

The total number of trials is:

$$n = 15 = \sum_{i=1}^8 x_i$$

3.3 Solutions

The multinomial probability is:

$$\begin{aligned} P &= \frac{n!}{x_1!x_2!x_3!x_4!x_5!x_6!x_7!x_8!} \cdot P_1^{x_1} P_2^{x_2} \cdot \dots \cdot P_8^{x_8} \\ &= P = \frac{15!}{3! \cdot 2! \cdot 3! \cdot 2! \cdot 1! \cdot 2! \cdot 1! \cdot 1!} \cdot (0.30)^3 \cdot (0.33)^2 \cdot (0.12)^3 \cdot (0.06)^2 \cdot (0.07)^1 \cdot (0.08)^2 \cdot (0.03)^1 \cdot (0.01)^1. \end{aligned}$$

Solution 3.7. The total number of possible selections is:

$$\binom{15}{3}$$

(a). Probability that at least one bulb is defective

This is the complement of selecting 3 non-defective bulbs:

$$\begin{aligned} P(\text{at least one defective}) &= 1 - P(\text{all non-defective}) \\ &= 1 - \frac{\binom{10}{3}}{\binom{15}{3}} \\ &= 1 - \frac{120}{455} \\ &= \frac{335}{455} \\ &= \frac{67}{91}. \end{aligned}$$

(b). Probability that all three bulbs are defective

We choose 3 defective bulbs from the 5 available:

$$P(\text{all defective}) = \frac{\binom{5}{3}}{\binom{15}{3}} = \frac{10}{455} = \frac{2}{91}.$$

(c). Probability that exactly one bulb is defective

We have:

$$P(\text{exactly one defective}) = \frac{\binom{5}{1} \cdot \binom{10}{2}}{\binom{15}{3}} = \frac{5 \cdot 45}{455} = \frac{225}{455} = \frac{45}{91}.$$

Solution 3.8.

1. Probabilities for different numbers of reviewed topics among the 3:

We define the following:

3.3 Solutions

- R : reviewed topics (60 topics)
- U : unreviewed topics (40 topics)
- Total number of possible draws: $\binom{100}{3}$

(a). Probability of selecting 3 reviewed topics:

$$P(3 \text{ reviewed}) = \frac{\binom{60}{3}}{\binom{100}{3}} = \frac{34220}{161700} = \frac{1711}{8085}.$$

(b). Probability of selecting exactly 2 reviewed topics:

$$P(2 \text{ reviewed}) = \frac{\binom{60}{2} \cdot \binom{40}{1}}{\binom{100}{3}} = \frac{1770 \cdot 40}{161700} = \frac{70800}{161700} = \frac{236}{539}.$$

(c). Probability of selecting 0 reviewed topics: (i.e., all 3 are unreviewed)

$$P(0 \text{ reviewed}) = \frac{\binom{40}{3}}{\binom{100}{3}} = \frac{9880}{161700} = \frac{494}{8085}.$$

2. Probability distribution and expected value:

Let X be the random variable representing the number of reviewed topics among the 3 selected.

Thus $X \in \{0, 1, 2, 3\}$

We already calculated:

$$P(X = 3) = \frac{1711}{8085}, \quad P(X = 2) = \frac{236}{539}, \quad P(X = 0) = \frac{494}{8085}.$$

Now compute $P(X = 1)$:

$$P(X = 1) = \frac{\binom{60}{1} \cdot \binom{40}{2}}{\binom{100}{3}} = \frac{60 \cdot 780}{161700} = \frac{46800}{161700} = \frac{156}{539}.$$

The probability distribution of X is tabulate as:

X	0	1	2	3
$P(X)$	$\frac{494}{8085}$	$\frac{156}{539}$	$\frac{236}{539}$	$\frac{1711}{8085}$

Expected Value $E(X)$:

3.3 Solutions

We compute:

$$\begin{aligned} E(X) &= 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3) \\ &= 1 \cdot \frac{156}{539} + 2 \cdot \frac{236}{539} + 3 \cdot \frac{1711}{8085} \\ &= \frac{156 + 472}{539} + \frac{5133}{8085} \\ &= \frac{94140 + 5133}{8085} \\ &= \frac{99273}{8085} \\ &\approx 1.228. \end{aligned}$$

Solution 3.9. We model the number of people taller than 1.90 m in a group using a Poisson distribution, as the event is rare and the number of trials is large.

Let X_n be the number of people taller than 1.90 m among n individuals.

$$X_n \sim \mathcal{P}(\lambda_n), \quad \lambda_n = n \cdot \frac{1}{80}.$$

1. For $n = 100$

$$\lambda = \frac{100}{80} = 1.25$$

We want the probability that at least one person is taller than 1.90 m:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1.25} \approx 1 - 0.2865 \approx 0.7135.$$

2. For $n = 300$

$$\lambda = \frac{300}{80} = 3.75.$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-3.75} \approx 1 - 0.0235 \approx 0.9765.$$

Solution 3.10. Let X denote the number of miles that the car can run before its battery wears out. Suppose the following holds:

$$P(X > s + t \mid X > s) = P(X > t).$$

3.3 Solutions

This means that the probability that the battery lasts more than $s + t$ miles, given that it has already lasted s miles, is the same as the probability that it lasts more than t miles. This property is known as the **memoryless property**, and it is a key characteristic of the exponential distribution.

Now, since we are given that $X \sim \mathcal{E}(\lambda)$, where the mean is $\mu = 10000$ miles, we have:

$$\lambda = \frac{1}{\mu} = \frac{1}{10000}.$$

The probability that the battery lasts more than x miles is given by the survival function of the exponential distribution:

$$P(X > x) = e^{-\lambda x}$$

So, the probability that the car can make a 5000-mile trip is:

$$P(X > 5000) = e^{-5000/10000} = e^{-0.5} \approx 0.6065.$$

Solution 3.11. Let $X \sim \mathcal{U}(-5, 15)$, i.e., a continuous uniform distribution over the interval $[-5, 15]$.

The density of a uniform distribution is:

$$f_X(x) = \frac{1}{b-a} = \frac{1}{15-(-5)} = \frac{1}{20}, \quad \text{for } x \in [-5, 15]$$

(a). $P(X \leq 2)$:

$$P(X \leq 2) = \int_{-5}^2 \frac{1}{20} dx = \frac{2-(-5)}{20} = \frac{7}{20}.$$

(b). $P(-1 \leq X \leq 1)$:

$$P(-1 \leq X \leq 1) = \int_{-1}^1 \frac{1}{20} dx = \frac{1-(-1)}{20} = \frac{2}{20} = \frac{1}{10}.$$

(c). **Conditional probability** $P_{(X \geq 0)}(-1 \leq X \leq 2)$:

3.3 Solutions

We have:

$$\begin{aligned}P_{(X \geq 0)}(-1 \leq X \leq 2) &= \frac{P(-1 \leq X \leq 2 \cap X > 0)}{P(X \geq 0)} \\&= \frac{P(0 \leq X \leq 2)}{P(X \geq 0)} \\&= \frac{2/20}{15/20} \\&= \frac{2}{15}.\end{aligned}$$

(d). **Conditional probability** $P_{(X \leq 10)}(Y \geq 1)$:

This is a linear transformation of X . Since $X \in [-5, 15]$, then:

$$Y = \frac{X + 5}{10} \Rightarrow Y \in [0, 2]$$

We rewrite the condition in terms of X :

$$Y \geq 1 \Rightarrow \frac{X + 5}{10} \geq 1 \Rightarrow X \geq 5$$

$$\begin{aligned}P_{(X \leq 10)}(Y \geq 1) &= P_{(X \leq 10)}(X \geq 5) \\&= \frac{P(X \leq 10 \cap X \geq 5)}{P(X \leq 10)} \\&= \frac{P(5 \leq X \leq 10)}{P(X \leq 10)} \\&= \frac{5/20}{15/20} \\&= \frac{1}{3}.\end{aligned}$$

Solution 3.12.

1. $P(X \leq 2)$:

Use the cumulative distribution function (CDF) of the exponential distribution:

$$P(X \leq x) = 1 - e^{-\lambda x}$$

$$P(X \leq 2) = 1 - e^{-3 \cdot 2} = 1 - e^{-6}.$$

2. $P(X \geq 4)$:

$$P(X \geq 4) = e^{-3 \cdot 4} = e^{-12}.$$

3.3 Solutions

3. $P(2 \leq X \leq 4)$:

$$P(2 \leq X \leq 4) = P(X \leq 4) - P(X \leq 2) = [1 - e^{-12}] - [1 - e^{-6}] = e^{-6} - e^{-12}.$$

4. **Conditional probability** $P_{(X \geq 2)}(x \geq 4)$:

$$P_{(X \geq 2)}(x \geq 4) = \frac{P(X \geq 4)}{P(X \geq 2)} = \frac{e^{-12}}{e^{-6}} = e^{-6}.$$

Solution 3.13. We determine the real number $u > 0$ such that:

$$P(200 - 2u \leq X \leq 200 + 2u) = 0.9.$$

Let $X \sim \mathcal{N}(200, 15^2)$. So, first we standardize the variable X using:

$$Z = \frac{X - 200}{15} \Rightarrow Z \sim \mathcal{N}(0, 1).$$

So the inequality becomes:

$$P(200 - 2u \leq X \leq 200 + 2u) = P\left(\frac{-2u}{15} \leq Z \leq \frac{2u}{15}\right) = 0.9.$$

Since the standard normal is symmetric, this is equivalent to:

$$P(-z \leq Z \leq z) = 0.9 \quad \text{with } z = \frac{2u}{15}.$$

So we need to find z such that:

$$P(-z \leq Z \leq z) = 0.9 \Rightarrow P(Z \leq z) = 0.95.$$

From standard normal distribution tables:

$$z = \Phi^{-1}(0.95) \approx 1.65$$

Thus:

$$\frac{2u}{15} = 1.65 \Rightarrow u = \frac{15 \cdot 1.65}{2} = 12.375.$$

3.3 Solutions

Solution 3.14. We standardize the variable:

$$P(100 \leq X \leq 140) = P\left(\frac{100 - 120}{\sigma} \leq Z \leq \frac{140 - 120}{\sigma}\right) = P\left(\frac{-20}{\sigma} \leq Z \leq \frac{20}{\sigma}\right).$$

Let

$$z = \frac{20}{\sigma} \Rightarrow P(-z \leq Z \leq z) = 0.92$$

Using symmetry of the normal distribution:

$$2\Phi(z) - 1 = 0.92 \Rightarrow \Phi(z) = 0.96$$

From standard normal tables:

$$z \approx 1.76 \Rightarrow \frac{20}{\sigma} = 1.76 \Rightarrow \sigma = \frac{20}{1.76} \approx 11.36.$$

Solution 3.15. Let the random variable $X \sim \mathcal{N}(20, 6^2)$ represent the monthly milk consumption (in liters) of a family of four.

1. Percentages of low and high consumers:

We standardize:

$$Z = \frac{X - 20}{6} \Rightarrow Z \sim \mathcal{N}(0, 1)$$

(a). **Low consumers:** $X < 10$

$$Z = \frac{10 - 20}{6} = -\frac{10}{6} \approx -1.67$$

$$P(X < 10) = P(Z < -1.67) = 1 - \Phi(1.67) = 1 - 0.9525 \approx 0.0475$$

So, about 4.75% are low consumers.

(b). **High consumers:** $X > 30$

$$Z = \frac{30 - 20}{6} = \frac{10}{6} \approx 1.67$$

$$P(X > 30) = P(Z > 1.67) = 1 - P(Z \leq 1.67) = 1 - \Phi(1.67) = 1 - 0.9525 \approx 0.0475$$

So, about 4.75% are high consumers as well.

2. 75% of consumers purchase less than how many liters?

We have:

$$P(X \leq x) = 0.75 \Rightarrow P\left(Z \leq \frac{x - 20}{6}\right) = 0.75$$

3.3 Solutions

From standard normal tables:

$$z = \Phi^{-1}(0.75) \approx 0.674 \Rightarrow \frac{x-20}{6} = 0.674 \Rightarrow x = 20 + 6 \cdot 0.674 \approx 24.04$$

So, 24 liters is the 75th percentile.

3. Median Consumption (50% threshold):

The median corresponds to:

$$P(X \leq x) = 0.5 \Rightarrow z = 0 \Rightarrow x = 20$$

So, half of consumers consume at most 20 liters.

4. Thresholds for Top 1/3 and 2/3 Consumers:

We want x such that:

- $P(X > x) = \frac{1}{3} \Rightarrow P(X \leq x) = \frac{2}{3}$
- $P(X > x) = \frac{2}{3} \Rightarrow P(X \leq x) = \frac{1}{3}$

(a). For 1/3 consuming more: $P(X \leq x) = 2/3$

$$z = \Phi^{-1}(0.67) \approx 0.43 \Rightarrow x = 20 + 6 \cdot 0.43 \approx 22.58$$

So, 22.58 liters is the 67th percentile.

(b). For 2/3 consuming more: $P(X \leq x) = 1/3$

$$z = \Phi^{-1}(0.33) \approx -0.43 \Rightarrow x = 20 + 6 \cdot (-0.43) \approx 17.41$$

So, 17.41 liters is the 33rd percentile.

Solution 3.16.

1. (a). The number of passengers who show up, X , follows a binomial distribution:

$$X \sim \mathcal{B}(n = 327, p = 0.95)$$

(b). Using the normal approximation to the binomial distribution:

$$\mu = np = 327 \times 0.95 = 310.65 \approx 311$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{327 \times 0.95 \times 0.05} \approx \sqrt{15.53} \approx 4$$

So we approximate $X \sim \mathcal{N}(311, 4^2)$.

3.3 Solutions

(c). We use the continuity correction:

$$P(X \leq 320) \approx P(Y \leq 320.5)$$

where $Y \sim \mathcal{N}(311, 4^2)$

$$Z = \frac{320.5 - 311}{4} = \frac{9.5}{4} = 2.375.$$

Using standard normal distribution tables:

$$P(Z \leq 2.375) \approx 0.9912.$$

So the probability that 320 or fewer passengers show up is about 99.12%. The risk of overbooking (more than 320 people showing up) is $1 - 0.9912 = 0.0088$, or 0.88%. This risk is low and may be considered acceptable by the company.

2. For 330 reservations:

$$\mu = 330 \times 0.95 = 313.5, \quad \sigma \approx \sqrt{330 \times 0.95 \times 0.05} \approx 4$$

$$Z = \frac{320.5 - 313.5}{4} = 1.75, \quad P(Z \leq 1.75) \approx 0.9599$$

Risk = $1 - 0.9599 = 0.0401$ or 4%.

For 335 reservations:

$$\mu = 335 \times 0.95 = 318.25, \quad \sigma \approx \sqrt{15.92} \approx 4$$

$$Z = \frac{320.5 - 318.25}{4} = 0.5625, \quad P(Z \leq 0.5625) \approx 0.713$$

Risk = $1 - 0.713 = 0.287$ or 28.7%.

Accepting 330 reservations has moderate risk. Accepting 335 involves a high risk of having more passengers than capacity.

3. With 337 reservations. We have

$$\mu = 337 \times 0.95 = 320.15, \quad \sigma = \sqrt{330 \times 0.95 \times 0.05} \approx 4$$

$$P_{(X \geq 310)}(X \leq 320) = \frac{P(310 \leq X \leq 320)}{P(X \geq 310)}$$

3.3 Solutions

with,

$$\begin{aligned}P(310 \leq X \leq 320) &\simeq P\left(\frac{310 - 320.15}{4} \leq \frac{X - 320.15}{4} \leq \frac{320 - 320.15}{4}\right) \\&\simeq P(-2.54 \leq Z \leq -0.04) \\&\simeq \Phi(-0.04) - \Phi(-2.54) \\&\simeq 1 - \Phi(0.04) - (1 - \Phi(2.54)) \\&\simeq \Phi(2.54) - \Phi(0.04) \\&\simeq 0.994 - 0.484 \\&\simeq 0.478\end{aligned}$$

and the same

$$\begin{aligned}P(X \geq 310) &\simeq P\left(\frac{X - 320.15}{4} \geq \frac{310 - 320.15}{4}\right) \\&\simeq P\left(\frac{X - 320.15}{4} \geq -2.54\right) \\&\simeq 1 - \Phi(-2.54) \\&\simeq 1 - (1 - \Phi(2.54)) \\&\simeq \Phi(2.54) \simeq 0.994\end{aligned}$$

Thus, we have

$$P_{(X \geq 310)}(X \leq 320) \simeq \frac{0.478}{0.994} \simeq 0.48.$$

Solution 3.17.

1. The variable X , number of sprouting bulbs, follows a binomial distribution:

$$X \sim \mathcal{B}(n = 100, p = 0.83)$$

Its probability mass function is:

$$P(X = k) = \binom{100}{k} (0.83)^k (0.17)^{100-k}$$

2. The probability that exactly 3 bulbs germinate is:

$$\begin{aligned}P(X = 3) &= \binom{100}{3} (0.83)^3 (0.17)^{97} \\&\approx 161700 \times 0.5718 \times 2.25 \times 10^{-75} \\&\approx 2.09 \times 10^{-70}\end{aligned}$$

3.3 Solutions

3. The probability that at least one bulb germinates is:

$$\begin{aligned}P(X \geq 1) &= 1 - P(X = 0) \\&= 1 - (0.17)^{100} \\&\approx 1 - 1.1088 \times 10^{-77} \\&\approx 0.1088 \times 10^{-77}\end{aligned}$$

4. The expected number of bulbs that germinate is:

$$E(X) = np = 100 \times 0.83 = 83$$

5. Since $n = 100$ is large and $p = 0.83$ is not too close to 0 or 1, we can approximate the binomial distribution with a normal distribution:

$$\mu = np = 83, \quad \sigma = \sqrt{np(1-p)} = \sqrt{100 \cdot 0.83 \cdot 0.17} \approx \sqrt{14.11} \approx 3.756$$

So:

$$X \sim \mathcal{N}(83, 3.756^2)$$

Because:

$$n = 100 > 30, \quad np = 83 > 5, \quad nq = 17 > 5$$

6. Using the normal approximation with continuity correction:

Let:

$$Z = \frac{X - 83}{\sqrt{14.11}} \approx \frac{X - 83}{3.756}$$

$$\begin{aligned}P(76 \leq X \leq 90) &\approx P\left(\frac{75.5 - 83}{3.756} \leq Z \leq \frac{90.5 - 83}{3.756}\right) \\&\approx P(-1.994 \leq Z \leq 1.997) \\&\approx \Phi(1.997) - \Phi(-1.994) \\&\approx \Phi(1.997) - 1 + \Phi(1.994) \\&\approx 0.9768 - 1 + 0.9766 \\&\approx 0.9536\end{aligned}$$

So the probability is approximately 95.36%.

3.3 Solutions

Solution 3.18. Let $X \sim \mathcal{N}(\mu = 11.5, \sigma = 3.2)$ represent the age in months at which a child says their first words.

1. Proportion before 10 months:

We standardize:

$$Z = \frac{10 - 11.5}{3.2} = \frac{-1.5}{3.2} \approx -0.468$$

$$P(X \leq 10) = P(Z \leq -0.468) = 1 - P(Z \leq 0.468) \approx 0.3192$$

So the proportion before 10 months is approximately 31.92%.

2. Proportion after 18 months:

$$Z = \frac{18 - 11.5}{3.2} = \frac{6.5}{3.2} \approx 2.031$$

$$P(X > 18) = P(Z > 2.03) = 1 - P(Z \leq 2.03) \approx 1 - 0.9788 \approx 0.0212$$

So the proportion after 18 months is approximately 2.11%

3. Proportion between 8 and 12 months:

$$Z_1 = \frac{8 - 11.5}{3.2} = -1.094, \quad Z_2 = \frac{12 - 11.5}{3.2} = 0.156$$

$$\begin{aligned} P(8 < X < 12) &= P(-1.094 < Z < 0.156) \\ &\approx \phi(0.156) - \phi(-1.094) \\ &\approx 0.5619 - 0.1372 \\ &\approx 0.4247 \end{aligned}$$

42.47% of children spoke their first words between 8 and 12 months.

4. Age above which one-third of children are found:

We seek the value x such that:

$$P(X > x) = \frac{1}{3} \Rightarrow P(X \leq x) = \frac{2}{3}$$

From the standard normal table:

$$z = \Phi^{-1}(0.67) \approx 0.43$$

Then:

$$x = \mu + z\sigma = 11.5 + 0.43 \times 3.2 \approx 12.876$$

So about 33.33% of children speak after approximately 12.88 months.

3.3 Solutions

Solution 3.19. Part I – First Adjustment

Let $X \sim \mathcal{B}(n = 5, p = 0.30)$.

1. **Expected value and standard deviation:**

$$E(X) = np = 5 \times 0.3 = 1.5$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{5 \times 0.3 \times 0.7} = \sqrt{1.05} \approx 1.025$$

2. **Probability of exactly 2 defective parts:**

$$P(X = 2) = \binom{5}{2} \cdot (0.3)^2 \cdot (0.7)^3 = 10 \cdot 0.09 \cdot 0.343 = 0.3087$$

3. **Probability of at most 1 defective part:**

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$P(X = 0) = (0.7)^5 = 0.16807$$

$$P(X = 1) = \binom{5}{1} \cdot 0.3 \cdot (0.7)^4 = 5 \cdot 0.3 \cdot 0.2401 = 0.36015$$

$$P(X \leq 1) = 0.16807 + 0.36015 = 0.52822$$

4. **Most probable value (mode) of X :**

$$\text{Mode} \approx \lfloor (n+1)p \rfloor = \lfloor 6 \cdot 0.3 \rfloor = \lfloor 1.8 \rfloor = 1$$

$$P(X = 1) = 0.36015 \quad (\text{already computed above})$$

Part II – Second Adjustment

Let $X \sim \mathcal{B}(n = 100, p = 0.05)$, and set $\lambda = np = 5$.

1. **Approximating distribution:**

Since n is large and p is small, we use the Poisson approximation:

$$X \sim \mathcal{P}(\lambda = 5)$$

2. **Probability of 0 defective parts:**

$$P(X = 0) = \frac{e^{-5} \cdot 5^0}{0!} = e^{-5} \approx 0.00674$$

3.3 Solutions

3. **Probability of exactly 2 defective parts:**

$$P(X = 2) = \frac{e^{-5} \cdot 5^2}{2!} = \frac{25}{2} \cdot e^{-5} \approx 12.5 \cdot 0.00674 = 0.08425$$

4. **Probability that the number of defective parts is between 2 and 4 (inclusive):**

$$P(2 \leq X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4)$$

$$P(X = 3) = \frac{5^3}{3!} \cdot e^{-5} = \frac{125}{6} \cdot e^{-5} \approx 20.833 \cdot 0.00674 = 0.1404$$

$$P(X = 4) = \frac{5^4}{4!} \cdot e^{-5} = \frac{625}{24} \cdot e^{-5} \approx 26.042 \cdot 0.00674 = 0.1755$$

$$\Rightarrow P(2 \leq X \leq 4) \approx 0.08425 + 0.1404 + 0.1755 = 0.40015$$

Solution 3.20. Let $X \sim \mathcal{N}(42.34, 18.22^2)$ be the score distribution.

1. **Proportion of students with a score below 30:**

$$Z = \frac{30 - 42.34}{18.22} \approx \frac{-12.34}{18.22} \approx -0.68.$$

From the standard normal table:

$$P(X < 30) = P(Z < -0.68) = 1 - \phi(0.68) \approx 0.2583.$$

So about 25.83% of students with a score below 30.

2. **Proportion of students with a score above 50:**

$$Z = \frac{50 - 42.34}{18.22} \approx \frac{7.66}{18.22} \approx 0.42.$$

From the standard normal table:

$$P(X > 50) = P(Z > 0.42) = 1 - P(Z < 0.42) = 1 - \phi(0.42) \approx 1 - 0.663 = 0.337.$$

So about 33.7% of students with a score above 50.

3. **Score thresholds for 80% and 30% of students:**

- We have $P(X > x) = 0.80 \Rightarrow P(X < x) = 0.20$ From the standard normal table:

$$z_{0.20} \approx -0.8416$$

$$x = \mu + z\sigma = 42.34 + (-0.8416)(18.22) \approx 42.34 - 15.34 = 27.$$

3.3 Solutions

- We have $P(X > x) = 0.30 \Rightarrow P(X < x) = 0.70$ From the standard normal table:

$$z_{0.70} \approx 0.5244$$

$$x = \mu + z\sigma = 42.34 + 0.5244 \cdot 18.22 \approx 42.34 + 9.55 = 51.89.$$

Solution 3.21.

1. Individual checkout waiting time:

(a). $P(T \leq t)$ and $P(T > t)$:

The cumulative distribution function (CDF) of the exponential distribution is:

$$P(T \leq t) = 1 - e^{-\lambda t}.$$

Hence, the survival function is

$$P(T > t) = e^{-\lambda t}.$$

With $\lambda = 0.1$, this becomes:

$$P(T > t) = e^{-0.1t}.$$

(b). **Conditional probability:**

The exponential distribution has the memoryless property:

$$P(T > t + s \mid T > t) = P(T > s).$$

So:

$$P(T > t + s \mid T > t) = e^{-\lambda s}.$$

With $\lambda = 0.1$, this gives:

$$P(T > t + s \mid T > t) = e^{-0.1s}.$$

(c). Given $T > 10$, we have:

$$P(T \leq 15 \mid T > 10) = 1 - P(T > 15 \mid T > 10).$$

Using the memoryless property:

$$P(T > 15 \mid T > 10) = P(T > 5) = e^{-0.1 \cdot 5} = e^{-0.5}.$$

So, the exact expression:

$$P(T \leq 15 \mid T > 10) = 1 - e^{-0.5}.$$

3.3 Solutions

and the approximate value:

$$P(T \leq 15 \mid T > 10) \approx 1 - 0.6065 \approx 0.39.$$

2. Multiple Checkouts

(a). Distribution of X :

Each checkout has probability:

$$P(T > 10) = e^{-0.1 \cdot 10} = e^{-1} \approx 0.3679.$$

Let $X \sim \mathcal{B}(n = 6, p = 0.3679)$, the number of checkouts with waiting time exceeding 10 minutes.

(b). Probability that at least 4 checkouts have $T > 10$:

We compute:

$$P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$$

Let $p = e^{-1} \approx 0.3679$, $q = 1 - p \approx 0.6321$.

$$P(X = 4) = \binom{6}{4} p^4 q^2 = 15 \cdot (0.0183) \cdot (0.3996) \approx 0.1096$$

$$P(X = 5) = \binom{6}{5} p^5 q = 6 \cdot (0.0067) \cdot (0.6321) \approx 0.0254$$

$$P(X = 6) = \binom{6}{6} p^6 = 1 \cdot (0.0024) = 0.0024$$

Thus

$$P(X \geq 4) \approx 0.1097 + 0.0254 + 0.0025 = 0.1376 \approx 0.14.$$

Solution 3.22.

(a). We compute:

$$P(T > 1200) = 1 - F(1200) = e^{-(1200/1000)^2} = e^{-1.44}.$$

$$P(T > 1200) \approx e^{-1.44} \approx 0.2369.$$

(b). The expected value of a Weibull-distributed random variable is:

$$E[T] = \lambda \cdot \Gamma\left(1 + \frac{1}{k}\right).$$

3.3 Solutions

For $k = 2$, we get:

$$E[T] = 1000 \cdot \Gamma\left(1 + \frac{1}{2}\right) = 1000 \cdot \Gamma\left(\frac{3}{2}\right)$$

where

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi} \approx 0.5 \cdot 1.77245 = 0.88623$$

thus

$$E[T] \approx 1000 \cdot 0.88623 = 886.23 \text{ hours.}$$

(c). The median t_m satisfies $P(T \leq t_m) = 0.5$. We have

$$0.5 = 1 - e^{-(t_m/1000)^2} \Rightarrow e^{-(t_m/1000)^2} = 0.5 \Rightarrow (t_m/1000)^2 = \ln(2)$$

then

$$t_m = 1000 \cdot \sqrt{\ln(2)} \approx 1000 \cdot \sqrt{0.6931} \approx 1000 \cdot 0.8326 = 832.6 \text{ hours.}$$

Solution 3.23.

1. The value of $\Gamma\left(\frac{7}{2}\right)$

We use the recursive property of the Gamma function:

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right) \cdot \Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right) \cdot \left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right) = \left(\frac{5}{2}\right) \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right).$$

Using the known value:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Thus:

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{15}{8} \sqrt{\pi}.$$

2. The value of $I = \int_0^{\infty} x^6 e^{-5x} dx$:

This is of the form:

$$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}.$$

Using property with $\alpha = 7$ and $\lambda = 5$:

$$I = \int_0^{\infty} x^6 e^{-5x} dx = \frac{\Gamma(7)}{5^7} = \frac{6!}{5^7} = \frac{720}{78,125} \approx 0.0092.$$

3.3 Solutions

Solution 3.24. We start by factoring out constants:

$$\int_0^{\infty} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx.$$

Now use the substitution property of the Gamma function:

$$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}.$$

Substituting this result back:

$$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\lambda^{\alpha}} = 1.$$

therefore

$$\int_0^{\infty} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = 1.$$

Solution 3.25. We are told that pumps are expected to fail once every 100 hours. That is, the failure rate is:

$$\lambda = \frac{1}{100}.$$

Since we are interested in the time until the "second failure", and assuming failures follow a Poisson process, the time until the second failure follows a Gamma distribution with shape parameter $\alpha = 2$ and rate parameter $\lambda = \frac{1}{100}$. The probability density function of the Gamma distribution is:

$$f(t) = \frac{\lambda^2 t^{2-1} e^{-\lambda t}}{\Gamma(2)} = \lambda^2 t e^{-\lambda t}, \quad t > 0.$$

We want the probability that the system fails before 50 hours, i.e.:

$$P(T < 50) = \int_0^{50} \lambda^2 t e^{-\lambda t} dt.$$

Let's evaluate this integral using integration by parts.

Let:

$$u = t \quad \Rightarrow \quad du = dt$$

$$dv = \lambda^2 e^{-\lambda t} dt \quad \Rightarrow \quad v = -\lambda e^{-\lambda t}.$$

Then:

$$\int \lambda^2 t e^{-\lambda t} dt = -\lambda t e^{-\lambda t} + \int \lambda e^{-\lambda t} dt$$

3.3 Solutions

$$= -\lambda t e^{-\lambda t} - e^{-\lambda t} + C.$$

Thus, the definite integral becomes:

$$\int_0^{50} \lambda^2 t e^{-\lambda t} dt = \left[-\lambda t e^{-\lambda t} - e^{-\lambda t} \right]_0^{50}$$

Substitute $\lambda = \frac{1}{100}$:

$$\begin{aligned} &= -\left(\frac{1}{100} \cdot 50 \cdot e^{-0.5} + e^{-0.5} \right) + (0 + 1) \\ &= 1 - \left(\frac{50}{100} + 1 \right) e^{-0.5} = 1 - (1.5) e^{-0.5}. \end{aligned}$$

Using $e^{-0.5} \approx 0.6065$:

$$P(T < 50) \approx 1 - 1.5 \cdot 0.6065 = 1 - 0.90975 = 0.09025$$

$$P(\text{System fails before 50 hours}) \approx 0.0903.$$

Solution 3.26. The upper fifth percentile is the chi-square value x such that the probability to the right of x is 0.05. That is:

$$P(X > x) = 0.05 \quad \Rightarrow \quad P(X \leq x) = 0.95.$$

To find this value using the chi-square table:

1. Locate the degrees of freedom $df = 10$ in the first column.
2. Find the column corresponding to the cumulative probability 0.95.
3. The intersection of the $df = 10$ row and the 0.95 column gives the chi-square value x .

Thus, the upper fifth percentile is the chi-square value x such that:

$$P(X \leq x) = 0.95 \quad \Rightarrow \quad x = \chi_{0.95,10}^2.$$

From the chi-square table:

$$\chi_{0.95,10}^2 = 18.31.$$

Thus $x = 18.31$.

3.3 Solutions

Solution 3.27. The tenth percentile is the chi-square value x such that the probability to the left of x is 0.10:

$$P(X \leq x) = 0.10.$$

To find this value using the chi-square table:

1. Locate the degrees of freedom $df = 10$ in the first column.
2. Find the column corresponding to the cumulative probability 0.10.
3. The intersection of the $df = 10$ row and the 0.10 column gives the chi-square value x .

From the chi-square distribution table:

$$\chi_{0.10,10}^2 = 4.86.$$

Thus $x = 4.86$.

Solution 3.28. We want to find:

$$P(X > 15.99), \quad \text{where } X \sim \chi^2(10)$$

To find this probability using the chi-square table, we read the table "in reverse" as follows:

1. Locate the degrees of freedom $df = 10$ in the first column.
2. Scan across the $df = 10$ row to find the value 15.99.
3. Identify the column heading under which the value 15.99 appears — this heading represents the cumulative probability to the left of 15.99, i.e., $P(X \leq 15.99)$.
4. Subtract this cumulative probability from 1 to get the desired tail probability.

From the chi-square table:

$$P(X \leq 15.99) = 0.90 \quad \Rightarrow \quad P(X > 15.99) = 1 - 0.90 = 0.10.$$

Thus

$$P(X > 15.99) = 0.10.$$

3.3 Solutions

Solution 3.29. Let $X \sim \text{Beta}(\alpha, \beta)$. We know that the expected value and variance of a Beta distribution are:

$$E[X] = \frac{\alpha}{\alpha + \beta}, \text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

We are given:

$$\mu = 0.04, \quad \sigma = 0.02 \Rightarrow \sigma^2 = 0.0004.$$

We want to solve the following system:

$$\frac{\alpha}{\alpha + \beta} = \mu \tag{3.1}$$

$$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \sigma^2. \tag{3.2}$$

From equation (3.1), we can express β in terms of α :

$$\mu = \frac{\alpha}{\alpha + \beta} \Rightarrow \beta = \alpha \left(\frac{1 - \mu}{\mu} \right). \tag{3.3}$$

Substituting equation (3.3) into equation (3.2):

$$\begin{aligned} \sigma^2 &= \frac{\alpha \cdot \alpha \left(\frac{1 - \mu}{\mu} \right)}{\left(\alpha + \alpha \left(\frac{1 - \mu}{\mu} \right) \right)^2 \left(\alpha + \alpha \left(\frac{1 - \mu}{\mu} \right) + 1 \right)} \\ &= \frac{\alpha^2 \left(\frac{1 - \mu}{\mu} \right)}{\alpha^2 \left(\frac{1}{\mu} \right)^2 \left(\frac{1}{\mu} + 1 \right)} \\ &= \frac{(1 - \mu)\mu^2}{1(1 + \mu)} \Rightarrow \alpha = \left(\frac{\mu^2(1 - \mu)}{\sigma^2} - \mu \right). \end{aligned}$$

Now plug in $\mu = 0.04$, $\sigma^2 = 0.0004$:

$$\alpha = \left(\frac{(0.04)^2 \cdot (1 - 0.04)}{0.0004} - 0.04 \right) = \left(\frac{0.001536}{0.0004} - 0.04 \right) = (3.84 - 0.04) = 3.80$$

Now from equation (3.3):

$$\beta = \alpha \left(\frac{1 - \mu}{\mu} \right) = 3.80 \cdot \frac{0.96}{0.04} = 3.80 \cdot 24 = 91.20$$

The parameters for the Beta distribution should be:

$$\alpha = 3.80, \quad \beta = 91.20.$$

3.3 Solutions

Solution 3.30.

1. Mean of the Weibull Distribution

Let $T \sim \text{Weibull}(k, \lambda)$. The mean is given by:

$$E[T] = \lambda \cdot \Gamma\left(1 + \frac{1}{k}\right)$$

Let $k = 1.5$, $\lambda = 27000$ hours (these parameters are assumed based on the claim that the mean is around 24000 hours). Then:

$$E[T] = 27000 \cdot \Gamma\left(1 + \frac{1}{1.5}\right) = 27000 \cdot \Gamma\left(\frac{5}{3}\right) \approx 27000 \cdot 0.9027 \approx 24073 \text{ hours}$$

So, the expected time to failure is approximately: 24000 hours.

2. The probability that the Bearing Lasts at Least 5000 Hours:

The cumulative distribution function (CDF) of the Weibull distribution is:

$$F(t) = 1 - \exp\left(-\left(\frac{t}{\lambda}\right)^k\right)$$

Therefore, the probability that the bearing lasts at least 5000 hours is:

$$\begin{aligned} P(T > 5000) &= 1 - F(5000) = \exp\left(-\left(\frac{5000}{27000}\right)^{1.5}\right) \\ &= \exp\left(-\left(\frac{5}{27}\right)^{1.5}\right) \\ &\approx \exp(-0.0803) \\ &\approx 0.9229. \end{aligned}$$

Solution 3.31.

1. Mean time until failure

The mean of a Weibull distribution is given by:

$$E[T] = \lambda \cdot \Gamma\left(1 + \frac{1}{k}\right) = 4000 \cdot \Gamma\left(1 + \frac{1}{1/3}\right) = 4000 \cdot \Gamma(4)$$

and we have:

$$\Gamma(4) = (4 - 1)! = 3! = 6.$$

Therefore:

$$E[T] = 4000 \cdot 6 = 24000 \text{ hours.}$$

3.3 Solutions

2. Probability the Bearing Lasts at Least 5000 Hours

The survival function of the Weibull distribution is:

$$P(T > t) = \exp\left(-\left(\frac{t}{\lambda}\right)^k\right).$$

Substitute $t = 5000$, $\lambda = 4000$, and $k = \frac{1}{3}$:

$$\begin{aligned} P(T > 5000) &= \exp\left(-\left(\frac{5000}{4000}\right)^{1/3}\right) = \exp\left(-1.25^{1/3}\right) \\ &\approx \exp(-1.0772) \approx 0.3405. \end{aligned}$$

Solution 3.32. Suppose the lifetime of a motor follows a lognormal distribution. That is, if T is the lifetime in hours, then:

$$\ln(T) \sim \mathcal{N}(\mu = 11, \sigma = 1.3)$$

We are asked to find the probability that the lifetime exceeds 12,000 hours:

$$P(T > 12000).$$

Take the natural logarithm of both sides:

$$P(T > 12000) = P(\ln(T) > \ln(12000)).$$

Calculate $\ln(12000)$:

$$\ln(12000) \approx 9.3927.$$

Standardize using the Z-score:

$$Z = \frac{\ln(12000) - \mu}{\sigma} = \frac{9.3927 - 11}{1.3} \approx -1.2341.$$

Now compute the upper tail probability:

$$P(T > 12000) = P(Z > -1.2341) = 1 - P(Z < -1.2341) = 1 - \Phi(-1.2341) = \Phi(1.2341).$$

Using standard normal tables or a calculator:

$$\Phi(1.2341) \approx 0.8918$$

There is approximately an 89.18% chance that the motor's lifetime exceeds 12000 hours.

3.3 Solutions

Solution 3.33. First, we compute the sample statistics:

$$\bar{x} = 8.2267, \quad s = 1.6722, \quad n = 15$$

where the degrees of freedom is:

$$df = n - 1 = 14.$$

We have the confidence level is 95%, so:

$$\alpha = 1 - 0.95 = 0.05, \quad \frac{\alpha}{2} = 0.025.$$

Using the t -distribution, we find the critical value $t_{\alpha/2}$ such that:

$$P(T > t_{0.025}) = 0.025, \quad \Rightarrow t_{0.025,14} = 2.14.$$

Now, we calculate the error bound for the mean (i.e., EBM):

$$EBM = t_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 2.14 \cdot \frac{1.6722}{\sqrt{15}} \approx 0.924.$$

Finally, we construct the confidence interval:

$$\bar{x} \pm EBM = 8.2267 \pm 0.924.$$

Confidence Interval: (7.30, 9.15).

Solution 3.34. From the data, compute the sample statistics:

$$\bar{x} = 127.45, \quad s = 25.965, \quad n = 20.$$

Degrees of freedom:

$$df = n - 1 = 19.$$

We have confidence level 0.90 thus

$$\alpha = 1 - 0.9 = 0.10 \Rightarrow \frac{\alpha}{2} = 0.05.$$

The critical value from the t -distribution:

$$t_{0.05,19} = 1.729 \quad (\text{from } t\text{-table})$$

$$EBM = t_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 1.729 \cdot \frac{25.965}{\sqrt{20}} \approx 10.038$$

We have

$$\bar{x} \pm EBM = 127.45 \pm 10.038$$

thus the confidence interval is: (117.412, 137.488).

3.3 Solutions

Solution 3.35. From the data, we compute the following statistics:

$$\bar{x} = 6.133, \quad s = 5.514, \quad n = 15.$$

$$df = n - 1 = 14.$$

We have the confidence Level 0.98 thus

$$\alpha = 1 - 0.98 = 0.02 \Rightarrow \frac{\alpha}{2} = 0.01.$$

From the t -distribution table we get:

$$t_{\alpha/2, 14} = t_{0.01, 14} = 2.624$$

and

$$\begin{aligned} EBM &= t_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 2.624 \cdot \frac{5.514}{\sqrt{15}} \approx 3.736 \\ \bar{x} \pm EBM &= 6.133 \pm 3.736 \end{aligned}$$

thus the confidence interval is (2.397, 9.869).

Solution 3.36. To estimate the ratio, we use the F -distribution with degrees of freedom:

$$df_1 = n_1 - 1 = 9, \quad df_2 = n_2 - 1 = 9.$$

Using the F -table:

$$F_{0.025, 9, 9} = 4.03, \quad F_{0.975, 9, 9} = \frac{1}{F_{0.025, 9, 9}} = \frac{1}{4.03} \approx 0.248.$$

Let the sample variances be such that:

$$\frac{s_1^2}{s_2^2} = R.$$

Then the 95% confidence interval for the ratio $\frac{\sigma_1^2}{\sigma_2^2}$ is:

$$\left(\frac{R}{F_{0.025, 9, 9}}, \frac{R}{F_{0.975, 9, 9}} \right)$$

Assuming we are given that $R = \frac{s_D^2}{s_M^2} = 1.742$ (as calculated from data), then:

$$\left(\frac{1.742}{4.03}, \frac{1.742}{0.248} \right) = (0.433, 7.033)$$

Since the interval contains the value 1, we cannot conclude that the population variances differ.

3.3 Solutions

Solution 3.37. First, we express the above probability in terms of the cumulative distribution function (CDF) of X :

$$P(1.5 < X < 5.9) = P(X < 5.9) - P(X < 1.5).$$

The values:

$$P(X < 5.9) \quad \text{and} \quad P(X < 1.5)$$

can be computed using a Chi-square distribution table for $df = 3$. From standard Chi-square CDF tables:

$$P(X < 5.9) \approx 0.857, \quad P(X < 1.5) \approx 0.198.$$

Therefore:

$$P(1.5 < X < 5.9) \approx 0.857 - 0.198 = 0.659.$$

Solution 3.38. We express the probability in terms of the cumulative distribution function (CDF) of X :

$$P(0.35 < X < 7.81) = P(X < 7.81) - P(X < 0.35).$$

Using chi-square CDF values for $df = 3$:

$$P(X < 7.81) \approx 0.95, \quad P(X < 0.35) \approx 0.05.$$

Therefore:

$$P(0.35 < X < 7.81) \approx 0.95 - 0.05 \approx 0.90.$$

Solution 3.39. We start by standardizing X_1 and X_2 :

$$X_1 = 4Z_1 \quad \text{and} \quad X_2 = 4Z_2$$

where Z_1 and Z_2 are independent standard normal random variables.

Then:

$$X_1^2 + X_2^2 = 16Z_1^2 + 16Z_2^2 = 16(Z_1^2 + Z_2^2).$$

Let:

$$Y = Z_1^2 + Z_2^2.$$

Then $Y \sim \chi^2(2)$, a chi-square distribution with 2 degrees of freedom.

Now the probability becomes:

$$P\left(X_1^2 + X_2^2 < \frac{1}{2}\right) = P\left(16Y < \frac{1}{2}\right) = P\left(Y < \frac{1}{32}\right).$$

3.3 Solutions

Let $F_{\chi^2(2)}(x)$ denote the CDF of the chi-square distribution with 2 degrees of freedom. Then:

$$P\left(Y < \frac{1}{32}\right) = F_{\chi^2(2)}\left(\frac{1}{32}\right).$$

Using statistical chi-square table:

$$F_{\chi^2(2)}\left(\frac{1}{32}\right) \approx 0.0156.$$

$$P\left(X_1^2 + X_2^2 < \frac{1}{2}\right) \approx 0.0156.$$

Solution 3.40. We can write:

$$X_1 = 4Z_1 \quad \text{and} \quad X_2 = 4Z_2,$$

where $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ are standard normal random variables.

Then:

$$X_1^2 + X_2^2 = 16Z_1^2 + 16Z_2^2 = 16(Z_1^2 + Z_2^2).$$

Note that $Z_1^2 + Z_2^2 \sim \chi^2(2)$, a chi-square distribution with 2 degrees of freedom.

Let $Y \sim \chi^2(2)$, then:

$$P(X_1^2 + X_2^2 > 8) = P(16Y > 8) = P\left(Y > \frac{8}{16}\right) = P(Y > 0.5)$$

Using the cumulative distribution function (CDF) for a chi-square distribution with 2 degrees of freedom:

$$P(Y > 0.5) = 1 - F_{\chi^2(2)}(0.5).$$

From chi-square tables:

$$F_{\chi^2(2)}(0.5) \approx 0.2212.$$

Therefore:

$$P(X_1^2 + X_2^2 > 8) = 1 - 0.2212 = 0.7788.$$

Solution 3.41. We use the linearity of expectation:

$$E[Y] = E[3X + 7] = 3E[X] + 7.$$

Since $X \sim \chi^2(5)$, the expected value of a Chi-square distribution with k degrees of freedom is:

$$E[X] = k = 5.$$

Thus:

$$E[Y] = 3 \cdot 5 + 7 = 15 + 7 = 22.$$

3.3 Solutions

Solution 3.42. We have the probability density function of the chi-square distribution with 5 degrees of freedom

$$f_X(x) = \frac{1}{2^{5/2}\Gamma(5/2)} x^{3/2} e^{-x/2}, \quad x > 0.$$

We want to compute:

$$\begin{aligned} E[Y] &= E[e^{1-X}] = \int_0^{\infty} e^{1-x} f_X(x) dx \\ &= e^1 \int_0^{\infty} e^{-x} \cdot \frac{1}{2^{5/2}\Gamma(5/2)} x^{3/2} e^{-x/2} dx \\ &= \frac{e}{2^{5/2}\Gamma(5/2)} \int_0^{\infty} x^{3/2} e^{-3x/2} dx. \end{aligned}$$

This is the form of the Gamma integral:

$$\int_0^{\infty} x^{k-1} e^{-\lambda x} dx = \frac{\Gamma(k)}{\lambda^k}.$$

Set $k = \frac{5}{2}$, $\lambda = \frac{3}{2}$, so:

$$\int_0^{\infty} x^{3/2} e^{-3x/2} dx = \frac{\Gamma(5/2)}{(3/2)^{5/2}}.$$

Therefore:

$$\begin{aligned} E[Y] &= \frac{e}{2^{5/2}\Gamma(5/2)} \cdot \frac{\Gamma(5/2)}{(3/2)^{5/2}} \\ &= e \cdot \left(\frac{1}{2} \cdot \frac{2}{3}\right)^{5/2} \\ &= e \cdot \left(\frac{1}{3}\right)^{5/2} \\ &= e \cdot 3^{-5/2} \\ &\approx 0.0573. \end{aligned}$$

Appendix: Statistical tables

Normal Distribution

Table 3.5: Table of cumulative normal distribution for negative z-values:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table 3.6: Table of cumulative normal distribution for positive z-values:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Poisson Distribution

Table 3.7: Table of Poisson distribution for a given value of λ , entry indicates the probability of a specified value of X .

		λ									
X	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679	
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659	0.3679	
2	0.0045	0.0164	0.0333	0.0536	0.0758	0.0988	0.1217	0.1438	0.1647	0.1839	
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494	0.0613	
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111	0.0153	
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020	0.0031	
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0005	
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	

		λ									
X	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	
0	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496	0.1353	
1	0.3662	0.3614	0.3543	0.3452	0.3347	0.3230	0.3106	0.2975	0.2842	0.2707	
2	0.2014	0.2169	0.2303	0.2417	0.2510	0.2584	0.2640	0.2678	0.2700	0.2707	
3	0.0738	0.0867	0.0998	0.1128	0.1255	0.1378	0.1496	0.1607	0.1710	0.1804	
4	0.0203	0.0260	0.0324	0.0395	0.0471	0.0551	0.0636	0.0723	0.0812	0.0902	
5	0.0045	0.0062	0.0084	0.0111	0.0141	0.0176	0.0216	0.0260	0.0309	0.0361	
6	0.0008	0.0012	0.0018	0.0026	0.0035	0.0047	0.0061	0.0078	0.0098	0.0120	
7	0.0001	0.0002	0.0003	0.0005	0.0008	0.0011	0.0015	0.0020	0.0027	0.0034	
8	0.0000	0.0000	0.0001	0.0001	0.0001	0.0002	0.0003	0.0005	0.0006	0.0009	
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0002	

λ

X	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672	0.0608	0.0550	0.0498
1	0.2572	0.2438	0.2306	0.2177	0.2052	0.1931	0.1815	0.1703	0.1596	0.1494
2	0.2700	0.2681	0.2652	0.2613	0.2565	0.2510	0.2450	0.2384	0.2314	0.2240
3	0.1890	0.1966	0.2033	0.2090	0.2138	0.2176	0.2205	0.2225	0.2237	0.2240
4	0.0992	0.1082	0.1169	0.1254	0.1336	0.1414	0.1488	0.1557	0.1622	0.1680
5	0.0417	0.0476	0.0538	0.0602	0.0668	0.0735	0.0804	0.0872	0.0940	0.1008
6	0.0146	0.0174	0.0206	0.0241	0.0278	0.0319	0.0362	0.0407	0.0455	0.0504
7	0.0044	0.0055	0.0068	0.0083	0.0099	0.0118	0.0139	0.0163	0.0188	0.0216
8	0.0011	0.0015	0.0019	0.0025	0.0031	0.0038	0.0047	0.0057	0.0068	0.0081
9	0.0003	0.0004	0.0005	0.0007	0.0009	0.0011	0.0014	0.0018	0.0022	0.0027
10	0.0001	0.0001	0.0001	0.0002	0.0002	0.0003	0.0004	0.0005	0.0006	0.0008
11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0002	0.0002
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

 λ

X	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
0	0.0450	0.0408	0.0369	0.0334	0.0302	0.0273	0.0247	0.0224	0.0202	0.0183
1	0.1397	0.1340	0.1217	0.1135	0.1057	0.0984	0.0915	0.0850	0.0789	0.0733
2	0.2165	0.2087	0.2008	0.1929	0.1850	0.1771	0.1692	0.1615	0.1539	0.1465
3	0.2237	0.2226	0.2209	0.2186	0.2158	0.2125	0.2087	0.2046	0.2001	0.1954
4	0.1734	0.1781	0.1823	0.1858	0.1888	0.1912	0.1931	0.1944	0.1951	0.1954
5	0.1075	0.1140	0.1203	0.1264	0.1322	0.1377	0.1429	0.1477	0.1522	0.1563
6	0.0555	0.0608	0.0662	0.0716	0.0771	0.0826	0.0881	0.0936	0.0989	0.1042
7	0.0246	0.0278	0.0312	0.0348	0.0385	0.0425	0.0466	0.0508	0.0551	0.0595
8	0.0095	0.0111	0.0129	0.0148	0.0169	0.0191	0.0215	0.0241	0.0269	0.0298
9	0.0033	0.0040	0.0047	0.0056	0.0066	0.0076	0.0089	0.0102	0.0116	0.0132
10	0.0010	0.0013	0.0016	0.0019	0.0023	0.0028	0.0033	0.0039	0.0045	0.0053
11	0.0003	0.0004	0.0005	0.0006	0.0007	0.0009	0.0011	0.0013	0.0016	0.0019
12	0.0001	0.0001	0.0001	0.0002	0.0002	0.0003	0.0003	0.0004	0.0005	0.0006
13	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

Binomial Distribution

Table 3.8: Table of Binomial distribution for all values ≤ 0.0005 have been left out:

<i>n</i>	<i>x</i>	<i>p</i>										
		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
2	0	0.902	0.810	0.640	0.490	0.360	0.250	0.160	0.090	0.040	0.010	0.002
	1	0.095	0.180	0.320	0.420	0.480	0.500	0.480	0.420	0.320	0.180	0.095
	2	0.002	0.010	0.040	0.090	0.160	0.250	0.360	0.490	0.640	0.810	0.902
3	0	0.857	0.729	0.512	0.343	0.216	0.125	0.064	0.027	0.008	0.001	
	1	0.135	0.243	0.384	0.441	0.432	0.375	0.288	0.189	0.096	0.027	0.007
	2	0.007	0.027	0.096	0.189	0.288	0.375	0.432	0.441	0.384	0.243	0.135
	3		0.001	0.008	0.027	0.064	0.125	0.216	0.343	0.512	0.729	0.857
4	0	0.815	0.656	0.410	0.240	0.130	0.062	0.026	0.008	0.002		
	1	0.171	0.292	0.410	0.412	0.346	0.250	0.154	0.076	0.026	0.004	
	2	0.014	0.049	0.154	0.265	0.346	0.375	0.346	0.265	0.154	0.049	0.014
	3		0.004	0.026	0.076	0.154	0.250	0.346	0.412	0.410	0.292	0.171
	4			0.002	0.008	0.026	0.062	0.130	0.240	0.410	0.656	0.815
5	0	0.774	0.590	0.328	0.168	0.078	0.031	0.010	0.002			
	1	0.204	0.328	0.410	0.360	0.259	0.156	0.077	0.028	0.006		
	2	0.021	0.073	0.205	0.309	0.346	0.312	0.230	0.132	0.051	0.008	0.001
	3	0.001	0.008	0.051	0.132	0.230	0.312	0.346	0.309	0.205	0.073	0.021
	4			0.006	0.028	0.077	0.156	0.259	0.360	0.410	0.328	0.204
	5				0.002	0.010	0.031	0.078	0.168	0.328	0.590	0.774
6	0	0.735	0.531	0.262	0.118	0.047	0.016	0.004	0.001			
	1	0.232	0.354	0.393	0.303	0.187	0.094	0.037	0.010	0.002		
	2	0.031	0.098	0.246	0.324	0.311	0.234	0.138	0.060	0.015	0.001	
	3	0.002	0.015	0.082	0.185	0.276	0.312	0.276	0.185	0.082	0.015	0.002
	4		0.001	0.015	0.060	0.138	0.234	0.311	0.324	0.246	0.098	0.031
	5			0.002	0.010	0.037	0.094	0.187	0.303	0.393	0.354	0.232
	6				0.001	0.004	0.016	0.047	0.118	0.262	0.531	0.735
7	0	0.698	0.478	0.210	0.082	0.028	0.008	0.002				
	1	0.257	0.372	0.367	0.247	0.131	0.055	0.017	0.004			
	2	0.041	0.124	0.275	0.318	0.261	0.164	0.077	0.025	0.004		
	3	0.004	0.023	0.115	0.227	0.290	0.273	0.194	0.097	0.029	0.003	
	4		0.003	0.029	0.097	0.194	0.273	0.290	0.227	0.115	0.023	0.004
	5			0.004	0.025	0.077	0.164	0.261	0.318	0.275	0.124	0.041
	6				0.004	0.017	0.055	0.131	0.247	0.367	0.372	0.257
	7					0.002	0.008	0.028	0.082	0.210	0.478	0.698
8	0	0.663	0.430	0.168	0.058	0.017	0.004	0.001				
	1	0.279	0.383	0.336	0.198	0.090	0.031	0.008	0.001			
	2	0.051	0.149	0.294	0.296	0.209	0.109	0.041	0.010	0.001		
	3	0.005	0.033	0.147	0.254	0.279	0.219	0.124	0.047	0.009		
	4		0.005	0.046	0.136	0.232	0.273	0.232	0.136	0.046	0.005	
	5			0.009	0.047	0.124	0.219	0.279	0.254	0.147	0.033	0.005
	6			0.001	0.010	0.041	0.109	0.209	0.296	0.294	0.149	0.051
	7				0.001	0.008	0.031	0.090	0.198	0.336	0.383	0.279
	8					0.001	0.004	0.017	0.058	0.168	0.430	0.663

<i>n</i>	<i>x</i>	<i>p</i>											
		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	
9	0	0.630	0.387	0.134	0.040	0.010	0.002						
	1	0.299	0.387	0.302	0.156	0.060	0.018	0.004					
	2	0.063	0.172	0.302	0.267	0.161	0.070	0.021	0.004				
	3	0.008	0.045	0.176	0.267	0.251	0.164	0.074	0.021	0.003			
	4	0.001	0.007	0.066	0.172	0.251	0.246	0.167	0.074	0.017	0.001		
	5		0.001	0.017	0.074	0.167	0.246	0.251	0.172	0.066	0.007	0.001	
	6			0.003	0.021	0.074	0.164	0.251	0.267	0.176	0.045	0.008	
	7				0.004	0.021	0.070	0.161	0.267	0.302	0.172	0.063	
	8					0.004	0.018	0.060	0.156	0.302	0.387	0.299	
	9						0.002	0.010	0.040	0.134	0.387	0.630	
10	0	0.599	0.349	0.107	0.028	0.006	0.001						
	1	0.315	0.387	0.268	0.121	0.040	0.010	0.002					
	2	0.075	0.194	0.302	0.233	0.121	0.044	0.011	0.001				
	3	0.010	0.057	0.201	0.267	0.215	0.117	0.042	0.009	0.001			
	4	0.001	0.011	0.088	0.200	0.251	0.205	0.111	0.037	0.006			
	5		0.001	0.026	0.103	0.201	0.246	0.201	0.103	0.026	0.001		
	6			0.006	0.037	0.111	0.205	0.251	0.200	0.088	0.011	0.001	
	7			0.001	0.009	0.042	0.117	0.215	0.267	0.201	0.057	0.010	
	8				0.001	0.011	0.044	0.121	0.233	0.302	0.194	0.075	
	9					0.002	0.010	0.040	0.121	0.268	0.387	0.315	
	10						0.001	0.006	0.028	0.107	0.349	0.599	
11	0	0.569	0.314	0.086	0.020	0.004							
	1	0.329	0.384	0.236	0.093	0.027	0.005	0.001					
	2	0.087	0.213	0.295	0.200	0.089	0.027	0.005	0.001				
	3	0.014	0.071	0.221	0.257	0.177	0.081	0.023	0.004				
	4	0.001	0.016	0.111	0.220	0.236	0.161	0.070	0.017	0.002			
	5		0.002	0.039	0.132	0.221	0.226	0.147	0.057	0.010			
	6			0.010	0.057	0.147	0.226	0.221	0.132	0.039	0.002		
	7			0.002	0.017	0.070	0.161	0.236	0.220	0.111	0.016	0.001	
	8				0.004	0.023	0.081	0.177	0.257	0.221	0.071	0.014	
	9				0.001	0.005	0.027	0.089	0.200	0.295	0.213	0.087	
	10					0.001	0.005	0.027	0.093	0.236	0.384	0.329	
	11							0.004	0.020	0.086	0.314	0.569	
12	0	0.540	0.282	0.069	0.014	0.002							
	1	0.341	0.377	0.206	0.071	0.017	0.003						
	2	0.099	0.230	0.283	0.168	0.064	0.016	0.002					
	3	0.017	0.085	0.236	0.240	0.142	0.054	0.012	0.001				
	4	0.002	0.021	0.133	0.231	0.213	0.121	0.042	0.008	0.001			
	5		0.004	0.053	0.158	0.227	0.193	0.101	0.029	0.003			
	6			0.016	0.079	0.177	0.226	0.177	0.079	0.016			
	7			0.003	0.029	0.101	0.193	0.227	0.158	0.053	0.004		
	8			0.001	0.008	0.042	0.121	0.213	0.231	0.133	0.021	0.002	
	9				0.001	0.012	0.054	0.142	0.240	0.236	0.085	0.017	
	10					0.002	0.016	0.064	0.168	0.283	0.230	0.099	
	11						0.003	0.017	0.071	0.206	0.377	0.341	
	12							0.002	0.014	0.069	0.282	0.540	

<i>n</i>	<i>x</i>	<i>p</i>											
		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	
13	0	0.513	0.254	0.055	0.010	0.001							
	1	0.351	0.367	0.179	0.054	0.011	0.002						
	2	0.111	0.245	0.268	0.139	0.045	0.010	0.001					
	3	0.021	0.100	0.246	0.218	0.111	0.035	0.006	0.001				
	4	0.003	0.028	0.154	0.234	0.184	0.087	0.024	0.003				
	5		0.006	0.069	0.180	0.221	0.157	0.066	0.014	0.001			
	6		0.001	0.023	0.103	0.197	0.209	0.131	0.044	0.006			
	7			0.006	0.044	0.131	0.209	0.197	0.103	0.023	0.001		
	8			0.001	0.014	0.066	0.157	0.221	0.180	0.069	0.006		
	9				0.003	0.024	0.087	0.184	0.234	0.154	0.028	0.003	
	10				0.001	0.006	0.035	0.111	0.218	0.246	0.100	0.021	
	11					0.001	0.010	0.045	0.139	0.268	0.245	0.111	
	12						0.002	0.011	0.054	0.179	0.367	0.351	
13							0.001	0.010	0.055	0.254	0.513		
14	0	0.488	0.229	0.044	0.007	0.001							
	1	0.359	0.356	0.154	0.041	0.007	0.001						
	2	0.123	0.257	0.250	0.113	0.032	0.006	0.001					
	3	0.026	0.114	0.250	0.194	0.085	0.022	0.003					
	4	0.004	0.035	0.172	0.229	0.155	0.061	0.014	0.001				
	5		0.008	0.086	0.196	0.207	0.122	0.041	0.007				
	6		0.001	0.032	0.126	0.207	0.183	0.092	0.023	0.002			
	7			0.009	0.062	0.157	0.209	0.157	0.062	0.009			
	8			0.002	0.023	0.092	0.183	0.207	0.126	0.032	0.001		
	9				0.007	0.041	0.122	0.207	0.196	0.086	0.008		
	10				0.001	0.014	0.061	0.155	0.229	0.172	0.035	0.004	
	11					0.003	0.022	0.085	0.194	0.250	0.114	0.026	
	12					0.001	0.006	0.032	0.113	0.250	0.257	0.123	
	13						0.001	0.007	0.041	0.154	0.356	0.359	
14							0.001	0.007	0.044	0.229	0.488		
15	0	0.463	0.206	0.035	0.005								
	1	0.366	0.343	0.132	0.031	0.005							
	2	0.135	0.267	0.231	0.092	0.022	0.003						
	3	0.031	0.129	0.250	0.170	0.063	0.014	0.002					
	4	0.005	0.043	0.188	0.219	0.127	0.042	0.007	0.001				
	5	0.001	0.010	0.103	0.206	0.186	0.092	0.024	0.003				
	6		0.002	0.043	0.147	0.207	0.153	0.061	0.012	0.001			
	7			0.014	0.081	0.177	0.196	0.118	0.035	0.003			
	8			0.003	0.035	0.118	0.196	0.177	0.081	0.014			
	9			0.001	0.012	0.061	0.153	0.207	0.147	0.043	0.002		
	10				0.003	0.024	0.092	0.186	0.206	0.103	0.010	0.001	
	11				0.001	0.007	0.042	0.127	0.219	0.188	0.043	0.005	
	12					0.002	0.014	0.063	0.170	0.250	0.129	0.031	
	13						0.003	0.022	0.092	0.231	0.267	0.135	
	14							0.005	0.031	0.132	0.343	0.366	
	15								0.005	0.035	0.206	0.463	

<i>n</i>	<i>x</i>	<i>P</i>															
		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95					
16	0	0.440	0.185	0.028	0.003												
	1	0.371	0.329	0.113	0.023	0.003											
	2	0.146	0.275	0.211	0.073	0.015	0.002										
	3	0.036	0.142	0.246	0.146	0.047	0.009	0.001									
	4	0.006	0.051	0.200	0.204	0.101	0.028	0.004									
	5	0.001	0.014	0.120	0.210	0.162	0.067	0.014	0.001								
	6		0.003	0.055	0.165	0.198	0.122	0.039	0.006								
	7			0.020	0.101	0.189	0.175	0.084	0.019	0.001							
	8			0.006	0.049	0.142	0.196	0.142	0.049	0.006							
	9			0.001	0.019	0.084	0.175	0.189	0.101	0.020							
	10				0.006	0.039	0.122	0.198	0.165	0.055	0.003						
	11				0.001	0.014	0.067	0.162	0.210	0.120	0.014	0.001					
	12					0.004	0.028	0.101	0.204	0.200	0.051	0.006					
	13						0.001	0.009	0.047	0.146	0.246	0.142	0.036				
	14							0.002	0.015	0.073	0.211	0.275	0.146				
	15								0.003	0.023	0.113	0.329	0.371				
	16									0.003	0.028	0.185	0.440				
17	0	0.418	0.167	0.023	0.002												
	1	0.374	0.315	0.096	0.017	0.002											
	2	0.158	0.280	0.191	0.058	0.010	0.001										
	3	0.041	0.156	0.239	0.125	0.034	0.005										
	4	0.008	0.060	0.209	0.187	0.080	0.018	0.002									
	5	0.001	0.017	0.136	0.208	0.138	0.047	0.008	0.001								
	6		0.004	0.068	0.178	0.184	0.094	0.024	0.003								
	7		0.001	0.027	0.120	0.193	0.148	0.057	0.009								
	8			0.008	0.064	0.161	0.185	0.107	0.028	0.002							
	9			0.002	0.028	0.107	0.185	0.161	0.064	0.008							
	10				0.009	0.057	0.148	0.193	0.120	0.027	0.001						
	11				0.003	0.024	0.094	0.184	0.178	0.068	0.004						
	12				0.001	0.008	0.047	0.138	0.208	0.136	0.017	0.001					
	13					0.002	0.018	0.080	0.187	0.209	0.060	0.008					
	14						0.005	0.034	0.125	0.239	0.156	0.041					
	15							0.001	0.010	0.058	0.191	0.280	0.158				
	16								0.002	0.017	0.096	0.315	0.374				
17									0.002	0.023	0.167	0.418					

<i>n</i>	<i>x</i>	<i>p</i>																
		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95						
18	0	0.397	0.150	0.018	0.002													
	1	0.376	0.300	0.081	0.013	0.001												
	2	0.168	0.284	0.172	0.046	0.007	0.001											
	3	0.047	0.168	0.230	0.105	0.025	0.003											
	4	0.009	0.070	0.215	0.168	0.061	0.012	0.001										
	5	0.001	0.022	0.151	0.202	0.115	0.033	0.004										
	6		0.005	0.082	0.187	0.166	0.071	0.015	0.001									
	7		0.001	0.035	0.138	0.189	0.121	0.037	0.005									
	8			0.012	0.081	0.173	0.167	0.077	0.015	0.001								
	9			0.003	0.039	0.128	0.185	0.128	0.039	0.003								
	10			0.001	0.015	0.077	0.167	0.173	0.081	0.012								
	11				0.005	0.037	0.121	0.189	0.138	0.035	0.001							
	12				0.001	0.015	0.071	0.166	0.187	0.082	0.005							
	13					0.004	0.033	0.115	0.202	0.151	0.022	0.001						
	14					0.001	0.012	0.061	0.168	0.215	0.070	0.009						
	15						0.003	0.025	0.105	0.230	0.168	0.047						
	16						0.001	0.007	0.046	0.172	0.284	0.168						
	17							0.001	0.013	0.081	0.300	0.376						
18								0.002	0.018	0.150	0.397							
19	0	0.377	0.135	0.014	0.001													
	1	0.377	0.285	0.068	0.009	0.001												
	2	0.179	0.285	0.154	0.036	0.005												
	3	0.053	0.180	0.218	0.087	0.017	0.002											
	4	0.011	0.080	0.218	0.149	0.047	0.007	0.001										
	5	0.002	0.027	0.164	0.192	0.093	0.022	0.002										
	6		0.007	0.095	0.192	0.145	0.052	0.008	0.001									
	7		0.001	0.044	0.153	0.180	0.096	0.024	0.002									
	8			0.017	0.098	0.180	0.144	0.053	0.008									
	9			0.005	0.051	0.146	0.176	0.098	0.022	0.001								
	10			0.001	0.022	0.098	0.176	0.146	0.051	0.005								
	11				0.008	0.053	0.144	0.180	0.098	0.071								
	12				0.002	0.024	0.096	0.180	0.153	0.044	0.001							
	13				0.001	0.008	0.052	0.145	0.192	0.095	0.007							
	14					0.002	0.022	0.093	0.192	0.164	0.027	0.002						
	15					0.001	0.007	0.047	0.149	0.218	0.080	0.011						
	16						0.002	0.017	0.087	0.218	0.180	0.053						
	17							0.005	0.036	0.154	0.285	0.179						
18							0.001	0.009	0.068	0.285	0.377							
19								0.001	0.014	0.135	0.377							

<i>n</i>	<i>x</i>	<i>p</i>																			
		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95									
20	0	0.358	0.122	0.012	0.001																
	1	0.377	0.270	0.058	0.007																
	2	0.189	0.285	0.137	0.028	0.003															
	3	0.060	0.190	0.205	0.072	0.012	0.001														
	4	0.013	0.090	0.218	0.130	0.035	0.005														
	5	0.002	0.032	0.175	0.179	0.075	0.015	0.001													
	6		0.009	0.109	0.192	0.124	0.037	0.005													
	7		0.002	0.055	0.164	0.166	0.074	0.015	0.001												
	8			0.022	0.114	0.180	0.120	0.035	0.004												
	9			0.007	0.065	0.160	0.160	0.071	0.012												
	10			0.002	0.031	0.117	0.176	0.117	0.031	0.002											
	11				0.012	0.071	0.160	0.160	0.065	0.007											
	12				0.004	0.035	0.120	0.180	0.114	0.022											
	13				0.001	0.015	0.074	0.166	0.164	0.055	0.002										
	14					0.005	0.037	0.124	0.192	0.109	0.009										
	15					0.001	0.015	0.075	0.179	0.175	0.032	0.002									
	16						0.005	0.035	0.130	0.218	0.090	0.013									
	17						0.001	0.012	0.072	0.205	0.190	0.060									
	18							0.003	0.028	0.137	0.285	0.189									
	19								0.007	0.058	0.270	0.377									
	20									0.001	0.012	0.122	0.358								

Chi-square Distribution

Table 3.9: Table of Chi-square distribution for different values of k :

df	χ^2 .995	χ^2 .990	χ^2 .975	χ^2 .950	χ^2 .900	χ^2 .100	χ^2 .050	χ^2 .025	χ^2 .010	χ^2 .005
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

Student Distribution

Table 3.10: Table of Student distribution for differents values of t :

cum. prob	t.50	t.75	t.80	t.85	t.90	t.95	t.975	t.99	t.995	t.999	t.9995
one-tail	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
two-tails	1.00	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.002	0.001
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	ConfidenceLevel										

Fisher Distribution

Table 3.11: Table of Fisher distribution for different values of α :

d.f.D.		$\alpha = 0.005$																			
		1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	
1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24224	24426	24630	24836	24940	25044	25148	25253	25359	25465		
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5	
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83		
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32		
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14		
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88		
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.65	7.53	7.42	7.31	7.19	7.08		
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95		
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19		
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64		
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44	4.34	4.23		
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90		
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87	3.76	3.65		
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66	3.55	3.44		
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26		
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33	3.22	3.11		
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.51	3.41	3.31	3.21	3.10	2.98		
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10	2.99	2.87		
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00	2.89	2.78		
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69		
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84	2.73	2.61		
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77	2.66	2.55		
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.71	2.60	2.48		
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43		
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.63	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61	2.50	2.38		
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.56	2.45	2.33		
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.52	2.41	2.25		
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.48	2.37	2.29		
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.76	2.66	2.56	2.45	2.33	2.24		
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18		
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93		
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69		
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43		
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00		

		$\alpha = 0.01$																			
		d.f.N.																			
d.f.D.	I	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞		
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366		
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50		
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13		
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46		
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02		
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88		
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65		
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86		
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31		
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91		
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60		
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36		
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17		
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00		
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87		
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75		
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65		
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57		
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49		
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42		
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36		
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31		
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26		
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21		
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17		
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13		
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10		
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06		
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03		
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01		
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80		
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60		
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38		
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00		

d.f.D.		$\alpha = 0.02$																			
		d.f.N.																			
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞		
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001	1006	1010	1014	1018		
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50		
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90		
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26		
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02		
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85		
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25	4.20	4.14		
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67		
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33		
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08		
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88		
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72		
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60		
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49		
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40		
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32		
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25		
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19		
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13		
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09		
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04		
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00		
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97		
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94		
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91		
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88		
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93	1.85		
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83		
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81		
30	5.57	4.18	3.69	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79		
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64		
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48		
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31		
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00		

$\alpha = 0.05$

		d.f.N.																	
d.f.D.	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.52	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

		$\alpha = 0.10$																			
		d.f.N.																			
d.f.D.		1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33	63.33	
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49	9.49	
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13	5.13	
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76	3.76	
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10	3.10	
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72	2.72	
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47	2.47	
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29	2.29	
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16	2.16	
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06	2.06	
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97	1.97	
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90	1.90	
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85	1.85	
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80	1.80	
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76	1.76	
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72	1.72	
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69	1.69	
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66	1.66	
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63	1.63	
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.78	1.77	1.74	1.71	1.68	1.64	1.61	1.61	
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59	1.59	
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57	1.57	
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55	1.55	
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53	1.53	
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52	1.52	
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54	1.50	1.50	
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49	1.49	
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48	1.48	
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47	1.47	
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46	1.46	
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38	1.38	
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29	1.29	
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19	1.19	
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.10	1.00	

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