

الجمهورية الجزائرية الديمقراطية الشعبية
PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
جامعة 20 أوت 1955 سكيكدة
UNIVERSITY OF 20 AUGUST 1955 -SKIKDA-



Faculty of Sciences
Department of Mathematics

Course handout Mathematics 1 ST

For first year university students in matter sciences and related disciplines

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Academic year: 2025-2026

The simplified version is adapted to the templates proposed by the Algerian Ministry of
Higher Education (MESRS).

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General Information

- Semester: 1
- Teaching Unit: UEF 1.1
- Subject: Mathematics 1
- Total Semester Hours (TSH): 67h30
 - Lecture: 3h00/week
 - Tutorials: 1h30/week
- Credits: 6
- Coefficient: 3

Course Objectives

This course is primarily intended to harmonize students' mathematical background at the beginning of their university studies. New concepts are introduced progressively to guide students towards more advanced mathematics. The topics covered are fundamental and widely used in Science and Technology fields.

Recommended Prerequisites

Basic mathematical knowledge from final-year secondary education (sets, functions, equations, etc.).

Course Content

1. Methods of Mathematical Reasoning (1 week)
 - Direct reasoning
 - Contrapositive reasoning
 - Proof by contradiction

- Counterexample
 - Mathematical induction
2. Sets, Relations, and Functions (2 weeks)
- Set theory
 - Order and equivalence relations
 - Functions: definition, direct image, inverse image
 - Injective, surjective, bijective functions
3. Real Functions of a Real Variable (3 weeks)
- Limits and continuity
 - Derivative and differentiability
4. Applications to Elementary Functions (3 weeks)
- Power functions
 - Logarithmic functions
 - Exponential functions
 - Hyperbolic functions
 - Trigonometric functions
 - Inverse functions
5. Taylor Expansions (2 weeks)
- Taylor's formula
 - Taylor expansions
 - Applications
6. Linear Algebra (4 weeks)
- Binary operations
 - Vector spaces: definitions, bases, dimension
 - Linear maps: kernel, image, rank

Assessment Methods

- Continuous assessment: 40% (assignments, quizzes, in-class work)
- Final exam: 60% (written examination at semester end)

Bibliographical references

(Depending on the availability of documentation at the establishment level, Websites...etc.)

Learning Outcomes: Mathematics 1

Upon completion of Mathematics 1, the student will be able to:

1. Understand and apply basic methods of mathematical proof.
2. Select the appropriate type of proof to solve given problems.
3. Identify and perform basic set operations: intersection, union, symmetric difference, Cartesian product, etc.
4. Classify relations into order relations and equivalence relations.
5. Apply fundamental concepts such as limits, continuity, differentiation, and derivation to study real-valued functions.
6. Analyse and study standard elementary functions.
7. Use Taylor expansions (limited development) to compute limits and analysed the position of a graph relative to its tangent line.
8. Understand the structure of vector spaces and the properties of linear maps.

Prerequisite Knowledge

In order to follow this course under the best conditions, it is essential for the student to review:

1. Basic functions studied in secondary school.
2. Set operations.
3. The definition of the derivative and its properties.

INTRODUCTION

This handout covers some of the mathematical concepts underlying the Algebra section of the Maths1 teaching unit in the first years of L.M.D Science and Technology and Mathematics and Computer science. It can also be used by students at other levels of study, whether in science and engineering, biology, economics or other subjects.

The chapters in this text break down as follows:

- In these lessons, we follow the official Mathematics syllabus for first-year students in the Technology stream for the first semester, commonly referred to as “Mathematics 1,” as outlined by the Ministry of Higher Education. This course is based on six main chapters that form the foundational core of university-level mathematics. It blends elements of algebra and analysis and introduces key mathematical concepts and terminology essential for students in the Technology program. The goal is not to dive deeply into theoretical aspects, but rather to emphasize the practical applications suited to the students’ specialization. That said, this material may also serve as a simplified reference for those specializing in mathematics — both students and instructors.
- The first chapter introduces fundamental proof techniques commonly used in mathematics. It begins with a brief introduction to mathematical logic to facilitate the understanding of these proof methods.
- The second chapter focuses on three main topics: set theory, relation theory, and function theory. Each concept is presented in a simplified way and accompanied by illustrative examples.
- The third chapter is dedicated to a special case of functions — real-valued functions of a real variable. It covers the foundational tools for analysing such functions: limits, continuity, differentiation, and derivation. Several important theorems are also introduced.

- ❑ The fourth chapter offers a comprehensive study of standard elementary functions. In addition to reviewing exponential, logarithmic, and power functions, it also includes hyperbolic, trigonometric, and inverse functions.
- ❑ Chapter five deals with Taylor expansions. It introduces the concept of limited expansions using Taylor's formula, operations on expansions, and how to use them to compute limits and study the behaviour of graphs in relation to their tangents.
- ❑ The sixth and final chapter serves as an introduction to linear algebra. It covers the basic properties of vector spaces and linear maps.
- ❑ This document is organized in accordance with the official syllabus established by the Ministry of Higher Education. It is characterized by its simplicity and the abundance of examples, making it easier to grasp the required mathematical concepts in each chapter.
- ❑ It is thanks to this indispensable personal work that we can go far in understanding and assimilating the mathematical concepts introduced. This is the only method known today for making progress in mathematics. The conscientious student will work on justifying each of his or her answers. Remember, in science, finding the right answer isn't enough you have to justify it too!
- ❑ As with any human effort, this work may contain errors or oversights. I would be grateful to anyone who points out mistakes or offers constructive corrections.

Chapter 1

Mathematical Reasoning Method

Euclid, often called the "Father of Axiomatic Method," is renowned for his work "The Elements," a foundational text in mathematics. In "The Elements," Euclid employed axioms, definitions, and deductive reasoning to establish geometric principles. His structured approach to proofs, building from established axioms and definitions to more complex theorems, remains the basis for mathematical reasoning today.



1.0.1 Logic, proposition and open sentence

Definition 1.1

- Logic is an analysis of ways of thinking.
- We call the case of every correct declarative sentence a linguistic one, and it is possible to judge whether its meaning is true or false.
- We call an open sentence every sentence that contains a variable to a specific group, and it becomes an issue whenever the variable is replaced by an element from this group.
- If the case is valid, it will be attached to the value **1** and if it is wrong attach the value **0**. These values are called the truth values of an issue.

1.0.2 Quotation and quantified sentences

Considering the open sentence, where \mathbf{x} student, $P(\mathbf{x})$, \mathbf{x} present.

- To express that all student are present we write $\forall \mathbf{x}; P(\mathbf{x})$ and read for all \mathbf{x} , $P(\mathbf{x})$.
- To express that some students are present we write $\exists \mathbf{x}; P(\mathbf{x})$ and you read there is \mathbf{x} at least in which $P(\mathbf{x})$.
- To express that there is only one student present , we write $\exists! \mathbf{x}, P(\mathbf{x})$, and we read there is \mathbf{x} and only one \mathbf{x} , $P(\mathbf{x})$.
- For two unspecified propositions P and Q , there are 2^2 possible truth attributions. Generally speaking, n propositions correspond to 2^n truth attribution possibilities.

1.0.3 Connectors logic

If P is a proposition and Q is another proposition, we'll define new propositions built from P and Q .

Negation of a proposition

The negation of a proposition P is a proposition denoted \bar{P} and defined from its truth table

P	\bar{P}
1	0
0	1

Conjunction "and"

The conjunction is logical connector "and" which associated the propositions (P, Q) .

" $P \wedge Q$ " is true if both P and Q with any pair of propositions (P, Q) .

$P \wedge Q$ is true if both P and Q are true at the same time: $P \wedge Q$ is true if P and Q are both true simultaneously, false otherwise.

This is summarized in the following truth table

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction "or"

The disjunction is logical connector that associates the proposition " P or Q " with any pair of propositions (P, Q) . $P \vee Q$ is false if both P and Q are simultaneously, true otherwise. This is summarized in the following truth table

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

Involvement " \Rightarrow "

Implication is the logical connector that associates the proposition " $(P$ implies $Q)$ " with any pair of propositions (P, Q) , \Rightarrow denoted as follows: $P \Rightarrow Q$ is false when P is true and Q is false, true in other cases.

This is summarized in the following truth table

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Equivalence " \Leftrightarrow "

Equivalence is the logical connector that associates the proposition " P equals Q " with any pair of propositions (P, Q) , denoted \Leftrightarrow and defined as follows $P \Leftrightarrow Q$ is true when P and Q have the same truth value, false otherwise.

This is summarized in the following truth table

P	Q	$P \Leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

1.0.4 Properties of logic connectors

Consider the proposition P . This proposition can take the truth value as true or false. Consider the compound proposition

$$R = P \vee \bar{P}.$$

This is a remarkable proposition. R is always true, regardless of P .

Let's check it:

P	\bar{P}	$P \vee \bar{P}$
1	0	1
0	1	1

The proposition R is then qualified as a tautology.

Definition 1.2 A proposition that is true whatever the truth value of its constituent propositions is called a Tautology.

Proposition 1.1 Whatever the truth values of the proposition P , Q and R , the following propositions are always true.

- $\bar{\bar{P}} \vee P$.
- $\bar{\bar{P}} \Leftrightarrow P$.
- $P \wedge P \Leftrightarrow P$.
- $P \vee P \Leftrightarrow P$.
- $P \wedge Q \Leftrightarrow Q \wedge P$ (The \wedge connector is commutative).
- $P \vee Q \Leftrightarrow Q \vee P$ (The \vee connector is commutative).

1.0.5 Properties of logic connectors

- $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ (The connector \wedge is associative).
- $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (The \wedge connector \vee is associative).
- $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ (The \wedge connector is distributive on \vee).
- $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ (The \vee connector is distributive on \wedge).
- $P \wedge (P \vee Q) \Leftrightarrow P$.
- $P \vee (P \wedge Q) \Leftrightarrow P$.
- $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$ (Transitivity of \Rightarrow).

- $(P \Leftrightarrow Q) \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$.
- $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$ (Morgan's Laws).
- $\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$ (Morgan's Laws).
- $[(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)] \Rightarrow (P \Leftrightarrow R)$ (Transitivity of \Leftrightarrow).
- $(P \Rightarrow Q) \Leftrightarrow (\overline{P} \vee Q)$.
- $(P \Rightarrow Q) \Leftrightarrow \overline{Q} \Rightarrow \overline{P}$ (Contrapositive).

Remark 1.1  These properties can be demonstrated by drawing up the truth table.

1.0.6 Reasoning or demonstration method

Logic operations \longrightarrow reasoning or demonstration method.

Aim: demonstrate involvement $H \Rightarrow C$.

Direct demonstration

logically deduce C from H, based on the transitivity of implication (deductive reasoning) .

Proof by absurd

In logic, proof by absurd is a form of proof that establishes the truth or the validity of a proposition, by showing that assuming the proposition to be false leads to a contradiction.

The principle may be formally expressed as the propositional formula $\overline{\overline{P}} = P$ which reads "if assuming P to be false implies false hood, then P is true".

Example 1.1 Let's prove that $\sqrt{2}$ is irrational number.

We suppose by absurd that $\sqrt{2}$ is a rational number so we can write it $\sqrt{2} = \frac{p}{q}; q \in \mathbb{Z}^*, p \in \mathbb{Z}$. where $\frac{p}{q}$ is irreducible fraction.

We have:

$$\sqrt{2} = \frac{p}{q} \Leftrightarrow 2 = \frac{p^2}{q^2} \Leftrightarrow p^2 = 2q^2. \quad (1.1)$$

So p^2 is an even number, because it's a product two numbers one of which is even so p is an even number, we can write it $p = 2k, k \in \mathbb{Z}$.

$$(2k)^2 = 2q^2 \Leftrightarrow 4k^2 = 2q^2 \Leftrightarrow q^2 = 2k^2.$$

We conclude that q^2 is an even number, so q is an even number. So $\frac{p}{q}$ is reducible fraction because $PGCD(p, q) \neq 1$.

We conclude that $\sqrt{2}$ is irrational number.

Proof by contrapositive

In logic, the contrapositive of a conditional statement is formed by negating both terms and reversing the direction of inference.

The principle may be formally expressed as the propositional formula $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$.

Example 1.2 Let $x \in \mathbb{R}$ prove that: $x \neq -8 \Rightarrow \frac{x+2}{x+5} \neq 2$.

Hence: $\left(x \neq -8 \Rightarrow \frac{x+2}{x+5} \neq 2\right) \Leftrightarrow \left(\frac{x+2}{x+5} = 2 \Rightarrow x = -8\right)$.

We can prove: $\frac{x+2}{x+5} = 2 \Rightarrow x = -8$.

We get: $\frac{x+2}{x+5} = 2 \Rightarrow x+2 = 2x+10 \Rightarrow x = -8$.

Proof by induction

A proof by induction is just like an ordinary proof in which every step must be justified. However it employs a neat trick which allows you to prove a statement about an arbitrary number n by first proving it is true when n is 1 and then assuming it is true for $n = k$ and showing it is true for $n = k + 1$.

Example 1.3 Prove that $\forall n \in \mathbb{N} : (3^{2n} - 2^n)$ is divisible by 7.

For $n = 0$ We have: $3^0 - 2^0 = 0 = 0.7$ so, $3^0 - 2^0$ is divisible by 7.

We prove that: $3^{2(n+1)} - 2^{n+1}$ is divisible by 7.

$$\begin{aligned} 3^{2(n+1)} - 2^{n+1} &= 3^{2n} \cdot 3^2 + 2^n \cdot 2 \\ &= 3^{2n}(7+2) - 2^n \cdot 2 \\ &= 2(3^{2n} - 2^n) + 7 \cdot 3^{2n} \\ &= 2 \cdot 7k + 7 \cdot 3^{2n} \\ &= 7(2k + 3^{2n}) = 7k'. \end{aligned}$$

Then $3^{2(n+1)} - 2^{n+1}$, is divisible by 7, hence $3^{2(n+1)} - 2^{(n+1)}$ is divisible by 7.

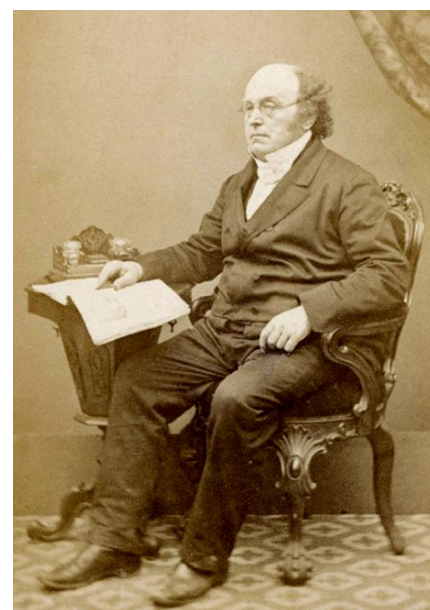
Chapter 2

Sets, Relations and Maps

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Augustus De Morgan (born June 27, 1806, in Madura, India, and died March 18, 1871, in London, England) was a prominent English mathematician and logician. His significant contributions to logic include formulating De Morgan's laws and his work that influenced the development of the theory of relations and symbolic logic.



2.1 Notions of Mathematical Logic

Mathematics is a set of abstract knowledge arrived from logic, adapted to complex phenomena to make accurate and real calculations.

2.2 Sets

Definition 2.1 A set is any collection or assembly of objects called set elements. In general, an element is denoted by a lowercase letter (the set E) note $x \in E$. If x is an element of E .

A set made up of a finite number of distinct elements can be defined by extension: by an explicit enumeration of all its elements, usually enclosed in braces $\{\dots\}$.

A set made up of an infinite (or even finite) number of elements can be given in understanding, i.e. by one or more properties defining its elements.

Example 2.1

- Sets given in extension $\{0, 1, 3\}$, $\{0, 1, 2, 3, \dots\} = \mathbb{N}$.
- Set given in comprehension $\{x \in \mathbb{R} : |x - 1| < 2\} =] - 1, 3[$.
- A set made up of one and only one element is called singleton.
For example $\{n \in \mathbb{N} : -0.1 < n < 0.1\} = \{0\}$.
- A special set is the empty set, denoted \emptyset , which is the set containing no elements.

2.2.1 Relation between sets

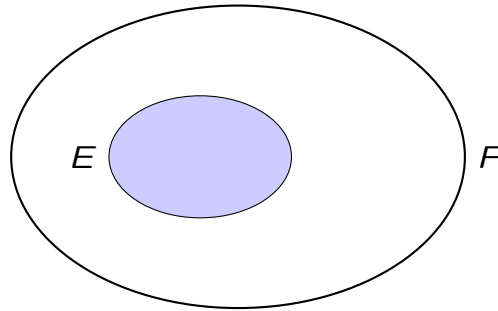
Inclusion

Definition 2.2 E and F are two sets. When any element of E is also an element of F , we say that E is a subset of F or F is a part of E . We also say that E is included in F , which we denote by $E \subset F$ and formally we have:

$$E \subset F \Leftrightarrow \forall x, x \in E \Rightarrow x \in F.$$

When E is not a part of F , we note $E \not\subset F$ and formally we have:

$$E \not\subset F \Leftrightarrow \exists x, (x \in E) \wedge (x \notin F).$$



Example 2.2

- \mathbb{Z} : set of relative integers.
- \mathbb{Q} : set of rational numbers .
- \mathbb{D} : set of decimal numbers.
- \mathbb{C} : set of complex numbers. We have the following inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

equality


Definition 2.3 Two sets E and F are said to be equal if and only if they contain the same elements.

$$(E = F) \Leftrightarrow (\forall x, x \in E \Leftrightarrow x \in F).$$

Or even

$$(E = F) \Leftrightarrow (E \subset F) \wedge (F \subset E).$$

Definition 2.4 Let E be a set. The set of all parts of E constitute a new set, denoted $P(E)$. It is sometimes called a Power set.

Remark 2.1 

- Let E be a set with n elements. Then the set (E) is finite, and has 2^n elements.
- For any set X , $P(X)$ is never empty because X and \emptyset belong to $P(X)$.
- $E \in P(X) \Leftrightarrow E \subset X$.
- $\{x\} \in P(X) \Leftrightarrow x \in X$.

Example 2.3 If $X = \{-1, 0, 1\}$, we get:

$$P(X) = \{\emptyset, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{-1, 1\}, \{0, 1\}, X\}$$

2.2.2 Operations on sets

Consider a set X .

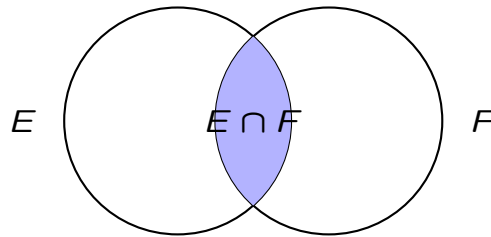
Intersection

Definition 2.5 Let E and F be two parts of X . The intersection of sets E and F is the set of all elements of E , that belong to both E and F . This is denoted $E \cap F = \emptyset$ (i.e. when E and F have no elements in common), E and F are said to be disjoint. We write

$$E \cap F = \{x \in X : (x \in E) \wedge (x \in F)\}$$

Or even

$$x \in E \cap F \Leftrightarrow (x \in E) \wedge (x \in F).$$



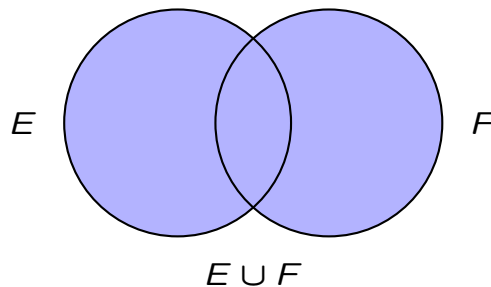
Union

Definition 2.6 Let E and F be two parts of X . The union of sets E and F is the set of elements belonging to E or F . This set is denoted $E \cup F$. The "or" is not exclusive: x can belong to both E and F at the same time.

$$E \cup F = \{x \in X : (x \in E) \vee (x \in F)\}.$$

or even

$$x \in E \cup F \Leftrightarrow (x \in A) \vee (x \in F).$$



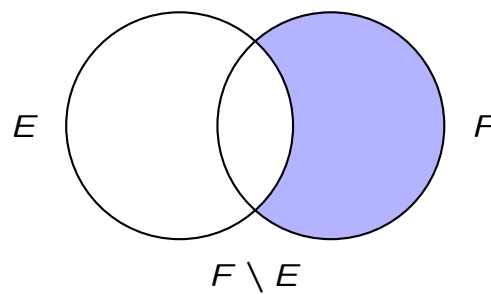
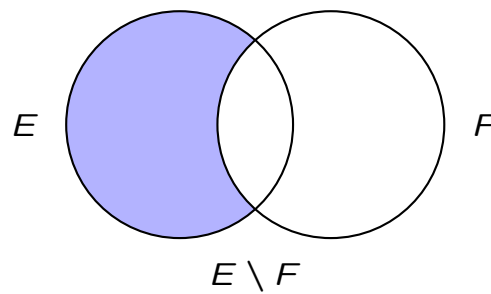
Complementary

Definition 2.7 Let E be a subset of X . We call the elements of X , and note C_X^E , the set of elements of X that do not belong to E .

$$X/E = C_X^E = \{x \in X, x \notin E\}.$$

or even

$$x \in C_X^E \Leftrightarrow x \notin E.$$



Example 2.4 Let $E = \{a, b, c\}$; $F = \{c, d, e\}$ we obtains:

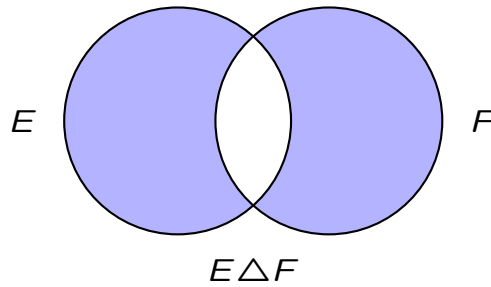
$$E/F = C_E^F = \{a, b\}.$$

$$F/E = C_F^E = \{d, e\}.$$

Symmetrical difference

Definition 2.8 Let E and F be two parts of a set X . We call the symmetric difference of E and F , and note $E\Delta F$, the set of elements of X belonging to E or F . The "or" is exclusive, i.e. an element must not belong to E and F simultaneously. We then have

$$E\Delta F = (E/F) \cup (F/E).$$



Example 2.5 Let be the following set $X = \mathbb{N}$, $E = \{0, 1, 2, 3, 4\}$; $F = \{1, 3, 5, 6\}$, then we have:

$$E\Delta F = (E/F) \cup (F/E) = \{0, 2, 4, 5, 6\}.$$

2.2.3 Properties of operations on sets


Whatever E , F and G are subsets of the same set X , we have the following properties.

- $E \cap F = F \cap E$ and $E \cup F = F \cup E$.
- $E \cap (F \cap G) = (E \cap F) \cap G$ and $(E \cup F) \cup G = E \cup (F \cup G)$.
- $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$ and $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$.

Operations \cap and \cup are mutually distributive.

- $A \cap A = A$ and $A \cup A = A$.
- Morgan's laws

$$C_X^{(E \cap F)} = C_X^E \cup C_X^F \text{ and } C_X^{(E \cup F)} = C_X^E \cap C_X^F.$$

Remark 2.2  The demonstration can be deduced directly from the properties of Morgan's laws for connectors \wedge and \vee .

Partition

Definition 2.9 Let X be a set and E_1, E_2, \dots, E_n subsets of X . These subsets are an aid to form a partition of X if the following three conditions are verified:

- Each of these sets is non-empty:

$$\forall i \in \{1, 2, \dots, n\}, E_i \neq \emptyset.$$

- They are two by two disjoint:

$$\forall i, j \in \{1, 2, \dots, n\}, i \neq j, E_i \cap E_j = \emptyset.$$

- Their union is equal to X :

$$\bigcup_{i=1}^{i=n} A_i = X.$$

Example 2.6 Here are some examples:

- $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Then the subsets $\{0, 1, 2\}, \{3, 5, 7, \}, \{4, 6\}, \{8\}$ constitute a partition of X .
- Let $X = \mathbb{N}$, A_1 the subset formed by even integers, A_2 the subset formed by odd integers. Then the subsets A_1 and A_2 form a partition of X .

2.2.4 Cartesian product

Definition 2.10 The Cartesian product of sets E and F is the set of elements (a, b) such that $a \in E$ and $b \in F$ in the order in which they are written. It is denoted by $E \times F$.

$$E \times F = \{(a, b) / (a \in E) \wedge (b \in F)\}.$$

The ordered system (a, b) is called a pair ; a is the first ordinate or component, b is the second ordinate or component of the pair (a, b) .

Definition 2.11 The Cartesian product of a family of sets $A_i, i = 1, \dots, n, n \in \mathbb{N}^*$, the set of systems (a_1, a_2, \dots, a_n) of n ordered elements $a_i \in A_i$, these ordered systems are called triplets for $n = 3$, quadruplets for $n = 4$ and n tuples. It is denoted by

$$A_1 \times A_2 \times \dots \times A_n \text{ or in abbreviation } \prod_{i=1}^{i=n} A_i.$$

Example 2.7 Let $A = \{0, 2, 4, 6\}, B = \{3, 5\}$.

$$A \times B = \{(0, 3), (2, 3), (4, 3), (6, 3), (0, 5), (2, 5), (4, 5), (6, 5)\}.$$

$$B \times A = \{(3, 0), (3, 2), (3, 4), (3, 6), (5, 0), (5, 2), (5, 4), (5, 6)\}.$$

2.3 Binary relations

Definition 2.12 Given two sets E and F . A relation from E to F is any assertion linking an element of E to an element of F that may or not may be verified. A relation is denoted by \mathcal{R} .

The set E is called the start set of \mathcal{R} , and F is called the end set of \mathcal{R} .

For any element $x \in E$ and any element $y \in F$ satisfying \mathcal{R} , x of E is said to be related by \mathcal{R} to y , which is denoted $x\mathcal{R}y$. If $E = F$, the relation \mathcal{R} is called a binary relation in E .

Example 2.8

- Consider the sets $E = \{1, 3, 4, 6, 8, 10, 11, 12, 17\}, F = \{0, 2, 3, 4, 5, 6, 7, 8\}$ and the relation \mathcal{R} defined from E into F with "...is double..."

1. Element 4 of E is the double of element 2 of F then $4\mathcal{R}2$.

2. Element 11 of E is not the double of element 5 of F then $11\not\mathcal{R}5$.

- $P(E)$ is the set of all parts of a set E . The relation \mathcal{R} in $P(E)$ is defined by:

$$\forall A, B \in P(E) : A\mathcal{R}B \Leftrightarrow A \subset B.$$

$$\forall A \in P(E), \emptyset \subset A, \text{ then } \forall A \in P(E), \emptyset\mathcal{R}A.$$

- Let P be the plane, the relation \mathcal{R} in P is defined by: for all straight lines $(D), (\Delta), (D) \parallel (\Delta)$.

2.3.1 Properties of binary relations in a set

Let E be a set and \mathcal{R} a relation defined in E .

1. Reflexivity: \mathcal{R} is said to be reflexive if

$$\forall x \in E, x\mathcal{R}x.$$

2. Symmetry: \mathcal{R} is said to be symmetrical if

$$\forall x, y \in E, x\mathcal{R}y \Rightarrow y\mathcal{R}x.$$

3. Transitivity: \mathcal{R} is said to be transitive if

$$\forall x, y, z \in E, (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow x\mathcal{R}z.$$

4. Antisymmetry: \mathcal{R} is said to be antisymmetric if

$$\forall x, y \in E, (x\mathcal{R}y) \wedge (y\mathcal{R}x) \Rightarrow x = y.$$

Example 2.9 1. Equality in any set is reflexive, symmetrical and transitive.

2. Inclusion in $P(E)$ is reflexive, non-symmetrical, anti symmetrical and transitive.

3. In \mathbb{R} , the relation "... \leq ..." is reflexive, non-symmetrical, anti symmetrical and transitive.

4. In the plane P the relation "...is parallel to..." is reflexive, symmetrical, non-symmetrical and transitive.


2.3.2 Relation of equivalence

Definition 2.13 Let \mathcal{R} a binary relation in a set E . We say that \mathcal{R} is an equivalence relation if \mathcal{R} is reflexive, symmetrical and transitive.

Example 2.10 The following relations are easily shown to be equivalence relations.

The equality relation in any set is an equivalence relation.

In the plane P the relation "...is parallel to..." is an equivalence relation.

 Given an equivalence relation, we identify the elements that are related introducing equivalence classes.

Definition 2.14 Let be \mathcal{R} an equivalence relation in a set E . For each x in E , the subset of E defined by :

$$\dot{x} = \{y \in E, y\mathcal{R}x\}.$$

If $y \in \dot{x}$, we say that y is a representative of \dot{x} . The set of equivalence classes is called the quotient set of E by \mathcal{R} and is denoted E/\mathcal{R} .

Proposition 2.1 Let E be a set and \mathcal{R} an equivalence relation. We have the following properties:

1. $\forall x, y \in E, \dot{x} = \dot{y} \Leftrightarrow x\mathcal{R}y$.
2. $\forall x, y \in E : \dot{x} = \dot{y} \vee \dot{x} \cap \dot{y} = \emptyset$.
3. The equivalence classes form a partition of E :

$$E = \bigcup_{x \in E} \dot{x}.$$

Example 2.11 In \mathbb{R} , the relation \mathcal{T} id defined by:

$$\forall x, y \in \mathbb{R} : x\mathcal{T}y \Leftrightarrow x^2 - 1 = y^2 - 1.$$

Show that \mathcal{T} is an equivalence relation and determine the set \mathbb{R}/\mathcal{T} .

- \mathcal{T} is an equivalence relation:

1. \mathcal{R} is reflexive because because we have:

$$\forall x \in \mathbb{R} : x^2 - 1 = x^2 - 1$$

So

$$\forall x \in \mathbb{R}, x\mathcal{T}x.$$

2. \mathcal{T} is symmetrical because we have:

$$\begin{aligned}\forall x, y \in \mathbb{R}, x\mathcal{T}y &\Leftrightarrow x^2 - 1 = y^2 - 1 \\ &\Leftrightarrow y^2 - 1 = x^2 - 1 \\ &\Leftrightarrow y\mathcal{T}x.\end{aligned}$$

3. \mathcal{T} is transitive because we have:

$$\begin{aligned}\forall x, y, z \in \mathbb{R} : (x\mathcal{T}y) \wedge (y\mathcal{T}z) &\Rightarrow (x^2 - 1 = y^2 - 1) \wedge (y^2 - 1 = z^2 - 1) \\ &\Rightarrow x^2 - 1 = z^2 - 1 \\ &\Rightarrow x\mathcal{T}z.\end{aligned}$$

From 1), 2), and 3), we deduce that \mathcal{T} is an equivalence relation.

- Let's determine the quotient set \mathbb{R}/\mathcal{T} . Let $x \in \mathbb{R}$, let's find $y \in \mathbb{R}$ such that $y\mathcal{T}x$?


$$\begin{aligned}y\mathcal{T}x &\Leftrightarrow y^2 - 1 = x^2 - 1 \\ &\Leftrightarrow (y - x)(y + x) = 0 \\ &\Leftrightarrow (y = x) \vee (y = -x).\end{aligned}$$

Therefore, $\forall x \in \mathbb{R}, \dot{x} = \{x, -x\}$, consequently:

$$\mathbb{R}/\mathcal{T} = \{x, -x, x \in \mathbb{R}\}.$$

2.3.3 Order relation

Definition 2.15 Let \mathcal{R} be a binary relation in a set E . We say that \mathcal{R} is an order relation if \mathcal{R} is reflexive, antisymmetric and transitive.

 An order relation is often denoted " \leq " or " \preceq ". The pair (E, \leq) , where E is a set and \leq an order relation, is called an ordered set. The relation $x \leq y \wedge x \neq y$ is denoted $x < y$.

Definition 2.16 Let (E, \leq) be an ordered set. The relation " \leq " is said to be of total order if any two elements of E are comparable:

$$\forall x, y \in E : (x \leq y \vee y \leq x).$$

Otherwise, the order is said to be partial.

Example 2.12 Here are some examples of order relations.

- The usual " $\dots \leq \dots$ " on \mathbb{R} is a total order relation.
- If E is a set with at least two elements, inclusion in $P(E)$ is a partial order relation.
- In $\mathbb{R} \times \mathbb{R}$, the relation \mathcal{R} is defined by:

$$\forall (x, y), (x', y') \in \mathbb{R} \times \mathbb{R} : (x, y)\mathcal{R}(x', y') \Leftrightarrow \begin{cases} x \leq x' \\ \wedge \\ y \leq y'. \end{cases}$$

it's easy to show that \mathcal{R} is an order relation. The order isn't total.

For $(x, y) = (1, 2)$ and $(x', y') = (3, 1)$, we have

$$1 \leq 3 \wedge 2 \not\leq 1,$$

so

$$(1, 2) \not\mathcal{R}(3, 1),$$

we also

$$3 \not\leq 1 \wedge 1 \leq 2,$$

have

$$(3, 1) \not\mathcal{R}(1, 2).$$

Which shows that the order is not total but partial.

2.3.4 Bounded sets in \mathbb{R}

Definition 2.17 Let $M, m \in \mathbb{R}$; given a set $X \subseteq \mathbb{R}$.


- We call M an upper bound for X if

$$M \geq x; \forall x \in X.$$

- We call m a lower bound for X if

$$m \leq x; \forall x \in X.$$

- X is bounded above if there exists an upper bound of X .
- X is bounded below if there exists a lower bound of X .
- X is bounded if there exists an upper and lower bound of X .

 One can define the smallest among the upper bounds and the largest among the lower bounds.

Definition 2.18 Suppose $X \subseteq \mathbb{R}$ is bounded above. The supremum of X , written $\sup X$ is the smallest upper bound for X ; that is, $\sup X$ satisfies:

- $\sup X \geq x; \forall x \in X$.
- $\forall y \leq \sup X, \exists x \in X$ such that $x > y$.

Definition 2.19 Suppose $X \subseteq \mathbb{R}$ is bounded below. The infimum of X written $\inf X$, is the greatest lower bound for X , that is if X satisfies:

- $\inf X \leq x; \forall x \in X$.
- $\forall y > \inf X; \exists x \in X$ such that $x < y$.


Remark 2.3  $\sup X$ and $\inf X$ need not be elements of X .

Definition 2.20 Given a set $X \subseteq \mathbb{R}$, we say a is a maximum for X (denoted $\max X$) if

$$a \in X \text{ and } a \geq x; \forall x \in X,$$

and we say b is a minimum for X if

$$b \in X \text{ and } b \leq x; \forall x \in X.$$

Remark 2.4 

- Maximum and minimum do not always exist even if the set is bounded, but the **sup** and the **inf** do always exist if the set is bounded.
- If **sup** and **inf** are also elements of the set, then they coincide with **max** and **min**.

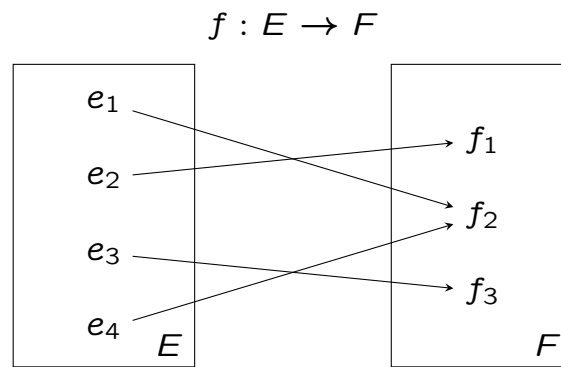
Example 2.13 Determine **min**, **max**, **sup**, **inf** of sets:

$$A =]3, 6[, B =] - 1, 3[, C = \left\{ \frac{1}{n}, n \in \mathbb{N}^* \right\}, D =] - \infty, 3[.$$

2.4 Maps

Definition 2.21 A map from a set E to a set F is any correspondence f that associates any element $x \in E$ with a single element $y \in F$.

E is called the starting set; F the ending set. The element y of F is associated with an element x of E denoted by $y = f(x)$.

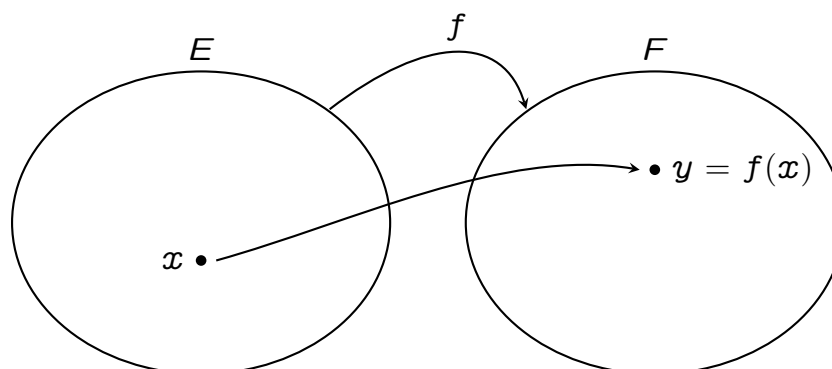


$y = f(x)$ is called the image of x and x is an antecedent of y . We write:

$$\begin{array}{ccc} f : E & \longrightarrow & F \\ x & \longrightarrow & y = f(x) \end{array}$$

Formally, a correspondence f between two non-empty sets is a map if and only if:

$$\forall x_1, x_2 \in E : x_1 = x_2 \Rightarrow f(x_1) = f(x_2).$$



Example 2.14 • The map:

$$\begin{array}{ccc} Id_E : E & \longrightarrow & E \\ x & \longrightarrow & y = x. \end{array}$$

is called an identity map on E .

- Let E and F be non-empty sets and a an element of F , then the correspondence f from E into F defined by:

$$\forall x \in E : f(x) = a.$$

is a map known as a constant map.

- Correspondence

$$\begin{array}{ccc} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & \frac{1}{x}. \end{array}$$

is not a map because the element 0 has no image in \mathbb{R} .

Definition 2.22 Two maps f and g are said to be equal if:

1. They have the same start set E and end set F .
2. $\forall x \in E : f(x) = g(x)$.

2.4.1 Direct and Reciprocal image

Direct image

Definition 2.23 Let $f : E \rightarrow F$ a map and $A \subseteq E$. The subset of F containing the image of the elements of A by f is called the direct image of A by f (or simply, the image of A by f).

$$f(A) = \{f(x) \in F, x \in A\}.$$

Formally we have:

$$\forall y \in F, (y \in f(A) \Leftrightarrow \exists x \in A, y = f(x))$$

Example 2.15 Consider the map:

$$\begin{array}{ccc} f : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & f(x) = 2 - x. \end{array}$$

$$f\left(\left[0, \frac{1}{2}\right]\right) = \left\{f(x) \in \mathbb{R}, x \in \left[0, \frac{1}{2}\right]\right\}.$$

We have:

$$\begin{aligned} 0 \leq x \leq \frac{1}{2} &\Rightarrow -\frac{1}{2} \leq -x \leq 0 \\ &\Rightarrow \frac{3}{2} \leq 2 - x \leq 2. \end{aligned}$$

Hence $f\left(\left[0, \frac{1}{2}\right]\right) = \left[\frac{3}{2}, 2\right]$.

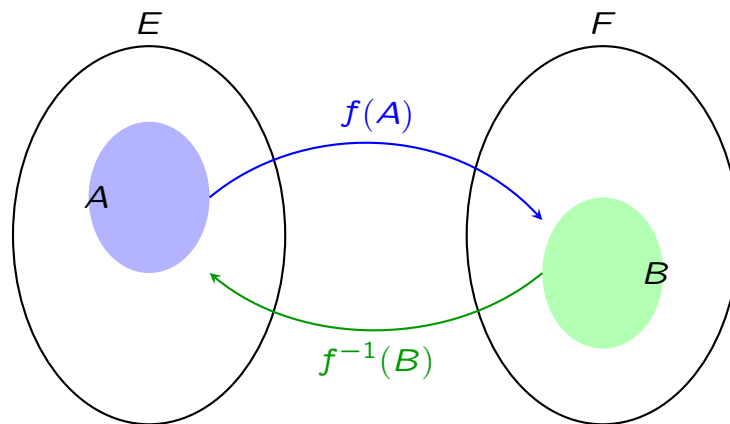
Reciprocal image

Definition 2.24 Let $f : E \rightarrow F$ be a map and $B \subseteq F$. The subset of E containing the antecedents of the elements of B is called the reciprocal image of B by f , denoted by $f^{-1}(B)$.

$$f^{-1}(B) = \{x \in E, f(x) \in B\}$$

Finally, we have:

$$\forall x \in E, x \in f^{-1}(B) \Leftrightarrow f(x) \in B.$$



Example 2.16 Consider the map

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto f(x) = (x - 1)^2. \end{aligned}$$

- $f^{-1}(0) = \{x \in \mathbb{R}, f(x) = 0\} = \{1\}$.

- $f^{-1} \left(\left] 0, \frac{1}{2} \right[\right) = \left\{ x \in \mathbb{R}, f(x) \in \left] 0, \frac{1}{2} \right[\right\}$.
Solving the inequation $0 < (x - 1)^2 < 1$ gives:

$$f^{-1} \left(\left] 0, \frac{1}{2} \right[\right) = \left] \frac{\sqrt{2}-1}{\sqrt{2}}, 1 \right[\cup \left] 1, \frac{\sqrt{2}+1}{\sqrt{2}} \right[.$$

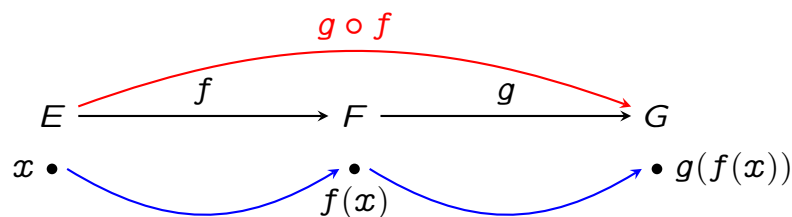
Proposition 2.2 Let $f : E \rightarrow F$ be a map $A, B \subset E$ and $M, N \subset F$. We have:

1. $A \subset B \Rightarrow f(A) \subset f(B)$.
2. $f(A \cup B) = f(A) \cup f(B)$.
3. $f(A \cap B) \subset f(A) \cap f(B)$.
4. $M \subset N \Rightarrow f^{-1}(M) \subset f^{-1}(N)$.
5. $f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N)$.
6. $f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N)$.

2.4.2 Composition of maps

Definition 2.25 Let $f : E \rightarrow F$ and $g : F \rightarrow G$ be two maps. The compound of the maps f and g is the map $g \circ f$ defined from E into G by:

$$\forall x \in E, g \circ f(x) = g(f(x)).$$



Example 2.17 Given the following map:

$$\begin{aligned}f, g : \mathbb{R} &\rightarrow \mathbb{R} \\x &\mapsto f(x) = x^2; \\x &\mapsto g(x) = 3x - 2.\end{aligned}$$

Then we have:

$$f \circ g : \mathbb{R} \rightarrow \mathbb{R}; (f \circ g)(x) = f[g(x)] = f(3x - 2) = (3x - 2)^2.$$

$$(g \circ f)(x) = g[f(x)] = g(x^2) = 3x^2 - 2.$$

Clearly: $(f \circ g)(x) \neq (g \circ f)(x)$.

Definition 2.26 Let $f : E \rightarrow F$ be a bijective map of E in F . We then define a map from F to E by associating any element y of F with its only antecedent. This map is called the reciprocal map of f and noted f^{-1} , thus verifies:

$$\forall x \in E, \forall y \in F, y = f(x) \Leftrightarrow x = f^{-1}(y).$$

2.4.3 Injection, Surjection, Bijection

Let E and F be two sets and f a map from E into F .

Injection

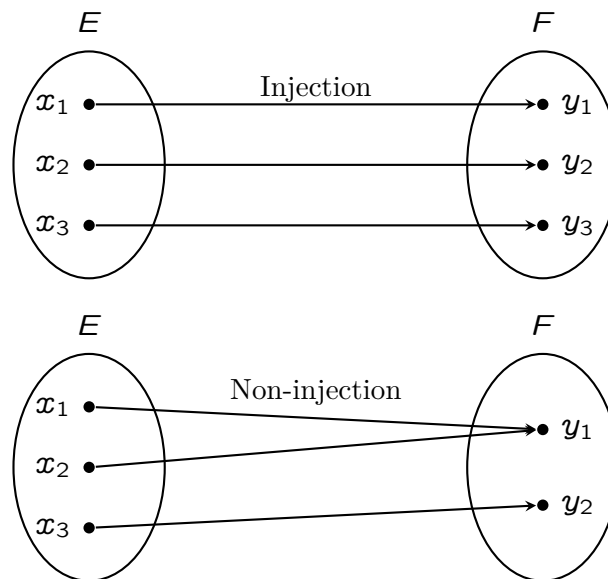
Definition 2.27 The map f is said to be injection if and only if:

$$\forall x, x' \in E, f(x) = f(x') \Rightarrow x = x'.$$

or by taking the contrapositive of the implication,

$$\forall x, x' \in E, x \neq x' \Rightarrow f(x) \neq f(x').$$

i-e: f is injective if and only if every element y of F has at most one antecedent.



Example 2.18 Consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$\forall x \in \mathbb{R}, f(x) = 2x + 1.$$

Is f injectif?.

Let x_1 and x_2 be two real numbers such that $f(x_1) = f(x_2)$.

$$f(x_1) = f(x_2) \Leftrightarrow 2x_1 + 1 = 2x_2 + 1 \Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2.$$

Thus f is injectif.

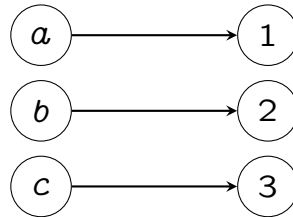
Surjection

Definition 2.28 The map f is said to be surjective if and only if:

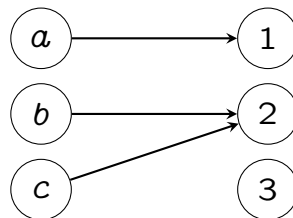
$$\forall y \in F, \exists x \in E, y = f(x),$$


i.e. f is surjective if and only if every element y of F has at least one antecedent in E .

Surjection



Non-surjection



Remark 2.5  The map f is surjective if and only if the equation $y = f(x)$ admits at least one solution x of E for any element y of F .

Example 2.19 Example (2.18) is repeated. Is f surjective?

Let $y \in \mathbb{R}$, solve the equation $y = f(x)$.

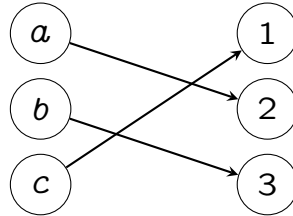
$$y = f(x) \Leftrightarrow y = 2x + 1 \Leftrightarrow y - 1 = 2x \Leftrightarrow x = \frac{y - 1}{2}.$$

Clearly, the expression $\frac{y - 1}{2}$ is defined for any real y . Thus f is surjective.

Bijection


Definition 2.29 The map f is said to be bijective if and only if it is both surjective and injective.

Bijection



Proposition 2.3 The map f is bijective if and only if every element y of F has a unique antecedent x of f in E :

$$\forall y \in F, \exists! x \in E : y = f(x).$$

 When a map is bijective, it is possible to introduce the notion of reciprocal map.

Definition 2.30 Let $f : E \rightarrow F$ be a bijective map of E in F . We then define a map from F to E by associating any element y of F with its only antecedent. This map is called the reciprocal map of f and noted f^{-1} , thus verifies:

$$\forall x \in E, \forall y \in F, y = f(x) \Leftrightarrow x = f^{-1}(y).$$

Or even:

$$\begin{aligned} f : E &\rightarrow F \\ x &\mapsto f(x) = y \end{aligned}$$

Then

$$\begin{aligned} f^{-1} : F &\rightarrow E \\ y &\mapsto x = f^{-1}(y). \end{aligned}$$

Example 2.20 1. If E is a set, Id_E is bijective and $Id_E^{-1} = Id_E$.

2. The map $f : \mathbb{R} \rightarrow \mathbb{R}_+^*$ defined by $f(x) = e^x$ is bijective and its reciprocal bijection is $f^{-1} : \mathbb{R}_+^* \rightarrow \mathbb{R}$ defined by $f^{-1}(y) = \ln y$.
3. Considering the map f in Example (2.18). We showed earlier that it is injective and surjective, so it is a bijection. Its reciprocal map is:

$$f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto f^{-1}(x) = x - \frac{1}{2}.$$

Proposition 2.4 Let E, F be sets and $f : E \rightarrow F$ a map.

1. The map f is bijective if and only if there exists a map $g : F \rightarrow E$ such that $f \circ g = Id_F$ and $g \circ f = Id_E$.
2. If f is bijective, then the map g is unique and also bijective. The map g is called the reciprocal bijection of f and is denoted f^{-1} . Furthermore, $(f^{-1})^{-1} = f$.

Proposition 2.5 Let $f : E \rightarrow F$ and $g : F \rightarrow G$ be a bijective map. The map $g \circ f$ is bijective and its reciprocal bijection is

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proposition 2.6 Let E, F, G be three sets, $f : E \rightarrow F$ and $g : F \rightarrow G$ two maps.

1. If f and g are injective, then $g \circ f$ is injective.
2. If f and g are surjective, then $g \circ f$ is surjective.
3. If $g \circ f$ is injective, then f is injective.
4. If $g \circ f$ is surjective, then g is surjective.

Chapter 3

Real functions of a real variable

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Guillaume François Antoine, Marquis de l'Hôpital; sometimes spelled L'Hospital; 7 June 1661 – 2 February 1704) was a French mathematician. His name is firmly associated with l'Hôpital's rule for calculating limits involving indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Although the rule did not originate with l'Hôpital, it appeared in print for the first time in his 1696 treatise on the infinitesimal calculus. This book was a first systematic exposition of differential calculus. Several editions and translations to other languages were published and it became a model for subsequent treatments of calculus.



3.1 Definitions and properties

Definition 3.1 Let $X \subset \mathbb{R}$, we call a real function of a real variable defined on the set X , any map f from X into \mathbb{R} , which to each point $x \in X$ maps a single element $y \in \mathbb{R}$, we note :

$$f : D_f \rightarrow \mathbb{R}$$

$$x \mapsto y = f(x).$$

D_f is domain the definition set of f .

↔ Graph of a function (representative curve):


The graph of a function $f : X \rightarrow \mathbb{R}$ is the set $\Gamma(f)$ defined as follows:

$$\Gamma(f) = \{(x, f(x)) / x \in D_f, \Gamma(f) \subset \mathbb{R}^2\}.$$

↔ Even and odd function: Let $f : D_f \rightarrow \mathbb{R}$, $D_f \subset \mathbb{R}$ then

1. f is said to be even $\Leftrightarrow \forall x, -x \in D_f; f(-x) = f(x)$.
2. f is said to be odd $\Leftrightarrow \forall x, -x \in D_f \in; f(-x) = -f(x)$.

3. f is said to be periodic $\Leftrightarrow, \forall x, (x + T) \in D_f; f(x) = f(x + T)$.

 The smallest positive value of T is called the period of f .

↔ Bounded functions:

First, let's look at the concepts of major and minor functions: if $f : I \rightarrow \mathbb{R}$, then $f(I)$ is the set of all values of f , then :

↔ The function f is said to be majored on $I \Leftrightarrow f(x)$ is majored, i.e. $\exists M \in \mathbb{R} : f(x) \leq M, \forall x \in I$.

↔ The function f is said to be minored on $I \Leftrightarrow f(x)$ is minored, i.e.

$$\exists m \in \mathbb{R} : f(x) \geq m, \forall x \in I.$$

↔ The function f is said to be bounded on $I \Leftrightarrow f(x)$ is both increased and decreased, i.e.

$$\exists m, M \in \mathbb{R} : m \leq f(x) \leq M, \forall x \in I.$$

Example 3.1

□ Cosine is an even function on $\mathbb{R} : \cos(-x) = \cos(x), \forall x \in \mathbb{R}$.

□ Sine is an odd function on \mathbb{R} , indeed: $\sin(-x) = -\sin(x), \forall x \in \mathbb{R}$.

□ $\sin(x)$ is a bounded function on \mathbb{R} , in fact $\forall x \in \mathbb{R} : -1 \leq \sin(x) \leq 1$ in other words: $\forall x \in \mathbb{R} : |\sin(x)| \leq 1$.

↔ Monotonic functions :

- f is increasing if : $\forall x, y \in I, x \leq y \Rightarrow f(x) \leq f(y)$;
- f is strictly increasing if : $\forall x, y \in I, x < y \Rightarrow f(x) < f(y)$;
- f is decreasing if : $\forall x, y \in I, x \leq y \Rightarrow f(x) \geq f(y)$;
- f is strictly decreasing if : $\forall x, y \in I, x < y \Rightarrow f(x) > f(y)$.

3.2 Limit of a function

Theorem 3.1 The limit of $f(x)$ at point x_0 , if it exists, is unique.

3.2.1 Properties

1. If $\lim_{x \rightarrow x_0^+} f(x) = L_1 \wedge \lim_{x \rightarrow x_0^-} f(x) = L_2 \wedge L_1 \neq L_2$, The function $f(x)$ has no limit at the point x_0 .

$$2. \text{ If } \begin{cases} f(x) \geq 0 \vee f(x) > 0 \\ \wedge \\ \lim_{x \rightarrow x_0} f(x) = L \end{cases} \Rightarrow L \geq 0$$

$$3. \text{ If } \begin{cases} f(x) \geq g(x) \vee f(x) > g(x) \\ \wedge \\ \lim_{x \rightarrow x_0} f(x) = L_1 \wedge \lim_{x \rightarrow x_0} g(x) = L_2 \end{cases} \Rightarrow L_1 \geq L_2.$$

$$4. \text{ If } \begin{cases} f(x) \leq h(x) \leq g(x) \vee f(x) < h(x) < g(x) \\ \wedge \\ \lim_{x \rightarrow x_0} f(x) = L_1 \wedge \lim_{x \rightarrow x_0} g(x) = L_2 \end{cases} \Rightarrow L_1 \leq \lim_{x \rightarrow x_0} h(x) \leq L_2.$$

$$5. \text{ If } \begin{cases} f(x) \leq h(x) \leq g(x) \\ \wedge \\ \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = L \end{cases} \Rightarrow \lim_{x \rightarrow x_0} h(x) = L.$$

6. If $f(x) \geq g(x) \lim_{x \rightarrow +\infty} g(x) = +\infty \Rightarrow \lim_{x \rightarrow +\infty} f(x) = +\infty$.

7. If $f(x) \leq g(x) \lim_{x \rightarrow +\infty} g(x) = -\infty \Rightarrow \lim_{x \rightarrow -\infty} f(x) = -\infty$.

Example 3.2 Calculate $\lim_{x \rightarrow +\infty} e^{-x} \cos x$, $\lim_{x \rightarrow \pm\infty} x + \sin x$.

$$\text{We have } \begin{cases} \lim_{x \rightarrow +\infty} e^{-x} = 0 \\ -1 \leq \cos x \leq 1 \Rightarrow -e^{-x} \leq e^{-x} \cos x \leq e^{-x} \end{cases} \Rightarrow \lim_{x \rightarrow +\infty} e^{-x} \cos x = 0$$

$$\text{We have } -1 \leq \sin x \leq 1 \Rightarrow -1 + x \leq \sin x \leq 1 + x.$$

Since $\lim_{x \rightarrow +\infty} (x - 1) = +\infty \wedge \lim_{x \rightarrow -\infty} (x + 1) = -\infty$,

then $\lim_{x \rightarrow -\infty} (x + \sin x) = -\infty$, $\lim_{x \rightarrow +\infty} (x + \sin x) = +\infty$.

3.2.2 Indeterminate form

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 \times \infty, -\infty + \infty, 1^\infty, 0^0, 0^\infty, \infty^0, \dots$$

3.2.3 Comparison of functions near a point x_0

Theorem 3.2 Let $f(x)$ and $g(x)$ be two functions defined in the near of the point x_0 .

1. We say that $f(x)$ is negligible in front of $g(x)$ in the near of the point x_0 if and only if $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$, we denote this by the symbol o and write it in the form $f = o(g)$.
2. The function $f(x)$ is said to be equivalent to the function $g(x)$ in the near of the point x_0 if and only if $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$, we denote this by the symbol \sim and write $g \sim f$.

3.3 Continuous functions

3.3.1 Continuity at one point

Definition 3.2 let f be a function defined at point x_0 of \mathbb{R} . We say that f is continuous at the point x_0 if and only if f is continuous to the left and the right of x_0 i.e

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) = f(x_0).$$

3.3.2 Continuity on an interval

Definition 3.3 A function f defined on an interval I is said to be continuous on this interval if it is continuous at any point $x_0 \in I$. The set of continuous functions on I is then denoted $C(I)$.

Remark 3.1



① When we study the continuity of a function on an interval I , at the limit points, we study continuity on the right or the left only.

② If the interval $I = [a, b]$, we don't study continuity at point b and neither to the right of point b nor to the left of point b . but at point a , we study the continuity of f to the right of point a when f is defined at this point and does not study continuity either at the point a or to the left of the point a , because f is not defined to the left of this point.

3.3.3 Properties of continuity

Each continuous function f on a closed interval $[a, b]$:

1. A function f is bounded on a interval $[a, b]$.
2. f has a lower bound and upper bound i.e :
 1. $f([a, b]) = [m, M]$.
 2. $\exists x_1 \in [a, b] : f(x_1) = \max f(x) = M; \exists x_2 \in [a, b] : f(x_2) = \min f(x) = m$.

3.3.4 Extension by continuity

Definition 3.4 Let f defined on interval $I \setminus \{x_0\}$ and $\lim_{x \rightarrow x_0} f(x) = \alpha$, we say that \tilde{f} is a extension by continuity of a function f at a point x_0 if :

$$\tilde{f}(x) = \begin{cases} f(x); \forall x \in I \setminus \{x_0\} \\ \lim_{x \rightarrow x_0} f(x) = \alpha; x = x_0 \end{cases}$$

Example 3.3 Let $f(x) = \frac{\sin x}{x}$ defined on \mathbb{R}^* , does the function f accept extension by continuity at a point $x_0 = 0$?

We have $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, then the function f accepts an extension by continuity \tilde{f} at a point $x_0 = 0$:

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x}; x \neq 0, \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; x = 0. \end{cases}$$

3.3.5 Intermediate value theorem

We use this theorem to solve equations of the form $f(x) = 0$. if f is a continuous function defined on the interval $[a, b]$, and $f(a) \times f(b) < 0$ then there exists a number c in the interval $]a; b[$ such that $f(c) = 0$.

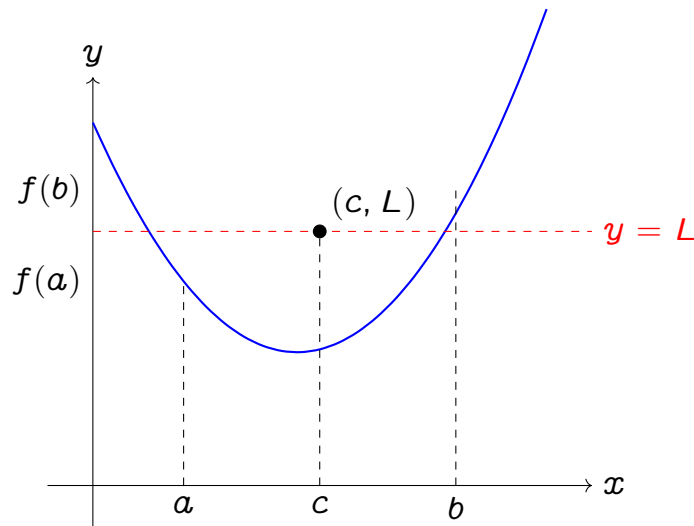


Figure 3.1: Intermediate value theorem

Remark 3.2



If f is also strictly monotone on $[a, b]$, then c is unique.

Example 3.4 Let's show that the equation $x^3 + x^2 - x - 5 = 0$ has at least one solution in $[-1, 2]$. Apply the intermediate value theorem (I.V.T) to the function $f(x) = x^3 + x^2 - x - 5$, on $[-1, 2]$. f is polynomial, so it is continuous on \mathbb{R} , especially on $[-1, 2]$. We can easily check that $f(-1)f(2) < 0$. The (I.V.T) ensures $:\exists c \in]-1, 2[/ f(c) = 0$.

3.4 Inverse functions

Let I an interval and $f : I \rightarrow \mathbb{R}$ is a continuous and strictly monotonic function, then there is a continuous inverse function $f^{-1} : f(I) \rightarrow \mathbb{R}$.

Remark 3.3



• If $f : I \rightarrow J$ is a bijective function then :

1. $\forall x \in I : f^{-1} \circ f(x) = x$.
2. $\forall y \in J : f \circ f^{-1}(y) = y$.
3. $f^{-1} \circ f = f \circ f^{-1} = Id$.

• The graph of a functions f and f^{-1} are symmetrical about the first bisector ($y = x$).

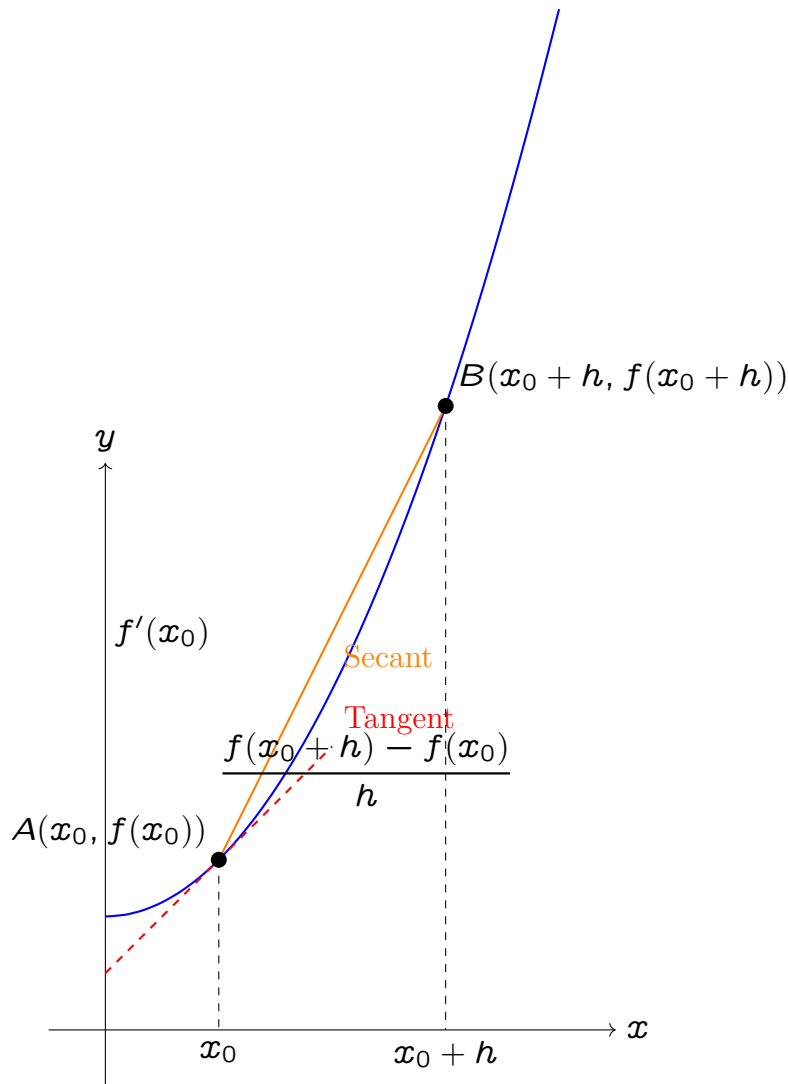


Figure 3.2: Geometric interpretation of the derivative number

3.5 Derivative functions

Definition 3.5 Let $f : I \rightarrow R, I \subset R$. f is derivable at x_0 if and only if:

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = l = f'(x_0).$$

exists and is finite.

This limit is called the "derivative of f at the point x_0 ", denoted $f'(x_0)$.

3.5.1 Geometric interpretation of the derivative number

Proposition 3.1 ① f is derivable on the right at the point x_0 if : $\lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$

exists and is finite, then $\lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f'_r(x_0)$ (right derivative).

② f is derivable on the left at the point x_0 if : $\lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite,

then $\lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = f'_l(x_0)$ (left derivative).

Remark 3.4



① f is derivable at $x_0 \Leftrightarrow f'_l(x_0) = f'_r(x_0) = f'(x_0)$.

② f is derivable at an interval I if it is derivable at each point of the interval I .

③ if f is not continuous at point x_0 then it is not derivable at that point.

Example 3.5 Let $f(x) = |x|$, defined on \mathbb{R} . Let's study its derivability at the point $x_0 = 0$.

Recall that

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Calculating the limits gives : $f'_r(x_0) = \lim_{x \rightarrow 0^+} \frac{x - 0}{x - 0} = 1$, $f'_l(x_0) = \lim_{x \rightarrow 0^-} \frac{-x - 0}{x - 0} = -1 \Rightarrow f'_d(0) \neq f'_l(0)$, then f is not derivable at the 0 point.

3.5.2 Derivative of a composite function

Let $f : I \rightarrow J$ and $g : J \rightarrow \mathbb{R}$, f be derivable at $x_0 \in I$, and g be derivable at $f(x_0)$,

Then

$$(g \circ f)'(x_0) = f'(x_0)g'(f(x_0)).$$

3.5.3 The derivative of an inverse function

Let $f : I \rightarrow J$ is a bijective function, $x_0 \in I$, $y_0 = f(x_0) \in J$, f^{-1} is derivable at point y_0 if and only if the function f is derivable at point x_0 , $f'(x_0) \neq 0$, f^{-1} is continuous at



point y_0 and

$$(f^{-1})'(y_0) = \frac{1}{(f' \circ f^{-1})(y)} = \frac{1}{f'(x_0)}; x_0 = f^{-1}(y_0).$$

Example 3.6 Calculate the derivative of the function e^x is an inverse function of $f(x) = \ln x$. We put : $y = f(x) = \ln x$, we have : $(\exp y)' = (f^{-1})'(y) = (f^{-1})'f(x) = \frac{1}{f'(x)} = \frac{1}{1/x} = x = \exp y$.

3.5.4 Class C^n function

Definition 3.6 A function f , defined on I , is said to be of class $C^1(I)$, if it is derivable on I and f' is continuous on I . f is said to be of class $C^n(I)$ or n times continuously derivable on I , if f is n times differentiable on I and $f^{(n)}$ is continuous on the same interval.

3.5.5 Fundamental theorems about differentiable functions

Theorem 3.3 (Role's theorem) Let be $f : [a, b] \rightarrow \mathbb{R}$, be a function satisfying the following conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$.
3. $f(a) = f(b)$.

Then $\exists c \in]a, b[$ such that $f'(c) = 0$.

Geometric interpretation

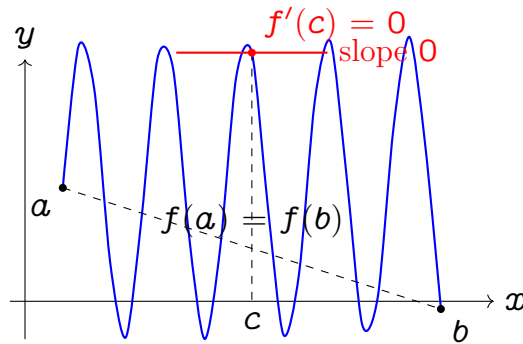
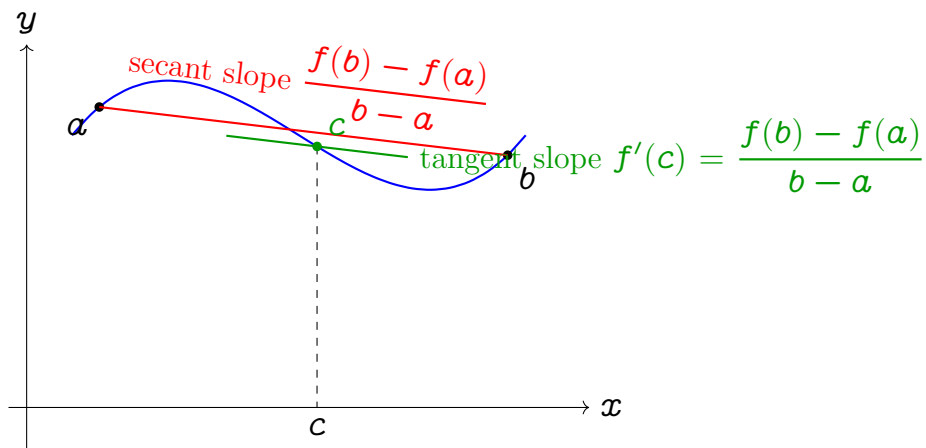


Figure 3.3: Geometric interpretation of Rolle's theorem

Theorem 3.4 Finite increase theorem: Let $f[a, b] \rightarrow \mathbb{R}$ be a function satisfying :

1. f is continuous on $[a, b]$.
2. f is derivable on $]a, b[$;

then : $\exists c \in]a, b[$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.




Finite Increase Theorem: $\exists c \in]a, b[$ with $f'(c) = \frac{f(b) - f(a)}{b - a}$

Figure 3.4: Geometric interpretation of finite increase theorem

Example 3.7 Using the Finite Increase Theorem for the function

$f(t) = \ln(1 + t); \forall t \in [0, x] \ x > 0$, to prove that

$$\frac{x}{1+x} < \ln(1+x) < x.$$

 We have $f(t)$ defined and continuous on $[0, x]$, and derivable on $]0, x[$ then $\exists c \in]0, x[$:

$$\begin{aligned} f(x) - f(0) &= f'(c)(x - 0) \Rightarrow \ln(1+x) = xf'(c); f'(c) = \frac{1}{1+c} \\ &\Rightarrow \ln(1+x) = \frac{x}{1+c}, c \in]0, x[\\ &\Rightarrow 0 < c < x \\ &\Rightarrow 1 < c+1 < 1+x \\ &\Rightarrow \frac{1}{1+x} < \frac{1}{1+c} < 1 \\ &\Rightarrow \frac{x}{1+x} < \frac{x}{1+c} < x \\ &\Rightarrow \frac{x}{1+x} < \ln(1+x) < x. \end{aligned}$$

Then the estimation is verified.

Theorem 3.5 Generalized finite increment theorem Let $f : I \rightarrow \mathbb{R}$, $g : [a, b] \rightarrow \mathbb{R}$ two functions are defined and continuous on interval $[a, b]$, and are derivable on interval $]a, b[$ if $g'(x) \neq 0$ then

$$\exists \alpha \in]a, b[: \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Hopital's rule

Let f, g two function derivable at au point x_0 and $\lim_{x \rightarrow x_0} \frac{f'(c)}{g'(c)}$ exist, then :

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}.$$

Example 3.8

$$\textcircled{1} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1.$$

$$\textcircled{2} \lim_{x \rightarrow +\infty} x e^{-x} = \lim_{x \rightarrow +\infty} \frac{x}{e^x} \stackrel{H}{=} \lim_{x \rightarrow +\infty} \frac{1}{e^x} = 0.$$

$$\textcircled{3} \lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{-x^2}{x} = 0.$$

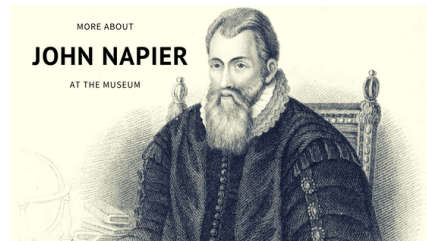
Chapter 4

Application to Elementary Functions

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John Napier (1550–1617), a Scottish mathematician, is renowned for inventing logarithms, which significantly simplified complex calculations like multiplication and division. He introduced this concept to reduce the burden of tedious arithmetic operations, particularly those encountered in astronomy and navigation. Napier’s work also included the popularization of decimal notation and the development of “Napier’s Bones,” a calculating tool.



4.1 Exponential Function, Logarithmic Function, and the General Exponential Function a^x with $a > 0$

After introducing real-valued functions and defining essential concepts and theorems, we now turn to the study of standard functions, some of which students may have encountered in secondary school. We focus on definitions, key properties, and graphical representations.

4.1.1 The Exponential Function

The exponential function is defined over \mathbb{R} and denoted by \exp . Its derivative is equal to the function itself:

$$(\exp(x))' = \exp(x)$$

It takes the value 1 at $x = 0$, and the value of the exponential function at $x = 1$ is denoted by e . This number e , known as the base of the natural exponential function, has the approximate value $e \approx 2.71828$. It is also called “Euler’s constant”¹ or “Napier’s constant”².

We also write:

$$\forall x \in \mathbb{R}, \quad \exp(x) = e^x$$

¹Leonhard Euler: A Swiss mathematician, physicist, astronomer, logician, and engineer who made important and influential discoveries in most fields of mathematics such as calculus and graph theory. He also contributed to topology and analytic number theory.

²John Napier: Born in 1550 near Edinburgh, Scotland, and died on April 14, 1617. He was a mathematician, physicist, and astronomer known for his work on logarithms.

Proposition 4.1 (Algebraic Properties of the Natural Exponential Function) For all real numbers x, y and all integers n :

1. $e^x > 0$.
2. $e^{x+y} = e^x \cdot e^y$.
3. $(e^x)^n = e^{nx}$.
4. $e^{-x} = \frac{1}{e^x}$.
5. $e^{x-y} = \frac{e^x}{e^y}$.

Proposition 4.2 (Limits and Graph) The exponential function is defined on $(-\infty, +\infty)$ and:

1. $\lim_{x \rightarrow +\infty} e^x = +\infty$.
2. $\lim_{x \rightarrow -\infty} e^x = 0$.

Its derivative is strictly positive over its domain, so the exponential function is strictly increasing.

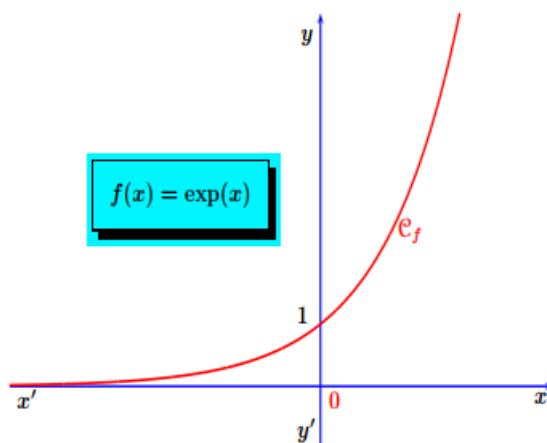


Figure 4.1: The graph of the function $\exp x$

Theorem 4.1 (Functions of the Form $e \circ u$) Let u be a real-valued function. Then:

1. The function is defined as:

$$(e \circ u)(x) = e^{u(x)}, \quad \forall x \in \mathcal{D}_u.$$

2. Derivative:

$$(e \circ u)'(x) = u'(x) \cdot e^{u(x)}, \quad \forall x \in \mathcal{D}_u.$$

3. The functions $e \circ u$ and u share the same monotonicity on each subinterval of \mathcal{D}_u .

4. Sign:

$$(e \circ u)(x) > 0, \quad \forall x \in \mathcal{D}_u.$$

4.1.2 The Natural Logarithmic Function

The natural logarithm is the inverse function of the exponential function, defined on the interval $(0, +\infty)$, and vanishes at $x = 1$. It is denoted by \ln .

Proposition 4.3 (Algebraic Properties of the Natural Logarithm) For all $x, y \in (0, +\infty)$ and $r \in \mathbb{Q}$:

1. $\ln(xy) = \ln(x) + \ln(y)$.

2. $\ln(x^r) = r \ln(x)$.

3. $\ln\left(\frac{1}{x}\right) = -\ln(x)$.

4. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$.

Theorem 4.2 (Limits and Graph) The natural logarithm is defined on $(0, +\infty)$, and:

1. $\lim_{x \rightarrow +\infty} \ln(x) = +\infty.$
2. $\lim_{x \rightarrow 0^+} \ln(x) = -\infty.$

The function is continuous and differentiable on its domain with:

$$(\ln(x))' = \frac{1}{x}.$$

Thus, \ln is strictly increasing.

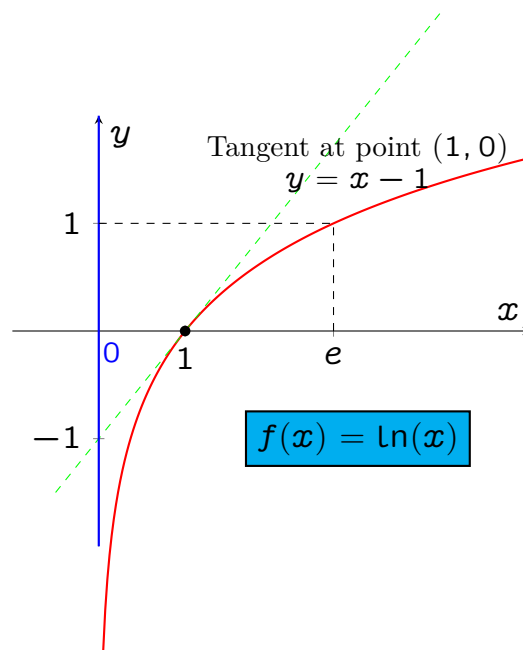


Figure 4.2: The graph of the function $\ln x$

Proposition 4.4 (Functions of the Form $\ln \circ u$) Let u be a real-valued function. Then:

1. Defined by:

$$(\ln \circ u)(x) = \ln(u(x)), \quad \forall x \in \mathcal{D}_u \text{ such that } u(x) > 0.$$

2. Derivative:

$$(\ln \circ u)'(x) = \frac{u'(x)}{u(x)}, \quad \forall x \in \mathcal{D}_{\ln \circ u}.$$

3. The functions $\ln \circ u$ and u have the same monotonicity on their domain.

4. Sign:

- (a). If $u(x) > 1$, then $\ln(u(x)) > 0$.
- (b). If $u(x) = 1$, then $\ln(u(x)) = 0$.
- (c). If $0 < u(x) < 1$, then $\ln(u(x)) < 0$.

4.1.3 The General Exponential Function a^x with $a > 0$

The exponential function with base $a > 0$, denoted by a^x , maps each real number x to a^x . It can also be defined using the natural exponential and logarithmic functions³ as:

$$a^x = e^{x \ln(a)}.$$

Proposition 4.5 For all $x \in \mathbb{R}$:

- 1. $\exp_a(x) = a^x$.
- 2. $a^x = e^{x \ln(a)}$.
- 3. $1^x = 1$.
- 4. $a^x > 0$.

³Note: When the domain is restricted to natural numbers, this defines a geometric sequence with base a and first term 1.

Proposition 4.6 (Study of the Function $f(x) = a^x$)

1. Domain: $\mathcal{D}_f = \mathbb{R} = (-\infty, +\infty)$.
2. Derivative:

$$f'(x) = \frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln(a)}) = \ln(a) \cdot a^x.$$

Hence:

- If $0 < a < 1$, then a^x is strictly decreasing.
 - If $a > 1$, then a^x is strictly increasing.
3. Limits:

- If $a > 1$:

$$\lim_{x \rightarrow +\infty} a^x = +\infty, \quad \lim_{x \rightarrow -\infty} a^x = 0.$$

- If $0 < a < 1$:

$$\lim_{x \rightarrow +\infty} a^x = 0, \quad \lim_{x \rightarrow -\infty} a^x = +\infty.$$

Below is the graph of the functions $f(x) = e^x$ (black), $h(x) = 10^x$ (red), and $g(x) = \left(\frac{1}{2}\right)^x$ (green):

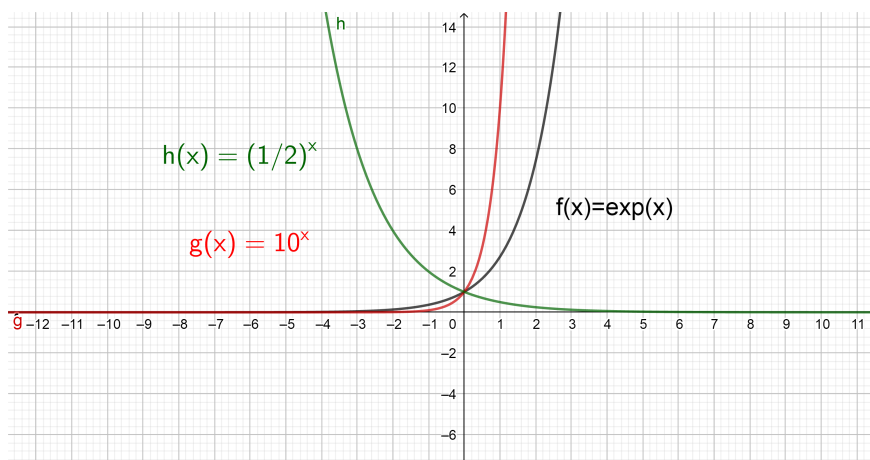


Figure 4.3: Graph of the real-based function

4.2 Hyperbolic functions

Let $x \in \mathbb{R}$, we define the functions :

1. hyperbolic cosine: $Shx = \frac{e^x - e^{-x}}{2}; x \in \mathbb{R}$.
2. hyperbolic sine: $Chx = \frac{e^x + e^{-x}}{2}; x \in \mathbb{R}$.
3. hyperbolic tangent: $Thx = \frac{Shx}{Chx} = \frac{e^{2x} - 1}{e^{2x} + 1}; x \in \mathbb{R}$.
4. hyperbolic cotangent: $Cothx = \frac{Chx}{Shx} = \frac{1}{Thx} = \frac{e^{2x} + 1}{e^{2x} - 1}; x \neq 0$.

Proposition 4.7 1. $Sh(-x) = -Sh(x)$.

2. $Ch(-x) = Ch(x)$.

3. $Chx + Shx = e^x$.

4. $Ch^2x - Sh^2x = 1$.

5. $Sh(x + y) = Shx.Chy + Chx.Shy$.

6. $Ch(x + y) = Chx.Chy + Shx.Shy$.

7. $1 - Th^2x = \frac{1}{Ch^2x}$.

4.2.1 Study of variations

Shx function

1. Definition domain : $D_f = \mathbb{R} =]-\infty, +\infty[$.
2. Shx is an odd function.
3. $(Shx)' = Chx > 0 \Rightarrow Shx$ is increasing on \mathbb{R} .
4. Limits :

x	$-\infty$	$+\infty$
$f'(x)$	+	
Shx	$-\infty$	$+\infty$

Chx function

1. definition domain : $D_f = \mathbb{R} =]-\infty, +\infty[$.
2. **Chx** is odd an function.
3. $(Chx)' = Shx > 0 \Rightarrow Chx$ is increasing on \mathbb{R}_+ and decreasing on \mathbb{R}_- .
4. Limits :

x	$-\infty$	0	$+\infty$
$f'(x)$	-	0	+
Chx	$+\infty$	1	$+\infty$

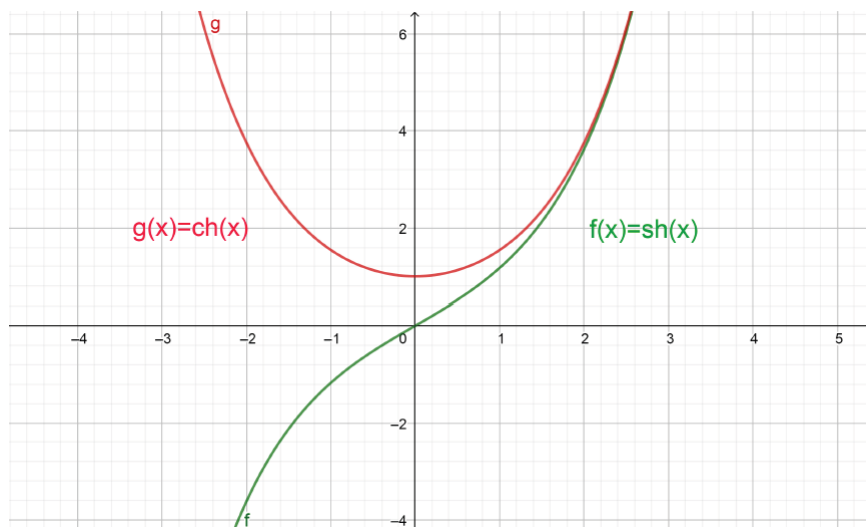


Figure 4.4: The graph of the functions **sh** and **ch**

Thx function

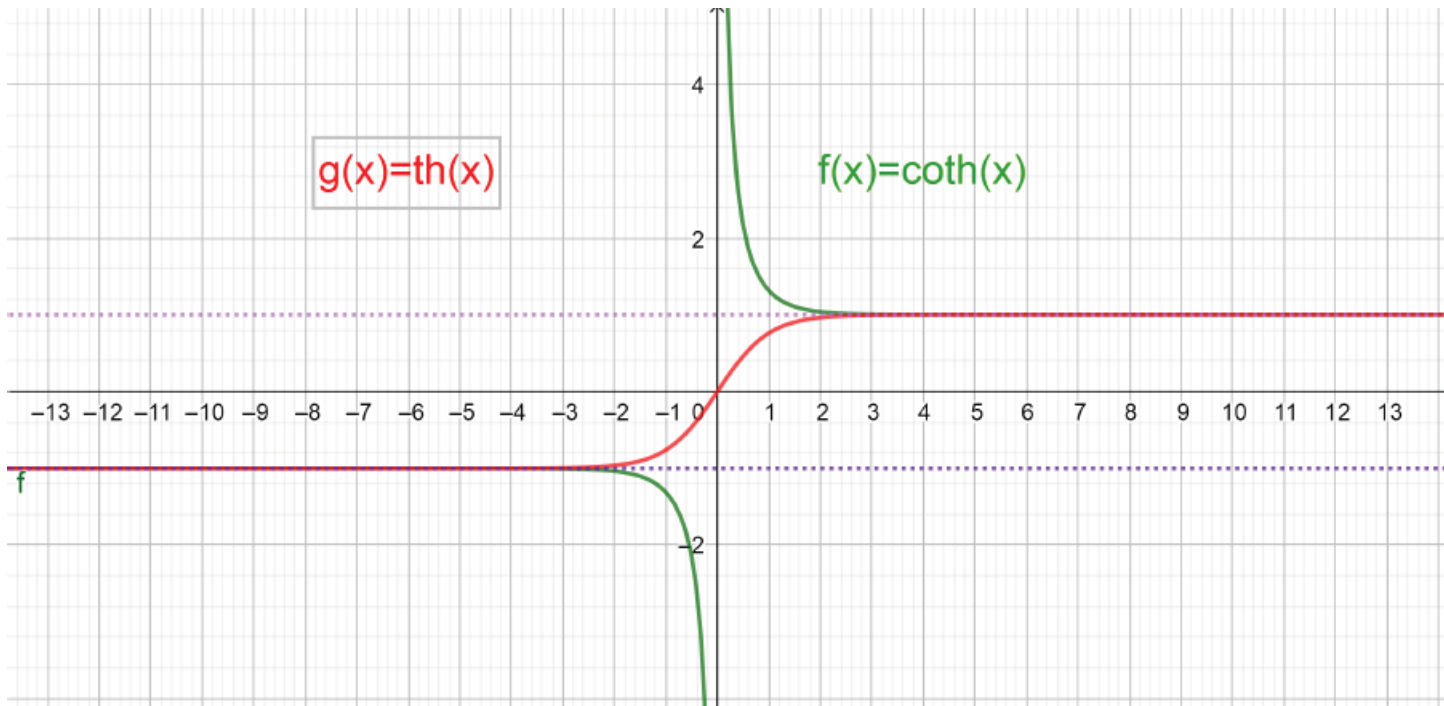
1. definition domain : $D_f = \mathbb{R} =]-\infty, +\infty[$.
2. *Thx* is an odd function.
3. $(Thx)' = Thx > 0 \Rightarrow Thx$ is increasing on \mathbb{R} .
4. Limits :

x	$-\infty$	$+\infty$
$f'(x)$	+	
<i>Thx</i>	-1	+1

Cothx function

1. Definition domain : $D_f = \mathbb{R}^+$.
2. *Cothx* is odd function.
3. $(Cothx)' = \frac{1}{Thx} = -\frac{1}{Sh^2x} < 0 \Rightarrow Cothx$ is decreasing on \mathbb{R}^+ .
4. Limits :

x	$-\infty$	0	$+\infty$
$f'(x)$	-	-	-
<i>Cothx</i>	-1	+∞	1

Figure 4.5: The graph of the functions th and $coth$

4.3 Inverse Hyperbolic Functions

The most common notation for naming inverse hyperbolic functions is to add the prefix *ar*- (from the English word *area*, since the inverse corresponds to the area of a hyperbolic sector).

Example: *arsh*, *arch*.

Other authors prefer the notation *argsh*, *argch*, *argth* (where the prefix *arg*- is an abbreviation of the Latin word *argumentum*, meaning "argument"). This notation in Arabic corresponds to "argument of the hyperbolic sine", "argument of the hyperbolic cosine", etc.⁴.

Inverse of the Hyperbolic Sine sh : the function *arsh*

The hyperbolic sine function $sh = \sinh$ is continuous and strictly increasing on \mathbb{R} , so it admits an inverse denoted *arsh* defined by

$$y = \operatorname{arsh}(x) \iff x = \sinh(y), \quad x \in \mathbb{R}, y \in \mathbb{R}.$$

⁴Some authors also use the simple notation th^{-1} , ch^{-1} , sh^{-1} , ...

Its explicit formula is

$$\operatorname{arsh}(x) = \ln(x + \sqrt{x^2 + 1}).$$

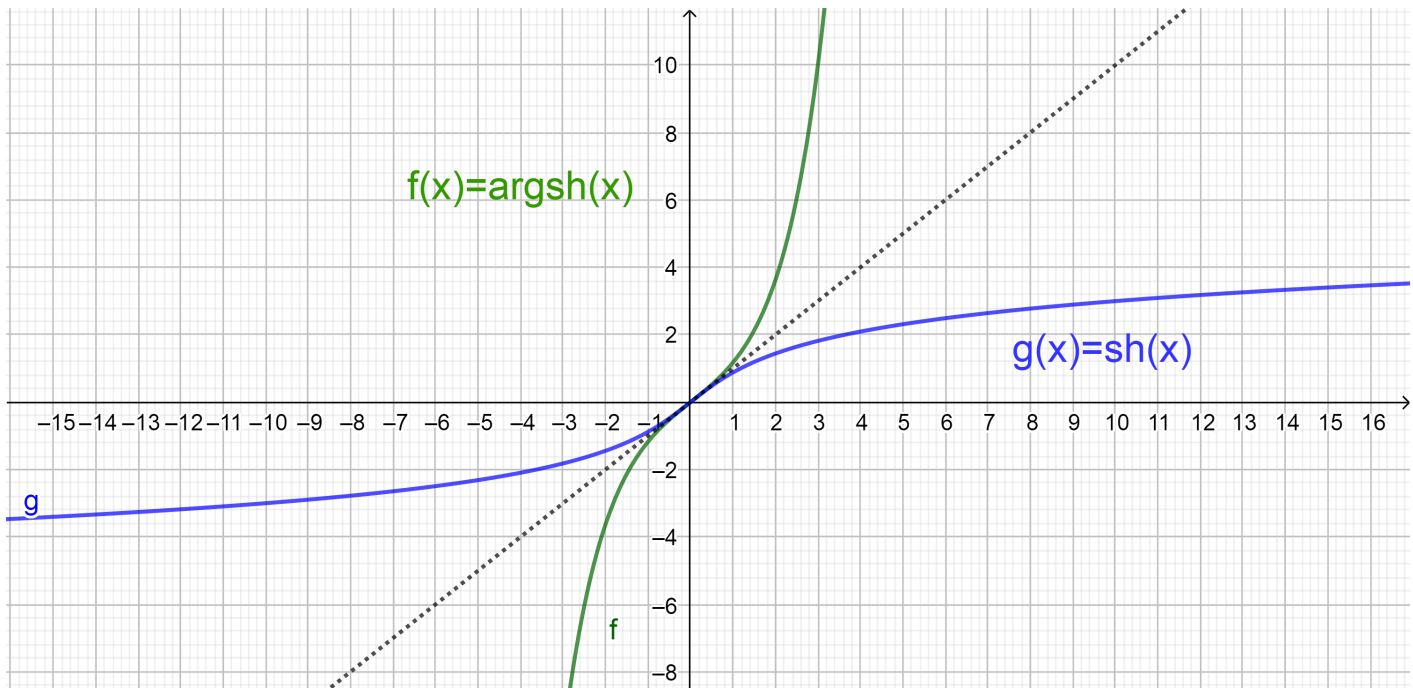


Figure 4.6: The graph of the functions sh and $argsh$

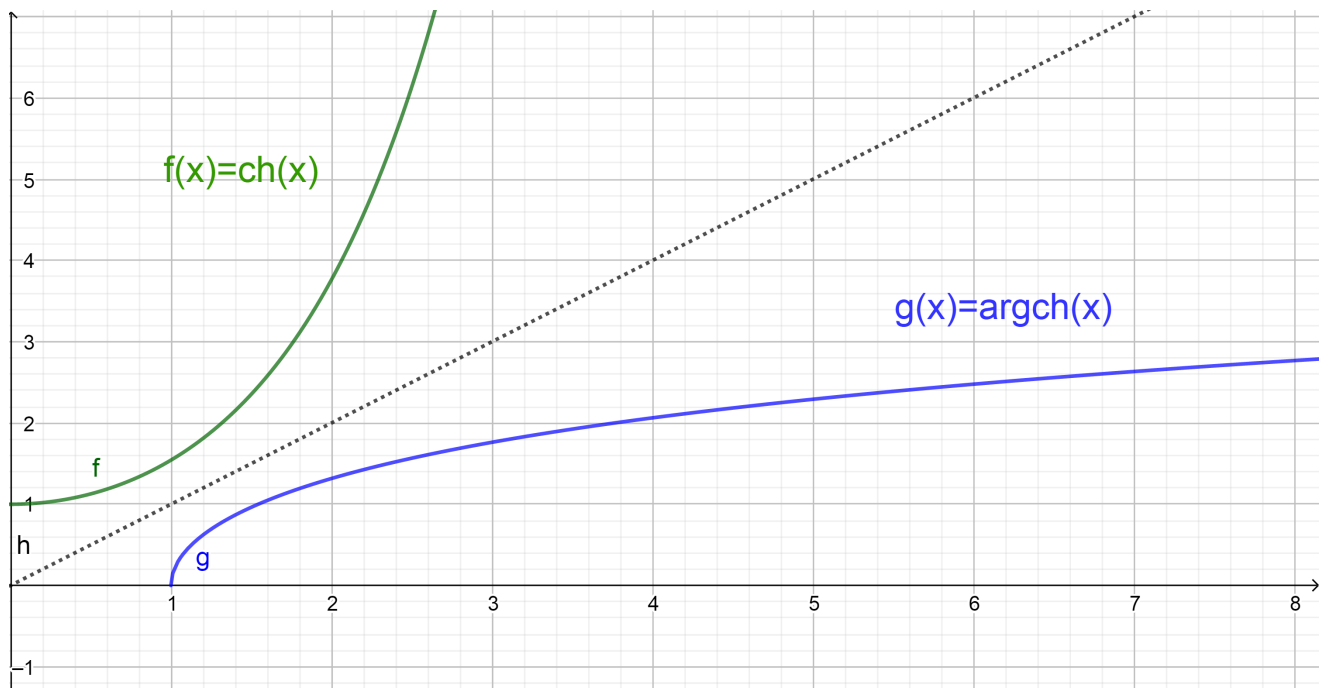
Inverse of the Hyperbolic Cosine ch : the function $arch$

The hyperbolic cosine function $ch = \cosh$ is continuous and strictly increasing on \mathbb{R}_+ , so it admits an inverse denoted $arch$ defined by

$$y = \operatorname{arch}(x) \iff x = \cosh(y), \quad x \geq 1, \quad y \geq 0.$$

Its explicit formula is

$$\operatorname{arch}(x) = \ln(x + \sqrt{x^2 - 1}).$$

Figure 4.7: The graph of the functions ch and $argch$

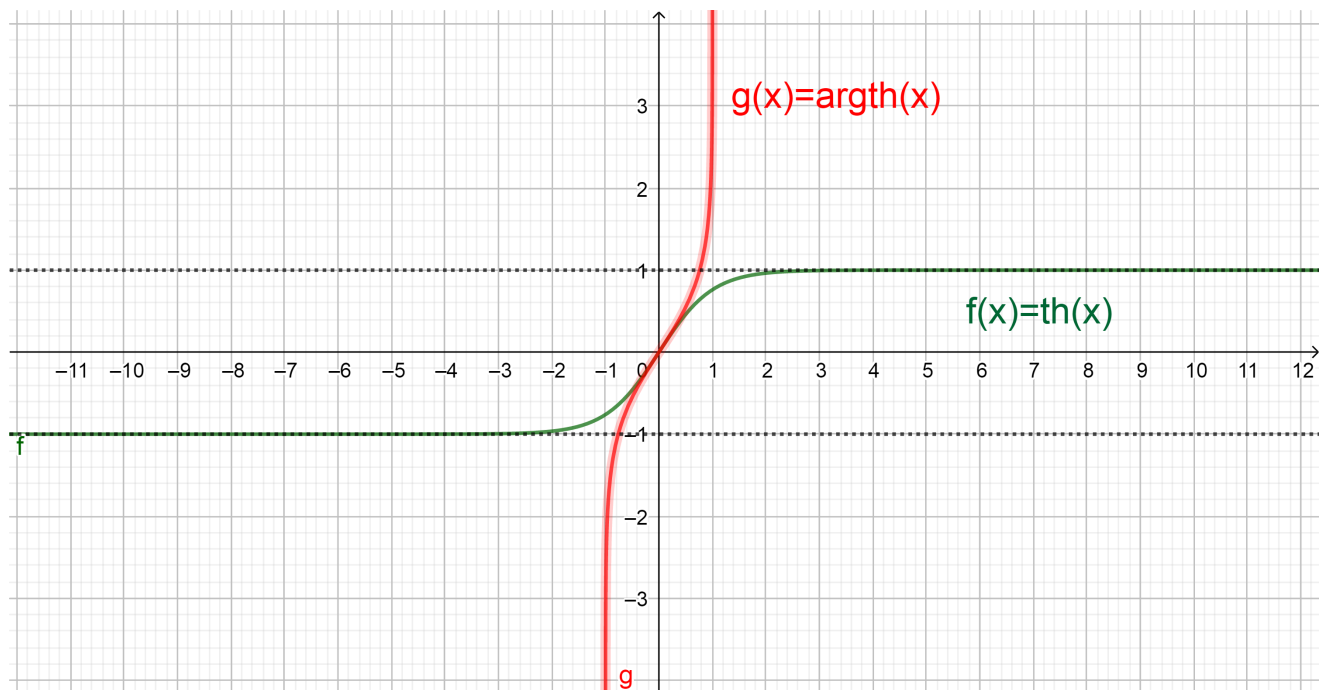
Inverse of the Hyperbolic Tangent th : the function $arth$

- The hyperbolic tangent function $th = \tanh$ is continuous and strictly increasing on \mathbb{R} , and it admits an inverse denoted $arth$ defined by

$$y = arth(x) \iff x = \tanh(y), \quad -1 < x < 1, \quad y \in \mathbb{R}.$$

Its explicit formula is

$$arth(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$



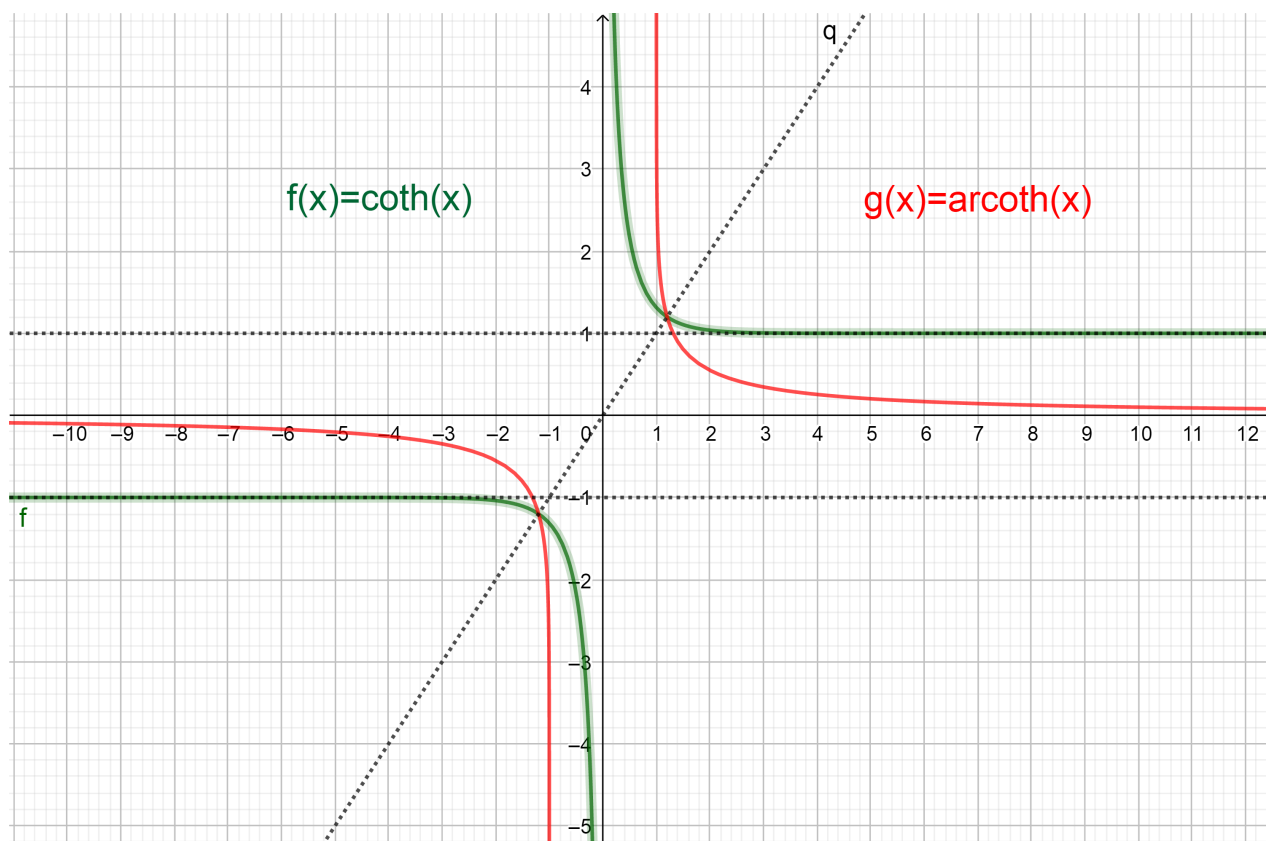
Inverse of the Hyperbolic Cotangent \coth : the function arcoth

The hyperbolic cotangent function $\coth = \operatorname{coth}$ is continuous and strictly increasing on $\mathbb{R}^* \setminus (-1, 1)$, and it admits an inverse denoted arcoth defined by

$$y = \operatorname{arcoth}(x) \iff x = \coth(y), \quad |x| > 1, \quad y \in \mathbb{R}^*.$$

Its explicit formula is

$$\operatorname{arcoth}(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right).$$

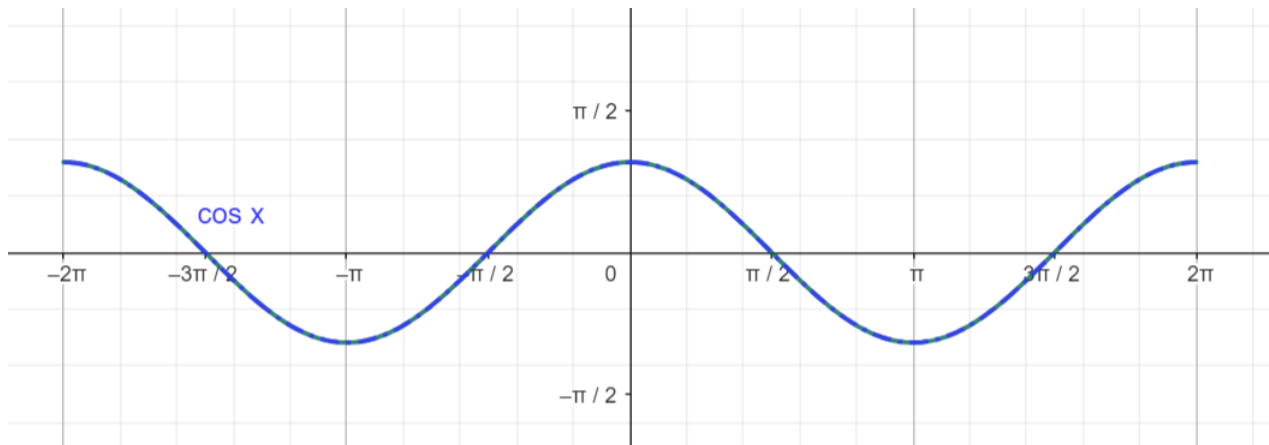
Figure 4.8: The graph of the functions \coth and $\operatorname{arccoth}$

4.4 Inverse trigonometric functions

Trigonometric functions, $\cos x$, $\sin x$, $\tan x$, $\cot x$ are periodic and continuous on their domain of definition, to define their reciprocal functions we must take only the domain where the functions are monotonic.

Functions $\cos x$, reciprocal function $\arccos x$

Consider the restriction of cosine to the interval $[0, \pi]$. On this interval the cosine function is continuous and strictly decreasing, so the restriction $\cos : [0, \pi] \rightarrow [-1, 1]$.

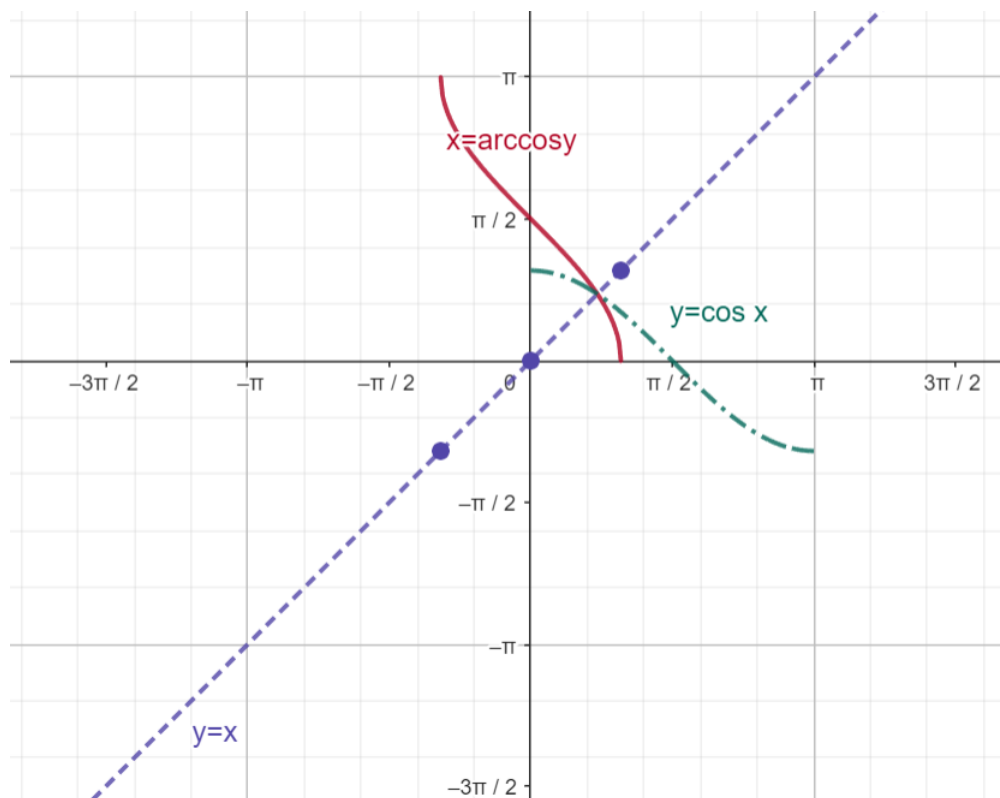
Figure 4.9: The graph of the function \cos

Its reciprocal (bijection) function is the arccosine function :

$$\arccos : [-1, 1] \longrightarrow [0, \pi]$$

$$y \longmapsto x = \arccos(y).$$

such that $y = \cos x$.

Figure 4.10: The graph of the functions \cos and \arccos

The function $\arccos x$ is continuous and strictly decreasing on interval $[-1, 1]$ and derivable on interval $] -1, 1[$.

According to the reciprocal function theorem such that : $y = \cos x = f(x) \iff y = \cos x = f(x) \iff x = \arccos y = f^{-1}(y)$

$$\begin{aligned} (\arccos y)' &= (f^{-1})'(y) = (f^{-1})'f(x) = \frac{1}{f'(x)} \\ &= \frac{-1}{\sqrt{1 - \cos^2 x}} = \frac{-1}{\sqrt{1 - y^2}}; |y| < 1. \end{aligned}$$

We have $x = \arccos y, -1 \leq y \leq 1 \iff y = \cos x; 0 \leq x \leq \pi$.

Proposition 4.8

1. $\cos(\arccos y) = y; \forall y \in [-1, 1]$.
2. $\arccos(\cos x) = x; \forall x \in \mathbb{R}$.

Function $\sin x$, reciprocal function $\arcsin x$

We know the sine function is continuous and strictly increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

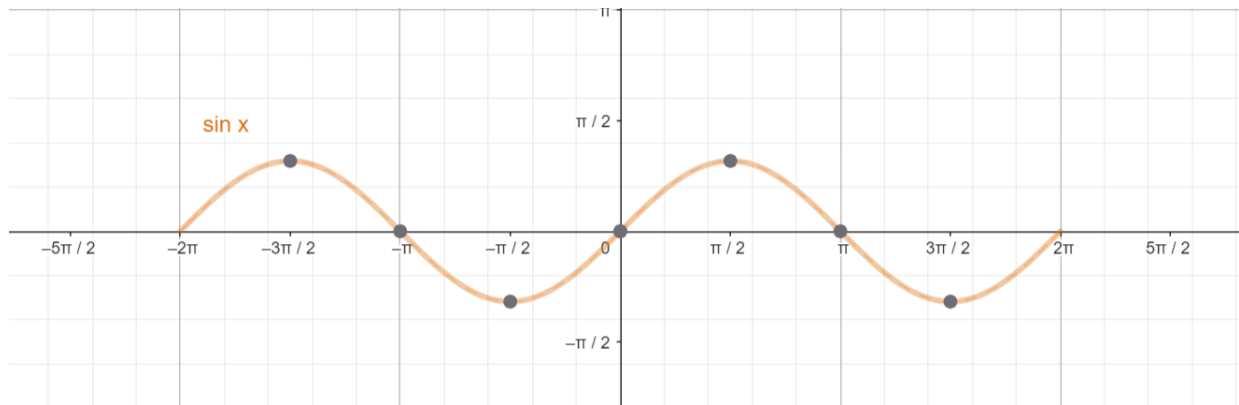
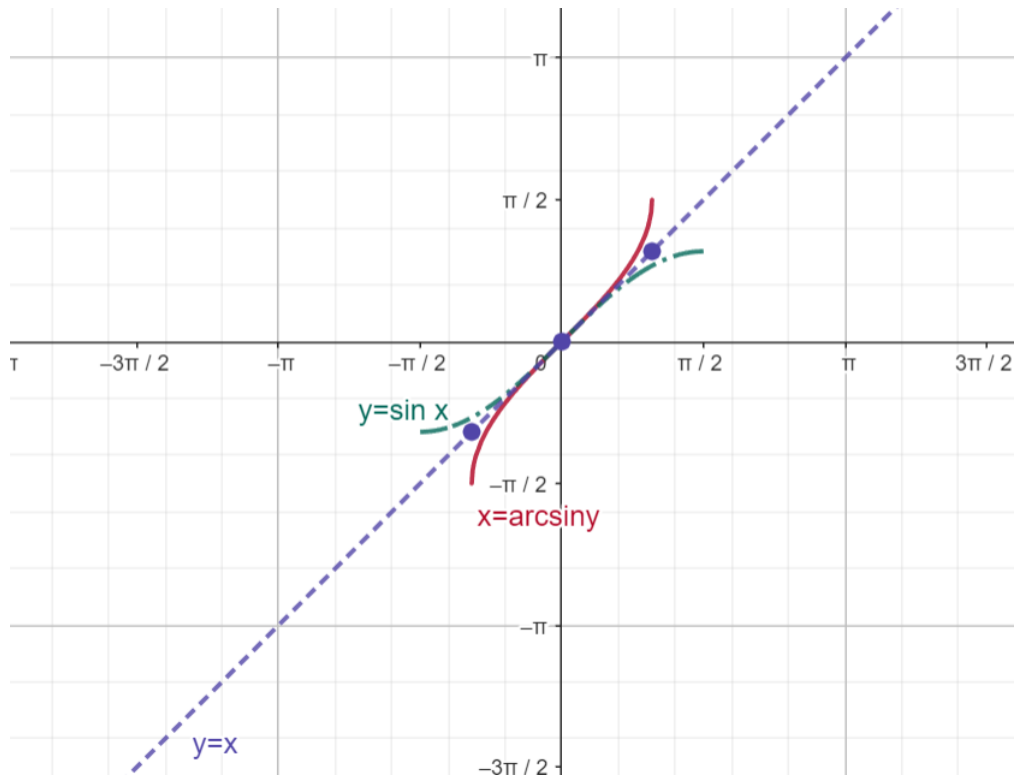


Figure 4.11: The graph of the function \sin

Its reciprocal (bijection) function is the arcsine function :

$$\begin{aligned} \arcsin : [-1, 1] &\longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ y &\longmapsto x = \arcsin(y), \end{aligned}$$

Figure 4.12: The graph of the functions \sin and \arcsin

such that $y = \sin x$.

The function $\arcsin x$ is continuous and strictly increasing on interval $[-1, 1]$. According to the reciprocal function theorem such that :

$$y = \sin x = f(x) \quad y = \sin x = f(x) \Leftrightarrow x = \arcsin y = f^{-1}(y)$$

$$\begin{aligned} (\arcsin y)' &= (f^{-1})'(y) = (f^{-1})'f(x) = \frac{1}{f'(x)} \\ &= \frac{-1}{\sqrt{1 - \sin^2 x}} = \frac{-1}{\sqrt{1 - y^2}}; |y| < 1. \end{aligned}$$

From there, the function $\arcsin y$ is derivable on interval $] -1, 1[$ and its derivative is given by the relation :

$$\arcsin y = \frac{-1}{\sqrt{1 - y^2}}; |y| < 1.$$

We have $x = \arcsin y, -1 \leq y \leq 1 \Leftrightarrow y = \sin x; \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$.

Proposition 4.9

1. $\cos(\arcsin y) = y; \forall y \in [-1, 1]$.
2. $\arcsin(\sin x) = x; \forall x \in \mathbb{R}$.

Function $\tan x$, reciprocal function $\arctan x$

We know the $\tan x$ function is continuous and strictly increasing on interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, Its reciprocal (bijection) function is the $\arctan x$ function :

$$\begin{aligned} \arctan : \mathbb{R} &\longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ y &\longmapsto x = \arctan(y), \end{aligned}$$

such that $y = \tan x$.

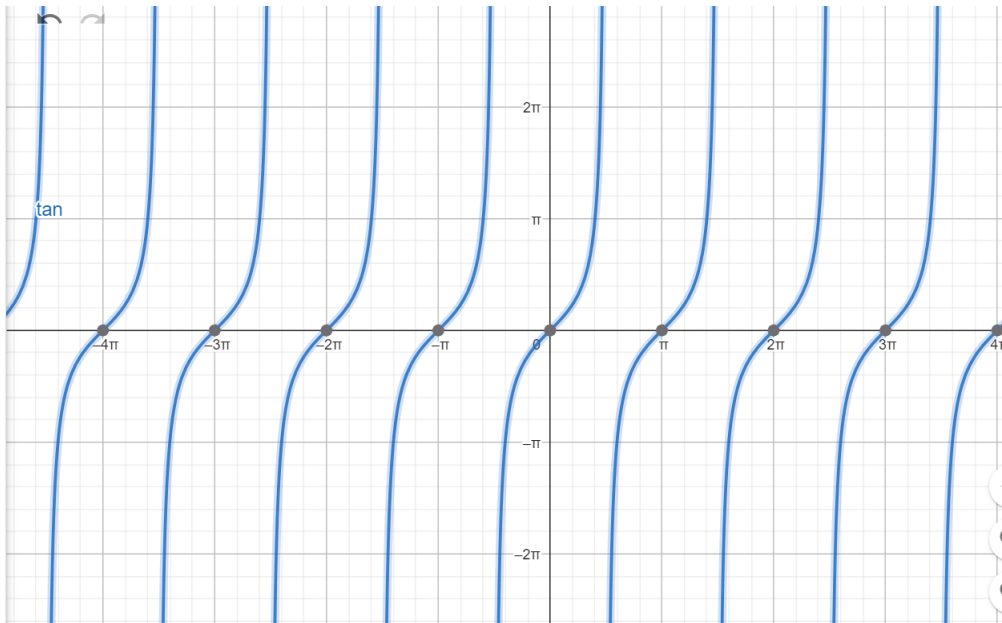


Figure 4.13: The graph of the function \tan

The function $\arctan x$ is continuous and strictly increasing on \mathbb{R} .

According to the reciprocal function theorem such that $y = \tan x$.

$$\begin{aligned} y = \tan x = f(x) &\Leftrightarrow x = \arctan y = f^{-1}(y)(\arctan y)' \\ &\Leftrightarrow (f^{-1})'(y) = (f^{-1})'f(x) = \frac{1}{f'(x)} \\ &\Leftrightarrow \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}. \end{aligned}$$

From there, the function $\arctan y$ is derivable on \mathbb{R} and its derivative is given by the relation:

$$\arctan y = \frac{1}{1 + y^2}, y \in \mathbb{R}.$$

We have $x = \arctan y, y \in \mathbb{R} \Leftrightarrow y = \tan x; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

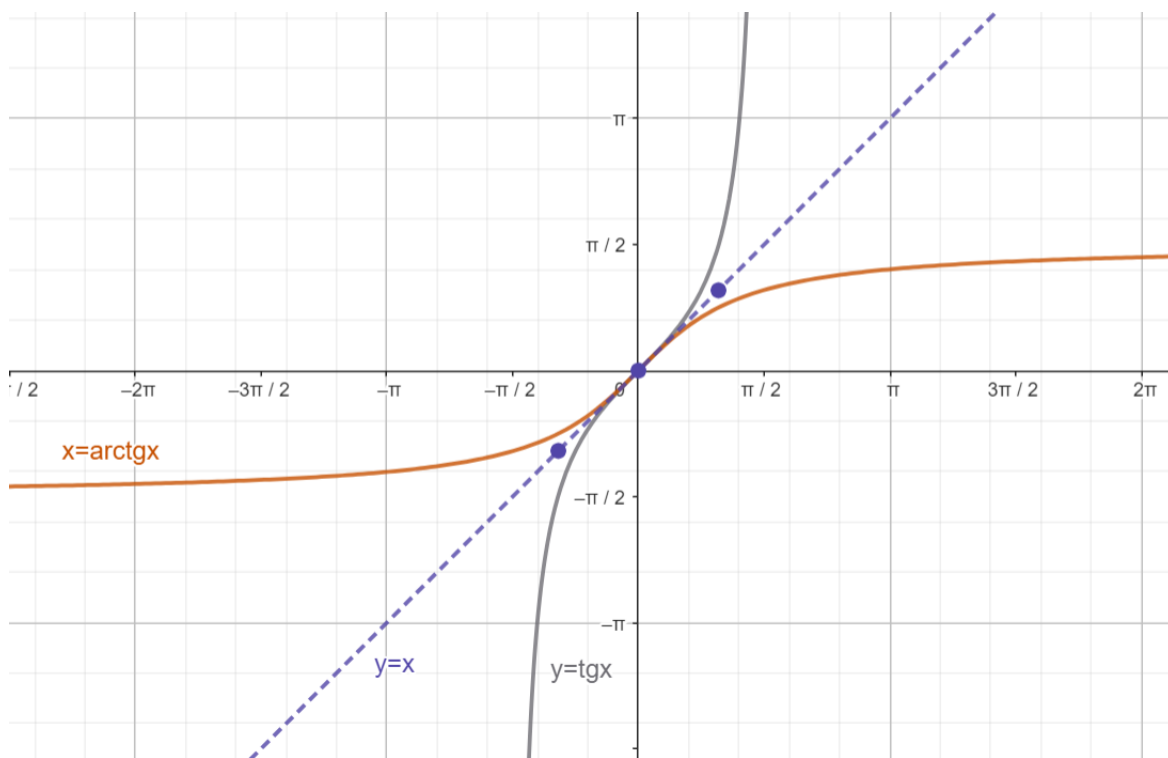


Figure 4.14: The graph of the functions \tan and \arctan

Proposition 4.10

1. $\tan(\arctan y) = y; \forall y \in \mathbb{R}.$
2. $\arctan(\tan x) = x; \forall x \in \left] \frac{-\pi}{2}, \frac{\pi}{2} \right[.$
3. $\forall y \in [-1, 1]; \arcsin y + \arccos y = \frac{\pi}{2}.$
4. $\forall y \in [-1, 1]; \arccos y + \arccos(-y) = \pi.$

Chapter 5

Taylor Series - Limited development

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Brook Taylor was an English mathematician born in 1685 and known for his foundational work in calculus and mathematical analysis. He is most famous for Taylor's theorem and the Taylor series, which provide fundamental tools for approximating functions using polynomials. Educated at St John's College, Cambridge, Taylor's work paved the way for further advances in analysis and the formalization of function approximation techniques. He was a member of the Royal Society and made significant contributions to



mathematical and scientific discussions of his time. Taylor's legacy endures through the widespread use of his theorem in diverse fields such as physics, engineering, and computer science.

5.1 Taylor Constitution

5.1.1 Preliminaries

Let $f(\mathbf{x})$ be a derivable function at point \mathbf{x}_0 then:

$$\begin{aligned}\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0)}{\mathbf{x} - \mathbf{x}_0} = f'(\mathbf{x}_0) &\Rightarrow \frac{f(\mathbf{x}) - f(\mathbf{x}_0)}{\mathbf{x} - \mathbf{x}_0} - f'(\mathbf{x}_0) = \varepsilon(\mathbf{x}) / \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \varepsilon(\mathbf{x}) = 0 \\ &\Rightarrow f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)f'(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)\varepsilon(\mathbf{x}).\end{aligned}$$

We have

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{(\mathbf{x} - \mathbf{x}_0)\varepsilon(\mathbf{x})}{\mathbf{x} - \mathbf{x}_0} = 0,$$

then $(\mathbf{x} - \mathbf{x}_0)\varepsilon(\mathbf{x})$ negligent in front of $(\mathbf{x} - \mathbf{x}_0)$ then: $(\mathbf{x} - \mathbf{x}_0)\varepsilon(\mathbf{x}) = o(\mathbf{x} - \mathbf{x}_0)$.

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)f'(\mathbf{x}_0) + o(\mathbf{x} - \mathbf{x}_0) = P_1(\mathbf{x} - \mathbf{x}_0) + R_1(\mathbf{x} - \mathbf{x}_0).$$

This relation is called approximation of the function $f(\mathbf{x})$ by a polynomial P_1 polynomial with an error of R_1 .

5.1.2 Taylor's formula

Taylor's formulas generalize the previous relation in that they show that the differentiable function n time we can approximate them in the near the point \mathbf{x}_0 with a polynomial of degree n i-e:

$$f(\mathbf{x}) = P_n(\mathbf{x} - \mathbf{x}_0) + R_n(\mathbf{x} - \mathbf{x}_0). \quad (5.1)$$

Where P_n is a polynomial of degree n with variable $(\mathbf{x} - \mathbf{x}_0)$ such that:

$$P_n(\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}_0) + f'(\mathbf{x}_0)\frac{\mathbf{x} - \mathbf{x}_0}{1!} + f''(\mathbf{x}_0)\frac{(\mathbf{x} - \mathbf{x}_0)^2}{2!} + \dots + f^{(n)}(\mathbf{x}_0)\frac{(\mathbf{x} - \mathbf{x}_0)^{(n)}}{n!}.$$

Approximates $f(x)$ by the error $R_n(x - x_0) = o(x - x_0)$ which is called the rest of the order n .

Example 5.1 Write the Taylor formula for the function $f(x) = \ln x \dots (1)$ in the near of the point $x_0 = 2$ of order 2. We have:

$$\begin{aligned}f(x) &= \ln x \implies f(2) = \ln 2. \\f'(x) &= \frac{1}{x} \implies f'(2) = \frac{1}{2}. \\f''(x) &= -\frac{1}{x^2} \implies f''(2) = -\frac{1}{4}.\end{aligned}$$

Substituting into expression (1), we find:

$$\ln x = \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + o(x - 2)^2.$$

Example 5.2 Write the Taylor formula for the function $f(x) = \sqrt{x}$ in the near of the point $x_0 = 1$ of order 1, then conclude $\sqrt{1.002}$.

We have:

$$\begin{aligned}f(x) &= \sqrt{x} \implies f(1) = 1. \\f'(x) &= \frac{1}{2\sqrt{x}} \implies f'(1) = \frac{1}{2}.\end{aligned}$$

Substituting into expression (1), we find:

$$\sqrt{x} = 1 + \frac{1}{2}(x - 1) + o(x - 1).$$

And

$$\sqrt{1.0021} \simeq 1 + \frac{1}{2}(1.002 - 1) = 1.001.$$


5.2 Maclaurin formula

We put $x_0 = 0$ in Taylor's formula, we get Maclaurin's formula in the order n ie:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + o(x^n).$$

Example 5.3 Write the Maclaurin formula for the function $f(x) = e^x$ of order n

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots + \frac{x^n}{n!} + o(x^n).$$



Remark 5.1  According to Taylor's formulas, derivable functions n times are approximated by polynomial n near point x_0 . However, there are non-derivable or even non-continuous functions at point x_0 . However, this was made possible almost by polynomial, which led us to address the concept of limited development.


5.3 Limited development near zero


Let $f(x)$ is a function defined at zero and let $n \in \mathbb{N}$, we say f has a limited development of order n in the near zero, and we write a function abbreviation DL_n if it exists $a_0, a_1, a_2, \dots, a_n$ a $(n + 1)$ real number so:


$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + o(x^n).$$

We call the part $P_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ with the regular part and $o(x^n)$ with the rest.

Remark 5.2   Each function accepts a limited development near zero, it accepts a finite limit if $x \rightarrow 0$.

 The existence of DL_n near zero for a function implies continuity for the equation at zéro.

 Limited development of a function if it exists, is unique.

 If $f(x)$ is an even function (odd) accepts DL_n in the near zero, then the regular part $P_n(x)$ is an even (odd) polynomial.

5.3.1 Operation on limited development

Let be $f(x)$ and $g(x)$ they have a limited development of order n in the near of zero, such that:

$$g(x) = Q_n(x) + o(x^n); \quad f(x) = P_n(x) + o(x^n).$$

Addition

The function $(f + g)$ accept a DL_n in the near of zero and it's regular part is $P_n(x) + Q_n(x)$.

Example 5.4 Write DL_n for the function Shx in the near of zero.

We have: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots + \frac{x^n}{n!} + o(x^n)$ and $e^{-x} = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots + (-1)^n \frac{x^n}{n!} + o(x^n)$,
as $Shx = \frac{e^x - e^{-x}}{2}$ then;

$$Shx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}.$$

Multiplication by a real number

The function λf accept a DL_n near zero and its regular part is $\lambda P_n(x)$.

Product

$f.g$ accept a limited development in the near zero, its regular part is $(Q_n(x).P_n(x))$ with keep the elements of a degree lower than or equal to the degree n .

Example 5.5 Find DL_3 in the near zero for the function $f(x) = e^x \cdot \sin x$ We have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$; $\sin x = x - \frac{x^3}{3!} + o(x^3)$. Then $f(x) = e^x \cdot \sin x = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}) \cdot (x - \frac{x^3}{3!})$, we develop and we keep the elements of a degree lower than or equal to the degree 3, we find:

$$f(x) = x + x^2 + \frac{1}{3}x^3 + o(x^3).$$

Quotient

If $\lim_{x \rightarrow 0} g(x) \neq 0$ then f/g accept a limited development in the near of zero, its regular part is a polynomial of order n we get it by the euclidean divide for the regular part $\frac{P_n(x)}{Q_n(x)}$ based on the increasing powers of x .

Example 5.6 Write DL_2 for the function $f(x) = \frac{e^x}{\cos x}$.

$$\text{We have } e^x = 1 + x + \frac{x^2}{2!} + o(x^2); \quad \cos x = 1 - \frac{x^2}{2!} + o(x^2)$$

Then:

$$f(x) = x + x^2 + o(x^2).$$

$$\begin{array}{r|l} 1 + x + \frac{x^2}{2} & 1 - \frac{x^2}{2} \\ \hline - (1 - \frac{x^2}{2}) & 1 + x + x^2 \\ & x + x^2 \\ - (x - \frac{x^3}{2}) & \\ & x^2 + \frac{x^3}{2} \\ - (x^2 - \frac{x^4}{2}) & \\ & \frac{x^3}{2} + \frac{x^4}{2} \end{array}$$

Composition

Let $f(x)$ and $g(x)$ be two functions accept a limited development in the near zero, if $g(0) = 0$ then the composite function $(f \circ g)$ has a limited development in the near zero, we obtain its regular part by substituting the regular part of the function $g(x)$ with the regular part of the function $f(x)$ and keeping only the terms of degree less than or equal to order n .

Example 5.7 Find DL_4 for the function $h(x) = e^{\sin x}$.

$$\text{We have } g(x) = \sin x \Rightarrow g(0) = 0$$

We put :

$$y = \sin x = x - \frac{x^3}{3!} + o(x^3) \Rightarrow e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + o(y^4).$$

$$e^{\sin x} = 1 + (x - \frac{x^3}{3!}) + \frac{1}{2!}(x - \frac{x^3}{3!})^2 + \frac{1}{3!}(x - \frac{x^3}{3!})^3 + \frac{1}{4!}(x - \frac{x^3}{3!})^4.$$

We develop and keep the elements of a degree lower than or equal to the degree 4, we find:

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4).$$

Example 5.8 Find DL_4 for the function $h(x) = e^{\cos x}$

We have $g(x) = \cos x \Rightarrow g(0) = \cos 0 = 1$; $e^{\cos x} = e^{\cos x - 1 + 1} = e \cdot e^{\cos x - 1} \Rightarrow$
 $g(x) = \cos x - 1$; $g(0) = 0$,
 and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$; $\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$,
 then $y = \cos x - 1 \Rightarrow e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + o(y^4)$.

$$\begin{aligned} e^{\cos x - 1} &= 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \frac{1}{2!} \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^2 \\ &\quad + \frac{1}{3!} \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^3 + \frac{1}{4!} \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^4. \end{aligned}$$

We develop and we keep the elements of a degree lower than or equal to the degree 4, we find:

$$e^{\cos x - 1} = 1 + \frac{x^2}{2} + \frac{x^4}{6} + o(x^4).$$

then

$$h(x) = e^{\cos x} = e \left(1 + \frac{x^2}{2} + \frac{x^4}{6} + o(x^4)\right).$$

Derivative of the limited development

Let $f : V \rightarrow \mathbb{R}$ is a derivable function and they accept a limited development in the near zero, f' have DL_{n-1} , then

$$f'(x) = a_1 + 2a_2x + \dots + na_nx^{n-1} + o(x^{n-1}).$$

Example 5.9 We have limited development in the near zero of order n for the function

$$f(x) = \frac{1}{1-x} \text{ is}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + o(x^n),$$

and $\left[\frac{1}{1-x}\right]' = \frac{1}{(1-x)^2}$, then the limited development in the near zero of order n is:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1} + o(x^{n-1}).$$

5.3.2 The limited development in the near point x_0 :

Definition 5.1 We say that a function $f(x)$ defined in the near point x_0 accept DL_n in the near this point if

$$f(x) = F(x - x_0) = F(y); x \rightarrow x_0; y \rightarrow 0,$$

have DL_n in the near zero i-e

$$F(y) = a_0 + a_1y + a_2y^2 + \dots + a_ny^n + o(y^n),$$

so

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + o(x - x_0)^n.$$

Example 5.10 Find DL_n in the near x_0 for the function $f(x) = e^x$.

We put $y = x - 1; x \rightarrow 1; y \rightarrow 0$

$$e^x = e^{x+1-1} = e \cdot e^{x-1} = e \cdot e^y; e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} \dots + \frac{y^n}{n!} + o(y^n).$$

$$e^y = e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} \dots + \frac{(x-1)^n}{n!} + o((x-1)^n).$$

Then in the near of $x_0 = 1$ we have:

$$e^x = e \left[x + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} \dots + \frac{(x-1)^n}{n!} + o((x-1)^n) \right].$$

5.3.3 The limited development in the near $+\infty$:

Let $f(x)$ defined on interval $[a, +\infty[$, we say that $f(x)$ accept a limited development in the near of $+\infty$, if the coefficients $c_0, c_1, c_2, \dots, c_n$ exist, such that :

$$f(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots + \frac{c_n}{x^n} + o\left(\frac{1}{x^n}\right).$$

Example 5.11 Write DL_4 in the near $+\infty$ for the function $f(x) = e^{\frac{1}{x}}$.

We put $y = \frac{1}{x}$; $x \rightarrow +\infty$; $y \rightarrow 0$ we have:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + o(y^4),$$

then

$$e^{\frac{1}{x}} = 1 + \frac{1}{x} + \frac{1}{2!x^2} + \frac{1}{3!x^3} + \frac{1}{4!x^4} + o(y^4).$$

Theorem 5.1 Let $f(x)$ accept a limited development in the near zero where the regular part is non zero. The non-zero term of the lowest degree for this development is equivalent to the function $f(x)$ in the near zero.

Example 5.12 1. $\sin x \simeq x + o(x^2)$.

2. $shx \simeq x + o(x^2)$.

3. $\cos x \simeq 1 - \frac{x^2}{2!} + o(x^3)$;

4. $chx \simeq 1 + \frac{x^2}{2!} + o(x^3)$.

5. $\tan x \simeq x + o(x^2)$

6. $Thx \simeq x + o(x^2)$.

5.3.4 Applications of limited development in calculating limits:

Using limited development, calculate the following limits:

$$\lim_{x \rightarrow 0} \frac{shx - \sin x}{x - \arctan x}; \quad \lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}; \quad \lim_{x \rightarrow +\infty} \left[\sqrt{x^2 + 3x + 2} - x \right].$$

Solution 1- $\lim_{x \rightarrow 0} \frac{\operatorname{sh} x - \sin x}{x - \arctan x}$.

We have

$$\operatorname{sh} x \simeq x + o(x^2); \quad \sin x \simeq x + o(x^2); \quad \arctan x \simeq x + o(x^2),$$

then

$$\lim_{x \rightarrow 0} \frac{\operatorname{sh} x - \sin x}{x - \arctan x} = \lim_{x \rightarrow 0} \frac{x + o(x^2) - x + o(x^2)}{x - x + o(x^2)} = \frac{0}{0}.$$

In this case, we add the term which follows the last term in the limited development, i.e:

$$\lim_{x \rightarrow 0} \frac{x + \frac{x^3}{3!} - x + \frac{x^3}{3!} + o(x^4)}{x - x + \frac{x^3}{3} + o(x^4)} = \lim_{x \rightarrow 0} \frac{2\frac{x^3}{6}}{\frac{x^3}{3}} = 1.$$

2- $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$.

We put $t = x - 1$; $x \rightarrow 1$; $t \rightarrow 0$

we have

$$\begin{aligned} \frac{\ln x}{x^2 - 1} &= \frac{\ln(1+t)}{(1+t)^2 - 1} = \frac{\ln(1+t)}{1+2t+t^2-1} = \frac{\ln(1+t)}{t(2+t)} \simeq \frac{t+o(t)}{t(2+t)} \\ &= \frac{1+o(t)}{2+t} \Rightarrow \lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{1+o(t)}{2+t} = \frac{1}{2}. \end{aligned}$$

3- $\lim_{x \rightarrow +\infty} [\sqrt{x^2 + 3x + 2} - x]$.

We put $y = \frac{1}{x}$; $x \rightarrow +\infty$; $y \rightarrow 0$

$$\begin{aligned} \sqrt{x^2 + 3x + 2} - x &= \sqrt{\frac{1}{y^2} + 3\frac{1}{y} + 2} - \frac{1}{y} = \sqrt{\frac{1 + 3y + 2y^2}{y^2}} - \frac{1}{y} \\ &= \frac{\sqrt{1 + 3y + 2y^2 - 1}}{y} - \frac{1}{y}; \quad y > 0; \quad (X = 3y + 2y^2 \rightarrow 0); \quad y \rightarrow 0 \\ \Rightarrow \sqrt{x^2 + 3x + 2} - x &= \frac{\sqrt{1 + 3y + 2y^2} - 1}{y} = \frac{\sqrt{1 + X} - 1}{y}. \end{aligned}$$

$$\sqrt{1 + X} \simeq 1 + \frac{1}{2}X.$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1 + 3y + 2y^2 - 1}{y} = \lim_{y \rightarrow 0} \frac{1 + \frac{1}{2}(3y + 2y^2) + o(y^2) - 1}{y} = \frac{3}{2}.$$

Then

$$\lim_{x \rightarrow +\infty} \left[\sqrt{x^2 + 3x + 2} - x \right] = \frac{3}{2}.$$

5.3.5 The limited development for some function in near zero

1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + o(x^n).$
2. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + o(x^n).$
3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}).$
4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}).$
5. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n).$
6. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots (-1)^{n-1} \frac{x^n}{n!} + o(x^n).$
7. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + o(x^n); m \in \mathbb{R}^*.$
8. $shx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}).$
9. $chx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}).$
10. $\frac{1}{1+x^2} = 1 - x^2 + x^4 + \dots + (-1)^n x^{2n} + o(x^{2n}).$
11. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}).$
12. $\arcsin x = x + \frac{1}{3} \frac{x^3}{3} + \dots + \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2 \times 4 \times 6 \times \dots (2n)} \frac{x^{2n+1}}{2n+1} + o(x^{2n+2}).$
13. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^6).$
14. $Thx = x - \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^6).$

Chapter 6

Linear Algebra

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Emmy Noether was a pioneering German mathematician, widely regarded as one of the most influential figures in the development of modern abstract algebra and theoretical physics. She was born on March 23, 1882, in Erlangen, Germany, into a mathematical family—her father was the mathematician Max Noether. Initially intending to teach languages, she turned to mathematics and earned her doctorate in 1907.



Despite facing significant gender-based discrimination, Noether made groundbreaking contributions to:

- Abstract Algebra: She developed the theory of rings, ideals, and modules, helping transform algebra into a structural, axiomatic field.
- Theoretical Physics: Her celebrated Noether's Theorem (1918) established a deep connection between symmetries and conservation laws, foundational to modern physics.

In 1933, she fled Nazi Germany and took a position at Bryn Mawr College in the United States. She died shortly after, on April 14, 1935, at the age of 53.

Emmy Noether is often hailed as the “mother of modern algebra,” and her legacy continues to shape both mathematics and physics to this day.

6.1 Internal laws of composition

Definition 6.1 Let E be a non empty set. An internal composition law $*$ on E is a map of $E \times E$ in E which associates any pair (a, b) of $E \times E$ with an element of E noted $a * b$:

$$\begin{aligned} * : E \times E &\rightarrow E \\ (a, b) &\longrightarrow a * b. \end{aligned}$$

Example 6.1

- The usual operations $+$ and \times constitute internal composition laws on $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \dots$
- Let $*$ be defined on \mathbb{Q} by:

$$a * b = \frac{a + b}{2},$$

then $*$ is a law of internal composition.

- Let $*$ be defined on \mathbb{R} by:

$$a * b = \frac{1}{a + b},$$

then $*$ is not a law of internal composition because $(-1, 1) \in \mathbb{R} \times \mathbb{R}$, does not admit an image.

6.1.1 Internal operations properties

Commutativity

Definition 6.2 $*$ is said to be commutative if

$$\forall (a, b) \in E^2 : a * b = b * a.$$

Example 6.2 Let $*$ be an internal composition law defined on \mathbb{R} by:

$$a * b = a + b - 1.$$

We have: $a * b = a + b - 1 = b + a - 1 = b * a$, then $*$ is commutative law.

Associativity

Definition 6.3 $*$ is said to be associative if:

$$\forall (a, b, c) \in E^3 : a * (b * c) = (a * b) * c.$$

Example 6.3 Let $*$ be an internal composition law defined on \mathbb{R} by:

$$a * b = a + b - 1.$$

We have

$$(a * b) * c = (a + b - 1) + c - 1 = a + b + c + 2 \dots \dots \dots (1)$$

$$a * (b * c) = a + (b + c - 1) - 1 = a + b + c - 2 \dots \dots \dots (2),$$

when (1) = (2) then $*$ is associative.

Identity element

Definition 6.4 The internal composition law \mathbf{x} admits a neutral element on E if:

$$\exists e \in E, \forall a \in E : e * a = a * e = a.$$

Remark 6.1 The neutral element, if it exists, is unique. Indeed, let another neutral element for $*$, then

$$e' = e' * e = e * e' = e.$$

Example 6.4 Let $*$ be an internal composition law defined on \mathbb{R} by:

$$a * b = a + b - 1,$$

we have: $a * e = a \Rightarrow a + e - 1 = a \Rightarrow e = 1$. Then $e = 1$ is an identity element.

Symmetric element

Definition 6.5 Assume that E admits an identity element e for $*$. Let a and a' be two elements from E . We say that a' is a symmetric of a (for the $*$ law) if:

$$\forall a \in E, \exists a' \in E : a * a' = a' * a = e.$$

Example 6.5 Let $*$ be an internal composition law defined on \mathbb{R} by:

$$a * b = a + b - 1.$$

We have: $a * a' = 1 \Rightarrow a + a' - 1 = 1 \Rightarrow a' = (2 - a) \in \mathbb{R}$. Then $a' = 2 - a$ is a symmetric element.

Distributivity

Definition 6.6 Let $*$ and \mathcal{T} be two internal laws defined from E :

- The law \mathcal{T} is said to be left distributive with respect to the law $*$ if:

$$\forall(a, b, c) \in E^3 : a\mathcal{T}(b * c) = (a\mathcal{T}b) * (a\mathcal{T}c).$$

- The law \mathcal{T} is said to be right distributive with respect to the law $*$ if:

$$\forall(a, b, c) \in E^3 : (b * c)\mathcal{T}a = (b\mathcal{T}a) * (c\mathcal{T}a).$$

The law \mathcal{T} considered distributive concerning the law $*$ if it is distributive to left and right with respect to $*$.

Example 6.6 Let $*$ be an internal composition law defined on \mathbb{R} by:

$$a * b = a + b - 1.$$

And \mathcal{T} be an internal composition law defined on \mathbb{R} by:

$$a\mathcal{T}b = a + b - ab.$$

Then the law \mathcal{T} is distributive concerning $*$. When \mathcal{T} is commutative then it can be shown that \mathcal{T} is left distributive concerning the law $*$.

$$a\mathcal{T}(b * c) = a\mathcal{T}(b + c - 1) = 2a + b + c - ab - ac - 1 \dots \dots (1)$$

$$(a\mathcal{T}b) * (a\mathcal{T}c) = (a + b - ab) * (a + c - ac) = 2a + b + c - ab - ac - 1 \dots \dots (2)$$

When (1) = (2) then \mathcal{T} is distributive concerning $*$ distribution.

6.1.2 Stability

Definition 6.7 Let E be a set with an internal law $*$. A subset F of E is said to be stable for this internal law if and only if:

$$\forall a, b \in F : a * b \in F.$$

Example 6.7 \mathbb{N} is a stable subset of \mathbb{R} for the internal composition laws $+$ and \times .

6.1.3 Group

Definition 6.8 Let the internal composition law $*$ be defined on a set G , the pair $(G, *)$ is said to be group if:

1. The law $*$ is associative

$$\forall (a, b, c) \in G^3 : a * (b * c) = (a * b) * c.$$

2. There is a neutral element

$$\exists e \in G, \forall a \in G : e * a = a * e = a.$$

3. Any element of G has a symmetric

$$\forall a \in G, \exists a' \in E : a' * a = a * a' = e,$$

we also say that the set G has a group structure for the law $*$.

Example 6.8

- (\mathbb{N}, \times) is not a group.
- $(\mathbb{Z}, +)$ is a group.
- (\mathbb{Z}, \times) is not a group.
- $(\mathbb{R}, +)$ is a group.

6.1.4 Subgroup

Definition 6.9 Let $(G, *)$ be a group. A non-empty part H of G is a subgroup of G if:

- $\forall (a, b) \in H \times H : a * b \in H.$
- $\forall a \in H : a' \in H.$

Example 6.9

- Let $\mathbb{Z}, +$ be a group, then $3\mathbb{Z}$ is a subgroup of \mathbb{Z} .

We have $3\mathbb{Z} = \{3\mathbb{Z}/\mathbb{Z} \in \mathbb{Z}\} = \{\dots, -6, -3, 0, 3, 6, \dots\}.$

1. Let $a, b \in 3\mathbb{Z}$, then $\exists z_1 \in \mathbb{Z}$ such that $a = 3z_1$ and $\exists z_2 \in \mathbb{Z}$ such that $b = 3z_2$, so $a + b = 3(z_1 + z_2) \in 3\mathbb{Z}.$
2. If $a \in 3\mathbb{Z}$, then $-a = -3z_1 = 3(-z_1) \in 3\mathbb{Z}.$

From (1) – (2), $3\mathbb{Z}$ is a subgroup of $\mathbb{Z}.$


Theorem 6.1 Let H be a non empty part of a group G , then a subgroup of G if and only if :

$$\forall (a, b) \in H \times H : a * b' \in H.$$

6.1.5 Rings

Definition 6.10 Let A be a set with two internal composition laws $*$ and \mathcal{T} , we say that $(A, *, \mathcal{T})$ is a ring if:

1. $A, *$ is a commutative group.
2. The \mathcal{T} law is associative.
3. The \mathcal{T} law is distributive with respect to the $*$ law.

Remark 6.2 

1. A ring $(A, *, \mathcal{T})$ is called commutative if the operation \mathcal{T} commutative.
2. A ring $A, *, \mathcal{T}$ is unitary if the operation \mathcal{T} has a neutral element.

Example 6.10

□ $(\mathbb{Z}, +, \times)$ is a commutative and unitary ring.

□ $(\mathbb{R}, +, \times)$ is a commutative and unitary ring.

6.1.6 Field

Definition 6.11 Let \mathbb{K} be a set with two internal composition laws $*$ and \mathcal{T} , we say that $(\mathbb{K}, *, \mathcal{T})$ is a field if:

1. $(\mathbb{K}, *, \mathcal{T})$ is a unit ring.
2. $(\mathbb{K} - \{e\}, \mathcal{T})$ is a group, where e is the neutral element of $*$.

Example 6.11

□ $(\mathbb{Z}, +, \times)$ is not a field.

□ $(\mathbb{R}, +, \times)$ is a commutative field.

6.2 Vector spaces

6.2.1 Vector space structure

Let \mathbb{K} be a field, and E be a non empty set equipped with

1. An internal operation (the vector addition):

$$\begin{aligned} + : E \times E &\longrightarrow E \\ (x, y) &\longrightarrow x + y \end{aligned}$$

2. An external operation (scalar multiplication) is defined as follows:

$$\begin{aligned} \cdot : \mathbb{K} \times E &\longrightarrow E \\ (\alpha, x) &\longrightarrow \alpha x \end{aligned}$$


check the following properties:

$$(i) \quad \forall \alpha, \beta \in \mathbb{K}, x \in E : (\alpha + \beta)x = \alpha x + \beta x.$$

$$(ii) \quad \forall \alpha \in \mathbb{K}, x, y \in E : \alpha(x + y) = \alpha x + \alpha y.$$

$$(3i) \quad \forall \alpha, \beta \in \mathbb{K}, (\alpha.\beta).x = \alpha.(\beta.x).$$

$$(4i) \quad 1.x = x, \forall x \in E.$$

Remark 6.3 

1. The elements of E are called vectors, and the members of the field \mathbb{K} are called scalars.
2. The identity element for $+$ in the set E is 0_E .
3. When $\mathbb{K} = \mathbb{R}$ we say that E is a real vector space.
4. When $\mathbb{K} = \mathbb{C}$ we say that E is a complex vector space.

6.2.2 Calculation rules in vector space

Let E be a vector space over the field \mathbb{K} , $\alpha, \beta \in \mathbb{K}$, $x, y \in E$, then:

- $\forall \alpha \in \mathbb{K}, \forall x \in E : \alpha.x = 0_E \Leftrightarrow (\alpha = 0_{\mathbb{K}}) \vee (x = 0_E).$
- $\forall \alpha \in \mathbb{K}, \forall x \in E : (-\alpha)x = (-\alpha.x) = \alpha(-x).$
- $\forall \alpha \in \mathbb{K}, \forall x, y \in E : \alpha(x - y) = \alpha(x + (-y)) = \alpha.x - \alpha.y.$
- $\forall \alpha, \beta \in \mathbb{K}, \forall x \in E : (\alpha - \beta).x = \alpha.x - \beta.y.$


Example 6.12 The triplet $(\mathbb{R}^2, +, \cdot)$ forms a vector space, and for general we take the triplet $(\mathbb{R}^n, +, \cdot)$ is a real vector space for $n \in \mathbb{N}^*$, also the triplet $(\mathbb{C}^n, +, \cdot)$ is a complex vector space for $n \in \mathbb{N}^*$.

6.2.3 Sub-vector spaces(Sub-spaces)

Let $(E, +, \cdot)$ be a vector space over the field \mathbb{K} .

Theorem 6.2 Let F be non-empty part from the vector space E over the field \mathbb{K} then we have the equivalence between:

1. $(F, +, \cdot)$ is sub-space from $(E, +, \cdot)$.
 - a. $\forall(x, y) \in F^2 : (x + y) \in F$.
 - b. $\forall\alpha \in \mathbb{K}, \forall x \in F : \alpha \cdot x \in F$.
2. $\forall(\alpha, \beta) \in \mathbb{K}^2, \forall(x, y) \in F^2 : (\alpha \cdot x) + (\beta \cdot y) \in F$.

Remark 6.4 

1. The identity element for E is $\{0_E\}$ which is a sub-space from E and belongs to every sub-space from it.
2. To prove that $F \neq \emptyset$ verify that $\{0_E\} \in F$.

Example 6.13 Let

$$F_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbb{R}^3, x, y \in \mathbb{R} \right\}; F_2 = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \in \mathbb{R}^3 \right\};$$

$$F_3 = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \in \mathbb{R}^3, x \leq 0, z \in \mathbb{R} \right\}$$

Does F_1, F_2, F_3 have a sub-space structure over \mathbb{R}^3 ?

$$1. \quad \square \text{ Since } 0_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in F_1, \text{ then } F_1 \neq \emptyset.$$

$$\square \forall \alpha, \beta \in \mathbb{R}, \forall X = \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} \in F_1, Y = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \in F_1 : \alpha X + \beta Y \stackrel{?}{\in} F_1.$$

$$\text{we have: } \alpha X + \beta Y = \alpha \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ 0 \end{pmatrix} \in F_1.$$

Hence F_1 is a sub-space over \mathbb{R}^3 .

2. \square Since $0_{\mathbb{R}^3} \notin F_2$, then F_2 is not a sub-space from \mathbb{R}^3 .

3. \square we have $0_{\mathbb{R}^3} \in F_3 \implies F_3 \neq \emptyset$.

$$\square \text{ For } X = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, Y = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}, \alpha = -2, \beta = -3$$

$$\text{we have: } \alpha X + \beta Y = -2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \notin F_3.$$

So F_3 is not a sub-space from \mathbb{R}^3 .

6.2.4 Operation on the sub spaces vector


The intersection

Let E_1, E_2 two sub-spaces from the vector space E over the field \mathbb{K} we have:

$$E_1 \cap E_2 = \{X \in E : X \in E_1 \wedge X \in E_2\}.$$

Example 6.14 Let $F_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : y = -2x, z = -x. \right\};$

$$F_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0. \right\}$$

 Prove that F_1 and F_2 are two subspaces from \mathbb{R}^3 , then find $F_1 \cap F_2$.

Theorem 6.3 The intersection of sub-spaces is a sub-space.

The union

Let E_1, E_2 two sub-spaces from the vector space E over the Field \mathbb{K} then

$$E_1 \cup E_2 = \{X \in E : X \in E_1 \vee X \in E_2\}.$$

Theorem 6.4 The union of two sub-spaces is not necessarily a sub-space.

The sum

Let E_1, E_2 two spaces from the vector space E over the field \mathbb{K} then:

$$E_1 + E_2 = \{X \in E : X = x_1 + x_2, x_1 \in E_1 \wedge x_2 \in E_2\}.$$

Theorem 6.5 The sum of two sub-spaces is a sub-space.

The direct sum

Let E_1, E_2 two sub spaces from the vector space E over the field \mathbb{K} , we say the E is the direct sum of E_1 and E_2 if and only if:

1. $E = E_1 + E_2$.
2. $E_1 \cap E_2 = \{0_E\}$,

and we write $E = E_1 \oplus E_2$ and in this case we say that E_1 and E_2 are complementary or additional.

Example 6.15 Let $E_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2, x \in \mathbb{R} \right\}; E_2 = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathbb{R}^2, y \in \mathbb{R} \right\}$

1. Prove that E_1 and E_2 are sub-space from \mathbb{R}^2 .
2. Prove that $\mathbb{R}^2 = E_1 \oplus E_2$.
1. Is clear.

2. We have $\mathbb{R}^2 = E_1 \oplus E_2 \Leftrightarrow \begin{cases} E_1 + E_2 = E \dots\dots (*) \\ E_1 \cap E_2 = \{0_{\mathbb{R}^2}\} \dots (**) \end{cases}$

(*) We have: $(\mathbb{R}^2 = E_1 + E_2) \Leftrightarrow ((E_1 + E_2) \subset \mathbb{R}^2 \wedge \mathbb{R}^2 \subset (E_1 + E_2))$.

(*) It's clear that $E_1 + E_2 \subset \mathbb{R}^2$, let's prove that $\mathbb{R}^2 \subset E_1 + E_2$.

we take $X \in \mathbb{R}^2 = \begin{pmatrix} x \\ y \end{pmatrix}$ we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \Rightarrow \mathbb{R}^2 \subset E_1 + E_2$.

Then (*) is verify.

(**) $E_1 \cap E_2 = \left\{ X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : X \in E_1 \wedge X \in E_2 \right\}$.

$X \in E_1 \Rightarrow y = 0, X \in E_2 \Rightarrow x = 0$. So $X = 0_{\mathbb{R}^2}$, then $E_1 \cap E_2 = 0_{\mathbb{R}^2}$, finally $\mathbb{R}^2 = E_1 \oplus E_2$.

6.2.5 Linear combination

Let E be a vector space over the field \mathbb{K} and $\alpha, \beta \in \mathbb{K}, \forall x, y \in E$

the vector $v = \alpha x + \beta y \in E$, called linear expression for the vectors x, y , and in general:

Let $(x_i)_{i \in I}$ is a set of vectors from E , we call linear expression for the vectors $(x_i)_{i \in I}$ every

expression in the form $\sum_{i=1}^n \alpha_i x_i$.

Example 6.16 Let $E = \mathbb{R}^2, v(1, -1), v_1(5, 2), v_2(1, 0)$, write v as a linear combination of v_1, v_2 .

We have: $\exists \alpha, \beta \in \mathbb{R} : v = \alpha v_1 + \beta v_2 \Rightarrow (1, -1) = \alpha(5, 2) + \beta(1, 0)$.


So $(1, -1) = (5\alpha + \beta, 2\alpha) \Rightarrow \begin{cases} 5\alpha + \beta = 1 \\ 2\alpha = -1 \end{cases} \Rightarrow \begin{cases} \alpha = \frac{-1}{2} \\ \beta = \frac{7}{2} \end{cases}$

$\Leftrightarrow v = \frac{-1}{2}v_1 + \frac{7}{2}v_2$.

6.2.6 Generator(spanning)set of vector space

Definition 6.12 Let E be a vector space over the field \mathbb{K} and $S = \{x_1, x_2, \dots, x_n\}$ is a subset vectors from E we say that S generate the space E if every vector of E is written as a linear combination of the vectors S it means:

$$\forall v \in E : \exists \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{K} / v = \sum_{i=1}^n \alpha_i x_i.$$

Remark 6.5  The sub space V from E which is composed of all linear combination $(x_i)_{i=1}^n$

is called the subspace generated (spanning) by the sub set V and we denote this by:

$$V = \text{span} \{x_1, x_2, \dots, x_n\} \text{ or } V = \langle \{x_i\}_{i=1}^n \rangle,$$

it means $V = \{x \in E : \exists \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{K} : x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\}$.

 The above V is the smallest subspace from E .

 To prove that any set is a subset from the space E we have just found the generated part.

Example 6.17 Let's $E = \mathbb{R}^2$ and $u(1, 2), v(0, -1)$ prove that $\{u, v\}$ generate \mathbb{R}^2 .

We have: $\forall u, v \in \mathbb{R}^2, \exists \alpha, \beta \in \mathbb{R} : w = \alpha u + \beta v / w(x, y)$.

$$w = \alpha(1, 2) + \beta(0, -1) \implies (x, y) = \alpha(1, 2) + \beta(0, -1)$$

$$\implies \begin{cases} x = \alpha \\ y = 2\alpha - \beta \end{cases} \implies \begin{cases} \alpha = x \\ \beta = 2x - y \end{cases}$$

Hence, $w = x(1, 2) + (-y + 2x)(0, -1)$.

6.2.7 Dependent and independent linear set

Definition 6.13 Let $(x_i)_{i=1}^n$ a set of vectors from the vector space E over the field \mathbb{K} , we say that $S = \{x_1, x_2, \dots, x_n\}$ is linear independent if:

$$\forall \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{K}, \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0_E \implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0_{\mathbb{K}},$$

we say that the set of vectors $(x_i)_{i=1}^n$ is linearly dependent if it is not linearly independent it means:

$$\exists \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{K}, \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0_E,$$

but $\alpha_1, \alpha_2, \dots, \alpha_n$ are not not null together.

Remark 6.6  The empty set is linearly independent in any vector space.

Example 6.18 1. The vector $v(3, 3, 1)$ is linearly dependent with the vectors

$$u_1(1, 1, 1), u_2(1, 1, 0) \text{ because } v = 2u_1 + u_2.$$

2. Does the vectors $v_1(1, 1, 0), v_2(0, 2, 2), v_3(3, 7, 1)$ are linearly independent?.

Let's prove that $\forall \alpha, \beta, \gamma \in \mathbb{R} : \alpha v_1 + \beta v_2 + \gamma v_3 = 0_{\mathbb{R}^3} \stackrel{?}{\implies} \alpha = \beta = \gamma = 0$.

$$\alpha(1, 0, 0) + \beta(0, 2, 2) + \gamma(3, 7, 1) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} \alpha + 3\gamma = 0 \\ 2\beta + 7\gamma = 0 \\ 2\beta + \gamma = 0. \end{cases} \Rightarrow \alpha = \beta = \gamma = 0.$$

Hence: $\alpha = \beta = \gamma = 0 \Rightarrow \{v_1, v_2, v_3\}$ linearly independent.

3. Find the value of x to make $\{u, v\}$ linearly dependent such that $u(3, -2), v(8, x)$.

6.2.8 Basis and dimension

Definition 6.14 The basis of a vector space E is the set of vectors that satisfy both properties:

- (i) Linearly independent.
- (ii) Vector space generator.

Definition 6.15 We call the number of elements of any basis of vector space E the dimension of the vector space, we denote it with the symbol $\dim E$.

Example 6.19 Let $v(0, 3), u(1, -1), E = \mathbb{R}^2$, does (u, v) form a basis for \mathbb{R}^2 , then determine $\dim \mathbb{R}^2$.

we have $v(0, 3), u(1, -1)$, form a basis for $\mathbb{R}^2 \Leftrightarrow \begin{cases} \{u, v\} \text{ linearly independent} \dots (*) \\ \{u, v\} \text{ generate } \mathbb{R}^2 \dots (**) \end{cases}$

(*) $\{u, v\}$ linearly independent?

$$\forall \alpha, \beta \in \mathbb{R} : \alpha u + \beta v = 0_{\mathbb{R}^2} \Rightarrow \alpha = \beta = 0.$$

$$\alpha(1, -1) + \beta(0, 3) = (0, 0) \Rightarrow \begin{cases} \alpha = 0 \\ -\alpha + 3\beta = 0 \end{cases} \Rightarrow \alpha = \beta = 0.$$


Hence $\{u, v\}$ are linearly independent.

(**) $\{u, v\}$ generate \mathbb{R}^2 ?

$$\exists \alpha, \beta \in \mathbb{R} : w = \alpha u + \beta v / w(x, y)$$

$$\text{We have } (x, y) = \alpha(1, -1) + \beta(0, 3) \Rightarrow \begin{cases} \alpha = x \\ -\alpha + 3\beta = y \end{cases} \Rightarrow \begin{cases} \alpha = x \\ \beta = 1/3(y + x). \end{cases}$$

We conclude that $\{u, v\}$ generate \mathbb{R}^2 , and $\dim \mathbb{R}^2 = 2$.

Remark 6.7 

(i) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a canonical basis of \mathbb{R}^2 .

(ii) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a canonical basis of \mathbb{R}^3 .

(iii) $\dim \mathbb{R}^n = n$.

Theorem 6.6 Let E be a vector space with a finite dimension and let $(\mathbf{x}_i)_{i=1}^n$ be a set of vectors from the vector E , then we have the equivalence:

- ① The set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ form a basis for E .
- ② Each vector of E is written in a single form as a linear combination of the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$.

The dimension of a subspace

Let E be a vector space $\dim E < \infty$ and $F \subset E$ is a subspace from E then we have:

- ① $\dim F \leq \dim E$.
- ② $\dim E = \dim F \Leftrightarrow E = F$.

The dimension theorem

Let E be a vector space ($\dim E < \infty$), F and G two sub-spaces from E then:

$$\dim(F + G) = \dim F + \dim G - \dim F \cap G.$$

Remark 6.8  If $E = F \oplus G$, then $\dim F + G = \dim F + \dim G$.

6.3 Linear maps

Definition 6.16 Let E, F two spaces over the the field \mathbb{K} and let the map $f : E \rightarrow F$, we sat that f is linear if:

1. $\forall x_1, x_2 \in E, f(x_1 + x_2) = f(x_1) + f(x_2)$.
2. $\forall x \in E, \forall \alpha \in \mathbb{K}, f(\alpha x) = \alpha f(x)$.

Definition 6.17 $\forall x_1, x_2 \in E, \forall \alpha, \beta \in \mathbb{K}; f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$.

Remark 6.9 Given two sets E and F . A relation from E to F is any assertion linking an element of E to an element of F that may or not may be verified. A relation is denoted by \mathcal{R} . The set E is called the start set of \mathcal{R} , and F is called the end set of \mathcal{R} .

For any element $x \in E$ and any element $y \in F$ satisfying \mathcal{R} , x of E is said to be related by \mathcal{R} to y , which is denoted $x\mathcal{R}y$. If $E = F$, the relation \mathcal{R} is called a binary relation in E .

- $(f \text{ linear}) \implies f(0_E) = 0_F$.
- $(f \text{ linear}) \implies \forall x \in E : f(-x) = -f(x)$.
- $f(0_E) \neq 0_F \implies (f \text{ is not linear})$.
- We denote the set of linear maps between the two vector spaces E and F by $\mathcal{L}(E, F)$.
- If f is linear and $F = \mathbb{R}$ we say that f is linear form.
- If f is linear and $E = F$ we say that f is endomorphism.
- If f is linear and bijective we say that f is isomorphism.
- If f is linear, bijective and $E = F$ we say that f is automorphism.

Example 6.20 State whether the following maps are linear or not.

- $f : \mathbb{R} \longrightarrow \mathbb{R}, x \longrightarrow f(x) = ax/a \in \mathbb{R}$.
- $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, X \longrightarrow f(X) = (x - y, y - z)$.
- $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, X \longrightarrow f(X) = (x + y - z, 1)$.
- $\Phi : \mathbb{R}[x] \longrightarrow \mathbb{R}[x], p \longrightarrow \Phi(p) = p'$.

- Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ the canonical basis for \mathbb{R}^3 we define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ a linear map realise:

$$f(e_1) = (1, 1), f(e_2) = (1, 3), f(e_3) = (2, 1).$$

Find the expression of f ?

Theorem 6.7 Let $f : E \longrightarrow F$ is a linear map

- E_1 is a subspace from $E \implies f(E_1)$ is a subspace from F .
- F_1 is a subspace from $F \implies f^{-1}(F_1)$ is a subspace from E .

6.3.1 Kernel and image of a linear map

Let $f : E \longrightarrow F$ be a linear map.

Definition 6.18 We call the set of elements x from E and realise $f(x) = 0_F$ the Kernel of the linear map f and we denoted it by $\ker f$:

$$\ker f = \{x \in E : f(x) = 0_F\}.$$

Remark 6.10  The kernel is a subspace from E .

Definition 6.19 We call the set of elements y from F which are images of E by the linear map f and we denoted it by $\text{Im} f$.

$$\text{Im} f = \{y \in F; \exists x \in E : y = f(x)\}.$$

Remark 6.11  The image is a subspace from F .

Theorem 6.8 Let E and F be two vector spaces over the field \mathbb{K} and $f : E \longrightarrow F$ is a linear map:

- f is injectif $\Leftrightarrow \ker f = \{0_E\}$.
- f is surjective $\Leftrightarrow \text{Im} f = F$.


Theorem 6.9 Let E and F be two vector spaces over the field \mathbb{K} and $\dim E = n$ and $f \in \mathcal{L}(E, F)$, we have:

$$\dim E = \dim \ker f + \dim \operatorname{Im} f$$

6.3.2 Linear map rank

We call the linear map rank the dimension of its image; it means $\operatorname{rank} f = \dim \operatorname{Im} f$.

Example 6.21 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x - y$.

 Prove that f is a linear map, find $\ker f$, $\operatorname{Im} f$ what do you conclude?.

We have $\ker f = \{X \in \mathbb{R}^2 : f(X) = 0\}$,

$\ker f = \{(x, y) \in \mathbb{R}^2, f(x, y) = 0\} \implies f(x, y) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

$\ker f = \{(x, x) \in \mathbb{R}^2\} = \{x(1, 1) / x \in \mathbb{R}\} = \langle (1, 1) \rangle$. So f is not injective.

$\operatorname{Im} f = \{z \in \mathbb{R}; \exists (x, y) \in \mathbb{R}^2 : f(x, y) = z\} \Leftrightarrow f(x, y) = z \Leftrightarrow x - y = z \Leftrightarrow$

$\operatorname{Im} f = \mathbb{R}$. So f is surjective.

Bibliography

- [1] J. M. Ferrard, " Dérivation, convexité Développements limités ", Cours de Mathématiques. Partie 1, Dérivabilité d'une fonction numérique.
- [2] J. M. Ferrard, " Limites, continuité, fonction usuelles ", Cours de Mathématiques. Partie 1, Fonctions numériques, généralités.
- [3] J. M. Ferrard, " Dérivation et intégration ", Cours de Mathématiques. Partie 1 , Dérivabilité d'une fonction vectorielle.
- [4] A . Bodin, B. Boutin, P. Romon, " Développements limités ", Applications des développements limités.
- [5] B. Ycart, " Développements limités ", Université Joseph Fourier, Grenoble.
- [6] B. Aoubiza, " Formules de Taylor et applications ", IUT Belfort-Montbéliard, Département GTR, 2013.
- [7] S. MILLES, " Algèbre 1 : Cours et exercices ", Polycopie de cours du module : Algebre01, Université Mohamed Boudiaf - M'sila, Département de Mathématiques, 2018-2019.
- [8] Bouchelaghem Fayçal. Algèbre RAPPEL ET EXERCICES RESOLUS, Ecole esst Annaba.
- [9] BENAÏSSA CHERIF Amin, Mathématiques 1, Cours et exercices Corrigés, Université des Sciences et de la Technologie d'Oran-M.Boudiaf, Faculté des mathématiques et l'informatique.
- [10] M. A. Henning, " An introduction to logic and proof techniques ", School of Mathematical Sciences University of KwaZulu-Natal.
- [11] P. Brown, " Limits and continuity ", University of NSW. 2011/2012.

- [12] S. S. Epp , ” Discrete Mathematics with Applications ”, Third Edition. Thomson Brooks/Cole. 2004.
- [13] E. Mendelson, ” Introduction to Mathematical Logic ”, Fourth Edition, Chapman and Hall, 1997.
- [14] ص. ركاب، م. يوشقوف، ن. عرعار، "دروس رياضيات 1" ، جامعة الإخوة منتوري قسنطينة، 2016 / 2017 .
- [15] ع. ك. سعيح، " محاضرات في مقياس الرياضيات 1 " ، جامعة أمحمد بوقرة بومرداس، 2016 / 2017 .
- [16] لطرش عبد الكريم، مطبوعة بيداغوجية تحت عنوان: رياضيات 1: دروس و تمارين، جامعة 20 أوت 1955 سكيكدة، 2023/2024 .
- [17] ص. ركاب، م. يوشقوف، ن. عرعار، " التحليل الرياضي : الإشتقاق و دساتير المتوسط و تالور "، المدرسة العليا للأساتذة (القبة).
- [18] ع. قوبا، " التحليل 2 "، منشورات المعهد العالي للعلوم التطبيقية والتكنولوجي، 2017 .