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Numerical Solutions of Some Partial Differential Equations Using Semi-Analytic Methods

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Numerical Solutions of Partial Differential Equations Using Semi-Analytical Methods

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بسم الله الرحمن الرحيم

Dedications

With love and respect, I dedicate this work :

To the man to whom I owe my life, my success, and all my respect—my dear father .

To the most precious person in this world, the one who raised me, nurtured me, and never stopped praying for me—my dear mother .

To my dear sisters and brothers for their continuous support and encouragement at every moment, especially my sister Meriem, my first teacher, my loyal companion, and my pillar throughout this journey.

To the joy and happiness of the family: Layan, Abd el-Raouf, Baha' El-Din, Amin.

To all my family and friends.

KHouder IKram

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KHouder IKram

ABSTRACT

This work addresses the numerical solution of partial differential equations (PDEs) using semi-analytical methods. We begin by introducing the fundamental principles of PDEs, laying the groundwork for the application of advanced solution techniques. A detailed investigation is then carried out on three widely used semi-analytical methods: the Homotopy Perturbation Method (HPM), the Variational Iteration Method (VIM), and the Adomian Decomposition Method (ADM). The convergence properties of each method are analyzed to assess their reliability and effectiveness in solving nonlinear problems. To demonstrate their practical utility, we apply these methods to the Burgers equation a canonical nonlinear PDE arising in fluid mechanics. The results confirm that semi-analytical methods provide accurate, efficient, and easily implementable solutions, offering a robust alternative to purely numerical approaches.

Keywords: partial differential equations, HPM, VIM, ADM, Burgers equation.

RÉSUMÉ

Ce travail porte sur la résolution numérique des équations aux dérivées partielles (EDP) à l'aide de méthodes semi-analytiques. Nous commençons par introduire les principes fondamentaux des EDP, servant de base à l'application de techniques avancées de résolution. Une étude détaillée est ensuite consacrée à trois méthodes semi-analytiques largement utilisées : la méthode de perturbation par homotopie (HPM), la méthode d'itération variationnelle (VIM) et la méthode de décomposition d'Adomian (ADM). Les propriétés de convergence de chaque méthode sont analysées afin d'évaluer leur fiabilité et leur efficacité dans le traitement des problèmes non linéaires. Pour illustrer leur utilité pratique, ces méthodes sont appliquées à l'équation de Burgers, une EDP non linéaire classique en mécanique des fluides. Les résultats confirment que ces approches offrent des solutions précises, efficaces et simples à mettre en oeuvre, représentant ainsi une alternative robuste aux méthodes purement numériques.

Mots-clé: Equations différentielles partielles, HPM, VIM, ADM, l'équation de Burgers.

ملخص

يتناول هذا العمل دراسة الحلول العددية للمعادلات التفاضلية الجزئية (PDEs) باستخدام الطرق شبه التحليلية. نبدأ بتقديم مبادئ أساسية للمعادلات، تمهيدًا لتطبيق تقنيات متقدمة لحلها. ثم نعرض دراسة مفصلة لثلاث طرق شبه تحليلية شائعة، وهي: طريقة الاضطراب بالتشابه(HPM) ، وطريقة التكرار التغايري(VIM) ، وطريقة التحليل إلى مركبات أدوميان(ADM) .

يتم تحليل خصائص التقارب لكل طريقة من أجل تقييم دقتها وكفاءتها في حل المعادلات غير الخطية ولتأكيد الجانب التطبيقي، تم استخدام هذه الطرق لحل معادلة بورغرز، وهي معادلة تفاضلية جزئية غير خطية تُستخدم كثيرًا في ميكانيكا الموائع. أظهرت النتائج أن هذه الطرق توفر حلولًا دقيقة وفعالة وسهلة الاستخدام، مما يجعلها بديلًا واعدًا عن الأساليب العددية التقليدية.

الكلمات المفتاحية: المعادلات التفاضلية الجزئية، طريقة الاضطراب التماثلي، طريقة التكرار التغايري، طريقة تحليل آدميان، معادلة برجرز، الحلول العددية.

Notations

- $\to$ designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- ∇ stands for the gradient operator.
- div is the divergence operator.
- $\dfrac{\partial}{\partial x}$ partial derivative.
- $\dfrac{\partial}{\partial n}$ outward normal derivative.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^n$ is the real space of dimension n.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.
- Ω^c the complement of Ω.
- $< .,. >$ denotes the scalar product.
- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.
- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.
- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.
- $H(\Omega)$ Hilbert space.
- $B(\Omega)$ Banach space.

CONTENTS

GENERAL INTRODUCTION

Partial differential equations (PDEs) play a vital role in modeling a wide range of phenomena in physics, biology, and engineering. These equations describe systems involving several variables, such as heat distribution, fluid flow, and wave propagation. However, the complexity of many PDEs −especially those that are non-linear or high-dimensional−makes finding exact analytical solutions extremely challenging. In such cases, numerical methods are often used, though they can be computationally expensive and may lack the precision of analytical approaches [9].

To overcome these limitations, semi-analytical methods have gained prominence as effective tools for approximating solutions to complex PDEs. These methods combine the accuracy of analytical techniques with the flexibility and efficiency of numerical procedures. Among the most widely used semi-analytical methods are the Homotopy Perturbation Method (HPM), the Variational Iteration Method (VIM), and the Adomian Decomposition Method (ADM). These techniques are especially effective for solving non-linear PDEs, providing series-based or iterative solutions that are both accurate and computationally efficient [11, 19, 22, 26].

The Homotopy Perturbation Method (HPM) is an iterative technique that constructs a homotopy to continuously transform a simple problem into a more complex one. It is particularly useful for solving non-linear problems without requiring linearization or discretization, making it a powerful and flexible tool [11]. HPM has been successfully applied in fields such as fluid dynamics and heat transfer [3].

The Variational Iteration Method (VIM), on the other hand, employs variational principles and Lagrange multipliers to iteratively improve approximate solutions. It is known for its fast convergence and its ability to handle boundary conditions and complex structures effectively [21].

The Adomian Decomposition Method (ADM) decomposes the solution into a series of functions or polynomials, allowing one to solve non-linear and time-dependent PDEs without requiring linearization. ADM is widely used because of its simplicity, accuracy, and ease of implementation [21].

Despite their success, these semi-analytical methods still face challenges when applied to real-world problems, particularly in improving convergence speed, accuracy, and general applicability. This thesis investigates the application of HPM, VIM, and ADM to a variety of non-linear PDEs. It focuses on improving these methods effectiveness and expanding their practical use in solving complex differential equations.

Main Objective: The main objective of this thesis is to present, analyze, and apply semi-analytical methods for solving some partial differential equations.

Plan of the Thesis

This manuscript consists of a general introduction and three chapters.

Chapter 1:

This chapter introduces essential preliminary concepts, including key definitions, theorems, and mathematical tools necessary for understanding the rest of the thesis. It also provides a brief historical background and classification of PDEs, along with a discussion of their applications in modeling real-world systems.

Chapter 2:

This chapter presents a detailed overview of several semi-analytical methods, specifically the Homotopy Perturbation Method (HPM), the Variational Iteration Method (VIM), and the Adomian Decomposition Method (ADM). The convergence properties and theoretical foundations of each method are also discussed.

Chapter 3:

In the final chapter, selected non-linear PDEs are solved using the HPM, VIM, and ADM methods. Numerical results and graphical illustrations are provided to evaluate and compare the performance of each method.

CHAPTER 1

PRIMARY CONCEPTS

The importance of both nonlinear and linear differential equations becomes clear when tackling problems in various fields, such as physical and chemical sciences, as well as engineering. Many of these problems are inherently nonlinear. It is widely recognized that solving such nonlinear problems presents significant challenges, requiring the application of efficient methods to obtain analytical, approximate, or numerical solutions.

This chapter is divided into two sections. The first section covers preliminaries, presenting key definitions and theorems that will be referenced in later chapters. The second section provides an overview of the history of PDEs and their various types, and discusses some of the applications and modeling techniques associated with PDE problems.

1.1 Functional Spaces

In this section, we present a preliminary discussion in which we review fundamental concepts and results from functional analysis theory, which serve as an essential tool in the theory of functional calculus.

1.1.1 Spaces of integrable functions

Definition 1.1 ([25]) Let $\Omega = [0, T]$ $(0 < T < +\infty)$ is a interval of $\mathbb{R}$ and $1 \leq p \leq \infty$.

1. For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the space of functions f real on Ω such that f is measurable and $\int_0^T |f(t)|^p dt < \infty$.
2. For $p = \infty$, the space $L^\infty(\Omega)$ is the space of measurable, the functions f is bounded almost every where $(p.p)$ on Ω.

Theorem 1.1 ([25]) *Let $\Omega = [0, T]$ $(0 < T < +\infty)$ is a finite interval of $\mathbb{R}$.*

1. *For $1 \leq p < \infty$, the space $L^p(\Omega)$ is a Banach space equipped with the norm:*

$$\|f\|_p = \left(\int_0^T |f(t)|^p dt \right)^{1/p}.$$

2. *The space $L^\infty(\Omega)$ is a Banach space equipped with the norm:*

$$\|f\|_{\infty} = \inf\{M \geq 0 : |f(t)| \leq M \ p.p \ On \ \Omega\}.$$

1.1.2 Spaces of continuous and absolutely continuous functions

Definition 1.2 ([27]) Let $\Omega = [0, T]$ $(0 < T < +\infty)$ is a finite interval of $\mathbb{R}$ and $n \in \mathbb{N}$. We denote by $C^n(\Omega)$ the space of functions f whose derivatives of order less than or equal to n are continuous on Ω, equipped with the norm:

$$\|f\|_{C^n(\Omega)} = \sum_{k=0}^{n} \left\| f^{(k)} \right\|_{C(\Omega)} = \sum_{k=0}^{n} \max_{t\in\Omega} \left| f^{(k)}(t) \right|, n \in \mathbb{N}.$$

In particular, if $n = 0, C^0(\Omega) = C(\Omega)$ the space of functions f continuous on Ω equipped with the norm:

$$\|f\|_{C(\Omega)} = \max_{t\in\Omega} |f(t)|.$$

Definition 1.3 ([27]) Let $\Omega = [0, T]$ $(0 < T < +\infty)$ is a finite interval of $\mathbb{R}$.
We denote by $AC(\Omega)$ the space of primitive integrable functions, that is to say:

$$AC(\Omega) = \left\{ f / \exists u \in L^1(\Omega) : f(t) = c + \int_0^t u(s)ds \right\},$$

and it is called $AC(\Omega)$ the space of absolutely continuous functions on Ω.

Definition 1.4 ([27]) For $n \in \mathbb{N}^*$, we denote by $C^n_\mu(\Omega)$ the space of functions f whose derivatives are continuous on Ω to order $(n-1)$ and such that $f^{(n-1)} \in AC(\Omega)$ i.e.:

$$AC^n(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{C}, f^{(k)} \in C(\Omega), k \in \{0, 1, \ldots, n-1\}, f^{(n-1)} \in AC(\Omega) \right\}$$

.
In particular $AC^1(\Omega) = AC(\Omega)$.

1.1.3 Fixed-point theorem

Definition 1.5 Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two normed vector spaces over the same field F.
A map, $A : X \longrightarrow Y$ is said to be Lipchitz continuous if there is a positive constant k such that

$$\forall x, y \in X : \|Ax - Ay\|_Y \leq k \|x - y\|_X.$$

If $k \in [0, 1[$, A is called a contraction mapping.

Theorem 1.2 *(Banach's fixed point theorem) [27] Let $(X, \|.\|_X)$ be a Banach space and let $A : X \longrightarrow X$ be a contraction on X. Then A has a unique fixed point $M \in X$ such that:*

$$A(M) = M.$$

Moreover, if $T^k, k \in \mathbb{N}$ is a sequence of operators defined by

$$T^1 = T \ et \ T^k = TT^{k-1}, k \in \mathbb{N}\backslash\{1\},$$

then for every $u_0 \in B$ the sequence $\left\{T^k u_0\right\}_{k=0}^{\infty}$ converges to the fixed point u^*.

$$\lim_{k\to\infty} \left\| T^k u_0 - u^* \right\| = 0.$$

1.2 Partial Differential Equations (PDEs)

PDE is an equation that contains the dependent variable, and its partial derivatives. In the ordinary differential equations, the dependent variable $v = v(x)$ depends only on one independent variable x. Unlike the ODEs, the dependent variable in the PDEs, such as $v = v(x, t)$ or $v = v(x, y, t)$, must depend on more than one independent variable. If $v = v(x, t)$, then the function v depends on the independent variable x and on the time variable t. In addition, it is well known that most of the phenomena that arise in mathematical, chemical, physics, and engineering fields can be described by partial differential equations. For example, the heat flow and the wave propagation phenomena, physical phenomena of fluid dynamics, and many other models are described by partial differential equations.
In this section, we provides a foundational understanding of PDEs, from their historical development to the types of equations and their broad range of applications.

1.2.1 Historical Development of PDEs

Partial Differential Equations (PDEs) have played a pivotal role in the development of modern mathematics, physics, and engineering. The study of PDEs has evolved over centuries, from early attempts at solving simple physical problems to the sophisticated theories and methods used today.

1.2.1.1 Early Beginnings

The history of PDEs begins with the need to describe physical phenomena. The earliest recorded use of PDEs can be traced back to the 17th century with the work of mathematicians and physicists such as **Isaac Newton** and **Gottfried Wilhelm Leibniz**, who formulated laws of motion and fluid dynamics that could be described by differential equations.

1.2.1.2 18th Century Developments

The 18th century saw substantial progress, especially in the areas of heat conduction and wave propagation. **Jean-Baptiste Joseph Fourier** (1768 -1830) made significant contributions with his work on heat conduction, leading to the *Fourier series* and the Fourier transform. These ideas helped formalize the study of solutions to the heat equation, a fundamental PDE in mathematical physics.

1.2.1.3 19th Century Advancements

In the 19th century, the study of PDEs was greatly advanced by **Augustin-Louis Cauchy**, **Bernhard Riemann**, and **William Rowan Hamilton**, who developed systematic approaches to solving and analyzing PDEs. The *Cauchy-Riemann equations*, for example, laid the foundation for complex analysis and the study of boundary value problems.

1.2.1.4 20th Century and Modern Developments

The 20th century brought more sophisticated methods for solving PDEs, such as the development of *functional analysis* by **David Hilbert** and the introduction of modern numerical methods for approximating solutions, including *finite element analysis.* The study of nonlinear PDEs, *chaos theory*, and *computational modeling* became key areas of interest, particularly in fields like fluid dynamics, electromagnetism, and materials science.

1.2.2 Types of PDEs and Examples

Partial Differential Equations can be classified into several categories based on their form and characteristics. The three primary types of PDEs are **elliptic**, **parabolic**, and **hyperbolic** equations.

1.2.2.1 Elliptic PDEs

The elliptic equation is an equation that satisfies the following property:

$$B^2 - 4AC < 0.$$

Elliptic PDEs generally describe steady-state systems, such as the distribution of temperature in a solid object. The most famous example is the *Laplace equation*:

$$\Delta u = 0,$$

where Δ is the Laplacian operator. Another important example is the *Poisson equation*:

$$\Delta u = f(x).$$

Elliptic equations are used in electrostatics, steady-state heat conduction, and potential theory.

1.2.2.2 Parabolic PDEs

The parabolic equation is an equation that satisfies the following property:

$$B^2 - 4AC = 0.$$

Parabolic PDEs describe systems that evolve over time but reach a steady state in the long run. The *heat equation* is the classic example:

$$u_t = \alpha \Delta u,$$

where u_t is the time derivative, and α is a constant related to the thermal conductivity. Parabolic equations are used to model heat diffusion, population dynamics, and other phenomena that evolve over time.

1.2.2.3 Hyperbolic PDEs

The hyperbolic equation is an equation that satisfies the following property:

$$B^2 - 4AC > 0.$$

Hyperbolic PDEs are used to describe wave propagation, such as sound waves, light waves, and fluid dynamics. The *wave equation* is a well-known example:

$$u_{tt} = c^2 \Delta u,$$

where c is the wave speed. Hyperbolic equations are important in acoustics, electromagnetics, and seismology.

1.2.2.4 Mixed-Type PDEs

Some equations do not neatly fit into one of the three categories above and may exhibit properties of multiple types. An example is the *Korteweg-de Vries equation*, which is used to model waves on shallow water surfaces.

1.2.3 The order of a PDE

Definition 1.6 The order of a partial differential equation (PDE) is the order of the highest partial derivative that appears in the equation.

Example 1.1 The following PDEs:

1. $u_x - u_y = 0$.
2. $u_{xx} - u_t = 0$.
3. $u_y - u_{xxx} = 0$.

The following are first-order, second-order, and third-order PDEs, respectively.

1.2.4 Linear and Non-linear PDEs

1. Partial differential equations are classified as linear or nonlinear.
2. If a partial differential equation is linear:
   (a) The power of the dependent variable and of each partial derivative contained in the equation is one.
   (b) The coefficients of the dependent variable and of each partial derivative are constants or depend only on the independent variables.
3. However, if one of these conditions is not met, the equation is said to be *nonlinear*.

Example 1.2 Let the following PDEs be:

$$xu_{xx} + yu_{yy} = 0, \quad (1.1)$$

$$uu_t + xu_x = 2. \quad (1.2)$$

The PDE (1.1) is linear because the power of each partial derivative, u_{xx} and u_{yy}, is equal to one. Moreover, the coefficients of the partial derivatives u_{xx} and u_{yy} are the independent variables x and y, respectively.

Although the PDE (1.2) is nonlinear because the coefficient of the partial derivative u_t depends on the variable u.

1.2.5 Homogeneous and non-homogeneous PDEs

Definition 1.7 A PDE of any order is said to be homogeneous if its right-hand side is zero, that is, if it is of the form where the differential operator L is a linear operator and u is the unknown function. Otherwise, it is called a non-homogeneous PDE.

Example 1.3 Let the following PDEs:

$$u_t = 4u_{xx}, \quad (1.3)$$

$$u_t = u_{xx} + x. \quad (1.4)$$

The PDE (1.3) is homogeneous because all the terms in the equation only contain partial derivatives of u.

The PDE (1.4) is non-homogeneous because one term in the equation contains the dependent variable x.

1.2.6 Applications and Modeling of PDEs

PDEs are essential for modeling and solving real-world problems in many fields, from physics to engineering, economics, and biology. Below are a few prominent applications.

1.2.6.1 Heat and Diffusion Processes

One of the most important applications of PDEs is in the modeling of heat transfer. The *heat equation* is a parabolic PDE, describes how heat diffuses through a medium over time. It is widely used in engineering to design systems for thermal management, such as cooling mechanisms in electronics or the heating and cooling of buildings.

1.2.6.2 Fluid Dynamics and Navier-Stokes Equations

The *Navier-Stokes equations* are a set of nonlinear PDEs that describe the motion of incompressible fluids. These equations are central to fluid mechanics and have applications in aerodynamics, meteorology, oceanography, and engineering. Solving the Navier-Stokes equations for turbulence remains one of the most challenging problems in mathematics.

1.2.6.3 Wave Propagation

The wave equation, a hyperbolic PDE, models the propagation of waves in various mediums, including sound waves in air, seismic waves in the Earth, and electromagnetic waves in space. The equation is used in fields such as acoustics, optics, and telecommunications.

1.2.6.4 Electromagnetic Fields

Maxwell's equations, which describe electromagnetism, can be written as a system of PDEs. These equations model how electric and magnetic fields evolve over time and are fundamental to the design of electrical circuits, wireless communication, and understanding phenomena such as light and radiation.

1.2.6.5 Biological Modeling

PDEs are used in biology to model the diffusion of substances within cells, the spread of diseases through populations, and the dynamics of ecosystems. For example, the *reaction-diffusion equations* model the interaction between chemical species in biological systems and can describe phenomena such as pattern formation in animal coats or the spread of a virus.

1.2.6.6 Economics and Finance

In economics, PDEs are used to model market behaviors and option pricing. The *Black-Scholes equation*, a famous example of a PDE in finance, is used to price options and derivatives in financial markets. The equation models the evolution of the price of financial assets over time and incorporates factors like interest rates and volatility.

1.2.6.7 Environmental Modeling

PDEs are also used to model environmental systems, such as the dispersion of pollutants in air or water, groundwater flow, and the spread of wildfires. Environmental scientists use these equations to predict the effects of different variables on ecosystems and to design sustainable solutions.

Partial Differential Equations (PDEs) provide a robust framework for modeling complex physical, biological, and financial systems. From their historical roots in classical mechanics to their modern-day applications in diverse fields, PDEs remain a central tool for understanding and solving real-world problems. By classifying PDEs into types such as elliptic, parabolic, and hyperbolic, mathematicians can choose the most appropriate methods for solving these equations and gain insight into the systems they model.

CHAPTER 2

SEMI ANALYTICAL METHODS

Semi-Analytical Methods are powerful techniques used to find approximate analytical solutions to nonlinear differential equations. Unlike purely numerical methods, they retain the functional form of the solution. In this chapter, we briefly present some semi-analytical methods: the homotopy perturbation method (HPM), the variational iteration method (VIM), the Adomian decomposition method (ADM), and then we study their convergences.

2.1 Homotopy perturbation method (HPM)

The homotopy perturbation method was proposed and developed by the Chinese mathematician **Ji-Haun He** in 1999 [12], [13],[14]. This method has been widely used to solve nonlinear boundary and initial value problems. The homotopy perturbation method is a powerful mathematical tool for studying a wide variety of problems arising in different fields. It is successfully obtained by combining the theory of homotopy in topology with perturbation theory. The key feature of the homotopy perturbation method is that it provides an almost exact solution to a wide range of linear and nonlinear problems, without the need for unrealistic assumptions, linearization, discretization, or the calculation of Adomian polynomials [23].

2.1.1 Description of the Method

The basic concepts of the homotopy perturbation method for the following nonlinear differential equation as follows

$$L(u) + N(u) = f(r),\ r \in \Omega, \tag{2.1}$$

with the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0,\ r \in \Gamma, \tag{2.2}$$

where L is linear differential operator and N is nonlinear differential operator, $f(r)$ is a known analytic function, B is a boundary operator and n is the unit outward normal and Γ is the boundary of the domain Ω. In this the method HPM, we defined the homotopy as

$$v(r,p) : \Omega \times [0, 1] \to \mathbb{R}, \tag{2.3}$$

which corresponds to

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{2.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $r \in \Omega$ and $p \in [0,1]$ are the attached parameters, u_0 is initial appoarch value which fulfil the initial condition. From equation (2.4) and (2.5), it is obtained

$$H(v,0) = L(v) - L(u_0) = 0, \tag{2.6}$$

and

$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{2.7}$$

He [15] assumes that the solutions of (2.3) and (2.4), can be expressed as the power series of p:

$$v = \sum_{i=0} p^i v_i = v_0 + pv_1 + p^2 v^2 + ... \tag{2.8}$$

The appoarch solution of (2.1) is

$$u = \lim_{p \to 1} \sum_{i=0} p^i v_i = v_0 + v_1 + v_2 + ... \tag{2.9}$$

2.1.2 Convergence of HPM

The convergence of the HPM method is studied in [6, 5].

We can rewrite relation (2.5) as follows:

$$L(v) - L(u_0) = p\left[f(r) - L(u_0) - N(v)\right]. \tag{2.10}$$

By substituting (2.8) into (2.5), we obtain:

$$L\left(\sum_{i=0}^{\infty} p^i v_i\right) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]. \tag{2.11}$$

Therefore

$$\sum_{i=0}^{\infty} L(v_i) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]. \tag{2.12}$$

According to the Maclaurin series expansion of $N\left(\sum_{i=0}^{\infty} p^i v_i\right)$ with respect to p, we have:

$$N\left(\sum_{i=0}^{\infty} p^i v_i\right) = \sum_{i=n=0}^{\infty} \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right)_{p=0} p^i. \tag{2.13}$$

According to [10], we have:

$$\left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right)_{p=0} = \left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i v_i\right)\right)_{p=0}.$$

Then

$$N\left(\sum_{i=0}^{\infty} p^i v_i\right) = \sum_{i=n=0}^{\infty} \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i v_i\right)\right)_{p=0} p^i.$$

Let :

$$H_n(v_0, v_1, ..., v_n) = \left(\frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i v_i \right) \right)_{p=0}, n = 0, 1, 2, ..., \tag{2.14}$$

where H_n are called He polynomials [10].
Then

$$N\left(\sum_{i=0}^{\infty} p^i v_i \right) = \sum_{i=0}^{\infty} H_i p^i. \tag{2.15}$$

By substituting (2.15) into (2.11), we obtain:

$$\sum_{i=0}^{\infty} L(v_i) - L(u_0) = p\left[f(r) - L(u_0) - \sum_{i=0}^{\infty} H_i p^i \right]. \tag{2.16}$$

By matching the terms with those of the same power of p, we obtain:

$$\begin{array}{rcl}
p^0 & : & L(v_0) - L(u_0) = 0, \\
p^1 & : & L(v_1) = f(r) - L(u_0) - H_0, \\
p^2 & : & L(v_2) = -H_1, \\
p^3 & : & L(v_3) = -H_2, \\
 & & . \\
 & & . \\
 & & . \\
p^{n+1} & : & L(v_{n+1}) = -H_n, \\
 & & . \\
 & & . \\
 & & .
\end{array}$$

Therefore, we conclude that:

$$\left\{ \begin{array}{lcc}
p^0 & : & v_0 = u_0, \\
p^1 & : & v_1 = L^{-1}(f(r)) - u_0 - L^{-1}(H_0), \\
p^2 & : & v_2 = -L^{-1}(H_1), \\
p^3 & : & v_3 = -L^{-1}(H_2), \\
 & & . \\
 & & . \\
 & & . \\
p^{n+1} & : & v_{n+1} = -L^{-1}(H_n), \\
 & & . \\
 & & . \\
 & & .
\end{array} \right. \tag{2.17}$$

Theorem 2.1 *The solution of the equation (2.1) obtained by the homotopy perturbation method (HPM) is equivalent to the determination of S_n given by:*

$$S_n = v_1 + v_2 + ... + v_n, S_0 = 0. \tag{2.18}$$

By using the iterative scheme

$$S_{n+1} = -L^{-1} N_n (S_n + v_0) - u_0 + L^{-1}(f(r)), \tag{2.19}$$

where

$$N_n\left(\sum_{i=0}^{n} v_i\right)=\sum_{i=0}^{n} H_i, n=0,1,2,.... \tag{2.20}$$

Proof. *For $n=0$, according to (2.19), we have:*

$$S_1=-L^{-1}N_0\left(S_0+v_0\right)-u_0+L^{-1}\left(f\left(r\right)\right)=-L^{-1}\left(H_0\right)-u_0+L^{-1}\left(f\left(r\right)\right). \tag{2.21}$$

Then

$$v_1=-L^{-1}\left(H_0\right)-u_0+L^{-1}\left(f\left(r\right)\right).$$

For

$$\begin{aligned} S_2 &= -L^{-1}N_1\left(S_1+v_0\right)-u_0+L^{-1}\left(f\left(r\right)\right) \\ &= -L^{-1}\left(H_1+H_0\right)-u_0+L^{-1}\left(f\left(r\right)\right) \\ &= -L^{-1}\left(H_1\right)+v_1. \end{aligned}$$

According to $S_2=v_1+v_2$, we obtain:

$$v_2=-L^{-1}\left(H_1\right).$$

The proof of this theorem will be done by induction.
Let us suppose that:

$$v_{k+1}=-L^{-1}\left(H_k\right), \quad \textit{for} \quad k=1,2,...,n-1,$$

therefore

$$\begin{aligned} S_{n+1} &= -L^{-1}N_n\left(S_n+v_0\right)-u_0+L^{-1}\left(f\left(r\right)\right) \\ &= -L^{-1}\left(\sum_{i=0}^{n} H_i\right)-u_0+L^{-1}\left(f\left(r\right)\right) \\ &= -\sum_{i=0}^{n} L^{-1}\left(H_i\right)-u_0+L^{-1}\left(f\left(r\right)\right) \\ &= v_1+v_2+...+v_n-L^{-1}\left(H_n\right). \end{aligned}$$

Then , from (2.18), we can find

$$v_{n+1}=-L^{-1}\left(H_n\right).$$

This result is identical to that of (2.17) obtained by the HPM method. ■

Theorem 2.2 *Let B be a Banach space.*

1. $\sum_{i=0}^{\infty} v_i$ *converges to $S\in B$,if*

$$\exists\left(0\leq\lambda<1\right) \quad \textit{such that} \quad \left(\forall n\in\mathbb{N}\implies\parallel v_n\parallel\leq\lambda\parallel v_{n+1}\parallel\right). \tag{2.22}$$

2. $S=\sum_{i=1}^{\infty} v_i$ *verify*

$$S=-L^{-1}N\left(S+v_0\right)-u_0+L^{-1}\left(f\left(r\right)\right). \tag{2.23}$$

Proof.

1. *we have:*

$$\|S_{n+1}-S_n\| = \|v_{n+1}\| \leq \lambda \|v_n\| \leq \lambda^2 \|v_{n-1}\| \leq ... \leq \lambda^{n+1}\|v_0\|. \tag{2.24}$$

For $n, m \in \mathbb{N}, n \geq m$, *we have:*

$$\begin{aligned}
\|S_n - S_m\| &= \|\left(S_n - S_{n-1}\right) + \left(S_{n-1} - S_{n-2}\right) + ... + \left(S_{m+1} - S_m\right)\| \\
&\leq \|\left(S_n - S_{n-1}\right)\| + \|\left(S_{n-1} - S_{n-2}\right)\| + ... + \|\left(S_{m+1} - S_m\right)\| \\
&\leq \lambda^n \|v_0\| + \lambda^{n-1}\|v_0\| + ... + \lambda^{m+1}\|v_0\| \\
&\leq \left(\lambda^n + \lambda^{n-1} + ... + \lambda^{m+1}\right)\|v_0\| \\
&\leq \left(\lambda^{m+1} + ... + \lambda^n + ...\right)\|v_0\| \\
&\leq \lambda^{m+1}\left(1 + \lambda + ... + \lambda^n + ...\right)\|v_0\| \\
&\leq \frac{\lambda^{m+1}}{1-\lambda}\|v_0\|.
\end{aligned}$$

Therefore

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

$(S_n)_{n\geq 0}$, *is a Cauchy sequence in the Banach space , and it is convergent , that is to say:*

$$\exists S \in B, with: \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} v_n = S.$$

2. *According to* (2.18), *we have:*

$$\begin{aligned}
\lim_{n\to\infty} S_{n+1} &= -L^{-1} \lim_{n\to\infty} N_n\left(S_n + v_0\right) - u_0 + L^{-1}\left(f\left(r\right)\right) \\
&= -L^{-1} \lim_{n\to\infty} N_n\left(\sum_{i=0}^{n} v_i\right) - u_0 + L^{-1}\left(f\left(r\right)\right) \\
S &= -L^{-1} \lim_{n\to\infty} \sum_{i=0}^{n} H_i - u_0 + L^{-1}\left(f\left(r\right)\right) \\
&= -L^{-1} \sum_{i=0}^{\infty} H_i - u_0 + L^{-1}\left(f\left(r\right)\right).
\end{aligned}$$

From (2.19) *and* (2.12), *for,* $p = 1$, *we get:*

$$\sum_{i=0}^{\infty} H_i = N_n\left(\sum_{i=0}^{\infty} v_i\right).$$

Therefore

$$\begin{aligned}
S &= -L^{-1}N\left(\sum_{i=0}^{\infty} v_i\right) - u_0 + L^{-1}\left(f\left(r\right)\right) \\
&= -L^{-1}N\left(S + v_0\right) - u_0 + L^{-1}\left(f\left(r\right)\right).
\end{aligned}$$

■

Lemma 2.1 *The equation (2.23) is equivalent to:*

$$L(u) + N(u) - f(r) = 0. \tag{2.25}$$

Proof. *We write equation (2.23) as follows:*

$$S + u_0 = -L^{-1}N(S + v_0) + L^{-1}(f(r)).$$

By applying the operator L to the previous equation, we obtain:

$$L(S + u_0) = -N(S + v_0) + f(r).$$

Since $u_0 = v_0$,

$$L(S + v_0) = -N(S + v_0) + f(r).$$

Let, $u = S + v_0 = \sum_{i=0}^{\infty} v_i$, the equation (2.22) becomes the original equation.
The solution of equation (2.23) is the same as the solution to $A(u) - f(r) = 0$. ■

Example 2.1 We consider the following nonlinear differential equation:

$$\begin{cases} u' + u^2 = 0, t \geq 0, t\epsilon\Omega, \\ u(0) = 1. \end{cases} \tag{2.26}$$

According to the HPM method, one can construct the following homotopy:

$$u : \Omega \times [0,1] \to \mathbb{R}n$$

$$(1-p)(v' - u_0') + p\left(v' + v^2\right) = 0, p\epsilon[0,1], t\epsilon\Omega, \tag{2.27}$$

with $u_0 = 1$.

The solutions of equation (2.26) can be written in the form of a series:

$$v = v_0 + pv_1 + p^2v_2 + ... \tag{2.28}$$

By substituting (2.28) into (2.27) and identifying the terms with those of the same powers of p, we obtain:

$$\begin{array}{lll} p^0 & : & v_0' = u_0', \\ p^1 & : & v_1' = -u_0 - v_0^2, v_1(0) = 0, \\ p^2 & : & v_2' = -2v_0v_1, v_2(0) = 0, \\ & & . \\ & & . \\ & & . \end{array}$$

Consequently, the first terms of the solution are given by:

$$\begin{aligned} p^0 &: v_0 = 1, \\ p^1 &: v_1 = -t, \\ p^2 &: v_2 = t^2, \\ & \quad . \\ & \quad . \\ & \quad . \end{aligned}$$

Thus, the solution to equation (2.26) is:

$$\begin{aligned} u &= \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... = 1 - t + t^2 - ... \\ &= \sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}. \end{aligned}$$

2.2 Variational Iteration Method (VIM)

The Variational Iteration Method (VIM) was proposed and developed by the Chinese mathematician **Je-Haun-He** in the early 1990s [16, 17, 18]. It was first introduced to solve problems in mechanics. This method has been employed to solve a linear and nonlinear problems using successive approximations that, quickly converge to the exact solution if it exists. The method is based on the optimal determination of the Lagrange multiplier through variational theory.

2.2.1 Description of the Method

To illustrate the basic ideas of this method, let us consider the following nonlinear differential equation:

$$L(u) + N(u) = g(t), \tag{2.29}$$

where L is a linear differential operator, N is a nonlinear operator, and g is a known function.
We can construct a functional correction using the following variational iteration method:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) \left[L\left(u_n(\tau)\right) + N\left(\tilde{u}_n(\tau)\right) - g(\tau) \right] d\tau, n \geq 0, \tag{2.30}$$

whereb λ is a general Lagrange multiplier. The index n represents the n^{ieme}approximation, and $\tilde{u}_n(\tau)$ is considered as a restricted variation, i.e. $\delta \tilde{u}_n(\tau) = 0$. For to solve equation (2.29) using the VIM method, one must first determine the Lagrange multiplier, which will be optimally identified through integration by parts. Then, the successive approximations u_n of the solution $u(t)$ will be obtained by using the Lagrange multiplier and a well-chosen function u_0 (which must at least satisfy the initial conditions). Consequently, the exact solution will be the limit:

$$\lim_{n \to \infty} u_n(t) = u(t).$$

2.2.2 Alternative Approach to VIM

In this paragraph, an alternative approach to the VIM is presented. This approach can be implemented reliably and effciently to solve the following nonlinear differential equation:

$$Lu(t) + Nu(t) = g(t), t > 0. \tag{2.31}$$

With the initial conditions:

$$u^{(k)}(0) = c_k, k = 0, 1, ..., m-1, \tag{2.32}$$

where L is a linear differential operator defined by $L = \frac{d^m}{dt}$, $m \in \mathbb{N}$, N is a nonlinear operator, g is a known function, and c_k are real numbers.
According to the variational iteration method, we can construct a functional correction formula for (2.31) as follows:

$$u_{k+1}(t) = u_k(t) + \int_0^t \left[\lambda(\tau)\left(Lu_k(\tau) + N\tilde{u}_k(\tau) - g(\tau)\right)\right] d\tau, \tag{2.33}$$

where λ is a general Lagrange multiplier, which can be optimally identified by the variational theory. Here, we apply constrained variations to the nonlinear term Nu, and in this case, we can easily determine the multiplier.

Then, make the functional correction(2.33) stationary by noting that $\delta\tilde{u}_k(\tau) = 0$, the equation:

$$\delta u_{k+1}(t) = \delta u_k(t) + \delta \int_0^t \left[\lambda(\tau)\left(Lu_k(\tau) - g(\tau)\right)\right] d\tau, \tag{2.34}$$

this gives the following Lagrange multipliers:

$$\lambda = -1, \quad for \quad m = 1,$$
$$\lambda = \tau - t, \quad for \quad m = 2,$$

and in general:

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}, \quad for \quad m \geq 1. \tag{2.35}$$

Therefore, by substituting (2.35) into the functional (2.33), we obtain the following iteration formula:

$$u_{k+1}(t) = u_k(t) + \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left(Lu_k(\tau) + Nu_k(\tau) - g(\tau)\right)\right] d\tau. \tag{2.36}$$

Now, we define the operator $A(u)$ as:

$$A(u) = \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left(Lu_k(\tau) + Nu_k(\tau) - g(\tau)\right)\right] d\tau, \tag{2.37}$$

and we define the components $v_k, k = 0, 1, 2, ...$, as follows:

$$\begin{cases} v_0 = u_0, \\ v_1 = A(v_0), \\ v_2 = A(v_0 + v_1), \\ . \\ . \\ . \\ v_{k+1} = A(v_0 + v_1 + ... + v_k). \end{cases} \tag{2.38}$$

So, we have

$$u(t) = \lim_{k\to\infty} u_k(t) = \sum_{k=0}^{\infty} v_k(t).$$

Finally, the solution of the problem (2.31) can be deduced using (2.37) and (2.38), in the form of a series:

$$u(t) = \sum_{k=0}^{\infty} v_k(t). \tag{2.39}$$

The initial approximation $v_0 = u_0$ can be chosen freely as long as it satisfies the initial conditions of the problem. The success of the method depends on the proper choice of the initial approximation v_0. In this alternative approach, we choose the initial approximation as follows:

$$v_0 = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k. \tag{2.40}$$

2.2.3 Convergence analysis

In this paragraph, we study the convergence of the variational iteration method, according to the alternative approach of the VIM method presented in the previous paragraph [28, 24].

Theorem 2.3 *Let H be a Hilbert space and $A : H \to H$, an operator defined by (2.37). The series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ is convergent if, $\exists 0 < \gamma < 1$, such that:*

$$\|A(v_0 + v_1 + ... + v_{k+1})\| \leq \gamma \|A(v_0 + v_1 + ... + v_k)\|,$$

then, we obtian

$$\|v_{k+1}\| \leq \gamma \|v_k\|, \forall k \epsilon \mathbb{N} \cup \{0\}.$$

Proof. *Let $(S_n)_{n\geq 0}$ be a sequence defined as follows:*

$$\begin{cases} S_0 = v_0, \\ S_1 = v_0 + v_1, \\ S_2 = v_0 + v_1 + v_2, \\ . \\ . \\ . \\ . \\ S_n = v_0 + v_1 + v_2, ..., v_n. \end{cases}$$

It is shown that $(S_n)_{n\geq 0}$ is a Cauchy sequence in the Hilbert space H. To do this, we have:

$$\|S_{n+1} - S_n\| = \|v_{n+1}\| \leq \gamma \|v_n\| \leq \gamma^2 \|v_{n-1}\| \leq ..... \leq \gamma^{n+1} \|v_0\|.$$

For $n, m \epsilon \mathbb{N}, n \geq m$, we have:

$$\begin{aligned}
\|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots.. + (S_{m+1} - S_m)\| \\
&\leq \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + \ldots.. + \|(S_{m+1} - S_m)\| \\
&\leq \gamma^n \|v_0\| + \gamma^{n-1}\|v_0\| + \ldots.. + \gamma^{m+1}\|v_0\| \\
&\leq \left(\gamma^n + \gamma^{n-1} + \ldots.. + \gamma^{m+1}\right)\|v_0\| \\
&\leq \left(\gamma^{m+1} + .. + \gamma^n + ..\right)\|v_0\| \\
&\leq \gamma^{m+1}(1 + \gamma + .. + \gamma^n + ..)\|v_0\| \\
&\leq \frac{\gamma^{m+1}}{1-\gamma}\|v_0\|,
\end{aligned}$$

and since $0 < \gamma < 1$, we find:

$$\lim_{n,m \to \infty} \|S_n - S_m\| = 0.$$

Therefore, $(S_n)_{n \geq 0}$ is a Cauchy sequence in the Hilbert space H, and this implies that the serie solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ is converge. ■

Theorem 2.4 *If the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ converges, then it is an exact solution to the nonlinear problem (2.31).*
Proof. *Suppose that the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ converges than we have :*

$$\begin{aligned}
\lim_{k \to \infty} v_k(t) &= 0, \\
\sum_{k=0}^{n} (v_{k+1} - v_k) &= v_{n+1} - v_0.
\end{aligned}$$

Therefore

$$\sum_{k=0}^{\infty} (v_{k+1} - v_k) = \lim_{k \to \infty} (v_{n+1} - v_0) = -v_0. \tag{2.41}$$

By applying the operator $L = \frac{d^m}{dt^m}, m\epsilon\mathbb{N}$, to both sides of equation (2.41) and then using the relation (2.40)), we obtain:

$$\sum_{k=0}^{\infty} L(v_{k+1} - v_k) = -L(v_0) = 0. \tag{2.42}$$

On the other hand, from relation (2.38), we have:

$$L(v_{k+1} - v_k) = L(A(v_0 + v_1 + \ldots. + v_k) - A(v_0 + v_1 + \ldots. + v_{k-1})), k \geq 1.$$

Using the defnition of the operator $A(u)$ defned by (2.37), we obtain:

$$L(v_{k+1} - v_k) = L\left(\int_0^t \left[\begin{array}{c} \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}(L(v_0 + v_1 + \ldots. + v_k) - L(v_0 + v_1 + \ldots. + v_{k-1}) \\ +N(v_0 + v_1 + \ldots. + v_k) - N(v_0 + v_1 + \ldots. + v_{k-1})) \end{array}\right] d\tau\right), k \geq 1. \tag{2.43}$$

Now, the operator $A(u)$, defned by (2.37), gives the integral of the m-th time of $Lu(t) + Nu(t) - g(t)$. Since the differential operator $L = \frac{d^m}{dt^m}$ of order m is the inverse of the m-th integral operator, equation (2.43) becomes:

$$L(v_{k+1} - v_k) = L(v_k) + N(v_0 + v_1 + \ldots. + v_k) - N(v_0 + v_1 + \ldots. + v_{k-1}), k \geq 1.$$

Therefore, we have:

$$\begin{aligned}\sum_{k=0}^{n} L\left(v_{k+1}-v_k\right) &= L\left(v_0\right)+N\left(v_0\right)-g(t)\\ &+L\left(v_1\right)+N\left(v_0+v_1\right)-N\left(v_0\right)\\ &+L\left(v_2\right)+N\left(v_0+v_1+v_2\right)-N\left(v_0+v_1\right)\\ &.\\ &.\\ &.\\ &+L\left(v_n\right)+N\left(v_0+v_1+....+v_n\right)-N\left(v_0+v_1+....+v_{n-1}\right).\end{aligned}$$

Therefore

$$\sum_{k=0}^{\infty} L\left(v_{k+1}-v_k\right)=L\left(\sum_{k=0}^{\infty} v_k\right)+N\left(\sum_{k=0}^{\infty} v_k\right)-g(t). \tag{2.44}$$

From (2.41) *and* (2.44), *we can observe that* $u(t)=\sum_{k=0}^{\infty} v_k\left(t\right)$ *is an exact solution to the problem* (2.31). ■

Example 2.2 Consider the following nonlinear differential equation:

$$\begin{aligned} u'\left(t\right) &= u^2\left(t\right)+1, 0<t<1, \\ u\left(0\right) &= 0. \end{aligned} \tag{2.45}$$

The functional correction of equation (2.45) according to the VIM method is given by:

$$u_{n+1}\left(t\right)=u_n\left(t\right)+\int_0^t \lambda\left(\tau\right)\left(u_n'\left(\tau\right)-\left(\tilde{u}_n\right)^2\left(\tau\right)-1\right)d\tau.$$

According to (2.35), the Lagrange multiplier $\lambda\left(\tau\right)$ can be identified as $\lambda\left(\tau\right)=-1$,so the iteration formula can be obtained as follows:

$$u_{n+1}\left(t\right)=u_n\left(t\right)-\int_0^t \left(u_n'\left(\tau\right)-\left(u_n\right)^2\left(\tau\right)-1\right)d\tau. \tag{2.46}$$

According to formula (2.46), we obtain the first terms of the approximate solution:

$$\begin{aligned} u_0\left(t\right) &= 0,\\ u_1\left(t\right) &= t,\\ u_2\left(t\right) &= t+\frac{1}{3}t^3,\\ u_3\left(t\right) &= t+\frac{1}{3}t^3+\frac{2}{15}t^6+\frac{1}{63}t^7,\\ &.\\ &.\\ &. \end{aligned}$$

and as

$$u\left(t\right)=\lim_{n\to\infty} u_n\left(t\right),$$

we can express the solution of equation (2.45) in the form of a series that converges to the exact solution given by:

$$\begin{aligned} u(t) &= t + \frac{1}{3}t^3 + \frac{2}{15}t^6 + \frac{1}{63}t^7 + ... \\ &= \tan(t). \end{aligned}$$

2.3 The Adomian Decomposition Method (ADM)

The Adomian decomposition is a semi-analytical method developed by the American mathematician George Adomian [2] during the second half of the 20th century. It is used to solve a wide range of problems, including algebraic, differential, integral, integro-differential problems, higher-order ordinary differential equations, and partial differential equations. The advantage of this method is that it allows the problem at hand to be solved through a direct scheme, providing the solution in the form of an infinite series, which converges rapidly to the exact solution if it exists [7].

2.3.1 Description of the method

To illustrate the basic ideas of this method, we consider the following functional equation:

$$Fu = g, \tag{2.47}$$

Where F represents a nonlinear differential operator, either ordinary or partial, including both linear and nonlinear terms, g is a known function. The linear part is typically decomposed into $L + R$, where L is a differential operator that is easily invertible and R represents the remainder of the linear operator. Under these conditions, the previous equation can be written in the following form:

$$Lu + Ru + Nu = g, \tag{2.48}$$

With N is a nonlinear differential operator.

We can write equation (2.48) as follows:

$$Lu = g - Ru - Nu. \tag{2.49}$$

By multiplying equation (2.49) by L^{-1}, we obtain:

$$L^{-1}(Lu) = L^{-1}g - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.50}$$

where $L^{-1} = \int\int ... \int (.)\,(dt)^n$ is the inverse of the operator L.

Because

$$L^{-1}(Lu) = u - U,$$

and U is the constant of integration.

Consequently, the equation (2.50) becomes:

$$u = U + L^{-1}g - L^{-1}(Ru) - L^{-1}(Nu). \tag{2.51}$$

The Adomian method consists of seeking the solution in the form of a series.

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.52}$$

Then, the nonlinear term $N(u)$ is decomposed into a series:

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{2.53}$$

The terms A_n are called Adomian polynomials and are obtained using the following relation:

$$A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ..., \tag{2.54}$$

where λ is a real parameter introduced for convenience.

By substituting equations (2.52) and (2.53) into (2.51), we obtain:

$$\sum_{n=0}^{\infty} u_n = U + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{2.55}$$

From which we deduce:

$$\begin{cases} u_0 = U + L^{-1}g \\ u_1 = -L^{-1}Ru_0 - L^{-1}A_0 \\ u_2 = -L^{-1}Ru_1 - L^{-1}A_1 \\ . \\ . \\ . \\ u_{n+1} = -L^{-1}[Ru_n - L^{-1}A_n],\ n \geq 0. \end{cases} \tag{2.56}$$

It should be noted that this identification is not unique, but it is the only one that allows for the explicit definition of the terms u_n. The relation (2.56) allows for the calculation of all the terms of the series without ambiguity, as the A_n terms depend solely on $u_0, u_1, ..., u_n$.

In practice, it is almost impossible to compute the sum of the infinite series $\sum_{n=0}^{\infty} u_n$ (except in very specific cases). Therefore, one generally settles for an approximate solution u_n, in the form of a truncated series:

$$u_n = \sum_{i=0}^{n-1} u_i, n \geq 1.$$

The question that can be asked is how to determine the$(A_n)_{n\geq 0}$ and under what conditions the method converges.

2.3.2 The Adomian polynomials

Definition 2.1 The Adomian polynomials are defined by the formula:

$$\begin{cases} A_0(u_0) = N(u_0) \\ A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}. \end{cases} \tag{2.57}$$

The formula proposed by G.

Adomian for calculating the Adomian polynomials $(A_n)_{n\geq 0}$ is as follows [1]:

$$\begin{aligned}
A_0(u_0) &= N(u_0)\\
A_1(u_0,u_1) &= u_1\frac{\partial}{\partial u}N(u_0)\\
A_2(u_0,u_1,u_2) &= u_2\frac{\partial}{\partial u}N(u_0)+\frac{1}{2!}u_1^2\frac{\partial^2}{\partial u^2}N(u_0)\\
A_3(u_0,u_1,u_2,u_3) &= u_3\frac{\partial}{\partial u}N(u_0)+u_1u_2\frac{\partial^2}{\partial u^2}N(u_0)+\frac{1}{3!}u_1^3\frac{\partial^3}{\partial u^3}N(u_0)\\
&.\\
&.\\
&.
\end{aligned}$$

This formula can be written in the following form:

$$A_n=\sum_{v=0}^{n}c(v,n)\,N^{(v)}(u_0)\,,n\geq 1,$$

where $c(v,n)$ represents the sum of all products (divided by $m!$) of the v terms u_i, where the sum of the indices i equals n, m, is the number of repetitions of the same terms in the product.

2.3.3 The convergence of the ADM

Important theorems have been given involving sufficient conditions for convergence. All these conditions focus on the nonlinear operator N.

Indeed, from the relation (2.56), we deduce:

Theorem 2.5 *If*

$$\sum_{n\geq 0}A_n<+\infty \text{ Then } \sum_{n\geq 0}u_n<+\infty, \tag{2.58}$$

and vice versa.

The first proofs of convergence were cited by Yves Cherruault. They are based on the fixed-point method.

Let's outline the main points of the proof (see [8] for more details).

First, note that the decomposition method applied to (2.47) reduces to the search for a sequence:

$$S_n=u_0+u_1+...+u_n,$$

with $S_0=0$ and satisfying the following recurrence relation:

$$S_{n+1}=N(u_0+S_n)\,,S_0=0,u_0=g,n=0,1,2,... \tag{2.59}$$

We deduce the following convergence result:

Theorem 2.6 *If the operator N is a contraction (i.e., satisfies $\|N\|<\delta<1$) , then the sequence $(S_n)_n$ satisfying the recurrence relation $S_{n+1}=N(u_0+S_n)$ with $S_0=0,n\geq 0$, converges to S, the solution of $S=N(u_0+S)$.*

Proof. *From the relation (2.59), we have:*

$$\begin{aligned}
\|S_n-S\| &= \|N(u_0+S_n)-N(u_0+S)\|\\
&\leq \|N\|\|S_n-S\|\leq\delta\|S_n-S\|\\
&\leq \delta^n\|S_1-S\|.
\end{aligned}$$

Hence, the sequence $(S_n)_n$ *converges towards* S.
Furthermore, we have:

$$\sum_{n\geq 0} A_n = \sum_{n\geq 1} u_n,$$

■ *And since* $\sum_{n\geq 1} u_n$ *is convergent according to Theorem* (2.47), *we have the following result:*

Corollary 2.1 *If* N *is a contraction, then the series of* u_n *and* A_n *are convergent. Moreover,* $\sum_{n\geq 0} u_n$ *is the solution of the equation:*

$$Fu = g.$$

Example 2.3 Let the following nonlinear differential equation be:

$$\begin{cases} u' + u^2 = 0, t \geq 0, \\ u(0) = 1. \end{cases} \tag{2.60}$$

We have:

$$Lu = u' \text{ and } Nu = u^2,$$

with: $L = \frac{d}{dt}(.)$.

L^{-1} represents a simple integration from 0 to t. We obtain:

$$u = \sum_{n=0}^{\infty} u_n = u(0) - L^{-1} \sum_{n=0}^{\infty} A_n. \tag{2.61}$$

The Adomian polynomials are:

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ &\vdots \end{aligned}$$

Therefore, we have:

$$\begin{aligned} u_0 &= 1, \\ u_1 &= -L^{-1}(A_0) = -t, \\ u_2 &= -L^{-1}(A_1) = t^2, \\ u_3 &= -L^{-1}(A_2) = -t^3, \\ u_4 &= -L^{-1}(A_3) = t^4, \\ &\vdots \end{aligned}$$

From (2.61), we have the solution of (2.60) given by:

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n = 1 - t + t^2 - t^3 + t^4 - .... \\ &= \sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}. \end{aligned}$$

CHAPTER 3

APPLICATIONS

In this chapter, we focus on the practical application of the semi-analytical methods discussed in the previous chapter namely, the Homotopy Perturbation Method (HPM), the Variational Iteration Method (VIM), and the Adomian Decomposition Method (ADM). These methods are applied to solve the one-dimensional Burgers equation, a well-known nonlinear partial differential equation that arises in fluid mechanics and other fields of applied science.

The purpose of this chapter is to demonstrate the effectiveness and accuracy of these techniques in handling nonlinear problems. For each method, we present the solution procedure, derive the corresponding series solution, and provide graphical representations to illustrate the behavior of the solution over time and space.

Finally, a comparison of the methods is discussed based on their convergence, computational efficiency, and ease of implementation. This highlights the strengths and potential limitations of each approach in solving nonlinear PDEs such as the Burgers equation.

3.1 Solve Burgers equation using HPM

In this section, we solve the following one dimensional Burgers equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - w\frac{\partial w}{\partial x}, \tag{3.1}$$

with conditions

$$w(x, 0) = 2x, \quad t > 0. \tag{3.2}$$

We write a homotopy of eq(3.1) by using the **HPM** as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial w_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2}\right) = 0, \tag{3.3}$$

or

$$\frac{\partial v}{\partial t} - \frac{\partial w_0}{\partial t} = p\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial w_0}{\partial t} - v\frac{\partial v}{\partial x}\right). \tag{3.4}$$

We can write eq (3.4) as the power series of p:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{3.5}$$

Comparing the coefficient of a similar intensity of p of eq(3.4) with the use of eq(3.5), it follows that

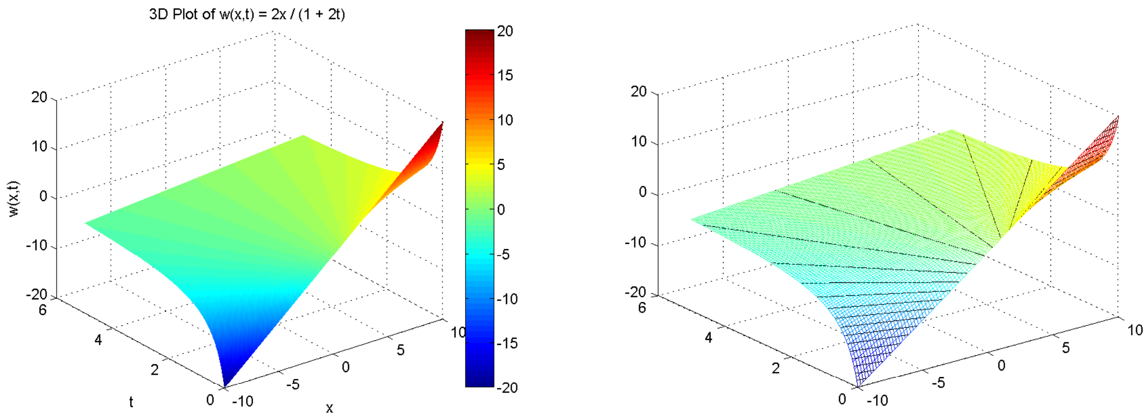
$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t}-\frac{\partial w_0}{\partial t}=0, && (3.6)\\
p^1 &: \frac{\partial v_1}{\partial t}=\frac{\partial^2 v_0}{\partial x^2}-v_0\frac{\partial v_0}{\partial x}-\frac{\partial v_0}{\partial t}, && (3.7)\\
p^2 &: \frac{\partial v_2}{\partial t}=\frac{\partial^2 v_1}{\partial x^2}-v_0\frac{\partial v_1}{\partial x}-v_1\frac{\partial v_0}{\partial x}, && (3.8)\\
p^3 &: \frac{\partial v_3}{\partial t}=\frac{\partial^2 v_2}{\partial x^2}-v_0\frac{\partial v_2}{\partial x}-v_1\frac{\partial v_1}{\partial x}-v_2\frac{\partial v_0}{\partial x}, && (3.9)\\
&\quad .\\
&\quad .\\
&\quad .\\
p^i &: \frac{\partial v_i}{\partial t}=\frac{\partial^2 v_{i-1}}{\partial x^2}-\sum_{i=0}^{i-1}v_j\frac{\partial v_{i-j-1}}{\partial x}, && (3.10)
\end{aligned}$$

Now, we can start with $v_0(x,0)=w(x,0)=2x$ and solve the above equations

$$\begin{aligned}
v_1 &= -4xt,\\
v_2 &= 8xt^2,\\
v_3 &= -16xt^3,\\
v_4 &= 32xt^4,\\
v_5 &= -64xt^5,\\
&\quad .\\
&\quad .\\
&\quad .\\
v_i &= (-1)^i\,2^{i+1}xt^i. && (3.11)
\end{aligned}$$

Then, the solution of eq(3.1) expresses a series form by HPM are given in the form:

$$\begin{aligned}
w(x,t) &= \lim_{p\to 1} v=\sum_{i=0}^{\infty}v_i=v_0+v_1+v_2+v_3+...\\
w(x,t) &= \sum_{i=0}^{\infty}(-1)^i\,2^{i+1}xt^i=2x\left(1-2t+4t^2-8t^3+16t^4-32t^5+......\right)\\
w(x,t) &= \frac{2x}{1+2t}. && (3.12)
\end{aligned}$$



(a) 3D Surface Plot (varying x and t) (b) 3D Surface and Mesh and Contours

Figure 3.1: The solution of Burgers equation (3.1) using HPM.

3.2 Solve Burgers equation using VIM

We consider the modified KdV (mKdV) equation

$$u_t + \frac{1}{2}\left(u^2\right)_x - u_{xx} = 0, u\left(x, 0\right) = x. \tag{3.13}$$

we construct a functional correction formula for (3.13) as follows:

$$u_{n+1}\left(x,t\right) = u_n\left(x,t\right) + \int_0^t \lambda\left(\tau\right)\left(\frac{\partial u_n\left(x,\tau\right)}{\partial \tau} + \frac{1}{2}\frac{\partial\left(\tilde{u}_n\right)^2\left(x,\tau\right)}{\partial x} - \frac{\partial^2\left(\tilde{u}_n\right)^2\left(x,\tau\right)}{\partial x^2}\right)d\tau. \tag{3.14}$$

This again yields

$$\lambda = -1. \tag{3.15}$$

Substituting this value of the Lagrangian multiplier into functional (3.14) gives the iteration formula

$$u_{n+1}\left(x,t\right) = u_n\left(x,t\right) - \int_0^t \left(\frac{\partial u_n\left(x,\tau\right)}{\partial \tau} + \frac{1}{2}\frac{\partial\left(\tilde{u}_n\right)^2\left(x,\tau\right)}{\partial x} - \frac{\partial^2\left(\tilde{u}_n\right)^2\left(x,\tau\right)}{\partial x^2}\right)d\tau. \tag{3.16}$$

Selecting the initial value $u_0(x,0) = x$ for $u_0(x,t)$ and using (3.16) we obtain the following successive approximations:

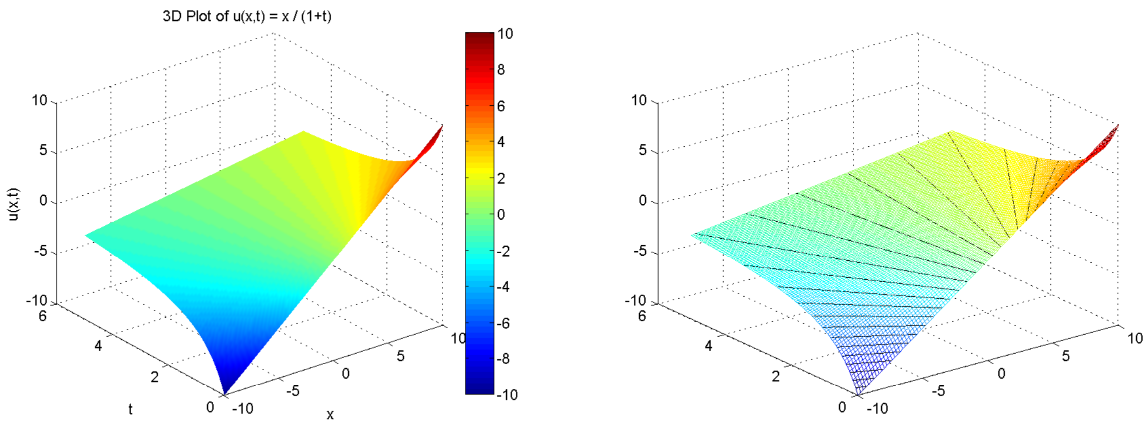
$$\begin{aligned}
u_0\left(x,t\right) &= x, \\
u_1\left(x,t\right) &= x\left(1-t\right), \\
u_2\left(x,t\right) &= x\left(1-t+t^2-\frac{1}{3}t^3\right), \\
u_3\left(x,t\right) &= x\left(1-t+t^2-t^3+t^4\right), \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
u_n\left(x,t\right) &= x\left(1-t+t^2-t^3+t^4-t^5+\ldots\right).
\end{aligned} \tag{3.17}$$

Recall that

$$u = \lim_{n\to\infty} u_n, \qquad , \tag{3.18}$$

hence the exact solution give as

$$u\,(x,t) = \tfrac{x}{1+t}. \tag{3.19}$$



(a) 3D Surface Plot (varying x and t). (b) 3D Surface and Mesh and Contours.

Figure 3.2: The solution of Burgers equation (3.13)using VIM.

3.3 Solve Burgers equation using ADM

W consider the Burgers equation defined as

$$u_t - uu_x = 0, \tag{3.20}$$

with initial condition:

$$u(x,0) = \frac{x}{10}. \tag{3.21}$$

The exact solution is:

$$u(x,t) = -\frac{x}{t-10}.$$

From (3.20) and (3.21) we have:

$$L = \frac{\partial}{\partial t} \quad ; \quad R\,(u) = 0 \quad ; \quad N\,(u) = -uu_x \quad ; \quad g\,(x,t) = 0 \;\; ; \quad f = \frac{x}{10},$$

by

$$\begin{cases} u_0 = \gamma \\ u_{n+1} = -L^{-1}\left[R\,(u_n) + A_n\right] \qquad n \geq 0, \end{cases}$$

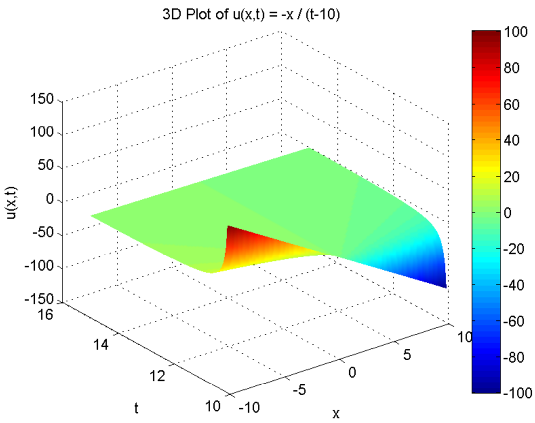
we get:

$$
\begin{aligned}
u_0 &= \frac{x}{10} \\
A_0 &= -u_0\left(u_{0x}\right) = -\frac{x}{10}.\frac{1}{10} = -\frac{x}{100} \\
u_1 &= -L^{-1}A_0 = -\int_0^t A_0 dt = -\int_0^t \left(-\frac{x}{100}\right) dt = \frac{x}{100}t \\
A_1 &= -\left(u_0 u_{1x} + u_1 u_{0x}\right) = -\frac{x}{100}.\frac{t}{100} - \frac{x}{100} + \frac{1}{10} = -\frac{xt}{500} \\
u_2 &= -L^{-1}A_1 = -\int_0^t \left(-\frac{xt}{500}\right) dt = \frac{xt^2}{1000} \\
A_2 &= -\left(u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}\right) = -\left(3\frac{xt^2}{10000}\right) \\
u_3 &= -L^{-1}A_2 = -\int_0^t \left(-3\frac{xt^2}{10000}\right) dt = \frac{xt^3}{10000} \\
A_3 &= -\left(u_0 u_{3x} + u_1 u_{2x} + u_2 u_{2x} + u_3 u_{0x}\right) = \frac{xt^3}{25000} \\
u_4 &= -L^{-1}A_3 = -\int_0^t A_3 dt = \frac{xt^4}{100000}.
\end{aligned}
$$

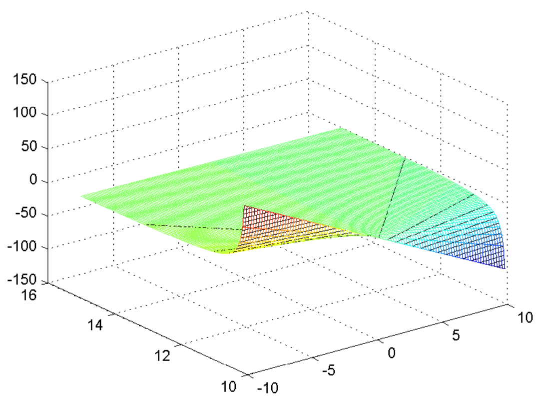
So, we have

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + u_3 + ... = \frac{x}{10} + \frac{xt}{100} + \frac{xt^2}{1000} + \frac{xt^3}{10000} + ... \\
&= \frac{x}{10}\left[1 + \frac{t}{10} + \left(\frac{t}{10}\right)^2 + \left(\frac{t}{10}\right)^3 + ...\right] = \frac{x}{10}.\frac{1}{\frac{t}{10} - 1} \\
&= -\frac{x}{t-10}.
\end{aligned}
$$

$$u(x,t) = -\frac{x}{t-10}.$$



(a) 3D Surface Plot (varying x and t).



(b) 3D Surface and Mesh and Contours.

Figure 3.3: The solution of Burgers equation (3.20) using ADM.

Conclusion

In this section, we took three Burger's equations and solved them using the HPM and VIM and ADM. These methods are considered direct methods (without the need for unrealistic assumptions, linearization, discretization). The VIM is considered to be very direct compared to HPM and ADM. The latter requires calculating the Adomian polynomial, which is a basic condition for finding the solution compared to the two previous methods.

GENERAL CONCLUSION

In this Thesis, we focused on studying some semi-analytical methods, specifically the (HPM, VIM, ADM), to solve partial differential equations. We discussed the analysis of the algorithms for these methods to find solutions to these equations and their convergence. We also provided three types of Burger's equations and solved them using these methods.

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