# MATHEMATICS 1

For first year university students in matter sciences and related disciplines

**Rabah Khemis**

2024

**PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA**
**MINISTRY OF HIGHER EDUCATIONAND SCIENTIFIC RESEARCH**
**20 AUGUST 1955 UNIVERSITY OF SKIKDA**

**Realised by: Rabah Khemis**

# MATHEMATICS 1

First Semester

For first year university students in matter sciences and related disciplines

**Domain : Matter sciences**

February 2024

# Contents

# Preface

*He who never starts, never finishes*

William Shakespeare (1564–1616)

Mathematics can be deemed as the language of science and technology due to the fact that the mathematical concepts and tools form integral parts of the vocabulary of several scholars working in various fields. For this, a knowledge of some basic mathematical concepts and techniques is crucial for an increasing number of university courses for a wide range of scientific disciplines such as mathematics, physics, chemistry, computer sciences, engineering and life sciences. Indeed, for physics and chemistry, mathematics has always been, and still is, one of the core tools since it promotes rigorous thinking, problem solving ability and help in expressing ideas, formulating theories, modelling and also getting a better understanding of a broad range of phenomena that appear almost in every facet of our lives.

The red thread of this course which can be considered as a first step towards further learning in mathematics, is to introduce some basic mathematical concepts that are assumed to be mastered by students in chemistry and physics. Most of them are presented in their simplest but rigorous forms so that students that take their first steps in the university can easily understand them especially those with little background in mathematics and often no motivation to learn more.

This course is not proof-based but it provides a scaffolded approach to learning main ideas and notions that will be required for applying mathematics in physics and chemistry. Whist it has been geared primarily towards first year students at the universities whose speciality is precisely matter sciences, the actual audience may be all students studying mathematics at the university whatever their speciality. Students are assumed to have a little prior knowledge especially knowledge of high school mathematics which should be a sufficient prerequisite. Furthermore, an acquaintance with some basic concepts of mathematical logic and some types of mathematical proof is an element of the knowledge required for this module.

The content here is divided into two main parts: The first part that is made up of three chapters, deals with analysis whereas the second one is algebra. The first three chapters cover a collection of topics such as sets, relations and functions while the other three chapters focus on groups, rings, fields, vector spaces and linear transformations.

Finally, it is our aspiration that this course will greatly simplify the work of the students and will also be a helpful resource for them - including those struggling with their mathematics.

# PART 1: ANALYSIS 1

# Chapter 1

## Sets, Relations, Functions

This chapter renders some notations, terminology and elementary operations of sets which are fundamental in all branches of mathematics. In addition, a considerable portion of it is concerned with introducing a detailed discussion of two of the most important concepts in mathematics which are the relation and the function.

### 1.1. Sets

**Definition 1.1.** A **set** is a well-defined collection of objects that have some characteristics in common, called its **elements** or **members**.

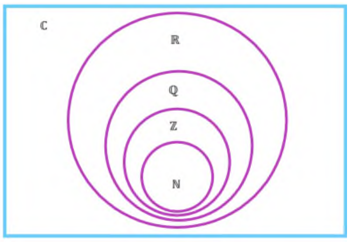
**Remark 1.1.**
**1**) Sets are usually denoted by capital letters $A, B, C,$ etc. while the elements are usually denoted by small letters $a, b, c,$ etc.
**2**) A set $A$ can be represented by listing all its elements between the conventional curly brackets (braces). For example $A = \{1,2,3,4,5\}$.
**3**) If $a$ is an element of a set $A$, then we write $a \in A$ and we can also say that $a$ **belongs** to $A$ or $a$ is in $A$. If $a$ does **not belong** to $A$, we write $a \notin A$.
**4)** The elements could be anything (animals, plants, fruits, people, objects, etc.) but for us they will be mathematical objects such as numbers, or sets of numbers.

### 1.1.1. Some important Sets

The fundamental sets of numbers are contained in one another according to the following proper subset relationships $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ where

**1.** $\mathbb{N} = \{0,1,2,3,\cdots\}$: The set of all **natural** numbers.
**2.** $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$: The set of all **integers**.
**3.** $\mathbb{Q} = \left\{\frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\right\}$: The set of all **rational** numbers (ratios of integers).
**4.** $\mathbb{R}$: The set of all **real** numbers.
**5.** $\mathbb{C} = \{x + iy, x, y \in \mathbb{R}\}$: The set of all **complex** numbers.

These sets come equipped with the familiar arithmetic operations of sum and product.



The letter Z comes from zahl (German for number) and the letter Q comes from quotient. It is worth noting here that the aforementioned symbols for these sets can be decorated with some superscripts such as $*$, $+$, and $-$ to designate the corresponding subcollections of non-zero, positive, and negative numbers, respectively. For instance

$$\mathbb{N}^* = \{1,2,3,4,\cdots\},\ \mathbb{Z}^* = \{\cdot\cdot\cdot, -3, -2, \mathbf{-1}, \mathbf{1}, 2, 3, \cdot\cdot\cdot\},$$

$$\mathbb{Z}^+ = \{0,1,2,3,\cdot\cdot\cdot\} \text{ and } \mathbb{Z}^- = \{\cdot\cdot\cdot, -3, -2, 0\}.$$

## 1.1.2. Types of Sets

### 1.1.2.1. Null Set

**Definition 1.2.** The **null empty** set (or **empty** set or **void set**) is a set that contains no elements. It is denoted a pair of curly brackets (braces) with nothing inside { } or by using the symbol ∅.

Example 1.1. Sets whose definition contains a contradiction or impossibility are often empty. For example, the set

$$A = \{x \in \mathbb{N}: 3 < x < 4\},$$

is an empty set, since there is no natural number between 3 and 4.

**Remark 1.2.**
**1**) ∅ is unique.
**2**) We cannot use ∅ and { } together to denote an empty set since {∅} is not the null set.

### 1.1.2.2. Singleton Set

**Definition 1.3.** A set having only one element is called a **singleton.**

Example 1.2. $A = \{1\}$, $A = \{\emptyset\}$.

### 1.1.2.3 Finite Set

**Definition 1.4.** A **finite** set is a set that contains a finite number of elements or no element.

**Example 1.3.**

**1**) $A = \{1,2,3,4\}$.

**2**) $A = \{\emptyset\}$.

**3**) $A = \emptyset$.

**4**) $A = \{x \in \mathbb{Z}: 1 < x < 10\}$ is a finite set because it has eight elements.

**5**) A set of all English alphabet is a finite set because it consists of 26 letters.

**6**) $A =$\{January, February, March, April, May, June, July, August, September, October, November, December\}.

### 1.1.2.4. Infinite Set

**Definition 1.5.** An **infinite** set is a set that contains an infinite number of elements.

**Example 1.4.**

**1**) $A = \mathbb{N}^* = \{1,2,3,\cdots\}$: The set of all non-zero natural numbers.

**2**) A set of all points on a line.

### 1.1.2.5. Equal Sets

**Definition 1.6.** Two sets $A$ and $B$ are said to be **equal**, if every element of $A$ is an element of $B$ and every element of $B$ is an element of $A$. Mathematically, we write $A = B$.

**Example 1.5.** If

$$A = \{x: x \text{ is a prime number and } 2 < x < 11\},$$

and

$$B = \{7,3,5\},$$

then $A$ and $B$ have exactly the same elements which means that $A = B$.

### 1.1.2.6. Subset, Super-set and Proper Subset

**Definition 1.7.** Let $A$ and $B$ be two sets. If every element of $A$ is an element of $B$, then $A$ is called a **subset** of $B$ whereas $B$ is called a **super-set** of $A$. We write $A \subseteq B$ or $B \supseteq A$.

If $A$ is a subset of $B$ and $A \neq B$, then $A$ is called a **proper subset** of $B$ and we write $A \subset B$ and for showing that $A \subset B$, it suffices to prove that $A \subseteq B$ but $B \nsubseteq A$.

**Remark 1.3.**

**1**) Every set is a subset of itself and the empty set is a subset of every set.

**2**) For showing that $A \subseteq B$, it suffices to pick an arbitrary $a \in A$, then prove that $a \in B$.

**3**) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
**4)** $A = B$ if and only if (iff) $A \subseteq B$ and $B \subseteq A$.
**5)** A super-set can be considered as the parent set that at least contains all the elements of the subset and may or may not contain some extra elements.

**Example 1.6.**
**1**) Let $A = \{1,2,3,4\}$ and $B = \{2,3\}$.
Here $B$ is a subset of $A$ since, all the elements of set the $B$ are contained in the set $A$. But $A$ is not a subset of $B$ since, all the elements of the set $A$ are not contained in the set $B$.
**2**) The set $\mathbb{N}$ of natural numbers is a subset of the set $\mathbb{Z}$ of integers.

### 1.1.3. Venn Diagram

**Definition 1.8.** A **Venn diagram** is employed to highlight and give a pictorial representation of the logical relationships between two or more sets of items where the sets can be represented by any closed figure whether it be a circle, an oval shape, a rectangle or any type of polygons but usually, the universal set is represented by a rectangular region and the other sets are represented by circles or oval shapes within this rectangular region.

### 1.1.4. Operations on Sets

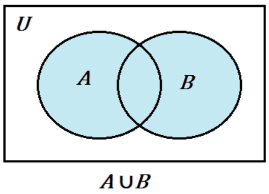
Let $U$ be a set and and let $A$ and $B$ be two subsets of it.

#### 1.1.4.1. Union of Sets

**Definition 1.9.** The **union** of the sets $A$ and $B$, denoted by $A \cup B$ is the set of all elements that belong to $A$ or $B$ or both $A$ and $B$; that is

$$A \cup B = \{x \in U:\ x \in A \text{ or } x \in B\}.$$

The union of $A$ and $B$ can be represented by the following Venn diagram:



The shaded blue part indicates the set $A \cup B$.

**Remark 1.4**.
**1**) $A \cup B = B \cup A$.
**2**) $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

**Example 1.7.** If

$$A = \{1,2,3,4\} \text{ and } B = \{2,3,6,8\},$$

then

$$A \cup B = \{1,2,3,4,6,8\}.$$

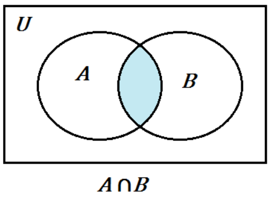
### 1.1.4.2. Intersection of Sets

**Definition 1.10**. The **intersection** of the sets $A$ and $B$, denoted by $A \cap B$, is the set of all elements that belong to both $A$ and $B$; that is

$$A \cap B = \{x \in U\colon\ x \in A \text{ and } x \in B\}.$$

$A$ and $B$ are said to be disjoint if $A \cap B = \emptyset$; that is, if $A$ and $B$ have no elements in common.

This intersection of $A$ and $B$ can be represented by the following Venn diagram:



The shaded blue part indicates the set $A \cap B$.

**Example 1.8.** If

$$A = \{1,2,3,4\} \text{ and } B = \{2,3,6,8\},$$

then

$$A \cap B = \{2,3\}.$$

### 1.1.4.3. Complement of a Set

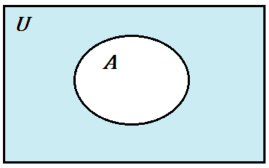
**Definition 1.11**. The **complement** of a set $A$ with respect to $U$ which can be denoted by $A^c$ or $U - A$ is the collection of elements in $U$ that are not in $A$; that is

$$A^c = \{x\colon x \in U \text{ and } x \notin A\},$$

or more compactly as

$$A^c = \{x\colon x \notin A\}.$$

$A^c$ can be represented by the following Venn diagram:



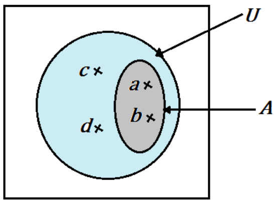
The shaded blue part indicates the set $A^c$.

**Example 1.9.** Let

$$U = \{a, b, c, d\},$$

and

$$A = \{a, b\}.$$



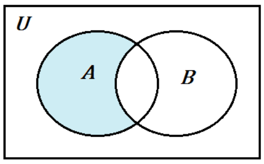
The gray region represents $A$, whereas the blue one represents $A^c$. So $A^c = \{c, d\}$.

### 1.1.4.4. Difference of Sets

**Definition 1.12.** For two sets $A$ and $B$, the **difference** $A - B$ is the set of all elements of $A$ which do not belong to $B$; that is

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

$A - B$ can be represented by the following Venn diagram:



$A-B$

The blue region represents the set $A - B$.

**Example 1.10.** Let

$$A = \{1,2,3,4\} \text{ and } B = \{2,4,6\}.$$

We have

$$A - B = \{x : x \in A \text{ and } x \notin B\} = \{1,3\},$$

and

$$B - A = \{x : x \in B \text{ and } x \notin A\} = \{6\}.$$

**Remark 1.5.**

**1)** $(A^c)^c = A$

**2)** $A \cap A^c = \emptyset$ and $A \cup A^c = U$

**3)** If $A \subset B$, then $B^c \subset A^c$

**4)** $U^c = \emptyset$ and $\emptyset^c = U$

**5)** $A - B \neq B - A$.

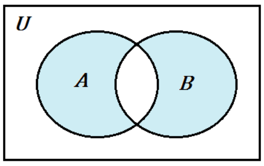
### 1.1.4.5. Symmetric Difference of two Sets

**Definition 1.13.** For two sets $A$ and $B$, the **symmetric difference** of $A$ and $B$ is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$ where

$$\begin{aligned} A \Delta B &= (A - B) \cup (B - A) \\ &= (A \cup B) - (A \cap B). \end{aligned}$$

In other words it is the set of all those elements which belong either to $A$ or to $B$ but not to both.

$A \Delta B$ can be represented by the following Venn diagram:



$A \Delta B$

The blue region represents the set $A \Delta B$.

**Example 1.11.** Let

$$A = \{1,2,3,4\} \text{ and } B = \{2,4,6\}.$$

Since

$$A - B = \{x : x \in A \text{ and } x \notin B\} = \{1,3\},$$

and

$$B - A = \{x : x \in B \text{ and } x \notin A\} = \{6\},$$

then

$$A \Delta B = (A - B) \cup (B - A) = \{1,3\} \cup \{6\} = \{1,3,6\}.$$

### 1.1.4.6. Cartesian Product

**Definition 1.14.** Any two elements $a$ an $b$, written in the form $(a, b)$ is called an **ordered pair**.

**Remark 1.6.** In an ordered pair $(a, b)$, the order of the elements is significant, that is, $a$ is the **first coordinate** and $b$ is the **second coordinate**. So, $(a, b)$ is different from $(b, a)$ unless $a = b$.

**Definition 1.15.** A **cartesian product** of two non-empty sets $A$ and $B$ is the set of all possible ordered pairs where the first component of the pair is from $A$, and the second component of the pair is from $B$. The set of ordered pairs thus obtained is denoted by $A \times B$ where

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

**Example 1.12.** Let
$$A = \{1,2,3\} \text{ and } B = \{4,6\}.$$
We have
$$A \times B = \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)\},$$
and
$$B \times A = \{(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)\}.$$

**Remark 1.7.** $A \times B \neq B \times A$, unless $A = B$.

### 1.1.4.7. Power Set

**Definition 1.16.** The **power set** $\mathcal{P}(A)$ of a set $A$ is the family of all the subsets of $A$.

**Remark 1.8.**
**1**) Notice that we always have $\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$.
**2**) If the original set consists of $n$ elements, then the power set consists of $2^n$ elements.

**Example 1.13.** Let $A = \{a, b, c\}$. Then
$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

### 1.1.4.8. Partition of a Set

**Definition 1.17.** A **partition** of a set $A$ is a subdivision of the set into subsets that are disjoint and exhaustive, i.e., every element of $A$ must belong to one and only one of the subsets.

### 1.1.4.9. Cardinal Number

**Definition 1.18.** The number of distinct elements in a finite set $A$ is called its **cardinal number**. It is denoted by $card(A)$ and read as the number of elements of the set $A$.

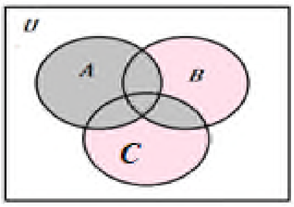
In the example 1.13,
$$card(A) = 3 \text{ and } card\mathcal{P}(A) = 2^3 = 8.$$

### 1.1.5. Distributive Properties of Union and Intersection of Sets

**Theorem 1.1.** Let $U$ be a set and let $A, B$ and $C$ be subsets of $U$. Then
**1**) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
**2**) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**Proof.**

**1**) First, let us use a Venn diagram as a visual help in constructing our proof.



The shaded gray part indicates the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

To prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, we will prove the following double inclusion: $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.
For the first inclusion, let $x \in A \cup (B \cap C)$, we have

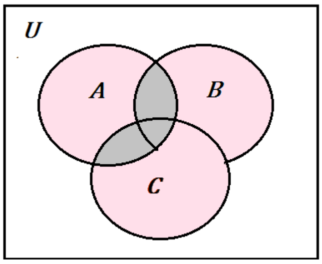
$$\begin{aligned} x \in A \cup (B \cap C) &\Longrightarrow (x \in A) \text{ or } (x \in B \cap C) \\ &\Longrightarrow (x \in A) \text{ or } \big((x \in B) \text{ and } (x \in C)\big) \\ &\Longrightarrow \big((x \in A) \text{ or } (x \in B)\big) \text{ and } \big((x \in A) \text{ or } (x \in C)\big) \\ &\Longrightarrow (x \in A \cup B) \text{ and } (x \in A \cup C) \\ &\Longrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

Therefore, $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
For the second inclusion, let $y \in (A \cup B) \cap (A \cup C)$, we get

$$\begin{aligned} y \in (A \cup B) \cap (A \cup C) &\Longrightarrow (y \in A \cup B) \text{ and } (y \in A \cup C) \\ &\Longrightarrow (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C) \\ &\Longrightarrow (y \in A) \text{ or } (y \in B \text{ and } y \in C) \\ &\Longrightarrow (y \in A) \text{ or } (y \in B \cap C) \\ &\Longrightarrow y \in A \cup (B \cap C). \end{aligned}$$

Therefore, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.
From these two inclusions, we infer that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ as required.

**2**) Let us illustrate the required set using a Venn diagram:



The shaded gray part indicates the sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

To prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we will prove the following double inclusion: $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. For the first inclusion, let $x \in A \cap (B \cup C)$, we have

$$\begin{aligned} x \in A \cap (B \cup C) &\Longrightarrow (x \in A) \text{ and } (x \in B \cup C) \\ &\Longrightarrow (x \in A) \text{ and } \big((x \in B) \text{ or } (x \in C)\big) \\ &\Longrightarrow \big((x \in A) \text{ and } (x \in B)\big) \text{ or } \big((x \in A) \text{ and } (x \in C)\big) \\ &\Longrightarrow (x \in A \cap B) \text{ or } (x \in A \cap C) \\ &\Longrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

So, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
For the second inclusion, let $y \in (A \cap B) \cup (A \cap C)$, we have

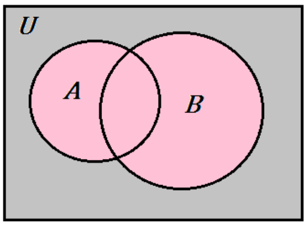
$$\begin{aligned} y \in (A \cap B) \cup (A \cap C) &\Longrightarrow (y \in A \cap B) \text{ or } (y \in A \cap C) \\ &\Longrightarrow (y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C) \\ &\Longrightarrow (y \in A) \text{ and } (y \in B \text{ or } y \in C) \\ &\Longrightarrow (y \in A) \text{ and } (y \in B \cup C) \\ &\Longrightarrow y \in A \cap (B \cup C), \end{aligned}$$

which means that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Finally, according to the above inclusions, we conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ as required. ■

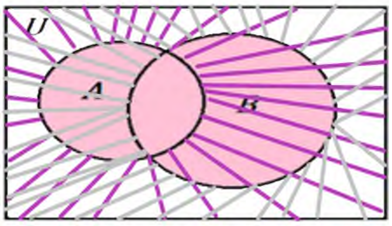
### 1.1.6. De Morgan's Laws for Sets

**Theorem 1.2.** Let $U$ be a set and let $A$ and $B$ be two subsets of $U$. Then
**1**) $(A \cup B)^c = A^c \cap B^c$ (which is a the De Morgan’s Law of union).
**2**) $(A \cap B)^c = A^c \cup B^c$ (which is a the De Morgan’s Law of intersection).

**Proof.**
1) Here we are going to see the proof of the De Morgan’s first law by a Venn diagram.



$(A \cup B)^c$

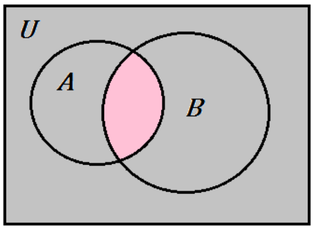


$A^c \cap B^c$

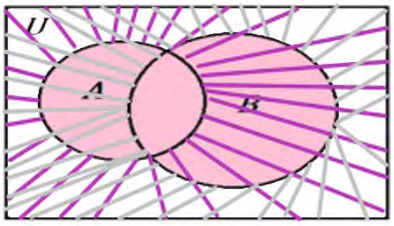
Now, let us prove this law using algebra of sets.

$$\begin{aligned} x \in (A \cup B)^c &\Leftrightarrow (x \in U) \text{ and } x \notin (A \cup B) \\ &\Leftrightarrow (x \in U) \text{ and } \big((x \notin A) \text{ and } (x \notin B)\big) \\ &\Leftrightarrow \big((x \in U) \text{ and } (x \notin A)\big) \text{ and } \big((x \in U) \text{ and } (x \notin B)\big) \\ &\Leftrightarrow (x \in A^c) \text{ and } (x \in B^c) \\ &\Leftrightarrow \; x \in A^c \cap B^c. \end{aligned}$$

**2**) Here we are going to see the proof of the De Morgan's second law by a Venn diagram.



$(A \cap B)^c$



$A^c \cup B^c$

Now, let us prove this law using algebra of sets.

$$\begin{aligned} x \in (A \cap B)^c &\Leftrightarrow (x \in U) \text{ and } x \notin (A \cap B) \\ &\Leftrightarrow (x \in U) \text{ and } \big((x \notin A) \text{ or } (x \notin B)\big) \\ &\Leftrightarrow \big((x \in U) \text{ and } (x \notin A)\big) \text{ or } \big((x \in U) \text{ and } (x \notin B)\big) \\ &\Leftrightarrow (x \in A^c) \text{ or } (x \in B^c) \\ &\Leftrightarrow \; x \in A^c \cup B^c. \end{aligned}$$

This completes the proof. ■

## 1.2. Relations

**Definition 1.19.** A binary **relation** $\mathcal{R}$ from a non-empty set $A$ to a non empty set $B$ is a subset of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$ and we write $a\mathcal{R}b$ or $(a, b) \in \mathcal{R}$ if $a \in A$ and $b \in B$ are related.

Moreover, the set of all first elements in a relation $\mathcal{R}$ denoted by $Dom\,\mathcal{R}$, is called the **domain** of the relation $\mathcal{R}$, and the set of all second elements in a relation $\mathcal{R}$ denoted by $Rang\,\mathcal{R}$, is called the **range** of $\mathcal{R}$ whereas $B$ is the **co-domain** of $\mathcal{R}$. In short

$$Dom\,\mathcal{R} = \{a: (a, b) \in \mathcal{R}\} \text{ and } Rang\,\mathcal{R} = \{b: (a, b) \in \mathcal{R}\}.$$

One can also define relations on more than two sets. From now on, we will consider only binary relations and refer to them simply as relations. Furthermore, if $A = B$, then we call $\mathcal{R}$ a relation on $A$.

**Example 1.14.**

**1**) Find the domain and the range of the relation $\mathcal{R}$ defined as follows:

$$\mathcal{R} = \{(45.5, 65.5), (48.2, 68.2), (41.8, 62.2), (46.6, 66.3), (50.4, 70.01)\}.$$

We have

$$Dom\ \mathcal{R} = \{45.5,48.2, 41.8, 46.6, 50.4\},$$

and

$$Rang\ \mathcal{R} = \{65.5,68.2, 62.2, 66.3,70.01\}.$$

**2**) Let $A = \{2,3,4\}$ and $B = \{3,4,5,6,7\}$. Define the relation $\mathcal{R}$ by

$$a\,\mathcal{R}\,b \text{ if and only if } a \text{ divides } b.$$

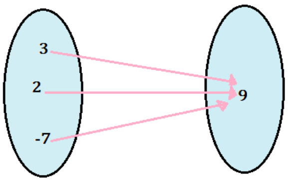
Write $\mathcal{R}$ as a set of ordered pairs. Also find $Dom\ \mathcal{R}$ and $Rang\ \mathcal{R}$ .

We have

$$\mathcal{R} = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}.$$

$Dom\ \mathcal{R}$=$\{2,3,4\}$ and $Rang\ \mathcal{R}$=$\{3, 4, 6\}$.

**3**) Find the domain and the range of the relation $\mathcal{R}$ defined by:



$Dom\ \mathcal{R}$=$\{2,3,-7\}$ and $Rang\ \mathcal{R}$=$\{9\}$.

### 1.2.1. Inverse Relations

**Definition 1.20.** Given a relation $\mathcal{R}$ from $A$ to $B$, the **inverse** of $\mathcal{R}$ (also called the **converse** of $\mathcal{R}$), denoted by $\mathcal{R}^{-1}$ is a relation from $B$ to $A$ defined by

$$\mathcal{R}^{-1} = \{(b, a): (a, b) \in \mathcal{R}\}.$$

The inverse of a relation is formed by interchanging or swapping the components of each of the ordered pairs in the given relation ($b\mathcal{R}^{-1}a$).

**Example 1.15.**

**1**) If $\mathcal{R}$ is the relation "being a son or daughter of", then $\mathcal{R}^{-1}$ is the relation "being a parent of".

**2**) Let $A = \{2,3,4\}$ and $B = \{3,4,5,6,7\}$. Define the relation $\mathcal{R}$ by

$$a\,\mathcal{R}\,b \text{ if and only if } a \text{ divides } b.$$

Find $\mathcal{R}^{-1}$, Domain of $\mathcal{R}^{-1}$, Range of $\mathcal{R}^{-1}$.
According to example 1.14, we have

$$\mathcal{R} = \{(2,4),(2,6),(3,3),(3,6),(4,4)\}.$$

So

$$\mathcal{R}^{-1} = \{(4,2),(6,2),(3,3),(6,3),(4,4)\},$$

where

$$b\,\mathcal{R}^{-1}a \text{ if and only if } b \text{ is a multiple of } a.$$

Thus $Dom\,\mathcal{R}^{-1} = \{3,4,6\}$ and $Rang\,\mathcal{R}^{-1} = \{2,3,4\}$.

## 1.2.2. Properties of Binary Relations

In this section, we would like to study different properties of relations on a set $A$.

**Definition 1.21.** A relation $\mathcal{R}$ on a set $A$ is called **reflexive** if for all $x \in A$, $x\mathcal{R}x$ holds, i.e.,

$$(\mathcal{R} \text{ is reflexive}) \Leftrightarrow (\forall x \in A: x\mathcal{R}x).$$

In simple words, $\mathcal{R}$ is reflexive if each element is related to itself.

**Example 1.16.**

**1**) The following relations: "is equal to" (equality:=), "is a subset of" (set inclusion: $\subseteq$), "divides" (divisibility: $\div$ or $/$), "is greater than or equal to" ($\geq$), "is less than or equal to" ($\leq$) are reflexive relations.

**2**) Let $A = \{1,2,3,4\}$ and let $\mathcal{R}$ be a relation on $A$ defined as follows:

$$\mathcal{R} = \{(1,1),(1,3),(2,1),(2,2),(3,3),(3,4),(4,4)\}.$$

Since

$$1\mathcal{R}1, 2\mathcal{R}2, 3\mathcal{R}3, \text{and } 4\mathcal{R}4,$$

then $\mathcal{R}$ is reflexive.

**Definition 1.22.** A relation $\mathcal{R}$ on a set $A$ is called **symmetric** if for all $x, y \in A$, $x\mathcal{R}y$ implies $y\,\mathcal{R}x$. i.e.,

$$(\mathcal{R} \text{ is symmetric}) \Leftrightarrow \big(\forall x, y \in A: (x\mathcal{R}y) \Rightarrow (y\mathcal{R}x)\big).$$

In simple words, $\mathcal{R}$ is symmetric if any element is related to any other element, then the second element is related to the first one.

**Example 1.17.**

**1)** On $\mathbb{Z}$, the equality $(=)$ is symmetric, but the strict inequality $(<)$ is not.

**2**) (Parallelism: $\parallel$) and (Perpendicularity: $\perp$) are symmetric relations.

**3**) Let $A = \{1, 2, 3, 4\}$ and let $\mathcal{R}$ be the relation on $A$ defined as follows:

$$\mathcal{R} = \{(1,1),(2,3),(3,2),(4,4)\}.$$

Since$1\mathcal{R}1, 4\mathcal{R}4$, $2\mathcal{R}3$ and $3\mathcal{R}2$, then $\mathcal{R}$ is symmetric.

**4**) Let $A = \{1, 2, 3, 4\}$ and let $\mathcal{R}$ a the relation defined by:

$$\mathcal{R} = \{(1,1),(2,3),(3,2),(4,4),(1,3)\}.$$

We have $1\mathcal{R}3$ but $3\mathcal{R}1$ is not true which means that $\mathcal{R}$ is not symmetric.

**Definition 1.23.** A relation $\mathcal{R}$ on a set $A$ is called **antisymmetric** if for all $x, y \in A$, $x\mathcal{R}y$ and $y\,\mathcal{R}x$ implies $x = y$, i.e.,

$$(\mathcal{R} \text{ is antisymmetric}) \Leftrightarrow \left(\forall (x,y) \in A \times A : \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}x \end{cases} \Longrightarrow x = y\right).$$

**Example 1.18.**

**1**) (The equality:=), (the set inclusion: $\subseteq$), (the divisibility: $\div$ or $/$), $(\geq)$ are antisymmetric relations.

**2**) Let $A = \{1, 2, 3\}$ and let $\mathcal{R}$ be a relation on $A$ defined by

$$\mathcal{R} = \{(1,1),(2,2),(3,3)\}.$$

$\mathcal{R}$ is antisymmetric.

**3**) Let $A = \{1, 2, 3\}$ and let $\mathcal{R}$ be the relation on $A$ defined by

$$\mathcal{R} = \{(1,1),(1,2),(2,1)\}.$$

$\mathcal{R}$ is not antisymmetric, because $1\mathcal{R}2$ and $2\mathcal{R}1$, but $1 \neq 2$.

**4**) The relation $\mathcal{R}$ on the set $\mathbb{N}$ of natural numbers given by

$$\forall x, y \in \mathbb{N} : x\mathcal{R}y \Leftrightarrow 5x + 4y = 15,$$

is antisymmetric.

Indeed, if $x, y \in \mathbb{N}$ such that $x\mathcal{R}y$ and $y\mathcal{R}x$, then

$$5x + 4y = 15 \ldots (1)$$

and

$$5y + 4x = 15 \ldots (2)$$

The subtraction of equation $(2)$ from equation $(1)$ yields $x - y = 0$ and hence $x = y$. So $\mathcal{R}$ is antisymmetric.

**Definition 1.24.** A relation $\mathcal{R}$ on a set $A$ is called **transitive** if for all $x, y \in A$, $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$. i.e.,

$$(\mathcal{R} \text{ is transitive }) \Leftrightarrow \left(\forall x, y, z \in A: \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}z \end{cases} \Longrightarrow x\mathcal{R}z\right).$$

In simple words, $\mathcal{R}$ is transitive if any one element is related to a second and that second element is related to a third one, then the first element is related to the third one.

**Example 1.19.**

**1**) (The equality:=), (the set inclusion: $\subseteq$), (the divisibility: $\div$ or $/$), (greater than or equal to: $\geq$),) are transitive relations.

**2**) Let $\mathcal{R}$ be a relation on $\mathbb{N}$ given by

$$\forall x, y \in \mathbb{N}: x\mathcal{R}y \Leftrightarrow x \text{ is a multiple of } y.$$

If $x, y, z \in \mathbb{N}$ such that $x\mathcal{R}y$ and $y\mathcal{R}z$, then

$$\exists k_1 \in \mathbb{N}: x = k_1 y \ldots (1)$$

and

$$\exists k_2 \in \mathbb{N}: y = k_2 z \ldots (2)$$

Substituting equation $(2)$ into equation $(1)$, we get

$$x = k_1(k_2 z) = (k_1 k_2)z.$$

By taking $k = k_1 k_2 \in \mathbb{N}$, we get

$$x = kz, k \in \mathbb{N}.$$

So, $x$ is a multiple of $z$. Therefore $x\mathcal{R}z$ holds and hence $\mathcal{R}$ is transitive.

### 1.2.3. Equivalence Relations

**Definition 1.25.** A relation $\mathcal{R}$ on a set $A$ is said to be an **equivalence** relation if it is reflexive, symmetric and transitive.

**Example 1.20.**

**1**) (The equality:=), (the set inclusion: $\subseteq$), (the divisibility: $\div$ or $/$), (greater than or equal to: $\geq$) are equivalence relations.

**2**) Let $L$ be the set of all lines in a plane and let $\mathcal{R}$ be a relation in $L$ defined as follows:

$$\forall l_1, l_2 \in L: l_1\mathcal{R}l_2 \Leftrightarrow l_1 \text{ is perpendicular to } l_2.$$

Determine whether $\mathcal{R}$ is reflexive, symmetric, transitive, or it is an equivalence relation.

**a**) $\mathcal{R}$ is not reflexive, because a line $l$ can not be perpendicular to itself.

**b**) $\mathcal{R}$ is symmetric because $\forall l_1, l_2 \in L$:

$$l_1 \text{ is perpendicular to } l_2 \Longrightarrow l_2 \text{ is perpendicular to } l_1.$$

So, $l_2 \mathcal{R} l_1$. Thus $\mathcal{R}$ is symmetric.

**c**) $\mathcal{R}$ is not transitive. Indeed, if $l_1$ is perpendicular to $l_2$ and $l_2$ is perpendicular to $l_3$, then $l_1$ can never be perpendicular to $l_3$. In fact, $l_1$ is parallel to $l_3$. So, $\mathcal{R}$ is not an equivalence relation.

**3**) The relation $\mathcal{R}$ on the set $\mathbb{Z}$ given by

$$\forall x, y \in \mathbb{Z}: x\mathcal{R}y \Leftrightarrow 2 \text{ divides } x - y,$$

is an equivalence relation.

Indeed, by definition

$$\big(\mathcal{R} \text{ is an equivalence relation}\big) \Leftrightarrow \left(\begin{cases} (\mathbf{a})\ \mathcal{R} \text{ is reflexive} \\ (\mathbf{b})\ \mathcal{R} \text{ is symmetric} \\ (\mathbf{c})\ \mathcal{R} \text{ is transitive} \end{cases}\right).$$

(**a)** <u>$\mathcal{R}$ is reflexive</u>

$$\big(\mathcal{R} \text{ is reflexive}\big) \Leftrightarrow \big(\forall x \in \mathbb{Z}: x\mathcal{R}x\big).$$

For $x \in \mathbb{Z}$, we have 2 divides $x - x = 0$. In fact, there exists $k = 0$ such that

$$x - x = 0 = 2k, k \in \mathbb{Z}.$$

So $x\mathcal{R}x$. Thus $\mathcal{R}$ is reflexive.

(**b**) <u>$\mathcal{R}$ is symmetric</u>

$$\big(\mathcal{R} \text{ is symmetric}\big) \Leftrightarrow \Big(\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}: \big(x\mathcal{R}y\big) \Longrightarrow \big(y\mathcal{R}x\big)\Big).$$

Let $x, y \in \mathbb{Z}$. If $x\mathcal{R}y$, then there exists $k \in \mathbb{Z}$ such that

$$x - y = 2k,$$

But

$$x - y = 2k \Longrightarrow y - x = -2k = 2(-k).$$

By taking $l = -k$ we get

$$y - x = 2l:\ l \in \mathbb{Z}.$$

So $y\mathcal{R}x$. Thus $\mathcal{R}$ is symmetric.

(**c**) <u>$\mathcal{R}$ is transitive</u>

$$\big(\mathcal{R} \text{ is transitive }\big) \Leftrightarrow \left(\forall x, y, z \in \mathbb{Z}: \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}z \end{cases} \Longrightarrow x\mathcal{R}z\right).$$

Let $x, y, z \in \mathbb{Z}$. If $x\mathcal{R}y$ and $y\mathcal{R}z$ then there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$x - y = 2k_1 \quad \ldots (1)$$

and

$$y - z = 2k_2 \quad \ldots (2)$$

Adding the two equations (1) to (2), we get

$$x - z = 2k_1 + 2k_2 = 2(k_1 + k_2).$$

By taking $k = k_1 + k_2$, we obtain

$$x - z = 2k, k \in \mathbb{Z}.$$

So $x\mathcal{R}z$. Thus $\mathcal{R}$ is transitive.

Since $\mathcal{R}$ is reflexive, symmetric and transitive, then $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

### 1.2.4. Equivalence Classes, Quotient Sets and Partitions

**Definition 1.26.** Given an equivalence relation $\mathcal{R}$ on a set $A$ and an element $x \in A$. The set denoted by $\dot{x}$ of all elements of $A$ that are related to $x$ is called the **equivalence class** of $x$. Mathematically, we write

$$\dot{x} = \{y \in A : y\mathcal{R}x\} = \{y \in A : x\mathcal{R}y\}.$$

The collection of equivalence classes, represented by

$$\frac{\mathcal{R}}{A} = \{\dot{x} : \ x \in A\},$$

is called the **quotient set** of $A$ by $\mathcal{R}$ .

**Remark 1.9.**

**1**) Be mindful that $\dot{x}$ is a subset of $A$ ($\dot{x} \subseteq A$), it is not an element of $A$ ($\dot{x} \notin A$).

**2**) Typically, the set $\dot{x}$ contains much more than just $x$. The element $x$ is called a **representative** of the equivalence class $\dot{x}$. Any element of an equivalence class can serve as a representative.

**3)** Since $\mathcal{R}$ is reflexive, then $x\mathcal{R}x$ . Thus $x \in \dot{x}$.

**Example 1.21.** Let $\mathcal{R}$ be a relation defined on $\mathbb{R}$ as follows:

$$\forall x, y \in \mathbb{R}: x\mathcal{R}y \Leftrightarrow x^2 + 2x = y^2 + 2y.$$

(**i**) Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$ .

(**ii**) Find the equivalence classes $\dot{0}$ and $\dot{1}$.

(**i**) (**a**) <u>$\mathcal{R}$ is reflexive</u>

$$(\mathcal{R} \text{ is reflexive}) \Leftrightarrow (\forall x \in \mathbb{R}: x\mathcal{R}x).$$

$\forall x \in \mathbb{R}: x^2 + 2x = x^2 + 2x \Leftrightarrow x\mathcal{R}x$ holds. So $\mathcal{R}$ is reflexive.

**(b)** $\mathcal{R}$ is symmetric

We have

$$(\mathcal{R}\text{ is symmetric}) \Leftrightarrow \big(\forall x, y \in \mathbb{R}: (x\mathcal{R}y) \Rightarrow (y\mathcal{R}x)\big).$$

Let $x, y \in \mathbb{R}$. Since the equality is symmetric, we get

$$x\mathcal{R}y \Leftrightarrow x^2 + 2x = y^2 + 2y \Rightarrow y^2 + 2y = x^2 + 2x \Rightarrow y\mathcal{R}x.$$

So $\mathcal{R}$ is symmetric.

**(c)** $\mathcal{R}$ is transitive

$$(\mathcal{R}\text{ is transitive}) \Leftrightarrow \left(\forall x, y, z \in \mathbb{R}: \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}z \end{cases} \Rightarrow x\mathcal{R}z\right).$$

Let $x, y, z \in \mathbb{R}$. If $x\mathcal{R}y$ and $y\mathcal{R}z$ then

$$x^2 + 2x = y^2 + 2y \quad \dots (1)$$

and

$$y^2 + 2y = z^2 + 2z \quad \dots (2)$$

Adding the two equations (1) and (2) together, we get $x^2 + 2x = z^2 + 2z$. So $x\mathcal{R}z$ holds. Thus $\mathcal{R}$ is transitive.

Since $\mathcal{R}$ is reflexive, symmetric and transitive, then $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.

**(ii)** We have

$$\dot{0} = \{y \in \mathbb{R}: y\mathcal{R}0\} = \{y \in \mathbb{R}: y^2 + 2y = 0^2 + 2(0)\}$$
$$= \{y \in \mathbb{R}: y^2 + 2y = 0\}.$$

$y^2 + 2y = y(y + 2) = 0$ implies that $y = 0$ or $y = -2$, then

$$\dot{0} = \{0, -2\}.$$

$$\dot{1} = \{y \in \mathbb{R}: y\mathcal{R}1\} = \{y \in \mathbb{R}: y^2 + 2y = 1^2 + 2(1)\}$$
$$= \{y \in \mathbb{R}: y^2 + 2y = 3\}.$$

$y^2 + 2y = 3$ implies that $y = 1$ or $y = -3$, then

$$\dot{1} = \{1, -3\}.$$

**Theorem 1.3.** Let $\mathcal{R}$ be an equivalence relation on a set $A$. Then the equivalence classes of $\mathcal{R}$ form a partition of $A$.

**Example 1.22.** Let $\mathcal{R}$ be a relation on the set of integers $\mathbb{Z}$ defined as follows:

$$\forall x, y \in \mathbb{Z}: x\mathcal{R}y \Leftrightarrow 2 \text{ divides } x - y.$$

$\mathcal{R}$ is an equivalence relation, and partitions the set integers into two equivalence classes, i.e., the **even** and **odd** integers.

## 1.2.5. Partial Orders

**Definition 1.27.** A relation $\mathcal{R}$ on a set $A$ is called a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive. A set $A$ together with a partial ordering $\mathcal{R}$ is called a **partially ordered** set, or **poset**, and it is denoted by $(A, \mathcal{R})$. Members of $A$ are called elements of the poset.

**Example 1.23.**

**1**) Let $\mathcal{R}$ be the relation defined on $\mathbb{Z}^+$ such that

$$\forall x, y \in \mathbb{Z}^+ : x\mathcal{R}y \Leftrightarrow x \text{ divides } y.$$

Show that $\mathcal{R}$ is a partial ordering relation on $\mathbb{Z}^+$ .

By definition

$$\left(\mathcal{R} \text{ is a partial ordering relation }\right) \Leftrightarrow \left(\begin{cases} (\mathbf{a})\ \mathcal{R} \text{ is reflexive} \\ (\mathbf{b})\ \mathcal{R} \text{ is antisymmetric} \\ (\mathbf{c})\ \mathcal{R} \text{ is transitive} \end{cases}\right).$$

(**a**) <u>$\mathcal{R}$ is reflexive</u>

One has

$$\left(\mathcal{R} \text{ is reflexive}\right) \Leftrightarrow \left(\forall x \in \mathbb{Z}^+ : x\mathcal{R}x\right).$$

$\forall\, x \in \mathbb{Z}^+$: $x$ divides $x \Leftrightarrow x\mathcal{R}x$. Hence the reflexive property is proved.

(**b**) <u>$\mathcal{R}$ is antisymmetric</u>

$$\left(\mathcal{R} \text{ is antisymmetric}\right) \Leftrightarrow \left(\forall x, y \in \mathbb{Z}^+ : \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}x \end{cases} \Rightarrow x = y\right).$$

If $x, y \in \mathbb{Z}^+$ with $x\mathcal{R}y$ and $y\mathcal{R}x$ , then there exist $m_1, m_2 \in \mathbb{Z}^+$ such that

$$y = m_1 x \quad \dots (1)$$

and

$$x = m_2 y \quad \dots (2)$$

By multiplying equations (1) and (2) together, we get

$$yx = m_1 m_2 xy.$$

So $m_1 m_2 = 1$. But $m_1, m_2 \in \mathbb{Z}^+$ implies $m_1 = m_2 = 1$. Thus $x = y$ and $\mathcal{R}$ is anti-symmetric.

(**c**) <u>$\mathcal{R}$ is transitive</u>

$$\left(\mathcal{R} \text{ is transitive }\right) \Leftrightarrow \left(\forall x, y, z \in \mathbb{Z}^+ : \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}z \end{cases} \Rightarrow x\mathcal{R}z\right).$$

Let $x, y, z \in \mathbb{Z}^+$. If $x\mathcal{R}y$ and $y\mathcal{R}z$ , then there exist $m_1, m_2 \in \mathbb{Z}^+$ such that

$$y = m_1 x \quad \dots (1)$$

and

$$z = m_2 y \quad \dots (2)$$

The multiplication of equations (1) and (2) together, yields

$$yz = m_1 m_2 xy.$$

So, $z = m_1 m_2 x$. Putting $m_1 m_2 = m \in \mathbb{Z}^+$ gives $z = mx$ which implies $x\mathcal{R}z$. Thus $\mathcal{R}$ is transitive.

Since $\mathcal{R}$ is reflexive, antisymmetric and transitive, then $\mathcal{R}$ is a partial ordering relation on $\mathbb{Z}^+$.

**2**) Let $\mathcal{R}$ be a relation on $\mathbb{R}$ defined by

$$\forall x, y \in \mathbb{R} : x\mathcal{R}y \Leftrightarrow x \le y.$$

(**a**) <u>$\mathcal{R}$ is reflexive</u>

One has

$$\big(\mathcal{R} \text{ reflexive}\big) \Leftrightarrow \big(\forall x \in \mathbb{R}: x\mathcal{R}x\big).$$

$\forall\, x \in \mathbb{R}$: $x \le x \Leftrightarrow x\mathcal{R}x$. So, $\mathcal{R}$ is reflexive.

(**b**) <u>$\mathcal{R}$ is antisymmetric</u>:

$$\big(\,\mathcal{R} \text{ is antisymmetric}\big) \Leftrightarrow \left(\forall (x, y) \in \mathbb{R} \times \mathbb{R}: \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}x \end{cases} \Rightarrow x = y\right).$$

If $x, y \in \mathbb{R}$ such that $x\mathcal{R}y$ and $y\mathcal{R}x$ , then

$$x \le y \quad \dots (1)$$

and

$$y \le x \quad \dots (2)$$

In view of equations (1) and (2), we get $x = y$. Thus $\mathcal{R}$ is antisymmetric.

(**c**) <u>$\mathcal{R}$ is transitive</u>

$$\big(\mathcal{R} \text{ is transitive }\big) \Leftrightarrow \left(\forall x, y, z \in \mathbb{R}: \begin{cases} x\mathcal{R}y \\ \text{and} \\ y\mathcal{R}z \end{cases} \Rightarrow x\mathcal{R}z\right).$$

Let $x, y, z \in \mathbb{R}$. If $x\mathcal{R}y$ and $y\mathcal{R}z$ , then

$$x \le y \quad \dots (3)$$

and

$$y \le z \quad \dots (4)$$

In view of equations (3) and (4), we get $x \le y$. So $x\mathcal{R}z$. Thus $\mathcal{R}$ is transitive.

Since $\mathcal{R}$ is reflexive, antisymmetric and transitive, then $\mathcal{R}$ is a partial ordering relation on $\mathbb{R}$.

### 1.2.6. Comparability

**Definition 1.28.** Let $x$ and $y$ be two elements belonging to the set $A$. Assume that $(A, \mathcal{R})$ is a poset. We say that $x$ and $y$ are **comparable** if exactly one of $x\mathcal{R}y$ and $y\mathcal{R}x$ holds. When $x$ and $y$ are elements of $A$ such that neither $x\mathcal{R}y$ nor $y\mathcal{R}x$, $x$ and $y$ are called **incomparable**.

**Definition 1.29.** If $(A, \mathcal{R})$ is a poset and every two elements of $A$ are comparable, $A$ is called a **totally ordered** or a **linearly ordered** set, and $\mathcal{R}$ is called a **total or linear order**. A totally ordered set is also called a chain.

**Example 1.24.**

**1**) Let $\mathcal{R}$ be a relation on $\mathbb{Z}^+$ such that

$$\forall x, y \in \mathbb{Z}^+ : x\mathcal{R}y \Leftrightarrow x \text{ divides } y.$$

We have $3, 7 \in \mathbb{Z}^+$, $3$ does not divide $7$ and $7$ does not divide $3$. So $3$ and $7$ are incomparable. Thus $\mathcal{R}$ is not a total order.

**2**) Let $A = \{2,4,8,16\}$ and let $\mathcal{R}$ be a relation on $A$ such that

$$\forall x, y \in A : x\mathcal{R}y \Leftrightarrow x \text{ divides } y.$$

We have all elements in $A$ are powers of $2$. So, $\forall x, y \in A$, $x$ divides $y$ or $y$ divides $x$. Thus $\mathcal{R}$ is a total order.

**3)** Let $\mathcal{R}$ be a relation defined on $\mathbb{R}$ such that

$$\forall x, y \in \mathbb{R} : x\mathcal{R}y \Leftrightarrow x \leq y.$$

$\forall x, y \in \mathbb{R}$, $x \leq y$ or $y \leq x$. Thus $\mathcal{R}$ is a total order.

## 1.3. Functions

**Definition 1.30.** Let $X$ and $Y$ be two nonempty sets. A relation $f$ from a set $X$ to a set $Y$ is called **function** if every element of the set $X$ has one and only one image in the set $Y$. In simple words, a function is a relation such that no two ordered pairs have the same first coordinates and different second coordinates.

**Remark 1.10.**

**1**) The notation $f: X \longrightarrow Y$ means that $f$ is a function from $X$ to $Y$. $X$ is called the **domain** of $f$ and $Y$ is called the **co-domain** of $f$.

**2**) Given an element $x \in X$, there is a unique element $y \in Y$ that is related to $x$. The unique element $y$ to which $f$ relates $x$ is denoted by $f(x)$ and is pronounced $f$ of $x$, or the value of $f$ at $x$, or the image of $x$ under $f$.

**3**) The set of all values of $f(x)$ taken together is called the range of $f$ or image of $X$ under $f$. Symbolically

$$\begin{aligned} Rang\, f &= \{y \in Y : \exists x \in X, f(x) = y\} \\ &= \{f(x) : x \in X\}. \end{aligned}$$

**4**) The definition of a function $f$ from a set $X$ to a set $Y$ requires that a relation must fulfill the following two conditions in order to qualify as a function:

(i) Every $x \in X$ must be related to $y \in Y$; that is the domain of $f$ must be $X$ and not merely a subset of $X$.

(ii) The second requirement of uniqueness can be expressed as follows:

$$\forall x, y \in X : \begin{cases} xfy \\ \text{and} \\ xfz \end{cases} \Longrightarrow y = z.$$

**5**) An element of $Y$ may have more than one element of $X$ associated with it.

**6**) Functions are sometimes also called mappings or transformations.

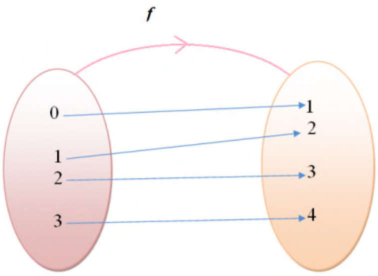
**Example 1.25.** Determine which of the relations define $y$ as a function of $x$.

**1**) Let $X = \{0,1,2,3\}$ and $Y = \{1,2,3,4\}$ and let $f$ be a relation from $X$ to $Y$ given by

$$xfy \Leftrightarrow y = x + 1.$$

In other words, $f : X \longrightarrow Y$ and

$$f(x) = x + 1.$$



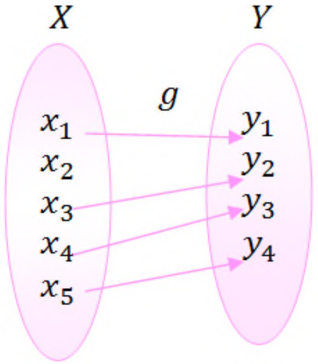
For each $x \in X$ there is only one corresponding $y \in Y$. Therefore, this relation is a function. Moreover

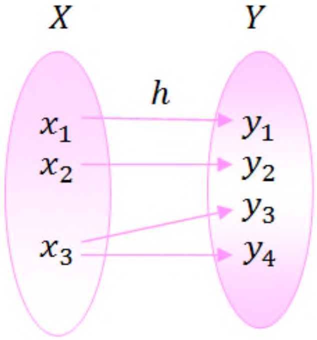
$$Dom\, f = \{0,1,2,3\},$$

and

$$Rang\, f = \{1,2,3,4\}.$$

**2)**





- $g$ is not a function, because the element $x_2 \in X$ has no image in the set $Y$.
- $h$ is not a function, because the element $x_3 \in X$ has two different images $y_3, y_4 \in Y$ such that $f(x_3) = y_3$ and $f(x_3) = y_4$.

**3)** Let $X$ be the set of children and $Y$ the set of fathers. The relation $f$ from $X$ to $Y$ given by

$$xfy \Leftrightarrow y \text{ is the father of } x,$$

is a function because no child has more than one father and no child has no father.

**4)** $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$ is not a function, because the element $0 \in \mathbb{R}$ has no image in the set $\mathbb{R}$.

### 1.3.1. Some types of Functions

Let $X$ and $Y$ be two nonempty sets and let $f: X \longrightarrow Y$ be a function.

**1)** If $X = Y$, the function $f: X \longrightarrow X$ defined by $y = f(x) = x$ for each $x \in X$ is called the **identity** function, and it is denoted by $i_d$. Symbolically $i_d: X \longrightarrow X$ and $i_d(x) = x$. Here, $Dom\ i_d = Rang\ i_d = X$.

**2)** If $X = Y = \mathbb{R}$, the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $y = f(x) = C, x \in \mathbb{R}$, where $C$ is a constant in $\mathbb{R}$, is called the **constant** function. Here, $Dom\ f = \mathbb{R}$ and $Rang\ f = \{C\}$.

**3)** If $X \subseteq Y$, the function $f: X \longrightarrow Y$ defined by $y = f(x) = x$ for each $x \in X$ is called the **inclusion** function from $X$ to $Y$.

**4)** Let $A, B$ be two sets, and let $f: X \longrightarrow Y$ and $g: A \longrightarrow B$ be two functions. We say that $f$ is **identically equal** to $g$ (denoted by $f \equiv g$) if the following requirements are met: $A = X$, $B = Y$ and $f(x) = g(x)$ for all $x \in X$.

**5)** Given a set $U \subseteq X$, the **restriction** of $f$ to $U$, denoted by $f_{/U}$, is the function $f_{/U}: U \longrightarrow Y$ that is defined by

$$f_{/U}(x) = f(x) \text{ for all } x \in U.$$

Function $f$ is called an **extension** of $f_{/U}$ over $U$.

**6)** If $X = Y = \mathbb{R}$, the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$

where $n \in \mathbb{N}$, and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in \mathbb{R}$, for each $x \in \mathbb{R}$, is called a **polynomial function**.

**7)** A **rational** function is a fraction of polynomials. That is, if $p(x)$ and $q(x)$ are polynomials, then $\frac{p(x)}{q(x)}$ is a rational function.

**8)** If $X = Y = \mathbb{R}$, the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$y = f\,(x) = |x| = \begin{cases} x, & \text{if} x \geq 0 \\ -x, & \text{if} x < 0 \end{cases}, \forall x \in \mathbb{R},$$

is called the **modulus function** or **absolute value function.** Here, $Dom\, f = \mathbb{R}$ and $Rang\, f = \mathbb{R}^+$.

### 1.3.2. One to one, Onto, and Bijective functions

**Definition 1.31.** A function $f: X \longrightarrow Y$ is called **one to one**, **one-one**, **injection** or **injective function** if each element of $Y$ is the image of at most one element of $X$. In brief, $f$ is said to be injective if

$$\forall x_1, x_2 \in X: x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2),$$

or

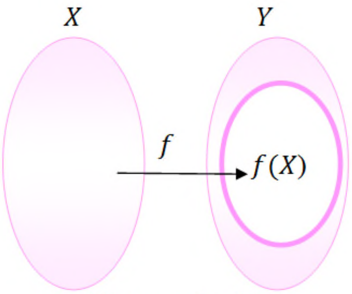
$$\forall x_1, x_2 \in X: f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

**Definition 1.32.** A function $f: X \longrightarrow Y$ is called **onto**, **surjection** or **surjective function** if every element of $Y$ is the image of some element of $X$. This condition can be expressed as

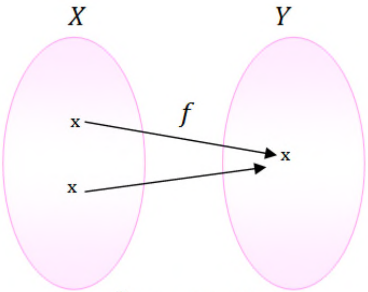
$$\forall y \in Y, \exists x \in X: y = f(x).$$

**Definition 1.33.** A function $f: X \longrightarrow Y$ is said to be **bijective** (or **bijection** or **one to one correspondence**) if it is both injective and surjective.

**Example 1.26.** **1)**



$f$ is not surjective



$f$ is not injective

**2**) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by

$$f(x) = 2x + 1.$$

If $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$, then

$$2x_1 + 1 = 2x_2 + 1 \Longrightarrow 2x_1 = 2x_2 \Longrightarrow x_1 = x_2.$$

Therefore, $f$ is injective.

For every $y \in \mathbb{R}$ we have to find a real number $x \in \mathbb{R}$ such that $f(x) = y$, which means that $2x + 1 = y$. Solving this equation for $x$ leads to

$$x = \frac{y-1}{2},$$

so

$f(\frac{y-1}{2}) = y$. Therefore $f$ is surjective.

Since $f$ is one to one and onto, then it is bijective.

**3**) Let $g: \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $g(x) = x^2$. If $x_1 = 1$ and $x_2 = -1$, then

$$g(x_1) = 1^2 = g(x_2) = (-1)^2 = 1,$$

which proves that $g$ is not injective.

If $y = -3$, then for no real number $x$ we can have $g(x) = x^2 = -3$, since the square of a real number is never negative. Therefore, $g$ is not surjective.

Now, if we change the codomain of $g$ by putting $\mathbb{R}^+$ instead of $\mathbb{R}$ ,i.e., $g: \mathbb{R} \longrightarrow \mathbb{R}^+$, then $g$ still is not one to one, but now it is onto, since every positive real number is a square number.

**Remark 1.11.** Let $f: X \longrightarrow Y$ be a function. By solving the equation $f(x) = y$ with respect to the variable $x$, we get

**1**) If for any $y \in Y$ the equation $f(x) = y$ has at most one solution, then $f$ is injective.

**2**) If for any $y \in Y$ the equation $f(x) = y$ has at least one solution, then $f$ is surjective.

**3**) If for any $y \in Y$ the equation $f(x) = y$ has one and only one solution, then $f$ is bijective.

### 1.3.3. Composition

**Definition 1.34.** Let $X, Y, U, W$ be sets, and let $f: X \longrightarrow Y$ and $g: U \longrightarrow W$ be two functions such that $Rang\ f \subseteq Dom\ g$.

The **composition** of $g$ and $f$, which is notated $g \circ f$, is the operation which combines them into the single function $g \circ f: X \longrightarrow W$ where

$$(g \circ f)(x) = g(f(x)).$$

**Remark 1.12**: In the previous definition, $Rang\ f \subseteq Dom\ g$ implies $Y \subseteq U$.

**Example 1.27.** Let $f:\mathbb{N} \longrightarrow \mathbb{N}$ and $g:\mathbb{N} \longrightarrow \mathbb{N}$ be two functions defined by $f(x) = x^2 + 2$ and $g(x) = x^2$. Then, $g \circ f, f \circ g:\mathbb{N} \longrightarrow \mathbb{N}$ such that

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^2 + 2) = (x^2 + 2)^2,$$

and

$$(f \circ g)(x) = f\big(g(x)\big) = f(x^2) = (x^2)^2 + 2 = x^4 + 2.$$

### 1.3.4 Inverse Functions

**Definition 1.35.** Let $f:X \longrightarrow Y$ be a function. We say that $f$ is **invertible** if there is a function $g:Y \longrightarrow X$ such that

$$g\big(f(x)\big) = i_d(x) = x \text{ for all } x \in X,$$

and

$$f\big(g(y)\big) = i_d(y) = y \text{ for all } y \in Y.$$

In other words

$$\forall x \in X, \forall y \in Y: f(x) = y \Leftrightarrow x = g(y).$$

Function $g:Y \longrightarrow X$ is called the **inverse** of $f:X \longrightarrow Y$ and denoted by $f^{-1}$.

**Theorem 1.4.** Let $f:X \longrightarrow Y$ be a function, $f$ is invertible if and only if $f$ is a bijection.

**Example 1.28.** Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ a function which is defined by $f(x) = 2x + 1$. In view of example 1.26, $f$ is bijective. Consequently, $f$ is invertible and $f^{-1}:\mathbb{R} \longrightarrow \mathbb{R}$ where

$$f^{-1}(y) = \frac{y-1}{2},$$

Often we write $f^{-1}(x) = \frac{x-1}{2}$ to denote the inverse function (also called anti function).

### 1.3.5. Functions and Subsets

**Definition 1.36.** Let $X, Y$ be two sets, and let $f:X \longrightarrow Y$ be a function. Given a subset $U \subseteq X$ and a subset $V \subseteq Y$.

**(i)** We define the image of $U$ in $Y$, denoted by $f(U)$, to be the subset of $Y$ given by

$$f(U) = \{y \in Y: \exists x \in U, f(x) = y\} = \{f(x): x \in U\}.$$

If $U = X$, the image of $U$ in $Y$ is also called the range of $f$.

**(ii)** We define the preimage of $V$ under $f$, denoted by $f^{-1}(V)$, to be the subset of $X$ given by

$$f^{-1}(V) = \{x \in X: f(x) \in V\}.$$

**Example 1.29.**

**1**) Define $f\colon \mathbb{Z} \longrightarrow \mathbb{N}$ by

$$f(x) = |x| + 1,$$

and let $U = \{-1,0,1,2,3\}$. We have

$$f(-1) = 2,\ f(0) = 1,\ f(1) = 2,\ f(2) = 3 \text{ and } f(3) = 4.$$

So

$$f(U) = \{1,2,3,4\}.$$

**2**) Define $g\colon \mathbb{Z} \longrightarrow \mathbb{N}$ by $g(x) = |2x|$ and let $V = \{0,1,2,3,4\}$. The preimage of $V$ is all those elements in $\mathbb{Z}$ that map to 0,1,2,3 and 4; that is, it is all possible values of $x$ for which

$$\begin{aligned}
&g(x) = |2x| = 0 \Longrightarrow x = 0,\\
&g(x) = |2x| = 1\text{: There are no such elements},\\
&g(x) = |2x| = 2 \Longrightarrow x = 1, x = -1,\\
&g(x) = |2x| = 3\text{: There are no such elements},\\
&g(x) = |2x| = 4 \Longrightarrow x = -2, x = 2.
\end{aligned}$$

Hence

$$g^{-1}(V) = \{-2,-1,0,1,2\}.$$

**Theorem 1.5.** Let $X, Y$ be sets, $A_1$ and $A_2$ be subsets of $X$, $B_1$ and $B_2$ be subsets of $Y$ and let $f\colon X \longrightarrow Y$ be a function. Then

**1**) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

**2**) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

**3**) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

**4**) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

**5**) $A_1 \subset A_2 \Longrightarrow f(A_1) \subset f(A_2)$.

**6**) $f^{-1}(Y - B_1) = X - f^{-1}(B_1)$.

## 1.4. The Order Structure of the Real Numbers

**Remark 1.13**.

**1**) As we have seen before, the set of real numbers $\mathbb{R}$ contains all of the following sets: $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$, such that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

**2**) An important property of the real numbers is that they are **totally ordered**, so we can compare any two real numbers, $x$ and $y$, and make a statement of the form $x \leq y$ or $y \leq x$, with a strict inequality if $x \neq y$.

### 1.4. 1. Upper and Lower Bounds

**Definition 1.37.** Let $A \subset \mathbb{R}$ be a set of real numbers.

**(i)** We say that $A$ is **bounded from above** if there exists a real number $M \in \mathbb{R}$, called an **upper bound** of $A$, such that $x \leq M$ for every $x \in A$. In simple words

$$\exists M \in \mathbb{R}:\ x \leq M, \forall x \in A.$$

**(ii)** We say that $A$ is **bounded from below** if there exists a real number $m \in \mathbb{R}$, called a **lower bound** of $A$, such that $x \geq m$ for every $x \in A$. In simple words

$$\exists m \in \mathbb{R}:\ x \geq m, \forall x \in A.$$

### 1.4. 2. Supremums and Infimums

**Definition 1.38.** Let $A \subset \mathbb{R}$ a set of real numbers.

**(i)** The **supremum** of a set $A$ is its **least upper bound**. We denote the supremum of $A$ by $supA$. Mathematically

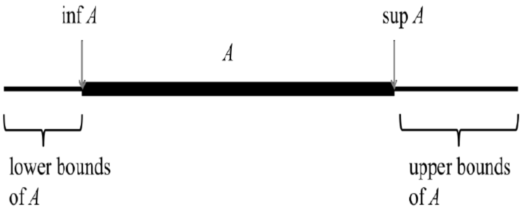
$$M = supA \Leftrightarrow \begin{cases} \forall x \in A:\ x \leq M \\ \qquad\text{and} \\ \forall \varepsilon > 0, \exists x_\varepsilon \in A:\ M - \varepsilon < x_\varepsilon \end{cases}.$$

In other words, no element of the set $A$ exceeds $M$, but if $\varepsilon$ is any positive quantity, however small, there is a element that exceeds $M - \varepsilon$.

**(ii)** The **infimum** of a set $A$ is its **greatest lawer bound**. We denote the infimum of $A$ by $infA$. Mathematically

$$m = infA \Leftrightarrow \begin{cases} \forall x \in A:\ x \geq m \\ \qquad\text{and} \\ \forall \varepsilon > 0, \exists x_\varepsilon \in A:\ m + \varepsilon > x_\varepsilon \end{cases}.$$

In other words, no element of the set is less than $m$, but if $\varepsilon$ is any positive quantity, however small, there is always one element that is less than $m + \varepsilon$.



### 1.4.3. Maximums and Minimums

**Definition 1.39.** Let $A \subset \mathbb{R}$ be a set of real numbers and $m, M \in \mathbb{R}$.

**i**) We say that $M$ is a **maximum** of $A$ if it is an element of $A$, (i.e., $M \in A$) such that $M$ is also the supremum. In simple words, if $supA \in A$, then we call it the maximum of $A$ and we denote it by $maxA$.

**ii**) We say that $m$ is a **minimum** of $A$ if it is an element of $A$, (i.e., $m \in A$) such that $m$ is also the infimum. In simple words, if $infA \in A$, then we call it the minimum of $A$ and we denote it by $minA$.

**Remark 1.14.** Note that not all sets that are bounded above have maximums and that not all sets that are bounded below have minimums.

Take a look at some examples.

**Example 1.30.** Let $A = [-2,5]$, $B = ]-2,5]$, $C = [-2,5[$ and $D = ]-2,5[$. We have

1) $infA = minA = -2$.

2) $infB = -2$. Since $-2 \notin B$, then $B$ has no minimum.

3) $supC = maxC = 5$.

4) $supD = 5$. Since $5 \notin C$, then $B$ has no maximum.

### 1.4.4. Absolute Value

**Definition 1.40.** The absolute value of a number $x$, denoted by $|x|$, is a positive number describing the distance from zero that a number $x$ is on the number line, without considering direction where

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

**Example 1.31.**

**1**) Suppose $x \geq 3$.

**a)** Rewrite $|x-1|$ without using the absolute value sign. Since $x \geq 3$, then $x - 1 > 0$. So,

$$|x-1| = x - 1.$$

**b)** Rewrite $|1-x|$ without using the absolute value sign. Since $x \geq 3$, then $1 - x < 0$. So, $|1-x| = -(1-x) = x - 1$.

**2**) Rewrite $f(x) = |3 - 2x|$ without using the absolute value sign. We have

$$3 - 2x = 0 \Leftrightarrow 3 = 2x \Leftrightarrow x = \frac{3}{2}.$$

So,

| $x$ | $-\infty$ |  | $\frac{3}{2}$ |  | $+\infty$ |
|---|---|---|---|---|---|
| $\lvert 3-2x \rvert$ |  | $3-2x$ | $0$ | -$(3-2x)$ |  |

Thus,

$$f(x) = |3 - 2x| = \begin{cases} -(3-2x), & \text{if } x \geq \frac{3}{2} \\ 3 - 2x, & \text{if } x < \frac{3}{2} \end{cases} = \begin{cases} -3 + 2x, & \text{if } x \geq \frac{3}{2} \\ 3 - 2x, & \text{if } x < \frac{3}{2} \end{cases}.$$

## 1.4.5. Intervals

**Definition 1.41.** Given any two points $x$ and $y$ on the real line, we call the set of points between $x$ and $y$ an **interval**.

**Remark 1.15.**

**1**) When discussing intervals, it is important to indicate whether we are including one or both endpoints.

**2**) We also have the concept of intervals that extend to the ends of the number line $-\infty$ and $+\infty$.

### 1.4.5.1. List of Notations for Intervals

$[a, b] = \{x \in \mathbb{R}:\ a \leq x \leq b\}$,

$]a, b[ = \{x \in \mathbb{R}:\ a < x < b\}$,

$[a, b[ = \{x \in \mathbb{R}:\ a \leq x < b\}$,

$]a, b] = \{x \in \mathbb{R}:\ a < x \leq b\}$,

$[a, +\infty[ = \{x \in \mathbb{R}:\ a \leq x\}$,

$]a, +\infty[ = \{x \in \mathbb{R}:\ a < x\}$,

$]-\infty, b] = \{x \in \mathbb{R}:\ x \leq b\}$,

$]-\infty, b[ = \{x \in \mathbb{R}:\ x < b\}$.

## 1.5. Mathematical Induction

The mathematical induction (or just **induction** for short) is a powerful and elegant tool for proving a mathematical proposition, theorem or formula which is thought to be true, for each and every natural number $n$. We see here the most frequently used kind of induction which is called simple induction or weak induction. There are other variants of induction such as double induction and strong induction which work on the same principle as simple induction, but are generally easier to prove theorems with.

### The principle of mathematical induction

The principle can be stated as follows:
A property of integers is called hereditary if, whenever any integer $n$ has the property, its successor has the property. If the integer 0 or 1 has a certain property and this property is hereditary, every positive integer has the property.

**Theorem 1.6.** Suppose that $P(n)$ is a proposition that it either true or false for any given natural numbers $n$. If

**(i)** $P(0)$ is true and,

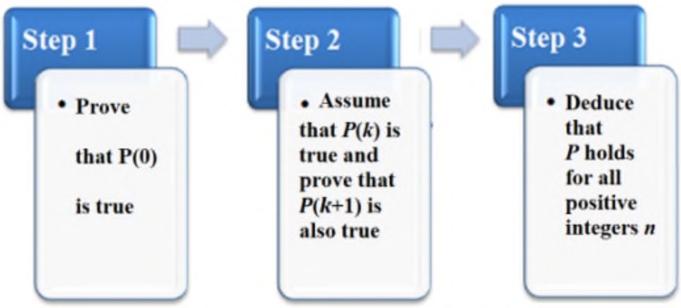
**(ii)** Given any $k \geq 0$, if $P(k)$ is true, then $P(k+1)$ is also true which means that the statement $P(k) \Rightarrow P(k+1)$ is true.

It follows by mathematical induction that $P(n)$ is true for any natural number.

In other words
Let P be a property (or conjecture) of positive integers such that:

- Basis step: $P(0)$ is true. (or $P(n)$ is true for the smallest possible value).
- Inductive step: if $P(k)$ is true, then $P(k+1)$ is also true.

Then $P(n)$ is true for all positive integers.



**Example 1.32.** Using induction, prove that

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Let $P(n)$ is the statement

$$1+2+\cdots+n=\frac{1}{2}n(n+1).$$

**Basis step:** Observe that if $n=1$, this statement is $1=\frac{1}{2}(1)(1+1)$, which is obviously true. So $P(1)$ is true.

**Inductive step:** We must now prove $P(k)\Rightarrow P(k+1)$ for any $k\geq 1$.

That is, we must show that $P(k)$ is true, i.e.,

$$1+2+\cdots+k=\frac{1}{2}k(k+1), \quad \ldots(*)$$

then $P(k+1)$ is also true.

By virtue of $(*)$, one has

$$1+2+\cdots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)=\frac{1}{2}\big(k(k+1)+2(k+1)\big)$$
$$=\frac{1}{2}(k+1)(k+2),$$

and so we have shown that if $P(k)$ is true then $P(k+1)$ is also true.
It follows by mathematical induction that $P(n)$ is true for all $n\geq 1$.

# Chapter 2

## Real Functions of one Real Variable

This chapter further explores the concept of a function of one real variable where the terms "function", "map", "mapping" and "transformation" mean the same thing; the choice of which word to use is usually determined by tradition and the mathematical background of the person using the term.

### 2.1. Basic Notions

### 2.1.1. Real-ValueFunction, Domain, and Range

**Definition 2.1.** A real-valued function $f: M \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a **function whose values are real numbers**. The set $M$ is the domain of $f$, denoted by $Domf$ or $D_f$, while the set

$$\{f(x) = y : x \in D_f\}$$

is called the range of $f$ and is denoted by $Rang\ f$.

**Remark 2.1.**

**1**) The set of images is inside $\mathbb{R}$, i.e., $Rang\ f \subseteq \mathbb{R}$ and the domain $D_f$ is also a subset of $\mathbb{R}$, i.e., $D_f \subseteq \mathbb{R}$.

**2**) When a function is given in the form $y = f(x)$, we do not say explicitly what its domain is; rather, we find out a largest subset of $\mathbb{R}$ which may serve as its domain. For example,

Polynomial function: $D_f = \mathbb{R}$ all real numbers.

Rational function: $D_f$ is all real numbers except those that cause the denominator to equal zero.

Absolute value function: $D_f = \mathbb{R}$ all real numbers.

Square root function: If $f(x) = \sqrt{x}$ , then $D_f$ is the values of $x$ for which the radicand (the value under the radical sign) is not negative, i.e., $x \geq 0$.

Radical function: A radical expression is an expression that includes a radical symbol and the domain of a radical function is the values of $x$ for which the radicand (the value under the radical sign) is not negative. In other words, if $f(x) = \sqrt{g(x)}$, then $D_f$ is all values of $x$ value such that $g(x) \geq 0$.

Root function: If $f(x) = \sqrt[n]{x}$, then $D_f$ is all real numbers if $n$ is odd or all nonnegative real numbers if $n$ is even.

Exponential function: Domain of an exponential function is $\mathbb{R}$.

Logarithmic function: Domain of logarithmic function $\ln x$ is any $x$ value for which $x > 0$. Moreover, $\ln f(x)$ is defined only when the input is positive ($f(x) > 0$).

Sine and cosine functions: The domain of the sine and cosine functions is all real numbers.

**Example 2.1.**

**1**) Find the domain of each of the following functions given by

**a**) $f_1(x) = \dfrac{1}{x-2}$, **b**) $f_2(x) = \sqrt{25 - x^2}$, **c**) $f_3(x) = \dfrac{3}{3 - \sqrt{x}}$,

**d**) $f_4(x) = \ln(x-1)$, **e**) $f_5(x) = \dfrac{1}{\sqrt{1 - \cos x}}$.

We have

**a**) $D_{f_1} = \{x \in \mathbb{R}: x - 2 \neq 0\}$. Since $x - 2 = 0 \Leftrightarrow x = 2$, then

$$D_{f_1} = \mathbb{R} - \{2\}.$$

**b**) $D_{f_2} = \{x \in \mathbb{R}: 25 - x^2 \geq 0\}$. Since $25 - x^2 \geq 0 \Leftrightarrow (5-x)(5+x) \geq 0$, then

$$D_{f_2} = [-5,5].$$

**c**) $D_{f_3} = \left\{x \in \mathbb{R}: 3 - \sqrt{x} \neq 0 \text{ and } x \geq 0\right\}$. Since $3 - \sqrt{x} = 0 \Leftrightarrow x = 9$, then

$$D_{f_3} = [0,9[ \cup ]9, +\infty[.$$

**d**) $D_{f_4} = \{x \in \mathbb{R}: x - 1 > 0\}$. Since $x - 1 > 0 \Leftrightarrow x > 1$, then

$$D_{f_4} = ]1, +\infty[.$$

**e**) $D_{f_5} = \left\{x \in \mathbb{R}: \sqrt{1 - \cos x} \neq 0 \text{ and } 1 - \cos x \geq 0\right\}$. Since $-1 \leq \cos x \leq 1$, then $1 - \cos x \geq 0$. So,

$$D_{f_5} = \left\{x \in \mathbb{R}: \sqrt{1 - \cos x} \neq 0\right\}.$$

We have

$$\sqrt{1-\cos x} = 0 \Leftrightarrow 1-\cos x = 0 \Leftrightarrow \cos x = 1 \Leftrightarrow x = 2\pi k, k \in \mathbb{Z}.$$

Thus

$$D_{f_5} = \mathbb{R} - \{2\pi k, k \in \mathbb{Z}\}.$$

**2**) Find the range of each of the following functions given by

$$f_1(x) = \sin x\,, \qquad f_2(x) = \cos x\,, \qquad f_3(x) = \tan x = \frac{\sin x}{\cos x}.$$

We have

$$Rang\ f_1 = Rang\ f_2 = [-1,1].$$

The domain of the function $f_3$ is all real numbers except those where $\cos x \neq 0$.Since

$$\cos x = 0 \Leftrightarrow x = \tfrac{\pi}{2} + k\pi, k \in \mathbb{Z},$$

Then,

$$D_{f_3} = \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}.$$

The range of $f_3$ is, obviously, all real numbers from $\mathbb{R}$, i.e., $Rang\ f_3 = \mathbb{R}$.

### 2.1.2. Monotone Functions

**Definition 2.2.** Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable and let $M \subseteq D_f$.

**1**) $f$ is said to be **increasing** (also monotonically increasing or non-decreasing) on $M$, if

$$\forall x_1, x_2 \in M: x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2).$$

If $f(x_1) < f(x_2)$ the function is said to be **strictly increasing**.

**2**) $f$ is said to be **decreasing** (also monotonically decreasing or non-increasing) on $M$, if

$$\forall x_1, x_2 \in M: x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

If $f(x_1) > f(x_2)$ the function is said to be **strictly decreasing.**

**3**) $f$ is said to be **monotone** (or **monotonic**) on $M$, if $f$ is increasing or decreasing on $M$.

**4**) $f$ is said to be **strictly monotone** on $M$, if $f$ is strictly increasing or strictly decreasing on $M$.

**5**) $f$ is said to be constant on $M$, if

$$\forall x_1, x_2 \in M: f(x_1) = f(x_2).$$

**Remark 2.2.**

Functions that are strictly monotone are one-to-one. This is because, in such a situation, for $x_1 \neq x_2$, either $x_1 < x_2$ or $x_1 > x_2$, either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$ which means that $f(x_1) \neq f(x_2)$.

**Example 2.2.**

**1**) Let $f_1$ be the function defined by

$$f_1(x) = e^x.$$

One has $D_{f_1} = \mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$ and assume that $x_1 < x_2$. We have

$$e^{x_2} = e^{x_2 - x_1 + x_1} = e^{x_1} e^{x_2 - x_1}.$$

But

$$x_1 < x_2 \Longrightarrow x_2 - x_1 > 0 \Longrightarrow e^{x_2 - x_1} > 1.$$

So,

$$f_1(x_1) = e^{x_1} < f_1(x_2) = e^{x_2}.$$

Thus $f_1$ is strictly increasing on $\mathbb{R}$.

**2**) Let $f_2$ be the function that is defined by

$$f_2(x) = x^2.$$

On has $D_{f_2} = \mathbb{R}$. If $x_1 = -3$ and $x_2 = 2$, then $x_1 < x_2$ and

$$f_2(x_1) = (-3)^2 = 6 > f_2(x_2) = (2)^2 = 4.$$

So, $f_2$ is nonincreasing on $\mathbb{R}$.

### 2.1.3. Bounded Functions

**Definition 2.3.** Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function of a real variable and let $M \subseteq D_f$.

**1**) We say that function $f$ is **bounded from above** on $M$ if

$$\exists K \in \mathbb{R}: f(x) \leq K.$$

**2**) We say that function $f$ is **bounded from below** on $M$ if

$$\exists k \in \mathbb{R}: f(x) \geq k.$$

**3**) The function $f$ is said to be **bounded** if it is bounded from below and above.

**4**) Let $a \in M$, $f$ has a **minimum** on $M$ at the point $x = a$ if

$$f(x) \geq f(a), \forall x \in M.$$

The value of the minimum is $f(a)$.

**5**) Let $b \in M$, $f$ has a **maximum** on $M$ at the point $x = b$ if

$$f(x) \leq f(b), \forall x \in M.$$

The value of the maximum is $f(b)$.

**Remark 2.3.** If $f$ has a maximum (minimum) on $M$, then $f$ is bounded from above (below) on $M$. If $f$ is not bounded from above (below) on $M$ then $f$ has no maximum (minimum) on $M$.

### 2.1.4. Even, and Odd Functions

**Definition 2.4.** A domain $D_f$ of a function $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be **symmetric** with respect to the origin if

$$x \in D_f \Leftrightarrow -x \in D_f.$$

**Definition 2.5.** Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable defined on a symmetric domain $D_f$.

**1**) We say that function $f$ is **even** if

$$\forall x \in D_f: f(-x) = f(x).$$

**2**) We say that function $f$ is **odd** if

$$\forall x \in D_f: f(-x) = -f(x).$$

**Remark 2.4.**

**1**) The graph of an even function is symmetric with respect to the $y$-axis.

**2**) The graph of an odd function is symmetric with respect to the origin.

**Example 2.3.**

**1**) $f_1(x) = x^2 + 5$. We have $D_{f_1} = \mathbb{R}$ which is a symmetric domain and

$$f_1(-x) = (-x)^2 + 5 = x^2 + 5 = f_1(x).$$

So, $f_1$ is an even function.

**2)** $f_2(x) = \cos x$. We have $D_{f_2} = \mathbb{R}$ which is a symmetric domain and

$$f_2(-x) = \cos(-x) = \cos x = f_2(x).$$

So, $f_2$ is an even function.

**3)** $f_3(x) = \frac{1}{x}$. We have $f_3 = \mathbb{R} - \{0\}$ which is a symmetric domain and

$$f_3(-x) = -\frac{1}{x} = -f_3(x).$$

So, $f_3$ is an odd function.

**4)** $f_4(x) = \sin x$. We have $D_{f_4} = \mathbb{R}$ which is a symmetric domain and

$$f_4(-x) = \sin(-x) = -\sin x = -f_4(x).$$

So, $f_4$ is an even function.

**5)** $f_5(x) = x^3 + 1$. We have $D_{f_5} = \mathbb{R}$ which is a symmetric domain and

$$f_5(-x) = (-x)^3 + 1 = -x^3 + 5 \neq f_5(x) \neq -f_5(x) = -x^3 - 1.$$

So, $f_5$ is neither odd nor even.

### 2.1.5. Periodic Functions

**Definition 2.6.** Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable. The function $f$ is called **periodic** if there exists $T > 0$ such that

**(i)** $x \in D_f \Rightarrow x + T \in D_f$.

**(ii)** $\forall x \in D_f: f(x + T) = f(x)$.

The number $T$ is called a **period** of $f$. The smallest amongst all periods is called the **fundamental (primitive) period**.

**Example 2.4.**

The functions $\cos \omega x$ and $\sin \omega x$ both have periods equal to $T = \frac{2\pi}{\omega}$. Indeed, we have

$$\cos \omega(x + T) = \cos \omega \left(x + \frac{2\pi}{\omega}\right) = \cos(\omega x + 2\pi) = \cos \omega x,$$

and

$$\sin \omega(x + T) = \sin \omega \left(x + \frac{2\pi}{\omega}\right) = \sin(\omega x + 2\pi) = \sin \omega x.$$

### 2.1.6. Arithmetic Operations on Functions

**Definition 2.7.** Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ and $g: D_g \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be real functions of one real variable. If $D_f \cap D_g \neq \Phi$, then $f+g$, $f-g$, and $f.g$ are defined on $D_f \cap D_g$ by

**1)** $(f+g)(x) = f(x) + g(x)$.

**2)** $(f-g)(x) = f(x) - g(x)$.

**3)** $(f.g)(x) = f(x).g(x)$.

4) The quotient $\frac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

for $x$ in $D_f \cap D_g$ such that $g(x) \neq 0$.

**Example 2.5.** Let

$$f(x) = \sqrt{4-x^2} \text{ and } g(x) = \sqrt{x-1}.$$

We have

$$D_f = \{x \in \mathbb{R}: 4 - x^2 \geq 0\} = [-2,2] \text{ and } D_g = \{x \in \mathbb{R}: x - 1 \geq 0\} = [1,+\infty[.$$

So,

$$D_f \cap D_g = [1,2] \neq \Phi.$$

Consequently, $f+g$, $f-g$, and $f.g$ are defined on $[1,2]$ by

$$(f+g)(x) = \sqrt{4-x^2} + \sqrt{x-1}, \qquad (f-g)(x) = \sqrt{4-x^2} - \sqrt{x-1},$$

and

$$(f.g)(x) = \left(\sqrt{4-x^2}\right)\left(\sqrt{x-1}\right) = \sqrt{(4-x^2)(x-1)}.$$

The quotient $\frac{f}{g}$ is defined on $]1,2]$ by

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{4-x^2}}{\sqrt{x-1}}.$$

## 2.2. Limits of Functions

In what follows, we will introduce the concept of limits which can be considered as one of the cornerstones of modern calculus. For instance, they are used to find out the

particle state at a particular position, and they can be used to calculate the time a chemical reaction takes to complete. Various physics concept s also use calculus limits as the base approach to solve complex equations. Even in the study of quantum control problems we can find many neat limits.
What is a limit and how a function behaves when the independent variable $x$ approaches a certain value?

### 2.2.1. Limit of a Function at a Point

**Definition 2.8.** A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing $x$.

**Definition 2.9.** Let $f$ be a real function of a real variable, defined on a neighbourhood of a point $x_0$, but not necessarily at $x_0$ (except perhaps at $x_0$, itself) and let $\ell$ be a real number. We say that $\ell$ is the **limit** of $f$ as $x$ approaches (goes to) $x_0$, and we write $\lim_{x \to x_0} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0: \left(x \in D_f \text{ and } 0 < |x - x_0| < \delta(\varepsilon)\right) \Longrightarrow (|f(x) - \ell| < \varepsilon).$$

The limit is sometimes denoted by a right arrow as in

$$f(x) \to \ell \text{ as } x \to x_0,$$

is read as $f$ of $x$ tends to $\ell$ as $x$ tends to $x$ sub zero (also $x$ subscript zero or $x$ – naught).

**Remark 2.5.** If the domain of $f$ is $\mathbb{R}$, we can omit $x \in D_f$ on the above definition.

**Example 2.6.**

**1**) Let $f(x) = 2x + 2$. We will prove that

$$\lim_{x \to 1} f(x) = 4.$$

Given $\varepsilon > 0$, we must find $\delta(\varepsilon) > 0$ such that

$$|f(x) - \ell| < \varepsilon \text{ if } |x - x_0| < \delta(\varepsilon).$$

A useful general rule is to write down $f(x) - \ell$ and then to express it in terms of $x - x_0$, as much as possible generally, for example (by writing $x = x - x_0 + x_0$). To prove this, we write

$$|f(x) - \ell| < \varepsilon \Leftrightarrow |(2x + 2) - 4| = 2|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \frac{\varepsilon}{2}.$$

This yields

$$|f(x) - 4| < \varepsilon \text{ if } |x - 1| < \delta(\varepsilon),$$

where $\delta(\varepsilon)$ is any number such that $\delta(\varepsilon) \leq \frac{\varepsilon}{2}$.

**2**) Let $g(x) = x \sin\frac{1}{x}$. We will prove that

$$\lim_{x\to 0} g(x) = 0.$$

We have $D_g = \mathbb{R} - \{0\}$. Even though $g$ is not defined at $x_0 = 0$, we have

$$|g(x) - \ell| = \left|x \sin\frac{1}{x} - 0\right| = \left|x \sin\frac{1}{x}\right| \leq |x|.$$

This yields

$$|g(x) - 0| < \varepsilon \text{ if } |x| < \delta(\varepsilon),$$

where $\delta(\varepsilon)$ is any number such that $\delta(\varepsilon) = \varepsilon$.

**Theorem 2.1.** If $\lim\limits_{x\to x_0} f(x)$ exists then it is unique.

#### 2.2.1.1. Basic Properties of Limits

**Theorem 2.2.** If

$$\lim_{x\to x_0} f(x) = \ell_1 \text{ and } \lim_{x\to x_0} g(x) = \ell_2,$$

then

**1**) $\lim\limits_{x\to x_0} (f + g)(x) = \ell_1 + \ell_2$.

**2**) $\lim\limits_{x\to x_0} (f - g)(x) = \ell_1 - \ell_2$.

**3**) $\lim\limits_{x\to x_0} (f.g)(x) = \ell_1.\ell_2$.

**4**) If $\ell_2 \neq 0$, and $g(x) \neq 0$, then

$$\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{\ell_1}{\ell_2}.$$

**Example 2.7.**

**1**) By using the sum and difference rules we obtain

$$\begin{aligned}\lim_{x\to 1}(x^3 + 4x^2 - 3) &= \lim_{x\to 1}(x^3) + \lim_{x\to 1}(4x^2) - \lim_{x\to 1}(3) \\ &= 1 + 4 - 3 = 2.\end{aligned}$$

**2**) By using the quotient rule, we obtain

$$\lim_{x\to 1} \frac{x^3 + 4x^2 - 3}{2 + 4x^2} = \frac{\lim\limits_{x\to 1}(x^3 + 4x^2 - 3)}{\lim\limits_{x\to 1}(2 + 4x^2)} = \frac{2}{6} = \frac{1}{3}.$$

**Theorem 2.3 (The sandwich or squeeze theorem).** Let $f, g$ and $h$ be functions such that

$$f(x) \leq g(x) \leq h(x),$$

for all numbers $x$ in some open interval containing $x_0$, except possibly at $x_0$ itself. If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell,$$

then

$$\lim_{x \to x_0} g(x) = \ell.$$

**Example 2.8.** We will show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Using the inequality

$$\sin x < x < \tan x, \text{ for } 0 < x < \frac{\pi}{2},$$

we get

$$\sin x < x < \tan x \Leftrightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \Leftrightarrow \cos x < \frac{\sin x}{x} < 1.$$

By virtue of the sandwich theorem and $\lim_{x \to 0^+} \cos x = 1$ we get

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

In the case when $x < 0$, it suffices to take $x = -t$ for $t > 0$. Then

$$\sin x = \sin(-t) = -\sin t.$$

So,

$$\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{x \to 0^-} \frac{-\sin t}{-t} = 1.$$

**Theorem 2.4.** Let $f, g$ be functions such that

$$f(x) \leq g(x),$$

for all numbers $x$ in some open interval containing $x_0$, except possibly at $x_0$ itself. Then

$$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x).$$

### 2.2.1.2. One-sided Limits ( Left- and Right-hand Limits)

**Definition 2.10.**

**1)** Let $f$ be a real function of a real variable, defined for all $x$ in an interval of the form $]x_0, b[$, and let $\ell$ be a real number. We say that $\ell$ is the limit of $f$ as $x$ approaches $x_0$ from the **right**, and we write $\lim\limits_{x\to x_0^+} f(x) = \ell$ if

$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0: \big(x_0 < x < x_0 + \delta(\varepsilon)\big) \Longrightarrow (|f(x) - \ell| < \varepsilon).$$

**2)** Let $f$ be a real function of a real variable, defined for all $x$ in an interval of the form $]a, x_0[$, and let $\ell$ be a real number. We say that $\ell$ is the limit of $f$ as $x$ approaches $x_0$ from the **left**, and we write $\lim\limits_{x\to x_0^-} f(x) = \ell$ if

$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0: (x_0 - \delta(\varepsilon) < x < x_0) \Longrightarrow (|f(x) - \ell| < \varepsilon).$$

**Example 2.9.**

**1**) Let

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x < 1 \\ 2x + 1, & x \geq 1 \end{cases}.$$

We will prove that

$$\lim_{x\to 1^+} f(x) = 3.$$

We want to show that for any $\varepsilon > 0$, we can find a $\delta(\varepsilon) > 0$ so that if

$$1 < x < 1 + \delta(\varepsilon),$$

then

$$|f(x) - 3| < \varepsilon.$$

If $x \geq 1$, then

$$f(x) = 2x + 1 \Longrightarrow f(x) - 3 = 2x - 2.$$

So, we have

$$|f(x) - 3| < \varepsilon \Leftrightarrow |2x - 2| < \varepsilon \Leftrightarrow 2x - 2 < \varepsilon \Leftrightarrow 2x < \varepsilon + 2 \Leftrightarrow x < 1 + \frac{\varepsilon}{2},$$

Therefore, we need to satisfy $1 < x < 1 + \frac{\varepsilon}{2}$, and it is clear that if we choose $\delta(\varepsilon) = \frac{\varepsilon}{2}$, then we will satisfy the right-hand limit condition.

**2**) Let

$$g(x) = \frac{x + |x|(1 + x)}{x} \sin\frac{1}{x}.$$

We will prove that

$$\lim_{x \to 0^-} g(x) = 0.$$

If $x < 0$, then

$$g(x) = \frac{x + |x|(1 + x)}{x} \sin\frac{1}{x} = \frac{x - x(1 + x)}{x} \sin\frac{1}{x} = -x \sin\frac{1}{x}.$$

We want to show that for any $\varepsilon > 0$, we can find a $\delta(\varepsilon) > 0$ such that if $0 - \delta(\varepsilon) < x < 0$, then $|g(x) - 0| < \varepsilon$. We have

$$|g(x) - 0| = \left|-x \sin\tfrac{1}{x}\right| \leq |x|.$$

Thus we need to satisfy $0 - \delta(\varepsilon) < x < 0$. Likewise, if we choose $\delta(\varepsilon) = \varepsilon$, then we will satisfy the left-hand limit condition.

**Theorem 2.5.** A function $f$ has a limit as $x$ approaches $x_0$ if and only if it has both a left- and a right-hand limit as $x$ approaches $x_0$ and these one-sided limits both equal the same value. More specifically,

$$\left(\lim_{x \to x_0} f(x) = \ell\right) \Leftrightarrow \left(\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = \ell\right).$$

### 2.2.2. Finite Limits as the variable tends to infinity

**Definition 2.11.**

**1)** Let $f$ be a real function whose domain contains an interval of the form $]a, +\infty[$. We say that $\ell$ is the limit of $f$ as $x$ approaches $+\infty$, and we write $\lim_{x \to +\infty} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists A > 0: (x > A) \Longrightarrow (|f(x) - \ell| < \varepsilon).$$

**2)** Let $f$ be a real function whose domain contains an interval of the form $]-\infty, a[$. We say that $\ell$ is the limit of $f$ as $x$ approaches $-\infty$, and write we $\lim_{x \to -\infty} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists A > 0: (x < -A) \Longrightarrow (|f(x) - \ell| < \varepsilon).$$

**Remark 2.6.** When

$$\lim_{x \to +\infty} f(x) = \ell \text{ or } \lim_{x \to -\infty} f(x) = \ell,$$

the line $y = \ell$ is called a **horizontal asymptote** of the graph $y = f(x)$.

**Example 2.10.** We will show that

$$\lim_{x\to+\infty}\frac{1}{x}=\lim_{x\to-\infty}\frac{1}{x}=0.$$

Let $\varepsilon>0$. We have

$$\left|\frac{1}{x}-0\right|<\varepsilon\Leftrightarrow\left|\frac{1}{x}\right|<\varepsilon\Leftrightarrow\frac{1}{|x|}<\varepsilon\Leftrightarrow|x|>\frac{1}{\varepsilon}.$$

Choose $A=\frac{1}{\varepsilon}$.

If $x>A$, then $x>0$ and $|x|>\frac{1}{\varepsilon}\Leftrightarrow x>\frac{1}{\varepsilon}\Leftrightarrow\frac{1}{x}<\varepsilon$. Hence, $\lim_{x\to+\infty}\frac{1}{x}=0$.

On the other hand, if $x<-A$, then $x<0$ and $|x|>\frac{1}{\varepsilon}\Leftrightarrow -x>\frac{1}{\varepsilon}\Leftrightarrow-\frac{1}{x}<\varepsilon$. Hence,

$$\lim_{x\to-\infty}\frac{1}{x}=0.$$

### 2.2.3. Infinite Limits

**Definition 2.12.**

(**a**) Let $f$ be a real function of a real variable, defined for all $x$ in an interval of the form $]x_0,b[$.

**1**) We say that $f(x)$ approaches $+\infty$ as $x$ approaches $x_0$ from the right, and we write $\lim_{x\to x_0^+}f(x)=+\infty$ if

$$\forall A>0,\exists\delta>0:(x_0<x<x_0+\delta)\Longrightarrow(f(x)>A).$$

**2**) We say that $f(x)$ approaches $-\infty$ as $x$ approaches $x_0$ from the right, and we write $\lim_{x\to x_0^+}f(x)=-\infty$ if

$$\forall A>0,\exists\delta>0:(x_0<x<x_0+\delta)\Longrightarrow(f(x)<-A).$$

(**b**) Let $f$ be a real function of a real variable, defined for all $x$ in an interval of the form $]a,x_0[$.

**1**) We say that $f(x)$ approaches $+\infty$ as $x$ approaches $x_0$ from the left, and we write $\lim_{x\to x_0^-}f(x)=+\infty$ if

$$\forall A>0,\exists\delta>0:(x_0-\delta<x<x_0)\Longrightarrow(f(x)>A).$$

**2)** We say that $f(x)$ approaches $-\infty$ as $x$ approaches $x_0$ from the left, and we write $\lim_{x\to x_0^-}f(x)=-\infty$ if

$$\forall A>0,\exists\delta>0:(x_0-\delta<x<x_0)\Longrightarrow(f(x)<-A).$$

(**c**) Let $f$ be a real function whose domain contains an interval of the form $]a, +\infty[$.

**1**) We say that $f(x)$ approaches $+\infty$ as $x$ approaches $+\infty$, and we write $\lim_{x \to +\infty} f(x) = +\infty$ if

$$\forall A > 0, \exists B > 0 : (x > B) \Longrightarrow (f(x) > A).$$

**2)** We say that $f(x)$ approaches $-\infty$ as $x$ approaches $+\infty$, and we write $\lim_{x \to +\infty} f(x) = -\infty$ if

$$\forall A > 0, \exists B > 0 : (x > B) \Longrightarrow (f(x) < -A).$$

(**d**) Let $f$ be a real function whose domain contains an interval of the form $]-\infty, a[$.

**1**) We say that $f(x)$ approaches $+\infty$ as $x$ approaches $-\infty$ and and we write $\lim_{x \to -\infty} f(x) = +\infty$ if

$$\forall \varepsilon > 0, \exists A > 0 : (x < -B) \Longrightarrow (f(x) > A).$$

**2)** We say that $f(x)$ approaches $-\infty$ as $x$ approaches $-\infty$ and we write $\lim_{x \to -\infty} f(x) = -\infty$ if

$$\forall \varepsilon > 0, \exists A > 0 : (x < -B) \Longrightarrow (f(x) < -A).$$

**Remark 2.7.**

**1**) We have

$$\left(\lim_{x \to x_0} f(x) = +\infty\right) \Leftrightarrow \left(\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = +\infty\right).$$

So,

$$\left(\lim_{x \to x_0} f(x) = +\infty\right) \Leftrightarrow \big(\forall A > 0, \exists \delta > 0 : (0 < |x - x_0| < \delta) \Longrightarrow (f(x) > A)\big).$$

**2**) We have

$$\left(\lim_{x \to x_0} f(x) = -\infty\right) \Leftrightarrow \left(\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = -\infty\right).$$

So,

$$\left(\lim_{x \to x_0} f(x) = -\infty\right) \Leftrightarrow \big(\forall A > 0, \exists \delta > 0 : (0 < |x - x_0| < \delta) \Longrightarrow (f(x) < -A)\big).$$

**Theorem 2.6 (End behaviour and asymptotes of rational functions).** Suppose that

$$f(x) = \frac{p(x)}{q(x)},$$

is a rational function, where

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{m-1}x^{m-1} + a_mx^m,$$

and

$$q(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_{n-1}x^{n-1} + b_nx^n,$$

with $a_m \neq 0$ and $b_n \neq 0$.

**1**) If the degree of the numerator is less than the degree of the denominator, i.e, $m < n$, then

$$\lim_{x\to\mp\infty} f(x) = 0,$$

and $y = 0$ is a horizontal asymptote of $f$.

**2** If the degree of the numerator equals the degree of the denominator. i.e, $m = n$, then

$$\lim_{x\to\mp\infty} f(x) = \frac{a_m}{a_n},$$

and $y = \frac{a_m}{a_n}$ is a horizontal asymptote of $f$.

**3**) If the degree of the numerator is greater than the degree of the denominator, i.e, $m > n$, then

$$\lim_{x\to\mp\infty} f(x) = \mp\infty,$$

and $f$ has no horizontal asymptote.

**Example 2.11.**

$$\lim_{x\to\mp\infty} \frac{2x^3 + x^2 - 1}{-x^4 + 2x^3 + 2} = 0.$$

$$\lim_{x\to\mp\infty} \frac{2x^3 + x^2 - 1}{-x^3 + 2x^3 + 2} = -2.$$

$$\lim_{x\to\mp\infty} \frac{2x^4 + x^2 - 1}{-x^3 + 2x^3 + 2} = \mp\infty.$$

### 2.2.4. Indeterminate Forms

The indeterminate form is a mathematical expression that means that we cannot be able to determine the original value even after the substitution of the limits. The possible indeterminate forms are given below:

$$\textbf{1)}\ \frac{0}{0},\quad \textbf{2)}\ \frac{\infty}{\infty},\quad \textbf{3)}\ \infty-\infty,\quad \textbf{4)}\ 0.\infty$$

**Remark 2.8.** The forms $0^0$, $\infty^0$, and $1^\infty$ can be converted to one of the previous forms.

### 2.2.5. Some Sstandard Limits

There are many fascinating mathematical limits that simplify calculations and are important to learn. Here are some basic limit formulas tabulated below.

**1)** $\lim\limits_{x\to 0}\frac{\sin x}{x}=1.$

**2)** $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=0,$

**3)** $\lim\limits_{x\to\mp\infty}\left(1+\frac{a}{x}\right)^x=e^a, a\in\mathbb{R},$

**4)** $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}, a\in\mathbb{R},$

**5)** $\lim\limits_{x\to 0}(1+x)^{\frac{a}{x}}=e^a, a\in\mathbb{R},$

**6)** $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1,$

**7)** $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1,$

**8)** $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2},$

**9)** $\forall\alpha>0\colon \lim\limits_{x\to+\infty}\frac{\ln x}{x^\alpha}=0,$

**10)** $\forall\alpha>0\colon \lim\limits_{x\to 0}x^\alpha\ln x=0.$

## 2.3. Continuous Functions

### 2.3.1. Continuity

#### 2.3.1.1. Continuity at a Point

**Definition 2.13.** Let $f\colon D_f\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a real function of a real variable and let $x_0\in D_f$. We say that $f$ is **continuous** at $x_0$ if

$$\lim_{x\to x_0} f(x)=f(x_0),$$

or, more precisely,

$$\forall\varepsilon>0,\exists\delta(\varepsilon)>0\colon \left(x\in D_f \text{ and } |x-x_0|<\delta(\varepsilon)\right)\Longrightarrow\left(|f(x)-f(x_0)|<\varepsilon\right).$$

**Remark 2.9.** If $f$ is continuous at $x_0$, then $f$ is defined at $x_0$.

**Definition 2.14.**

**1**) Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable, defined on a domain $D_f$ of the form $[x_0, b]$ or $[x_0, b[$ or $[x_0, +\infty[$. We say that $f$ is continuous at **the left end-point** $x_0$ if

$$\lim_{x \to x_0^+} f(x) = f(x_0),$$

or, more precisely,

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \big(0 \leq x - x_0 < \delta(\varepsilon)\big) \Longrightarrow (|f(x) - f(x_0)| < \varepsilon).$$

**2**) Let $f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable, defined on a domain $D_f$ of the form $[a, x_0]$ or $]a, x_0]$ or $]-\infty, x_0]$. We say that $f$ is continuous at the **right** end-point $x_0$ if

$$\lim_{x \to x_0^-} f(x) = f(x_0),$$

or, more precisely,

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \big(0 \leq x_0 - x < \delta(\varepsilon)\big) \Longrightarrow (|f(x) - f(x_0)| < \varepsilon).$$

**Theorem 2.7.**

$$(f \text{ is continuous at } x_0) \Leftrightarrow \left(\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)\right).$$

**Example 2.12.**

**1**) $f_1(x) = \frac{1}{x-2}$. Since $D_{f_1} = \mathbb{R} - \{2\}$, then $f_1(2)$ is not defined ($x_0 = 2 \notin D_f$). Thus $f_1$ is discontinuous at $x_0 = 2$.

**2**) Let

$$f_2(x) = \begin{cases} 2x + 1 \text{ if } 0 \leq x < 2 \\ 7 - x \quad \text{if } 2 \leq x < 4 \\ x \qquad\quad \text{if } 4 \leq x \leq 6 \end{cases}.$$

At $x_0 = 2$, $f_2(2) = 5$ and the left and right limits are equal:

$$\lim_{x \to 2^-} f_2(x) = \lim_{x \to 2^-} (2x + 1) = 5 \text{ and } \lim_{x \to 2^+} f_2(x) = \lim_{x \to 2^-} (7 - x) = 5$$

and their common limit matches the value of the function at $x_0 = 2$:

$$\lim_{x \to 2} f_2(x) = 5 = f_2(2).$$

So, $f_2$ is continuous at $x_0 = 2$.

At $x_0 = 4$, $f_2(4) = 4$, but the left and right limits are not equal:

$$\lim_{x\to 4^-} f_2(x) = \lim_{x\to 4^-} (7 - x) = 3.$$

and

$$\lim_{x\to 4^+} f_2(x) = \lim_{x\to 4^+} (x) = 4.$$

So $f_2$ is not continuous at $x_0 = 4$. (We can, however, say that $f_2$ is continuous from the right at $x_0 = 4$).

### 2.3.1.2. Continuity on an Interval

**Definition 2.15.** A function $f$ is continuous over an interval $I$ if it is continuous at every point in the interval. More precisely,

**1**) $f$ is continuous on the open interval $]a, b[$ if $f$ is continuous at every point in $]a, b[$.

**2**) $f$ is continuous on the closed interval $[a, b]$ if $f$ is continuous on $]a, b[$, continuous from the right at $a$, and continuous form the left at $b$.

**Example 2.13.**

**1**) Every polynomial function is continuous on $\mathbb{R}$.

**2**) Every rational function is continuous on its domain.

**3**) The sine ($\sin x$) and cosine ($\cos x$) functions are continuous on $\mathbb{R}$.

**4**) The tangent ($\tan x$) and cotangent ($\cot x$) are continuous at all points in their respective domains.

**5**) Let $f$ be defined on $[0,2]$ by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ x + 1 & \text{if } 1 \leq x \leq 2 \end{cases},$$

We have $f(0) = 0$, $f(1) = 2$, $f(2) = 3$,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2) = 0 = f(0),$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2) = 1 \neq f(1) = 2,$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x + 1) = 2 = f(1) = 2,$$

and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x + 1) = f(2) = 3.$$

So, $f$ is continuous from the right at $0$ and $1$ and continuous from the left at $2$, but not at $1$. Therefore, $f$ is continuous on the interval $[0,1[\cup]1,2]$.

### 2.3.2. Properties of Continuous Functions

**Theorem 2.8.** Let $f(x)$ and $g(x)$ be functions continuous at $x_0$ and let $k \in \mathbb{R}$. Then

**1**) $f+g$, $f-g$, $k.g$, and $f.g$ are continuous at $x_0$.

**2**) $\frac{f}{g}$ is continuous at $x_0$ provided that $g(x_0) \neq 0$.

**3**) If $g$ is continuous at $x_0$, $g(x_0) \in D_f$ and $f$ is continuous at $g(x_0)$. Then $(f \circ g)$ is continuous at $x_0$.

**Example 2.14.** Let

$$f(x) = \left|\frac{(2x+1)\cos x}{x^4+3x^2+2}\right|.$$

We have

$$f_1(x) = 2x+1,$$

is continuous on $\mathbb{R}$ as a polynomial function and

$$f_2(x) = \cos x,$$

is continuous on $\mathbb{R}$.

Thus

$$f_3(x) = (2x+1)\cos x = f_1(x).f_2(x),$$

is continuous on $\mathbb{R}$.

Furthermore, $f_4(x) = x^4+3x^2+2$, is continuous on $\mathbb{R}$ as a polynomial function that cannot equal to $0$. So,

$$f_5(x) = \frac{(2x+1)\cos x}{x^4+3x^2+2} = \frac{f_3(x)}{f_4(x)},$$

is continuous on $\mathbb{R}$.

Since $f_6(x) = |x|$, is continuous on $\mathbb{R}$. Then,

$$f(x) = \left|\frac{(2x+1)\cos x}{x^4+3x^2+2}\right| = (f_6 \circ f_5)(x),$$

is continuous on $\mathbb{R}$.

### 2.3.3. Discontinuity

**Definition 2.16.** Let $f : D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable and let $x_0 \in D_f$. We say that $f$ is **discontinuous** at $x_0$ if $f$ is not continuous at $x_0$.

#### 2.3.3.1. Types of Discontinuities

There are three main types of discontinuity:

- Removable discontinuity,
- Jump discontinuity,
- Infinite discontinuity.

**Definition 2.17.** Let $f : D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function of a real variable and let $x_0 \in \mathbb{R}$ (the point $x_0$ may or may not belong to $D_f$)

**1**) We say that the function $f$ has a removable discontinuity at $x_0$ if $\lim\limits_{x \to x_0} f(x)$ exists but

$$\lim_{x \to x_0} f(x) \neq f(x_0),$$

or $f(x_0)$ is not defined.

**2**) We say that the function $f$ has a jump discontinuity at $x_0$ if

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x).$$

That is, the left-hand and the right-hand limits exist but are not equal.

**3**) We say that the function $f$ has an infinite discontinuity at $x_0$ if

$$\lim_{x \to x_0^-} f(x) = \mp\infty \text{ or } \lim_{x \to x_0^+} f(x) = \mp\infty.$$

**Example 2.15.**

**1**) Let $f$ be a function defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}.$$

We have $D_f = \mathbb{R}$ and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left( \frac{x^2 - 4}{x - 2} \right) = \lim_{x \to 2^-} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2^+}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-}(x+2) = 4.$$

So,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4.$$

Thus $\lim_{x\to 2} f(x) = 4$ exists. Since

$$f(2) = 0 \neq 4,$$

then $f$ has a removable discontinuity at $x_0 = 2$.

**2**) Let $g$ be a function defined by

$$g(x) = \begin{cases} x+1 & \text{if } x \le 3 \\ x^3 & \text{if } x > 3 \end{cases}'$$

We have $D_g = \mathbb{R}$ and

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(x+1) = 4,$$

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 2^+}(x^3) = 27.$$

So,

$$\lim_{x\to 3^-} g(x) = 4 \neq \lim_{x\to 3^+} g(x) = 27.$$

Thus $g$ has a jump discontinuity at $x_0 = 3$.

**3**) Let $h$ be a function defined by

$$h(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

We have have $D_h = \mathbb{R}$ and

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty,$$

and

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\left(\frac{1}{x}\right) = +\infty.$$

Thus $h$ has an infinite discontinuity at $x_0 = 0$.

**Theorem 2.9.** Let $I$ be an interval, $x_0 \in I$ and let $f$ be a real function defined on $I - \{x_0\}$ such that $\lim\limits_{x \to x_0} f(x)$ exists and equal $\ell$ ($\ell \in \mathbb{R}$), i.e.,

$$\lim_{x \to x_0} f(x) = \ell,$$

then

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in I - \{x_0\} \\ \ell & \text{if } x = x_0 \end{cases},$$

is continuous at $x_0$ and called an extension of $f$. Moreover $\tilde{f}$ is defined at $x_0$.

**Example 2.16.**

**1**) The function

$$f(x) = x \sin \frac{1}{x},$$

is not defined at $x_0 = 0$, and therefore certainly not continuous there, but

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0.$$

So, $\lim_{x \to 0} f(x) = 0$. Consequently, $f$ has a removable discontinuity at $0$. Moreover,

$$\tilde{f}(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

is defined and is continuous at $0$.

**2**) The function

$$g(x) = \sin \frac{1}{x},$$

is not defined at 0 and its discontinuity is not removable, since $\lim\limits_{x \to 0} g(x)$ does not exist.

**3**) The function

$$h(x) = \frac{x}{|x|},$$

is not defined at $x_0 = 0$, and therefore certainly not continuous there, Furthermore,

$$\lim_{x \to 0^-} h(x) = -1 \neq \lim_{x \to 0^+} h(x) = 1.$$

Consequently, the discontinuity of $h$ at 0 is not removable.

### 2.3.4. Continuous Functions on Closed Intervals

**Theorem 2.10.** Let $f$ be a continuous function on a closed bounded interval $[a,b]$. Then

**1**) $f([a,b])$ is a closed bounded interval of $\mathbb{R}$. In other words,

$$\exists \alpha,\beta \in \mathbb{R}: f([a,b]) = \{f(x): x \in [a,b]\} = [\alpha,\beta].$$

**2**) $f$ is bounded on $[a,b]$ and $f$ attains its maximum and minimum values on $[a,b]$. In other words, if $\alpha = inf_{a\leq x\leq b} f(x)$ and $\beta = sup_{a\leq x\leq b} f(x)$, then $\alpha$ and $\beta$ are respectively the minimum and maximum of $f$ on $[a,b]$; that is there are points $x_1$ and $x_2$ in $[a,b]$ such that

$$f(x_1) = \alpha \text{ and } f(x_2) = \beta.$$

**Example 2.17.**

**1**) Define the function $f: [1,2] \longrightarrow \mathbb{R}$ by $f(x) = x^2 - 2$. The function $f$ is continuous on the closed bounded interval $[1,2]$. Since $f$ is strictly increasing, $f(2) = 2$, and $f(1) = -1$, then

$$f([1,2]) \subseteq [-1,2].$$

Using the above theorem, we conclude that

$$f([1,2]) = [-1,2].$$

Moreover,

$$inf_{\leq x\leq 2} f(x) = f(1) = -1 \text{ and } sup_{1\leq x\leq 2} f(x) = f(2) = 2.$$

**2**) Define the function $g: ]0,1[ \longrightarrow \mathbb{R}$ by $g(x) = x$. We have

$$inf_{0\leq x\leq 1} g(x) = 0,\ sup_{0\leq x\leq 1} g(x) = 1,$$

but $g(x) \neq 0$ and $g(x) \neq 1$ for all $x \in ]0,1[$. Thus, even if a continuous function on a non closed bounded interval is bounded, it needn't attain its supremum or infimum.

**3**) Define a function $h: [0,1] \longrightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then $h$ is unbounded on the closed bounded interval $[0,1]$ and has no maximum value since $h$ is discontinuous at $0$.

### 2.3.4. 1. Intermediate Value Theorem

**Theorem 2.10.** Let $f$ be a continuous function on a closed bounded interval $[a,b]$. If $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$, then

$$\exists c \in ]a,b[: f(c) = 0.$$

**Remark 2.9:** Theorem 2.11 fails for discontinuous functions.

**Example 2.18.**

**1**) We will show that the equation $x^5 - 3x + 1 = 0$ has at least one root in $]0,1[$. For this, we define the continuous function $f: [0,1] \longrightarrow \mathbb{R}$ as follows:

$$f(x) = x^5 - 3x + 1.$$

It follows that,

$f(0) = 1 > 0$ and $f(1) = -1 < 0$. So, as a result of the intermediate value theorem,

$$\exists c \in ]0,1[: f(c) = c^5 - 3c + 1 = 0.$$

As a consequence, equation $x^5 - 3x + 1 = 0$ has at least one root in $]0,1[$.

**2**) Define a function $g: [-1,1] \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} -1 & \text{if} -1 \leq x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \end{cases}.$$

Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$, but there does not exist a $c \in ]0,1[$such that $f(c) = 0$ due to the fact that $g$ is not continuous on $[-1,1]$. Indeed,

$$\lim_{x \to 0^-} g(x) = -1 \neq \lim_{x \to 1^+} g(x) = 1.$$

Which proves that $g$ is not continuous at 0 and hence $g$ is not continuous on $[-1,1]$.

### 2.3.5. Continuous Inverse Theorem

**Theorem 2.12.**

**1**) If $I \subset \mathbb{R}$ is an interval and $f: I \longrightarrow \mathbb{R}$ is monotone and not constant, then $f(I)$ is an interval if and only if $f$ is continuous.

**2**) If $I \subset \mathbb{R}$ is an interval and $f: I \longrightarrow \mathbb{R}$ is a strictly monotone and continuous function, then the inverse function $f^{-1}: f(I) \longrightarrow I$ exists and it is continuous.

# Chapter 3

## Inverses of Trigonometric and Hyperbolic Functions

Inverse trigonometric and hyperbolic functions are defined as the inverse functions of the basic trigonometric and hyperbolic functions that have major applications in physics and chemistry. For instance, they play a crucial role in the study of molecular geometry, many vibratory phenomena, sound, light, electricity, etc. Due to their promising applications in many areas, this chapter treats the most important properties and relations among trigonometric and hyperbolic functions.

## 3.1. Overview

### 3.1.1. Domain and Range of Trigonometric Functions

| Function | Notation | Domain | Range |
|---|---|---|---|
| sine function | sin | $\mathbb{R}$ | $[-1,1]$ |
| cosine function | cos | $\mathbb{R}$ | $[-1,1]$ |
| tangent function | tan | $\mathbb{R}-\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}.$ | $\mathbb{R}$ |
| cotangent function | cot | $\mathbb{R}-\{k\pi\}, k\in\mathbb{Z}.$ | $\mathbb{R}$ |

### 3.1.2. Some Formulae regarding Compound Angles

**1**) $\sin(\alpha+\beta)=\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta$

**2**) $\sin(\alpha-\beta)=\sin\alpha\ \cos\beta-\cos\alpha\ \sin\beta$

**3**) $\cos(\alpha+\beta)=\cos\alpha\ \cos\beta-\sin\alpha\ \sin\beta$

**4**) $\cos(\alpha-\beta)=\cos\alpha\ \cos\beta+\sin\alpha\ \sin\beta$

**5**) $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\ \tan\beta}$

**6**) $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\ \tan\beta}$

**7**) $\sin^2\alpha+\cos^2\alpha=1$

**8**) $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

**9**) $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

**10**) $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

**11**) $\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$.

### 3.1.3. Solution of Trigonometric Equations

**1**) If $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin x = \sin\alpha \Longrightarrow x = \alpha + 2k\pi$ or $x = \pi - \alpha + 2k\pi, k \in \mathbb{Z}$.

**2**) If $\alpha \in [0, \pi]$, then $\cos x = \cos\alpha \Longrightarrow x = 2k\pi \pm \alpha, k \in \mathbb{Z}$

**3**) $\tan x = \tan\alpha \Longrightarrow x = k\pi + \alpha, k \in \mathbb{Z}$

**4**) If $\sin^2 x = \sin^2\alpha$ or $\cos^2 x = \cos^2\alpha$ or $\tan^2 x = \tan^2\alpha$, then $x = k\pi \pm \alpha, k \in \mathbb{Z}$

## 3.2. Inverses of Trigonometric Functions

The inverse trigonometric functions (also called arcus functions, antitrigonometric functions, inverse trig functions https://en.wikipedia.org/wiki/Inverse_trigonometric_functions - cite_note-Hall_1909-6 or cyclometric functions) are the inverse functions of the trigonometric functions on suitably restricted domains.

### 3.2.1. The arcsine Function

**Definition 3.1.** The restriction of the sine function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is continuous and strictly increasing and takes values on $[-1,1]$, i.e.,

$$\sin\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1,1].$$

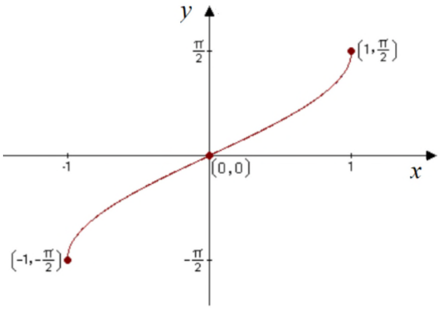
As a result, it has an inverse function defined on $[-1,1]$ that takes values on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, called the arcsine function and denoted by **arcsin**.

$$\text{arcsin}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where

$$\forall x \in [-1,1]: y = \arcsin x \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \text{and} \\ \sin y = x \end{cases}.$$

In addition, this inverse function is strictly increasing and continuous on $[-1,1]$.



**Example 3.1.**

**1**) We have

$$\arcsin 0 = y \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \text{and} \\ \sin y = 0 \end{cases} \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \text{and} \\ y = k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then $y = 0$. Thus $\arcsin 0 = 0$.

**2**) We have

$$\arcsin 1 = y \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \text{and} \\ \sin y = 1 \end{cases} \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \text{and} \\ y = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then $y = \frac{\pi}{2}$. Thus $\arcsin 1 = \frac{\pi}{2}$.

**3**) We have

$$\arcsin\frac{\sqrt{3}}{2} = y \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \text{and} \\ \sin y = \frac{\sqrt{3}}{2} \end{cases} \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \text{and} \\ y = \frac{\pi}{3} + 2k\pi \text{ or } \frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then $y = \frac{\pi}{3}$. Thus $\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

**Remark 3.1.**

**1**) We have

$$\forall x \in [-1,1]: \sin(\arcsin x) = x.$$

**2**) The function arcsin is not the inverse function of the sine function but of its restriction to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. So, we don't always have $\arcsin(\sin y) = y$, but

$$\forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]: \arcsin(\sin y) = y.$$

For example,

**a**) Since $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\arcsin\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$. Indeed, we have

$$\arcsin\left(\sin\frac{\pi}{6}\right) = \arcsin\frac{1}{2},$$

and

$$\arcsin\frac{1}{2} = y \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \text{and} \\ \sin y = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \text{and} \\ y = \frac{\pi}{6} + 2k\pi \text{ or } \frac{5\pi}{6} + 2k\pi\,, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $y = \frac{\pi}{6}$. Thus $\arcsin\left(\sin\frac{\pi}{6}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$.

**b**) One has $\arcsin(\sin \pi) = 0 \neq \pi$. Here, $\pi \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which gives

$$\arcsin(\sin \pi) = \arcsin 0 = 0.$$

**3**) arcsin is an odd function.

$$\forall x \in [-1,1]: \arcsin(-x) = -\arcsin x.$$

**4**) If $y = \arcsin x$, then $x \in [-1,1]$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin y = x$. We have

$$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos y \geq 0,$$

and

$$\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \sqrt{1 - \sin^2 y}.$$

Since $y = \arcsin x$ and $\sin y = x$, we get

$$\forall x \in ]-1,1[: \cos(\arcsin x) = \sqrt{1 - x^2}.$$

Similarly, we get

$$\forall x \in ]-1,1[: \tan(\arcsin x) = \frac{x}{\sqrt{1 - x^2}}.$$

**5**) arcsin is differentiable and has a derivative not null on $]-1,1[$ where

$$\forall x \in \left]-1,1\right[: (\arcsin x)' = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos\big((\arcsin x)\big)} = \frac{1}{\sqrt{1-x^2}}.$$

Or

$$\forall x \in \left]-1,1\right[: (\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

More generally,

$$\forall x \in \left]-1,1\right[: \Big(\arcsin\big(f(x)\big)\Big)' = \frac{\big(f(x)\big)'}{\sqrt{1-\big(f(x)\big)^2}}.$$

### 3.2.2. The arccosine Function

**Definition 3.2.** The restriction of the cosine function to $[0,\pi]$ is continuous and strictly decreasing and takes values on $[-1,1]$, i.e.,

$$\cos([0,\pi]) = [-1,1].$$

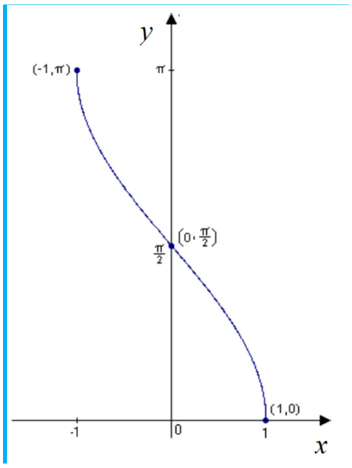
As a result, it has an inverse function defined on $[-1,1]$ that takes values on $[0,\pi]$ called the arcsine function and denoted by **arccos**.

$$\arccos: [-1,1] \to [0,\pi],$$

where

$$\forall x \in [-1,1]: y = \arccos x \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \quad \text{and} \\ \cos y = x \end{cases}.$$

In addition, this function is strictly decreasing and continuous on $[-1,1]$.



**Example 3.2.**

**1**) We have

$$\arccos 0 = y \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \text{and} \\ \cos y = 0 \end{cases} \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \text{and} \\ y = \frac{\pi}{2} + 2k\pi \text{ or } -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in [0,\pi]$, then $y = \frac{\pi}{2}$. Thus $\arccos 0 = \frac{\pi}{2}$.

**2**) We have

$$\arccos 1 = y \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \text{and} \\ \cos y = 1 \end{cases} \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \text{and} \\ y = 2k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in [0,\pi]$, then $y = 0$. Thus $\arccos 1 = 0$.

**3**) We have

$$\arccos \frac{\sqrt{3}}{2} = y \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \text{and} \\ \cos y = \frac{\sqrt{3}}{2} \end{cases} \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \text{and} \\ y = \mp\frac{\pi}{6} + 2k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in [0,\pi]$, then $y = \frac{\pi}{6}$. Thus $\arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

**Remark 3.2.**

**1**) We have

$$\forall x \in [-1,1]: \cos(\arccos x) = x.$$

**2**) The function arccos is not the inverse function of the cosine function but of its restriction to $[0,\pi]$. So, we don't always have $\arccos(\cos y) = y$, but

$$\forall y \in [0,\pi]: \arccos(\cos y) = y.$$

For example,

**a**) Since $\frac{\pi}{6} \in [0,\pi]$, then $\arccos\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$. Indeed, we have

$$\arccos\left(\cos\frac{\pi}{6}\right) = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}.$$

**b**) One has

$$\arccos(\cos 2\pi) = 0 \neq 2\pi.$$

Indeed, $2\pi \notin [0,\pi]$, and

$$\arccos(\cos 2\pi) = \arccos 1 = 0.$$

**3**) We have

$$\arccos\frac{1}{2} = y \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \quad \text{and} \\ \cos y = \dfrac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \quad \text{and} \\ y = 2k\pi \mp \dfrac{\pi}{3} +, k \in \mathbb{Z} \end{cases}.$$

Since $y \in [0,\pi]$, then $y = \frac{\pi}{3}$. Thus $\arccos\frac{1}{2} = \frac{\pi}{3}$. On the other hand

$$\arccos\left(-\frac{1}{2}\right) = y \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \quad \text{and} \\ \cos y = -\dfrac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y \in [0,\pi] \\ \quad \text{and} \\ y = 2k\pi \pm \dfrac{2\pi}{3}, k \in \mathbb{Z} \end{cases}.$$

Since $y \in [0,\pi]$, then $y = \frac{2\pi}{3}$. Thus $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. So

$$\arccos\frac{1}{2} = \frac{\pi}{3} \neq -\arccos\frac{1}{2} = -\frac{\pi}{3} \neq \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

Thus arccos is neither an odd nor even function.

**4**) If $y = \arccos x$, then $x \in [-1,1]$, $y \in [0,\pi]$ and $\cos y = x$.

We have

$$y \in [0,\pi] \Longrightarrow \sin y \geq 0,$$

and

$$\cos^2 y + \sin^2 y = 1 \Longrightarrow \sin y = \sqrt{1 - \cos^2 y}.$$

Since $y = \arccos x$ and $\cos y = x$, we get

$$\forall x \in [-1,1]: \sin(\arccos x) = \sqrt{1 - x^2}.$$

Similarly, if $x \neq 0$, we can get

$$\forall x \in \,]-1,1[: \tan(\arccos x) = \frac{\sqrt{1 - x^2}}{x}.$$

**5**) We have

$$\forall x \in [-1,1]: \pi - \arccos x = \arccos(-x).$$

Indeed, if $x \in \,]-1,1[$, then $\pi - \arccos x \in [0,\pi]$ and $\arccos(-x) \in [0,\pi]$. Furthermore

$$\cos(\pi - \arccos x) = \cos(\pi).\cos(\arccos x) + \sin(\pi).\sin(\arccos x) = -x,$$

and

$$\cos\big(\arccos(-x)\big) = -x.$$

Thus,

$$\cos(\pi - \arccos x) = \cos\big(\arccos(-x)\big) \Rightarrow \pi - \arccos x = \arccos(-x).$$

**6**) arccos is differentiable and has a derivative not null on $]-1,1[$ where

$$\forall x \in\, ]-1,1[ : (\arccos x)' = \frac{1}{\cos'(\arccos x)} = \frac{1}{-\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}}.$$

Or

$$\forall x \in\, ]-1,1[ : (\arccos x)' = \frac{1}{(\cos y)'} = \frac{-1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

More generally,

$$\forall x \in\, ]-1,1[ : \Big(\arccos\big(f(x)\big)\Big)' = -\frac{\big(f(x)\big)'}{\sqrt{1-\big(f(x)\big)^2}}.$$

### 3.2.3. The arc tangent Function

**Definition 3.3.** The restriction of the tangent function to $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ is continuous and strictly increasing and takes values on $\mathbb{R}$, i.e.,

$$\tan\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right) = \mathbb{R}.$$

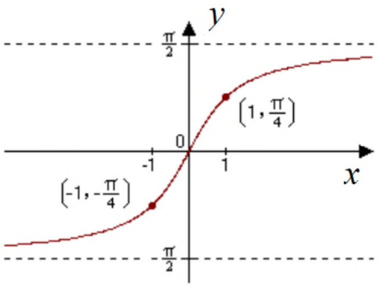
As a result, it has an inverse function defined on $\mathbb{R}$ that takes values on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ called the arc tangent function and denoted by **arctan**.

$$\arctan : \mathbb{R} \to \left]-\frac{\pi}{2},\frac{\pi}{2}\right[,$$

where

$$\forall x \in \mathbb{R} : y = \arctan x \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[ \\ \quad \text{and} \\ \tan y = x \end{cases}.$$

In addition, this function is strictly increasing and continuous on $\mathbb{R}$.



**Example 3.3.**

**1**) We have

$$\arctan 0 = y \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \\ \text{and} \\ \tan y = 0 \end{cases} \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \\ \text{and} \\ y = k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $y = 0$. Thus $\arctan 0 = 0$.

**2**) We have

$$\arctan 1 = y \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \\ \text{and} \\ \tan y = 1 \end{cases} \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \\ \text{and} \\ y = \frac{\pi}{4} + k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $y = \frac{\pi}{4}$. Thus $\arctan 1 = \frac{\pi}{4}$.

**3**) We have

$$\arctan\sqrt{3} = y \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \\ \text{and} \\ \tan y = \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \\ \text{and} \\ y = \frac{\pi}{3} + k\pi\,, k \in \mathbb{Z} \end{cases}.$$

Since $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $y = \frac{\pi}{3}$. Thus $\arctan\sqrt{3} = \frac{\pi}{3}$.

**Remark 3.3.**

**1**) We have

$$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2} \text{ and } \lim_{x \to +\infty} \arctan x = \frac{\pi}{2}.$$

**2**) We have

$$\forall x \in \mathbb{R} : \tan(\arctan x) = x.$$

**3**) The function arctan is not the inverse function of the tangent function but of its restriction to $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$. So, we don't always have $\arctan(\tan y) = y$, but

$$\forall y \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[ : \arctan(\tan y) = y.$$

For example,

**a**) Since $\frac{\pi}{4} \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, then $\arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$. Indeed, we have

$$\arctan\left(\tan\frac{\pi}{4}\right) = \arctan(1) = \frac{\pi}{4}.$$

**b**) One has

$$\arctan(\tan\pi) = 0 \neq \pi.$$

Indeed, $\pi \notin \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, and

$$\arctan(\tan\pi) = \arctan(0) = 0.$$

**4**) arctan is odd as it is the inverse function of an odd function.

$$\forall x \in \mathbb{R}: \arctan(-x) = -\arctan x.$$

**5**) If $y = \arctan x$, then $x \in \mathbb{R}$, $y \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and $\tan y = x$. We have

$$y \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[ \Longrightarrow \cos y > 0,$$

and

$$\cos^2 y + \sin^2 y = 1 \Longrightarrow \frac{\cos^2 y}{\cos^2 y} + \frac{\sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} \Rightarrow 1 + \tan^2 y = \frac{1}{\cos^2 y}$$

$$\Longrightarrow \cos y = \frac{1}{\sqrt{1+\tan^2 y}}.$$

Since $y = \arctan x$ and $\tan y = x$, we get

$$\forall x \in \mathbb{R}: \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

Similarly, we can get

$$\forall x \in \mathbb{R}: \sin(\arctan x) = \frac{x}{\sqrt{1-x^2}}.$$

**6**) arctan is differentiable and has a derivative not null on ℝ where

$$\forall x \in \mathbb{R}: (\mathrm{arctan}x)' = \frac{1}{\tan'(\mathrm{arctan}x)} = \frac{1}{1 + (\tan(\mathrm{arctan}x))^2} = \frac{1}{1 + x^2}.$$

Or

$$\forall x \in \mathbb{R}: (\mathrm{arctan}x)' = \frac{1}{(\tan y)'} = \frac{1}{1 + (\tan y)^2} = \frac{1}{1 + x^2}.$$

More generally,

$$\forall x \in \mathbb{R}: \left(\mathrm{arctan}\big(f(x)\big)\right)' = \frac{\big(f(x)\big)'}{1 + \big(f(x)\big)^2}.$$

### 3.2.4. The arc-cotangent Function

**Definition 3.4:** The restriction of the cotangent function to $]0, \pi[$ is continuous and strictly decreasing and takes values on ℝ, i.e.,

$$\cot(]0, \pi[) = \mathbb{R}.$$

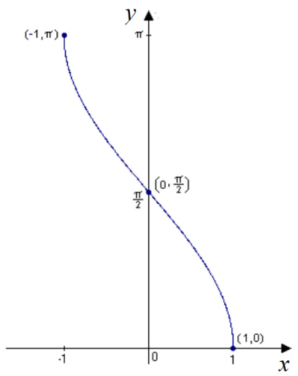
As a result, it has an inverse function defined on ℝ that takes values on $]0, \pi[$ called the arc tangent function and denoted by **arccot**.

$$\mathrm{arccot}: \mathbb{R} \to ]0, \pi[,$$

where

$$\forall x \in \mathbb{R}: y = \mathrm{arccot}x \Leftrightarrow \begin{cases} y \in ]0, \pi[ \\ \quad \text{and} \\ \cot y = x \end{cases}.$$

In addition, this function is strictly decreasing and continuous on ℝ.



**Example 3.4.**

**1**) One has

$$\operatorname{arccot}0 = y \Leftrightarrow \begin{cases} y \in ]0,\pi[ \\ \quad\text{and} \\ \cot y = 0 \end{cases} \Leftrightarrow \begin{cases} y \in ]0,\pi[ \\ \quad\text{and} \\ y = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in ]0,\pi[$, then $y = \frac{\pi}{2}$. Thus $\arctan 0 = \frac{\pi}{2}$.

**2**) We have

$$\operatorname{arccot}1 = y \Leftrightarrow \begin{cases} y \in ]0,\pi[ \\ \quad\text{and} \\ \cot y = 1 \end{cases} \Leftrightarrow \begin{cases} y \in ]0,\pi[ \\ \quad\text{and} \\ y = \frac{\pi}{4} + k\pi, k \in \mathbb{Z} \end{cases}.$$

Since $y \in ]0,\pi[$, then $y = \frac{\pi}{4}$. Thus $\operatorname{arccot}1 = \frac{\pi}{4}$.

**3**) We have

$$\operatorname{arccot}\sqrt{3} = y \Leftrightarrow \begin{cases} y \in ]0,\pi[ \\ \quad\text{and} \\ \cot y = \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} y \in ]0,\pi[ \\ \quad\text{and} \\ y = \frac{\pi}{6} + k\pi\,, k \in \mathbb{Z} \end{cases}.$$

Since $y \in ]0,\pi[$, then $y = \frac{\pi}{6}$. Thus $\operatorname{arccot}\sqrt{3} = \frac{\pi}{6}$.

**Remark 3.4.**

**1**) We have

$$\lim_{x\to-\infty} \operatorname{arccot}x = \pi \text{ and } \lim_{x\to+\infty} \operatorname{arccot}x = 0.$$

**2**) We have

$$\forall x \in \mathbb{R}: \cot(\operatorname{arccot}x) = x.$$

**3**) The function arccot is not the inverse function of the cotangent function but of its restriction to $]0,\pi[$. So, we don't always have $\operatorname{arcot}(\cot y) = y$, but

$$\forall y \in \ ]0,\pi[ \ : \operatorname{arcot}(\cot y) = y.$$

For example,

**a**) Since $\frac{\pi}{4} \in \ ]0,\pi[$, then $\operatorname{arcot}\left(\cot\frac{\pi}{4}\right) = \frac{\pi}{4}$. Indeed, we have

$$\operatorname{arcot}\left(\cot\frac{\pi}{4}\right) = \operatorname{arcot}(\cot 1) = \frac{\pi}{4}.$$

**b**) One has

$$\operatorname{arccot}\left(\cot\left(\frac{3\pi}{2}\right)\right)=\frac{\pi}{2}\neq\frac{3\pi}{2}.$$

Indeed, $\frac{3\pi}{2}\notin\ ]0,\pi[$, and

$$\operatorname{arccot}\left(\cot\frac{3\pi}{2}\right)=\operatorname{arccot}0=\frac{\pi}{2}.$$

**4**) We have

$$\operatorname{arccot}\frac{1}{\sqrt{3}}=y\Leftrightarrow\begin{cases}y\in\ ]0,\pi[\\ \text{and}\\ \cot y=\frac{1}{\sqrt{3}}\end{cases}\Leftrightarrow\begin{cases}y\in\ ]0,\pi[\\ \text{and}\\ y=\frac{\pi}{3}+k\pi,k\in\mathbb{Z}\end{cases}.$$

Since $y\in\ ]0,\pi[$, then $y=\frac{\pi}{3}$. Thus $\operatorname{arccot}\frac{1}{\sqrt{3}}=\frac{\pi}{3}$. On the other hand

$$\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)=y\Leftrightarrow\begin{cases}y\in\ ]0,\pi[\\ \text{and}\\ \cot y=-\frac{1}{\sqrt{3}}\end{cases}\Leftrightarrow\begin{cases}y\in\ ]0,\pi[\\ \text{and}\\ y=-\frac{\pi}{3}+k\pi,k\in\mathbb{Z}\end{cases}.$$

Since $y\in\ ]0,\pi[$, then $y=\frac{2\pi}{3}$. Thus, $\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)=\frac{2\pi}{3}$. So

$$\operatorname{arccot}\frac{1}{\sqrt{3}}=\frac{\pi}{3}\neq-\operatorname{arccot}\frac{1}{\sqrt{3}}=-\frac{\pi}{3}\neq\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)=\frac{2\pi}{3}.$$

Thus arccot is neither an odd nor even function.

**5**) If $y=\operatorname{arccot}x$, then $x\in\mathbb{R}$, $y\in\ ]0,\pi[$ and $\cot y=x$. We have

$$y\in\ ]0,[\ \Longrightarrow\sin y>0,$$

and

$$\cos^2 y+\sin^2 y=1\Longrightarrow\frac{\cos^2 y}{\sin^2 y}+\frac{\sin^2 y}{\sin^2 y}=\frac{1}{\sin^2 y}\Rightarrow\cot^2 y+1=\frac{1}{\sin^2 y}$$

$$\Longrightarrow\sin y=\frac{1}{\sqrt{1+\cot^2 y}}.$$

Since $y=\operatorname{arccot}x$ and $\cot y=x$, we get

$$\forall x\in\mathbb{R}:\cos(\operatorname{arccot}x)=\frac{1}{\sqrt{1+x^2}}.$$

**6**) arccot is differentiable and has a derivative not null on ℝ where

$$\forall x \in \mathbb{R}: (\mathrm{arccot} x)' = \frac{1}{\cot'(\mathrm{arccot} x)} = -\frac{1}{1+(\cot(\mathrm{arccot} x x))^2} = -\frac{1}{1+x^2}.$$

More generally,

$$\forall x \in \mathbb{R}: \Big(\mathrm{arccot}\big(f(x)\big)\Big)' = -\frac{\big(f(x)\big)'}{1+\big(f(x)\big)^2}.$$

## 3.3. Hyperbolic Functions

### 3.3.1. Hyperbolic sine and cosine Functions

**Definition 3.5.**

**1**) The hyperbolic sine function, denoted by **sinh** is defined for all real values of $x$ by the relation

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

**2**) The hyperbolic cosine function, denoted by **cosh** is defined for all real values of $x$ by the relation

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

**Remark 3.5.**

**1**) We have

$$\sinh x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x.$$

Thus

$$e^x = \sinh x + \cosh x.$$

**2**) We have

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}.$$

Thus

$$e^{-x} = \cosh x - \sinh x.$$

**3**) We have

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1.$$

Thus

$$\cosh^2 x - \sinh^2 x = 1.$$

**4**) sinh and cosh are continuous and differentiable on $\mathbb{R}$ where

$$\forall x \in \mathbb{R}: (\sinh x)' = \frac{e^x + e^{-x}}{2} = \cosh x \text{ and } (\cosh x)' = \frac{e^x - e^{-x}}{2} = \sinh x.$$

**5**) For all $x \in \mathbb{R}$, we have

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x,$$

and

$$\cosh(-x) = \frac{e^x + e^{-x}}{2} = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

So, sinh is odd, while cosh is even.

**6**) We have

$$(\sinh x)' > 0, \lim_{x \to -\infty} \sinh x = -\infty, \lim_{x \to +\infty} \sinh x = +\infty,$$

and

$$(\cosh x)' = 0 \Leftrightarrow x = 0, \ (\cosh x)' > 0 \Leftrightarrow x > 0, (\cosh x)' < 0 \Leftrightarrow x < 0,$$
$$\cosh 0 = 1, \lim_{x \to -\infty} \cosh x = +\infty, \text{and} \lim_{x \to +\infty} \cosh x = +\infty.$$

So

$$\text{Rang}\ (\sinh x) = \mathbb{R},$$

and

$$\text{Rang}\ (\cosh x) = [1, +\infty[.$$

**7**) sinh is strictly increasing on $\mathbb{R}$ while cosh is strictly decreasing on $]-\infty, 0]$ and strictly increasing on $[0, +\infty[$.

### 3.3.2. Hyperbolic tangent and cotangent Functions

**Definition 3.6.**

**1**) The hyperbolic tangent function, denoted by **tanh** is defined for all real values of $x$ by

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

**2**) The hyperbolic cotangent function, denoted by **coth** is defined for all real values of $x \neq 0$ by

$$\coth x = \frac{\cosh x}{\sinh x}.$$

**Remark 3.6.**

**1**) We have

$$\frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})} = \frac{e^{2x} - 1}{e^{2x} - 1}.$$

Thus

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

**2**) For $x \neq 0$, we have

$$\coth x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \Longrightarrow \coth x = \frac{1 + e^{-2x}}{1 - e^{-2x}}.$$

**4**) tanh is continuous and differentiable on $\mathbb{R}$ where

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x.$$

**5**) coth is continuous and differentiable on $\mathbb{R} - \{0\}$ where

$$(\coth x)' = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = 1 - \coth^2 x.$$

**6**) Since sinh is odd and cosh is even, then tanh and coth are odd functions

$$\tanh(-x) = -\tanh x, \forall\, x \in \mathbb{R} \text{ and } \coth(-x) = -\coth x, \forall\, x \in \mathbb{R} - \{0\}.$$

**7**) For all $x \in \mathbb{R}$, we have

$$(\tanh x)' > 0, \lim_{x \to -\infty} \tanh x = -1, \lim_{x \to +\infty} \tanh x = 1,$$

and

$$e^{2x} - 1 < e^{2x} + 1 \Longrightarrow \tanh x < 1.$$

On the other hand

$$\tanh(-x) = -\tanh x < 1 \Longrightarrow \tanh x > -1.$$

So, $-1 < \tanh x < 1,\ \forall\, x \in \mathbb{R}$ and

$$\text{Rang}(\tanh x) = ]-1,1[.$$

**8**) For all $x \in \mathbb{R} - \{0\}$, we have $(\coth x)' < 0$ and

$$\lim_{x\to 0^-} \coth x = -\infty,\ \lim_{x\to 0^+} \coth x = +\infty,\ \lim_{x\to -\infty} \coth x = -1,\ \lim_{x\to +\infty} \coth x = 1.$$

So

$$\text{Rang}(\coth x) = ]-\infty, -1[\, \cup\, ]1, +\infty[.$$

**9**) tanh is strictly increasing on $\mathbb{R}$ while coth is strictly decreasing on $\mathbb{R} - \{0\}$.

### 3.3.3. Hyperbolic Function Identities

**1**) $\sinh(x+y) = \sinh(x)\ \cosh(y) + \cosh(x)\ \sinh(y)$

**2**) $\sinh(x-y) = \sinh(x)\ \cosh(y) - \cosh(x)\ \sinh(y)$

**3**) $\cosh(x+y) = \cosh(x)\ \cosh(y) + \sinh(x)\ \sinh(y)$

**4**) $\cosh(x-y) = \cosh(x)\ \cosh(y) - \sinh(x)\ \sinh(y)$

**5**) $\tanh(x+y) = \dfrac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\ \tanh(y)}$

**6**) $\tanh(x-y) = \dfrac{\tanh(x) - \tanh(y)}{1 - \tanh(x)\ \tanh(y)}$

**7**) $1 - (\tanh x)^x = \dfrac{1}{(\cosh x)^2}.$

**8**) $\sinh(x) + \sinh(y) = 2\sinh\left(\dfrac{x+y}{2}\right)\ \cosh\left(\dfrac{x-y}{2}\right)$

**9**) $\sinh(x) - \sinh(y) = 2\cosh\left(\dfrac{x+y}{2}\right)\ \sinh\left(\dfrac{x-y}{2}\right)$

**10**) $\cosh(x) + \cosh(y) = 2\cosh\left(\dfrac{x+y}{2}\right)\ \cosh\left(\dfrac{x-y}{2}\right)$

**11**) $\cosh(x) - \cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$.

## 3.4. Inverses of Hyperbolic Functions

### 3.4.1. Inverse Hyperbolic sine Function

**Definition 3.7.** The hyperbolic sine function is a bijection from $\mathbb{R}$ to $\mathbb{R}$ as it is continuous and strictly increasing from $\mathbb{R}$ to $\mathbb{R}$, i.e.,

$$\sinh(\mathbb{R}) = \mathbb{R}.$$

Thus it has an inverse function defined on $\mathbb{R}$ that takes values on $\mathbb{R}$ called inverse hyperbolic sine and denoted by **arsinh** or $\mathbf{sinh^{-1}}$

$$\operatorname{arsinh}: \mathbb{R} \to \mathbb{R},$$

where

$$\forall x \in \mathbb{R}: y = \operatorname{arsinh}(x) \Leftrightarrow \begin{cases} y \in \mathbb{R} \\ \text{ and} \\ \sinh(y) = x \end{cases}.$$

In addition, this function is strictly increasing and continuous on $\mathbb{R}$.

**Remark 3.7.**
**1**) We have

$$\forall x \in \mathbb{R}: \sinh\big(\operatorname{arsinh}(x)\big) = x,$$

and

$$\forall y \in \mathbb{R}: \operatorname{arsinh}\big(\sinh(y)\big) = y.$$

**2**) arsinh is odd

$$\forall x \in \mathbb{R}: \operatorname{arsinh}(-x) = -\operatorname{arsinh}(x).$$

**3**) If $y = \operatorname{arsinh}(x)$, then $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $\sinh(y) = x$. We have $\cosh y > 0$ and

$$\cosh^2 y - \sinh^2 y = 1 \Longrightarrow \cosh y = \sqrt{1 + \sinh^2 y}.$$

Since $y = \operatorname{arsinh}(x)$ and $\sinh(y) = x$, we get

$$\forall x \in \mathbb{R}: \cosh\big(\operatorname{arsinh}(x)\big) = \sqrt{1 + x^2}.$$

**4**) arsinh is differentiable on $\mathbb{R}$ where

$$\forall x \in \mathbb{R}: \big(\operatorname{arsinh}(x)\big)' = \frac{1}{\sinh'\big(\operatorname{arsinh}(x)\big)} = \frac{1}{\cosh\big(\operatorname{arsinh}(x)\big)} = \frac{1}{\sqrt{1 + x^2}}.$$

Or

$$\forall x \in \mathbb{R}: \big(\operatorname{arsinh}(x)\big)' = \frac{1}{(\sinh y)'} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1-x^2}}.$$

**5**) If $y = \operatorname{arsinh}(x)$, then $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $\sinh(y) = x$.

Since $\operatorname{cosh} y = \sqrt{1+x^2}$, then

$$e^y = \sinh y + \cosh y = x + \sqrt{1+x^2} \Rightarrow \ln(e^y) = y = \ln\left(x + \sqrt{1+x^2}\right).$$

So,

$$\forall x \in \mathbb{R}: y = \operatorname{arsinh}(x) \Leftrightarrow y = \ln\left(x + \sqrt{1+x^2}\right).$$

### 3.4.2. Inverse Hyperbolic cosine Function

**Definition 3.8.** The restriction of the hyperbolic cosine function to $[0, +\infty[$ is continuous and strictly decreasing and takes values on $[1, +\infty[$ , i.e.,

$$\cosh([0, +\infty[) = [1, +\infty[.$$

Thus it has an inverse function defined on $[1, +\infty[$ that takes values on $[0, +\infty[$ called inverse hyperbolic cosine and denoted by **arcosh** or $\mathbf{cosh^{-1}}$

$$\operatorname{arcosh}: [1, +\infty[ \to [0, +\infty[,$$

where

$$\forall x \in [1, +\infty[: y = \operatorname{arcosh} x \Leftrightarrow \begin{cases} y \in [0, +\infty[ \\ \quad \text{and} \\ \cosh y = x \end{cases}$$

In addition, this function is strictly decreasing and continuous on $[1, +\infty[$.

**Remark 3.8.**

**1**) We have

$$\forall x \in [1, +\infty[: \cosh\big(\operatorname{arcosh}(x)\big) = x.$$

**2**) The function arcosh is not the inverse function of the cosh function but of its restriction on $[0, +\infty[$. So,

$$\forall y \in [0, +\infty[: \operatorname{arcosh}(\cosh y) = y.$$

**3**) The domain of arcosh is non symmetric. Thus, arcosh is neither an odd nor even function.

**4**) If $y = \operatorname{arcosh} x$, then $x \in [1, +\infty[$, $y \in [0, +\infty[$ and $\cosh y = x$. We have

$$\cosh^2 y - \sinh^2 y = 1 \Longrightarrow \sinh y = \sqrt{\cosh^2 y - 1}.$$

Since $y = \operatorname{arcosh} x$ and $\cosh y = x$, we get

$$\sinh y = \sinh(\operatorname{arcosh} x) = \sqrt{x^2 - 1}.$$

**5**) If $y = \operatorname{arcosh} x$, then $x \in [1, +\infty[$, $y \in [0, +\infty[$ and $\cosh y = x$.

Since $\sinh y = \sqrt{x^2 - 1}$, then

$$e^y = \sinh y + \cosh y = x + \sqrt{x^2 - 1} \Longrightarrow \ln(e^y) = y = \ln\left(x + \sqrt{x^2 - 1}\right).$$

Consequently,

$$\forall x \in [1, +\infty[ : y = \operatorname{arcosh}(x) \Leftrightarrow y = \ln\left(x + \sqrt{x^2 - 1}\right).$$

**6**) arsinh is differentiable on $]1, +\infty[$ where, $\forall x \in ]1, +\infty[$

$$\left(\operatorname{arcosh}(x)\right)' = \frac{1}{\cosh'\left(\operatorname{arcosh}(x)\right)} = \frac{1}{\sinh\left(\operatorname{arcosh}(x)\right)} = \frac{1}{\sqrt{x^2 - 1}}.$$

Or

$$\forall x \in ]1, +\infty[ : \left(\operatorname{arcosh}(x)\right)' = \frac{1}{(\cos y)'} = \frac{1}{\sinh y} = \frac{1}{\sqrt{x^2 - 1}}.$$

### 3.4.3. Inverse Hyperbolic tangent Function

**Definition 3.9.** The hyperbolic tangent function is a bijection from $\mathbb{R}$ to $]-1,1[$ as it is continuous and strictly increasing from $\mathbb{R}$ to $]-1,1[$, i.e.,

$$\tanh(\mathbb{R}) = ]-1,1[.$$

Thus it has an inverse function defined on $]-1,1[$ that takes values on $\mathbb{R}$ called inverse hyperbolic tangent and denoted by **artanh** or $\mathbf{tanh^{-1}}$

$$\operatorname{artanh} : ]-1,1[ \to \mathbb{R},$$

where

$$\forall x \in ]-1,1[ : y = \operatorname{artanh}(x) \Leftrightarrow \begin{cases} y \in \mathbb{R} \\ \text{ and } \\ \tanh(y) = x \end{cases}.$$

In addition, this function is strictly increasing and continuous on $]-1,1[$.

**Remark 3.9.**

**1**) We have

$$\forall x \in \left]-1,1\right[ : \tanh\big(\operatorname{artanh}(x)\big) = x,$$

and

$$\forall y \in \mathbb{R} : \operatorname{artanh}\big(\tanh(y)\big) = y.$$

**2**) artanh is odd as it is the inverse function of an odd function.

$$\forall x \in \left]-1,1\right[ : \operatorname{artanh}(-x) = -\operatorname{artanh}(x).$$

**3**) If $y = \operatorname{artanh}(x)$, then $x \in \left]-1,1\right[$, $y \in \mathbb{R}$ and $\tanh(y) = x$. We have $\cosh y > 0$ and

$$\cosh^2 y - \sinh^2 y = 1 \Longrightarrow \frac{\cosh^2 y - \sinh^2 y}{\cosh^2 y} = \frac{1}{\cosh^2 y} \Longrightarrow 1 - \tanh^2 y = \frac{1}{\cosh^2 y}$$

$$\Longrightarrow \cosh y = \frac{1}{\sqrt{1 - \tanh^2 y}}.$$

Since $y = \operatorname{artanh}(x)$ and $\tanh(y) = x$, we get

$$\forall x \in \left]-1,1\right[ : \cosh\big(\operatorname{artanh}(x)\big) = \frac{1}{\sqrt{1 - x^2}}.$$

**4**) artanh is differentiable on $\left]-1,1\right[$ where

$$\forall x \in \left]-1,1\right[ : \big(\operatorname{artanh}(x)\big)' = \frac{1}{\tanh'\big(\operatorname{artanh}(x)\big)} = \frac{1}{1 - \Big(\tanh\big(\operatorname{artanh}(x)\big)\Big)^2}$$

$$= \frac{1}{\frac{1}{\cosh^2 y}} = \frac{1}{1 - x^2}.$$

Or

$$\forall x \in \left]-1,1\right[ : \big(\operatorname{artanh}(x)\big)' = \frac{1}{(\tanh y)'} = \frac{1}{1 - x^2}.$$

**5**) If $y = \operatorname{artanh}(x)$, then $x \in \left]-1,1\right[$, $y \in \mathbb{R}$ and $\tanh(y) = x$. We have

$$x = \tanh(y) \Leftrightarrow x = \frac{e^{2y} - 1}{e^{2y} + 1} \Leftrightarrow e^{2y}(x - 1) = -1 - x \Leftrightarrow e^{2y} = \frac{1 + x}{1 - x}.$$

So,

$$\ln(e^{2y}) = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Thus,

$$\forall x \in ]-1,1[: y = \operatorname{artanh}(x) \Leftrightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

### 3.4.4. Inverse Hyperbolic cotangent Function

**Definition 3.10.** The hyperbolic cotangent function is a bijection from $\mathbb{R}^*$ to $]-\infty,-1[\,\cup\,]1,+\infty[$ as it is continuous and strictly decreasing from $\mathbb{R}^*$ to $]-\infty,-1[\,\cup\,]1,+\infty[$, i.e.,

$$\coth(\mathbb{R}^*) = ]-\infty,-1[\,\cup\,]1,+\infty[.$$

Thus it has an inverse function defined on $]-\infty,-1[\,\cup\,]1,+\infty[$ that takes values on $\mathbb{R}^*$ called inverse hyperbolic cotangent and denoted by **arcoth** or $\mathbf{coth^{-1}}$

$$\operatorname{arcoth}: ]-\infty,-1[\,\cup\,]1,+\infty[ \to \mathbb{R}^*,$$

where

$$\forall x \in ]-\infty,-1[\,\cup\,]1,+\infty[: y = \operatorname{arcoth}(x) \Leftrightarrow \begin{cases} y \in \mathbb{R}^* \\ \text{and} \\ \coth(y) = x \end{cases}.$$

In addition, this function is strictly decreasing and continuous on $]-\infty,-1[\,\cup\,]1,+\infty[$.

**Remark 3.10.**

**1**) We have

$$\forall x \in ]-\infty,-1[\,\cup\,]1,+\infty[: \coth\big(\operatorname{arcoth}(x)\big) = x,$$

and

$$\forall y \in \mathbb{R}^*: \operatorname{arcoth}\big(\coth(y)\big) = y.$$

**2**) arcoth is odd as it is the inverse function of an odd function.

$$\forall x \in ]-\infty,-1[\,\cup\,]1,+\infty[: \operatorname{arcoth}(-x) = -\operatorname{arcoth}(x).$$

**3**) If $y = \operatorname{arcoth}(x)$, then $x \in ]-\infty,-1[\,\cup\,]1,+\infty[$, then $y \in \mathbb{R}^*$ and $\coth(y) = x$. We have

$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \frac{\cosh^2 y - \sinh^2 y}{\sinh^2 y} = \frac{1}{\sinh^2 y} \Rightarrow \coth^2 y - 1 = \frac{1}{\sinh^2 y}.$$

So,

$$\operatorname{sinh} y = \frac{1}{\sqrt{\coth^2 y - 1}}.$$

Since $y = \operatorname{arcoth}(x)$ and $\coth(y) = x$, we get

$$\forall x \in ]-\infty, -1[ \cup ]1, +\infty[ : \sinh\big(\operatorname{arcoth}(x)\big) = \frac{1}{\sqrt{x^2 - 1}}.$$

**4**) $\operatorname{arcoth}(x)$ is differentiable on $]-\infty, -1[ \cup ]1, +\infty[$ where

$\forall x \in ]-\infty, -1[ \cup ]1, +\infty[$:

$$\big(\operatorname{arcoth}(x)\big)' = \frac{1}{\coth'\big(\operatorname{arcoth}(x)\big)} = \frac{1}{1 - \Big(\coth\big(\operatorname{arcoth}(x)\big)\Big)^2} = \frac{1}{-\frac{1}{\sinh^2 y}} = \frac{1}{1 - x^2}.$$

Or

$$\forall x \in ]-\infty, -1[ \cup ]1, +\infty[ : \big(\operatorname{arcoth}(x)\big)' = \frac{1}{(\coth y)'} = \frac{1}{1 - x^2}.$$

**5**) If $y = \operatorname{arcoth}(x)$, then $x \in ]-\infty, -1[ \cup ]1, +\infty[$, then $y \in \mathbb{R}^*$ and $\coth(y) = x$. We have

$$x = \coth(y) \Leftrightarrow x = \frac{e^{2y} + 1}{e^{2y} - 1} \Leftrightarrow e^{2y}(x - 1) = x + 1 \Leftrightarrow e^{2y} = \frac{x + 1}{x - 1}.$$

So,

$$\ln(e^{2y}) = \ln\left(\frac{x + 1}{x - 1}\right) \Leftrightarrow y = \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right).$$

Thus,

$$\forall x \in ]-\infty, -1[ \cup ]1, +\infty[ : y = \operatorname{artanh}(x) \Leftrightarrow y = \tfrac{1}{2}\ln\left(\tfrac{x+1}{x-1}\right).$$

# PART 2: ALGEBRA 1

# Chapter 4

## Groups, Rings and Fields

To the best of our knowledge, the concept of groups have a history of more than three centuries where it appeared for the first time in the nineteenth century in connection with the solution of equations. Recently this concept can be found in a bewildering number of subjects in various disciplines including physics and chemistry. So, the key aim of this chapter is to help students to master the primary concepts of groups, rings and fields.

### 4.1. Internal Binary Operations

**Definition 4.1.** Let $S$ be a nonempty set. An internal binary operation $*$ on $S$ is a function from $S \times S$ to $S$. So, $*$ assigns to each ordered pair of elements of $S$ a uniquely determined element of $S$. The element assigned to the ordered pair $(x, y)$ with $x, y \in S$ is denoted by $x * y$. Notationally, $*: S \times S \longrightarrow S$ such that

$$x * y \in S, \forall x, y \in S.$$

**Example 4.1.**

**1**) Addition, " $+$ ", multiplication, " $.$ " are internal binary operations on each of the following sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. While, " $+$ ", and " $.$ " are not internal binary operations on the set $S = \{1,2,3\}$. For example, $2 \in S$, $3 \in S$, but $2 + 3 = 5 \notin S$ and $2.3 = 6 \notin S$.
**2**) Subtraction, " $-$ " is an internal binary operation on each of the following sets: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, but not an internal binary operation on $\mathbb{N}$. For example, we have
$x = 5 \in \mathbb{N}$, $y = 7 \in \mathbb{N}$, but $x - y = -2 \notin \mathbb{N}$.
**3**) The vector product or the cross product on $\mathbb{R}^2$ is an internal binary operation on $\mathbb{R}^2$ (the inputs are two vectors in $\mathbb{R}^2$ and the output is another vector in $\mathbb{R}^2$), whereas the scalar product on $\mathbb{R}^2$ is not an internal binary operation on $\mathbb{R}^2$ (the inputs are two vectors in $\mathbb{R}^2$ but the output is a real number).
**4**) In $\mathbb{R} - \{2\}$, we define an operation $*$ by

$$\forall x, y \in \mathbb{R} - \{2\}: \; x * y = x\,y - 2(x + y) + 6.$$

We will determine whether $*$ is an internal binary operation on $\mathbb{R} - \{2\}$.

Clearly, if $x, y \in \mathbb{R}$, then $x * y \in \mathbb{R}$. But to prove $x * y \in \mathbb{R} - \{2\}$ we have to prove $x * y \neq 2$. To this end, we take two elements $x, y \in \mathbb{R}$ such that $x \neq 2$, $y \neq 2$ and discuss whether the value $x * y = 2$ or not. By contradiction, suppose that $x * y = 2$, then

$$\begin{aligned} x * y = 2 &\Leftrightarrow x\,y - 2(x + y) + 6 = 2 \\ &\Leftrightarrow x\,y - 2x - 2y + 6 = 2 \\ &\Leftrightarrow x(y - 2) = 2(y - 2) \\ &\Leftrightarrow x = 2\frac{y-2}{y-2} = 2 \text{ (since } y \neq 2). \end{aligned}$$

This contradict the fact that $x \neq 2$.Therefore, our assumption is wrong. So, $x * y \neq 2$. Thus

$$x * y \in \mathbb{R} - \{2\}, \forall x, y \in \mathbb{R} - \{2\},$$

and hence $*$ is an internal binary operation on $\mathbb{R} - \{2\}$.

**Remark 4.1.** It's worth mentioning that a nonempty set $S$ can be equipped with more than one internal binary operation.

## 4.2. Groups

**Definition 4.2.** Let $G$ be a nonempty set and let $*$ be an internal binary operation on $G$. Then $(G,*)$ is a group if the following axioms are fulfilled:

(**G1**) **Associativity**:

$$\forall x, y, z \in G: x * (y * z) = (x * y) * z.$$

(**G2**) **Identity element**: $G$ has exactly one identity element, which means that

$$\exists\, e \in G, \forall\, x \in G: x * e = e * x = x.$$

(**G3**) **Inverse**: Every element of $G$ has exactly one inverse, which means that

$$\forall\, x \in G, \exists\, x^{-1} \in G :\ x * x^{-1} = x^{-1} * x = e.$$

**Remark 4.2.** In the above definition

**1**) The fact that $*$ is an internal binary operation is an essential part of the definition.

**2**) The identity element $e \in G$ is called also a neutral element, unit element, two-sided identity or identity for short and $x^{-1} \in G$ is said to be the inverse of $x \in G$. Note that (**G2**) must precede (**G3**) because (**G3**) refers back to the identity element $e$.

**Remark 4.3.** If the internal binary operation $*$ is commutative, which means that

$$\forall x, y \in G: x * y = y * x,$$

then $(G,*)$ is called an abelian group, or simply a commutative group.

**Remark 4.4.** Let $G$ be a nonempty set and let $*$ be an internal binary operation on $G$.

**1)** $e \in G$ is said to be the left identity element for the internal binary operation $*$ if

$$e * x = x, \forall\, x \in G.$$

**2)** $e \in G$ is said to be the right identity element for the internal binary operation $*$ if

$$x * e = x, \forall\, x \in G.$$

**3)** Let $e \in G$ be the identity element for the internal binary operation $*$, then $x^{-1} \in G$ is said to be left inverse of $x \in G$ if

$$x^{-1} * x = e, \forall\, x \in G.$$

**4)** Let $e \in G$ be the identity element for the internal binary operation $*$, then $x^{-1} \in G$ is said to be right inverse of $x \in G$ if

$$x * x^{-1} = e, \forall\, x \in G.$$

**5)** For commutative internal binary operations, every left identity element is also a right identity element. So, it suffices to use one of the two aforementioned conditions $e * x = x$ or $x * e = x$ for determining the identity element of $G$.
Likewise, it suffices to use $x^{-1} * x = x$ or $x * x^{-1} = x$ for determining the inverse of each element $x \in G$.

**Example 4.2.**

**1)** $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are abelian groups.
**2)** In $\mathbb{R} - \{2\}$, we define an operation $*$ by

$$\forall x, y \in \mathbb{R} - \{2\}:\ x * y = x\,y - 2(x + y) + 6,$$

is $(G,*)$ an abelian group.
We have $\mathbb{R} - \{2\} \neq \phi$ and $*$ is an internal binary operation on $\mathbb{R} - \{2\}$ (see example 4.1).

**Commutativity**

We will try to prove

$$\forall x, y \in \mathbb{R} - \{2\}: x * y = y * x.$$

If $x, y \in \mathbb{R} - \{2\}$, then

$$x * y = x\,y - 2(x + y) + 6 = y\,x - 2(y + x) + 6 = y * x.$$

Therefore, $*$ is commutative.

**Associativity**

We will try to prove

$$\forall x, y, z \in \mathbb{R} - \{2\}: \underbrace{x * (y * z)}_{M_1} = \underbrace{(x * y) * z}_{M_2}.$$

If $x, y, z \in \mathbb{R} - \{2\}$, then

$$\begin{aligned} M_1 &= x * (y * z) = x * (y\,z - 2(y + z) + 6) \\ &= x\,(y\,z - 2(y + z) + 6) - 2(x + y\,z - 2(y + z) + 6) + 6 \\ &= x\,y\,z - 2x\,y - 2x\,z - 2y\,z + 4x + 4y + 4z - 6, \end{aligned}$$

and

$$\begin{aligned} M_2 &= (x * y) * z = (x\,y - 2(x + y) + 6) * z \\ &= (xy - 2(x + y) + 6)\,z - 2(x\,y - 2(x + y) + 6 + z) + 6 \\ &= \ x\,yz - 2xz - 2yz - 2xy + 4x + 4y + 4z - 6. \end{aligned}$$

Since $M_1 = M_2$, then $*$ is associative.

**Identity element**

We will try to prove

$$\exists\, e \in \mathbb{R} - \{2\}, \forall\, x \in \mathbb{R} - \{2\}: x * e = e * x = x.$$

Since $*$ is commutative it suffices to solve $x * e = x$. We have

$$x * e = x \Leftrightarrow xe - 2(x + e) + 6 = x \Leftrightarrow e(x - 2) = 3x - 6$$

$$\Leftrightarrow e = \frac{3x - 6}{x - 2} = 3\frac{x - 2}{x - 2} = 3 \text{ (it is possible since } x \neq 2).$$

Thus, there exists an identity element $e = 3 \in \mathbb{R} - \{2\}$ for the internal binary operation $*$.

(**G3**) Inverse

$$\forall\, x \in \mathbb{R} - \{2\}, \exists\, x^{-1} \in G:\ x * x^{-1} = x^{-1} * x = e.$$

Since $*$ is commutative it suffices to solve $x * x^{-1} = e$. We have

$$x * x^{-1} = e \Leftrightarrow x.x^{-1} - 2(x + x^{-1}) + 6 = 3$$

$$\Leftrightarrow\ x^{-1}(x - 2) = 2x - 3 \Leftrightarrow\ x^{-1} = \frac{2x - 3}{x - 2}.$$

Clearly, $x^{-1} = \frac{2x-3}{x-2} \in \mathbb{R}$. We show that $x^{-1} = \frac{2x-3}{x-2} \neq 2$. By contradiction, suppose that $x^{-1} = 2$, then

$$x^{-1} = \frac{2x - 3}{x - 2} = 2 \Leftrightarrow 2x - 3 = 2(x - 2) = 2x - 4 \Leftrightarrow 3 = 4.$$

This a contradiction. Then, our assumption is wrong and hence

$$x^{-1} = \frac{2x-3}{x-2} \neq 2.$$

Thus, every element $x$ of $\mathbb{R} - \{2\}$ has an inverse $x^{-1} = \frac{2x-3}{x-2} \in \mathbb{R} - \{2\}$ for $*$.

Since $\mathbb{R} - \{2\} \neq \phi$, $*$ is an internal binary operation on $\mathbb{R} - \{2\}$, $*$ is commutative, $*$ is associative, there exists an identity element $e = 3$ for $*$ in $\mathbb{R} - \{2\}$, and every element $x$ of $\mathbb{R} - \{2\}$ has an inverse $x^{-1} = \frac{2x-3}{x-2} \in \mathbb{R} - \{2\}$ for $*$. Then $(\mathbb{R} - \{2\}, *)$ is an abelian group.

**Theorem 4.1.** Let $(G,*)$ be a group. Then

**1**) The identity element is unique.

**2**) The inverse of any element $x \in G$ is unique.

**Example 4.3.**

**1**) 0 is the unique identity element of $\mathbb{R}$ where the internal binary operation $*$ is the addition, i.e.,($* = +$) and $-x$ is the unique inverse of every element $x \in \mathbb{R}$.

**2**) 1 is the unique identity element of $\mathbb{R} - \{0\}$ where the internal binary operation $*$ is the multiplication, i.e., ($* = .$) and $\frac{1}{x}$ is the unique inverse of every element $x \in \mathbb{R} - \{0\}$.

### 4.2.1. Subgroups

**Definition 4.3.** Let $(G,*)$ be a group. A nonempty subset $H$ of $G$ is called a subgroup of $G$ if the following conditions are satisfied:

(**a**) $\forall x, y \in H: x * y \in H$.

(**b**) $e \in H$, where $e$ is the identity of $G$.

(**c**) $\forall x \in H: x^{-1} \in H$.

**Remark 4.5.** We do not need to check the associative property in $H$, because it is inherited directly from $G$.

**Example 4.4.**

**1**) $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$, $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$ and $(\mathbb{R}, +)$ is a subgroup of $(\mathbb{C}, +)$.

**2**) Consider the group $(\mathbb{R}^3, +)$ where the the internal binary operation $+$ is defined by: $\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_1, z_1 + z_2).$$

Let $H$ a subset of $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : x - y + 2z = 0\}.$$

Show that $(\mathrm{H}, +)$ is a subgroup of $(\mathbb{R}^3, +)$.

(**a**) Let $X = (x_1, y_1, z_1), Y = (x_2, y_2, z_2) \in H$. We have

$$X = (x_1, y_1, z_1) \in H \Longrightarrow x_1 - y_1 + 2z_1 = 0 \dots (1)$$

and

$$Y = (x_2, y_2, z_2) \in H \Longrightarrow x_2 - y_2 + 2z_2 = 0 \dots (2)$$

Adding these two equations together, we get

$$x_1 + x_2 - (y_1 + y_1) + 2(z_1 + z_2) = 0.$$

So, $X + Y = (x_1 + x_2, y_1 + y_1, z_1 + z_2) \in H$.

(**b**) $e = (0,0,0)$ is the identity of $(\mathbb{R}^3, +)$. So, $e \in H$ since $0 - 0 + 2(0) = 0$.

(**c**) Let $X = (x, y, z) \in \mathbb{R}^3$, then $X^{-1} = (-x, -y, -z)$. If $X \in H$, then

$$x - y + 2z = 0.$$

Since

$$-x - (-y) + 2(-z) = -(x - y + 2z) = 0,$$

then $X^{-1} \in H$.

From **(a)**, **(b)**, and **(c)**, $(H, +)$ is a subgroup of $(\mathbb{R}^3, +)$.

## 4.2. Rings

**Definition 4.4.** Let $R$ be a nonempty set and let $*$ and $\perp$ be two internal binary operations defined on $R$. Then $(R, *, \perp)$ is a ring if the following axioms are satisfied:

(**A1**) $(R, *)$ is an abelian group; that is,

**(i)** $\forall x, y, z \in R: x * (y * z) = (x * y) * z$.

**(ii)** $\exists\, e \in R, \forall\, x \in R: x * e = e * x = x$.

**(iii)** $\forall\, x \in R, \exists\, x^{-1} \in R :\ x * x^{-1} = x^{-1} * x = e$.

**(iv)** $\forall x, y \in R: x * y = y * x$.

**(A2)** $\perp$ is associative:

$$\forall x,y,z \in R: x \perp (y \perp z) = (x \perp y) \perp z.$$

**(A3)** $\perp$ is distributive over $*$:

$$\forall x,y,z \in R: \begin{cases} x \perp (y * z) = (x \perp y) * (x \perp z) \\ \text{and} \\ (y * z) \perp x = (y \perp x) * (z \perp x) \end{cases}.$$

**Remark 4.6.**

**1**) We say that $(R,*,\perp)$ is a commutative ring if the internal binary operation $\perp$ is commutative, which means that

$$\forall x,y \in R: x \perp y = y \perp x.$$

**2**) We say that $(R,*,\perp)$ is a unital ring if there is an identity element $\mathbf{1}_R$ for $\perp$, which means that

$$\exists\ \mathbf{1}_R \in R, \forall\ x \in R: x \perp \mathbf{1}_R = \mathbf{1}_R \perp x = x.$$

**3**) We denote by $e \in R$ the identity element for the internal binary operation $*$ and by $\mathbf{1}_R$ the identity element for the internal binary operation $\perp$.

**Example 4.5.** $(\mathbb{Z},+,.)$, $(\mathbb{Q},+,.)$, $(\mathbb{R},+,.)$ and $(\mathbb{C},+,.)$ are all commutative rings where

$e = 0$ : is the identity element for the addition operation,

and

$\mathbf{1}_R = 1$: is the identity element for the multiplication operation.

### 4.2.1. Basic Properties of Operations in a Ring

**Theorem 4.1.** Suppose that $(R,*,\perp)$ a ring. Then

**1)** $e \perp x = x \perp e = e, \forall x \in R.$

**2)** $x^{-1} \perp y = (x \perp y)^{-1} = x \perp y^{-1}, \forall x,y \in R.$

**3)** $x^{-1} \perp y^{-1} = x \perp y, \forall x,y \in R.$

**Proof.**

**1**) Since $e = e * e$ we have

$$e \perp x = (e * e\ ) \perp x = (e \perp x) * (e \perp x)$$

$$\Longrightarrow e \perp x = e.$$

In the same way, we get

$$x \perp e = x \perp (e * e\,) = (x \perp e) * (x \perp e) \Longrightarrow x \perp e = e.$$

**2**) To show that $x^{-1} \perp y = (x \perp y)^{-1}$, it suffices to prove that

$$(x \perp y) * (x^{-1} \perp y) = e.$$

We have

$$(x \perp y) * (x^{-1} \perp y) = (x * x^{-1}) \perp y = e \perp y = e.$$

Likewise, we have

$$(x \perp y^{-1}) * (x \perp y) = x \perp (y^{-1} * y) = x \perp e = e.$$

**3**) Using the second property twice, we obtain the last property as follows:

$$x^{-1} \perp y^{-1} = (x \perp y^{-1})^{-1} = x \perp y. \qquad \blacksquare$$

## 4.2. 2. Subring

**Definition 4.5.** Let $(R,*,\perp)$ be a ring. A nonempty subset $S$ of $R$ is called a subring of $R$ if the following axioms are satisfied:
**(a)** $(S,*)$ is a subgroup of $(R,*)$.

**(b)** $x \perp y \in S, \forall x, y \in S$.

## 4.3. Field

**Definition 4.6.** Let $F$ ne a nonempty set and let $*$ and $\perp$ be two internal binary operations defined on $F$. Then $(F,*,\perp)$ is a field if the following axioms are satisfied:

**(F1)** $(F,*,\perp)$ is a ring.

**(F2)** $(F - \{e\}, \perp)$ is a group, where $e$ is the identity element for $*$.

**Example 4.6.**

**1**) $(\mathbb{R}, +, .)$ is a field.
**2**) In $\mathbb{R}$, we define two internal binary operations $*$ and $\perp$ as follows:

$$\forall\, x, y \in \mathbb{R} : \; x * y = x + y - \frac{1}{2},$$

and

$$\forall\, x, y \in \mathbb{R} : \; x \perp y = x + y - 2xy.$$

Then $(\mathbb{R},*,\perp)$ is a field.

# Chapter 5

## Vector Spaces

The concept of a vector space dates back to the year 1844. The first timid attempts have been made by the German mathematician Hermann Grassmann but it has became well-established with the work of the Polish mathematician Stephan Banach. this concept which appears in many contexts is an algebraic structure consisting of a set on which are defined a binary operation referred to as addition, and an operation of multiplication by scalars.

This chapter introduces vector spaces, their basic properties and the associated notions such as linear combinations, linear independence, bases and dimension.

### 5.1. External Binary Operations

**Definition 5.1.** Let $F$ and $V$ be two nonempty sets. An **external binary operation** $*$ on $V$ is a function from $F \times V$ to $V$. So, $*$ assigns to each ordered pair of elements of $F \times V$ a uniquely determined element of $V$. The element assigned to the ordered pair $(x, y)$ with $x \in F, y \in V$ is denoted by $x * y$. Notationally, $*: F \times V \longrightarrow V$ such that

$$x * y \in V, \forall (x, y) \in F \times V.$$

### 5.2. Vector Space

**Definition 5.2.** Let $(F, *, \perp)$ be a field. A nonempty set $V$ is called a **vector space** (also called a **linear space**) over $F$ if there is an internal binary operation

$$+: V \times V \longrightarrow V,$$

called **vector addition** and an external binary operation

$$\cdot : F \times V \longrightarrow V,$$

called **scalar multiplication**, such that the following properties are satisfied:

**(V1)** $(V, +)$ is an abelian group, that is,

**(i)** $\forall x, y, z \in V: x + (y + z) = (x + y) + z.$

**(ii)** $\exists\, e \in V, \forall\, x \in V: x + e = e + x = x.$

**(iii)** $\forall\, x \in V, \exists\, x^{-1} \in V:\ x + x^{-1} = x^{-1} + x = e.$

**(iv)** $\forall x, y \in V: x + y = y + x.$

**(V2)** $\forall \alpha, \beta \in F, \forall x \in V, (\alpha * \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$.

**(V3)** $\forall \alpha \in F, \forall x, y \in \mathrm{V}, \alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$.

**(V4)** $\forall \alpha, \beta \in F, \forall x \in V, (\alpha \perp \beta) \cdot x = \alpha \cdot (\beta \cdot x)$.

**(V5)** $\forall x \in V, \mathbf{1}_F \cdot x = x$ ($\mathbf{1}_F$ is the identity element for the operation $\perp$).

**Remark 5.1.**
**1**) A vector space needs two nonempty sets $V$ and $F$, and four operations (two internal operations for the field $F$ and two operations, internal and external for the vector space $V$).
**2**) Let $V$ a vector space over the field $F$. The elemens of $V$ are called **vectors**, while the elements of $F$ are called **scalars**.

**3**) A vector space over the field $\mathbb{R}$ is called a **real vector space** and a vector space over the field $\mathbb{C}$ is called a **complex vector space**.

**Example 5.1.** The set $(\mathbb{R}^n, +, \cdot)$ of all ordered ***n*-tuples** $(x_1, x_2, \dots, x_\mathrm{n})$ where $x_1, x_2, \dots, x_\mathrm{n}$ are all real numbers is a vector space over the field $\mathbb{R}$ where the operation vector addition " $+$ " is defined as follows:

$\forall X_1 = (x_1, x_2, \dots, x_\mathrm{n}), X_2 = (y_1, y_2, \dots, y_\mathrm{n}) \in \mathbb{R}^n$:

$$(x_1, x_2, \dots, x_\mathrm{n}) + (y_1, y_2, \dots, y_\mathrm{n}) = (x_1 + y_1, x_2 + y_2, \dots, y_\mathrm{n} + y_\mathrm{n}),$$

and the operation scalar multiplication " $\cdot$ " is defined as follows:

$\forall \alpha \in \mathbb{R}, \forall X = (x_1, x_2, \dots, x_\mathrm{n}) \in \mathbb{R}^n$:

$$\alpha \cdot (x_1, x_2, \dots, x_\mathrm{n}) = (\alpha \cdot x_1, \alpha \cdot x_2, \dots, \alpha \cdot x_\mathrm{n}).$$

An $n$-tuples $(x_1, x_2, \dots, x_\mathrm{n})$ is called a vector of the vector space $\mathbb{R}^n$, and $x_1, x_2, \dots, x_\mathrm{n}$ are called **components** of the vector.

The identity element of this vector space (for the operation of vector addition '+') is $0_{\mathbb{R}^n} = \underbrace{(0,0, \dots, 0)}_{\text{n times}}$.

## 5.3. Subspaces

**Definition 5.3.** Let $(V, +, \cdot)$ be a vector space over a field $F$. A subset $W$ of $V$ is a **sub-space** of $V$ if
**(a)** $W \neq \Phi$.

**(b)** $\forall x, y \in W: x + y \in W$.

**(c)** $\forall \alpha \in F, \forall x \in W: \alpha \cdot x \in W$.

**Theorem 5.1.** Let $(V,+,\cdot)$ be a vector space over a field $F$. A subset $W$ of $V$ is a subspace of $V$ if

**(a)** $W \neq \Phi$.

**(b)** $\forall \alpha, \beta \in F, \forall x, y \in W: (\alpha \cdot x + \beta \cdot y) \in W$.

**Remark 5.2.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $0_V$ be the identity element of $V$, i.e., the identity element for the operation of vector addition '+'.

**1**) $V$ and $\{0_V\}$ are subspaces of $V$.

**2**) If $W$ is a subspace of $V$, then $0_V \in W$. In other words, if $W \neq \Phi$ and $0_V \notin W$, then $W$ is not a subspace of $V$.

**Example 5.2.**

**1**) In $\left(\mathbb{R}^2,+,.\right)$ we define a subset $W_1$ by

$$W_1 = \left\{X = (x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}.$$

**(a)** We have $0_{\mathbb{R}^2} = (0,0) \in W_1$ (since the second component is zero). Thus $W_1 \neq \Phi$.

**(b)** Let $\alpha, \beta \in \mathbb{R}$ and $X_1 = (x_1, y_1), X_2 = (x_2, y_2) \in W_1$. Then

$$X_1 = (x_1, y_1) \in W_1 \Rightarrow y_1 = 0,$$

and

$$X_2 = (x_2, y_2) \in W_1 \Rightarrow y_2 = 0.$$

We have

$$\begin{aligned}\alpha \cdot X_1 + \beta \cdot X_2 &= \alpha \cdot (x_1, y_1) + \beta \cdot (x_2, y_2)\\ &= (\alpha \cdot x_1, \alpha \cdot y_1) + (\beta \cdot x_2, \beta \cdot y_2)\\ &= (\alpha \cdot x_1, 0) + (\beta \cdot x_2, 0)\\ &= (\alpha \cdot x_1 + \beta \cdot x_2, 0) \in W_1.\end{aligned}$$

From **(a)** and **(b)**, $W_1$ is a subspace of $\mathbb{R}^2$.

**2**) In $\left(\mathbb{R}^2,+,.\right)$ we define a subset $W_2$ by

$$W_2 = \left\{X = (x,1) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}.$$

We have $0_{\mathbb{R}^2} = (0,0) \notin W_2$. So $W_2$ is not a subspace of $\mathbb{R}^2$.

### 5.3.1. Intersection of Subspaces

**Theorem 5.2.** Let $(V,+,\cdot)$ be a vector space over a field $F$. If $W_1$ and $W_2$ are two subspaces of $V$, then $W_1 \cap W_2$ is a subspace of $V$.

**Proof.**
(**a**) We have $0_V \in W_1$ and $0_V \in W_2$, so $0_V \in W_1 \cap W_2$. Thus $W_1 \cap W_2 \neq \Phi$.
(**b**) Let $x, y \in W_1 \cap W_2$, then $x, y \in W_1$ and $x, y \in W_2$. Since $W_1$ and $W_2$ are two subspaces of $V$, then

$$\forall \alpha, \beta \in F: (\alpha \cdot x + \beta \cdot y) \in W_1,$$

and

$$\forall \alpha, \beta \in F: (\alpha \cdot x + \beta \cdot y) \in W_2.$$

So

$$\forall \alpha, \beta \in F, \forall x, y \in W_1 \cap W_2: (\alpha \cdot x + \beta \cdot y) \in W_1 \cap W_2.$$

From (**a**) and (**b**), $W_1 \cap W_2$ is a subspace of $V$. ■

### 5.3.2. Union of Subspaces

**Remark 5.3.** Let $(V,+,\cdot)$ be a vector space over a field $F$. The union $W_1 \cup W_2$ of two subspaces $W_1$ and $W_2$ of $V$ is not necessarily a subspace of $V$. Indeed, in $\left(\mathbb{R}^2,+,.\right)$ we define two subspaces $W_1$ and $W_2$ by

$$W_1 = \left\{X = (x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\right\} \text{ and } W_2 = \left\{X = (0,y) \in \mathbb{R}^2 : y \in \mathbb{R}\right\}.$$

(**a**) We have $0_V \in W_1$ and $0_V \in W_2$, so $0_V \in W_1 \cup W_2$. Thus $W_1 \cup W_2 \neq \Phi$.

(**b**) Let $\alpha = \beta = 1 \in \mathbb{R}$, $X_1 = (2,0)$ and $X_2 = (0,4)$. We see that $X_1 \in W_1 \cup W_2$ and $X_2 \in W_1 \cup W_2$ (since $X_1 \in W_1$and $X_2 \in W_2$), while

$$\alpha \cdot X_1 + \beta \cdot X_2 = 1 \cdot (2,0) + 1 \cdot (0,4) = (2,0) + (0,4) = (2,4).$$

Since $(2,4) \notin W_1$ and $(2,4) \notin W_2$, then $(2,4) \notin W_1 \cup W_2$.

So, $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$.

### 5.3.3. Sums and Direct Sums of Subspaces

**Definition 5.4.** Let $W_1$ and $W_2$ be two subspaces of a vector space $(V,+,\cdot)$. The **sum** of $W_1$ and $W_2$ is defined by

$$W_1 + W_2 = \{x_1 + x_2 : x_1 \in W_1, x_2 \in W_2\}.$$

**Theorem 5.3.** If $W_1$ and $W_2$ are two subspaces of a vector space $(V,+,\cdot)$, then $W_1 + W_2$ is a subspace of $V$.

**Definition 5.5.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $W_1$ and $W_2$ be subspaces of $V$. Then $V$ is said to be the **direct sum** of $W_1$ and $W_2$, and we write

$V = W_1 \oplus W_2$, if

**1)** $V = W_1 + W_2$.

**2)** $W_1 \cap W_2 = \{0_V\}$.

**Theorem 5.4.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $W_1$ and $W_2$ be subspaces of $V$. Then $V = W_1 \oplus W_2$ if and only if for every $x \in V$ there exist unique vectors $x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$.

**Example 5.3.** In the vector space $\left(\mathbb{R}^2,+,.\right)$ over $\mathbb{R}$ we define two subspaces $W_1$ and $W_2$ by

$$W_1 = \left\{X = (x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\right\},$$

and

$$W_2 = \left\{X = (0,y) \in \mathbb{R}^2 : y \in \mathbb{R}\right\}.$$

**(1)** Let $X = (x,y) \in \mathbb{R}^2$. Then

$$X = (x,y) = \underbrace{(x,0)}_{\in W_1} + \underbrace{(0,y)}_{\in W_2}.$$

So, $V = W_1 + W_2$.

**(2)** We have

$$W_1 \cap W_2 = \left\{X = (x,y) \in \mathbb{R}^2 : (x,y) \in W_1 \text{ and } (x,y) \in W_2\right\}.$$

But

$$(x,y) \in W_1 \Longrightarrow x = 0 \text{ and } (x,y) \in W_2 \Longrightarrow y = 0$$

So

$$W_1 \cap W_2 = \{(0,0)\} = \left\{0_{\mathbb{R}^2}\right\}.$$

From **(1)** and **(2)**, $\mathbb{R}^2 = W_1 \oplus W_2$.

## 5.4. Linear Combination

**Definition 5.6.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ and $x$ be vectors in $V$. We say that $x$ is a **linear combination** of $x_1, x_2, \dots, x_n$ if

$$\exists \alpha_1, \alpha_2, \dots, \alpha_n \in F: x = \alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \cdots + \alpha_n \cdot x_n.$$

We call $\alpha_1, \alpha_2, \dots, \alpha_n$ the **coefficients** of the linear combination.

**Remark 5.4.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$. If $0_V$ is the identity element of $V$ (for the operation of vector addition '+'). Then $0_V$ is always a linear combination of $x_1, x_2, \dots, x_n$. Indeed,

$$\exists \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0_F \in F: 0_V = 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n.$$

### 5.4.1. Span

**Definition 5.7.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$. The **span** of $x_1, x_2, \dots, x_n$, denoted $\operatorname{span}(x_1, x_2, \dots, x_n)$ is the set of all linear combinations of $x_1, x_2, \dots, x_n$. Notationally,

$$\operatorname{span}(x_1, x_2, \dots, x_n) = \{\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \cdots + \alpha_n \cdot x_n : \alpha_1, \alpha_2, \dots, \alpha_n \in F\}.$$

**Remark 5.5.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$.

**1)** $\operatorname{span}(\phi) = \{0_V\}$.

**2)** $\operatorname{span}(x_1, x_2, \dots, x_n)$ is a subspace of $V$.

**3)** $\{x_1, x_2, \dots, x_n\} \subseteq \operatorname{span}(x_1, x_2, \dots, x_n)$.

**4)** If $W$ is any subspace of $V$, then

$$\{x_1, x_2, \dots, x_n\} \subseteq W \Longrightarrow \operatorname{span}(x_1, x_2, \dots, x_n) \subseteq W.$$

In other terms, $\operatorname{span}(x_1, x_2, \dots, x_n)$ is the **smallest subspace** of $V$ which contains $\{x_1, x_2, \dots, x_n\}$.

**Definition 5.8.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$. We say that the set $\{x_1, x_2, \dots, x_n\}$ **spans** or **generates** $V$ if

$$\operatorname{span}(x_1, x_2, \dots, x_n) = V.$$

**Example 5.4.**

**1)** In the vector space $\left(\mathbb{R}^2, +, .\right)$ over $\mathbb{R}$ we consider the vectors $e_1 = (1,0)$ and $e_2 = (0,1)$. On the one hand,

$$\operatorname{span}(e_1, e_2) \subseteq \mathbb{R}^2.$$

On the other hand, if $(x, y) \in \mathbb{R}^2$, then

$$(x, y) = (x, 0) + (0, y) = x \cdot (1,0) + y \cdot (0,1).$$

So,

$$\forall X = (x, y) \in \mathbb{R}^2, \exists \alpha_1 = x, \alpha_2 = y \in \mathbb{R}: (x, y) = \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2.$$

Thus $\mathbb{R}^2 \subseteq \operatorname{span}(e_1, e_2)$. Consequently,

$$\operatorname{span}(e_1, e_2) = \mathbb{R}^2,$$

which means the set $\{e_1, e_2\}$ spans $\mathbb{R}^2$.

**2**) Similarly, if $x_1 = (1,1)$ and $x_2 = (2,0)$, then $\operatorname{span}(x_1, x_2) \subseteq \mathbb{R}^2$. Furthermore, if $(x, y) \in \mathbb{R}^2$, then

$$\begin{aligned}(x, y) = \alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 &\Leftrightarrow (x, y) = \alpha_1 \cdot (1,1) + \alpha_2 \cdot (2,0)\\ &\Leftrightarrow (x, y) = (\alpha_1 + 2\alpha_2, \alpha_1),\end{aligned}$$

which implies that

$$\begin{cases}\alpha_1 + 2\alpha_2 = x\\ \alpha_1 = y\end{cases} \Rightarrow \begin{cases}\alpha_2 = \frac{x}{2} - \frac{y}{2}.\\ \alpha_1 = y\end{cases}$$

So,

$$\forall X = (x, y) \in \mathbb{R}^2, \exists \alpha_1 = y, \alpha_2 = \frac{x}{2} - \frac{y}{2} \in \mathbb{R}: (x, y) = \alpha_1 \cdot x_1 + \alpha_2 \cdot x_2.$$

Thus $\mathbb{R}^2 \subseteq \operatorname{span}(x_1, x_2)$. Consequently, $\operatorname{span}(x_1, x_2) = \mathbb{R}^2$,and the set $\{x_1, x_2\}$ spans $\mathbb{R}^2$.

**Remark 5.6.** Usually there are many different subsets which are able to generate the same vector space.

### 5.4.2. Linear Independence

**Definition 5.9.** Let $(V, +, \cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$. We say that vectors $x_1, x_2, \dots, x_n$ are **linearly independent** if for all $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ the equation

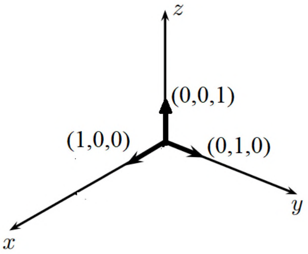
$$\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \cdots + \alpha_n \cdot x_n = 0_V,$$

can only be satisfied if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0_F$.

**Remark 5.7.** All set consists of a single vector $x \neq 0_V$ is linearly independent.

**Example 5.5.**

**1)** In the vector space $(\mathbb{R}^3, +, .)$ over $\mathbb{R}$, we consider the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.



Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ . If

$$\alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \alpha_3 \cdot e_3 = 0_{\mathbb{R}^3},$$

then

$$\alpha_1 \cdot (1,0,0) + \alpha_2 \cdot (0,1,0) + \alpha_3 \cdot (0,0,1) = (0,0,0) \Leftrightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0).$$

So,

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence, $e_1, e_2$ and $e_3$ are linearly independent.

**2)** In the vector space $(\mathbb{R}^3, +, .)$ over $\mathbb{R}$, we consider the vectors $v_1 = (0,1,-1)$, $v_2 = (1,0,-1)$ and $v_3 = (1,1,0)$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ . If

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0_{\mathbb{R}^3},$$

then

$$\alpha_1 \cdot (0,1,-1) + \alpha_2 \cdot (1,0,-1) + \alpha_3 \cdot (1,1,0) = (0,0,0),$$

gives

$$(\alpha_2 + \alpha_3,\ \alpha_1 + \alpha_3, -\alpha_1 - \alpha_2) = (0,0,0).$$

So,

$$\begin{cases} \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ -\alpha_1 - \alpha_2 = 0 \end{cases} \Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Thus $v_1, v_2$ and $v_3$ are linearly independent.

### 5.4.3. Linear Dependence

**Definition 5.10.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$. We say that vectors $x_1, x_2, \dots, x_n$ are **linearly dependent** if they are not linearly independent; that is, if there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that at least one $\alpha_i \neq 0,\ i = \overline{1,n}$ and

$$\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \dots + \alpha_n \cdot x_n = 0_V.$$

**Example 5.6.**

In the vector space $\left(\mathbb{R}^2, +, .\right)$ over $\mathbb{R}$ we consider the vectors $v_1 = (1,0)$, $v_2 = (0,1)$ and $v_3 = (2,3)$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. If

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0_{\mathbb{R}^2},$$

then

$$\alpha_1 \cdot (1,0) + \alpha_2 \cdot (0,1) + \alpha_3 \cdot (2,3) = (0,0),$$

leads to

$$(\alpha_1 + 2\alpha_3, \alpha_2 + 3\alpha_3) = (0,0).$$

So,

$$\begin{cases} \alpha_1 + 2\alpha_3 = 0 \\ \alpha_2 + 3\alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -2\alpha_3 \\ \alpha_2 = -3\alpha_3 \end{cases}.$$

If $\alpha_3 = 1$, then $\alpha_1 = -2$, $\alpha_2 = -3$ and

$$\begin{aligned} \alpha_1 \cdot (1,0) + \alpha_2 \cdot (0,1) + \alpha_3 \cdot (2,3) &= -2 \cdot (1,0) - 3 \cdot (0,1) + 1 \cdot (2,3) \\ &= (0,0). \end{aligned}$$

Thus $v_1, v_2$ and $v_3$ are linearly dependent.

**Remark 5.8.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$.

**1**) If one of the vectors $x_1, x_2, \dots, x_n$ equals $0_V$ then $x_1, x_2, \dots, x_n$ are linearly dependent.

**2**) $x_1, x_2, \dots, x_n$ are linearly dependent if and only if at least one of the vectors $x_1, x_2, \dots, x_n$ can be written as a linear combination of the remaining vectors of $x_1, x_2, \dots, x_n$.

**Example 5.7.**

In the vector space $\left(\mathbb{R}^3,+,.\right)$ over $\mathbb{R}$ we consider the vectors

$$v_1 = (2,-2,-1)\ ,\ v_2 = (1,1,1),\ v_2 = (1,2,3),\text{ and } v_4 = (2,-1,1).$$

We have

$$(2,-2,-1) = 1\cdot(1,1,1) - 1\cdot(1,2,3) + 1\cdot(2,-1,1).$$

So, $v_1$ is a linear combination of $v_2$, $v_3$ and $v_4$. Hence, the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

## 5.5. Bases and Dimension

### 5.5.1. Bases

**Definition 5.11.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let $x_1, x_2, \dots, x_n$ be vectors in $V$. We say that the set $\{x_1, x_2, \dots, x_n\}$ forms a **basis** of the vector space $V$ if

**1**) $\{x_1, x_2, \dots, x_n\}$ is linearly independent.

**2**) $\{x_1, x_2, \dots, x_n\}$ generates $V$, i.e.,

$$\operatorname{span}(x_1, x_2, \dots, x_n) = V.$$

**Example 5.8.**

In the vector space $(\mathbb{R}^n,+,\cdot)$, the set

$$\mathcal{B} = \{e_1 = (1,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_n = (0,0,\dots,0,1)\},$$

forms a basis of $\mathbb{R}^n$. Indeed,

**1**) Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ . If

$$\alpha_1\cdot e_1 + \alpha_2\cdot e_2 + \cdots + \alpha_n\cdot e_n = 0_{\mathbb{R}^n},$$

then

$$\alpha_1\cdot(1,0,\dots,0) + \alpha_2\cdot(0,1,0,\dots,0) + \cdots + \alpha_n\cdot(0,0,\dots,1) = (0,0,\dots,0),$$

which implies that

$$(\alpha_1, \alpha_2, \dots, \alpha_n) = (0,0,\dots,0).$$

So,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Thus $\{e_1, e, \dots, e_n\}$ is linearly independent.

**2**) Since $\{e_1, e, \dots, e_n\} \subset \mathbb{R}^n$, then

$$\operatorname{span}(e_1, e, \dots, e_n) \subseteq \mathbb{R}^n.$$

On the other hand, if $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$\begin{aligned}(x_1, x_2, \dots, x_n) &= (x_1, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, 0, \dots, x_n)\\ &= x_1 \cdot (1, 0, \dots, 0) + x_2 \cdot (0, 1, 0, \dots, 0) + \dots + x_n \cdot (0, 0, \dots, 1).\end{aligned}$$

So, for all $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\exists \alpha_1 = x_1, \alpha_2 = x_2, \dots, \alpha_n = x_n \in \mathbb{R}$ such that

$$\alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \dots + \alpha_n \cdot e_n = (x_1, x_2, \dots, x_n).$$

Thus $\mathbb{R}^n \subseteq \text{span}(e_1, e, \dots, e_n)$.

Since $\text{span}(e_1, e, \dots, e_n) \subseteq \mathbb{R}^n$ and $\mathbb{R}^n \subseteq \text{span}(e_1, e, \dots, e_n)$, then

$$\text{span}(e_1, e, \dots, e_n) = \mathbb{R}^n,$$

and the set $\{e_1, e, \dots, e_n\}$ spans $\mathbb{R}^n$.

Since $\{e_1, e, \dots, e_n\}$ is linearly independent and generates $\mathbb{R}^n$, then $\{e_1, e, \dots, e_n\}$ forms a basis of the vector space $\mathbb{R}^n$.

**Remark 5.9.** The basis

$$\mathcal{B} = \{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0, 0), \dots, e_n = (0, 0, \dots, 0, 1)\},$$

of the vector space $\mathbb{R}^n$ is called the **standard basis** (also **canonical** or **natural** basis) for $\mathbb{R}^n$.

**Remark 5.10.**

**1**) Usually there are many different subsets which are able to form a basis of the same vector space.

**2**) If a vector space $V$ has a basis consisting of $n$ elements, then any other basis for $V$ has $n$ elements.

**Example 5.9.**

According to Example 5.4., if $x_1 = (1,1)$ and $x_2 = (2,0)$, then the set $\{x_1, x_2\}$ spans $\mathbb{R}^2$. Furthermore, if $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ then

$$\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 = 0_{\mathbb{R}^2} \Leftrightarrow \alpha_1 \cdot (1,1) + \alpha_2 \cdot (2,0) = (0,0),$$

And hence

$$(\alpha_1 + 2\alpha_2, \alpha_2) = (0,0) \Longrightarrow \alpha_1 = \alpha_2 = 0.$$

So, $\{x_1, x_2\}$ is linearly independent. Since $\{x_1, x_2\}$ is linearly independent and generates $\mathbb{R}^2$, then $\mathcal{B}_1 = \{x_1, x_2\}$ forms a basis of the vector space $\mathbb{R}^2$.

According to Example 5.4., if $e_1 = (1,0)$ and $e_2 = (0,1)$, then

$$\mathcal{B}_2 = \{e_1, e_2\},$$

form a basis of the vector space $\mathbb{R}^2$.

Thus

$$\mathcal{B}_1 = \{x_1, x_2\} \text{ and } \mathcal{B}_2 = \{e_1, e_2\},$$

are two different basis of the same vector space $\mathbb{R}^2$.

### 5.5.2. Dimension

**Definition 5.12.** Let $(V, +, \cdot)$ be a vector space over a field $F$ and let $\mathcal{B}$ be a basis for $V$. The **dimension** of the vector space $V$, denoted by $\dim(V)$, is the number of the vectors of $\mathcal{B}$.

**Remark 5.11.** Let $(V, +, \cdot)$ be a vector space over a field $F$. We have

$$\dim\{0_V\} = 0.$$

**Remark 5.12.** According to Remark 5.9., even if a vector space $V$ has more than one basis, the dimension does not change.

**Example 5.10.**

$$\dim\left(\mathbb{R}^n\right) = n.$$

**Theorem 5.5.** Let $(V, +, \cdot)$ be a vector space of dimension $n$ ($\dim(V) = n$) over a field $F$.

**1**) Every linearly independent subset of $V$ with $n$ elements is a basis for $V$.

**2**) If $V$ is generated by a subset of $V$ with $n$ elements, then this subset is a basis for $V$.

**Example 5.11.**

In the vector space $\left(\mathbb{R}^2, +, .\right)$ let the set

$$\mathcal{B} = \{u_1 = (2,0), u_2 = (-1,1)\}.$$

Since $\{(2,0), (-1,1)\} \subset \mathbb{R}^2$, then

$$\operatorname{span}(u_1, u_2) \subseteq \mathbb{R}^2.$$

On the other hand, if $(x_1, x_2) \in \mathbb{R}^2$, then

$$(x_1, x_2) = \alpha_1 \cdot u_1 + \alpha_2 \cdot u_2 \Leftrightarrow (x_1, x_2) = \alpha_1 \cdot (2,0) + \alpha_2 \cdot (-1,1)(x, y)$$

$$\Leftrightarrow (x_1, x_2) = (2.\alpha_1 - \alpha_2, \alpha_2).$$

Consequently

$$\begin{cases} 2\cdot\alpha_1-\alpha_2=x \\ \alpha_2=y \end{cases} \Longrightarrow \begin{cases} \alpha_1=\dfrac{x+y}{2} \\ \alpha_2=y \end{cases}.$$

So,

$$\forall(x_1,x_2)\in\mathbb{R}^2, \exists\alpha_1=\frac{x+y}{2}, \alpha_2=y\in\mathbb{R}: (x_1,x_2)= \ \alpha_1\cdot u_1+\alpha_2\cdot u_2$$

Therefore,

$$\mathbb{R}^2\subseteq \operatorname{span}(u_1,u_2).$$

Since $\operatorname{span}(u_1,u_2)\subseteq\mathbb{R}^2$ and $\mathbb{R}^2\subseteq\operatorname{span}(u_1,u_2)$, then

$$\operatorname{span}(u_1,u_2)=\mathbb{R}^2,$$

and the set $\{u_1,u_2\}$ generates $\mathbb{R}^2$.

Since $\{u_1,u_2\}$ generates $\mathbb{R}^2$ and $\dim\left(\mathbb{R}^2\right)=2$ then

$$\mathcal{B}=\{u_1=(2,0),u_2=(-1,1)\},$$

is a basis of $\mathbb{R}^2$.

In on the other way, let $\alpha_1,\alpha_2\in\mathbb{R}$ . If $\alpha_1\cdot u_1+\alpha_2\cdot u_2=0_{\mathbb{R}^2}$, then

$$\alpha_1\cdot(2,0)+\alpha_2\cdot(-1,1)=(0,0)\Leftrightarrow(\,2.\alpha_1-\alpha_2,\alpha_2)=(0,0).$$

So,

$$\begin{cases} 2.\alpha_1-\alpha_2=0 \\ \alpha_2=0 \end{cases} \Longrightarrow \ \alpha_1=\alpha_2=0.$$

Thus $\{u_1,u_2\}$ is linearly independent.

Since $\{u_1,u_2\}$ is linearly independent and $\dim\left(\mathbb{R}^2\right)=2$ then

$$\mathcal{B}=\{u_1=(2,0),u_2=(-1,1)\},$$

is a basis of $\mathbb{R}^2$.

### 5.5.3. Coordinates Relative to a Basis

**Definition 5.11.** Let $(V,+,\cdot)$ be a vector space over a field $F$ and let

$$\mathcal{B}=\{x_1,x_2,\dots,x_n\},$$

be a basis of $V$. If $x\in V$ then $x$ has a unique representation

$$x=\alpha_1\cdot x_1+\alpha_2\cdot x_2+\cdots+\alpha_n\cdot x_n:\alpha_1,\alpha_2,\dots,\alpha_n\in F.$$

The coefficients $\alpha_1,\alpha_2,\dots,\alpha_n$ are called the **coordinates** of $x$ with respect to the basis $\mathcal{B}=\{x_1,x_2,\dots,x_n\}$.

**Example 5.12.**

We consider the basis

$$\mathcal{B}=\{u_1=(2,0),u_2=(-1,1)\},$$

of the vector space $\left(\mathbb{R}^2,+,.\right)$. Find coordinates of the vector $X=(-3,1)$ relative to the basis $\mathcal{B}$.

According to Example 5.11. we have

$$\forall(x_1,x_2)\in\mathbb{R}^2,\exists\alpha_1=\frac{x+y}{2},\alpha_2=y\in\mathbb{R}:(x_1,x_2)=\ \alpha_1\cdot u_1+\alpha_2\cdot u_2.$$

So

$$X=(-3,1)=\frac{-3+1}{2}\cdot u_1+1\cdot u_2=-u_1+u_2.$$

Hence, the coordinates of $X$ relative to the basis $\mathcal{B}$ are $(-1,1)$.

**Theorem 5.6.** Let $(V,+,\cdot)$ be a vector space of dimension $n$ $(\dim(V)=n)$ over a field $F$ and let $U$ and $W$ be subspaces of $V$. Then

$$\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W).$$

**Theorem 5.7.** Let $(V,+,\cdot)$ be a vector space of dimension $n$ $(\dim(V)=n)$ over a field $F$ and let $U$ and $W$ be subspaces of $V$.

**1**) If $U\subseteq W$, then $\dim(U)\leq\dim(W)$.

**2**) If $U\subseteq W$ and $\dim(U)=\dim(W)$, then $U=W$.

**Corollary 5.1.** Let $(V,+,\cdot)$ be a vector space of dimension $n$ $(\dim(V)=n)$ over a field $F$ and let $U$ be subspace of $V$. If $\dim(U)=\dim(V)$, then $U=V$.

**Example 5.13.**

In the vector spaces $\left(\mathbb{R}^3, +, .\right)$ let

$$U = \left\{X = (x, y, z) \in \mathbb{R}^3 : x + y = x - z = 0\right\},$$

and

$$W = \left\{X = (x, y, z) \in \mathbb{R}^3 : x + y + z = 0\right\},$$

two subspaces of it

**1**) Find a basis of $U$ and $W$.

**2**) Find $U \cap W$ and calculate $\dim(U \cap W)$.

**3**) Conclude that $\mathbb{R}^3 = U \oplus W$.
One has

**1) a**) If $X = (x, y, z) \in U$, then

$$\begin{cases} x + y = 0 \\ x - z = 0 \end{cases} \Longrightarrow \begin{matrix} y = -x \\ z = x \end{matrix},$$

which gives

$$\begin{aligned} U &= \left\{X = (x, y, z) \in \mathbb{R}^3 : x + y = x - z = 0\right\} \\ &= \left\{X = (x, -x, x) \in \mathbb{R}^3 : x \in \mathbb{R}\right\} = \left\{X = x \cdot (1, -1, 1) \in \mathbb{R}^3 : x \in \mathbb{R}\right\}. \end{aligned}$$

So, $U \subseteq \operatorname{span}\big((1, -1, 1)\big)$. On the other hand,

$$\begin{aligned} 1 + (-1) = 1 - (1) = 0 &\Longrightarrow (1, -1, 1) \in U \\ &\Longrightarrow \operatorname{span}\big((1, -1, 1)\big) \subseteq U. \end{aligned}$$

Thus $U = \operatorname{span}\big((1, -1, 1)\big)$.

Since $(1, -1, 1) \neq (0, 0, 0)$, then $(1, -1, 1)$ is linearly independent.

Because $U = \operatorname{span}\big((1, -1, 1)\big)$ and $(1, -1, 1)$ is linearly independent, then

$$\mathcal{B}_1 = \{u_1 = (1, -1, 1)\},$$

forms a basis of $U$. Hence, $\dim(U) = 1$.

**b**) If $X = (x, y, z) \in W$, then

$$x + y + z = 0 \Longrightarrow z = -x - y,$$

which implies that

$$
\begin{aligned}
W &= \left\{X = (x, y, z) \in \mathbb{R}^3 : x + y + z = 0\right\} \\
&= \left\{(x, y, -x - y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\} \\
&= \left\{(x, 0, -x) + (0, y, -y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\} \\
&= \left\{x(1,0,-1) + y(0,1,-1) \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\}.
\end{aligned}
$$

So, $W \subseteq \operatorname{span}\big((1,0,-1),(0,1,-1)\big)$. On the other hand,

$$
\begin{cases} 1 + 0 + 1 = 0 \\ 0 + 1 + 1 = 0 \end{cases} \Longrightarrow \{(1,0,-1),(0,1,-1)\} \subseteq W
$$

$$
\Longrightarrow \operatorname{span}\big((1,0,-1),(0,1,-1)\big) \subseteq W.
$$

Consequently $W = \operatorname{span}\big((1,0,-1),(0,1,-1)\big)$.

If $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ , then

$$
\alpha_1 \cdot (1,0,-1) + \alpha_2 \cdot (0,1,-1) = 0_{\mathbb{R}^3} \Leftrightarrow (\alpha_1, \alpha_2, -\alpha_1 - \alpha_2) = (0,0,0).
$$

It follows that

$$
\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ -\alpha_1 - \alpha_2 = 0 \end{cases} \Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.
$$

Thus $\{(1,0,-1),(0,1,-1)\}$ is linearly independent.

Because $W = \operatorname{span}\big((1,0,-1),(0,1,-1)\big)$ and $\{(1,0,-1),(0,1,-1)\}$ is linearly independent, we infer that

$$
\mathcal{B}_2 = \{u_2 = (1,0,-1), u_3 = (0,1,-1)\},
$$

forms a basis of $W$. Thus, $\dim(W) = 2$.

**2**) If $X = (x, y, z) \in U \cap W$, then

$$
\begin{cases} x + y = 0 \\ x - z = 0 \\ x + y + z = 0 \end{cases} \Longrightarrow x = y = z = 0.
$$

So,

$$
U \cap W = \left\{0_{\mathbb{R}^3}\right\} = \{(0,0,0)\},
$$

and $\dim(U \cap W) = 0$.

**3**) Since

$$\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W),$$

then

$$\dim(U+W)=1+2-0=3.$$

Now,

$$\begin{cases}\dim(U+W)=3=\dim\left(\mathbb{R}^3\right)\\ \qquad\text{and}\\ U+W\subseteq\mathbb{R}^3\end{cases}\Longrightarrow U+W=\mathbb{R}^3.$$

Because $U+W=\mathbb{R}^3$ and $U\cap W=\left\{0_{\mathbb{R}^3}\right\}$, then $\mathbb{R}^3=U\oplus W$.

# Chapter 6

## Linear Transformations

This chapter treats some elementary properties of a special class of functions, known as linear transformations, that map vectors in one vector space to those in another.

### 6.1. Linear Transformation

**Definition 6.1.** Let $V$ and $W$ be vector spaces over $F$. A function $T: V \to W$ is a linear transformation from $V$ to $W$ if

**1**) $T(x+y) = T(x) + T(y)$, for all $x, y \in V$.

**2**) $T(\alpha x) = \alpha T(x)$, for all $x \in V$, $\alpha \in F$.

**Theorem 6.1.** Let $V$ and $W$ be vector spaces over $F$. A function $T: V \to W$ is a linear transformation from $V$ to $W$ if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in V, \forall \alpha, \beta \in F.$$

**Example 6.1.**

**1**) Define $f_1: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f_1(x) = mx, m \text{ is a real fixed number.}$$

Show that $f_1$ is a linear transformation.

Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}$. We have

$$f_1(\alpha x + \beta y) = m(\alpha x + \beta y) = \alpha m x + \beta m y = \alpha f_1(x) + \beta f_1(y).$$

Thus $f_1$ is a linear transformation.

**2**) Define $f_2: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f_2(x, y) = x - y.$$

Show that $f_2$ is a linear transformation.

Let $\alpha, \beta \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We have

$$\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2).$$

So,

$$\begin{aligned} f_2\big(\alpha(x_1,y_1)+\beta(x_2,y_2)\big) &= f_2(\alpha x_1+\beta x_2,\alpha y_1+\beta y_2) \\ &= \alpha x_1+\beta x_2-(\alpha y_1+\beta y_2)=\alpha(x_1-y_1)+\beta(x_2-y_2) \\ &= \alpha f_2(x_1,y_1)+\beta f_2(x_2,y_2). \end{aligned}$$

Thus $f_2$ is a linear transformation.

**3**) Define $f_1:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$f_3(x)=m_1x+m_2, m_1, m_2 \text{ are real fixed numbers.}$$

Show that $f_3$ is not a linear transformation.

Let $\alpha,\beta\in\mathbb{R}$ and $x,y\in\mathbb{R}$. We have

$$f_3(\alpha x+\beta y)=m_1(\alpha x+\beta y)+m_2=\alpha m_1x+\beta m_1y+m_2.$$

However

$$\alpha f_3(x)+\beta f_3(y)=\alpha(m_1x+m_2)+\beta(m_1y+m_2)=\alpha m_1x+\beta m_1y+(\alpha+\beta)m_2.$$

Since $\alpha$ and $\beta$ are arbitrary numbers, then $\alpha+\beta\neq 1$ is not always true. So,

$$f_3(\alpha x+\beta y)\neq\alpha f_3(x)+\beta f_3(y).$$

Thus $f_3$ is not a linear transformation.

**4**) Let

$$\mathcal{B}=\{e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1)\},$$

and

$$\mathcal{B}'=\{e_1'=(1,0),e_2'=(0,1)\},$$

the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively and define a linear transformation $f_4:\mathbb{R}^3\longrightarrow\mathbb{R}^2$ by

$$f_4(e_1)=(1,1), f_4(e_2)=(1,3), f_4(e_3)=(2,1).$$

Give the expression of $f_4$.

Let $X=(x,y,z)\in\mathbb{R}^3$, we have

$$\begin{aligned} (x,y,z) &= (x,0,0)+(0,y,0)+(0,0,z) \\ &= x(1,0,0)+y(0,1,0)+z(0,0,1) \\ &= xe_1+ye_2+ze_3, \end{aligned}$$

so

$$f_4(x,y,z) = f_4(xe_1 + ye_2 + ze_3).$$

Since $f_4$ is a linear transformation then

$$\begin{aligned} f_4(xe_1 + ye_2 + ze_3) &= xf_4(e_1) + yf_4(e_2) + zf_4(e_3) \\ &= x(1,1) + y(1,3) + z(2,1) = (x + y + 2z, x + 3y + z). \end{aligned}$$

**Remark 6.1.**

**1**) Let $V$ and $W$ be vector spaces over $F$. If $T: V \to W$ is a linear transformation from $V$ to $W$, then

$$T(0_V) = 0_W,$$

where $0_V$ and $0_W$ are the identity elements of $V$ and $W$ respectively (with respect to the vector addition operation). In other words, if $T(0_V) \neq 0_W$, then $T$ is not a linear transformation. For example, if $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a function given by

$$f(x,y) = (2x + y, -1 - x - y).$$

We have

$$f(0_{\mathbb{R}^2}) = f(0,0) = \big(2(0) + (0), -1 - (0) - (0)\big) = (0,-1) \neq (0,0) = 0_{\mathbb{R}^2}.$$

Thus $f$ is not a linear transformation.

**2**) $T(-x) = -T(x)$ for all $x \in V$. Indeed, we have

$$T(-x) = T\big((-1)x\big) = (-1)T(x) = -T(x).$$

## 6.2. Kernel and Image

**Definition 6.2.** Let $V$ and $W$ be vector spaces over $F$ and suppose that $T: V \to W$ is a linear transformation.

**1**) The kernel of $T$ is the set of all elements $x \in V$ such that $T(x) = 0_W$. We denote this set by **Ker*T***. In other words,

$$\text{Ker}T = \{x \in V : T(x) = 0_W\} = T^{-1}\{0_W\}.$$

**2**) The range (or image) of $T$ is the set of all elements $y \in W$ that have the form $y = T(x)$ for some $x \in V$. We denote this set by **Im*T***. In other words,

$$\text{Im}T = \{y \in W : \exists x \in V, y = T(x)\} = \{T(x) : x \in V\}.$$

**Theorem 6.2.** Let $V$ and $W$ be vector spaces over $F$ and suppose that $T: V \to W$ is a linear transformation. Then

**1**) Ker$T$ is a subspace of $V$.

**2**) Im$T$ is a subspace of $W$.

**Example 6.2.**

**1**) Find Ker$f_1$ and Im$f_1$ , where $f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation defined by

$$f_1(x_1,x_2) = (x_1 + x_2, x_1 - x_2).$$

Ker$f_1$: By definition

$$\mathrm{Ker}f_1 = \{(x_1,x_2) \in \mathbb{R}^2 : f_1(x_1,x_2) = 0_{\mathbb{R}^2}\}.$$

We have

$$f_1(x_1,x_2) = 0_{\mathbb{R}^2} \Leftrightarrow (x_1 + x_2, x_1 - x_2) = (0,0).$$

For the left-hand side to be equal to the right-hand side, we have

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \Longrightarrow 2x_1 = 0 \Longrightarrow x_1 = 0 \Longrightarrow x_2 = 0.$$

Thus

$$\mathrm{Ker}f_1 = \{(0,0)\} = \{0_{\mathbb{R}^2}\}.$$

Im$f_1$: By definition

$$\begin{aligned} \mathrm{Im}f_1 &= \{(y_1,y_2) \in \mathbb{R}^2 : \exists (x_1,x_2) \in \mathbb{R}^2, (y_1,y_2) = f_1(x_1,x_2)\} \\ &= \{f_1(x_1,x_2) : (x_1,x_2) \in \mathbb{R}^2\}. \end{aligned}$$

We have

$$f_1(x_1,x_2) = (y_1,y_2) \Leftrightarrow (x_1 + x_2, x_1 - x_2) = (y_1,y_2).$$

For the left-hand side to be equal to the right-hand side, we have

$$\begin{cases} x_1 + x_2 = y_1 \\ x_1 - x_2 = y_2 \end{cases}.$$

Thus

$$\begin{aligned} \mathrm{Im}f_1 &= \{(x_1 + x_2, x_1 - x_2) : x_1, x_2 \in \mathbb{R}\} = \{(x_1,x_1) + (x_2,-x_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \{x_1(1,1) + x_2(1,-1) : x_1, x_2 \in \mathbb{R}\} = \mathrm{span}\big((1,1),(1,-1)\big). \end{aligned}$$

**2**) Find Ker$f_2$ and Im$f_2$ , where $f_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation defined by

$$f_2(x_1,x_2,x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3).$$

Ker$f_2$: By definition

$$\mathrm{Ker}f_2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : f_2(x_1,x_2,x_3) = 0_{\mathbb{R}^3}\}.$$

We have

$$f_2(x_1,x_2,x_3) = 0_{\mathbb{R}^3} \Leftrightarrow (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3) = (0,0,0).$$

For the left-hand side to be equal to the right-hand side, we have

$$\begin{cases} x_1 - x_2 + x_3 = 0 & \dots(1) \\ 2x_1 + x_2 - x_3 = 0 & \dots(2). \\ -x_1 - 2x_2 + 2x_3 = 0 & \dots(3) \end{cases}$$

Adding equations $(1)$ and $(2)$ together we get $3x_1 = 0 \Longrightarrow x_1 = 0$.

By substituting the value of $x_1$ into the third equation we obtain

$$x_2 = x_3.$$

So,

$$\mathrm{Ker} f_2 = \{(0, x_2, x_2), x_2 \in \mathbb{R}\} = \{x_2(0,1,1), x_2 \in \mathbb{R}\} = \mathrm{span}\big((0,1,1)\big).$$

<u>$\mathrm{Im} f_2$</u>: By definition

$$\begin{aligned} \mathrm{Im} f_2 &= \{(y_1, y_2, y_3) \in \mathbb{R}^3 : \exists (x_1, x_2, x_3) \in \mathbb{R}^3, (y_1, y_2, y_3) = f_2(x_1, x_2, x_3)\} \\ &= \{f_2(x_1, x_2, x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}. \end{aligned}$$

If $f_2(x_1, x_2, x_3) = (y_1, y_2, y_3)$, then

$$(x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3) = (y_1, y_2, y_3).$$

For the left-hand side to be equal to the right-hand side, we have

$$\begin{cases} x_1 - x_2 + x_3 = y_1 & \dots(1) \\ 2x_1 + x_2 - x_3 = y_2 & \dots(2). \\ -x_1 - 2x_2 + 2x_3 = y_3 & \dots(3) \end{cases}$$

By subtracting equation $(1)$ from the sum of equations $(1)$ and $(3)$, we get

$$(2) + (3) - (1) \Longrightarrow -y_1 + y_2 + y_3 = 0 \Longrightarrow y_1 = y_2 + y_3.$$

Thus,

$$\begin{aligned} \mathrm{Im} f_2 &= \{(y_2 + y_3, y_2, y_3) : y_2, y_3 \in \mathbb{R}\} = \{(y_2, y_2, 0) + (y_3, 0, y_3) : y_2, y_3 \in \mathbb{R}\} \\ &= \{y_2(1,1,0) + y_3(1,0,1) : y_2, y_3 \in \mathbb{R}\} = \mathrm{span}\big((1,1,0), (1,0,1)\big). \end{aligned}$$

## 6.3. Dimension Formula

**Theorem 6.3.** Let $V$ and $W$ be vector spaces over $F$ and suppose that $T: V \to W$ is a linear transformation. Then

$$\dim(V) = \dim(\mathrm{Ker} T) + \dim(\mathrm{Im} T).$$

**Example 6.3.**

Find Ker$f$ and Im$f$, where $f:\mathbb{R}^{\mathbf{2}}\longrightarrow\mathbb{R}$ is a linear transformation defined by

$$f(x_1,x_2)=x_1-x_2.$$

Ker$f$: By definition

$$\text{Ker}f=\{(x_1,x_2)\in\mathbb{R}^2: f_1(x_1,x_2)=0_{\mathbb{R}}\}.$$

We have

$$f(x_1,x_2)=0_{\mathbb{R}}\Leftrightarrow x_1-x_2=0\Longrightarrow x_1=x_2.$$

So,

$$\text{Ker}f=\{(x_1,x_1): x_1\in\mathbb{R}\}=\{x_1(1,1): x_1\in\mathbb{R}\}=\text{span}\big((1,1)\big).$$

Thus

$$\dim(\text{Ker}f)=1.$$

Im$f$: By means of the dimension formula we get

$$\dim(\text{Im}f)=\dim(\mathbb{R}^2)-\dim(\text{Ker}f)=2-1=1.$$

So

$$\begin{cases}\dim(\text{Im}f)=\dim(\mathbb{R})=1\\ \quad\text{and}\\ \text{Imf}\subseteq\mathbb{R}\end{cases}\Longrightarrow \text{Im}f=\mathbb{R}.$$

## 6.4. Injective and Surjective Linear Transformations

**Theorem 6.4.** Let $V$ and $W$ be vector spaces over $F$ and suppose that $T: V\rightarrow W$ is a linear transformation. Then

**1**) $T$ is injective if and only if Ker$T=\{0_V\}$.

**2**) $T$ is surjective if and only if Im$T=W$.

**Example 6.4.**

Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be a linear transformation defined by

$$f(x_1,x_2)=(x_1+x_2,x_1).$$

Determine whether $f$ is injective or surjective.

Ker$f$: By definition

$$\text{Ker}f=\{(x_1,x_2)\in\mathbb{R}^2: f_1(x_1,x_2)=0_{\mathbb{R}^2}\}.$$

We have

$$f(x_1,x_2)=0_{\mathbb{R}^2}\Leftrightarrow(x_1+x_2,x_1)=(0,0).$$

For the left-hand side to be equal to the right-hand side, we have

$$\begin{cases}x_1+x_2=0\\ x_1=0\end{cases}\Longrightarrow x_1=x_2=0.$$

So,

$$\mathrm{Ker}f=\{(0,0)\}=\{0_{\mathbb{R}^2}\}.$$

and

$$\dim(\mathrm{Ker}f)=0.$$

<u>$\mathrm{Im}f$</u>: By means of the dimension formula we get

$$\dim(\mathrm{Im}f)=\dim(\mathbb{R}^2)-\dim(\mathrm{Ker}f)=2-0=2.$$

So

$$\begin{cases}\dim(\mathrm{Im}f)=\dim(\mathbb{R}^2)=2\\ \quad\text{and}\\ \mathrm{Imf}\subseteq\mathbb{R}^2\end{cases}\Longrightarrow Imf=\mathbb{R}^2.$$

Since $\mathrm{Ker}f=\{0_{\mathbb{R}^2}\}$, then $f$ is injective.

Since $Imf=\mathbb{R}^2$, then $f$ is surjective.

# Bibliographie

[1] Allab Kada, *Éléments d'analyse: fonction d'une variable réelle; 1re et 2e année d'université, ELLIPSES* 2012

[2] Allen Gehret, *Math115 Lecture notes*

*https://www.math.ucla.edu/~chernikov/teaching/19S-MATH115B/115aF18_lecture_notes.pdf*

[3] Arindama Singh, Functions of One Real Variable.

https://home.iitm.ac.in/asingh/papers/one-var-primer.pdf

[4] Baba Hamed, *Algebre 1: Rappels de cours et exercices avec solutions.* OPU Alger 2012

[5] Baba Hamed, *Analyse I Rappels de cours exercices solutions.* OPU Alger 1992

[6] Benjamin Baumslak, Bruce Chandler, *Schaum's Outline of Group Theory , 1st ed*. McGraw-Hill 1968

[7] Bourbaki N, *Théorie des Ensembles*. Paris, Hermann 1963

[8] Charles C. Pinter, *A book of set Theory*. Dover Publications 2014

https://vcarrion.people.uic.edu/Pinter%20-%20A%20Book%20of%20Set%20Theory.pdf

[9] Demidovich B, *Problems in Mathematical Analysis (Translated from the Russian by G. YANKOVSKV).* Mir Publishers, Moscow 1970.

https://ia902803.us.archive.org/9/items/problemsinmathem031405mbp/problemsinmathem031405mbp.pdf

[10] Frank Ayres, Elliott Mendelson, *Schaum's Outline of Calculus, 5th ed*. McGraw-Hill 2008

[11] Gémes Margit, Szentmiklossy Zoltan, *Mathematical analysis- Exercises I.* 2014

http://etananyag.ttk.elte.hu/FiLeS/downloads/4a_GemesSzentm_MathAnExI.pdf

[12] Guegand Jean, Roque Jean Louis, Leboeuf Christian, *Cours d'analyse : classes préparatoires aux grandes écoles, 1ers cycles de l'enseignement supérieur.* Paris : Édition Marketing 1981

[13] Ivo Düntsch, Günther Gediga, *Sets, Relations, Functions*. Methodos Publishers (UK) 2000

https://www.cosc.brocku.ca/~duentsch/archive/methprimer1.pdf

[14] Janko Böhm, Magdaleen Marais, *Introduction to Algebraic Structures Lecture Notes 2019*

https://agag-jboehm.math.rptu.de/~boehm/lehre/1920_MfI/introduction_to_algebraic_structures.pdf

[15] Jerrold Marsden, Alan Weinstein, *Calculus II, 2nd ed*. Springer 1985

https://escholarship.org/content/qt5jh3x2h2/qt5jh3x2h2.pdf

[16] John B. Fraleigh, Victor J. Katz, *A first course in abstract algebra, 7th ed*. Addison-Wesley, Boston 2003

[17] JOHN T. Scheick, *Linear Algebra with Applications*. McGraw-Hill 1997

[18] PETER J. Cameron, *Notes on Algebraic Structures*.

https://webspace.maths.qmul.ac.uk/p.j.cameron/notes/algstr.pdf

[19] PISKUNOV N, *Differntial and integral calculus.* Mir Publishers, Moscow 1969

https://pisethsok.files.wordpress.com/2017/11/differential-and-integral-calculus-n-piskunov-editado.pdf

[20] ROBERT Howlett, *Vector Space Theory: A course for second year students*

https://www.maths.usyd.edu.au/u/bobh/UoS/MATH2902/vswhole.pdf

[21] ROBERT Wrede, MURRRAY R. Spiegel, *Schaum's Outline of Theory and Problems of ADVANCED CALCULUS, 2nd ed*. McGraw-Hill 2002

[22] SEYMOUR Lipshutz, *Schaum's Outline of set Theory and related topics, 2nd ed*. McGraw-Hill 2019

[23] WILLIAM F. Trench, *Introduction to real Analysis.* 2014

https://digitalcommons.trinity.edu/cgi/viewcontent.cgi?article=1006&context=mono