République Algérienne Démocratique et Populaire
وزارة التعليم العالي والبحث العلمي
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université 20 Août 1955 -Skikda
Faculté des Sciences
Département de Mathématiques
Ref :………………

جامعة 20 أوت 1955 –سكيكدة
كلية العـلوم
قسم الرياضيات
المرجع:.........

Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option : ***Mathématique***

Etude mathématique de quelques problèmes d'évolutions à retard

Présentée par :

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Soutenue publiquement : .. /../….

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Année Universitaire : 2023/2024

DEDICATION

I am immensely proud to dedicate the culmination of my studies to my beloved parents whose unwavering support and encouragement have been the foundation of my academic journey. Your sacrifices and belief in my potential have fueled my determination. This achievement is a tribute to your love and guidance.

I also dedicate this work to my siblings, whose presence provided motivation and joy during the ups and downs of this scholarly endeavor.

Finally, to all my friends and mentors who have shared their wisdom and supported me along this scholarly path.

ACKNOWLEDGEMENT

All praise is due to Allah (Subhanahu wa ta'ala), the One by whose blessing and favor all good works are accomplished and perfected.

This Ph.D. thesis is the result of the effort and support of several people, to whom I am extremely grateful.

My heartfelt thanks go to my family for their unwavering support, encouragement, and understanding during the challenging phases of this academic endeavor. Their love and belief in me have been my pillars of strength.

Dr. L. BOUZETTOUTA, I express my appreciation for all your support and guidance. I am grateful for the time you devoted to leading my way and for the knowledge and experiences you shared to enhance the quality of my research. Working with you was a great pleasure that, and I hope will continue. Your great availability, valuable advice, and the patience you granted me throughout this work have led to ideal supervision. I also highly appreciate your assistance in drafting this manuscript and preparing for the defense. For all this, I will never thank him enough.

I am profoundly grateful to my thesis co-supervisor, Pr. A. GUESMIA, for his guidance and mentorship. His insights and encouragement have been instrumental in shaping the direction of this research.

I extend my appreciation to the esteemed jury: Dr. A. Bouakkaz, who honored me by chairing the jury of my defense, Pr. S. ZITOUNI, Pr. N. BOUMAZA, and Dr. HE. KHOCHEMANE. They dedicated their time and expertise to evaluate and contribute to the development of this thesis.

A special thank you to Dr. CH. KAREK and Dr. S. LECHHEB for their support and assistance, Dr. H. ATOUI and all my teachers who played a crucial role in shaping my academic foundation. Great thanks to Dr. A. LLALOUCHE for all the administrative help offered throughout the doctoral enrollments.

I conclude these acknowledgments by thanking all the members of LAMAHIS.

THESIS ABSTRACT

This thesis focuses on the study of the well-posedness and the stability of the solutions to some evolution problems with the presence of different mechanisms of damping and various types of delay. We are interested in proving the existence and uniqueness of solutions using the semi-group theory and showing the exponential stability using the energy method by constructing a suitable Lyapunov functional equivalent to the system's energy. In this regard, we study the following three problems: The first is concerned with a transmission problem for waves with history, time varying delay, and non-constant weights, the second is related to the thermoelastic system of swelling porous elastic soils with second sound and distributed delay term, and the last is devoted to the study of the Bresse system with microtemperature effects and a delay term.

Keywords: Evolution problem, Transmission problem, Swelling porous system, Bresse system, Semi-group theory, Lyapunov functional, Exponential stability.

RÉSUMÉ

Cette thèse se concentre sur l'étude de l'existence, l'unicité et la stabilité des solutions pour les problèmes d'évolution en présence de différents mécanismes d'amortissement et sous les effets de différents types de retard. Nous nous intéressons à l'existence et à l'unicité des solutions en utilisant la théorie des semi-groupes et à l'établissement de la stabilité exponentielle à l'aide de la méthode d'énergie par la construction d'une fonctionnelle de Lyapunov appropriée équivalente à l'énergie du système. À cet égard, nous étudions les trois problèmes suivants : Le premier concerne un problème de transmission pour des ondes avec un terme de mémoire, un terme de retard variable et des poids non constants, le deuxième est lié au système thermoélastique de gonflement des sols poreux élastique avec un deuxième son et terme de retard distribué, et le dernier est consacré à l'étude du système de Bresse avec effets de microtempérature et terme de retard.

Mots-clés: Problèmes d'évolution, Retard, Systèmes poreux gonflés, Problème de transmission, Système de Bresse, Théorie des semi-groupes, Fonctionnelle de Lyapunov, Stabilité exponentielle.

الملخص:

تركز هذه الأطروحة على دراسة وجود و وحدانية واستقرار الحلول لمسائل التطور في وجود آليات تخميد مختلفة وتحت تأثير أنواع مختلفة من التأخير. نحن مهتمون بوجود وتفرد الحلول باستخدام نظرية أنصاف الزمر وتحقيق الاستقرار الأسي باستخدام طريقة الطاقة من خلال بناء مناسب لدالة ليابونوف مكافئة لطاقة النظام المدروس. في هذا الصدد، ندرس المسائل الثلاث التالية: الأولى تتعلق بمشكلة انتقال الموجات مع حد الذاكرة، حد التأخير المتغير بالنسبة للزمن والأوزان غير الثابتة، والثانية تتعلق بنظام حراري لتضخم التربة المسامية المرنة مع الصوت الثاني وحد التأخير الموزع، أما الأخيرة فهو مخصص لدراسة نظام بريس مع تأثيرات درجات الحرارة الدقيقة وحد التأخيرالثابث.

الكلمات المفتاحية :مشاكل التطور، حد التأخير، نظام مسامي منتفخ، مسألة النقل، نظام بريس، نظرية أنصاف الزمر، لدالة ليابونوف، الاستقرار الأسي.

CONTENTS

CHAPTER 1

INTRODUCTION

Mathematical thinking permeates virtually every scientific field, from physics and chemistry to biology and economics. It serves as a universal language for enabling scientists to model complex phenomena. Furthermore, employing mathematical descriptions becomes imperative for delivering reliable predictive analyses for the underlying process of nature. Consequently, mathematical models have been meticulously formulated to understand complex processes was set up.

Many of those mathematical models are described by a (system of) partial differential equation(s) (PDEs). Well established laws of nature and their mathematical counterparts have led to most of the development of suitable PDEs, exemplified in domains like heat conduction, fluid dynamics, or the deformation of solids. Within this framework, hyperbolic-type problems emerge as a class of PDEs that model propagation phenomena. A perfect example of this category is the wave equation, which defines the behavior of waves generated when a vibrating source disturbs the medium. To control and restrain these vibrations, various damping mechanisms can be introduced, including second sound, time delay, and past history damping, (see [14, 15, 36, 37, 38]).

In this thesis, we are interested in studying the behavior of solutions of certain hyperbolic systems where the dissipation is induced by the presence of different mechanisms of damping and under the effects of various types of delay. A time delay refers to the lag or interval between an event or action and its corresponding effect or response within the system. It represents the time it takes for an influence to propagate through the system, influencing its state or behavior. The presence of time delays introduces a unique set of challenges and behaviors in evolutionary problems. These effects can significantly impact the system's dynamics, stability, and overall behavior.

The delay differential equations (DDEs) are differential equations in which the derivatives of some unknown functions at present time depend on the values of the functions at previous times. Mathematically, we mention some types of delay for a simple differential equation where the time delay is denoted by τ:

Constant time delay

A constant delay refers to a fixed time lag between an event and its effect within a dynamic system, it is denoted by a constant τ, its equation is written as follows:

$$\frac{d}{dt}x(t) = f(t, x(t-\tau)).$$

Varying Time Delay

Varying delay refers to a time delay that changes dynamically with time. It is often represented as $\tau(t)$, indicating that the delay can vary as the system evolves, its equation is written as follows:

$$\frac{d}{dt}x(t) = f(t, x(t-\tau(t))).$$

Distributed Time Delay

Distributed time delay refers to a situation where the system's current state depends on a range of past states, each with a different time lag. Unlike a constant delay, where a fixed time lag is considered, its equation is written as follows:

$$\frac{d}{dt}x(t) = f(t, \int_0^\tau \mu(s)x(t-s)ds).$$

Another type of delay called neutral delay is referred to in earlier research [58, 21].

Recently, there has been a surge in research activity regarding partial differential equations (PDEs) with time delay effects, this can be observed in works such as [1, 56] and the associated references. The presence of delay may be a source of instability, as demonstrated in studies [20], where it was proved that an arbitrarily small delay may destabilize a system, which is uniformly asymptotically stable in the absence of delay.

Regarding the wave equation system with a linear frictional damping and a constant delay on the boundary:

$$\begin{cases} v_{tt}(x,t) - \Delta v(x,t) = 0, \quad x \in \Omega, t > 0, \\ v(x,t) = 0, \quad x \in \Gamma_0, t > 0, \\ \frac{\partial v(x,t)}{\partial \nu} = -\mu_1 v_t(x,t) - \mu_2 v_t(x,t-\tau), \quad x \in \Gamma_0, t > 0. \end{cases} \tag{1.1}$$

In the case where μ_1 is positive and μ_2 is null, meaning that there is no delay, several researchers [40, 60] have verified that the system (1.1) is exponentially stable.

In 2006, Serge Nicaise and Cristina Pignotti [44] demonstrated based on the hypothesis that $\mu_2 < \mu_1$, that the energy is stable in the exponentially sense for the wave equation with a delay term in the boundary or internal feedbacks. On the contrary, they showed instability in the case $\mu_2 \geq \mu_1$, for the problem (1.1). In 2008, they obtained exponential stability results in the case where the distributed delay replaces the constant delay in the system's (1.1) in [45]:

$$\int_{\tau_1}^{\tau_2} \mu_2(s) v_t(x, t-s) ds,$$

provided that

$$\int_{\tau_1}^{\tau_2} \mu_2(s) < \mu_1.$$

They verified the well-posedness of the analyzed problem using semigroup theory and achieved exponential stability. They also provided an example of instability when the following assumption was not verified:

$$||\alpha||_\infty \int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1.$$

On the other hand, in collaboration with J. Valein [47], they obtained a similar result when we replace the constant delay term in (1.1) on the boundary with a varying delay term of the form:

$$\mu_2 v_t(x, t-\tau(t)),$$

assuming that

$$\mu_2 < \sqrt{1-d}\, \mu_1, \tag{1.2}$$

where d is a constant such that

$$\tau'(t) < d < 1, \quad t > 0. \tag{1.3}$$

For further information on the same case with conditions (1.2,1.3) in one space dimension, the reader can refer to reference [46], where the authors studied the exponential stability of the wave equations with time-varying delays.

Results Description

The objective of this thesis is to establish the well-posedness of solutions and demonstrate exponential decay by using the energy method. The specific systems addressed in this study are:

The transmission problem for waves

The transmission problem for waves is a mathematical and physical concept that deals with the behavior of waves as they travel within a domain containing two or several materials. This problem focuses on the study of how waves interact with boundaries or interfaces between different media, involving the analysis of wave equations and boundary conditions.

Recently, researchers have been interested in transmission problems, particularly in the realm of material component design, as seen in their applications in analyzing damping mechanisms within the metallurgical industry and the development of smart materials technology, for instance, the authors in [8] presented an application to an automotive floor panel and in [54] they described how to control noise and vibration in vehicles and commercial airplanes. More applications in [33, 51] and the references therein. As far as our knowledge extends, the initial contribution in the literature regarding transmission problems with a time delay was made by A. Benseghir in [12], he addressed the transmission problem with delay:

$$\begin{aligned} &u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \text{ in } \Omega\times]0,+\infty[,\\ &v_{tt} - b v_{xx} = 0 \text{ in }]L_1, L_2[\times]0,+\infty[, \end{aligned}$$

with constant weights μ_1, μ_2 and time delay $\tau(t) > 0$ was studied, under an appropriate assumption on the weights of the two feedbacks ($\mu_1 < \mu_2$), and under this condition

$$\max\left\{1, \frac{a}{b}\right\} < \frac{L_1 + L_3 - L_2}{2\,(L_2 - L_1)}, \tag{1.4}$$

it was proved the well-posedness of the system and they established an exponential decay result.

In [49], the authors studied a transmission problem for waves with non-constant delay and non-constant weights:

$$\begin{aligned} &u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1(t) u_t(x,t) + \mu_2(t) u_t(x,t-\tau(t)) = 0 \text{ in } \Omega\times]0,+\infty[,\\ &v_{tt} - b v_{xx} = 0 \text{ in }]L_1, L_2[\times]0,+\infty[, \end{aligned}$$

under suitable assumptions on $\mu_1(t)$, $\mu_2(t)$ and $\tau(t)$, and using semigroup arguments by Kato, they obtained a well-posedness result, and established exponential stability by introducing a suitable Lyapunov functional.

A similar result has been shown in [59], where the authors examined the following transmission problem in the presence of infinite history and varying delay

$$\begin{aligned} & u_{tt}(x,t) - au_{xx}(x,t) + \int_0^{+\infty} g(s)u_{xx}(x,t-s)ds \\ & + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) = 0 \text{ in } \Omega\times]0,+\infty[, \\ & v_{tt} - bv_{xx} = 0 \text{ in }]L_1,L_2[\times]0,+\infty[, \end{aligned}$$

under suitable assumptions on the weight of the damping and the weight of the delay, for more informations see [5].

Our main results in this part can be summarized as follows: we consider the transmission problem for the wave equation in the presence of history, non-constant delay, and non-constant weights terms

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \int_0^{+\infty} g(s)u_{xx}(x,t-s)ds \\ +\mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 \text{ in } \Omega\times]0,+\infty[, \\ v_{tt} - bv_{xx} = 0 \text{ in }]L_1,L_2[\times]0,+\infty[. \end{cases}$$

We use some semigroup theory techniques based on Kato's variable norm technique to prove well posdeness then we give an exponential stability result for the energy using a suitable Lyapunov functional. This work has been recently published in [7].

The swelling porous-elastic system

Research on the behavior of swelling soils, falling under porous media theory during fluid saturation has attracted significant attention. Porous elastic soil undergoes notable volume changes based on moisture fluctuations expanding when gaining moisture and contracting when losing it. This phenomenon results from unique clay minerals with a high capacity for attracting and retaining water molecules. As these minerals absorb water, they cause the surrounding soil to swell, see [9, 13, 30, 31, 32] for a brief description of the development details/historical insights related to the general theory of mixtures. Moreover, Eringen [25] established a relationship between the continuum theory of swelling porous elastic soils and classical diffusion theories.

Numerous studies have explored the asymptotic behavior of the mathematical model describing this theory. Ieşan [28] initially formulated this model, which can be summarized as follows:

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} = 0, \quad \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x = 0, \quad \text{in } (0,L)\times(0,\infty), \end{cases}$$

with φ and ψ representing respectively, the displacement of the fluid and the elastic solid material. The parameters ρ_1 and ρ_2 are the densities of each constituent, which are assumed to be strictly positive constants. The other coefficients are positive constants that have a physical meaning, defining the coupling between the different components of the materials.

Among the many studies that discuss the influence of the time delay on the asymptotic behavior of solutions of the swelling porous elastic system we have the work [53], where the authors established the exponential stability of the swelling porous elastic soils with fluid saturation and delay time terms

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau) = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x = 0, & \text{in } (0,L)\times(0,\infty). \end{cases}$$

M. Douib and S. Zitouni [22] obtained the well-posedness of swelling porous-heat system with single time delay, based on the semi-group method and Lumer-Philips theory , and used the energy method and construct Lyapunov functional to prove the exponential stability. For more results concerning the study of the asymptotic behavior of solution for the swelling porous elastic system under effects of other types of dissipation, we refer the reader to [3, 27, 52] and the references therein

We briefly mention that our main results in this part are as follows: we consider a one-dimensional swelling porous-elastic system with second sound and distributed delay term

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2}\mu_2(s)\varphi_t(x,t-s)\,ds = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - a_3\psi_{xx} + a_2\varphi_{xx} + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t - q_x + \delta\psi_{tx} = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1)\times(0,\infty). \end{cases} \tag{1.5}$$

We prove that the combination of the frictional damping with the heat flux effect is strong enough to provoke an exponential decay of the energy even if the delay is a source of destabilization. This work has been recently published in [6].

The Bresse system

The Bresse system referred to as the problem of the arc and the linearly shearable curved beam serves as a mathematical model for capturing the vibration phenomena in planar structures. These elastic structures have diverse applications in technical field, structural design, naval engineering, Aeronautics and other relevant disciplines (see [41]).

The Bresse system takes into account the deformations that occur in the circular arc when it is subjected to the vertical shear angle and longitudinal displacements represented by φ, ψ and ω, respectively. In 1859 Bresse [17] initially formulated this model as follows:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - lk_0\left(\omega_x - l\varphi\right) = F_1 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) = F_2, & \text{in } (0,1)\times(0,\infty), \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\omega\right) = F_3, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{1.6}$$

where $F_i = 1, 2, 3$ denote the external forces exerted on the object and the coefficients $\rho_1, \rho_2, k, k_0, l, b$ reflecting various physical parameters.

We mention that it has been observed that the system described by (1.6) is conservative and do not represent a stable system when $F_i = 0$ for $i = 1 - 3$.

Researchers have shown significant interest in the mathematical modeling of the Bresse system in recent years. Many authors have explored the use of flexible systems to effectively control dissipation within the Bresse system.

Concerning to asymptotic behavior of solutions for Bresse systems, we mention in the following some important results. In 2010, Santos and Almeida Júnior [55] proved the exponential decay of the solution for the Bresse system with frictional dissipative terms without imposing any constraints on the coefficients. They achieved this by setting $F_1 = \gamma_1 \varphi_t$, $F_2 = \gamma_2 \psi_t$ and $F_3 = \gamma_3 \omega_t$, under initial and Dirichlet boundary conditions.

Another significant reference [43], Zhuangyi Liu and Bopeng Rao demonstrated that in a thermoelastic Bresse system with two dissipative processes, the solution decays exponentially to zero if and only if the conditions on wave propagation velocities satisfied

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}, \qquad k = k_0. \tag{1.7}$$

Additionally, in [4] the authors presented more precise findings, they showed that the Bresse system with $F_1 = F_3 = 0$ and $F_2 = \gamma \psi_t$ lacks exponential stability, when the condition (1.7) is not verified.

Regarding the asymptotic behavior of solutions within a Bresse system in the presence of delay, the authors in [10] establish the global existence of solutions by employing semigroup theory in Sobolev spaces under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay, and the multiplier method for energy decay in the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - lk_0(\omega_x - l\varphi) + \mu_1 \varphi_t(x,t) + \mu_2 \varphi_t(x, t - \tau_1) = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \bar{\mu}_1 \psi_t(x,t) + \bar{\mu}_2 \psi_t(x, t - \tau_2) = 0, \\ \rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + lk(\varphi_x + \psi + l\omega) + \bar{\bar{\mu}}_1 \omega_t(x,t) + \bar{\bar{\mu}}_2 \omega_t(x, t - \tau_3) = 0, \end{cases}$$

where $(x, t) \in (0, 1) \times (0, \infty)$.

Our main results in this part can be summarized as follows: The primary objective of our study is to investigate the thermal effects and demonstrate the efficacy of microtemperatures in achieving stability for the following Bresse system with micro-temperatures and delay term

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - lk_0(\omega_x - l\varphi) + \mu_1 \varphi_t(x,t) + \mu_2 \varphi_t(x, t - \tau) = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + dy_x = 0, \\ \rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + lk(\varphi_x + \psi + l\omega) = 0, \\ \alpha y_t - \gamma_1 y_{xx} + d\psi_{tx} + \gamma_2 y = 0, \end{cases}$$

where $(x, t) \in (0, 1) \times (0, \infty)$, we examine the well-posedness of the system and we prove exponential stability when the speeds of wave propagation are equal.

Thesis structure

The rest of this thesis is divided as follows:

Chapter 2: Preliminaries

We recall some basic notions which will be useful to us later.

Chapter 3: Transmission problem for waves with history, time-varying delay and non-constant weights

We study the existence, uniqueness and asymptotic stability of a Transmission problem for waves with history, time-varying delay and non-constant weights. It consists of three sections: preliminaries, result of existence and uniqueness and result of general stability of this system.

Chapter 4: Well-posedness and exponential stability of swelling porous elastic soils with a second sound and disrtibuted delay term

Chapter 4 deals with energy decay for a swelling porous elastic soils with a second sound and disrtibuted delay term, where the thermal conduction is given by Cattaneo's law.

Chapter 5: Well-possedness and energy decay for Bresse system with microtemperatures and delay term

we consider a one-dimensional Bresse system with microtemperatures and delay term, and we will show that the system is exponentially stable if and only if when the speeds of wave propagation are equal.

Publications

The following results were Published/Submitted from our research:

1. S. Baibeche, L. Bouzettouta, A. Guesmia, M. Abdelli, Well-posedness and exponential stability of swelling porous elastic soils with a second sound and distributed delay term, Journal of Mathematical and Computational Science, 12-82, (2022), https://doi.org/10.28919/jmcs/7106.

2. S. Baibeche, L. Bouzettouta, A. Guesmia, Transmission problem for waves with history, time-varying delay and nonlinear weight, Mathematics in Engineering, Science and Aerospace MESA, (4);1141-1163, (2022).

3. L. Bouzettouta, S. Baibeche, M. Abdelli, A. Guesmia, stability result for a thermoelastic Bresse system with delay term in the feedback, Mathematica Bohemica, 1-26, (2022), https://doi.org/10.21136/MB.2022.0154-21.

CHAPTER 2

PRELIMINARIES

This chapter is devoted to the reminders of some notions and preliminary results which will be used throughout the rest. Although it will be stated without proof, but we will provide the relevant references.

2.1 Some reminders on functional spaces

In what follows, we will denote by Ω an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 2.1 (Hilbert Space) [19] A Hilbert space H is a vectorial space equipped with inner product $\langle u, v\rangle$, and is complete with respect to the norm

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Definition 2.2 ($L^p(\Omega)$ spaces) [19] Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we set

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ \text{is measurable and } \int_\Omega |u|^p \, dx < +\infty \right\},$$

with

$$||u||_{L^p(\Omega)} = \left[\int_\Omega |u|^p \, dx \right]^{\frac{1}{p}}.$$

Definition 2.3 [19] We set

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R}; /\exists C \in \mathbb{R}^+, \quad |u(x)| \leq C \text{ a.e. } \text{ in } \Omega \right\}.$$

For $u \in L^\infty(\Omega)$, we define the norm

$$||u||_{L^\infty(\Omega)} = \inf \left\{ C \in \mathbb{R}^+; |u(x)| \leq C, \text{a.e. on } \Omega \right\}.$$

The Sobolev Space $W^{k,p}(\Omega)$

Definition 2.4 [29] For all $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega); D^\alpha u \in L^p(\Omega) \quad \text{for } 1 \leq |\alpha| \leq k\}.$$

With this definition, the Sobolev spaces $W^{k,p}(\Omega)$ equipped with the following norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{1 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, for\ p < +\infty.$$

Definition 2.5 [19] Whene $p = 2$, we denote by

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.$$

The $\mathbf{H}_0^k$ spaces

We define the H_0^k by the following set

$$\{u \in H^k(\Omega), u = u' = ... = u^{(k-1)} = 0 \text{ on } \partial\Omega\}, \tag{A}$$

the characterization (A) it is essential to notice the distinction between

$$\{u \in H^2(\Omega),\ u = u' = 0 \text{ on } \partial\Omega\},$$

and

$$H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega),\ u = 0 \text{ on } \partial\Omega\}.$$

2.2 Some useful algebraic inequalities

In this section, we present the inequalities used throughout this work:

Young's inequality

Let $a, b \in \mathbb{R}^+$ and $p, q \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Specifically, if $u, v \in L^2(\Omega)$, then

$$\int_\Omega |uv|\, dx \leq \varepsilon \int_\Omega |u|^2\, dx + \frac{1}{4\varepsilon} \int_\Omega |v|^2\, dx, \forall \varepsilon > 0.$$

Hölder's inequality

Let $p, q \in \left]1, \infty\right[$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$. Then $u.v \in L^1(\Omega)$ and

$$\|u.v\|_1 \leq \|u\|_p \|v\|_q .$$

If $p = q = 2$, we obtain the Cauchy-Schwarz inequality:

$$\int_\Omega |u.v| \, dx \leq \left(\int_\Omega |u|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 \, dx\right)^{\frac{1}{2}} .$$

Poincaré's inequality

There exists a constant C such that

$$\|u\|_{H^1(I)} \leq C \|u'\|_{L^2(I)}, \text{ for all } u \in H_0^1(I) . \tag{2.1}$$

2.3 Lax Milgram theorem

Theorem 2.1 *[19] Let H be a real Hilbert space, B a bilinear form on H, and La linear form on H. Suppose that:*

1. *B is continuous:*

$$\forall u, v \in H, |B(u,v)| \leq C_1 \|u\|_H \|v\|_H, C_1 > 0.$$

2. *B is coercive: there exists $C_2 > 0$ such that*

$$|B(u,u)| \geq C_2 \|u\|_H^2, \ \forall u \in H.$$

3. *L is continuous:*

$$|L(v)| \leq C_3 \|v\|_H, \ \forall v \in H.$$

Then, there exists a unique $u \in H$ that satisfies

$$\forall v \in H, B(u,v) = L(v) .$$

2.4 M-dissipative operator

Definition 2.6 [18] An unbounded linear operator in X is a pair $(A, D(A))$, where $D(A)$ is a subspace of X and A is a linear mapping from $D(A)$ to X. The subspace $D(A)$ is the domain of A.

Definition 2.7 [18] An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if

1. A is dissipative.
2. The operator $(\lambda I - A)$ is surjective i.e:

$$\forall f \in X, \quad \forall \lambda > 0, \quad \exists x \in D(A) \text{ such that } \lambda x - Ax = f.$$

M-dissipative operators in a Hilbert space

In this section, we assume that X is a Hilbert space.

Definition 2.8 [23] An unbounded linear operator $(A, D(A))$ on X, is said to be dissipative if and only if

$$(Ax, x)_X \leq 0, \quad \forall x \in D(A).$$

Theorem 2.2 *[23] If A is m-dissipative, then $D(A)$ is dense in X.*

2.5 Semigroups of unbounded linear operators

Let X be a Banach space.

Definition 2.9 [50] A family $(S(t))_{t\geq 0}$ of bounded linear operations in X is called a semigroup if it satisfies the following properties:

- $S(0) = I$ (I is the identity operator on X).
- $S(t+s) = S(t)S(s), \quad \forall t, s \geq 0$

Remark 2.1 When the second property is true for all t, s in $\mathbb{R}$, then we say that $(S(t))_{t\geq 0}$ is a group.

Definition 2.10 [50] A family of operators $(S(t))_{t\geq 0}$ is a strongly continuous semigroup in X if

$$\lim_{t\to 0} \|S(t)x - x\| = 0, \text{ for all } x \in X.$$

Definition 2.11 [50] The linear operator A defined on the set

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists} \right\},$$

by

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}, \quad \forall x \in D(A),$$

is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$, and $D(A)$ represents the domain of A.

Theorem 2.3 *(See Theorem 2.2 in [50]) A C_0-semigroup $(S(t))_{t\geq 0}$ on X is said to be of contractions if*

$$\|S(t)\| \leq 1, \forall t \geq 0$$

Definition 2.12 [19] An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is said to be monotone if it satisfies

$$(\mathcal{A}u, u) \geq 0, \ \forall u \in D(\mathcal{A}),$$

It is called maximal monotone if, in addition $R(\mathcal{I} + \mathcal{A}) = \mathcal{H}$, i.e.,

$$\forall f \in \mathcal{H}, \exists u \in D(\mathcal{A}) \text{ such that } u + Au = f.$$

Hille-Yosida theorem

Theorem 2.4 *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0,\infty[, \mathcal{H}) \cap C([0,\infty[, D(\mathcal{A}))$$

satisfying

$$\begin{cases} u' + \mathcal{A}u = 0 & \text{on } [0,\infty[, \\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0|, \text{ and } |\frac{du}{dt}(t)| = |Au(t)| \leq |Au_0|, \forall t > 0.$$

Theorem 2.5 *[24](Hille-Yosida 2) An unbounded linear operator $(A, D(A))$ in X is the infinitesimal generator of a strongly continuous contraction semigroup on X if and only if A is m-dissipative and has a dense domain in X.*

Lumer-Phillips theorem

Theorem 2.6 *[50](Lumer-Phillips) Let A be a unbounded linear operator with dense domain $D(A)$ in X.*

a. If A is dissipative and there is a $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.

b. If A is the infinitesimal generator of a C_0-semigroup of contractions on X then

$$\mathcal{R}(\lambda I - A) = X, \ \forall \lambda > 0,$$

and A is dissipative.

Consequently, A is maximal dissipative on a Hilbert space H if and only if it generates a C_0-semigroup of contractions on H, and thus the existence of the solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 2.1 *Let H be a Hilbert space and let A be a linear operator defined from $D(A) \subset H$ into H. If A is maximal dissipative, then the initial value problem (Cauchy problem) has a unique weak solution $U \in C([0, +\infty); H)$, for each initial data $U_0 \in H$. Moreover, if $U_0 \in D(A)$, then $U \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$.*

CHAPTER 3

TRANSMISSION PROBLEM FOR WAVES WITH HISTORY, TIME-VARYING DELAY AND NON-CONSTANT WEIGHTS

3.1 Presentation of the problem

In this chapter, we investigate the asymptotic behavior of a transmission problem for the wave equation with history, non-constant delay and non-constant weights terms, we intend to extend further the result in [59] where, for constant weights $\mu_1 = \mu_1(t)$, $\mu_2 = \mu_2(t)$.

In fact, the appearance of the past history term simultaneously with non-constant weights and time-varying delay makes the problem more difficult, which requires a detailed study in order to determine the appropriate assumptions on the weight of the damping and the weight of the delay, we prove the existence and the uniqueness of the solution using the Kato's variable norm technique together with semigroup theory to show that the system is well-posed. Also, we show the exponential stability of the solution by introducing a suitable Lyapunov functional.

We consider the following problem:

$$\begin{aligned}
&u_{tt}(x,t) - au_{xx}(x,t) + \int_0^{+\infty} g(s)u_{xx}(x,t-s)ds \\
&+ \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 \text{ in } \Omega\times]0,+\infty[, \\
&v_{tt} - bv_{xx} = 0 \text{ in }]L_1, L_2[\times]0,+\infty[,
\end{aligned} \tag{3.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\cup]L_2, L_3[$ and a, b are positive constants, $g \in \mathcal{C}^1(\mathbb{R}_+)$ is a positive function, $\tau(t) > 0$ is a time-varying delay and $\mu_1(t)$, $\mu_2(t)$ are non-constant weights.

The boundary and transmission conditions given by

$$\begin{aligned} & u(0,t) = u(L_3,t) = 0, \\ & u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ & au_x(L_i,t) - \int_0^{+\infty} g(s)u_x(L_i,t-s)ds = bv_x(L_i,t), \quad i = 1,2, \end{aligned} \tag{3.2}$$

and the initial conditions

$$\begin{aligned} & u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ on } \Omega, \\ & u(x,t-\tau(0)) = f_0(x,t-\tau(0)) \text{ in } \Omega\times]0,\tau(0)[, \\ & v(x,0) = v_0(x), v_t(x,0) = v_1(x) \text{ on }]L_1,L_2[. \end{aligned} \tag{3.3}$$

As in [48] , we assume that

$$\tau(t) \in W^{2,+\infty}([0,T]), \text{ for } T > 0, \tag{3.4}$$

and there exist positive constants τ_0, τ_1 and d satisfying

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0, \tag{3.5}$$

and

$$\tau'(t) \leq d < 1, \quad \forall t > 0. \tag{3.6}$$

For the problem (3.1)-(3.3), we are interested in demonstrating the exponential stability, in order to get this we will suppose that

$$\max\left\{1, \frac{l}{b}\right\} < \frac{L_1 + L_3 - L_2}{2\left(L_2 - L_1\right)}, \tag{3.7}$$

hold, as described in [12], the assumption (3.7) gives the relationship between the boundary regions and the transmission permitted.

The structure of this chapter is as follows:

In section (3.2), we introduce some assumptions and transformations related to our problem, in section (3.3), we use semigroup theory techniques based on Kato's variable norm technique to prove well-posedness. Finally, in section (3.4), we establish an exponential stability result for the energy by using a suitable Lyapunov functional.

3.2 Assumptions and transformations

This section will introduce some assumptions necessary for getting our results. Additionally, we will transform (3.1)-(3.3) into an equivalent problem through the introduction of new variables.

We assume the following assumptions:

(*H*1) $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $\mathcal{C}^1(\mathbb{R}_+)$ satisfying

$$\Big|\frac{\mu_1^{'}(t)}{\mu_1(t)}\Big| \leq M_1, \quad 0 < \alpha_0 \leq \mu_1(t), \quad \forall t \geq 0, \tag{3.8}$$

where α_0 and M_1 are constants such that $M_1 > 0$.

(*H*2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $\mathcal{C}^1(\mathbb{R}_+)$, which is not necessarily positives or monotones, such that

$$|\mu_2(t)| \leq \beta \mu_1(t), \tag{3.9}$$

and

$$|\mu_2^{'}(t)| \leq M_2 \mu_1(t), \tag{3.10}$$

for some $0 < \beta < \sqrt{1-d}$ and $M_2 > 0$.

We assume also that the function g satisfies the following:

(*H*3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is of class C^1 such that:

$$g(0) > 0, \quad a - \int_0^\infty g(t)dt = a - g_0 = l > 0. \tag{3.11}$$

(*H*4) There exists a positive constant δ

$$g'(s) \leq \delta g(s), \quad \forall s \in \mathbb{R}_+. \tag{3.12}$$

We add the new variable, similar to Nicaise and Pignotti [48]:

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Omega, \rho \in]0, 1[, t > 0, \tag{3.13}$$

then

$$\tau(t)z_t(x, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, \rho, t) = 0. \tag{3.14}$$

We set

$$\eta^t(x, s) = u(x, t) - u(x, t - s), \quad x \in \Omega, t > 0, s > 0, \tag{3.15}$$

then

$$\eta^t_t(x, s) + \eta^t_s(x, s) = u_t(x, t). \tag{3.16}$$

Proof. we have

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Omega, \rho \in]0, 1[, t > 0,$$

taking the derivative of $z(x, \rho, t)$ with respect to t, we obtain

$$\begin{aligned} z_t(x, \rho, t) &= \frac{\partial}{\partial t}[u_t(x, t - \tau(t)\rho] \\ &= u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho). \end{aligned} \tag{3.17}$$

then, taking the derivative of $z(x,\rho,t)$ with respect to ρ, we obtain

$$\begin{aligned} z_\rho(x,\rho,t) &= \frac{\partial}{\partial\rho}[u_t(x,t-\tau(t)\rho)] \\ &= u_{tt}(x,t-\tau(t)\rho)(-\tau(t)). \end{aligned} \tag{3.18}$$

Multiplying the equations (3.17) and (3.18) by $\tau(t)$ and $(1-\tau'(t)\rho)$ respectively, we get

$$\tau(t)z_t(x,\rho,t) = \tau(t)(1-\tau'(t)\rho)u_{tt}(x,t-\tau(t)\rho),$$

and

$$(1-\tau'(t)\rho)z_\rho(x,\rho,t) = -\tau(t)(1-\tau'(t)\rho)u_{tt}(x,t-\tau(t)\rho),$$

by addition the last two equations, we get the results (3.14).
Then, we have

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad x\in\Omega, t>0, s>0, \tag{3.19}$$

taking the derivative of $\eta^t(x,s)$ with respect to t, we obtain

$$\begin{aligned} \eta^t_t(x,s) &= \frac{\partial}{\partial t}[\eta^t(x,s)] = \frac{\partial}{\partial t}[u(x,t)-u(x,t-s)] \\ &= u_t(x,t) - u_t(x,t-s), \end{aligned}$$

taking the derivative of $\eta^t(x,s)$ with respect to s, we obtain

$$\begin{aligned} \eta^t_s(x,s) &= \frac{\partial}{\partial s}[\eta^t(x,s)] = \frac{\partial}{\partial s}[u(x,t)-u(x,t-s)] \\ &= 0 - (-u_t(x,t-s)) \\ &= u_t(x,t-s), \end{aligned}$$

by addition we obtain the results (3.16).
Also

$$\begin{aligned} \int_0^{+\infty} g(s)u_{xx}(x,t-s)ds &= \int_0^{+\infty} g(s)[u_{xx}(x,t) - \eta^t_{xx}(x,t)]\,ds \\ &= \int_0^{+\infty} g(s)u_{xx}(x,t)ds - \int_0^{+\infty} g(s)\eta^t_{xx}(x,t)\,ds \\ &= \left(\int_0^{+\infty} g(s)ds\right)u_{xx}(x,t) - \int_0^{+\infty} g(s)\eta^t_{xx}(x,s)\,ds, \end{aligned}$$

take $l = 1 - \int_0^{+\infty} g(s)ds$. ■

3.2. Assumptions and transformations

Thus the system (3.1) becomes

$$
\begin{aligned}
& u_{tt}(x,t) - lu_{xx}(x,t) - \int_0^{+\infty} g(s)\eta^t_{xx}(x,s)\,ds \\
& + \mu_1(t)u_t(x,t) + \mu_2(t)z(x,1,t) = 0, \text{ in } \Omega\times]0,+\infty[, \\
& v_{tt}(x,t) - bv_{xx}(x,t) = 0, \text{ in }]L_1,L_2[\times]0,+\infty[, \\
& \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, \text{ in } \Omega\times]0,1[\times]0,+\infty[, \\
& \eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), \text{ in } \Omega\times]0,+\infty[\times]0,+\infty[.
\end{aligned} \tag{3.20}
$$

The boundary and transmission conditions (3.2) become

$$
\begin{aligned}
& u(0,t) = u(L_3,t) = 0 \text{ on }]0,+\infty[, \\
& u(L_i,t) = v(L_i,t), \quad i=1,2, \\
& lu_x(L_i,t) - \int_0^{+\infty} g(s)\eta^t_x(L_i,s)ds = bv_x(L_i,t),\ i=1,2, \text{ in } (0,\infty),
\end{aligned} \tag{3.21}
$$

and the initial conditions (3.3) become

$$
\begin{aligned}
& u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{on } \Omega, \\
& v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{on }]L_1,L_2[, \\
& z(x,\rho,t) = u_t(x,-\tau(0)\rho) = f_0(x,-\tau(0)\rho) \quad \text{in } \Omega\times]0,\tau(0)[, \\
& \eta^t(x,0) = 0 \quad \text{on } \Omega, \\
& \eta^t(0,s) = \eta^t(L_3,s) = 0, \quad \text{on }]0,+\infty[, \\
& \eta^0(x,s) = \eta_0(s) \quad \text{in } \Omega\times]0,+\infty[.
\end{aligned} \tag{3.22}
$$

3.3 Well Posedness of the problem

In this section, we will prove the existence and uniqueness of the solution to problem (3.20)-(3.22), using semigroup theory techniques based on Kato's variable norm technique. To do this, we begin by transforming the system (3.20)-(3.22) into a Cauchy problem. Indeed, we let $u_t = \varphi$, $v_t = \psi$ and $\eta^t = \omega$, which leads to writing the system (3.20)-(3.22) as follows:

$$
\begin{cases}
U_t &= \mathcal{A}(t)U, \\
U(0) &= U_0 = (u_0, v_0, u_1, v_1, f_0(x,-\tau(0)\rho), \eta_0)^T.
\end{cases} \tag{3.23}
$$

Where $U = (u, v, \varphi, \psi, z, \omega)^T$ and the operator $\mathcal{A}$ defined by

$$\mathcal{A}(t)U = \begin{pmatrix} \varphi \\ \psi \\ lu_{xx} + \int_0^{+\infty} g(s)\omega_{xx}(s)ds - \mu_1(t)\varphi - \mu_2(t)z(x,1,t) \\ bv_{xx} \\ -\frac{1-\tau'(t)\rho}{\tau(t)} z_\rho \\ -\omega_s + \varphi \end{pmatrix}. \tag{3.24}$$

So

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ l\partial_{xx} & 0 & -\mu_1(t)I & 0 & -\mu_2(t)I & \int_0^\infty g(s)\partial_{xx}ds \\ 0 & b\partial_{xx} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho & 0 \\ 0 & 0 & I & 0 & 0 & -\partial_s \end{pmatrix}.$$

We define the energy $E(t)$ by

$$E(t) = \frac{1}{2}\int_\Omega [u_t^2(x,t) + lu_x^2(x,t)]\ dx + \frac{1}{2}\int_{L_1}^{L_2} \left[v_t^2(x,t) + bv_x^2(x,t)\right]\ dx$$
$$+ \frac{\tau(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho\ dx + \frac{1}{2}\int_\Omega\int_0^1 g(s)|\eta_x^t(x,s)|^2\ ds\ dx, \tag{3.25}$$

where $\xi(t)$ is a non-increasing function of class $\mathcal{C}^1(\mathbb{R}_+)$ such that

$$\xi(t) = \overline{\xi}\mu_1(t), \tag{3.26}$$

and $\overline{\xi}$ is positive constant satisfying

$$\frac{\beta}{\sqrt{1-d}} < \overline{\xi} < 2 - \frac{\beta}{\sqrt{1-d}}, \tag{3.27}$$

and we will show later in section (4.4) how to calculate this quantity.

We define the phase space as

$$\mathcal{H} = X_\star \times L^2(\Omega) \times L^2(]L_1, L_2[) \times L^2(\Omega\times]0,1[) \times L_g^2(\mathbb{R}_+, H^1(\Omega)), \tag{3.28}$$

where

$$X_\star = \{(u,v) \in H^1(\Omega) \times H^1(]L_1, L_2[) : u(0,t) = u(L_3,t) = 0, u(L_i,t) = v(L_i,t),$$
$$lu_x(L_i,t) + \int_0^\infty g(s)\eta_x^t(L_i,s)ds = bv(L_i,t), i = 1,2\},$$

3.3. Well Posedness of the problem

and $L_g^2(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of H^1-valued function on $\mathbb{R}_+$ equipped with the inner product

$$\langle \phi, \psi \rangle_{L_g^2(\mathbb{R}_+, H^1(\Omega))} = \int_\Omega \int_0^{+\infty} g(s)\phi_x(s)\psi_x(s)dsdx.$$

Set $U := (u, v, \varphi, \psi, z, \omega)^T, \bar{U} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\psi}, \bar{z}, \bar{\omega})^T$, we define the inner product in $\mathcal{H}$

$$\begin{aligned}\langle U, \bar{U} \rangle_{\mathcal{H}} = &\int_\Omega (lu_x\bar{u}_x + \varphi\bar{\varphi})\, dx + \int_{L_1}^{L_2} (bv_x\bar{v}_x + \psi\bar{\psi})\, dx + \tau(t)\xi(t) \int_\Omega \int_0^{+\infty} z\bar{z}d\rho dx \\ &+ \int_\Omega \int_0^{\infty} g(s)\omega_x(s)\bar{\omega}_x(s)dsdx.\end{aligned} \tag{3.29}$$

The domain of $\mathcal{A}(t)$ is

$$D(\mathcal{A}(t)) = \left\{ \begin{array}{l} U := (u, v, \varphi, \psi, z, \omega)^T \in \mathcal{H} : (u, v) \in \left\{ (H^2(\Omega) \times H^2(]L_1, L_2[)) \cap X_\star \right\}, \\ \varphi \in H^1(\Omega), \psi \in H^1(]L_1, L_2[), \omega \in L_g^2(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)), \\ \omega_s \in L_g^2(\mathbb{R}_+, H^1(\Omega)), z_\rho \in L^2\left(]0, 1[, L^2(\Omega)\right), \omega(x, 0) = 0, z(x, 0) = \varphi(x). \end{array} \right\}. \tag{3.30}$$

Notice that the domain of the operator $\mathcal{A}(t)$ does not dependent on time t.i.e

$$D\left(\mathcal{A}(t)\right) = D\left(\mathcal{A}(0)\right), \qquad \forall t > 0. \tag{3.31}$$

Next, we will introduce and prove the existence and uniqueness of the solution to problem (3.20)-(3.22) in the following theorem:

Theorem 3.1 *(see [34, 35]) Assume that*

(i) $Y = D(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$.

(ii) $D\left(\mathcal{A}(t)\right) = D\left(\mathcal{A}(0)\right), \qquad \forall t > 0.$

(iii) For all $t \in [0, T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ *is stable with stability constants* C *and* m *independent of* t.

(iv) $\partial_t\mathcal{A}$ *belongs to* $L_\star^\infty([0, T], B(Y, \mathcal{H}))$*, the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0, T]$ *into the set* $B(Y, \mathcal{H})$ *of bounded operators from* Y *into* $\mathcal{H}$.

Then, problem (3.23) has a unique solution $U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H}))$ *for any initial datum in* $D(\mathcal{A}(0))$.

Using the time-dependent inner product (3.29) and the Theorem 3.1, we get the following result of existence and uniqueness of global solutions to the problem (3.23)

Theorem 3.2 *For any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution U satisfying $U \in C([0,+\infty[,\mathcal{H})$ for the problem (3.20)-(3.22). Moreover, if $U_0 \in D(\mathcal{A}(0))$, then $U \in C([0,+\infty[, D(\mathcal{A}(0)) \cap C^1(U \in C([0,+\infty[,\mathcal{H})$.*

Proof. Let $(\overline{u},\overline{v},\overline{\varphi},\overline{\psi},\overline{z},\overline{\omega})^T$ be orthogonal to $D(\mathcal{A}(0))$

$$\begin{aligned}0 &= \left\langle (u,v,\varphi,\psi,z,\omega)^T, (\overline{u},\overline{v},\overline{\varphi},\overline{\psi},\overline{z},\overline{\omega})^T \right\rangle_{\mathcal{H}} \\ &= \int_\Omega l u_x \overline{u}_x dx + \int_{L_1}^{L_2} b v_x \overline{v}_x dx + \int_\Omega \varphi\overline{\varphi} dx + \int_{L_1}^{L_2} \psi \overline{\psi}_x dx \\ &\quad + \tau(t)\xi(t) \int_\Omega \int_0^1 z\overline{z} \quad d\rho dx + \int_\Omega \int_0^{+\infty} g(s)\omega_x(s)\overline{\omega}_x(s) ds dx,\end{aligned} \tag{3.32}$$

for all $(u,v,\varphi,\psi,z,\omega)^T \in D(\mathcal{A}(0))$.

For $u = v = \varphi = \psi = z = 0$ and $\omega \in D(\mathbb{R}_+,\Omega)$, then $\omega_x \in D(\mathbb{R}_+,\Omega)$. As $(0,0,0,0,0,\omega) \in D(\mathcal{A}(0))$, then from (3.32), we get

$$\int_\Omega \int_0^{+\infty} g(s)\omega_x(s)\overline{\omega}_x(s) ds dx = 0.$$

Since $D(\mathbb{R}_+,\Omega)$ is dense in $L^2_g(\mathbb{R}_+, H^1(\Omega))$ equipped with the inner product $\langle .,. \rangle_{L^2_g(\mathbb{R}_+,H^1(\Omega))}$ we deduce that $\overline{\omega} = 0$.

For $u = v = \varphi = \psi = \omega = 0$ and $z \in D(\Omega \times]0,1[)$. As $(0,0,0,0,z,0) \in D(\mathcal{A}(0))$, we obtain

$$\int_\Omega \int_0^1 z\overline{z}\, d\rho dx = 0.$$

Since $D(\Omega \times]0,1[)$ is dense in $L^2(\Omega \times (0,1))$, we deduce that $\overline{z} = 0$.

For $u = v = \psi = z = \omega = 0$ and $\varphi \in D(\Omega)$. As $(0,0,\varphi,0,0,0) \in D(\mathcal{A}(0))$, by (3.32) we have

$$\int_\Omega \varphi\overline{\varphi}\, dx = 0.$$

Since $D(\Omega)$ is dense in $L^2(\Omega)$, we deduce that $\overline{\varphi} = 0$.

In the same way, by taking $\psi \in D(]L_1,L_2[)$ in (3.32) we get

$$\int_{L_1}^{L_2} \psi\overline{\psi}\, dx = 0.$$

The density of $D(]L_1,L_2[)$ in $L^2(]L_1,L_2[)$ implies that $\overline{\psi} = 0$.

Finally, for $(u,v) \in D(\Omega \times (]L_1,L_2[))$, then $(u_x,v_x) \in D(\Omega \times (]L_1,L_2[))$, we obtain from (3.32) that

$$\int_\Omega l u_x \overline{u}_x\, dx + \int_{L_1}^{L_2} b v_x \overline{v}_x\, dx = 0.$$

Since $D(\Omega \times]L_1,L_2[)$ is dense in $L^2(\Omega \times]L_1,L_2[)$, we deduce that $(\overline{u}_x,\overline{v}_x) = (0,0)$.

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Consequently,

$$D\left(\mathcal{A}(0)\right) \text{ is dense in } \mathcal{H}. \tag{3.33}$$

Let $U = (u, v, \varphi, \psi, z, \omega)^T \in D\left(\mathcal{A}(t)\right)$, then

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t =& \int_\Omega l\varphi_x u_x dx + \int_\Omega (lu_{xx} + \int_0^\infty g(s)\omega_{xx}(s)ds - \mu_1(t)\varphi - \mu_2(t)z(x,1,t))\varphi dx \\ &+ \int_{L_1}^{L_2} bv_x\psi_x dx + \int_{L_1}^{L_2} bv_{xx}\psi dx + \tau(t)\xi(t)\int_\Omega\int_0^1 \left(-\frac{1-\tau'^{(}t)\rho}{\tau}(t)z_\rho\right) zd\rho dx \\ &+ \int_\Omega\int_0^1 g(s)\left(-\omega_s + \rho\right)_x \omega_x ds dx.\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t =& -\mu_1(t)\int_\Omega \varphi^2 dx - \mu_2(t)\int_\Omega z(x,1,t)\varphi dx + [bv_x\psi]_{L_1}^{L_2} \\ &+ \left[lu_x\varphi + \left(\int_0^\infty g(s)\omega_x(s)ds\right)\varphi\right]_{\partial\Omega} - \int_\Omega\int_0^\infty g(s)\omega_{xs}\omega_x ds dx \\ &- \frac{\xi(t)}{2}\int_\Omega\int_0^\infty (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)d\rho dx.\end{aligned}$$

Since

$$(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho) = \frac{\partial}{\partial\rho}\left((1-\tau'(t)\rho)z^2(x,\rho)\right) + \tau'(t)z^2(x,\rho),$$

we get

$$\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)d\rho = (1-\tau'(t)\rho)z^2(x,1) - z^2(x,0) + \tau'(t)\int_0^1 z^2(x,\rho)d\rho,$$

so we have

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t =& -\mu_1(t)\int_\Omega \varphi^2 dx - \mu_2(t)\int_\Omega z(x,1)\varphi dx + \frac{1}{2}\int_\Omega\int_0^1 g'(s)\omega_x^2 ds dx \\ &+ \frac{\xi(t)}{2}\int_\Omega z^2(x,0)dx - \frac{\xi(t)(1-\tau'(t))}{2}\int_\Omega z^2(x,1)dx \\ &- \frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho)d\rho dx.\end{aligned}$$

By using (3.61) and (3.62) we have

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t \leq& -\mu_1(t)\left(1 - \frac{\overline{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega \varphi^2 dx \\ &- \mu_1(t)\left(\frac{\overline{\xi}(1-\tau'(t))}{2} - \frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1)dx \\ &+ \frac{1}{2}\int_\Omega\int_0^1 g'(s)\omega_x^2 ds dx \\ &- \frac{\xi(t)|\tau'(t)|}{2\tau(t)}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho)dx.\end{aligned}$$

Then, we conclude that

$$\begin{aligned}\langle \mathcal{A}(t)U,U\rangle_t - k(t)\langle U,U\rangle_t \leq & -\mu_1(t)\left(1-\frac{\overline{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega \varphi^2 dx \\ & -\mu_1(t)\left(\frac{\overline{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2 dx \\ & +\frac{1}{2}\int_\Omega\int_0^1 g'(s)\omega_x^2 ds dx \leq 0,\end{aligned}$$

where $\kappa(t)=\frac{\sqrt{1+\tau'(t)^2}}{2\tau(t)}$. Thus the operator $\tilde{\mathcal{A}}(t)=\mathcal{A}(t)-k(t)I$ is dissipative.

Next, we will prove that $\lambda I-\mathcal{A}(t)$ is surjective for fixed $t>0$ and $\lambda>0$. Let $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in\mathcal{H}$. We look for $U=(u,v,\varphi,\psi,z,\omega)^T\in D(\mathcal{A}(t))$ solution of

$$(\lambda I-\mathcal{A}(t))\,U=F,$$

satisfying,

$$\begin{cases}\lambda u-\varphi=f_1,\\ \lambda v-\psi=f_2,\\ \lambda\varphi-lu_{xx}-\int_0^\infty g(s)\omega_{xx}(s)-\mu_1(t)\varphi-\mu_2 z(x,1,t)=f_3,\\ \lambda\psi-bv_{xx}=f_4,\\ \lambda z+\frac{1-\tau'(t)\rho}{\tau(t)}z_\rho=f_5,\\ \lambda\omega+\omega_s-\varphi=f_6.\end{cases} \tag{3.34}$$

Assume that we have found u and v with the the appropriated regularity, then

$$\varphi=\lambda u-f_1. \tag{3.35}$$

$$\psi=\lambda v-f_2. \tag{3.36}$$

It is not hard to see that $\varphi\in H^1(\Omega)$ and $\psi\in H^1(L_1,L_2)$. Furthermore we can find z such that

$$z(x,0)=\varphi(x)\qquad ,x\in\Omega. \tag{3.37}$$

From the fifth equation in (3.34), we find if $\tau'(t)\neq 0$

$$z(x,\rho)=\varphi(x)e^{\varrho(\rho,t)}+\tau(t)e^{\varrho(\rho,t)}\int_0^\rho\frac{f_5(x,s)}{1-s\tau'(s)}e^{-\varrho(s,t)}ds, \tag{3.38}$$

where,

$$\varrho(\rho,t)=\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\rho\tau'(t)).$$

If $\tau'(t) = 0$

$$z(x,\rho) = \varphi(x)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho f_5(x,s)e^{\lambda\tau(t)s}ds. \tag{3.39}$$

Then we have, if $\tau'(t) \neq 0$

$$z(x,1) = \lambda u e^{\varrho(1,t)} - f_1 e^{\varrho(1,t)} + \tau(t)e^{\varrho(1,t)}\int_0^1 \frac{f_5(x,s)}{1 - s\tau'(s)}e^{-\varrho(s,t)}ds, \tag{3.40}$$

and if $\tau(t) = 0$

$$z(x,1) = \lambda u(x)e^{-\lambda\tau(t)} - f_1(x,1)e^{-\lambda\tau(t)} + \tau(t)e^{-\lambda\tau(t)}\int_0^1 f_5(x,s)e^{\lambda\tau(t)s}ds. \tag{3.41}$$

From the last equation in (3.34), we get

$$\begin{aligned} \omega(x,s) &= \left(\int_0^s e^{\lambda y}(f_6(x,y) + \varphi(x))dy\right)e^{-\lambda s} \\ &= \left(\int_0^s e^{\lambda y}(f_6(x,y) + \lambda u(x) - f_1(x))dy\right)e^{-\lambda s}. \end{aligned} \tag{3.42}$$

From (3.34), (3.35), (3.36) and (3.42), the functions u and v satisfies

$$\begin{gathered} \left(\lambda^2 + \lambda\mu_1(t) + \lambda N_1\right)u - \bar{l}u_{xx} = \mu_1(t)f_1 + \mu_2(t)N_2 + \lambda f_1 + f_3 \\ \lambda^2 v - bv_{xx} = f_4 + \lambda f_2, \end{gathered} \tag{3.43}$$

where

$$\bar{l} = l + \lambda\int_0^\infty g(s)e^{-\lambda s}\left(\int_0^s e^{\lambda y}dy\right)ds,$$

and

$$N_1 = \begin{cases} \mathrm{e}^{\varrho(1,t)}, & \tau'(t) \neq 0 \\ \mathrm{e}^{-\lambda\tau(t)}, & \tau'(t) = 0, \end{cases}$$

and

$$N_2 = \begin{cases} f_1 e^{\varrho(1,t)} - \tau(t)e^{\varrho(1,t)}\int_0^1 \frac{f_5(x,s)}{1-s\tau'(s)}e^{-\varrho(s,t)}ds, & \tau'(t) \neq 0 \\ f_1 e^{-\lambda\tau(t)} - \tau(t)e^{-\lambda\tau(t)}\int_0^1 f_5(x,s)e^{\lambda\tau(t)s}ds, & \tau'(t) = 0. \end{cases}$$

The problem (3.43) is equivalent to

$$\Phi((u,v),(\overline{u},\overline{v})) = L(\overline{u},\overline{v}), \tag{3.44}$$

where the bilinear form $\Phi : (X_\star, X_\star) \longrightarrow \mathbb{R}$ and the linear form $L : X_\star \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} \Phi((u,v),(\overline{u},\overline{v})) = &\int_\Omega \left[\left(\lambda^2 + \lambda\mu_1(t) + \lambda N_1\right)u\overline{u} + \bar{l}u_x(\overline{u})_x\right]dx \\ &- \left[\bar{l}u_x\overline{u}\right]_{\partial\Omega} + \int_{L_1,}^{L_2}\left(\lambda^2 v\overline{v} + bv_x(\overline{v})_x\right)dx - [bv_x\overline{v}]_{L_1,}^{L_2} \end{aligned}$$

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and

$$L(\overline{u},\overline{v}) = (\mu_1(t)f_1 + \mu_2(t)N_2)\,\overline{u}dx + \int_\Omega (f_3+\lambda f_1)\overline{u}dx + \int_{L_1,}^{L_2} (f_4+\lambda f_2)\overline{v}dx.$$

It is easy to see that Φ continuous and coercive and L is continuous. Applying the Lax-Milgram theorem the problem (3.23) has unique solution $(u,v)\in X_\star$. It follows from the classical elliptic regularity and (3.43) that

$$(u,v)\in H^2(\Omega)\times H^2(]L_1,L_2[).$$

Therefore the operator $\lambda I-\mathcal{A}(t)$ is surjective for any $\lambda>0$ and $t>0$. Since $\kappa(t)>0$, then

$$\lambda I-\tilde{\mathcal{A}}(t) = (\lambda+\kappa(t))I-\mathcal{A}(t) \text{ is surjective, for all } \lambda>0, \text{ and } t>0. \tag{3.45}$$

Let $\Psi=(u,v,\varphi,\psi,z,\omega)^T$, then

$$\begin{aligned}||\Psi||_t^2 = &\int_\Omega lu_x^2dx + \int_{L_1}^{L_2} bv_x^2dx + \int_\Omega \varphi^2dx + \int_{L_1}^{L_2}\psi^2dx\\ &+\tau(t)\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx + \int_\Omega\int_0^{+\infty} g(s)\omega_x^2(s)dsdx.\end{aligned}$$

For all t,$r\in[0,T]$, we have

$$\begin{aligned}||\Psi||_t^2-||\Psi||_t^2\exp^{\frac{c}{\tau_0}|t-r|} = &\left(1-\exp^{\frac{c}{\tau_0}|t-r|}\right)\times\left(\int_\Omega lu_x^2dx + \int_{L_1}^{L_2} bv_x^2dx + \int_\Omega\varphi^2dx\right.\\ &\left.+\int_{L_1}^{L_2}\psi^2dx + \int_\Omega\int_0^{+\infty} g(s)\omega_x^2(s)dsdx\right)\\ &+\left(\xi(t)\tau(t)-\xi(r)\tau(r)\exp^{\frac{c}{\tau_0}|t-r|}\right)\int_\Omega\int_0^{+\infty} z^2(x,\rho,t)d\rho dx.\end{aligned}$$

It is claire that $1-\exp^{\frac{c}{\tau_0}|t-r|}\leq 0$.

Moreover, we have

$$\tau(t)=\tau(r)+\tau'(\theta)(t-r),\theta\in(r,t).$$

Since ξ positive and non increasing function , we get

$$\xi(t)\tau(t)\leq\xi(r)\tau(r)+\xi(r)\tau'(\theta)(t-r),$$

so that

$$\frac{\xi(t)\tau(t)}{\xi(r)\tau(r)}\leq 1+\frac{c}{\tau_0}|t-s|\leq\exp^{\frac{c}{\tau_0}|t-r|},$$

which implies

$$\xi(t)\tau(t)-\xi(r)\tau(r)\exp^{\frac{c}{\tau_0}|t-r|}\leq 0.$$

Thus

$$\frac{||\Psi||_t^2}{||\Psi||_r^2} \leq \exp^{\frac{c}{\tau_0}|t-r|}, \tag{3.46}$$

which proves condition (iii) in Theorem (3.1).

Furthermore, since $\kappa'(t) = \frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+\tau'(t)^2}} - \frac{\tau'(t)\sqrt{1+\tau'(t)^2}}{2\tau(t)^2}$ is bounded on $[0,T]$, for all $T>0$, and

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ -\mu_1'(t)\varphi - \mu_2'(t)z(;1) \\ 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{1+\tau(t)^2} z_\rho \\ 0 \end{pmatrix},$$

with $\frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{1+\tau(t)^2}$ bounded on [0; T] by (3.4) and (3.27). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_\star^\infty([0,T], B(D(\mathcal{A}(0)),\mathcal{H})). \tag{3.47}$$

The space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(\mathcal{A}(0)),\mathcal{H})$.

Then, (3.46), (3.34) and (3.45) imply that the family $\tilde{\mathcal{A}} = \left\{\tilde{\mathcal{A}}(t) : t \in [0,t]\right\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t, by theorem 1.9 from [34]. Therefore, the assumptions (i) (iv) of Theorem (3.1) are verified by (3.31), (3.46), (3.34), (3.47), (3.45) and (3.33), and thus, the problem

$$\begin{cases} \tilde{U}_t &= \tilde{\mathcal{A}}(t)\tilde{U}, \\ \tilde{U}(0) &= U_0 = (u_0, v_0, u_1, v_1, f_0(x, -\tau(0)\rho), \omega_0)^T, \end{cases} \tag{3.48}$$

has a unique solution $U \in C([0,+\infty[, D(\mathcal{A}(0)) \cap C^1([0,+\infty[,\mathcal{H})$ for $U_0 \in D(\mathcal{A}(0))$. The solution of (3.23) is then given by

$$U(t) = \exp\left(\int_0^t \kappa(s)ds\right)\tilde{U}(t),$$

then

$$\begin{aligned} \frac{d}{dt}U(t) &= \kappa(t)\exp\left(\int_0^t \kappa(s)ds\right)\tilde{U}(t) + \exp\left(\int_0^t \kappa(s)ds\right)\frac{d}{dt}\tilde{U}(t) \\ &= \left(\kappa(t) + \tilde{\mathcal{A}}(t)\right)\exp\left(\int_0^t \kappa(s)ds\right)\tilde{U}(t) \\ &= \mathcal{A}(t)\exp\left(\int_0^t \kappa(s)ds\right)\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which complete the proof. ■

3.3. Well Posedness of the problem

3.4 Stability result of solution

This section is dedicated to study of the asymptotic behavior. The main goal of this section is to study the stability of solutions to the system (3.20)-(3.22). This is the content of Theorem (3.3) where we show that the solution of problem (3.20)-(3.22) is exponentially stable. Our effort consists in building a suitable Lyapunov functional by the energy method. For this goal we present several technical lemmas.

Lemma 3.1 *Let* (u, v, η, z) *be a solution to the system* (3.20)-(3.22)*. Then the energy functional satisfies*

$$\begin{aligned}\frac{d}{dt}E(t) \leq & -\mu_1(t)\left(1-\frac{\overline{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega u_t^2\ dx \\ & -\mu_1(t)\left(\frac{\overline{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\ dx \\ & +\frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)\left|\eta_x^t(x,s)\right|^2\ ds\ dx \\ & \leq 0.\end{aligned} \tag{3.49}$$

Proof. Multiplying the first equation in (3.20) by $u_t(x,t)$ and integrating on Ω, we get

$$\begin{aligned}\int_\Omega u_{tt}(x,t)u_t(x,t)\ dx - l\int_\Omega u_{xx}(x,t)u_t(x,t)\ dx - \int_\Omega\left(\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)\ ds\right)u_t(x,t)\ dx \\ +\int_\Omega \mu_1(t)u_t^2(x,t)\ dx + \int_\Omega \mu_2(t)z(x,1,t)u_t(x,t)\ dx = 0.\end{aligned} \tag{3.50}$$

Since

$$\int_\Omega u_{tt}(x,t)u_t(x,t)\ dx = \frac{d}{dt}\left\{\frac{1}{2}\int_\Omega u_t^2(x,t)\ dx\right\},$$

and by using integrating by parts in the second term in (3.50)

$$\int_\Omega u_{xx}(x,t)u_t(x,t)\ dx = \frac{d}{dt}\left\{-\frac{1}{2}\int_\Omega u_x^2(x,t)\right\}\ dx, \tag{3.51}$$

by replacing (3.4) and (3.51) in (3.50), we obtain

$$\begin{aligned}\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega u_t^2(x,t)\ dx + \frac{l}{2}\int_\Omega u_x^2(x,t)\ dx\right\} = \int_\Omega\left(\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)\ ds\right)u_t(x,t)\ dx \\ -\mu_1(t)\int_\Omega u_t^2(x,t)\ dx - \mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\ dx.\end{aligned} \tag{3.52}$$

Again by multiplying the second equation in (3.20) by $v_t(x,t)$, integrating on $]L_1, L_2[$ and by using integrating by parts, we get

$$\int_{L_1}^{L_2} v_{tt}(x,t)v_t(x,t)\ dx - b[v_x(x,t)v_t(x,t)]_{L_1}^{L_2} + b\int_{L_1}^{L_2} v_x(x,t)v_{xt}(x,t)dx = 0,$$

we obtain the result

$$\frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2} v_t^2(x,t)dx - b[v_x(x,t)v_t(x,t)]_{L_1}^{L_2} + \frac{b}{2}\frac{d}{dt}\int_{L_1}^{L_2} v_x^2(x,t)dx = 0, \tag{3.53}$$

so

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{L_1}^{L_2}\left(v_t^2(x,t) + bv_x^2(x,t)\right)dx\right\} = b[v_x(x,t)v_t(x,t)]_{L_1}^{L_2}. \tag{3.54}$$

Now we multiply the third equation in (3.20) by $\xi(t)z(x,\rho,t)$ and integrating on $\Omega\times(0,1)$ we get

$$\int_\Omega\int_0^{+\infty}\tau(t)\xi(t)z(x,\rho,t)z_t(x,\rho,t)d\rho dx \\ + \int_\Omega\int_0^1(1-\tau'(t)\rho)\xi(t)z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx = 0, \tag{3.55}$$

so

$$\frac{d}{dt}\left\{\frac{\tau(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\right\} - \frac{\tau'(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\ - \frac{\tau(t)\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx + \frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\delta}{\delta\rho}(z(x,\rho,t))^2 d\rho dx = 0,$$

and

$$\frac{d}{dt}\left\{\frac{\tau(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\right\} = \frac{\tau'(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\ + \frac{\tau(t)\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\ - \frac{\xi(t)}{2}\left\{\int_\Omega\left([(1-\tau'(t)\rho)z^2(x,\rho,t)]_0^1 + \int_0^1\tau'(t)z^2(x,\rho,t)\right)dx\right\},$$

then

$$\frac{d}{dt}\left\{\frac{\tau(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\right\} = \frac{\tau(t)\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\ - \frac{\xi(t)}{2}\left[\int_\Omega((1-\tau'(t))z^2(x,\rho,t) - z^2(x,0,t))dx\right],$$

we find

$$\frac{d}{dt}\left\{\frac{\tau(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\right\} = \frac{\tau(t)\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\ + \frac{\tau'(t)\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx + \frac{\xi(t)}{2}\int_\Omega(z^2(x,0,t) - z^2(x,1,t))dx. \tag{3.56}$$

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Finally by mutiplying the fourth equation in (3.20) by $\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds$, and integrating over Ω, we get

$$\underbrace{\int_\Omega \left(\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds\right)\eta_t^t(x,s)dsdx}_{I_1} + \underbrace{\int_\Omega \left(\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds\right)\eta_s^t(x,s)dsdx}_{I_2}$$
$$= \int_\Omega \left(\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds\right)u_t(x,t)dx, \quad (3.57)$$

using integrating by parts, we find

$$\begin{aligned} I_1 &= \left[\left(\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)\eta_t^t(x,s)\right]_{\delta\Omega} - \int_\Omega \left(\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)\eta_{tx}^t(x,s)dx \\ &= -\frac{d}{dt}\left[\frac{1}{2}\int_\Omega\int_0^{+\infty} g(s)|\eta_x^t(x,s)|^2 dsdx\right], \end{aligned} \quad (3.58)$$

and

$$\begin{aligned} I_2 &= \left[\left(\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)\eta_s^t(x,s)\right]_{\partial\Omega} - \int_\Omega \left(\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)\eta_{xs}^t(x,s)dx \\ &= -\frac{1}{2}\int_\Omega \frac{\partial}{\partial s}\left(\int_0^{+\infty} g(s)|\eta_x^t(x,s)|^2 ds\right) + \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx \\ &= -\frac{1}{2}\int_\Omega \left[g(s)|\eta_x^t(x,s)|^2\right]_0^{+\infty} dx + \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx, \end{aligned} \quad (3.59)$$

thus

$$\begin{aligned} \frac{d}{dt}\left\{\frac{1}{2}\int_\Omega\int_0^{+\infty} g(s)|\eta_x^t(x,s)|^2 dsdx\right\} &= \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx \\ &\quad - \int_\Omega \left(\int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds\right)u_t dx. \end{aligned} \quad (3.60)$$

From (3.52),(3.54),(3.56) and (3.60) together with conditions (3.26) and (3.27), we have since $z(x,0,t) = u_t(x,t)$

$$\begin{aligned} \frac{d}{dt}E(t) =& \frac{\xi(t)}{2}\int_\Omega u_t^2 dx - \frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)dx \\ &+ \frac{\tau'(t)\xi(t)}{2}\int_\Omega z^2(x,1,t)dx + \frac{\tau(t)\xi'(t)}{2}\int_\Omega\int_0^{+\infty} z^2(x,\rho,t)d\rho dx \\ &- \mu_1(t)\int_\Omega u_t^2 dx - \mu_2(t)\int_\Omega z(x,1,t)u_t dx \\ &+ \frac{1}{2}\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx. \end{aligned} \quad (3.61)$$

3.4. Stability result of solution

Due to young's inequality , we obtain

$$\mu_2(t)\int_\Omega z(x,1,t)u_t dx \leq \frac{|\mu_2(t)|}{2\sqrt{1-d}}\int_\Omega u_t^2 dx + \frac{|\mu_2(t)|\sqrt{1-d}}{2}\int_\Omega z^2(x,1,t)dx. \tag{3.62}$$

Inserting (3.62) into (3.61), we get

$$\begin{aligned}\frac{d}{dt}E(t) \leq &-\left(\mu_1(t)-\frac{\xi(t)}{2}-\frac{|\mu_2(t)|}{2\sqrt{1-d}}\right)\int_\Omega u_t^2 dx\\ &-\left(\frac{\xi(t)}{2}-\frac{\tau'(t)\xi(t)}{2}-\frac{|\mu_2(t)|\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)dx\\ &+\frac{\tau(t)\xi'(t)}{2}\int_\Omega\int_0^{+\infty} z^2(x,\rho,t)d\rho dx\\ &+\frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 ds dx.\end{aligned} \tag{3.63}$$

Take in account (3.26) and (3.27), then (3.49) is verified. ■

We define the functional

$$\mathcal{D}(t)=\int_\Omega u u_t dt+\int_{L_1}^{L_2} v v_t dt. \tag{3.64}$$

Lemma 3.2 *The functional $\mathcal{D}(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) \leq &\left(1+\frac{1}{4\epsilon_1}\right)\int_\Omega u_t^2 dx+\left(\epsilon+2\epsilon_1\mu_1^2(0)c^2-l\right)\int_\Omega u_x^2 dx+\int_{L_1}^{L_2} v_t^2 dx\\ &-b\int_{L_1}^{L_2} v_x^2 dx+\frac{g_0}{4\epsilon}\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx+\frac{1}{4\epsilon_1}\int_\Omega z^2(x,1,t)dx.\end{aligned} \tag{3.65}$$

Proof. *Taking the derivative of $\mathcal{D}(t)$ with respect to t and using* (3.20)

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) &= \int_\Omega u_t^2 dx+\int_\Omega u u_{tt} dx+\int_{L_1}^{L_2} v_t^2 dx+\int_{L_1}^{L_2} v v_{tt} dx\\ &=\int_\Omega u_t^2 dx+\int_\Omega u\left[l u_{xx}+\int_0^\infty g(s)\eta_{xx}^t(x,s)ds-\mu_1(t)u_t-\mu_2(t)z(x,1,t)\right]\\ &\quad+\int_{L_1}^{L_2} v_t^2 dx+\int_{L_1}^{L_2} v(b v_{xx})dx,\end{aligned}$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) =& \int_\Omega u_t^2 dx + \left[u(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds)\right]_{\partial\Omega} \\ &- l\int_\Omega u_x^2 dx - \int_\Omega u_x \int_0^\infty g(s)\eta_x^t(x,s)dsdx - \mu_1(t)\int_\Omega uu_t dx \\ &- \mu_2(t)\int_\Omega uz(x,1,t)dx + \int_{L_1}^{L_2} v_t^2 dx + [bvv_x]_{L_1}^{L_2} - b\int_{L_1}^{L_2} v_x^2 dx \\ =& \int_\Omega u_t^2 dx - [bv_x v_t]_{L_1}^{L_2} - l\int_\Omega u_x^2 dx - \int_\Omega u_x \int_0^\infty g(s)\eta_x^t(x,s)dsdx \\ &- \mu_1(t)\int_\Omega uu_t dx - \mu_2(t)\int_\Omega uz(x,1,t)dx + \int_{L_1}^{L_2} v_t^2 dx - b\int_{L_1}^{L_2} v_x^2 dx.\end{aligned}$$

Where we use that

$$\begin{aligned}\left[u(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds)\right]_{\partial\Omega} =& u(L_1,t)(lu_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)ds) \\ &- u(L_2,t)(lu_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)ds) \\ =& -[bv_x v_t]_{L_1}^{L_2},\end{aligned} \tag{3.66}$$

then

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) \leq& \int_\Omega u_t^2 dx - l\int_\Omega u_x^2 dx + \left|\int_\Omega u_x \int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx\right| \\ &+ \left|\mu_1(t)\int_\Omega uu_t dx\right| + \left|\mu_2(t)\int_\Omega uz(x,1,t)dx\right| + \int_{L_1}^{L_2} v_t^2 dx - b\int_{L_1}^{L_2} v_x^2 dx.\end{aligned} \tag{3.67}$$

By applying Young's inequality, we have

$$\left|\int_\Omega u_x \int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx\right| \leq \epsilon\int_\Omega u_x^2 dx + \frac{g_0}{4\epsilon}\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \tag{3.68}$$

From $(\boldsymbol{H1})$, $(\boldsymbol{H2})$ *and* $(\boldsymbol{H4})$ *together, we have*

$$\left|\mu_1(t)\int_\Omega uu_t dx\right| + \left|\mu_2(t)\int_\Omega uz(x,1,t)dx\right| \leq \mu_1(0)\int_\Omega |uu_t|\, dx + \beta\mu_1(0)\int_\Omega |uz(x,1,t)|\, dx, \tag{3.69}$$

we replace (3.68) *and* (3.69) *in* (3.67), *we obtain*

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) \leq& \int_\Omega u_t^2 dx - l\int_\Omega u_x^2 dx + \epsilon\int_\Omega u_x^2 dx + \frac{g_0}{4\epsilon}\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\ &+ \mu_1(0)\int_\Omega |uu_t|\, dx + \beta\mu_1(0)\int_\Omega |uz(x,1,t)|\, dx + \int_{L_1}^{L_2} v_t^2 dx - b\int_{L_1}^{L_2} v_x^2 dx.\end{aligned} \tag{3.70}$$

3.4. Stability result of solution

By the boundary condition (3.21), we have

$$u^2(x,t) = \left(\int_0^x u_x(x,t)dx \right)^2 \le L_1 \int_0^x u_x^2(x,t)dx, \quad x \in [0, L_1] \,,$$

$$u^2(x,t) \le (L_3 - L_2) \int_{L_2}^{L_3} u_x^2(x,t)dx,$$

which implies

$$\int_\Omega u^2(x,t)dx \le c^2 \int_\Omega u_x^2(x,t)dx, \quad x \in \Omega, \tag{3.71}$$

where $c = \max\{L_1, L_3 - L_2\}$ *is the Poincarés constant. Using Young's inequality and* (3.71)*, we have .*

$$\mu_1(0) \int_\Omega |uu_t|\, dx \le \epsilon_1 \mu_1^2(0) c^2 \int_\Omega u_x^2 dx + \frac{1}{4\epsilon_1} \int_\Omega u_t^2 dx, \tag{3.72}$$

and

$$\beta\mu_1(0) \int_\Omega |uz(x,1,t)|\, dx \le \epsilon_1 \mu_1^2(0) c^2 \int_\Omega u_x^2 dx + \frac{\beta^2}{4\epsilon_1} \int_\Omega z^2(x,1,t)dx. \tag{3.73}$$

Substituting (3.72) *and* (3.73) *in* (3.70) *we get* (3.65)*.* ■

Now, we introduce the functional

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x \in [0, L_1], \\ \\ \frac{L_1}{2} - \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}(x - L_1), & x \in (L_1, L_2), \\ \\ x - \frac{L_2 + L_3}{2}, & x \in [L_2, L_3]. \end{cases} \tag{3.74}$$

It is easy to see that $q(x)$ is bounded, i.e. $|q(x)| \le M$, where

$$M = max\left\{ \frac{L_1}{2}, \frac{L_3 - L_2}{2} \right\}.$$

We define the functionals

$$\mathcal{F}_1(t) = -\int_\Omega q(x) u_t \left(l u_x + \int_0^\infty g(s) \eta_x^t(x,s)ds \right) dx, \tag{3.75}$$

$$\mathcal{F}_2(t) = -\int_{L_1}^{L_2} q(x) v_t v_x dx, \tag{3.76}$$

then we have the following results.

3.4. Stability result of solution

Lemma 3.3 *The functionals $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$ satisfy*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \leq & -\left[\frac{q(x)}{2}\left(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)^2 dx\right]_{\partial\Omega} + (l^2 + 2l^2\epsilon_1)\int_\Omega u_x^2 dx \\ & + (g_0 + 2g_0\epsilon_1)\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\ & + \left(\frac{l+g_0}{2} + \epsilon_1 M^2 + \frac{\mu_1^2(0)M^2}{4\epsilon_1}\right)\int_\Omega u_t^2(x,t)dx \\ & + \frac{\beta^2\mu_1^2(0)M^2}{4\epsilon_1}\int_\Omega z^2(x,1,t) - \left[\frac{l+g_0}{2}q(x)u_t^2 dx\right]_{\partial\Omega} \\ & - \frac{g(0)}{4\epsilon_1}\int_\Omega\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx,\end{aligned} \tag{3.77}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) \leq & -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\left(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} bv_x^2 dx\right) + \frac{L_1}{4}v_t^2(L_1) \\ & + \frac{L_3 - L_2}{4}v_t^2(L_2) + \frac{b}{4}\left((L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right).\end{aligned} \tag{3.78}$$

Proof. Differentiating $\mathcal{F}_1(t)$ with respect to t and using (3.20), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) = & -\int_\Omega q(x)u_{tt}\left(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx \\ & -\int_\Omega q(x)u_t\left(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)ds\right)dx \\ = & -\int_\Omega q(x)\left(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)ds - \mu_1(t)u_t(x,t) - \mu_2(t)z(x,1,t)\right) \\ & \times\left(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx - \int_\Omega q(x)u_t\left(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)ds\right)dx,\end{aligned} \tag{3.79}$$

then,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) = & -\int_\Omega q(x)\left(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)ds\right)\left(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx \\ & + \int_\Omega q(x)\left(\mu_1(t)u_t(x,t) + \mu_2(t)z(x,1,t)\right)\left(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx \\ & - \int_\Omega q(x)u_t\left(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)ds\right)dx.\end{aligned} \tag{3.80}$$

Integrating by parts the first term in (3.80), we obtain

$$\begin{aligned}
&-\int_\Omega q(x)\left(lu_{xx}+\int_0^\infty g(s)\eta^t_{xx}(x,s)ds\right)\left(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)dx\\
=&-\left[\frac{q(x)}{2}\left(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)^2 dx\right]_{\partial\Omega}\\
&+\frac{1}{2}\int_\Omega q'(x)\left(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)ds\right)^2 dx.
\end{aligned} \tag{3.81}$$

Similarly for the last term

$$\begin{aligned}
&-\int_\Omega q(x)u_t\left(lu_{xt}+\int_0^\infty g(s)\eta^t_{xt}(x,s)ds\right)dx\\
=&-l\int_\Omega q(x)u_t u_{xt}dx-\int_\Omega q(x)u_t\int_0^\infty g(s)\eta^t_{xt}(x,s)dsdx.
\end{aligned} \tag{3.82}$$

Using (3.16), (3.11) and integration by parts, thus the equation (3.83) becomes

$$\begin{aligned}
&=-\left[\frac{l}{2}q(x)u_t^2dx\right]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2dx-\int_\Omega q(x)u_t\int_0^\infty g(s)\left(u_t^t-\eta^t_s\right)_x dsdx\\
=&-\left[\frac{l}{2}q(x)u_t^2dx\right]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2dx\\
&-g_0\int_\Omega q(x)u_tu_{tx}dx+\int_\Omega q(x)u_t\int_0^\infty g(s)\eta^t_{sx}dsdx\\
=&-\left[\frac{l}{2}q(x)u_t^2dx\right]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2dx-\left[\frac{g_0}{2}q(x)u_t^2dx\right]_{\partial\Omega}+\frac{g_0}{2}\int_\Omega q'(x)u_t^2dx\\
&-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x dsdx\\
=&-\left[\frac{l+g_0}{2}q(x)u_t^2\right]_{\partial\Omega}+\frac{l+g_0}{2}\int_\Omega q'(x)u_t^2dx\\
&-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x dsdx,
\end{aligned} \tag{3.83}$$

where we use that

$$-\left[\int_\Omega q(x)u_tg(s)\eta^t_xdx\right]_0^\infty=0.$$

3.4. Stability result of solution

Inserting (3.81) and (3.83) in (3.80), we get

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t) = &-\left[\frac{q(x)}{2}\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)^2\right]_{\partial\Omega} \\
&+\frac{1}{2}\int_\Omega q'(x)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)^2 dx \\
&+\int_\Omega q(x)\left(\mu_1(t)u_t(x,t)+\mu_2(t)z(x,1,t)\right)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx \\
&-\left[\frac{l+g_0}{2}q(x)u_t^2\right]_{\partial\Omega}+\frac{l+g_0}{2}\int_\Omega q'(x)u_t^2dx \\
&-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta_x^t dsdx.
\end{aligned}
\tag{3.84}
$$

Using Minkowski and Young's inequalities, we have

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)^2 dx \\
&\leq l^2\int_\Omega u_x^2dx+g_0\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.
\end{aligned}
\tag{3.85}
$$

Young's inequality gives us that for any $\epsilon_1>0$,

$$
\begin{aligned}
&\left|\int_\Omega q(x)\mu_1(t)u_t(x,t)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\right| \\
&\leq \mu_1(0)\left|\int_\Omega q(x)u_t(x,t)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\right| \\
&\leq \mu_1(0)M\left|\int_\Omega u_t(x,t)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\right| \\
&\leq l^2\epsilon_1\int_\Omega u_x^2(x,t)dx+\frac{\mu_1^2(0)M^2}{4\epsilon_1}\int_\Omega u_t^2(x,t)dx \\
&+g_0\epsilon_1\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx,
\end{aligned}
\tag{3.86}
$$

and

$$
\begin{aligned}
&\left|\int_\Omega q(x)\mu_2(t)z(x,1,t)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\right| \\
&\leq \beta\mu_1(0)\left|\int_\Omega q(x)z(x,1,t)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\right| \\
&\leq \beta\mu_1(0)M\left|\int_\Omega z(x,1,t)\left(lu_x+\int_0^\infty g(s)\eta_x^t(x,s)ds\right)dx\right| \\
&\leq l^2\epsilon_1\int_\Omega u_x^2(x,t)dx+\frac{\beta^2\mu_1^2(0)M^2}{4\epsilon_1}\int_\Omega z^2(x,1,t) \\
&+g_0\epsilon_1\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.
\end{aligned}
\tag{3.87}
$$

It is clear that

$$\begin{aligned}&\left|\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta_x^t(x,s)dsdx\right|\\&\leq \epsilon_1 M^2\int_\Omega u_t^2dx-\frac{g(0)}{4\epsilon_1}\int_\Omega\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned}\tag{3.88}$$

Then by (3.85), (3.86), (3.87) and (3.88), we have (3.77).

Taking the derivative of $\mathcal{F}_2(t)$ and following the same arguments as in $\mathcal{F}_1(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t)=&\frac{d}{dt}\left[-\int_{L_1}^{L_2}q(x)v_tv_xdx\right]\\=&-\int_{L_1}^{L_2}q(x)v_{tt}v_xdx-\int_{L_1}^{L_2}q(x)v_tv_{xt}dx,\end{aligned}$$

using (3.20) and integration by parts, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t)&=\int_{L_1}^{L_2}q(x)v_xbv_{xx}dx-\int_{L_1}^{L_2}q(x)v_tv_{xt}dx\\&=\left[-\frac{b}{2}q(x)v_x^2dx\right]_{L_1}^{L_2}+\frac{1}{2}\int_{L_1}^{L_2}bq'(x)v_x^2dx+\left[-\frac{1}{2}q(x)v_t^2\right]_{L_1}^{L_2}+\frac{1}{2}\int_{L_1}^{L_2}q'(x)v_t^2dx,\end{aligned}$$

then by (3.74), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t)\leq&-\frac{L_1+L_3-L_2}{4(L_2-L_1)}\left(\int_{L_1}^{L_2}v_t^2dx+\int_{L_1}^{L_2}bv_x^2dx\right)+\frac{L_1}{4}v_t^2(L_1)\\&+\frac{L_3-L_2}{4}v_t^2(L_2)+\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right).\end{aligned}$$

Which complete the proof of Lemma 3. ■

We define the functional

$$\mathcal{F}_3(t)=\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,\tag{3.89}$$

then we have the following estimate.

Lemma 3.4 *(see [39]). The functional $F_3(t)$ satisfies*

$$\frac{d}{dt}\mathcal{F}_3(t)\leq -2\mathcal{F}_3+\overline{\xi}\int_\Omega u_t^2dx.\tag{3.90}$$

Proof. Taking the derivative of $\mathcal{F}_3(t)$ with respect to t , we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_3(t)=&\overline{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\&-2\overline{\xi}\tau(t)\tau'(t)\rho\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\&+2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z_t(x,\rho,t)z(x,\rho,t)d\rho dx,\end{aligned}\tag{3.91}$$

by using the third equation in (3.20), the last term in (3.91) can be rewritten as follows

$$\begin{aligned}
&2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z_t(x,\rho,t)z(x,\rho,t)d\rho dx\\
&=2\overline{\xi}\int_\Omega\int_0^1 e^{-2\rho\tau(t)}(\tau'(t)\rho-1)z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx\\
&=\overline{\xi}\int_\Omega\int_0^1(\tau'(t)\rho-1)2e^{-2\rho\tau(t)}z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx,
\end{aligned}\tag{3.92}$$

also, one can see that

$$\begin{aligned}
&2\overline{\xi}\int_\Omega\int_0^1 e^{-2\rho\tau(t)}(\tau'(t)\rho-1)z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx\\
&=\overline{\xi}\int_\Omega\int_0^1(\tau'(t)\rho-1)\left[\frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)+2\tau(t)e^{-2\rho\tau(t)}z^2(x,\rho,t)\right]d\rho dx\\
&=\overline{\xi}\int_\Omega\int_0^1(\tau'(t)\rho-1)\frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)d\rho dx\\
&+\overline{\xi}\int_\Omega\int_0^1(\tau'(t)\rho-1)2\tau(t)e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,
\end{aligned}\tag{3.93}$$

so

$$\begin{aligned}
&2\overline{\xi}\int_\Omega\int_0^1 e^{-2\rho\tau(t)}(\tau'(t)\rho-1)z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx\\
&=\overline{\xi}\int_\Omega\int_0^1\tau'(t)\rho\frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)d\rho dx\\
&-\overline{\xi}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)d\rho dx\\
&+2\overline{\xi}\tau'(t)\tau(t)\int_\Omega\int_0^1\rho e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\
&-2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,
\end{aligned}\tag{3.94}$$

using (3.92) and (3.94), equation (3.91) takes the form

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_3(t) =&\overline{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\&-2\overline{\xi}\tau(t)\tau'(t)\rho\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\&+\overline{\xi}\int_\Omega\int_0^1 \tau'(t)\rho\frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)d\rho dx\\&-\overline{\xi}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)d\rho dx\\&+2\overline{\xi}\tau'(t)\tau(t)\int_\Omega\int_0^1 \rho e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx+\\&-2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,\end{aligned}$$

by integration by parts, we find

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_3(t) =&\overline{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\&+\overline{\xi}\tau'(t)\int_\Omega\left[\rho(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right]_0^1 dx\\&-\overline{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\\&-\overline{\xi}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\rho\tau(t)}z^2(x,\rho,t)\right)d\rho dx\\&-2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,\end{aligned}$$

so

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_3(t) =&\overline{\xi}\tau'(t)\int_\Omega e^{-2\tau(t)}z^2(x,1,t)dx\\&-\overline{\xi}\int_\Omega e^{-2\tau(t)}z^2(x,1,t)d\rho dx+\overline{\xi}\int_\Omega z^2(x,0,t)d\rho dx\\&-2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,\end{aligned}$$

using (3.13), we find

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_3(t) =&-\overline{\xi}\left(1-\tau'(t)\right)\int_\Omega e^{-2\tau(t)}z^2(x,1,t)dx\\&+\overline{\xi}\int_\Omega u_t^2(x,t)d\rho dx\\&-2\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx,\end{aligned}$$

3.4. Stability result of solution

then

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_3(t) =& -\overline{\xi}\,(1-\tau'(t))\int_\Omega e^{-2\tau(t)}z^2(x,1,t)dx\\&+\overline{\xi}\int_\Omega u_t^2(x,t)d\rho dx\\&-2\mathcal{F}(t),\end{aligned}$$

then, we obtain the following result (3.90)

$$\frac{d}{dt}\mathcal{F}_3(t)\leq\overline{\xi}\int_\Omega u_t^2(x,t)d\rho dx-2\mathcal{F}_3(t).$$

■

We define the functional

$$\mathcal{F}_4(t)=-\int_\Omega u_t\int_0^\infty g(s)(u(t)-u(t-s))dsdx. \tag{3.95}$$

Then we have the following estimate.

Lemma 3.5 *The functional $\mathcal{F}_4(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_4(t)\leq&\epsilon_2 l^2\int_\Omega u_x^2dx+(|\mu_1(0)|^2\epsilon_2-g_0+\epsilon_2)\int_\Omega u_t^2dx\\&+(\frac{g_0}{4\epsilon_2}+g_0+\frac{\beta g_0c^2}{4\epsilon_2}+\frac{g_0c^2}{4\epsilon_2})\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx\\&+\beta^2|\mu_1(0)|^2\epsilon_2\int_\Omega z^2(x,1,t)dx-\frac{g(0)c^2}{4\epsilon_2}\int_\Omega\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned} \tag{3.96}$$

Proof. Differentiating $\mathcal{F}_4(t)$ with respect to t, we find

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_4(t)=&\frac{d}{dt}\left[-\int_\Omega u_t\int_0^\infty g(s)(u(t)-u(t-s))dsdx\right]\\=&-\int_\Omega u_{tt}\int_0^\infty g(s)(u(t)-u(t-s))dsdx-\int_\Omega u_t\int_0^\infty g(s)(u_t(t)-u_t(t-s))dsdx,\end{aligned} \tag{3.97}$$

using (3.20), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_4(t)=&-\int_\Omega\left(lu_{xx}+\int_0^\infty g(s)\eta_{xx}^t(x,s)ds-\mu_1(t)u_t(x,t)-\mu_2(t)z(x,1,t)\right)\\&\times\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx-\int_\Omega u_t\int_0^\infty g(s)(u_t(t)-u_t(t-s))dsdx,\end{aligned}$$

then

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_4(t) =& -\int_\Omega lu_{xx}\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx\\ &-\int_\Omega\left(\int_0^\infty g(s)\eta^t_{xx}(x,s)ds\right)\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx\\ &+\int_\Omega \mu_1(t)u_t(x,t)\left(\int_0^\infty g(s)(u(t)-u(t-s))dsdx\right)\\ &+\int_\Omega \mu_2(t)z(x,1,t)\left(\int_0^\infty g(s)(u(t)-u(t-s))dsdx\right)\\ &-\int_\Omega u_t\int_0^\infty g(s)(u_t(t)-u_t(t-s))dsdx,\end{aligned}$$

by integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_4(t) =& \int_\Omega lu_x\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)dx\\ &+\int_\Omega\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)^2dx\\ &+\int_\Omega \mu_1(t)u_t(x,t)\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx\\ &+\int_\Omega \mu_2(t)z(x,1,t)\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx\\ &-g_0\int_\Omega u_t^2dx+\int_\Omega u_t\int_0^\infty g(s)\eta^t_s(x,s)dsdx.\end{aligned} \tag{3.98}$$

Using Young's inequality and (3.71), we obtain for any $\epsilon_2>0$

$$\begin{aligned}&\int_\Omega lu_x\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)dx\\ &\le \epsilon_2 l^2\int_\Omega u_x^2dx+\frac{1}{4\epsilon_2}\int_\Omega\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s)ds)\right)^2dx\\ &\le \epsilon_2 l^2\int_\Omega u_x^2dx+\frac{g_0}{4\epsilon_2}\int_\Omega\int_0^\infty g(s)(\left|\eta^t_x(x,s)\right|^2dsdx,\end{aligned} \tag{3.99}$$

and

$$\begin{aligned}&\int_\Omega \mu_1(t)u_t(x,t)\left(\int_0^\infty g(s)(u(t)-u(t-s))dsdx\right)\\ &\le \left|\mu_1(0)\right|^2\epsilon_2\int_\Omega u_t^2dx+\frac{1}{4\epsilon_2}\int_\Omega\left(\int_0^\infty g(s)(u(t)-u(t-s)ds)\right)^2dx\\ &\le \left|\mu_1(0)\right|^2\epsilon_2\int_\Omega u_t^2dx+\frac{g_0}{4\epsilon_2}\int_\Omega\int_0^\infty g(s)\left|\eta^t(x,s)\right|^2dsdx\\ &\le \left|\mu_1(0)\right|^2\epsilon_2\int_\Omega u_t^2dx+\frac{g_0c^2}{4\epsilon_2}\int_\Omega\int_0^\infty g(s)\left|\eta^t_x(x,s)\right|^2dsdx.\end{aligned} \tag{3.100}$$

Similarly

$$\begin{aligned}
&\int_\Omega \mu_2(t)z(x,1,t)\left(\int_0^\infty g(s)(u(t)-u(t-s))dsdx\right)\\
&\le \beta\left|\mu_1(0)\right|^2\epsilon_2\int_\Omega z^2(x,1,t)dx+\frac{1}{4\epsilon_2}\int_\Omega\left(\int_0^\infty g(s)(u(t)-u(t-s)ds)\right)^2dx\\
&\le \beta^2\left|\mu_1(0)\right|^2\epsilon_2\int_\Omega z^2(x,1,t)dx+\frac{g_0}{4\epsilon_2}\int_\Omega\int_0^\infty g(s)\left|\eta^t(x,s)\right|^2dsdx\\
&\le \beta^2\left|\mu_1(0)\right|^2\epsilon_2\int_\Omega z^2(x,1,t)dx+\frac{g_0c^2}{4\epsilon_2}\int_\Omega\int_0^\infty g(s)\left|\eta^t_x(x,s)\right|^2dsdx.
\end{aligned}\tag{3.101}$$

Notice that

$$\begin{aligned}
\int_\Omega\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)^2dx &= \int_\Omega\left(\int_0^\infty \sqrt{g(s)}\sqrt{g(s)}(u_x(t)-u_x(t-s))ds\right)^2dx\\
&\le \int_\Omega\int_0^\infty g(s)ds\left(\int_0^\infty g(s)\left|\eta^t_x(x,s)\right|^2ds\right)dx\\
&\le g_0\int_\Omega\int_0^\infty g(s)\left|\eta^t_x(x,s)\right|^2dsdx,
\end{aligned}\tag{3.102}$$

and

$$\begin{aligned}
\int_\Omega u_t\int_0^\infty g(s)\eta^t_s(s)dsdx &= -\int_\Omega u_t\int_0^\infty g'(s)\eta^t(s)dsdx\\
&\le \epsilon_2\int_\Omega u_t^2dx-\frac{g(0)c^2}{4\epsilon_2}\int_\Omega\int_0^\infty g'(s)\left|\eta^t_x(x,s)\right|^2dsdx.
\end{aligned}\tag{3.103}$$

Inserting (3.99), (3.100), (3.101), (3.102) and (3.103) in (3.98), we get (3.96). ■

Now we are in position to prove our results of stability.

Theorem 3.3 *Let $U=(u,v,\varphi,\psi,z,\omega)$ be the solution of (3.20)-(3.22) with initial data $U_0\in D(\mathcal{A}(0))$ and $E(t)$ the energy of U. Assume that the hypothesis (3.4), (3.5), (3.6),* **(*H*1)**, **(*H*2)** , **(*H*3)**, **(*H*4)** *and*

$$\max\left\{1,\frac{l}{b}\right\}<\frac{L_1+L_3-L_2}{2\left(L_2-L_1\right)}\tag{3.104}$$

hold. Then there exist positive constants c_1 and α such that

$$E(t)\le c_1E(0)e^{-\alpha t},\quad \forall t\ge 0.\tag{3.105}$$

Proof. We define the Lyapunov functional

$$\mathcal{L}(t)=NE(t)+N_1\mathcal{D}(t)+\mathcal{F}_1(t)+N_2\mathcal{F}_2(t)+\mathcal{F}_3(t)+N_4\mathcal{F}_4(t),\tag{3.106}$$

where N, N_1, N_2, N_4 are positive real numbers which will be chosen later. By the Lemma (3.1), there exists a positive constant R such that

$$\frac{d}{dt}E(t) \leq -R\left[\int_\Omega u_t^2dx + \int_\Omega z^2(x,1,t)dx\right] + \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \quad (3.107)$$

It follows from the transmission conditions (3.21) that

$$l^2u_x^2(L_i,t) = b^2v_x^2(L_i,t), \quad i = 1,2. \quad (3.108)$$

Using the estimates (3.49), (3.65) (3.77), (3.78), (3.90) and (3.96), we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq{}& N\frac{d}{dt}E(t) + N_1\frac{d}{dt}\mathcal{D}(t) + \frac{d}{dt}\mathcal{F}_1(t) + N_2\frac{d}{dt}\mathcal{F}_2(t) + \frac{d}{dt}\mathcal{F}_3(t) + N_4\frac{d}{dt}\mathcal{F}_4(t)\\
\leq{}& N\left[-R\left(\int_\Omega u_t^2dx + \int_\Omega z^2(x,1,t)dx\right) + \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx\right]\\
&+ N_1\left[\left(1+\frac{1}{4\epsilon_1}\right)\int_\Omega u_t^2dx + \left(\epsilon + 2\epsilon_1\mu_1^2(0)c^2 - l\right)\int_\Omega u_x^2dx + \int_{L_1}^{L_2} v_t^2dx\right.\\
&\left.-b\int_{L_1}^{L_2} v_x^2dx + \frac{g_0}{4\epsilon}\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \frac{1}{4\epsilon_1}\int_\Omega z^2(x,1,t)dx\right]\\
&+\left[\left(\frac{l+g_0}{2} + \epsilon_1M^2 + \frac{\mu_1^2(0)M^2}{4\epsilon_1}\right)\int_\Omega u_t^2(x,t)dx + (l^2 + 2l^2\epsilon_1)\int_\Omega u_x^2dx\right.\\
&+(g_0 + 2g_0\epsilon_1)\int_\Omega\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \frac{\beta^2\mu_1^2(0)M^2}{4\epsilon_1}{}_1\int_\Omega z^2(x,1,t)\\
&-\frac{g(0)}{4\epsilon_1}\int_\Omega\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx - \left(\frac{l+g_0}{2}q(x)u_t^2dx\right)_{\partial\Omega}\\
&\left.-\left(\frac{q(x)}{2}\left(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds\right)^2 dx\right)_{\partial\Omega}\right]\\
&+ N_2\left[-\frac{L_1+L_3-L_2}{4(L_2-L_1)}\left(\int_{L_1}^{L_2} v_t^2dx + \int_{L_1}^{L_2} bv_x^2dx\right) + \frac{L_1}{4}v_t^2(L_1)\right.\\
&\left.+\frac{L_3-L_2}{4}v_t^2(L_2) + \frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t) + L_1v_x^2(L_1,t)\right)\right]\\
&+\left[-2\left(\bar{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\right) + \bar{\xi}\int_\Omega u_t^2dx\right]\\
&+ N_4\left[\epsilon_2l^2\int_\Omega u_x^2dx + \left(\left|\mu_1(0)\right|^2\epsilon_2 - g_0 + \epsilon_2\right)\int_\Omega u_t^2dx\right.\\
&+\beta^2\left|\mu_1(0)\right|^2\epsilon_2\int_\Omega z^2(x,1,t)dx\\
&+\left(\frac{g_0}{4\epsilon_2} + g_0 + \frac{g_0c^2}{4\epsilon_2} + \frac{\beta g_0c^2}{4\epsilon_2}\right)\int_\Omega\int_0^\infty g(s)(\left|\eta_x^t(x,s)\right|^2 dsdx\\
&\left.-\frac{g(0)c^2}{4\epsilon_2}\int_\Omega\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx\right],
\end{aligned}$$

then,

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq &-\left[NR - N_1\left(1+\frac{1}{4\epsilon_1}\right) - \left(\frac{l+g_0}{2} + \epsilon_1 M^2 + \frac{\mu_1^2(0)M^2}{4\epsilon_1}\right)\right.\\
&\left.-\overline{\xi} + \left(g_0 - |\mu_1(0)|^2\epsilon_2 - \epsilon_2\right)\right]\int_\Omega u_t^2 dx\\
&-\left[NR - \frac{N_1}{4\epsilon_1} - \frac{\beta^2\mu_1^2(0)M^2}{4\epsilon_1} - N_4\beta^2|\mu_1(0)|^2\epsilon_2\right]\int_\Omega z^2(x,1,t)dx\\
&-\left[N_1\left(l-\epsilon-\epsilon\mu_1^2(0)c^2\right) - (l^2+2l^2\epsilon_1) - N_4\epsilon_2 l^2\right]\int_\Omega u_x^2 dx\\
&-\left[N_1 - N_2\frac{L_2-L_3-L_1}{4(L_2-L_1)}\right] b\int_\Omega v_x^2 dx\\
&+\left[N_1 + N_2\frac{L_2-L_3-L_1}{4(L_2-L_1)}\right]\int_\Omega v_t^2 dx\\
&+\left[\frac{N_1 g_0}{4\epsilon} + (g_0+2g_0\epsilon_1) + N_4\left(\frac{g_0}{4\epsilon_2} + g_0 + \frac{g_0c^2}{4\epsilon_2} + \frac{\beta g_0 c^2}{4\epsilon_2}\right)\right]\int_\Omega\int_0^\infty g(s)(\left|\eta_x^t(x,s)\right|^2 dsdx\\
&-(a-N_2)\left[\frac{L_1}{4}v_t^2(L_1) + \frac{L_3-L_2}{4}v_t^2(L_2)\right]\\
&-(b-N_2)\left[\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right)\right]\\
&-2\left(\overline{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)d\rho dx\right)\\
&+\left[\frac{N}{2} - \frac{g(0)}{4\epsilon_1} - N_4\frac{g(0)c^2}{4\epsilon_2}\right]\int_\Omega\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx.
\end{aligned} \tag{3.109}$$

We observe that under assumption (3.104), we can always find real constants N_1 and N_2 in such way that

$$N_1 + N_2\frac{L_2-L_3-L_1}{4(L_2-L_1)} < 0, \quad a - N_2 \geq 0, \quad b - N_2 \geq 0,$$

$$N_1 > \max\left\{1, \frac{l}{b}\right\} N_2.$$

After that, we pick positive constants ϵ_1 and ϵ_2 small enough and $\epsilon_2 < \frac{1}{2N_4}$ such that

$$N_1\left(l-\epsilon-\epsilon\mu_1^2(0)c^2\right) - 2l^2\epsilon_1 > \frac{3l^2}{2}.$$

Now, we choose N large enough in (3.109) such that $\frac{d}{dt}\mathcal{L}(t)$ satisfy

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq -\gamma_1 E(t) + \gamma_2\int_\Omega\int_0^\infty g(s)(\left|\eta_x^t(x,s)\right|^2 dsdx\\
&\leq -\gamma_1 E(t) - \frac{\gamma_2}{\delta}\int_\Omega\int_0^\infty g'(s)(\left|\eta_x^t(x,s)\right|^2 dsdx\\
&\leq -\gamma_1 E(t) - \gamma_3 E'(t),
\end{aligned} \tag{3.110}$$

for some positive constants γ_1, γ_2 and γ_3. Thus we have

$$\frac{d}{dt}(\mathcal{L}(t) + \gamma_3 E(t)) \leq -\gamma_1 E(t). \tag{3.111}$$

Let

$$\mathcal{G}(t) = \mathcal{L}(t) + \gamma_3 E(t), \tag{3.112}$$

then it is not hard to see that

$$\mathcal{G}(t) \sim E(t), \tag{3.113}$$

this implies that there exists tow positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \leq \mathcal{G}(t) \leq \alpha_2 E(t). \tag{3.114}$$

Combining (3.111) and (3.114), we get

$$\frac{d\mathcal{G}(t)}{dt} \leq -\alpha \mathcal{G}(t), \tag{3.115}$$

for some constant $\alpha > 0$, which gives

$$\mathcal{G}(t) \leq \mathcal{G}(0)e^{-\alpha t}, \tag{3.116}$$

Using estimates (3.114)and (3.116), we obtain

$$E(t) \leq \gamma_1^{-1}\mathcal{G}(0)e^{-\alpha t},$$

which complete the proof. ■

CHAPTER 4

WELL-POSEDNESS AND EXPONENTIAL STABILITY OF SWELLING POROUS ELASTIC SOILS WITH A SECOND SOUND AND DISRTIBUTED DELAY TERM

4.1 Presentation of the problem

In this section, we are concerned with the following thermoelastic system of swelling porous elastic soils with second sound and distributed delay term, where the heat flux is given by Cattaneo's law:

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t\left(x, t-s\right) ds = 0 \ \text{ in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x = 0, \ \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \ \text{ in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, \ \text{ in } (0,1)\times(0,\infty), \\ \varphi\left(x,0\right) = \varphi_0\left(x\right), \varphi_t\left(x,0\right) = \varphi_1\left(x\right), \theta(x,0) = \theta_0(x) \ \text{ in } (0,1), \\ \psi\left(x,0\right) = \psi_0\left(x\right), \ \psi_t\left(x,0\right) = \psi_1\left(x\right), \ q\left(x,0\right) = q_0\left(x\right) \ \text{ in } (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0 \ \text{ in } (0,\infty), \\ \varphi(x,-t) = f_0(x,t) = 0 \ \text{ in } (0,1)\times(0,\tau_2), \end{cases} \tag{4.1}$$

where the functions φ, ψ, θ and q are the displacement of the fluid, the elastic solid material, the difference temperature and the heat flux, respectively, the parameters ρ_1 and ρ_2 are densities of each constituent, the constants $\rho_3, a_1, a_2, a_{3,}, \tau, \delta$ and β are positive constants, reflecting various physical parameters, τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$, μ_1 is a positive constant, $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is L^∞ function, $\mu_2 \geq 0$ almost everywhere and the initial data $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0$ and f_0 belongs to the suitable functional space.

We prove the well-posedness and stability results for problem on the following parameter, under the assumption $\mu_1 \geq \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds$.

The remainder of this chapter is organized as follows. In section (4.2), we introduce some transformations and assumptions needed in this chapter. In Section (4.3), we use the Lumer-Phillips theorem to prove the well-posedness of problem (4.1). Next, in Section (4.4), we present the result of exponential stability based on the energy method by constructing an appropriate Lyapunov functional equivalent to the system's energy using the multipliers method.

4.2 Assumptions and transformations

In this section, we present some materials needed in the proof of our results.

We will assume that

$$\mu_1 \geq \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds. \tag{4.2}$$

We transform (4.1) into an equivalent problem by introducing the new variable, as in [45]

$$z(x,\rho,s,t) = \varphi_t(x, t-\rho s) \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{4.3}$$

Taking the derivative of $z(x,\rho,s,t)$ with respect to t, we obtain

$$\begin{aligned} z_t(x,\rho,s,t) &= \frac{\partial}{\partial t}\left[\varphi_t(x,t-\rho s)\right] \\ &= \varphi_{tt}(x,t-\rho s), \end{aligned} \tag{4.4}$$

taking the derivative of $z(x,\rho,s,t)$ with respect to ρ, we obtain

$$\begin{aligned} z_\rho(x,\rho,s,t) &= \frac{\partial}{\partial \rho}\left[\varphi_t(x,t-\rho s)\right] \\ &= \varphi_{tt}(x,t-\rho s)(-s), \end{aligned} \tag{4.5}$$

multipling (4.4) by s, and by combining it with (4.5), it is straight forward to check that z satisfies

$$s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Consequently, problem (4.1) is equivalent to the following system

$$\begin{cases} \rho_1\varphi_{tt} - a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)z\left(x,1,s,t\right)ds = 0, & \text{in } (0,1)\times(0,\infty),\\ \rho_2\psi_{tt} - a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x = 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \text{ in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, \text{ in } (0,1)\times(0,\infty), \\ s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \\ \varphi\left(x,0\right) = \varphi_0\left(x\right), \varphi_t\left(x,0\right) = \varphi_1\left(x\right), \theta(x,0) = \theta_0(x) \text{ in } (0,1), \\ \psi\left(x,0\right) = \psi_0\left(x\right), \; \psi_t\left(x,0\right) = \psi_1\left(x\right), \; q\left(x,0\right) = q_0\left(x\right) \text{ in } (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0 \text{ in } (0,\infty), \\ z\left(x,0,t,s\right) = \varphi_t\left(x,t\right) \text{ in } (0,1)\times(0,\infty)\times(\tau_1,\tau_2), \\ z\left(x,\rho,0,s\right) = f_0\left(x,\rho,s\right) \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{4.6}$$

4.3 Well-posedness of the problem

In this section, we will study the existence and uniqueness of the solution to the problem under consideration, based on the theory of semigroups and more precisely the Lumer-Phillips theorem. To do this, we begin by transforming the system (4.6) into a Cauchy problem. Indeed, we pose $u = \varphi_t$ and $v = \psi_t$, which leads to writing the system (4.6) as follows:

$$\begin{cases} \phi'(t) + \mathcal{A}\phi\left(t\right) = 0, & t > 0, \\ \phi(0) = \phi_0 = \left(\varphi_0, u_0, \psi_0, v_0, \theta_0, q_0, z_0\right)^T, \end{cases} \tag{4.7}$$

where $\phi = (\varphi, u, \psi, v, \theta, q, z)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is an unbounded linear operator on $\mathcal{H}$ defined by the following expression

$$\mathcal{A}\phi = \begin{pmatrix} -u \\ \frac{1}{\rho_1}\left(-a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1 u + \int_{\tau_1}^{\tau_2} \mu_2(s)z\left(x,1,s,t\right)ds\right) \\ -v \\ \frac{1}{\rho_2}(-a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x) \\ \frac{1}{\rho_3}(q_x + \delta v_x) \\ \frac{1}{\tau}(\beta q + \theta_x) \\ \frac{1}{s}z_\rho \end{pmatrix}.$$

So

$$\mathcal{A}=\begin{pmatrix} 0 & -I & 0 & 0 & 0 & 0 & 0 \\ \frac{-a_1}{\rho_1}\partial_{xx} & \frac{\mu_1}{\rho_1} & \frac{-a_2}{\rho_1}\partial_{xx} & 0 & 0 & 0 & \frac{1}{\rho_1}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ -\frac{a_2}{\rho_2}\partial_{xx} & 0 & \frac{-a_3}{\rho_2}\partial_{xx} & 0 & \frac{\delta}{\rho_2}\partial_x & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta}{\rho_3}\partial_x & 0 & \frac{1}{\rho_3}\partial_x & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\tau}\partial_x & \frac{\beta}{\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s}\partial_\rho \end{pmatrix}.$$

With $\mathcal{H}$ is the energy space of the system (4.6), which is obtained after computing the energy using the multiplicative method.

So, we define the energy of our system as follows:

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_3\theta^2+\tau q^2+\left(a_1-\frac{a_2^2}{a_3}\right)\varphi_x^2\right. \\ &\left.+\left(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x\right)^2\right\}dx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z^2(x,\rho,s,t)dsd\rho dx, \end{aligned} \tag{4.8}$$

and we will show later in section (4.4) how to calculate this quantity.

We have reserved the following spaces

$$\begin{aligned} L_*^2(0,1) &= \left\{\psi\in L^2(0,1): \int_0^1 \psi(x)dx=0\right\}, \\ H_*^1(0,1) &= H^1(0,1)\cap L_*^2(0,1), \\ H_*^2(0,1) &= \{\psi\in H^2(0,1):\psi_x(0)=\psi_x(1)=0\}, \end{aligned}$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned} \mathcal{H} \;=\; & H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L_*^2(0,1)\times L^2(0,1) \\ & \times L_*^2(0,1)\times L^2\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right). \end{aligned}$$

For any

$$\begin{aligned} \phi &= (\varphi,u,\psi,v,\theta,q,z)^T, \\ \widetilde{\phi} &= \left(\widetilde{\varphi},\widetilde{u},\widetilde{\psi},\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{z}\right)^T, \end{aligned}$$

we equip $\mathcal{H}$ with the inner product defined by

$$\langle \phi, \widetilde{\phi} \rangle_{\mathcal{H}} = \rho_1 \int_0^1 u\widetilde{u}\, dx + \rho_2 \int_0^1 v\widetilde{v}\, dx + \rho_3 \int_0^1 \theta\widetilde{\theta} + \tau \int_0^1 q\widetilde{q} + \left(a_1 - \frac{a_2}{a_3}\right)$$
$$\int_0^1 \varphi_x \widetilde{\varphi}_x\, dx + \int_0^1 \left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)\left(\frac{a_2}{\sqrt{a_3}}\widetilde{\varphi}_x + \sqrt{a_3}\widetilde{\psi}_x\right)\, dx$$
$$+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z(x,\rho,s)\widetilde{z}(x,\rho,s) ds\, d\rho\, dx, \tag{4.9}$$

for which $\mathcal{H}$ is a Hilbert space.

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \phi \in \mathcal{H} : \varphi \in H^2(0,1) \cap H_0^1(0,1), \psi \in H_*^2(0,1) \cap H_*^1(0,1), \\ v, q \in H_*^1(0,1), u, \theta \in H_0^1(0,1), \\ z, z_\rho \in L^2((0,L) \times (0,1) \times (\tau_1,\tau_2)), u(x) = (x,0,s) \text{ in } (0,L). \end{array} \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Next, we will introduce and prove the existence and uniqueness of the solution to problem (4.6) in the following theorem:

Theorem 4.1 *Let $\phi = (\varphi, u, \psi, v, \theta, q, z)^T$ be a solution to problem (4.6), and assume that (4.2) hold. Then, if $\phi_0 \in \mathcal{H}$ there exists a unique solution of problem (4.6) $\phi \in C([0,\infty), \mathcal{H})$. Moreover, if $\phi_0 \in D(\mathcal{A})$ there exists a unique solution of problem (4.6)*

$$\phi \in C([0,\infty), D(\mathcal{A})) \cap C^1[0,\infty), \mathcal{H}).$$

The demonstration is divided into two parts. In the first one, we show that operator $\mathcal{A}$ is monotone and in the second one, we show that operator $\mathcal{A} + I$ is surjective.

Lemma 4.1 *The operator $\mathcal{A}$ defined by (4.6) is monotone.*

Proof. we show that

$$\langle A\phi, \phi \rangle_{\mathcal{H}} \geq 0.$$

For this purpose, we use formula (4.9) to calculate the following inner product.

$$\langle A\phi, \phi \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} -u \\ \frac{1}{\rho_1}\left(-a_1\varphi_{xx} - a_2\psi_{xx} + \mu_1 u + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s)\, ds\right) \\ -v \\ \frac{1}{\rho_2}(-a_3\psi_{xx} - a_2\varphi_{xx} + \delta\theta_x) \\ \frac{1}{\rho_3}(q_x + \delta v_x) \\ \frac{1}{\tau}(\beta q + \theta_x) \\ \frac{1}{s} z_\rho(x,\rho,s) \end{pmatrix}, \begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ \theta \\ q \\ z \end{pmatrix} \right\rangle,$$

we use integration by parts, taking into account the boundary conditions, which yields the following result:

$$\begin{aligned}\langle A\phi,\phi\rangle_{\mathcal{H}} =& -a_1\int_0^1 \varphi_{xx}u\,dx - a_2\int_0^1 \psi_{xx}u\,dx + \mu_1\int_0^1 u^2\,dx + \int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s)\,ds\,dx\\ &- a_3\int_0^1 \psi_{xx}v\,dx - a_2\int_0^1 \varphi_{xx}v\,dx + \delta\int_0^1 \theta_x v\,dx + \int_0^1 q_x\theta\,dx + \delta\int_0^1 v_x\theta\,dx\\ &+\beta\int_0^1 q^2\,dx + \int_0^1 \theta_x q\,dx - (a_1 - \frac{a_2}{a_3})\int_0^1 u_x\varphi_x\,dx\\ &- \int_0^1\left(\frac{a_2}{\sqrt{a_3}}u_x + \sqrt{a_3}v_x\right)\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)\,dx\\ &+ \int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_0^1 z(x,\rho,s)\,z_\rho(x,\rho,s)\;d\rho\,ds\,dx,\end{aligned}$$

using integrating by parts, we obtain

$$\begin{aligned}\langle A\phi,\phi\rangle_{\mathcal{H}} =& \beta\int_0^1 q^2 + \mu_1\int_0^1 u^2\,dx + \int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s)\,ds\,dx\\ &+ \int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_0^1 z(x,\rho,s)\,z_\rho(x,\rho,s)\;d\rho\,ds\,dx\\ =& \beta\int_0^1 q^2 + \mu_1\int_0^1 u^2\,dx + \int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s)\,ds\,dx\\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s)\;ds\,dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,0,s)\;ds\,dx,\end{aligned}$$

then

$$\begin{aligned}\langle A\phi,\phi\rangle_{\mathcal{H}} =& \beta\int_0^1 q^2 + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s)\,ds\,dx\\ &+ \left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^1 u^2\,dx + \int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s)\,ds\,dx.\end{aligned} \tag{4.10}$$

Using Young's inequality, the last term in (4.10), we have

$$\begin{aligned}&-\int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s)\,ds\,dx\\ &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^1 u^2\,dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s)\,ds\,dx,\end{aligned} \tag{4.11}$$

substitute (4.11) in (4.10) yield,

$$\langle A\phi,\phi\rangle_{\mathcal{H}} \geq \beta\int_0^1 q^2\,dx + \left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^1 u^2\,dx.$$

By (4.2), $\langle A\phi,\phi\rangle_{\mathcal{H}} \geq 0$, then we conclude that $\mathcal{A}$ is monotone. ■

4.3. Well-posedness of the problem

Lemma 4.2 *The operator $\mathcal{A}+I$ is surjective*

Proof. It is sufficient to prove that $\mathcal{A}$ is maximal:

$\mathcal{A}$ is maximal $\Leftrightarrow \mathcal{R}(\mathcal{A}+I)=\mathcal{H} \Longleftrightarrow \forall G=(g_1,g_2,g_3,g_4,g_5,g_6,g_7)\in\mathcal{H}$, $\exists\ \phi\in D(A)$ such that

$$(\mathcal{A}+I)\,\phi=G. \tag{4.12}$$

We can write (4.12) as follows:

$$\begin{cases} \varphi-u=g_1,\\ \rho_1 u-a_1\varphi_{xx}-a_2\psi_{xx}+\mu_1 u+\int_{\tau_1}^{\tau_2}\mu_2(s)z\,(x,1,s,t)\,ds=\rho_1 g_2,\\ \psi-v=g_3,\\ \rho_2 v-a_3\psi_{xx}-a_2\varphi_{xx}+\delta\theta_x=\rho_2 g_4,\\ \rho_3\theta+q_x+\delta v_x=\rho_3 g_5,\\ (\tau+\beta)\,q+\theta_x=\tau g_6,\\ z+\frac{1}{s}z_\rho\,(x,\rho,t)=g_7. \end{cases} \tag{4.13}$$

Suppose we have found u and v. Therefore, the first and the third equation in (4.13) given

$$\begin{cases} u=\varphi-g_1,\\ v=\psi-g_3. \end{cases} \tag{4.14}$$

It is clear that $u\in H_0^1(0,1), v\in H_0^1(0,1)$.

Following the same approach as in [45], the seventh equation in (4.13) together with (4.14) and the fact that

$$z(x,0,s)=u(x),\ \text{ if } x\in(0,1)\,, s\in(\tau_1,\tau_2)\,, \tag{4.15}$$

yield

$$z\,(x,\rho,s)+s^{-1}z_\rho\,(x,\rho,s)=g_7\,(x,\rho,s)\,,\ \text{ if } x\in(0,1)\,, s\in(\tau_1,\tau_2)\,. \tag{4.16}$$

Then by (4.15) and (4.16),

$$z\,(x,\rho,s)=e^{-\rho s}u(x)+se^{-\rho s}\int_0^\rho g_7\,(x,\tau,s)\,e^{\tau s}d\tau. \tag{4.17}$$

From $(4.13)_6$ we have

$$\theta_x=\tau g_6-(\beta+\tau)q. \tag{4.18}$$

Integrating (4.18), we obtain

$$\theta=\tau\int_0^x g_6(y)\,dy-(\beta+\tau)\int_0^x q(y)\,dy. \tag{4.19}$$

Then $\theta(0)=\theta(1)=0$.

Using (4.14) and (4.19) in (4.13), we get

$$\begin{cases} -a_1\varphi_{xx}-a_2\psi_{xx}+\mathcal{M}\varphi=h_1\in L^2(0,1),\\ -a_3\psi_{xx}-a_2\varphi_{xx}+\rho_2\psi-(\beta+\tau)\delta q=h_2\in L^2(0,1),\\ -q_x+\rho_3(\beta+\tau)\int_0^x q(y)\,dy-\delta\psi_x=h_3\in L^2(0,1), \end{cases} \tag{4.20}$$

where

$$\begin{aligned} \mathcal{M} &= \mu_1 + \rho_1 + \int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}\, ds, \\ h_1 &= \mathcal{M}g_1 + \rho_1 g_2 - \int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s} \int_0^1 e^{s\tau} g_7(x,\tau,s)\, d\tau\, ds, \\ h_2 &= \rho_2(g_3+g_4) - \tau\delta g_6, \\ h_3 &= -\delta g_{3x} - \rho_3(g_5 - \tau \int_0^x g_6(y)\, dy). \end{aligned} \tag{4.21}$$

To demonstrate that the system (4.20) admits a unique solution, we consider the following variational formulation.

$$B\left((\varphi,\psi,q),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{q}\right)\right) = F\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{q}\right), \tag{4.22}$$

such that $B : V^2 \to \mathbb{R}$ is the bilinear form, and $F : V \to \mathbb{R}$ is the linear form where

$$V = \left[H_0^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1)\right].$$

To obtain B and F, we multiply $(4.20)_1$, $(4.20)_2$, $(4.20)_3$, respectively, by φ, ψ, q and integrate them by parts over $(0,1)$, resulting in:

$$\begin{aligned} B\left((\varphi,\psi,q),\left(\tilde{\varphi},\tilde{\psi},\tilde{q}\right)\right) &= \rho_1 \int_0^1 \varphi\tilde{\varphi}dx + a_1 \int_0^1 \varphi_x\tilde{\varphi}_x dx + a_2 \int_0^1 \left(\psi_x\tilde{\varphi}_x + \varphi_x\tilde{\psi}_x\right) dx \\ &+ \rho_2 \int_0^1 \psi_x\tilde{\psi}_x dx + \alpha_3 \int_0^1 \psi\tilde{\psi}dx - \delta\left(\gamma+\beta\right) \int_0^1 q\tilde{\psi}dx \\ &+ \left(\tau+\beta\right) \int_0^1 q\tilde{q}dx + \delta\left(\tau+\beta\right) \int_0^1 \psi\tilde{q}dx \\ &+ \rho_3\left(\tau+\beta\right)^2 \int_0^1 \left(\int_0^x q\left(y\right)dy\right)\left(\int_0^x \tilde{q}\left(y\right)dy\right)dx, \end{aligned}$$

and

$$F\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{q}\right) = \int_0^1 h_1\widetilde{\varphi}\, dx + \int_0^1 h_2\widetilde{\psi}\, dx + (\tau+\beta)\int_0^1 h_3 \int_0^x \widetilde{q}(y)\, dy\, dx.$$

Now, we equip the space V with the following norm:

$$\|(\varphi,\psi,q\|_V^2 := \|\varphi_x\|_2^2 + \|\psi_x\|_2^2 + \left\|\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right\|_2^2 + \|\psi\|_2^2 + \|q\|_2^2,$$

where $||.||$ is the usual norm, using integration by parts, we have for $C > 0$

$$\begin{aligned} B\left((\varphi,\psi,q),(\varphi,\psi,q)\right)_V &= \rho_1 \int_0^1 \varphi^2 dx + \left(a_1 - \frac{a_2^2}{a_3}\right)\int_0^1 \varphi_x^2 dx + \rho_2 \int_0^1 \psi^2 dx \\ &+ \rho_3(\beta+\tau)^2 \int_0^1 \left(\int_0^1 q(y)\, dy\right)^2 dx + (\beta+\tau)\int_0^1 q^2 dx \\ &\geq C\|(\varphi,\psi,q)\|_V^2, \end{aligned}$$

thus B is coercive.

To show that B is continuous, it suffices to prove that B is bounded on V. This is due to the fact that $h_1, h_2, h_3 \in L^2(0,1)$ and $\varphi, \psi, q, \tilde{\varphi}, \tilde{\psi}, \tilde{q} \in V$, then a simple application of Young's inequality leads to the desired result. We can establish that F is bounded on V by the fact that $\varphi, \psi, q \in V$. According to the Lax-Milgram theorem, there exists a unique solution $\varphi, \psi, q \in H_0^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1)$ satisfied

$$B\left((\varphi, \psi, q), \left(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{q}\right)\right) = F\left(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{q}\right), \forall \left(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{q}\right) \in V. \tag{4.23}$$

Therefore, according to the Lax-Milgram theorem, the system (4.20) admits a unique solution

$$\varphi \in H_*^1(0,1), \ \psi \in H_0^1(0,1) \text{ and } q \in L_*^2(0,1).$$

By replacing φ in $(4.13)_1$, ψ in $(4.13)_3$ and q in (4.18) we get

$$u \in H_0^1(0,1), \ v \in H_*^1(0,1), \ \theta \in H_0^1(0,1).$$

Similarly, the compensation of u in (4.17) with $(4.13)_7$, gives

$$z, z_\rho \in L^2\left((0,1) \times (0,1) \times (\tau_1, \tau_2)\right).$$

If we take $(\widetilde{\psi}, \widetilde{q}) = (0,0) \in H_*^1(0,1) \times L_*^2(0,1)$, then (4.22) becomes

$$\rho_1 \int_0^1 \varphi\tilde{\varphi} dx + a_1 \int_0^1 \varphi_x \tilde{\varphi}_x dx + a_2 \int_0^1 \psi_x \tilde{\varphi}_x dx = \int_0^1 h_1 \tilde{\varphi}\ dx\ , \forall \tilde{\varphi} \in H_0^1(0,1), \tag{4.24}$$

which implies

$$\int_0^1 (a_1\varphi_x + a_2\psi_x)\, \tilde{\varphi}_x dx = -\int_0^1 (\rho_1\varphi - h_1)\, \tilde{\varphi}\ dx\ , \forall \tilde{\varphi} \in H_0^1(0,1). \tag{4.25}$$

The latter is also true for any function $\tilde{\varphi}_1 \in C^1(0,1)$ which is in $H_0^1(0,1)$, then

$$a_1\varphi_{xx} + a_2\psi_{xx} = \rho_1\varphi - h_1 \in L^2(0,1). \tag{4.26}$$

If we take $(\tilde{\varphi}, \tilde{q}) \equiv (0,0) \in H_0^1(0,1) \times L^2(0,1)$, we obtain from (4.22)

$$\begin{aligned} &\rho_2 \int_0^1 \psi\tilde{\psi}\ dx + a_3 \int_0^1 \psi_x \tilde{\psi}_x\ dx + a_2 \int_0^1 \varphi_x \tilde{\psi}_x\ dx - \delta\,(\beta + \tau) \int_0^1 q\tilde{\psi}\ dx \\ = &\int_0^1 h_2 \tilde{\psi}\ dx,\ \forall \tilde{\psi} \in H_*^1(0,1), \end{aligned} \tag{4.27}$$

or

$$\int_0^1 (a_3\psi_x + a_2\varphi_x)\, \tilde{\psi}_x dx = -\int_0^1 (\rho_2\psi + \delta\,(\beta+\tau)\, q - h_2)\, \tilde{\psi} dx,\ \forall \tilde{\psi} \in H_*^1(0,1). \tag{4.28}$$

The latter is also true for any function $\tilde{\psi}_1 \in C^1(0,1)$, $\tilde{\psi}_1(0) \equiv 0$, which is in $H_*^1(0,1)$, this leads to

$$a_3\psi_{xx} + a_2\varphi_{xx} = \rho_2\psi + \delta\,(\beta+\tau)\, q - h_2 \ \in L^2(0,1). \tag{4.29}$$

By combining (4.26) and (4.29), using the fact that $\alpha_1\alpha_3 - \alpha_2^2 \neq 0$ and by the theory of regularity for linear equations, it follows that

$$\varphi \in H^2(0,1) \cap H_0^1(0,1), \quad \psi \in H_*^2(0,1) \cap H_*^1(0,1).$$

If we take $(\tilde{\varphi}, \tilde{\psi}) \equiv (0,0) \in H_0^1(0,1) \times H_*^1(0,1)$, equation (4.22) is written as follows

$$(\tau+\beta)\int_0^1 (q+\delta\psi)\tilde{q}dx = -\int_0^1 \left[\rho_3(\tau+\beta)^2\left(\int_0^x q(y)\ dy\right) - (\tau+\beta)h_3\right]\left(\int_0^x \tilde{q}(y)dy\right),$$

or again

$$(\tau+\beta)(q_x+\delta\psi_x) = -\rho_3(\tau+\beta)^2\left(\int_0^x q(y)\ dy\right) + (\tau+\beta)h_3,$$

from where

$$q_x = -\delta\psi_x - \rho_3(\tau+\beta)\left(\int_0^x q(y)\ dy\right) + h_3 \in L^2(0,1).$$

This is equivalent to saying that

$$q \in H_*^1(0,1).$$

From relation (4.18), we conclude that

$$\theta \in H_0^1(0,1).$$

Finally, we ensure the existence of unique $\phi \in D(\mathcal{A})$ such that (4.12) is satisfied.

Consequently, the operator $\mathcal{A}$ is a maximal monotone, and according to the Lumer-Phillips theorem (see [42] and [50]), we deduce that $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup on $\mathcal{H}$, the result of Theorem (4.1) follows. ■

4.4 Exponential Stability

In this section, we show that, under the assumption $\mu_1 \geq \int_{\tau_1}^{\tau_2} \mu_2(s)\,ds$, the solution of problem (4.6) decays exponentially to the study state. For this purpose, we need some necessary results that will help us later to achieve our goal.

Lemma 4.3 *Let $(\varphi, \psi, \theta, q, z)$ be a solution (4.6) and assume (4.2) holds. Then the energy functional defined by (4.8) satisfied*

$$E'(t) \leq -\beta\int_0^1 q^2dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\right)\int_0^1 \varphi_t^2 dx \ \leq 0. \tag{4.30}$$

Proof. Multiplying the first equation of (4.6) by φ_t and we integrate along [0.1], we find

$$\rho_1\int_0^1 \varphi_t\varphi_{tt}dx - a_1\int_0^1 \varphi_t\varphi_{xx}dx - a_2\int_0^1 \varphi_t\psi_{xx}dx + \mu_1\int_0^1 \varphi_t^2 dx$$
$$+\int_0^1 \varphi_t\int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\,dsdx = 0,$$

using integration by parts and the boundary conditions in (4.6) allow to write the last equality in the form

$$\begin{aligned}\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \frac{a_1}{2}\frac{d}{dt}\int_0^1 \varphi_x^2 dx + a_2\int_0^1 \varphi_{xt}\psi_x dx + \mu_1\int_0^1 \varphi_t^2 dx \\ + \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds dx = 0.\end{aligned} \tag{4.31}$$

In addition, multiplying the second equation of (4.6) by ψ_t and we integrate along [0.1], we find

$$\rho_2\int_0^1 \psi_t\psi_{tt} dx - a_3\int_0^1 \psi_t\psi_{xx} dx - a_2\int_0^1 \psi_t\varphi_{xx} dx + \delta\int_0^1 \psi_t\theta_x dx = 0,$$

and thanks to the formula of integration by parts with the boundary conditions in (4.6), we deduce

$$\frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx + \frac{a_3}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx + a_2\int_0^1 \psi_{xt}\varphi_x dx + \delta\int_0^1 \psi_t\theta_x dx = 0. \tag{4.32}$$

Similarly, multiplying the third equation of (4.6) by θ and we integrate along [0.1], we obtain

$$\rho_3\int_0^1 \theta\theta_t dx + \int_0^1 \theta q_x dx + \delta\int_0^1 \theta\psi_{tx} dx = 0, \tag{4.33}$$

the formula of integration by parts and the boundary conditions imply

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \int_0^1 \theta q_x dx - \delta\int_0^1 \theta_x\psi_t dx = 0. \tag{4.34}$$

Multiplying the fourth equation of (4.6) by q and we integrate along [0.1], we find

$$\tau\int_0^1 q q_t dx + \beta\int_0^1 q^2 dx + \int_0^1 q\theta_x dx = 0,$$

then, by integrating by parts the term $\int_0^1 q\theta_x dx$, we deduce that

$$\int_0^1 q_x\theta dx = \frac{\tau}{2}\frac{d}{dt}\int_0^1 q^2 dx + \beta\int_0^1 q^2 dx. \tag{4.35}$$

By replacing (4.35) in (4.34), we obtain

$$\delta\int_0^1 \theta_x\psi_t dx = \frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \frac{\tau}{2}\frac{d}{dt}\int_0^1 q^2 dx + \beta\int_0^1 q^2 dx,$$

then, we replace the last equality in (4.32), we find

$$\begin{aligned}0 =& \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx + \frac{a_3}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx + a_2\int_0^1 \psi_{xt}\varphi_x dx \\ &+ \frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \frac{\tau}{2}\frac{d}{dt}\int_0^1 q^2 dx + \beta\int_0^1 q^2 dx,\end{aligned} \tag{4.36}$$

by adding (4.31) and (4.36), we arrive at

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx + \frac{a_1}{2}\frac{d}{dt}\int_0^1 \varphi_x^2 dx + \frac{a_3}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx \\ &\quad + a_2\int_0^1 (\varphi_{xt}\psi_x + \psi_{xt}\varphi_x)\,dx + \frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \frac{\tau}{2}\frac{d}{dt}\int_0^1 q^2 dx \\ &\quad + \mu_1\int_0^1 \varphi_t^2 dx + +\beta\int_0^1 q^2 dx + \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z\,(x,1,s,t)\,dsdx = 0, \end{aligned} \tag{4.37}$$

so

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx + \frac{a_1}{2}\frac{d}{dt}\int_0^1 \varphi_x^2 dx + \frac{a_3}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx \\ &\quad + a_2\frac{d}{dt}\int_0^1 \varphi_x\psi_x dx + \frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \frac{\tau}{2}\frac{d}{dt}\int_0^1 q^2 dx \\ &\quad + \mu_1\int_0^1 \varphi_t^2 dx + \beta\int_0^1 q^2 dx + \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z\,(x,1,s,t)\,dsdx = 0, \end{aligned} \tag{4.38}$$

then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{ \rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\theta^2 + \tau q^2 + \left(a_1 - \frac{a_2^2}{a_3}\right)\varphi_x^2 + \left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 \right\} dx \\ &= -\beta\int_0^1 q^2 dx - \mu_1\int_0^1 \varphi_t^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) dsdx, \end{aligned} \tag{4.39}$$

where

$$2a_2\psi_x\varphi_x + a_1\varphi_x^2 + a_3\psi_x^2 = \left(a_1 - \frac{a_2^2}{a_3}\right)\varphi_x^2 + \left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2. \tag{4.40}$$

Now, multiplying the fifth equation in (4.6) by $|\mu_2(s)|\,z$, integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, with respect to ρ and x respectively, we have

$$\begin{aligned} &\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z z_t(x,\rho,s,t) dsd\rho dx \\ &\qquad + \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z z_\rho(x,\rho,s,t) dsd\rho dx = 0, \end{aligned} \tag{4.41}$$

then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) dsd\rho dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)| \frac{\partial}{\partial\rho} z^2(x,\rho,s,t) d\rho dsdx \\ &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[z^2(x,1,s,t) - z^2(x,0,s,t)\right] dsdx, \end{aligned}$$

and recall that $z(x,0,s,t) = \varphi_t(x,t)$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) dsd\rho dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) dsdx \\ &\quad + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi_t^2(x,t) dsdx. \end{aligned} \tag{4.42}$$

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Summing (4.39) and (4.42), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \{ \rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_3\theta^2+\tau q^2+\left(a_1-\frac{a_2^2}{a_3}\right)\varphi_x^2+\left(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x\right)^2$$
$$+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho\Big\}dx$$
$$=-\beta\int_0^1 q^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx$$
$$-\int_0^1\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\varphi_t^2dx-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx, \tag{4.43}$$

which gives (4.8), and its derivative verify

$$E'(t)=-\beta\int_0^1 q^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx$$
$$-\int_0^1\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\varphi_t^2dx-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx. \tag{4.44}$$

On the other hand, applying Young's inequality to the last term in (4.44), we find

$$-\int_0^1\varphi_t\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\,ds\right)dx\le\frac{1}{2}\int_0^1\varphi_t^2\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)dx$$
$$+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx. \tag{4.45}$$

Inserting (4.45) in (4.44), we get (4.30). ■

In order to construct the Lyapunov functional to achieve our objective, we need the following lemmas

Lemma 4.4 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (4.6). Then the functional*

$$F_1(t):=\rho_1\int_0^1\varphi_t\varphi dx-\frac{a_2}{a_3}\rho_2\int_0^1\psi_t\varphi dx, \tag{4.46}$$

satisfies the estimate

$$F_1'(t)\quad\le-\tfrac{1}{2}\left(a_1-\tfrac{a_2^2}{a_3}\right)\int_0^1\varphi_x^2dx+(\rho_1+C_{\varepsilon_0})\int_0^1\varphi_t^2dx+\varepsilon_0\int_0^1\psi_t^2dx$$
$$+C_0\int_0^1\theta^2dx+C_0\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx. \tag{4.47}$$

Proof. Multiplying $(4.6)_1$ by φ and we integrate along $[0.1]$, we find

$$\rho_1 \int_0^1 \varphi\varphi_{tt}dx - a_1 \int_0^1 \varphi\varphi_{xx}dx - a_2 \int_0^1 \varphi\psi_{xx}dx + \mu_1 \int_0^1 \varphi\varphi_t dx + \int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\,dsdx = 0.$$

After the formula of integration by parts, and under the boundary conditions in (4.6), we deduce that

$$\rho_1 \int_0^1 \varphi\varphi_{tt}dx + a_1 \int_0^1 \varphi_x^2 dx + a_2 \int_0^1 \varphi_x\psi_x dx + \mu_1 \int_0^1 \varphi\varphi_t dx + \int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\,dsdx = 0,$$

therefore

$$\begin{aligned} \rho_1 \frac{d}{dt}\int_0^1 \varphi\varphi_t dx &= \rho_1 \int_0^1 \varphi_t^2 dx + \rho_1 \int_0^1 \varphi\varphi_{tt}dx \\ &= \rho_1 \int_0^2 \varphi_t^2 dx - a_1 \int_0^1 \varphi_x^2 dx - a_2 \int_0^1 \varphi_x\psi_x dx - \mu_1 \int_0^1 \varphi\varphi_t dx \\ &- \int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\,dsdx. \end{aligned} \tag{4.48}$$

Multiplying $(4.6)_2$ by φ and we integrate along $[0.1]$, we find

$$\rho_2 \int_0^1 \varphi\psi_{tt}dx - a_3 \int_0^1 \varphi\psi_{xx}dx - a_2 \int_0^1 \varphi\varphi_{xx}dx + \delta \int_0^1 \varphi\theta_x dx = 0,$$

using first the integration by parts, then, we use the boundary conditions, we conclude that

$$\rho_2 \int_0^1 \varphi\psi_{tt}dx + a_3 \int_0^1 \varphi_x\psi_x dx + a_2 \int_0^1 \varphi_x^2 dx - \delta \int_0^1 \varphi_x\theta dx = 0,$$

the result is that

$$\begin{aligned} -\frac{a_2}{a_3}\rho_2 \frac{d}{dt}\int_0^1 \varphi\psi_t dx &= -\frac{a_2}{a_3}\rho_2 \int_0^1 \varphi\psi_{tt}dx - \frac{a_2}{a_3}\rho_2 \int_0^1 \varphi_t\psi_t dx \\ &= a_2 \int_0^1 \varphi_x\psi_x dx + \frac{a_2^2}{a_3}\int_0^1 \varphi_x^2 dx - \frac{a_2}{a_3}\delta \int_0^1 \varphi_x\theta dx - \frac{a_2}{a_3}\rho_2 \int_0^1 \varphi_t\psi_t dx, \end{aligned} \tag{4.49}$$

by adding (4.48) and (4.49) , on obtain

$$\begin{aligned} \rho_1 \frac{d}{dt}\int_0^1 \varphi\varphi_t dx - \frac{a_2}{a_3}\rho_2 \frac{d}{dt}\int_0^1 \varphi\psi_t dx &= \rho_1 \int_0^1 \varphi_t^2 dx - a_1 \int_0^1 \varphi_x^2 dx - \mu_1 \int_0^1 \varphi\varphi_t dx \\ &- \int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\,dsdx \\ &+ \frac{a_2^2}{a_3}\int_0^1 \varphi_x^2 dx - \frac{a_2}{a_3}\rho_2 \int_0^1 \varphi_t\psi_t dx - \frac{a_2}{a_3}\delta \int_0^1 \varphi_x\theta dx. \end{aligned} \tag{4.50}$$

Now, we estimate the terms in the right hand side of (4.50). Using Young's inequality, we have

$$-\frac{a_2}{a_3}\delta\int_0^1 \varphi_x\theta dx \leq \frac{1}{4}\left(a_1-\frac{a_2^2}{a_3}\right)\int_0^1 \varphi_x^2 dx + C_0\int_0^1 \theta^2 dx, \tag{4.51}$$

$$-\frac{a_2}{a_3}\rho_2\int_0^1 \psi_t\varphi_t dx \leq \varepsilon_0\int_0^1 \psi_t^2 dx + C_{\varepsilon_0}\int_0^1 \varphi_t^2 dx. \tag{4.52}$$

By using the Cauchy-Schwarz's, Young's and Poincaré inequalities, we obtain

$$\begin{aligned} -\int_0^1 \varphi\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\right) dx &\leq C_0\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, dsdx \\ &\quad + \frac{1}{4}\left(a_1-\frac{a_2^2}{a_3}\right)\int_0^1 \varphi_x^2 dx. \end{aligned} \tag{4.53}$$

We deduce then that

$$\begin{aligned} F_1'(t) &= \rho_1\frac{d}{dt}\int_0^1 \varphi\varphi_t dx - \frac{a_2}{a_3}\rho_2\frac{d}{dt}\int_0^1 \varphi\psi_t dx \\ &\leq -\frac{1}{2}\left(a_1-\frac{a_2^2}{a_3}\right)\int_0^1 \varphi_x^2 dx + (\rho_1 + C_{\varepsilon_0})\int_0^1 \varphi_t^2 dx + \varepsilon_0\int_0^1 \psi_t^2 dx \\ &\quad + C_0\int_0^1 \theta^2 dx + C_0\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, dsdx, \end{aligned}$$

from where (4.47). ■

Lemma 4.5 *Let be the solution of (4.6). Then the functional*

$$\begin{aligned} F_2(t) &= \rho_1 a_2\int_0^1 \varphi_t\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right) dx - \frac{a_2^2}{a_3}\rho_2\int_0^1 \psi_t\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right) dx \\ &\quad + \frac{\mu_1 a_2^2}{2\sqrt{a_3}}\int_0^1 \varphi^2 dx + \mu_1 a_2\sqrt{a_3}\int_0^1 \psi\varphi dx, \end{aligned} \tag{4.54}$$

satisfies the estimate

$$\begin{aligned} F_2'(t) &\leq -\frac{a_2^2\rho_2}{4\sqrt{a_3}}\int_0^1 \psi_t^2 dx + C_{\varepsilon_1}\int_0^1 \varphi_x^2 dx + C_1\int_0^1 \varphi_t^2 dx \\ &\quad + C_{\varepsilon_1}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\int_0^1\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds\right) dx \\ &\quad + 3\varepsilon_1\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 dx + \frac{a_2^4\delta^2}{4\varepsilon_1 a_3^2}\int_0^1 \theta^2 dx. \end{aligned} \tag{4.55}$$

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Proof. we have

$$F_2'(t) = \rho_1 a_2 \int_0^1 \varphi_{tt}\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)dx + \rho_1 a_2 \int_0^1 \varphi_t\left(\frac{a_2}{\sqrt{a_3}}\varphi_t + \sqrt{a_3}\psi_t\right)dx$$
$$-\frac{a_2^2}{a_3}\rho_2 \int_0^1 \psi_{tt}\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)dx - \frac{a_2^2}{a_3}\rho_2 \int_0^1 \psi_t\left(\frac{a_2}{\sqrt{a_3}}\varphi_t + \sqrt{a_3}\psi_t\right)dx \qquad (4.56)$$
$$+\frac{\mu_1 a_2^2}{\sqrt{a_3}} \int_0^1 \varphi\varphi_t dx + \mu_1 a_2\sqrt{a_3} \int_0^1 \psi_t\varphi dx + \mu_1 a_2\sqrt{a_3} \int_0^1 \psi\varphi_t dx,$$

we replace $(4.6)_1$ and $(4.6)_2$ in (4.56)

$$\begin{aligned} F_2'(t) &= a_2 \int_0^1 \left[a_1\varphi_{xx} + a_2\psi_{xx} - \mu_1\varphi_t - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds\right]\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)dx \\ &\quad +\rho_1 a_2 \int_0^1 \varphi_t\left(\frac{a_2}{\sqrt{a_3}}\varphi_t + \sqrt{a_3}\psi_t\right)dx - \frac{a_2^2}{a_3}\int_0^1 [a_3\psi_{xx} + a_2\varphi_{xx} - \delta\theta_x] \times \\ &\quad \left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)dx - \frac{a_2^2}{a_3}\rho_2 \int_0^1 \psi_t\left(\frac{a_2}{\sqrt{a_3}}\varphi_t + \sqrt{a_3}\psi_t\right)dx + \frac{\mu_1 a_2^2}{\sqrt{a_3}}\int_0^1 \varphi\varphi_t dx \\ &\quad +\mu_1 a_2\sqrt{a_3}\left(\int_0^1 \psi_t\varphi dx + \int_0^1 \psi\varphi_t dx\right), \end{aligned}$$

using the integration by parts and the boundary conditions, we find

$$\begin{aligned} F_2'(t) &= -a_1 a_2 \int_0^1 \varphi_x\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx \\ &\quad -a_2 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)\left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds\right)dx \\ &\quad +\rho_1\frac{a_2^2}{\sqrt{a_3}}\int_0^1 \varphi_t^2 dx + \rho_1 a_2\sqrt{a_3}\int_0^1 \varphi_t\psi_t dx \\ &\quad +\frac{a_2^3}{a_3}\int_0^1 \varphi_x\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx - \frac{a_2^2}{a_3}\delta\int_0^1 \theta\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx \\ &\quad -\frac{a_2^3\rho_2}{a_3\sqrt{a_3}}\int_0^1 \psi_t\varphi_t dx - \frac{a_2^2\rho_2}{\sqrt{a_3}}\int_0^1 \psi_t^2 dx + \mu_1 a_2\sqrt{a_3}\int_0^1 \psi_t\varphi dx, \end{aligned}$$

then,

$$\begin{aligned} F_2'(t) &= -a_2\left(a_1 - \frac{a_2^2}{a_3}\right)\int_0^1 \varphi_x\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx \\ &\quad -a_2\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\,ds\right)dx \\ &\quad +\rho_1\frac{a_2^2}{\sqrt{a_3}}\int_0^1\varphi_t^2dx + \rho_1 a_2\sqrt{a_3}\int_0^1\varphi_t\psi_t dx - \frac{a_2^2}{a_3}\delta\int_0^1\theta\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx \\ &\quad -\frac{a_2^3\rho_2}{a_3\sqrt{a_3}}\int_0^1\psi_t\varphi_t dx - \frac{a_2^2\rho_2}{\sqrt{a_3}}\int_0^1\psi_t^2dx + \mu_1 a_2\sqrt{a_3}\int_0^1\psi_t\varphi dx, \end{aligned} \tag{4.57}$$

using Young's and Poincaré inequalities in (4.57), we get

$$\begin{aligned} &-a_2\left(a_1 - \frac{a_2^2}{\sqrt{a_3}}\right)\int_0^1\varphi_x\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx \\ &\le \epsilon_1\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2dx + C_{\epsilon_1}\int_0^1\varphi_x^2dx, \end{aligned} \tag{4.58}$$

$$-\frac{a_2^2}{a_3}\delta\int_0^1\theta\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)dx \le \frac{a_2^4\delta^2}{4\varepsilon_1 a_3^2}\int_0^1\theta^2dx + \varepsilon_1\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2dx, \tag{4.59}$$

$$-\frac{a_2^3\rho_2}{a_3\sqrt{a_3}}\int_0^1\psi_t\varphi_t dx \le \frac{a_2^2\rho_2}{4\sqrt{a_3}}\int_0^1\psi_t^2dx + C_1\int_0^1\varphi_t^2dx, \tag{4.60}$$

$$\rho_1 a_2\sqrt{a_3}\int_0^1\varphi_t\psi_t dx \le \frac{a_2^2\rho_2}{4\sqrt{a_3}}\int_0^1\psi_t^2dx + C_1\int_0^1\varphi_t^2dx, \tag{4.61}$$

$$\mu_1 a_2\sqrt{a_3}\int_0^1\psi_t\varphi dx \le \frac{a_2^2\rho_2}{4\sqrt{a_3}}\int_0^1\psi_t^2dx + C_1\int_0^1\varphi_x^2dx. \tag{4.62}$$

By using the Cauchy-Schwarz inequality and those of Young and Poincaré in (4.57), we obtain:

$$\begin{aligned} &-a_2\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi + \sqrt{a_3}\psi\right)\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\,ds\right)dx \\ \le\ &\varepsilon_1\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2dx \\ &+C_{\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^1\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,ds\right)dx. \end{aligned} \tag{4.63}$$

Finally, substuting (4.58),(4.59), (4.60), (4.61), (4.62) and (4.63) in (4.57), we obtain the estimate (4.55). ■

Lemma 4.6 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.6). Then the functional*

$$F_3(t) = \rho_2 \int_0^1 \psi\psi_t dx + \rho_1 \int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx, \tag{4.64}$$

satisfies

$$\begin{aligned} F_3'(t) \leq\; & -\frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right)\int_0^1 \varphi_x^2 dx - \frac{1}{2}\int_0^1 \left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 dx \\ & + C_2 \int_0^1 \theta^2 dx + |\mu_1| \int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds\right) dx \\ & + \rho_1 \int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \psi_t^2 dx. \end{aligned} \tag{4.65}$$

Proof. By differentiating F_3, taking in account the second and the third equations in (4.6), and integrating by parts, we obtain

$$\begin{aligned} F_3'(t) =\; & \rho_2 \int_0^1 \psi_t^2 dx - \alpha_3 \int_0^1 \psi_x^2 dx - \alpha_2 \int_0^1 \varphi_x\psi_x dx + \delta\int_0^1 \theta\psi_x dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ & - \alpha_1 \int_0^1 \varphi_x^2 dx - \alpha_2 \int_0^1 \varphi_x\psi_x dx - \mu_1 \int_0^1 \varphi\varphi_t dx \\ & - \int_0^1 \varphi\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\right) dx + \mu_1 \int_0^1 \varphi\varphi_t dx, \end{aligned}$$

then

$$\begin{aligned} F_3'(t) =\; & \rho_2 \int_0^1 \psi_t^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx - \alpha_3 \int_0^1 \psi_x^2 dx - 2\alpha_2 \int_0^1 \varphi_x\psi_x dx - \alpha_1 \int_0^1 \varphi_x^2 dx \\ & + \delta \int_0^1 \theta\psi_x dx - \int_0^1 \varphi\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\right) dx, \end{aligned} \tag{4.66}$$

by using Young's and Poincaré inequalities, we obtain

$$\delta\int_0^1 \theta\psi_x dx \leq C_2 \int_0^1 \theta^2 dx + \frac{1}{2}\alpha_3 \int_0^1 \psi_x^2 dx,$$

$$\int_0^1 \varphi\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\right) dx \leq \frac{1}{2}\alpha_1 \int_0^1 \varphi^2 dx + |\mu_1| \int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds\right) dx,$$

Finally, the estimate (4.65) is established. ■

4.4. Exponential Stability

Lemma 4.7 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.6). Then the functional*

$$F_4(t) = \tau\rho_3 \int_0^1 \theta \left(\int_0^x q(y)\, dy \right) dx, \tag{4.67}$$

satisfies the estimate

$$F_4'(t) \leq -\frac{\rho_3}{2} \int_0^1 \theta^2 dx + \varepsilon_2 \int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right) \int_0^1 q^2 dx. \tag{4.68}$$

Proof. *Differentiating F_4, using the equation in (4.6) and integrating by parts, we obtain*

$$\begin{aligned} F_4'(t) &= \tau\rho_3 \int_0^1 \theta_t \left(\int_0^x q(y)\, dy \right) dx + \tau\rho_3 \int_0^1 \theta \frac{d}{dt} \left(\int_0^x q(y)\, dy \right) dx \\ &= \tau \int_0^1 (-q_x - \delta\psi_{tx}) \left(\int_0^x q(y) dy \right) dx + \tau\rho_3 \int_0^1 \theta \int_0^x q_t dy dx \\ &= \tau \int_0^1 q^2 dx + \delta\tau \int_0^1 \psi_t q dx - \beta\rho_3 \int_0^1 \theta \left(\int_0^x q dy \right) dx - \rho_3 \int_0^1 \theta \int_0^x \theta_x dy dx. \end{aligned} \tag{4.69}$$

Now we use Young's inequalities

$$\delta\tau \int_0^1 \psi_t q dx \leq \varepsilon_2 \int_0^1 \psi_t^2 dx + \frac{\delta\tau}{4\varepsilon_2} \int_0^1 q^2 dx, \tag{4.70}$$

$$-\beta\rho_3 \int_0^1 \theta \left(\int_0^x q dy \right) dx \leq \frac{\rho_3}{2} \int_0^1 \theta^2 dx + \frac{\beta^2}{2} \int_0^1 \left(\int_0^x q dy \right)^2 dx, \tag{4.71}$$

By Cauchy-Schwartz inequality, it is clear that

$$\left(\int_0^x q dy \right)^2 \leq \left(\int_0^1 q dy \right)^2 \leq \int_0^1 q^2 dy, \tag{4.72}$$

by replacing (4.70), (4.71) and (4.72) in (4.69), we get

$$\begin{aligned} F_4'(t) \leq\ & \tau \int_0^1 q^2 dx + \varepsilon_2 \int_0^1 \psi_t^2 dx + \frac{\delta\tau}{4\varepsilon_2} \int_0^1 q^2 dx - \rho_3 \int_0^1 \theta^2 dx \\ & + \frac{\rho_3}{2} \int_0^1 \theta^2 dx + \frac{\beta_2}{2} \int_0^1 q^2 dx, \end{aligned} \tag{4.73}$$

then

$$F_4'(t) \leq \left(\tau + \frac{\delta\tau}{4\varepsilon_2} + \frac{\beta_2}{2} \right) \int_0^1 q^2 dx + \left(\frac{\rho_3}{2} - \rho_3 \right) \int_0^1 \theta^2 dx + \varepsilon_2 \int_0^1 \psi_t^2 dx. \tag{4.74}$$

Thus we get the estimate (4.68). ■

4.4. Exponential Stability

Lemma 4.8 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.6). Then the functional*

$$F_5(t) := \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx, \tag{4.75}$$

satisfies, for some positive constant m_1, the following estimate

$$\begin{aligned} F_5'(t) \leq & -m_1 \int_0^1 \int_{\tau_2}^{\tau_1} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\ & -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx + \mu_1 \int_0^1 \varphi_t^2 dx. \end{aligned} \tag{4.76}$$

Proof. Differentiating F_5 with respect to t and using $(4.6)_5$, we obtain

$$\begin{aligned} F_5'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} e^{-s\rho} |\mu_2(s)| z(x, \rho, s, t) z_\rho(x, \rho, s, t) ds d\rho dx \\ &= -\frac{d}{d\rho} \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \\ & \quad - \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx, \end{aligned}$$

and thus

$$\begin{aligned} F_5'(t) &= - \int_0^1 \int_{\tau_2}^{\tau_1} |\mu_2(s)| \left[e^{-s} z^2(x, 1, s, t) - z^2(x, 0, s, t) \right] ds dx \\ & \quad - \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Since $z(x, 0, s, t) = \varphi_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $\rho \in [0.1]$, we get

$$\begin{aligned} F_5'(t) \leq & - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int \varphi_t^2 dx \\ & - \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Using the fact that $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$. Finally, setting $m_1 = e^{-\tau_2}$ and by (4.2), we obtain (4.76).

Which completes the proof. ■

Now we define the Lyapunov functional $L(t)$ as follows:

$$L(t) = NE(t) + N_1 F_1(t) + N_2 F_2(t) + 2F_3(t) + N_3 F_4(t) + N_4 F_5(t), \tag{4.77}$$

where N, N_1, N_2, N_3 and N_4 are positive constants.

Before establishing the result of exponential stability, we first demonstrate the relationship between the Lyapunov functional and the energy functional.

Theorem 4.2 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.6). Then, there exist two positive constants η_1 and η_2 such that the Lyapunov functional $L(t)$ satisfies:*

$$\eta_1 E(t) \leq L(t) \leq \eta_2 E(t), \ \forall t \geq 0, \tag{4.78}$$

and

$$L'(t) \leq -\sigma_0 E(t), \quad \sigma_0 \geq 0. \tag{4.79}$$

Proof. From equation (4.77), we can write:

$$\begin{aligned}
|L(t) - NE(t)| &\leq N_1 |F_1(t)| + N_2 |F_2(t)| + 2 |F_3(t)| + N_3 |F_4(t)| + N_4 |F_5(t)| \\
&\leq N_1 \rho_1 \int_0^1 |\varphi_t| |\varphi| \, dx + N_1 \frac{a_2}{a_3} \rho_2 \int_0^1 |\psi_t| |\varphi| \, dx \\
&+ N_2 \rho_1 a_2 \int_0^1 |\varphi_t| \left| \frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right| dx \\
&+ N_2 \frac{a_2^2}{a_3} \rho_2 \int_0^1 |\psi_t| \left| \frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right| dx \\
&+ N_2 \frac{\mu_1 a_2^2}{2\sqrt{a_3}} \int_0^1 \left| \varphi^2 \right| dx + N_2 \mu_1 a_2 \sqrt{a_3} \int_0^1 |\psi| |\varphi| \, dx \\
&+ 2\rho_2 \int_0^1 |\psi| |\psi_t| \, dx + 2\rho_1 \int_0^1 |\varphi| |\varphi_t| \, dx + \mu_1 \int_0^1 \left| \varphi^2 \right| dx \\
&+ N_3 \tau \rho_3 \int_0^1 |\theta| \left| \int_0^x q(y) \, dy \right| dx \\
&+ N_4 \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} \left| s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) \right| ds d\rho dx.
\end{aligned}$$

Applying Young's, Poincaré's, Cauchy-Schwarz inequalities, and the fact that $e^{-s\rho} \geq 1$ for all $\rho \in [0, 1]$, gives

$$\begin{aligned}
|L(t) - NE(t)| &\leq \kappa \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \theta^2 + q^2 + \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 + \varphi_x^2 \right) \\
&+ \kappa \int_0^1 \int_0^1 \int_{\tau_2}^{\tau_1} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \\
&\leq \kappa E(t),
\end{aligned}$$

this leads to

$$(N - \kappa) E(t) \leq L(t) \leq (N + \kappa) E(t).$$

Consequently, by choosing N large enough, we obtain (4.78).

By differentiating $L(t)$ and recalling (4.30), (4.47), (4.55), (4.65), (4.68) and (4.76), we find

$$\begin{aligned}
L'(t) \leq & -\left[N\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)-N_1(\rho_1+C_{\varepsilon_0})-C_1N_2-2\rho_1-N_4\mu_1\right]\int_0^1\varphi_t^2dx \\
& -\left[\frac{1}{2}\left(a_1-\frac{a_2^2}{a_3}\right)N_1-N_2C_{\varepsilon_1}+\left(a_1-\frac{a_2^2}{a_3}\right)\right]\int_0^1\varphi_x^2dx \\
& -[1-3\varepsilon_1N_2]\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x\right)^2dx-\left[\beta N-N_3c\left(1+\frac{1}{\varepsilon_2}\right)\right]\int_0^1q^2dx \\
& -\left[\frac{\rho_3}{2}N_3-C_0N_1-2C_2-\frac{a_2^4\delta^2}{4\varepsilon_1a_3^2}N_2\right]\int_0^1\theta^2dx-\left[\frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2-\varepsilon_0N_1-2\rho_2-\varepsilon_2N_3\right]\int_0^1\psi_t^2dx \\
& -[N_1C_0\mu_1-C_{\varepsilon_1}N_2\mu_1-2\mu_1+m_1N_4]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx \\
& -m_1N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,z^2(x,\rho,s,t)\,dsd\rho dx.
\end{aligned}$$

We take

$$\varepsilon_0=\frac{1}{N_1},\quad \varepsilon_1=\frac{1}{4N_2},\quad ,\varepsilon_2=\frac{1}{N_3}.$$

We get

$$\begin{aligned}
L'(t) \leq & -\left[N\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)-N_1(\rho_1+C_{\varepsilon_0})-C_1N_2-2\rho_1-N_4\mu_1\right]\int_0^1\varphi_t^2dx \\
& -\left[\frac{1}{2}\left(a_1-\frac{a_2^2}{a_3}\right)N_1-N_2C_{\varepsilon_1}+\left(a_1-\frac{a_2^2}{a_3}\right)\right]\int_0^1\varphi_x^2dx \\
& -\frac{1}{2}\int_0^1\left(\frac{a_2}{\sqrt{a_3}}\varphi_x+\sqrt{a_3}\psi_x\right)^2dx-[\beta N-N_3c(1+N_3)]\int_0^1q^2dx \\
& -\left[\frac{\rho_3}{2}N_3-C_0N_1-2C_2-\frac{a_2^4\delta^2}{a_3^2}N_2^2\right]\int_0^1\theta^2dx-\left[\frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2-2\rho_2-2\right]\int_0^1\psi_t^2dx \\
& -[N_1C_0\mu_1-C_{\varepsilon_1}N_2\mu_1-2\mu_1+m_1N_4]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx \\
& -m_1N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,z^2(x,\rho,s,t)\,dsd\rho dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose N_2 large enough such that

$$\alpha_1 = \frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2 - 2\rho_2 - 2 > 0.$$

Next, we select N_1 large enough such that

$$\alpha_2 = \frac{N_1}{2}(a_1 - \frac{a_2^2}{a_3}) - N_2C_{\varepsilon_1} + \left(a_1 - \frac{a_2^2}{a_3}\right) > 0.$$

Then, we choose N_3 sufficiently large so that

$$\alpha_3 = \frac{\rho_3}{2}N_3 - C_0N_1 - 2C_2 - \frac{a_2^4\delta^2}{a_3^2}N_2^2 > 0.$$

After that, we pick N_4 very large so that

$$\alpha_4 = N_1C_0\mu_1 - C_{\varepsilon_1}N_2\mu_1 - 2\mu_1 + m_1N_4 > 0.$$

Finally, we choose N large enough (even larger so that (4.78) remains valid) so that

$$\alpha_5 = N\left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|\right) - N_1(\rho_1 + C_{\varepsilon_0}) - C_1N_2 - 2\rho_1 - N_4\mu_1 > 0,$$

and

$$\alpha_6 = \beta N - c(1 + N_3) > 0.$$

Take, $\alpha_0 = m_1N_4$, we obtain

$$L'(t) \le -\int_0^1 \left(\alpha_5\varphi_t^2 + \alpha_1\psi_t^2 + \alpha_3\theta^2 + \alpha_6 q^2 + \frac{1}{2}\left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 + \alpha_2\varphi_x^2\right)dx$$
$$-\alpha_0\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx.$$

We get to

$$L'(t) \le -\xi_1\int_0^1 \left(\varphi_t^2 + \psi_t^2 + \theta^2 + q^2 + (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)^2 + \varphi_x^2\right)dx$$
$$-\xi_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx. \tag{4.80}$$

Where $\xi_1 = \min\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} > 0$.

We assume that $\xi_2 = \max\left\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{\rho_3}{2}, \frac{\tau}{2}, \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right), \frac{1}{2}, \frac{s}{2}\right\} > 0$. Then

$$\int_0^1 \left(\varphi_t^2 + \psi_t^2 + \theta^2 + q^2 + \left(\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x\right)^2 + \varphi_x^2\right.$$
$$\left. + \int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho\right)dx \ge \frac{1}{\xi_2}E(t),$$

hence

$$-\xi_1 \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \theta^2 + q^2 + (\frac{a_2}{\sqrt{a_3}}\varphi_x + \sqrt{a_3}\psi_x)^2 + \varphi_x^2 \right) dx$$
$$-\xi_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx. \le -\frac{\xi_1}{\xi_2} E(t),$$

by (4.80), we get (4.79) with $\sigma_0 = \frac{\xi_1}{\xi_2} > 0$. ■

Now, we can state and prove the result of exponential stability.

Lemma 4.9 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (4.6). Assume that $\mu_1 \ge \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds$ hold. Then, for every $\phi_0 \in D(A)$, there exist two positive constants γ_1 and γ_2 such that*

$$E(t) \le \gamma_2 e^{-\gamma_1 t}, \forall t \ge 0. \tag{4.81}$$

Proof. According to (4.79), we have

$$L'(t) \le -\sigma_0 E(t), t \ge 0. \tag{4.82}$$

Taking into account the equivalence between $E(t)$ and $L(t)$, we can write

$$L'(t) \le -\gamma_1 L(t), t \ge 0, \tag{4.83}$$

where $\gamma_1 = \frac{\sigma_0}{\eta_2}$. A simple integration of (4.83) yields

$$L(t) \le L(0) e^{-\gamma_1 t}, t \le 0, \tag{4.84}$$

which leads to the desired result (4.81) with $\gamma_2 = \frac{L(0)}{\eta_1}$ and using once again the left-hand side of the equivalence relation (4.78). ■

CHAPTER 5

WELL-POSSEDNESS AND ENERGY DECAY FOR BRESSE SYSTEM WITH MICROTEMPERATURES EFFECTS IN THE PRESENCE OF DELAY

5.1 Presentation of the problem

In this chapter, we are concerned with the Bresse system with microtemperatures and delay term which has the form

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - lk_0\left(\omega_x - l\varphi\right) + \mu_1\varphi_t\left(x,t\right) + \mu_2\varphi_t\left(x,t-\tau\right) = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + dy_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\omega\right) = 0, & \text{in } (0,1)\times(0,\infty), \\ \alpha y_t - \gamma_1 y_{xx} + d\psi_{tx} + \gamma_2 y = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{5.1}$$

with initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ y(x,0) = y_1(x) & x \in (0,\infty), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x) & x \in (0,\infty), \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) & x \in (0,\infty), \\ \varphi(0,t) = \psi_x(0,t) = \omega_x(0,t) = y(0,t) = 0 & \forall t \geq 0, \\ \varphi_x(1,t) = \psi(1,t) = \omega(1,t) = y(1,t) = 0, & \forall t \geq 0, \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau). \end{cases} \tag{5.2}$$

The variables φ, ω and ψ correspond to the vertical, longitudinal and shear angle displacements of the beam, respectively. Additionally, the microtemperature vector is represented by y. The positive constants $\rho_1, \rho_2, k, k_0, l, b, \alpha, \gamma_1, \gamma_2$ and d are utilized as coefficients in the system reflecting various physical parameters. Moreover, $\tau > 0$ is a the time delay , and μ_1 represent positive constant and μ_2 is a real number. Furthermore, the initial data $\varphi_0, \varphi_1, f_0, \psi_0, \psi_1, \omega_0, \omega_1$ and y_0 are assumed to belong to an appropriate functional space.

The remainder of the chapter is structured as follows: in section (5.2), we provide specific assumptions and transformations, in section (5.3) we prove by using the semigroup theory the existence and uniqueness of the solution. Section (5.4) focuses on examining the exponential stability of the system if and only if

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}, \quad k = k_0, \tag{5.3}$$

holds, and using Lyapunov functionals.

5.2 Assumptions and Transformations

This part presents the assumptions and transformations required later to demonstrate our main result, we convert problem (5.1)-(5.2) into an equivalent problem by adding a new variable, also the following hypothesis will be used

$$|\mu_2| < \mu_1. \tag{5.4}$$

We present the next new variable as in [44]

$$z(x,\rho,t) = \varphi_t\left(x, t-\tau\rho\right), x \in (0,1)\,, \rho \in (0,1)\,, t > 0, \tag{5.5}$$

then, we find

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } \ (0,1) \times (0,1) \times (0,\infty)\,. \tag{5.6}$$

Hence, by substituting (5.5) and (5.6) in problem (5.1)-(5.2), we get

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - lk_0\left(\omega_x - l\varphi\right) + \mu_1\varphi_t\left(x,t\right) + \mu_2 z(x,1,t) = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + dy_x = 0, & \text{in } (0,1)\times(0,\infty) \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\omega\right) = 0, & \text{in } (0,1)\times(0,\infty) \\ \alpha y_t - \gamma_1 y_{xx} + d\psi_{tx} + \gamma_2 y = 0, & \text{in}(0,1)\times(0,\infty) \end{cases} \tag{5.7}$$

with the initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ y(x,0) = y_1(x) & x \in (0,\infty)\,, \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x) & x \in (0,\infty)\,, \\ \omega(x,0) = \omega_0(x), \ \omega_t(x,0) = \omega_1(x) & x \in (0,\infty)\,, \\ \varphi(0,t) = \psi_x(0,t) = \omega_x(0,t) = y(0,t) = 0 & \forall t \geq 0 \\ \varphi_x(1,t) = \psi(1,t) = \omega(1,t) = y(1,t) = 0, & \forall t \geq 0, \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau) & \\ \varphi_t(x,-t) = f_0(x,t) & \text{in } \ (0,1)\times(0,\tau)\,, \\ z(x,1,t) = f_0(x,t-\tau) & \text{in } \ (0,1)\times(0,\tau)\,. \end{cases} \tag{5.8}$$

5.3 Well-posedness of the problem

Using a semigroup method, we dedicate this section to the existence and uniqueness result of system (5.7)-(5.8). By the vector function $U = (\varphi, \varphi_t, z, \psi, \psi_t, \omega, \omega_t, y)^T$, where $\varphi_t = u$, $\psi_t = v$, $\omega_t = \varpi$, we transform (5.7)-(5.8) to

$$\begin{cases} U'(t) - AU(t) = 0, & t > 0, \\ U(0) = (\varphi_0, \varphi_1, f_0(.-,\tau), \psi_0, \psi_1, \omega_0, \omega_1, y_0), \end{cases} \tag{5.9}$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ an unbounded linear operator on $\mathcal{H}$ defined by the following expression:

$$A \begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ \omega \\ \varpi \\ y \end{pmatrix} = \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{lk_0}{\rho_1}(\omega_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(x,1,t) \\ -\left(\frac{1}{\tau}\right) z_\rho \\ v \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + l\omega) - \frac{d}{\rho_2}\psi_x \\ \varpi \\ \frac{k_0}{\rho_1}(\omega_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + l\omega) \\ \frac{\gamma_1}{\alpha}y_{xx} - \frac{d}{\alpha}v_x - \frac{\gamma_2}{\alpha}y \end{pmatrix}, \tag{5.10}$$

where $\mathcal{H}$ is the energy space of the system (5.7)-(5.8), which is obtained after computing the energy using the multiplicative method.

So, first we define the energy of our system as follows:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + b\psi_x^2 + \alpha y^2 \right. \\ &\quad \left. + k(\varphi_x + lw + \psi)^2 + k_0(w_x - l\varphi)^2\right] dx \\ &\quad + \tau|\mu_2| \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho dx, \end{aligned} \tag{5.11}$$

and we will show later in section (4.4) how to calculate this quantity.

We suppose the following spaces

$$H^1_\star(0,1) = \{f \in H^1(0,1) : f(0) = 0\},$$

$$H^1_\dagger(0,1) = \{f \in H^1(0,1) : f(1) = 0\},$$

$$H^2_\star(0,1) = H^2(0,1) \cap H^1_\star(0,1),$$

$$H^2_\dagger(0,1) = H^2(0,1) \cap H^1_\dagger(0,1),$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned} \mathcal{H} &= H^1_\star(0,1) \times L^2(0,1) \times L^2\left((0,1), L^2(0,1)\right) \\ &\quad \times H^1_\dagger(0,1) \times L^2(0,1) \times H^1_\dagger(0,1) \times L^2(0,1) \\ &\quad \times L^2(0,1). \end{aligned}$$

For any

$$U = (\varphi, u, z, \psi, v, \omega, \varpi, y)^T,$$
$$\bar{U} = \left(\bar{\varphi}, \bar{u}, \bar{z}, \bar{\psi}, \bar{v}, \bar{\omega}, \bar{\varpi}, \bar{y}\right)^T,$$

we define the inner product on $\mathcal{H}$ by the following form

$$\begin{aligned}\langle U, \bar{U}\rangle_{\mathcal{H}} \;=\; & k\int_0^1 (\varphi_x + \psi + l\omega)\left(\bar{\varphi}_x + \bar{\psi} + l\bar{\omega}\right)dx \\ & + k_0\int_0^1 (\omega_x - l\varphi)\left(\bar{\omega}_x - l\bar{\varphi}\right)\, dx \\ & + \rho_1\int_0^1 u\bar{u}dx + \rho_2\int_0^1 v\bar{v}dx + \rho_1\int_0^1 \varpi\bar{\varpi}dx \\ & + b\int_0^1 \psi_x\bar{\psi}_x dx + \alpha\int_0^1 y\bar{y}dx + \tau\mu_2\int_0^1\int_0^1 z\bar{z}d\rho dx. \end{aligned} \tag{5.12}$$

It is clear that $\mathcal{H}$ is a Hilbert space.

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} \;/\; \varphi \in H_\star^2(0,1); \psi, \omega \in H_\dagger^2(0,1); y \in H^2(0,1) \cap H_0^1(0,1), u \in H_\star^1(0,1); \\ v, \varpi \in H_\dagger^1(0,1), z_\rho \in L^2\left((0,1); L^2(0,1)\right), \\ \varphi_x(1) = 0, \omega_x(0) = \psi_x(0) = 0. \end{array} \right\}. \tag{5.13}$$

and it is dense in $\mathcal{H}$.

Now, we present and prove the result of existence and uniqueness of the solution of the problem (5.7)-(5.8) in the following theorem.

Theorem 5.1 *Let $U = (\varphi, u, \psi, v, \theta, q, z)^T$ be a solution to problem (5.7) – (5.8), and assume that (5.4) holds. Then, if $U_0 \in \mathcal{H}$ there exists a unique solution of problem (5.7) – (5.8) $U \in C([0,\infty), \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$ there exists a unique solution of problem (5.7) – (5.8)*

$$U \in C([0,\infty), D(\mathcal{A})) \cap C^1[0,\infty), \mathcal{H}).$$

The demonstration is divided into two parts. In the first one, we show that operator $\mathcal{A}$ is dissipative and in the second one, we show that operator $\mathcal{A} + I$ is surjective.

Lemma 5.1 *The operator $\mathcal{A}$ is dissipative.*

Proof. we show that

$$\langle AU, U\rangle_{\mathcal{H}} \leq 0.$$

To do this, we use formula (5.12) to calculate the following scalar product

$$\langle AU,U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x+\psi+l\omega)_x + \frac{lk_0}{\rho_1}(\omega_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(x,1,t) \\ -\left(\frac{1}{\tau}\right)z_\rho \\ v \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x+\psi+l\omega) - \frac{d}{\rho_2}y_x \\ \varpi \\ \frac{k_0}{\rho_1}(\omega_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x+\psi+l\omega) \\ \frac{\gamma_1}{\alpha}y_{xx} - \frac{d}{\alpha}\psi_x - \frac{\gamma_2}{\alpha}y \end{pmatrix}, \begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ \omega \\ \varpi \\ y \end{pmatrix} \right\rangle,$$

for any $U \in D(A)$, we find

$$\begin{aligned} \langle AU,U\rangle_{\mathcal{H}} &= k\int_0^1 (u_x+v+l\varpi)(\varphi_x+\psi+l\omega)\,dx + k_0\int_0^1(\varpi_x - lu)(\omega_x - l\varphi)\,dx \\ &+k\int_0^1(\varphi_x+\psi+l\omega)_x u dx + lk_0\int_0^1(\omega_x - l\varphi)u dx - \mu_1\int_0^1 u^2 dx \\ &-\mu_2\int_0^1 z(x,1,t)u dx + b\int_0^1 v_x\psi_x dx + b\int_0^1\psi_{xx}v dx - k\int_0^1(\varphi_x+\psi+l\omega)v dx \\ &-d\int_0^1 y_x v dx + k_0\int_0^1(\omega_x - l\varphi)_x\varpi dx - kl\int_0^1(\varphi_x+\psi+l\omega)\varpi dx \\ &+\gamma_1\int_0^1 y y_{xx} dx - d\int_0^1 y v_x dx - \gamma_2\int_0^1 y^2 dx - \mu_2\int_0^1\int_0^1 z z_\rho d\rho dx, \end{aligned}$$

by integrating by parts, taking into account the boundary conditions, which yields the following result:

$$\begin{aligned} \langle AU,U\rangle_{\mathcal{H}} &= -\left(\mu_1 - \frac{|\mu_2|}{2}\right)\int_0^1 u^2 dx - \gamma_1\int_0^1 y_x^2 dx - \gamma_2\int_0^1 y^2 dx \\ &-\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)dx - \mu_2\int_0^1 z(x,1,t)u dx, \end{aligned} \tag{5.14}$$

using Young's inequality, the last term in (5.14) gives

$$-\mu_2\int_0^1 z(x,1,t)u dx \le \frac{|\mu_2|}{2}\int_0^1 u^2 dx + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx \tag{5.15}$$

Substituting (5.15) into (5.14) we obtained

$$\langle AU,U\rangle_{\mathcal{H}} \le -(\mu_1 - |\mu_2|)\int_0^1 u^2 dx - \gamma_1\int_0^1 y_x^2 dx - \gamma_2\int_0^1 y^2 dx.$$

Assume that (5.4) hold, we deduce that

$$\langle AU,U\rangle_{\mathcal{H}} \le 0,$$

that is, the operator A is dissipative. ■

5.3. Well-posedness of the problem

Lemma 5.2 *The operator $I - \mathcal{A}$ is surjective.*

Proof. It is sufficient to prove that $\mathcal{A}$ is maximal:

$\mathcal{A}$ is maximal $\Leftrightarrow \mathcal{R}(I-\mathcal{A}) = \mathcal{H} \Longleftrightarrow \forall G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8) \in \mathcal{H}$, $\exists\, U \in D(A)$ such that

$$(I - \mathcal{A})\, U = G. \tag{5.16}$$

we can write (5.16) as follows:

$$\begin{cases} -u + \varphi = g_1 \in H_\star^1(0,1) \\ -k\left(\varphi_x + \psi + l\omega\right)_x - lk_0\left(\omega_x - l\varphi\right) + \rho_1 u + \mu_1 u + \mu_2 z\left(x,1,t\right) = \rho_1 g_2 \in L^2(0,1) \\ z + \frac{1}{\tau} z_\rho = g_3 \in L^2\left((0,1), L^2(0,1)\right) \\ -v + \psi = g_4 \in H_\dagger^1(0,1) \\ -b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + dv_x + dy_x + \rho_2 v = \rho_2 g_5 \in L^2(0,1) \\ -\varpi + \omega = g_6 \in H_\dagger^1(0,1) \\ -k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \rho_1 \varpi = \rho_1 g_7 \in L^2(0,1) \\ -\gamma_1 y_{xx} + dv_x + \gamma_2 y + \alpha y = \alpha g_8 \in L^2(0,1)\,. \end{cases} \tag{5.17}$$

Assuming that φ, ψ and ω are obtained with the appropriate regularity, we can deduce from equations $(5.17)_1$, $(5.17)_4$ and $(5.17)_6$ the following:

$$u = \varphi - g_1, \tag{5.18}$$

$$v = \psi - g_4, \tag{5.19}$$

$$\varpi = \omega - g_6. \tag{5.20}$$

Inserting (5.18), (5.19) and (5.20) into $(5.17)_2$, $(5.17)_5$, $(5.17)_7$, we get

$$\begin{cases} -k\left(\varphi_x + \psi + l\omega\right)_x - lk_0\left(\omega_x - l\varphi\right) + \left(\rho_1 + \mu_1 + \mu_2 e^{-\tau}\right)\varphi = h_1 \in L^2(0,1)\,, \\ -b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \rho_2\psi + dy_x = h_2 \in L^2(0,1)\,, \\ -k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \rho_1\omega = h_3 \in L^2(0,1)\,, \\ \left(\alpha + \gamma_2\right) y - \gamma_1 y_{xx} + d\psi_x = h_5, \end{cases} \tag{5.21}$$

where

$$\begin{cases} h_1 = \rho_1\left(g_1 + g_2\right) + \left(\mu_1 g_1 + \mu_2 z_0\right), \\ h_2 = \rho_2\left(g_4 + g_5\right), \\ h_3 = \rho_1\left(g_6 + g_7\right), \\ h_5 = \alpha g_8 + d g_{6x}. \end{cases} \tag{5.22}$$

As in [44], the third equation in (5.17) with $z(x,0) = u(x)$ has a unique solution given by

$$z(x,\rho) = u(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho g_3(x,p)e^{-\tau\rho p}dp,$$

from (5.17), we get

$$z(x,\rho) = \varphi(x)e^{-\tau\rho} - g_1 e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho g_3(x,p)e^{-\tau\rho p}dp,$$

then,

$$z(x,1) = \varphi(x)e^{-\tau} + z_0(x),$$

such that

$$z_0(x) = -g_1 e^{-\tau} + \tau e^{-\tau}\int_0^1 g_3(x,p)e^{\tau p}dp. \tag{5.23}$$

Taking the form of the variational formulation associated with (5.21)

$$a\left((\varphi,\psi,\omega,y),\left(\tilde{\varphi},\tilde{\psi},\tilde{\omega},\tilde{y}\right)\right) = L\left(\tilde{\varphi},\tilde{\psi},\tilde{\omega},\tilde{y}\right), \tag{5.24}$$

where a represents the bilinear form of $\left[H^1_\star(0,1)\times H^1_\dagger(0,1)\times H^1_\dagger(0,1)\times H^1_0(0,1)\right]^2 \longrightarrow \mathbb{R}$ defind by

$$L\left(\tilde{\varphi},\tilde{\psi},\tilde{\omega},\tilde{y}\right) = \int_0^1 h_1\tilde{\varphi}dx + \int_0^1 h_2\tilde{\psi}dx + \int_0^1 h_3\tilde{\omega}dx + \int_0^1 h_5\tilde{y}dx.$$

To show that the problem (5.24) admit a unique solution, we must prove that a is continuous and coercive and that L is continuous.
Now, let $V = H^1_\star(0,1)\times H^1_\dagger(0,1)\times H^1_\dagger(0,1)\times H^2(0,1)$ with the norm

$$||(\varphi,\psi,\omega,y)||^2_V = ||(\varphi_x+\psi+l\omega)||^2_2 + ||(\omega_x - l\varphi)||^2_2 + ||\psi_x||^2_2 + ||y_x||^2_2 + ||y||^2_2,$$

and using the following lemma

Lemma 5.3 *[26] There exists a positive constant c such that the following inequality holds for every* $(\varphi,\psi,\omega)\in\left[H^1_0(0,L)\right]^3$

$$\int_0^L(\varphi_x^2+\psi_x^2+\omega_x^2)dx \le c_1\int_0^L\left[b\psi_x^2 + k(\varphi_x+\psi_x+\omega_x)^2 + k_0(\omega_x - l\varphi)^2\right]dx, \tag{5.25}$$

where c_1 is a positive constant.

we can see that the forms a and L are bounded for l small enough. In addition, for $c_2>0$, we find

$$\begin{aligned} a\left((\varphi,\psi,\omega,y),(\varphi,\psi,\omega,y)\right) &= k\int_0^1(\varphi_x+\psi+l\omega)^2dx + k_0\int_0^1(\omega_x - l\varphi)^2dx + b\int_0^1\psi_x^2dx \\ &\quad +\rho_2\int_0^1\psi^2dx + \rho_1\int_0^1\omega^2dx + (\rho_1+\mu_1+\mu_2e^{-\tau})\int_0^1\varphi^2dx \\ &\quad +d\int_0^1 y\omega_xdx + (\alpha+\gamma_2)\int_0^1 y^2dx + \gamma_1\int_0^1 y_x^2dx, \\ &\ge c_2\,||(\varphi,\psi,\omega,y)||^2_V, \end{aligned}$$

hence, a is coercive.

As a result, using the Lax-Milgram Lemma, we can conclude that system (5.21) has a unique solution

$$\varphi\in H^1_\star(0,1),\ \psi\in H^1_\dagger(0,1),\ \omega\in H^1_\dagger(0,1)\ \text{ and } y\in H^1_0(0,1).$$

By substituting the values of φ, ψ and ω from (5.23), (5.18), (5.19) and (5.20), respectively we obtain

$$u \in H^1_\star(0,1),\ v \in H^1_\dagger(0,1),\ w \in H^1_\dagger(0,1) \text{ and } z, z_\rho \in L^2\left((0,1);L^2(0,1)\right).$$

Furthermore, if we take $\left(\tilde{\psi}, \tilde{\omega}, \tilde{y}\right) = (0,0,0)$ in (5.24), then for any $\tilde{\varphi} \in H^1_\star(0,1)$

$$k\int_0^1 (\varphi_x + \psi + l\omega)\tilde{\varphi}_x dx = \int_0^1 \left[k_0 l(\omega_x - l\varphi) - \varphi\left(\rho_1 + \mu_1 + \mu_2 e^{-\tau}\right) + h_1\right]\tilde{\varphi} dx, \quad \tilde{\varphi} \in H^1_\star(0,1),$$

which yields

$$k\varphi_{xx} = -k\psi_x - l(k-k_0)\omega_x - \left(\rho_1 + \mu_1 + \mu_2 e^{-\tau} + k_0 l^2\right)\varphi - h_1 \in L^2(0,1).$$

Hence, according to the regularity theory for linear elliptic equations, it can be concluded that

$$\varphi \in H^2_\star(0,1).$$

Similarly, if we put $(\tilde{\varphi}, \tilde{\omega}, \tilde{y}) = (0,0,0)$ in (5.24), we find

$$\psi \in H^2_\dagger(0,1).$$

If $\left(\tilde{\varphi}, \tilde{\psi}, \tilde{y}\right) = (0,0,0)$, then

$$\omega \in H^2_\dagger(0,1).$$

In a similar manner, when $\left(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}\right) = (0,0,0)$, we get

$$y \in H^2_0(0,1).$$

Finally, we have shown that there exists a unique $U \in D(\mathcal{A})$ such that (5.16) is satisfied. Consequently, the operator $\mathcal{A}$ is maximally dissipative, and according to the Lumer-Phillips theorem, the operator $\mathcal{A}$ generates a strongly continuous contraction semi-group. ■

5.4 Exponential Stability

In this section, we show that, given the assumption (5.3) and (5.4), then the solution to problem (5.7)-(5.8) decays exponentially. To accomplish this objective, we employ the energy method to produce an appropriate Lyapunov functional.

To prove this result, we will state and prove in the following some useful lemmas.

Lemma 5.4 *Let $(\varphi, \psi, \omega, y, z)$ is the solution of (5.7)-(5.8) and assume (5.4) holds. Then, the energy functional given by (5.11) satisfies:*

$$E'(t) \leq -\int_0^1 (\mu_1 - |\mu_2|)\varphi_t^2 dx - \gamma_1 \int_0^1 y_x^2 dx - \gamma_2 \int_0^1 y^2 dx \leq 0. \tag{5.26}$$

Proof. When $(5.7)_1$,$(5.7)_3$,$(5.7)_4$ and $(5.7)_5$ are simply multiplied by $\varphi_t, \psi_t, \omega_t$ and y, respectively, and by integrating over the interval $(0,1)$ using the technique of integration by parts and (5.8), we find

$$\begin{aligned}
&\frac{d}{2dt}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+b\psi_x^2+\alpha y^2+k\left(\varphi_x+l\omega+\psi\right)^2dx+k_0\left(\omega_x-l\varphi\right)^2\right)dx\\
=&-\int_0^1\mu_1\varphi_t^2dx-\int_0^1\mu_2 z\left(x,1,t\right)\varphi_t dx-\gamma_1\int_0^1 y_x^2dx-\gamma_2\int_0^1 y^2dx. \qquad (5.27)
\end{aligned}$$

Multiplying equation $(5.7)_2$ by $|\mu_2|\, z$ with respect to ρ and x, and integrating the resulting equation over the interval $(0,1)\times(0,1)$, we get

$$\begin{aligned}
|\mu_2|\,\tau\frac{d}{2dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx &= -\left|\mu_2\right|\int_0^1\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx\\
&= -\left|\mu_2\right|\int_0^1\frac{\partial}{2\partial\rho}z^2(x,\rho,t)d\rho dx\\
&= \left[-\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)dx+\frac{|\mu_2|}{2}\int_0^1 z^2(x,0,t)dx\right]\\
&= -\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)dx+\frac{|\mu_2|}{2}\int_0^1\varphi_t^2dx. \qquad (5.28)
\end{aligned}$$

When adding (5.27) with (5.28), we get

$$\begin{aligned}
&\frac{d}{2dt}\int_0^1\Big(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+b\psi_x^2+\alpha y^2+k\left(\varphi_x+l\omega+\psi\right)^2dx+k_0\left(\omega_x-l\varphi\right)^2\\
&+\tau\left|\mu_2\right|\int_0^1 z^2(x,\rho,t)d\rho dx\Big)\\
=&-\gamma_1\int_0^1 y_x^2dx-\gamma_2\int_0^1 y^2dx-\mu_2\int_0^1\varphi_t z\left(x,1,t\right)dx-\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)dx\\
&-\int_0^1\left(\mu_1-\frac{|\mu_2|}{2}\right)\varphi_t^2dx,
\end{aligned}$$

it results in (5.11) and

$$\begin{aligned}
E'(t) =& -\left(\mu_1-\frac{|\mu_2|}{2}\right)\int_0^1\varphi_t^2dx-\gamma_1\int_0^1 y_x^2dx-\gamma_2\int_0^1 y^2dx-\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)dx\\
&-\mu_2\int_0^1\varphi_t z\left(x,1,t\right)dx, \qquad (5.29)
\end{aligned}$$

in the last term of (5.29), we apply the inequality of Young, we find

$$-\mu_2\int_0^1\varphi_t z\left(x,1,t\right)dx\le\frac{|\mu_2|}{2}\int_0^1\varphi_t^2dx+\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)dx. \qquad (5.30)$$

We obtain (5.26) by substituting (5.30) into (5.29). The proof is finished. ■

To create the Lyapunov functional for the purpose of achieving our goal, we require the subsequent lemmas.

5.4. Exponential Stability

Lemma 5.5 *The functional*

$$F_1(t) := \rho_2 \int_0^1 \psi\psi_t dx, \tag{5.31}$$

verify

$$F_1'(t) \leq -\frac{b}{2}\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \frac{k^2}{b}\int_0^1 (\varphi_x + \psi + l\omega)^2 + c\int_0^1 y^2 dx. \tag{5.32}$$

Proof. By using the third equation from (5.7), finding the derivative of F_1, and using the technique of integration by parts and (5.8) , we get

$$F_1'(t) = \rho_2 \int_0^1 \psi_t^2 dx - b\int_0^1 \psi_x^2 - k\int_0^1 \psi(\varphi_x + \psi + l\omega) + d\int_0^1 \psi_x y. \tag{5.33}$$

Using Young's inequalities to establish (5.32). ■

Lemma 5.6 *The functional*

$$F_2(t) := \rho_2 \int_0^1 \psi_t(\varphi_x + \psi + l\omega)\, dx + \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_x dx + \frac{bl}{k_0\rho_1}\int_0^1 \psi\omega_t dx, \tag{5.34}$$

for any $\varepsilon_1, \varepsilon_2 > 0$, *(5.34) satisfies*

$$\begin{aligned} F_2'(t) \leq & -\left(\frac{k}{2} + b\right)\int_0^1 (\varphi_x + \psi + l\omega)^2 dx + \left(\rho_2 + \frac{l\chi^2}{2\varepsilon_1}\right)\int_0^1 \psi_t^2 dx \\ & + 2\varepsilon_1 \int_0^1 \omega_t^2 dx + \left(\frac{b^2 l^2}{k} + \frac{\mu_1\varepsilon_2}{k^2}\right)\int_0^1 \psi_x^2 dx + \frac{d}{2k}\int_0^1 y_x^2 dx \\ & + \frac{b^2}{4\varepsilon_2}\int_0^1 \varphi_t^2 dx + \mu_2^2 \int_0^1 z^2(x,1,t) dx. \end{aligned} \tag{5.35}$$

Proof. By differentiating F_2, using $(5.7)_1$, $(5.7)_3$ and $k = k_0$, we find

$$\begin{aligned} F_2'(t) = & \; b\int_0^1 \psi_{xx}(\varphi_x + \psi + l\omega)\, dx - k\int_0^1 (\varphi_x + \psi + l\omega)^2 dx - d\int_0^1 (\varphi_x + \psi + l\omega)\, y_x dx \\ & + \rho_2 \int_0^1 \psi_t (\varphi_x + \psi + l\omega)_t\, dx + b\int_0^1 (\varphi_x + \psi + l\omega)_x \psi_x dx + bl\int_0^1 (\omega_x - l\varphi)\, \psi_x dx \\ & - \frac{b\mu_1}{k}\int_0^1 \varphi_t\psi_x dx - \frac{b\mu_2}{k}\int_0^1 z(x,1,t)\psi_x dx + \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_{tx} dx \\ & + \frac{bl}{k\rho_1}\int_0^1 \psi_t\omega_t dx + bl\int_0^1 \psi(\omega_x - l\varphi)_x dx + bl^2 \int_0^1 \psi(\varphi + \psi + l\omega)\, dx, \end{aligned}$$

by integration by parts, we find

$$\begin{aligned} F_2'(t) = & -k\int_0^1 (\varphi_x + \psi + l\omega)^2 dx - d\int_0^1 (\varphi_x + \psi + l\omega)\, y_x dx + \rho_2 \int_0^1 \psi_t^2 dx \\ & + l\left(\rho_2 + \frac{b}{k\rho_1}\right)\int_0^1 \psi_t\omega_t dx - \frac{b\mu_1}{k}\int_0^1 \varphi_t\psi_x dx - \frac{b\mu_2}{k}\int_0^1 z(x,1,t)\psi_x dx \\ & - bl^2 \int_0^1 \psi(\varphi_x + \psi + l\omega)\, dx + \left(\rho_2 - \frac{b\rho_1}{k}\right)\int_0^1 \varphi_{tx}\psi_t dx, \end{aligned}$$

under the conditions $\frac{k}{\rho_1} = \frac{b}{\rho_2}$, we get

$$\begin{aligned} F_2'(t) &= -k\int_0^1 (\varphi_x+\psi+l\omega)^2 dx - d\int_0^1 (\varphi_x+\psi+l\omega) y_x dx + \rho_2 \int_0^1 \psi_t^2 dx \\ &+l\left(\rho_2+\frac{b}{k\rho_1}\right)\int_0^1 \psi_t\omega_t dx - \frac{b\mu_1}{k}\int_0^1 \varphi_t\psi_x dx - \frac{b\mu_2}{k}\int_0^1 z(x,1,t)\psi_x dx \\ &-bl^2\int_0^1 \psi(\varphi_x+\psi+l\omega)\, dx. \end{aligned} \tag{5.36}$$

By using Young's inequality, we obtain

$$-d\int_0^1 (\varphi_x+\psi+l\omega) y_x dx \le \frac{k}{2}\int_0^1 (\varphi_x+\psi+l\omega)^2 dx + \frac{d}{2k}\int_0^1 y_x^2,$$

$$l\left(\rho_2+\frac{b}{k\rho_1}\right)\int_0^1 \psi_t\omega_t dx \le 2\varepsilon_1 \int_0^1 \omega_t^2 dx + \frac{l\chi^2}{2\varepsilon_1}\int_0^1 \psi_t^2 dx,$$

where $\chi = \left(\rho_2 + \frac{b}{k\rho_1}\right)$.

$$-\frac{b\mu_1}{k}\int_0^1 \varphi_t\psi_x dx \le \frac{b^2}{4\varepsilon_2}\int_0^1 \varphi_t^2 dx + \frac{\mu_1\varepsilon_2}{k^2}\int_0^1 \psi_x^2 dx,$$

$$-\frac{b\mu_2}{k}\int_0^1 z(x,1,t)\psi_x dx \le \frac{b^2l^2}{32k}\int_0^1 \psi_x^2 dx + \mu_2^2\int_0^1 z^2(x,1,t) dx,$$

$$bl^2\int_0^1 \psi(\varphi_x+\psi+l\omega)\, dx \le \frac{b^2l^2}{32k}\int_0^1 \psi_x^2 dx + b^2\int_0^1 (\varphi_x+\psi+l\omega)^2 dx.$$

We obtain the result by inserting these inequalities in (5.36). ■

Lemma 5.7 *The functional*

$$F_3(t) := -\rho_1\int_0^1 \varphi_t(\omega_x - l\varphi)\, dx - \rho_1\int_0^1 \omega_t(\varphi_x+\psi+l\omega)\, dx, \tag{5.37}$$

satisfies

$$\begin{aligned} F_3'(t) &\le -\frac{lk}{2}\int_0^1 (\omega_x - l\varphi)^2 dx - \frac{l\rho_1}{2}\int_0^1 \omega_t^2 dx + c\int_0^1 \varphi_t^2 dx \\ &+lk\int_0^1 (\varphi_x + l\omega+\psi)^2 dx + c\int_0^1 \psi_t^2 dx + \frac{\mu_2}{4}\int_0^1 z^2(x,1,t) dx. \end{aligned} \tag{5.38}$$

Proof. By differentiating F_3 and using $(5.7)_1$ and $(5.7)_4$, we get

$$\begin{aligned} F_3'(t) &= -k\int_0^1 (\varphi_x+\psi+l\omega)_x(\omega_x - l\varphi)\, dx - k_0 l\int_0^1 (\omega_x - l\varphi)^2 dx \\ &+\mu_1\int_0^1 \varphi_t(\omega_x - l\varphi)\, dx + \mu_2\int_0^1 z(x,1,t)(\omega_x - l\varphi)\, dx \\ &-\rho_1\int_0^1 \varphi_t(\omega_x - l\varphi)_t\, dx - k_0\int_0^1 (\omega_x - l\varphi)_x(\varphi_x + l\omega+\psi)\, dx \\ &+lk\int_0^1 (\varphi_x + l\omega+\psi)^2 dx - \rho_1\int_0^1 \omega_t(\varphi_x + l\omega+\psi)_t\, dx, \end{aligned}$$

using integration by parts and $k=k_0$, we find

$$\begin{aligned} F_3'(t) &= -kl\int_0^1 (\omega_x - l\varphi)^2 dx - \rho_1 l\int_0^1 \omega_t^2 dx + \rho_1 l\int_0^1 \varphi_t^2 dx + lk\int_0^1 (\varphi_x + l\omega + \psi)^2 dx \\ &\quad +\mu_1\int_0^1 \varphi_t(\omega_x - l\varphi)\,dx + \mu_2\int_0^1 z(x,1,t)(\omega_x - l\varphi)\,dx - \rho_1\int_0^1 \omega_t\psi_t dx, \end{aligned} \tag{5.39}$$

using Young's inequality, we find

$$\mu_1\int_0^1 \varphi_t(\omega_x - l\varphi)\,dx \le \frac{kl}{4}\int_0^1 (\omega_x - l\varphi)^2 dx + \frac{\mu_1}{4}\int_0^1 \varphi_t^2 dx,$$

$$\mu_2\int_0^1 z(x,1,t)(\omega_x - l\varphi)\,dx \le \frac{kl}{4}\int_0^1 (\omega_x - l\varphi)^2 dx + \frac{\mu_2}{4}\int_0^1 z^2(x,1,t)dx,$$

$$-\rho_1\int_0^1 \omega_t\psi_t dx \le \frac{\rho_1 l}{2}\int_0^1 \omega_t^2 dx + \frac{\rho_1}{2l}\int_0^1 \psi_t^2 dx,$$

by substituting these inequalities in (5.39), we get the result. ■

Lemma 5.8 *The functional*

$$F_4(t) = -\frac{\alpha\rho_2}{d}\int_0^1 y\left(\int_0^x \psi_t(y,t)\,dy\right)dx, \tag{5.40}$$

satisfies, for any $\varepsilon_3, \varepsilon_4 > 0$

$$\begin{aligned} F_4'(t) &\le -\frac{\rho_2}{2}\int_0^1 \psi_t^2 dx + \frac{\gamma_1^2}{d}\int_0^1 y_x^2 dx + c\left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)\int_0^1 y^2 dx \\ &\quad +\varepsilon_3\int_0^1 \psi_x^2 dx + \varepsilon_4\int_0^1 (\varphi_x + \psi + l\omega)^2 dx. \end{aligned} \tag{5.41}$$

Proof. Taking the derivative of $F_4(t)$, using $(5.7)_5$, $(5.7)_3$, and by integration by parts, we obtain

$$\begin{aligned} F_4'(t) &= -\frac{\alpha\rho_2}{d}\int_0^1 y_t\left(\int_0^x \psi_t(y,t)\,dy\right)dx - \frac{\alpha\rho_2}{d}\int_0^1 y\left(\int_0^x \psi_{tt}(y,t)\,dy\right)dx \\ &= \frac{\gamma_1\rho_2}{d}\int_0^1 y_x\psi_t dx - \rho_2\int_0^1 \psi_t^2 dx + \frac{\gamma_2\rho_2}{d}\int_0^1 y\left(\int_0^x \psi_t(y,t)\,dy\right)dx \\ &\quad -\frac{\alpha b}{d}\int_0^1 y\psi_x dx + \frac{\alpha k}{d}\int_0^1 y\int_0^x (\varphi_x + \psi + l\omega)\,dydx + \alpha^2\int_0^1 y^2 dx. \end{aligned} \tag{5.42}$$

By using Young's inequality, we obtain

$$\frac{\gamma_1\rho_2}{d}\int_0^1 y_x\psi_t dx \le \frac{\gamma_1^2}{d}\int_0^1 y_x^2 dx + \frac{\rho_2}{4}\int_0^1 \psi_t^2 dx,$$

$$-\frac{\alpha b}{d}\int_0^1 y\psi_x dx \le \frac{(\alpha b)^2}{4d\varepsilon_3}\int_0^1 y^2 dx + \varepsilon_3\int_0^1 \psi_x^2 dx,$$

and

$$\frac{\gamma_2\rho_2}{d}\int_0^1 y\left(\int_0^x \psi_t(y,t)\,dy\right)dx \le \frac{\gamma_2^2}{d}\int_0^1 y^2 dx + \frac{\rho_2}{4}\int_0^1 \left(\int_0^x \psi_t(y,t)\,dy\right)^2 dx,$$

$$\frac{\alpha k}{d}\int_0^1 y\left(\int_0^x (\varphi_x+\psi+l\omega)\,dy\right)dx \le \frac{(\alpha k)^2}{4d\varepsilon_4}\int_0^1 y^2 dx+\varepsilon_4\int_0^1\left(\int_0^x (\varphi_x+\psi+l\omega)\,dy\right)^2 dx.$$

Applying Cauchy-Schwartz inequality,

$$\left(\int_0^x \psi_t(y,t)\,dy\right)^2 \le \left(\int_0^1 \psi_t(x,t)\,dx\right)^2 \le \int_0^1 \psi_t^2(x,t)\,dx,$$

$$\left(\int_0^x (\varphi_x+\psi+l\omega)(y,t)\,dy\right)^2 \le \left(\int_0^1 (\varphi_x+\psi+l\omega)(x,t)\,dx\right)^2 \le \int_0^1 (\varphi_x+\psi+l\omega)^2(x,t)\,dx,$$

by inserting these inequalities in (5.42), we obtain the estimates (5.41). ■

Lemma 5.9 *The functional*

$$F_5(t)=\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)\,d\rho dx, \tag{5.43}$$

satisfies,

$$F_5'(t)\le -n_1\int_0^1 z^2(x,1,t)dx - n_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx+\int_0^1 \varphi_t^2 dx. \tag{5.44}$$

Proof. Differentiating (5.43), with respect to t and employing (5.6) , we get

$$\begin{aligned}
F_5'(t) &= 2\tau\int_0^1\int_0^1 e^{-\tau\rho}zz_t(x,\rho,t)\,d\rho dx\\
&= -2\int_0^1\int_0^1 e^{-\tau\rho}zz_\rho(x,\rho,t)d\rho dx\\
&= -\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left[e^{-\tau\rho}z^2(x,\rho,t)\right]d\rho dx-\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx\\
&= -\int_0^1\left[e^{-\tau}z^2(x,1,t)-z^2(x,0,t)\right]dx-\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx\\
&= -\int_0^1 e^{-\tau}z^2(x,1,t)dx+\int_0^1 z^2(x,0,t)dx-\tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx.
\end{aligned}$$

Using for all $\rho\in[0,1]$, the fact that

$$z(x,0,t)=\varphi_t(x,t),$$

and $e^{-\tau}\le e^{-\tau\rho}\le 1$, we obtain

$$F_5'(t)\le -n_1\int_0^1 z^2(x,1,t)dx-n_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx+\int_0^1\varphi_t^2 dx,$$

where $n_1=e^{-\tau}$ is a positive constant. ■

We proceed to establish the Lyapunov functional L(t) in the following form:

Lemma 5.10 *We present a Lyapunov functional by*

$$L(t)=NE(t)+\sum_{i=1}^{5}N_iF_i(t), \tag{5.45}$$

where N_i, $i=1,2,3,4,5$ are positive constants to be selected later and for $N>0$ large enough, Before to proving the exponential stability result, we first show the relation between the Lyapunov functional and the energy functional.

Theorem 5.2 *Let (φ, ψ, w, y, z) be the solution of (5.7)-(5.8). So, there exist two positive constants, δ_1 and δ_2, such that the Lyapunov functional $L(t)$ satisfies*

$$\delta_1 E(t) \leq L(t) \leq \delta_2 E(t) \tag{5.46}$$

and

$$L'(t) \leq -\nu_1 E(t), \quad \nu_1 > 0. \tag{5.47}$$

Proof. We put

$$\mathcal{L}(t) = \sum_{i=1}^{5} N_i F_i(t). \tag{5.48}$$

From (5.31), (5.34), (5.37), (5.40) and (5.43), we have

$$\begin{aligned} |\mathcal{L}(t)| \leq\ & N_1\rho_2 \int_0^1 |\psi\psi_t|\, dx + N_2\rho_2 \int_0^1 |\psi_t(\varphi_x + \psi + l\omega)|\, dx + N_2 \frac{b\rho_1}{k} \int_0^1 |\varphi_t\psi_x|\, dx \\ & + N_2 \frac{bl}{k_0} \int_0^1 |\psi\omega_t|\, dx + N_3\rho_1 \int_0^1 |\varphi_t(\omega_x - l\varphi)|\, dx + N_3\rho_1 \int_0^1 |\omega_t(\varphi_x + \psi + l\omega)|\, dx \\ & + N_4 \frac{\alpha\rho_2}{d} \int_0^1 \left| y \left(\int_0^x \psi_t(y,t)\, dy \right) \right| dx + N_5\tau \int_0^1 \int_0^1 e^{-\tau\rho} z^2(x,\rho,t)\, d\rho dx. \end{aligned}$$

Using for all $0 < \rho < 1$ Young's, Poincaré's and Cauchy-Schwartz inequalities, as well as the fact that $e^{-\tau\rho} \leq 1$, we find

$$\begin{aligned} |\mathcal{L}(t)| &\leq M \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \omega_t^2 + \psi_x^2 + y^2 + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2 \right) \\ &\quad + M \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho dx, \\ &\leq M E(t), \end{aligned}$$

where

$$\begin{aligned} M = \max \Bigg\{ & \rho_1 \left(N_2 \frac{b}{k} + N_3 \right), \rho_2 \left(N_1 + N_2 + N_4 \frac{\alpha}{d} \right), N_2 \frac{bl}{k_2} + N_3\rho_1, \\ & \left(N_1\rho_2 + N_2 \frac{bl}{k_0} + N_2 \frac{b\rho_1}{k} \right), N_4 \frac{\alpha\rho_2}{d}, N_2\rho_2 + N_3\rho_1, N_3\rho_1, N_5\tau \Bigg\}. \end{aligned}$$

Thus,

$$|L(t) - NE(t)| \leq ME(t),$$

that is

$$(N - M)\, E(t) \leq L(t) \leq (N + M)\, E(t). \tag{5.49}$$

By selecting a sufficiently large value for N such that $N - M > 0$, we get (5.46).

5.4. Exponential Stability

We differentiate (5.45), and we use the estimates (5.26), (5.32), (5.35), (5.38), (5.41), (5.44) and by setting $N_3 = 1$, we find

$$\begin{aligned}
L'(t) \leq & -\left[N\left(\mu_1 - |\mu_2|\right) - N_2\frac{b^2}{4\varepsilon_2} - c - N_5\right]\int_0^1 \varphi_t^2 dx \\
& -\left[-N_1\rho_2 - N_2\left(\rho_2 + \frac{l\chi^2}{2\varepsilon_1}\right) - c + N_4\frac{\rho_2}{2}\right]\int_0^1 \psi_t^2 dx \\
& -\left[N_1\frac{b}{2} - N_2\left(\frac{b^2l^2}{16k} + \frac{\mu_1\varepsilon_2}{k^2}\right) - N_4\varepsilon_3\right]\int_0^1 \psi_x^2 dx \\
& -\left[-2N_2\varepsilon_1 + \frac{l\rho_1}{2}\right]\int_0^1 \omega_t^2 dx \\
& -\left[-N_1\frac{k^2}{b} + N_2\left(\frac{k}{2} + b\right) - lk - N_4\varepsilon_4\right]\int_0^1 \left(\varphi_x + l\omega + \psi\right)^2 dx \\
& -\left[\frac{lk}{2}\right]\int_0^1 \left(\omega_x - l\varphi\right)^2 dx \\
& -\left[N\gamma_2 - N_1 c - N_4 c\left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)\right]\int_0^1 y^2 dx \\
& -\left[N\gamma_1 - N_2\frac{d}{2k} - N_4\frac{\gamma_1^2}{d}\right]\int_0^1 y_x^2 dx \\
& -\left[-N_2\mu_2^2 - \frac{\mu_2}{4} + n_1 N_5\right]\int_0^1 z^2(x,1,t) dx \\
& -N_5\left[n_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)\, d\rho\, dx\right].
\end{aligned}$$

Now, by introducing

$$\varepsilon_1 = \frac{l\rho_1}{8N_2}, \quad \varepsilon_2 = \frac{lk_0}{8N_2}, \quad \varepsilon_3 = \frac{1}{N_4}, \quad \varepsilon_4 = \frac{kN_2}{8N_4}, \quad N_1 = \frac{b}{6k}N_2,$$

we get

$$
\begin{aligned}
L'(t) \leq & -\left[N\left(\mu_1-|\mu_2|\right)-\frac{2b^2}{lk}N_2^2-c-N_5\right]\int_0^1 \varphi_t^2dx \\
& -\left[-\frac{b}{6k}N_2\rho_2-N_2\left(\rho_2+\frac{4\chi^2}{\rho_1}N_2\right)+N_4\frac{\rho_2}{2}-c\right]\int_0^1 \psi_t^2dx \\
& -\left[\frac{b^2}{16k}\left(1-l^2\right)N_2-\frac{\mu_1 l}{8k}-1\right]\int_0^1 \psi_x^2dx \\
& -\left[\frac{l\rho_1}{4}\right]\int_0^1 \omega_t^2dx-\left[\frac{lk}{2}\right]\int_0^1\left(\omega_x-l\varphi\right)^2dx \\
& -\left[N_2\left(\frac{5k}{24}+b\right)-lk\right]\int_0^1\left(\varphi_x+l\omega+\psi\right)^2dx \\
& -\left[N\gamma_2-\frac{bc}{6k}N_2-N_4c\left(1+N_4+\frac{8N_4}{kN_2}\right)\right]\int_0^1 y^2dx \\
& -\left[N\gamma_1-N_2\frac{d}{2k}-N_4\frac{\gamma_1^2}{d}\right]\int_0^1 y_x^2dx \\
& -\left[-N_2\mu_2^2-\frac{\mu_2}{4}+n_1N_5\right]\int_0^1 z^2(x,1,t)dx \\
& -N_5\left[n_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx\right].
\end{aligned} \tag{5.50}
$$

Now, we must carefully select the constants, we begin by choosing $l>0$ small enough and selecting N_2 large enough that

$$\alpha_0=N_2\left(\frac{5k}{24}+b\right)-lk>0, \tag{5.51}$$

$$\alpha_1=\frac{b^2}{16k}\left(1-l^2\right)N_2-\frac{\mu_1 l}{8k}-1>0. \tag{5.52}$$

We fixed N_2 large enough, then we selected N_5 suffciently large so that

$$\alpha_2=n_1N_5-N_2\mu_2^2-\frac{\mu_2}{4}>0. \tag{5.53}$$

Following that, we fixed N_2, before selectding N_4 sufficiently large so that

$$\alpha_3=N_4\frac{\rho_2}{2}-\frac{b}{6k}N_2\rho_2-N_2\left(\rho_2+\frac{4\chi^2}{\rho_1}N_2\right)-c>0. \tag{5.54}$$

Finally, we fixed N_2, N_4 and N_5 before selecting N sufficiently large.

$$\alpha_4=N\gamma_2-\frac{bc}{6k}N_2-N_4c\left(1+N_4+\frac{8N_4}{kN_2}\right)>0, \tag{5.55}$$

$$\alpha_5=N\gamma_1-N_2\frac{d}{2k}-N_4\frac{\gamma_1^2}{d}>0, \tag{5.56}$$

$$\alpha_6=N\left(\mu_1-|\mu_2|\right)-\frac{2b^2}{lk}N_2^2-c-N_5>0. \tag{5.57}$$

5.4. Exponential Stability

Therefore, the relations (5.50) and these choices, result in

$$\begin{aligned} L'(t) \leq & -\left[\frac{l\rho_1}{4}\right]\int_0^1 \omega_t^2 dx - \left[\frac{lk}{2}\right]\int_0^1 (\omega_x - l\varphi)^2 dx - \alpha_6 \int_0^1 \varphi_t^2 dx - \alpha_3 \int_0^1 \psi_t^2 dx \\ & -\alpha_1 \int_0^1 \psi_x^2 dx - \alpha_0 \int_0^1 (\varphi_x + l\omega + \psi)^2 dx - \alpha_4 \int_0^1 y^2 dx - \alpha_5 \int_0^1 y_x^2 dx \\ & -\alpha_2 \int_0^1 z^2(x,1,t) dx - \alpha_7 \tau \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx. \end{aligned} \tag{5.58}$$

We take, $\alpha_7 = \eta_1 N_5$.
Therefore, the relations (5.50) and these choices, result in

$$\begin{aligned} L'(t) \leq & -\xi_1 \int_0^1 \left(\omega_t^2 + (\omega_x - l\varphi)^2 + \varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + l\omega + \psi)^2 + y^2 + y_x^2\right) dx \\ & -\xi_1 \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx, \end{aligned} \tag{5.59}$$

where $\xi_1 = \min \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} > 0$.
we put $\xi_2 = \max \left\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{b}{2}, \frac{\alpha}{2}, \frac{k}{2}, \frac{k_0}{2}, \frac{\tau}{2}\right\} > 0$. Then

$$\begin{aligned} & \int_0^1 \left(\omega_t^2 + (\omega_x - l\varphi)^2 + \varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + l\omega + \psi)^2 + y^2 + y_x^2\right) dx \\ & + \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx \\ & \geq \frac{1}{\xi_2} E(t), \end{aligned}$$

hence

$$\begin{aligned} & -\xi_1 \int_0^1 \left(\omega_t^2 + (\omega_x - l\varphi)^2 + \varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + l\omega + \psi)^2 + y^2 + y_x^2\right) dx \\ & -\xi_1 \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx \\ & \leq -\frac{\xi_1}{\xi_2} E(t), \end{aligned}$$

By (5.59), we get (5.47) with $\nu = \frac{\xi_1}{\xi_2}$. ■

Now we can state and prove the result of exponential stability.

Lemma 5.11 *Let (φ, ψ, w, y, z) be the solution of (5.7)-(5.8). Suppose that (5.3) and (5.4) are hold. Then, there exists positive constants κ_1 and κ_2 such that the energy $E(t)$ associated with problem (5.7)-(5.8) satisfies*

$$E(t) \leq \kappa_2 e^{-\kappa_1 t}, \quad \forall t \geq 0. \tag{5.60}$$

Proof. According to (5.47), we have

$$L'(t) \leq -\nu_1 E(t), t \geq 0, \tag{5.61}$$

Taking into account the equivalence between $E(t)$ and $L(t)$, we can write

$$L'(t) \leq -\kappa_1 L(t), t \geq 0, \tag{5.62}$$

where $\kappa_1 = \frac{\nu_1}{\delta_2}$. A simple integration of (5.62) yields

$$L(t) \leq L(0)e^{-\kappa_1 t}, t \leq 0, \tag{5.63}$$

which leads to the desired result (5.60) with $\kappa_2 = \frac{L(0)}{\delta_1}$ and using once again the left-hand side of the equivalence relation (5.46). ■

5.4. Exponential Stability

CONCLUSION

In the present work, a qualitative study of three evolution problems under the effect of damping mechanisms and various types of delay has been presented in an efficient and regurgitative way under suitable assumptions and by using mathematical tools involved in the theory of semi-groups concerning the study of existence and uniqueness. On the other hand, the behaviour was analysed asymptotic by the energy method based on an appropriate construction of the Lyapunov functional by exploiting the multiplicative method. This, leads to prove the exponential decrease of energy.

We intend in the future to make numerical simulations of the different problems studied in this thesis.

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