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Dedication To:

My parents

My husband

My siblings

My teachers

Amani and my friends

All my family

All the shining stars we lost

We dedicate this modest world

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Abstract

This dissertation addresses mathematical notion and the properties concerning stochastic processes and some importante example that we needed to build a irreplaceable concepts and here we are talking about stocastic calculus which depends primarily on Itô calculus including Itô integral and Itô formula, down to one of the most essential points stochastic differential equation, these equation are a generalization of deterministic differential equations that incorporate some randomness into continuous-time systems, and for discrete-time systems we deal with stochastic difference equations, and by integrating stochastic elements, these systems can achieve stability under conditions where deterministic models might fail.

keywords: Stochastic processes, stochastic calculus, Brownian motion, martingales, stochastic differential equation, stochastic difference equation, stochastic stabilization.

Résumé

Cette thèse aborde les notions mathématiques et les propriétés concernant les processus stochastiques ainsi que quelques exemples importants nécessaires à la construction de concepts indispensables. Nous traitons ici du calcul stochastique qui dépend principalement du calcul d'Itô, incluant l'intégrale d'Itô et la formule d'Itô, jusqu'à l'un des points les plus essentiels : l'équation différentielle stochastique. Ces équations sont une généralisation des équations différentielles déterministes qui intègrent une certaine aléa dans les systèmes à temps continu. Pour les systèmes à temps discret, nous abordons les équations de différences stochastiques. En intégrant des éléments stochastiques, ces systèmes peuvent atteindre la stabilité dans des conditions où les modèles déterministes pourraient échouer.

mots-clés : Processus stochastiques, calcul stochastique, mouvement Brownien, martingales, équation différentielle stochastique, équation de différence stochastique, stabilisation stochastique

Contents

Basic Notation

ϕ	The empty set
$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integers
$\mathbb{R}$	The set of all real numbers
ω	Event, outcome of a random experiment
Ω	The set of outcomes
A	A subset in Ω
A^c	The complement of A in Ω
$\mathcal{F}$	Sigma-field
$\mathcal{B}$	The Borel sigma-field
$\mathcal{B}(E)$	The Borel sigma-field generated by E
$\mathbb{P}$	Probability measure
$\mathbb{P}(A)$	Probability of the event A
$\mathbb{P}(A \cap B)$	Probability of A and B
$\mathbb{P}(B\|A)$	The conditional probability of B given A
$(\Omega, \mathcal{F})$	Measurable space
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
X or $X(\omega)$	Random variable
$\mathbb{P}_X$	Law of the random variable X
$f(x)$	Probability density(mass function)
CDF	cumulative distribution function
$\mathbb{E}[X]$	Expectation of X
$\mathbb{E}(X\|\mathcal{F})$	Conditional expectation of the randam variable X given $\mathcal{F}$
$Var(X)$	Variance of X
$Cov(X, Y)$	Covariance of X and Y
$X \sim N(\mu, \sigma^2)$	X is normally distributed with mean μ and variance σ^2
X_t or $X(t, \omega)$	Stochastic process

B_t or $B(t, \omega)$	Brownian motion
$:=$	The equal by definition sign
$i.e.$	Abbreviation for id est (= Latin for "that is")
$w.p.1$	With probability 1
Π	Partition of the interval $[0, t]$
$V_{\Pi}(f)[0, t]$	The first variation of f over $[0, t]$ with respect to Π
SDE	Stochastic Differential Equation
$\mathcal{L}^p([a, b]; \mathbb{R})$	the family of $\mathbb{R}$-valued $\mathcal{F}$-adapted processes $\{f(t)\}_{a \leq t \leq b}$ in$\mathcal{L}^p([a, b]; \mathbb{R})$

such that $\mathbb{E}(\int_a^b |f(t)|^p) < \infty$

Introduction

Stochastic calculus, a branch of mathematics that deals with analyzing and modeling systems influenced by randomness, finds wide applications in finance, physics, biology, and engineering. It extends traditional calculus to handle stochastic processes, enabling integration and differentiation of functions with respect to random variables. Here, we explore the historical development of key components of stochastic calculus. The study of stochastic processes began with the development of probability theory, laying the mathematical foundation for understanding random phenomena. Robert Brown (1827) observed the erratic movement of pollen particles in water, later termed "Brownian motion." Albert Einstein (1905) provided a theoretical explanation, linking it to the kinetic theory of heat and explaining random particle motion in fluids. Norbert Wiener (1923) constructed a rigorous mathematical framework for Brownian motion, leading to the term "Wiener process," characterized by continuous paths and stationary, independent increments.

Stochastic processes expanded beyond Brownian motion. Andrey Kolmogorov (1931) developed the foundations of modern stochastic processes, formalizing concepts like Markov processes, which aid in understanding dynamic systems, decision-making, and probabilistic phenomena across disciplines. Markov chains exemplify the enduring relevance of probabilistic models, from their inception in probability theory to applications in genetics, economics, and artificial intelligence, illustrating their impact on scientific inquiry and technological advancements.

In 1934, Paul Lévy introduced martingales, originally emerging as gambling strategies in games of chance like roulette and coin flipping. The term "martingale" derives from a strap used in horse racing, reflecting its early association with betting systems. Joseph Doob (1942) further developed martingale theory, formalizing their properties and applications in stochastic processes.

Since we are discussing prominent researchers in stochastic calculus, we cannot ignore the role of the Japanese mathematician Kiyoshi Itô. And his most notable achievement is the development of Itô calculus, which extends classical calculus to stochastic processes. This includes the formulation of the stochastic integral and Itô's lemma, especially his groundbreaking paper in 1944, the main breakthrough of the Ito integral is its capability to handle processes that involve randomness. It does this by including the differential (small change) of a stochastic process relative to Brownian motion. This differential, also known as the Ito differential or stochastic differential, captures the tiny changes in the process caused by random fluctuations. The Itô Lemma is also considered a result from the research that Itô published in 1944.This lemma represents a significant extension of the chain rule of calculus to stochastic processes, allowing mathematicians to compute partial differentials for functions dependent on stochastic processes like Brownian motion.

On the other hand, Stochastic differential equations (SDEs) are mathematical models for processes influenced by random variables. These equations include ordinary differential elements along with additional terms related to random changes, making them essential for understanding and predicting the dynamics of systems affected by unexpected or variable factors.

Moving from a stochastic differential equation (SDE) to a stochastic difference equation involves discretizing the continuous-time stochastic process described by the SDE into discrete-time steps. This discretization is necessary when dealing with situations where data or observations are only available at discrete time intervals, or when computational feasibility or numerical methods favor discrete-time formulations. Studying the stability of Stochastic Difference Equations is crucial for several reasons, as it provides insights into the long-term behavior of systems influenced by random perturbations. The stochastic stabilization of difference equations involves the incorporation of random processes into discrete-time systems to enhance their stability properties. By integrating stochastic elements, these systems can achieve stability under conditions where deterministic models might fail.For many applications, knowing whether a system will remain stable under stochastic influences is essential for predictability and control. Stability analysis allows for the design of control strategies that can maintain or restore stability in the presence of randomness.

This dissertation consists of three chapters. The first prerequisite probability measures, random variables, expectation, independence, conditional probability. The only other prerequisite is calculus. This covers limits, series, the notion of continuity, differentiation and the Riemann–Stieltjes integral. Familiarity with differential equations would be a bonus.

Probability theory in the first class and measure theore in the second are presented in the first chapter. This is primarily aimed at correcting terminology and reminder of information we may need later: However, conditional expectation is treated in the second chapter, including propositions designed to develop the necessary skills and intuition.

The second chapter contains six sections, in the first section, we come into the concept of stochastic processes, in the next one we mentionned popular and important examples of stochastic processes starting with the Markov chain, the Brownian motion, and Martingales. The third and the fourth sections are devoted to both stochastic integration and differentiation including Itô integral and Itô lemma. These are introduced and explained with several illustrative examples. in the last two sections, we discuss stochastic differential equation and stochastic difference equation and there applications passing through other concepts that help in reaching the essence of these equations.

In the third chapter, the stabillity of stochastic difference equations are studied using martingale method for Analyzing Stability

I Preliminaries and probability notions

On this chapter we shall recall some basic notions and facts from probability theory. Here is a short list of what needs to be reviewed:

1) Measure space
2) Probability space
3) Conditional Probalitity
4) Random variables
5) Moment of Random variable

I.1 Measure space

I.1.1 Sigma-Field

We begin generally with a set Ω whose elements are called points. For orientation, one might think of Ω as a subset of $\mathbb{R}^n$, but it might be a much more general set than that.

Definition I.1.1 [5]
Let Ω be a non-empty set. A σ-field $\mathcal{F}$ on Ω is a family of subsets of Ω such that :

(*i*) *the empty set ϕ belongs to $\mathcal{F}$;*

(*ii*) *if A belongs to $\mathcal{F}$, then so does the complement A^c (the complement of $A \in \Omega$);*

(*iii*) *If $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then their union $\bigcup_{i=1}^{\infty} A_i$ also belong $\mathcal{F}$.*

I.1.2 Sub-Sigma-Field

Let $\mathcal{F}$ be a sigma-field.

Definition I.1.2 [?] *A sigma-field $\mathcal{B}$ is a sub-sigma-field of $\mathcal{A}$ if $F \in \mathcal{A}$ for every $F \in \mathcal{B}$.*

Example I.1.1 *Consider the set* $\Omega = \{1, 2, a, b\}$, *one possible sigma-field on* Ω *is*

$\mathcal{A} = \{\phi, \{1\}, \{a\}, \{1, a\}, \{2, b\}, \{1, 2, b\}, \{2, a, b\}, \{1, 2, a, b\}\}$.

Then, the sigma-field $\mathcal{B} = \{\phi, \{1, a\}, \{2, b\}, \{1, 2, a, b\}\}$ *is a sub-sigma-field of* $\mathcal{A}$.

I.1.3 Generated Sigma-Field

It is a trivial fact that any family F of subsets of Ω can be extended to a sigma-field (just take the sigma-field consisting of all subsets of Ω).
Among all these extensions, there is a special one. Consider all the sigma-fields that contain F and take their intersection, which we call $\mathcal{F}$, i.e., a subset $A \subset \Omega$ is in $\mathcal{F}$ if and only if A is in every sigma-field containing F. It is easy to check that $\mathcal{F}$ is indeed a sigma-field. Indeed it is the smallest sigma-field containing F, it is also called the sigma-field generated by F. An important example is the sigma-field $\mathcal{B}$ of Borel sets of $\mathbb{R}$ which is generated by the open subsets of $\mathbb{R}$ [21]

I.1.4 Measure space

Definition I.1.3 [29]
the map μ *is called countably additive (or* σ *additive) if* $\mu(\phi) = 0$ *and whenever* $(A_n : n \in \mathbb{N})$ *is a sequence of disjoint sets in* $\mathcal{F}$ *, then :*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{I.1}$$

Definition I.1.4 [29] *let* $(\Omega, \mathcal{F})$ *be a measurable space, so that* $\mathcal{F}$ *is a* σ*-aglebra on* Ω.
A map

$$\mu : \mathcal{F} \longrightarrow [0, \infty], \tag{I.2}$$

is called a measure on $(\Omega, \mathcal{F})$ *if* μ *is countably additive. the triple* $(\Omega, \mathcal{F}, \mu)$ *is then called a measure space.*

Definition I.1.5 [4] *Let* $(\Omega, \mathcal{F})$ *and* $(\Omega', \mathcal{F}')$ *be two measurable spaces. For a mapping* $T : \Omega \longrightarrow \Omega'$ *consider the inverse images* $T^{-1}(A') = \{\omega \in \Omega : T_\omega \in A'\}$ *for* $A' \in \Omega'$. *The mapping* T *is measurable if* $T^{-1}(A') \in \mathcal{F}$ *for each* $A' \in \mathcal{F}$.

In simpler terms, it's a function where the inverse image of any measurable set is also measurable.

I.2 Probability Space

Definition I.2.1 [5, 7] *Let* $\mathcal{F}$ *be a* σ*-field on* Ω *A probability measure* $\mathbb{P}$ *is a function*

$$\mathbb{P} : \mathcal{F} \longrightarrow [0, 1], \tag{I.3}$$

such that

1 $\mathbb{P}(\Omega) = 1$;

2 *if* A_1,A_1, ... *are pairwise disjoint sets (that is,* A_i *n* $A_j = \phi$ *for* $i \neq j$*) belonging to*$\mathcal{F}$*, then*

$$\mathbb{P}(A_1 \bigcup A_2) \bigcup ... = \mathbb{P}(A_1) + \mathbb{P}(A_2) + ... \tag{I.4}$$

Remarks I.2.1

- *The triple* $(\Omega,\mathcal{F}, \mathbb{P})$ *is called a probability space. The sets belonging to* $\mathcal{F}$ *are called events. An event is a subset of the sample space(the set of all possible outcomes), an event* A *is said to occur almost surely (a.s.) whenever* $\mathbb{P}(A) = 1$.

- *Probability measures have the following interpretation. Let* A *be a subset of* $\mathcal{F}$*. Imagine that* Ω *is the set of all possible outcomes of some random experiment. There is a certain probability, between* 0 *and* 1*, that when that experiment is performed, the outcome will lie in the set* A*. We think of* $\mathbb{P}(A)$ *as this probability.*

Example I.2.1 *consider a bag with three colored balls: one red, one green, one white. The sample space* Ω*= red, green, white and* $\mathcal{F}$*=*ϕ*, red , green, white,* Ω*.Each ball has an equal chance of being drawn, so the probability space would be* $\mathbb{P}(\phi)$*=0,* $\mathbb{P}(red)$*=1/3,* $\mathbb{P}(green)$*=1/3,* $\mathbb{P}(white)$*=1/3,* $\mathbb{P}(\Omega)$*=1.*

Definition I.2.2 ***Independence*** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\{B_1, B_2, ..., B_n\} \subset \mathcal{F}$ *be a finite collection of events.*

- *(i)* $B_1, B_2, ..., B_n$ *are called independent w.r.t.* $\mathbb{P}$*, if*

$$\mathbb{P}\left(\bigcap_{j=1}^{k} (B_{i_j})\right) = \Pi_{j=1}^{k} \mathbb{P}(B_{i_j}). \tag{I.5}$$

for all $i_1, i_2, ..., i_k \subset 1, 2, ..., n$*,* $1 \leq k \leq n$*.*

- *(ii)* $B_1, B_2, ..., B_n$ *are called pairwise independent w.r.t.* $\mathbb{P}$ *if* $\mathbb{P}(B_i \cap B_j = \mathbb{P}(B_i \mathbb{P}(B_j$ *for all* $i, j, i \neq j$*.*

Note that a collection $B_1, B_2, ..., B_n$ *of events may be independent with respect to one probability measure* $\mathbb{P}$ *but not with respect to another measure* $\mathbb{P}'$*. Note also that pairwise independence does not imply independence[3].*

I.3 Conditional Probability

Suppose we are told that the event A with $\mathbb{P}(A) > 0$ occurs. Then the sample space is reduced from Ω to A and the probability that B will occur given that A has occurred is

$$\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(A), \tag{I.6}$$

to explain this formula, note that the only the part of B that lies in A can possibly occur, and since the sample space is now A, we have to divide by $\mathbb{P}(A)$ to make $\mathbb{P}(A|A) = 1$. Multiplying on each side of $I.6$ by P.A/ gives us the multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(A). \tag{I.7}$$

Intuitively, we think of things occurring in two stages. First we see if A occurs, then we see what the probability B occurs given that A did. In many cases these two stages are visible in the problem [13].

Example I.3.1 *We throw two fair dice. Let A be the event that the first dice is 6 and let B be the event that the sum of two dices is 11. Because $\mathbb{P}(B) = 2/36 = 1/18$ and $\mathbb{P}[A \cap B] = 1/36$ (we need to throw a 6 and then a 5),we have $\mathbb{P}[A|B] = (1/16)/(1/18) = 1/2$. The interpretation is that since we know that the event B happens, we have only two possibilities: $(5,6)$ or $(6,5)$. On this space of possibilities, only the second is compatible with the event B [18].*

I.4 Random Variables

Definition I.4.1 [3]
Let $(\Omega,\mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \longrightarrow \mathbb{R}$ be $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable (or $\mathcal{F}$-measurable), that is $X^{-1}(A) \in$ for all A $\mathcal{F} \in \mathcal{B}(\mathbb{R}))$ Then, X is called a random variable on $(\Omega,\mathcal{F}, \mathbb{P})$.

Recall that $X : \Omega \longrightarrow (\mathbb{R})$ is $\mathcal{F}$-measurable iff for all $x \in \mathbb{R}$, $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$.

Definition I.4.2 *Let X be a random variable on $(\Omega,\mathcal{F}, \mathbb{P})$. Let*

$$F_X(x) = \mathbb{P}(\omega : X(\omega) \leq x), \quad x \in \mathbb{R}. \tag{I.8}$$

Then $F_X(.)$ is called the cumulative distribution function (cdf) of X.

Definition I.4.3 *Let X be a random variable on $(\Omega,\mathcal{F}, \mathbb{P})$. Let*

$$\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A)), \quad for\ all A \in \mathcal{B}(\mathbb{R}). \tag{I.9}$$

Then the probability measure $\mathbb{P}_X$ is called the probability distribution of X.
Intuitive idea: A random variable consists of an experiment with a probability measure $\mathbb{P}$ defined on a sample space Ω and a function that assigns a real number to each outcome in the sample space of the experirnent. $X : \Omega \longrightarrow \mathbb{R}$, such that the values taken by X are known to someone who has access to the information $\mathcal{F}$.[6, 30]

Remarks I.4.1

1) *The set of possible values of X is the range of X.*

2) *X is an $\mathcal{F}$-measurable function.*

Example I.4.1 *consider a financial investment scenario.Suppose you're investing in stocks, and X represents the daily return on your investment, measured as a percentage.*
for instance:
if you invest Ł1000 *and the stock increases by* 2 *in a day, then* $(X = 20)$.
if the stock decreases by 1.5 *in a day, then* $(X = -15)$.
In this scenario X is a random variable representing the daily return on your investment, which can vary depending on the preformance of the stock market

I.4.1 Types of Random Variables

Random variables are key concepts in probability theory and statistics, they are used to model the outcomes os random experments and observations in various fields,includuind physics, finance, biology, and engineering. Understanding the different types of random variables is curail for analyzing and modeling various phenomena. There are two main types : discrete and continunous .
the primary difference between discrete random variable and continunous random variable lies in the nature of the value they can take, the discrete ones have distinct, separable outcomes while cpntinunous random variable can take on a continuous range of values without distinct separations.

Discrete Random Variables [26]

A random variable is called discrete if it takes values in a countable subset $\{x_1, x_2, ...,\} \subset \mathbb{R}$. hat is, $X_\omega \in \{x_1, x_2, ...,\}$ from each $\omega \in \Omega$.
mass funtion:
The probability mass function f of a discrete random variable X is the function f : $\{x_1, x_2, ...,\} \longrightarrow [0, 1]$ given by :

$$f(x) = \mathbb{P}(X = x).$$

Notice that for a discrete random variable

$$F_X(x) = \Sigma_{x_i \leq x} f(x_i).$$

Examples of Discrete Random Variables

1. Binomial Distribution A random variable X follows a binomial distribution with parameters n (number of trials) and p (probability of success in each trial), denoted as $X \sim \text{Binomial}(n, p)$, if its probability mass function (PMF) is given by:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n,$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient.

2. Geometric Distribution A random variable X follows a geometric distribution with parameter p (probability of success in each trial), denoted as $X \sim \text{Geometric}(p)$, if its PMF is given by:

$$P(X = k) = (1 - p)^{k-1} p \quad \text{for } k = 1, 2, 3, \ldots$$

Discrete Uniform Distribution A random variable X follows a discrete uniform distribution over the set $\{a, a + 1, \ldots, b\}$, denoted as $X \sim \text{Uniform}(a, b)$, if its PMF is given by:

$$P(X = k) = \frac{1}{b - a + 1} \quad \text{for } k = a, a + 1, \ldots, b$$

Continuous Random Variables [26]

A random variable is called continuous if there exists a piecewise continuous nonnegative function $f(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f(s) d(s).$$

In this case, $f(x)$ is called the probability density function of X. Notice that

$$\mathbb{P}(a \leq X \leq b) = \int_{a}^{b} f(s) ds. \tag{I.10}$$

Examples of Continuous Random Variables

Uniform Distribution A random variable X is uniformly distributed over the interval $[a, b]$ if its PDF is:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

The CDF is:

$$F_X(x) = \begin{cases} 0 & \text{for } x < a, \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b, \\ 1 & \text{for } x > b. \end{cases}$$

2. Normal Distribution A random variable X is normally distributed with mean μ and variance σ^2, denoted as $X \sim N(\mu, \sigma^2)$, if its PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The CDF does not have a closed-form expression and is usually computed using numerical methods or tables.

3. Exponential Distribution A random variable X is exponentially distributed with rate parameter λ, denoted as $X \sim \text{Exp}(\lambda)$, if its PDF is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The CDF is:

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

I.5 Independence

The independence of random variables refers to a scenario where the outcome of one variable doesn't affect the outcome of another. In mathematical terms, two random variables X and Y are independent if the occurrence of an event related to X does not affect the probability of an event related to Y, and vice versa, i.e., if for any two open intervals $A, B \subset \mathbb{R}$, the events

$$E = \{\omega,\ X(\omega) \in A\},\ F = \{\omega,\ Y(\omega) \in B\}$$

are independent, so $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$, then X and Y are called independent random variables [6].

I.6 Mathematical Expectation

Mathematical expectation, also known as = expected value and denoted by $\mathbb{E}[X]$, represents the average of outcome of a random variable over many trials. If X is a random variable on $(\Omega, \mathcal{F})$ and $\mathbb{P}$ is a probability, then the expectation of X with respect to $\mathbb{P}$ is :

$$\mathbb{E}[X] = \sum X(\omega)\mathbb{P}(\omega).$$

where the sum is taken over all elementary outcomes ω. It can be shown that the expectation can be calculated by using the probability distribution of X.

The Discrete Case [26]

If X is a discrete random variable having a probability mass function $\mathbb{P}(x)$, then the expected value of X is defined by

$$\mathbb{E}[X] = \sum x\mathbb{P}(x).$$

The Continuous Case [26]

If X is a continuous random variable having a probability density function $f(x)$, then the expected value of X is defined by

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

I.7 Variance and Covariance

Definition I.7.1 [11] *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable. Assume that the mathematical expectation $\mathbb{E}[X]$ exists and is finite. The variance is defined as the mean of the square of the difference $X - \mathbb{E}[X]$ i.e.*

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

The standard deviation is defined as

$$\sigma = \sigma(X) = \sqrt{Var(X)}.$$

Definition I.7.2 [1, 11] *Let $X, Y : \Omega \to \mathbb{R}$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and having finite means $\mathbb{E}[X]$, $\mathbb{E}[Y]$. The number*

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

is called the covariance of X and Y.

I.8 Limits of Sequences of Random Variables

Consider a sequence $(X_n)_{n\geq 1}$ of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There are several ways of making sense of the limit expression $X = \lim_{n\to\infty} X_n$, some of them will be discussed in the following sections[6].

I.8.1 Almost Sure Limit

The sequence X_n converges almost surely to X, if for all states of the world ω, except a set of probability zero, we have

$$\lim_{n\to\infty} X_n(\omega) = X(\omega).$$

More precisely, this means

$$\mathbb{P}\left(\omega; \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1,$$

and we shall write $as - \lim_{n\to\infty} X_n = X$ [6].

I.8.2 Mean Square Limit

We say that X_n converges to X in the mean square if

$$\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0.$$

This limit will be abbreviated by $ms - \lim_{n\to\infty} X_n = X$. The mean square convergence is useful when defining the Itô integral [6].

II Stochastic Calculus

Stochastic calculus is a branch of mathematics that deals with processes that evolve randomly over time. It provides tools to analyze and model these random phenomena,Stochastic calculus combines calculus and probability theory to study random processes, It extends traditional calculus to handle random variables, enabling the analysis of complex systems affected by random fluctuations. Key concepts include stochastic integration, Itô calculus, and the dynamics of Brownian motion, pivotal for pricing derivatives in financial markets and modeling diverse phenomena with inherent uncertainty[17].

II.1 Stochastic Processes

Definition II.1.1 *A stochastic process is a family of random variables* $\{X(t) : t \in \tau\}$ *defined on a probability space*$(\Omega, \mathcal{F}, \mathbb{P})$ *and indexed by a parameter* t *where* t *varies over a set* τ *. If the set* τ *is discrete, the stochastic process is called discrete. If the set* τ *is continuous, the stochastic process is called continuous.The random variables are defined on a common state space* S*. The parameter* t *usually plays the role of time and the rondam variables can be discrete valued or continuous-valued at each value of* t*. For example, a continuous stochastic process can be discrete-valued. For modeling purposes, it is useful to understand both continuous and discrete stochastic processes and how they are related.*[1]
A stochasticprocess $X = \{X(t) : t \in \tau\}$ *is thus a function of two variables* $X : T \times \Omega \longrightarrow \mathbb{R}$ *where* $X(t) = X(t, .)$ *is a random variable for each* $t \in T$*. For each* $\omega \in \Omega$ $X(., \omega) : T \longrightarrow \mathbb{R}$ *a realization, a sample path or a trajectory of the stochastic process*[11].

II.2 Some Stochastic Processes

II.2.1 Markov Chain

The notion of what is nowadays called a Markov chain was devised by the Russian mathematician A.A. Markov when, at the beginning of the twentieth century, he developed a probability model in which the outcomes of successive trials are allowed to be dependent on each other such that each trial depends only on its immediate predecessor. In other words, the future probabilistic behaviour of the process depends only on the present state of the process and is not influenced by its past history [28].

Definition II.2.1 [12, 25] *A Markov process X_t is a stochastic process with the property that, given the value of X_t, the values og X_s for $s > t$ are not influenced by the values of X_u for $u > t$. In words, the probability of any particular future bahavior of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behavior.*
A discrete-time Markov chain is a MArkov process whose state space is finite or countable set, and whose (time) index set is $T = \{0, 1, 2, ...\}$. In formal terms, the Markove property is that

Definition II.2.2

$$\mathbb{P}(X_{n+1} = j | X_0 = x_0, ..., X_{n-1} = x_{n-1}, X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i),$$

for all $x_0, \ldots, x_{n-1}, i, j \in S$ and $n \geq 0$. The set S is the state space of the Markov chain.

Example II.2.1 ***Weather Model***
Consider the weather model with states: Sunny (S), Cloudy (C), and Rainy (R). A stochastic process $\{X_n\}_{n\geq 0}$ (where X_n represents the weather state on day n) satisfies the Markov property if for any n, the conditional probability of the future state depends only on the current state:

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

Probabilities for Tomorrow's Weather
If it is Sunny today, the probabilities for tomorrow's weather are:

$$\begin{aligned} &\textit{Sunny tomorrow:} && P(X_{n+1} = S \mid X_n = S) = 0.6 \\ &\textit{Cloudy tomorrow:} && P(X_{n+1} = C \mid X_n = S) = 0.3 \\ &\textit{Rainy tomorrow:} && P(X_{n+1} = R \mid X_n = S) = 0.1 \end{aligned}$$

These probabilities are directly provided and depend only on the current state $X_n = S$.

Example II.2.2 (*Monopoly*) *The popular board game Monopoly can be modeled as a Markov chain stochastic process. A player starts at square 1. At each turn, the player rolls a die and moves around the board by the number of spaces shown on the face of the die. The player keeps moving around and around the board according to the roll of the die. Let X_k be the number of squares the player lands on after k moves, with $X_0 = 1$. The state space is $\{1, \ldots, 40\}$ denoting the 40 squares of a Monopoly board. The index set is $\{0, 1, 2, \ldots\}$. Both the index set and the state space are discrete. Assume that the player successively rolls 2, 1 and 4. The first four positions are*

$$(X_0, X_1, X_2, X_3) = (1, 3, 4, 8).$$

Given this information, what can be said about the player's next location X_4? Even though we know the player's full past history of moves, the only information relevant for predicting their future position is their most recent location X_3. Since $X_3 = 8$, then necessarily $X_4 \in \{9, 10, 11, 12, 13, 14\}$, with equal probability. Formally,

$$\mathbb{P}(X_4 = j | X_0 = 1, X_1 = 3, X_2 = 4, X_3 = 8) = \mathbb{P}(X_4 = j | X_3 = 8) = \frac{1}{6},$$

for $j = 9, 10, 11, 12, 13, 14$. Given the player's most recent location X_3, their future position X_4 is independent of the past history X_0, X_1, X_2.

II.2.2 Brownian Motion

The Brownian motion stochastic process arose early in 1827, when Robert Brown observed that miscroscopic grains suspended in a drop pf water moved constantly in haphazard zigzag trajectories. sililair Brownian motion was apparent whenever very small particles were suspended in a fluid medium, e.g., smoke particles in air. over time, it was established that finer particles move more rapidly, that the motion is stimulated by heat, ans that the mouvement becomes more active with a decrease in fluid viscosity.Today , the Brownian motion process and its many generalization occur in numerous and diverse areas of pure and applied science such as economics, communication theory, biology, and mathematical science.[25] to the definition of the Brownian motion, which is formalized in the following:

Definition II.2.3 [11] *A stochastic process $B(t, \omega)$ is called a Brownian motion if it satisfies the following conditions:*

1. $\mathbb{P}\{\omega; B(0, \omega) = 0\} = 1$;
2. *For any $0 \leq s \leq t$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, i.e., for any $a < b$*

$$\mathbb{P}\{a \leq B_t - B_s \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{\frac{-x^2}{2(t-s)}} dx;$$

3. *$B(t, \omega)$ has independent increments, i.e., for any $0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the random variables*

$$B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$$

are independent;

4. *Almost all sample paths of $B(t, \omega)$ are continuous functions, i.e.,*

$$\mathbb{P}\{\omega; B(., \omega) is\, continuous\} = 1$$

.

Remarks II.2.1 [6, 7]

(i) *It is worth noting that even if B_t is continuous, it is nowhere differentiable.*

(ii) *From condition 4 we get that B_t is normally distributed with mean $\mathbb{E}[B_t] = 0$ and $Var(B_t) = t$*

$$B_t \sim N(0, t).$$

Example II.2.3 ***Brownian Motion with Drift***
Consider the stochastic process $\{B(t)_{t \geq 0}\}$ defined by:

$$B(t) = \mu t + \sigma W(t),$$

where μ and σ are constants, and $W(t)_{t\geq 0}$ is a standard Brownian motion .
Proving that $B(t)$is a Brownian Motion
To show that $\{B(t)\}$ satisfies the properties of a Brownian motion:

1. $B(0) = 0$

   By definition:
$$B(0) = \mu \cdot 0 + \sigma \cdot W(0) = 0.$$
   This satisfies the first property of Brownian motion.

2. *Normal Distribution of Increments*

   For $0 \leq s < t$, consider the increment $B(t) - B(s)$:
$$B(t) - B(s) = \mu(t-s) + \sigma(W(t) - W(s)).$$
   Since $W(t) - W(s) \sim N(0, t-s)$ (normal distribution with mean 0 and variance $t-s$), the increment $B(t) - B(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

3. *Independent Increments*

   The increments $B(t) - B(s)$ are independent because $W(t) - W(s)$ is independent of $W(u)$ for $u < s$.

4. *Continuity of Paths*

   The paths of $B(t)$ are continuous because $W(t)$ is almost surely continuous, and $B(t)$ is a linear combination of t and $W(t)$.

Therefore, the process $\{B(t)\}_{t\geq 0}$ defined by $B(t) = \mu t + \sigma W(t)$, where $W(t)$ is a standard Brownian motion, satisfies all the properties required for a Brownian motion.

II.2.3 Martingales

This section deals with a class of stochastic processes called martingales. Margtingales arise in a natural way in many problems in probability and satistics. It provides a more general farmework than the cas of independent random variables where results ca be established. Much of the discrete parameter martingale theory was developed by the great American mathematician J. L. Doob, whose book (Doob (1953)) has been very influential. [3].

We begin with a definition that is important for this chapter and the next one

Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{A} \subset \mathcal{F}$ be a o-algebra and let $\xi : \Omega \longrightarrow \mathbb{R}$ is an integrable random variable ($\mathbb{E}(|\xi|) < \infty$)

Definition II.2.4 [11] *a funtion $\varphi : \Omega \longrightarrow \mathbb{R}$ is said to be a representative of the conditional expectation of the random variable ξ with respect to $\mathcal{A}$ ($\mathbb{E}[\xi|\mathcal{A}] = \varphi$), if:*

1/ φ *is* $\mathcal{A}$ *measurable.*

2/ for any $A \in \mathcal{A}$

$$\int_A \mathbb{E}[\xi|\mathcal{A}]d\mathbb{P} = \int_A \xi d\mathbb{P}.$$

It can be shown that there is at least one such function φ and that ifψ also satisfies conditions (1) and (2), then $\varphi = \psi$, w.p.1.
The family of all functions satisfying (1) and (2) is denoted by $\mathbb{E}[\xi|\mathcal{A}]$ and is called the conditional expectation of ξ with respect to $\mathcal{A}$. Here too, we usually identify the representatives of the conditional expectation with the conditional expectation itself.

Proposition II.2.1 [5] *Conditional expectation has the following properties:*

1) $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$ (*Linearity*);

2) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$;

3) $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$ *if X is* $\mathcal{G}$*-measurable* (*Taking out what is known*);

4) $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$ *if X is independent of* $\mathcal{G}$ (*An independent condition drops out*);

5) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ *if* $\mathcal{H} \subset \mathcal{G}$ (*Tower property*);

6) *if* $X \geq 0$*, then* $\mathbb{E}(X|\mathcal{G}) \geq 0$ (*Positivity*).

Here a, b are arbitrary real numbers, X, Y are integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\mathcal{G}$, $\mathcal{H}$ *are sigma-fields on* Ω *contained in* $\mathcal{F}$*. In* 3 *we also assume that the product XY is integrable.*

Lemma II.2.1 ***Jensen's Inquality***[5]
let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *be a convex function and let* ξ *be an integrable random variable on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\varphi(\xi)$*is also integrable . Then*

$$\varphi(\mathbb{E}[\xi|\mathcal{G}]) \leq \mathbb{E}[\varphi(\xi|\mathcal{G})] \quad a.s$$

for any a-field $\mathcal{G}$ *on* Ω *contained in* $\mathcal{F}$*.*

Martingale

In some sense the martingale conception can be explained by a fair game. Let us interprete it as follows:
In a game suppose that a person at the present time n has wealth X_n, for the game, and at the future time $n+1$ he will have the wealth X_{n+1}. The expected money for this person at the future time t is naturally expressed as $\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right]$, where $\mathbb{E}(.)$ means the expectation value of ., $\mathcal{F}_n$ means the information up to time n, which is known by the gambler, and $\mathbb{E}(.|\mathcal{F}_n)$ is the conditional expectation value of . under given $\mathcal{F}_n$, Obviously, if the game is fair, then it should be:

$$\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = X_n.$$

This is exactly the definition of a martingale for a random process X_n, $n \geq 0$. Let us make it more explicit for later developement.[27]

Definition II.2.5 [3] *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and let* $\mathbb{N} = 1, ..., n_0$ *be a nonempty subset of* $\mathbb{N} = 1, 2, ..., n_0 \leq \infty$ *.*

(A) *A collection* $\{\mathcal{F}_n : n \in N\}$*of sub*$\sigma - algebras$ *of F is called a filtration if* $\mathcal{F}_\backslash \subset \mathcal{F}_{\backslash+\infty}$ *for all* $1 \leq n < n_0$ *.*
We say that $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ *is a filtered probability space.*

(b) *A collection of random variables* $X_n : n \in N$ *is said to be adapted to the filtration* $\{\mathcal{F}_n : n \in N\}$ *if* X_n *is* $\mathcal{F}_n$*-measurable for all* $n \in N$*.*

(c) *Given a filtration* $\{\mathcal{F}_n : n \in N\}$ *and random variables* $X_n : n \in N$*, the collection* $\{(X_n, \mathcal{F}_n) : n \in N\}$ *is called a martingale if :*

(i) $\{X_n : n \in N\}$ *is adapted to* $\mathcal{F}_n : n \in N$*,*

(ii) $\mathbb{E}(|Xn|) < \infty$ *for all* $n \in N$*,*

(iii) *for all* $1 \leq n < n_0$*:*

$$\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = X_n.$$

Remark II.2.1 [19] *To each process* $\{X_t, t \geq 0\}$ *and for each* t*, we can associate a sigma-field denoted by* $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$*, which is the sigma-field generated by the process* X *up to time* t*, i.e., the smallest sigma-field of* $\mathcal{F}$ *that makes* $X(s,\omega)$ *measurable for every* $0 \leq s \leq t$*.*

Example II.2.4 *Let* ξ *be an integrable random variable and let* $\mathcal{F}_1, \mathcal{F}_2, ...$ *be a filtration. We put*

$$\xi_n = \mathbb{E}\left[\xi|\mathcal{F}_n\right],$$

for $n = 1, 2, ...$ *Then,* ξ_n *is*$\mathcal{F}_n$*-measurable,*

$$|\xi_n| = |\mathbb{E}\left[\xi|\mathcal{F}_n\right]| \leq \mathbb{E}\left[|\xi||\mathcal{F}_n\right], \tag{II.1}$$

which implies that

$$\mathbb{E}[|\xi_n|] \leq \mathbb{E}\left[\mathbb{E}\left[|\xi||\mathcal{F}_n\right]\right] = \mathbb{E}[|\xi|] \leq \infty, \tag{II.2}$$

and

$$\begin{aligned}\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\mathbb{E}\left[\xi|\mathcal{F}_{n+1}\right]|\mathcal{F}_n] \\ &= \mathbb{E}\left[\xi|\mathcal{F}_n\right] \\ &= \xi_n\end{aligned}$$

since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ *(the tower property of conditional expectation) . Therefore* ξ_n *is a martingale with respect to* $\mathcal{F}_n$ *.*

Supermartingale, Submartingale

Definition II.2.6 [29] *let* $\{\mathcal{F}_n : n \in N\}$ *be a filtration and* $\{X_n : n \in N\}$*be a collection of random variables in* $L^1(\Omega, \mathcal{F}, \mathbb{P})$ *adapted to* $\{(X_n, \mathcal{F}_n) : n \in N\}$ *Then is called a sub-martingale if*

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n \quad forall 1 \leq n < n0, \tag{II.3}$$

and a super-martingale if

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n \quad forall 1 \leq n < n0. \tag{II.4}$$

Remark II.2.2 *For the stochastic process*$\{X_n\}_{n=1}^{\infty}$ *and the random process* $\{X_t\}_{t\in[0,T]}$ *with continuous time similar definitions can be given.*

Stopping Time

In probability theory, a stopping time is a random variable whose value is interpreted as the time at which a given stochastic process exhibits a certain behavior of interest. A stopping time is often defined by a stopping rule, a mechanism for deciding whether to continue or stop a process on the basis of the present position and past events.

Definition II.2.7 [20] *A map* $\tau : \Omega \longrightarrow 0, 1, 2, ..., \infty$ *is a stopping time if,*

$$\{T \leq n\} = \{\omega : T(\omega) \leq n\} \in \mathcal{F}_n, \quad \forall n \leq \infty. \tag{II.5}$$

Note that T can be ∞

Intuitive idea : T is time when you can decide to stop playing our game, whether or not you stop immediately after the n^{th} game depends only on the history up to (and including) time $n : \{T \leq n\} \in \mathcal{F}_n$.

Example II.2.5 *Consider a sequence of independent coin tosses. Define a stopping time T as the first time we get a heads* (H).
Let $\{X_n\}_{n\geq 1}$ *be a sequence where* X_n *represents the outcome of the n-th coin toss, with:*

$$X_n = \begin{cases} 1 & \text{if the n-th toss is heads } (H), \\ 0 & \text{if the n-th toss is tails } (T). \end{cases}$$

Define the stopping time T as:

$$T = \inf\{n \geq 1 : X_n = 1\}.$$

Example II.2.6 *Suppose that* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and* $\mathbb{T} = \{0, 1, 2, 3\}$. *The stochastic process* $X = \{X_t, t \in \{0, 1, 2, 3\}\}$ *represents the evolution of stock price,* X_t = *stock price as soon as the stock market is closed at the* t^{th} *day, instant* $t = 0$ *representing today.*

$\omega \setminus X_t(\omega)$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$
ω_1	1	0.5	1	0.5
ω_2	1	0.5	1	0.5
ω_3	1	2	1	1
ω_4	1	2	2	2

Table II.1 : Values of the stock price X_t for each ω at each moment.

We had determined that the filtration containing the infomation revealed by the process at each moment is:

$$\mathcal{F}_0 = \sigma\{X_0\} = \{\phi, \Omega\};$$
$$\mathcal{F}_1 = \sigma\{X_0, X_1\} = \sigma\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\};$$
$$\mathcal{F}_2 = \sigma\{X_0, X_1, X_2\} = \sigma\{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\};$$
$$\mathcal{F}_3 = \sigma\{X_0, X_1, X_2, X_3\} = \sigma\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.$$

a) *We are not selling our shares today* $(t = 0)$ *but we will sell them as soon as the price is greater than or equal to 1 .*

- *The random time representing this situation is:* $\tau(\omega_1) = 2$, $\tau(\omega_2) = 2$, $\tau(\omega_3) = 1$, $\tau(\omega_4) = 1$.
  This random variable is indeed a stopping time since:

$$\{\omega \in \Omega : \tau(\omega) = 0\} = \phi \in \mathcal{F}_0;$$
$$\{\omega \in \Omega : \tau(\omega) = 1\} = \{\omega_3, \omega_4\} \in \mathcal{F}_1;$$
$$\{\omega \in \Omega : \tau(\omega) = 2\} = \{\omega_1, \omega_2\} \in \mathcal{F}_2;$$
$$\{\omega \in \Omega : \tau(\omega) = 3\} = \phi \in \mathcal{F}_3.$$

b) *Now consider random time* τ^* *modeling the following situation: we will buy shares as soon as we are able to make a profit later.*

- *This random variable takes the values* $\tau^*(\omega_1) = 1$, $\tau^*(\omega_2) = 1$, $\tau^*(\omega_3) = 0$, $\tau^*(\omega_4) = 0$ *is not a stopping time since :*

$$\{\omega \in \Omega : \tau^*(\omega) = 0\} = \{\omega_3, \omega_4\} \notin \mathcal{F}_0.$$

II.3 Stochastic Integration

For a dynamic system the simplest continuous stochastic perturbation is naturally considered to be a Brownian motion, since it is a Normal process (Guassian process) with independent increments which are also normally distributed. In general, a continuous stochastic perturbation will be modeled as some stochastic integral with respect to the Brownian motion. However, the Brownian motion.has the strange property that even though its trajectory is continuous in t ,it is not differentiable for all t. In this section we will discuss how to define such stochastic integral. This integralwas first defined by K.Itô in 1949 and now iknown as Itô stochastic integral which has the form $\int_0^t X(s)dB(s)$, where $B(t)$ is Brownian motion, and $X(t)$ is a stochastic nonanticipating process,The term "nonanticipating" used by Itô is nowadays commonly called "adapted". We sall start to define the stochastic integral step by step.[27, 22]

II.3.1 Riemann–Stieltjes

Let g be a function of bounded variation on a finite closed interval $[a, b]$. A countinous function f defined on $[a, b]$ is said to be **Riemann–Stieltjes** integrable with respect to g if the following limit exists:

$$\int_a^b f(t)dg(t) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\tau_i)(g(t_i) - g(t_{i-1}))$$

where $\Delta_n = \{t_0, t_1, ..., t_{n-1}, t_n\}$ is a partition of $[a, b]$ with the convention $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, $||\Delta_n|| = \max_{1 \le i \le n}(t_i - t_{i-1})$, and τ_i is an evaluation point in the interval $[t_{i-1}, t_i]$.
unlike the Riemann–Stieltjes integral, which requires the integrand to be of bounded variation,the Itô integral can handle integrands with respect to Brownian motion which have unbounded variaton.

II.3.2 unbounded variation of Brownian motion

Proposition II.3.1 [10] *With probability* 1*, the paths of Brownian motion* $B(t)$ *are not of bounded variation*

$$\mathbb{P}(V(B)[0, t]) = \mathbb{P}(\sum_{k=1}^{n} |(B(t_k) - B(t_{k-1}))| = \infty) = 1, \quad \textit{for all fixed} \ \ t > 0.$$

where partition Π *of* $[0, t]$ *is a set* $\Pi = \{t_0, t_1, ..., t_n\}$ *of points of* $[0, t]$ *such that* $0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$*. For any partition* Π *we set*

$$|\Pi| = \max_k(t_{k+1} - t_k).$$

II.3.3 The Itô Integral

[8] Let $\{B_t; t \geq 0\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let$\{\mathcal{F}_t; t \geq 0\}$ be the associated filtration. The Ito integral defined by:

$$I_n = \int_0^t f(B_t, t) dB_t = \sum_{i=0}^{n-1} f(t_i)(B(t_{i+1}) - B(t_i)).$$

where f is a simple process and $t_i = \frac{it}{n}, 0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = t, n \in \mathbb{N}$ has the following properties

Properties of the Itô Integral

We shall start with some properties which are similar with those of the Riemannian integral.

Proposition II.3.2 [24, 6] *Let $f(t, B_t)$, $g(t, B_t)$ be nonanticipating processes and $c \in \mathbb{R}$. Then we have*

1. *Isometry:*
$$\mathbb{E}\left[\left(\int_a^b f(t, B_t) dB_t\right)^2\right] = \mathbb{E}\left[\int_a^b f(t, B_t)^2 dt\right]. \tag{II.6}$$

2. *The Itô integral $\int_0^t f(s, B_s) dB_s$ is $\mathcal{F}_t$-martingale.*

3. *Zero mean:*
$$\mathbb{E}\left[\int_a^b f(t, B_t) dB_t\right] = 0. \tag{II.7}$$

4. *Predictability: $\int_0^T f(t, B_t) dB_t$ is $\mathcal{F}_T$-measurable.*

5. *Additivity:*
$$\int_0^T [f(t, B_t) + g(t, B_t)] dB_t = \int_0^T f(t, B_t) dB_t + \int_0^T g(t, B_t) dB_t.$$

6. *Partition property:*
$$\int_0^T f(t, B_t) dB_t = \int_0^u f(t, B_t) dB_t + \int_u^T f(t, B_t) dB_t, \quad \forall\, 0 < u < T.$$

7. *Homogeneity:*
$$\int_0^T c f(t, B_t) dB_t = c \int_0^T f(t, B_t) dB_t.$$

8. *Covariance:*

$$\mathbb{E}\left[\left(\int_a^b f(t,B_t)dB_t\right)\left(\int_a^b g(t,B_t)dB_t\right)\right] = \mathbb{E}\left[\int_a^b f(t,B_t)g(t,B_t)dt\right]. \quad \text{(II.8)}$$

From II.7 and II.6 it follows that the random variable $\int_a^b f(t,B_t)dB_t$ has mean zero and variance

$$Var\left[\int_a^b f(t,B_t)dB_t\right] = \mathbb{E}\left[\int_a^b f(t,B_t)^2 dt\right].$$

Example II.3.1 [6] *Let $X_t = c$, (constant), then the partial sums can be computed as follows*

$$S_n = \sum_{i=0}^{n-1} X(t_i)(B(t_{i+1}) - B(t_i)) = \sum_{i=0}^{n-1} c(B(t_{i+1}) - B(t_i)) = c(B(b) - B(a)),$$

and since the answer does not depend on n, we have

$$\int_a^b c dB(t) = c(B(b) - B(a)).$$

In particular, taking $c = 1$, $a = 0$, and $b = T$, since the Brownian motion starts at 0, we have the following formula:

$$\int_0^T dB(t) = B(T).$$

II.4 Stochastic Differentiation

In mathematics, Ito's formula (or lemma) is used in stochastic calculus to find the differential of a function of a particular type of stochastic process. In essence, it is the stochastic calculus counterpart of the chain rule in ordinary calculus via a Taylor series expansion. The formula is widely employed in mathematical finance and its best-known application is in the derivation of the Black–Scholes equation used to value options. The following is a formal result of Ito's formula. We know from the last section that derivatives like $\frac{dB_t}{dt}$ do not make sense in stochastic calculus. The only quantities allowed to be used are the infinitesimal changes of the process between instances t and $t + dt$, in our case, dB_t [6].

II.4.1 Itô's formula

Theorem II.4.1 [6](***Itô's formula, first version***) *Consider X_t a stochastic process satisfying*

$$dX_t = b_t dt + \sigma_t dB_t,$$

with $b_t(\omega)$ and $\sigma_t(\omega)$ "good behaving" processes. Let $F_t = f(X_t)$, with f twice continuously differentiable. Then

$$dF_t = \left[b_t f'(X_t) + \frac{\sigma_t^2}{2} f''(X_t)\right] dt + \sigma_t f'(X_t) dB_t. \tag{II.9}$$

or in integrated form,

$$f(X_t) = f(X_0) + \int_0^t \left[b_s f'(X_s) + \frac{\sigma_s^2}{2} f''(X_s)\right] dt + \int_0^t \sigma_s f'(X_s) dB_s. \tag{II.10}$$

Theorem II.4.2 {***Ito's Formula, second version***}[8] *Let $\{B_t; t \geq 0\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let$\{\mathcal{F}_t; t \geq 0\}$ be the associated filtration. Consider a stochastic process X_t satisfying the following SDE*

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t \tag{II.11}$$

or in integrated form,

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s \tag{II.12}$$

with $\int_0^t \left[|\mu(X_s, s)| + \sigma(X_s, s)^2\right] ds < \infty$. Then for any twice differentiable function $g(X_t, t)$, the stochastic process $Y_t = g(X_t, t)$ satisfies

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial t}(X_t, t)dt + \frac{\partial g}{\partial X_t}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial X_t^2}(X_t, t)\,(dX_t)^2 \\ &= \left[\frac{\partial g}{\partial t}(X_t, t) + \mu(X_t, t)\frac{\partial g}{\partial X_t}(X_t, t) + \frac{1}{2}\sigma(X_t, t)^2\frac{\partial^2 g}{\partial X_t^2}(X_t, t)\right] dt + \sigma(X_t, t)\frac{\partial g}{\partial X_t}(X_t, t)dB_t \end{aligned} \tag{II.13}$$

where (dX_t^2) is computed according to the rule

$$(dB_t)^2 = dt, \quad (dt)^2 = dB_t dt = dt dB_t = 0$$

In integrated form,

$$\begin{aligned} dY_t &= Y_0 + \int_0^t \frac{\partial g}{\partial t}(X_s, s)ds + \int_0^t \frac{\partial g}{\partial X_t}(X_s, s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial X_t^2}(X_s, s)d\left\langle X, X\right\rangle_s \\ &= Y_0 + \int_0^t \left[\frac{\partial g}{\partial t}(X_s, s) + \mu(X_s, s)\frac{\partial g}{\partial X_s}(X_s, s) + \frac{1}{2}\sigma(X_s, s)^2\frac{\partial^2 g}{\partial X_t^2}(X_s, s)\right] ds \\ &+ \int_0^t \sigma(X_s, s)\frac{\partial g}{\partial X_t}(X_s, s)dB_s \end{aligned} \tag{II.14}$$

where$\left\langle X, X\right\rangle_t = \int_0^t \sigma(X_s, s)^2 ds$.

II.5 stochastic differential equation

Stochastic differential equation (SDE) is a differential equation in which one or more of the terms has a random component. Within the context of mathematical finance, SDEs are frequently used to model diverse phenomena such as stock prices, interest rates or volatilities to name but a few. Typically, SDEs have continuous paths with both random and non-random components and to drive the random component of the model they usually incorporate a Brownian motion process. To enrich the model further, other types of random fluctuations are also employed in conjunction with the a Brownian motion process, such as the Poisson process when modelling discontinuous jumps. In this chapter we will concentrate solely on SDEs having only a Brownian motion.

A stochastic differential equation on the interval $[0, T]$ has the form

$$dX(t,\omega) = f(t, X(t,\omega))dt + g(t, X(t,\omega))dB(t,\omega) \tag{II.15}$$

where $0 \leq t \leq T$, with value $x(0) = x_0$. By the definition of stocastic diferential, this equation is equivalent to the following stochastic integral equation:

$$X(t,\omega) = X(0,\omega) + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dB(s,\omega) \tag{II.16}$$

where $0 \leq t \leq T$. The function f is often called the drift coefficient of the stochastic differential equation while g is referred to as the diffusion coefficient. It is assumed that the functions f and$g$ are nonanticipating and satisfy the following conditions (1) and (2) for some constant $k \geq 0$ [1, 16]

(1) $|f(t,x) - f(s,y)|^2 \leq k(|t-s| + |x-y|^2$ for $0 \leq s, t \leq T$ and $x, y \in \mathbb{R}$;

(2) $|f(t,x)|^2 \leq k(1+|x|^2)$ for $0 \leq t \leq T$ and $x \in \mathbb{R}$.

Let us first give a definition to the solution

Definition II.5.1 [22, 16] *An $\mathbb{R}$- valued stochastic process $X(t)_{0\leq t\leq T}$ is called a solution of equation $II.15$ if it has the followinf properties*

(i) *$X(t)$ a is continuous and $\mathcal{F}_t$- adapted;*

(ii) *$\{f(x(t),t)\} \in \mathcal{L}([0,T], \mathbb{R})$ and $\{g(x(t),t)\} \in \mathcal{L}^2([0,T], \mathbb{R})$;*

(iii) *equation $II.16$ holds for every $t \in [0,T]$ with probability 1 .*

A solution $X(t)$ is said to be unique if only other solution $\bar{X}(t)$ is indistinguishable from $X(t)$, that is

$$\mathbb{P}\{X(t) = \bar{X}(t) \ \ for \ \ all \ \ 0 \leq t \leq T\} = 1$$

Example II.5.1 *consider the following SDE :*

$$dX(t) = (\alpha - \beta X_t)dt + (\sigma\sqrt{X_t})dB_t, \tag{II.17}$$

in this equation :

- X_t *represents the stochastic process,*
- α, β *and* σ *are constants,*
- B_t *is a brownian motion.*

Let's break down the components:
$(\alpha - \beta X_t)$*: This term represents the deterministic drift component.*
$(\sigma\sqrt{X_t})$ *:This term represents the stochastic part of SDE.*
This type of stochastic differential equation is commonly in modeling biological population, whereX_t *could represents the population size, and the equation describes how it evolves over time under deterministis growth and stochastic fluctuations. The calculus involved here includes understanding how differential components affects the overall behavior of* X_t *over time.*

II.6 stochastic difference equation

II.6.1 difference equation

The field of difference equations has wide applications. The modern developement of calculus of differences began with a memoir by Poincaré published in 1885. The theory of difference equations, the methods used, and their wide application have progressed to such an extent that they occupy a central position in applicable analysis. In fact, in the last 12 years, hundreds of research articles and several monographs have been published and many international conferences and numerous special sessions have been convened. However, it seems that only minimal progress has been made in the development of a systematic theory of difference equations.[15]
Difference equations are equations that involve discrete changes or differences of the unknown function. This is in contrast to differential equations, which involve instantaneous rates of changes, or derivatives, of the unknown function. Difference equations are the discrete analogs of differential equations; they appear as mathematical models in situations where the variable takes or is assumed to take only a discrete set of values.[14]

Definition II.6.1 *A difference equation over the set of k-values* $0, 1, 2, ...$*is an equation of the form*

$$F(k, Y_k, Y_{k+1}, ..., Y_{k+n}) = 0, \tag{II.18}$$

where F is a given function, n is some positive integer, and $k = 0, 1, 2, ...$

The order of a difference equation is the difference between the highest and the lowest indices that appear in the equation.
The following are examples of difference equations over the set of k-values $0, 1, 2, ...$

$$\begin{aligned}&3Y_{k+1} - 8Y_k = 0,\\&Y_{k+2} + 5Y_{k+1} + 6Y_k = 0,\\&Y_{k+3} - 3Y_{k+2} + 6Y_{k+1} - 4Y_k = -2k + 5,\\&Y_{k+1}^2 - 2Y_{k+1}Y_k = k^3.\end{aligned}$$

Definition II.6.2 *A solution of the difference equation*

$$F(k, Y_k, Y_{k+1}, ..., Y_{k+n}) = 0, \tag{II.19}$$

is a sequence Y_k which satisfies $II.19$for $k = 0, 1, 2, ...$

For example, the sequence

$$Y_k = 2^k, \quad k = 0, 1, 2, ...$$

is a solution of the difference equation

$$Y_{k+1} - 2Y_k = 0.$$

In fact

$$Y_{k+1} - 2Y_k = 2^{k+1} - 2 \cdot 2^k = 2^{k+1} - 2^{k+1} = 0. \quad fork = 0, 1, 2, ...$$

Example II.6.1 *Let us show that the function Y given by:*

$$Y_n = 2^n(3 + 4n), \quad n = 0, 1, ... \tag{II.20}$$

is a solution of:

$$Y_{n+2} - 4Y_{n+1} + 4Y_n = 0 \quad n = 0, 1, ... \tag{II.21}$$

For $II.20$ to be a solution of $II.21$, $II.20$ should satisfy $II.21$, that is, substituting $II.20$ into $II.21$, we should show that the result is 0. Doing that, we have:

$$2^{n+2}[3 + 4(n + 2)] - 4 \cdot 2^{n+1}[3 + 4(n + 1)] + 4 \cdot 2^n[3 + 4(n)] = 0$$

Thus, it is true that $II.20$ is a solution of $II.21$.

Definition II.6.3 *A difference equation over the set of k-values $0, 1, 2, ...$ is said to be linear if it can be written in the form*

$$a_n(k)Y_{k+n} + a_{n-1}(k)Y_{k+n-1} + ... + a_1(k)Y_{k+1} + a_0(k)Y_k = f(k), \tag{II.22}$$

where n is some positive integer and the coefficients $a_n, a_{n-1}, ..., a_0$, together with the function f, are given functions of k defined for $k = 0, 1, 2, ...$

The following are examples of linear difference equations over the set of k-values 0, 1, 2, ...

$$5Y_{k+1} - 3Y_k = 0,$$
$$Y_{k+2} + 2Y_{k+1} + Y_k = k^2.$$

On the other hand, none of the following difference equations can be written in the form of $II.22$.

$$Y_{k+} - Y_k^2 = 0,$$
$$Y_k Y_{k+3} - Y_{k+1} = 3k - 1.$$

Such equations are called nonlinear.
As in the case of differential equations, when the function f is identically zero, Eq.$II.22$ is called homogeneous. When f is not identically zero, Eq.$II.22$ is called nonhomogeneous.

Example II.6.2 {***The Malthusian law of population growth***} *in the case of some insect populations, where one generation dies out before the next generation hatches. A simple model for such a population will be to assume that the increase in size, from one generationto the next, is proportional to the size of the former generation. Let N_k denote the size of the population of the kth generation. Then*

$$N_{k+1} - N_k = aN_k, \tag{II.23}$$

where a is the constant of proportionality. Equation$II.23$can be written in the form

$$N_{k+1} - (1 + a)N_k = 0, \tag{II.24}$$

which is a linear homogeneous difference equation of order 1. Its solution is

$$N_k = N_0(1 + a)^k, \tag{II.25}$$

where No is the initial size of the population. Equation $II.24$ assumes that the size of the population depends on the population in the previous generation.

II.6.2 stochastic difference equation

It is of great interest to see how robust the deterministic models are. One way to gauge robustness is to introduce a stochastic perturbation into the system. Some features remain, others undergo a change and some are so 'frail' that they disappear altogether. Of course, this is not the only motivation to investigate stochastic models. In all realistic modeling, we have to take the possibility of random disturbances into account. In addition, many models in economics, biology, chemistry, physics or other fields are stochastic at heart, but the mass action of a huge number of agents, individuals, molecules or particles made it natural to go over to a deterministic formulation. But what if the number is small or moderate? Then a stochastic formulation is called for. In this special issue, we have collected a number of papers illustrating various aspects of the theory of stochastic difference equations and essential applications. A stochastic difference equation on a state space S may come in many forms, but most of them can be formulated as

$$X_{n+1} = F(X_n, \xi_{n+1}), \tag{II.26}$$

i.e., the state X_{n+1} at time $n+1$ is a function F of the state X_n at the previous time point and a random variable, independent of X_n .
Theξ are often independent and identically distributed (i.i.d.), so the natural theoretical way to see the process is as a Markov chain on the state space S.
Time is discrete in stochastic difference equations and it is often a crucial feature. For example, in many branching models time is measured in generations and hence cannot easily be subdivided into smaller parts. Nevertheless, there are many natural methodological similarities between discrete and continuous time models.

Example II.6.3 ***Markov chains*** *are a specific type of stochastic difference equation where the future state depends only on the current state, not on the sequence of events that preceded it. General form*

$$\mathbb{P}(X_{n+1}=j|X_0=x_0,...,X_{n-1}=x_{n-1},X_n=i)=\mathbb{P}(X_{n+1}=j|X_n=i), \qquad \text{(II.27)}$$

A simple random walk:

$$X_{k+1}=X_k+\xi_{n+1}, \qquad \text{(II.28)}$$

where ξ_{n+1} can be $+1$ or -1 with equal probability.

Example II.6.4 *let's consider a simple example of a stochastic difference equation:*

$$X_{n+1}=aX_n+b\xi_{n+1},$$

where X_n represents the value of an asset's price at time t, X_{n+1} is the asset's price at the next time step $t+1$, ξ_{n+1} is a random shock term at time $t+1$, typically assumed to be drawn from a normal distribution $\mathcal{N}(0,1)$ This equation can be used to model the price of a financial asset, such as a stock or commodity, where the next period's price depends on the current price X_n and a random shock ξ_{n+1}, This example illustrates how a stochastic difference equation combines deterministic (the term X_n) and stochastic (the term $b\xi_{n+1}$) elements to model the evolution of a variable over time in the presence of randomness.

Example II.6.5 *To convert the given deterministic difference equation$II.24$ of the* **{*The Malthusian law of population growth*}** *example into a stochastic difference equation, we introduce a random shock term. Here's how we can modify the equation:*
Given deterministic difference equation:

$$N_{k+1}-(1+a)N_k=0, \qquad \text{(II.29)}$$

This equation suggests a deterministic relationship where the population size at the next generation N_{k+1} is determined solely by the current population size N_k scaled by $(1+a)$.
To introduce stochasticity, we can add a random shock term ξ_{n+1}

$$N_{k+1}=(1+a)N_k+\xi_{n+1}, \qquad \text{(II.30)}$$

ξ_{n+1} introduces stochasticity into the model. It could represent random fluctuations in birth rates, death rates, environmental conditions affecting population growth, or other unpredictable factors influencing population dynamics.

Some type of Stochastic difference equation

Stochastic difference equations (SDEs) can be categorized based on their structure, the nature of the random components, and their applications. Here are some common types of stochastic difference equations:

1. **Linear Stochastic Difference Equations** These equations have a linear relationship between the variables and the stochastic components. General Form:

$$X_{k+1} = AX_k + B\xi_{n+1}, \tag{II.31}$$

where A and B are linear functions, and $\xi_{n+1}$is a random variable.

2. **NonLinear Stochastic Difference Equations** These equations involve nonlinear functions of the state variables and the stochastic components. General Form:

$$X_{k+1} = f(X_k, \xi_{n+1}), \tag{II.32}$$

where f is nonlinear functions, and $\xi_{n+1}$is a random variable.

3. **Additive Noise Stochastic Difference Equations** In these equations, the stochastic component is added to the deterministic part. General Form:

$$X_{k+1} = f(X_k) + \xi_{n+1}, \tag{II.33}$$

where $f$is function of X_k , and $\xi_{n+1}$is a random noise term.

4. **Multiplicative Noise Stochastic Difference Equations** Here, the stochastic component multiplies the deterministic part. General Form:

$$X_{k+1} = f(X_k) \cdot \xi_{n+1}, \tag{II.34}$$

where $f$is function of X_k , and $\xi_{n+1}$is a random noise term.

III On stochastic stabilization of difference equation

III.0.1 Introduction

In this chapter we deal an unstable determnistic difference equation and show how this equation can be stabilise by adding the noise term $\sigma_n g(x)\xi_{n+1}$, where ξ_n takes the value 1 or -1 withprobability $\frac{1}{2}$. We discuss the folling question: how can we add a random noise term to an unstable deterministic difference equation to make it almost surely asymptotically stable? . We consider the deterministic difference equation

$$x_{n+1} = x_n \left(1 + a_n f(x_n)\right), \quad n \geq 1, \quad x_0 = a, \tag{III.1}$$

with a solution x_n which is not globally asymptotically stable. we do not devote much attention to the case when equation $III.1$ satisfies the following conditons:

$$a_n f(x_n) \leq 0, \tag{III.2}$$

$$\sum_{n=1}^{\infty} a_n = \infty \tag{III.3}$$

since these conditons assure that the previous equation has asymptotically stable solution. so we attempt to exhibit that equation $III.1$ may be stabilised even when the conditons are violated by adding the random noise term to the equation and we obtain a new equation :

$$x_{n+1} = x_n \left(1 + a_n f(x_n) + \sigma_n g(x_n)\xi_{n+1}\right), \quad n \geq 1, \quad x_0 = a, \tag{III.4}$$

where ξ_{n+1} are independent random variables, $\mathbb{E}\xi_{n+1} = 0$, $\mathbb{E}\xi^2_{n+1} = 1$.

III.1 Stability of stochastic difference equation

We consider

$$x_{n+1} = x_n \left(1 + a_n f(x_n) + \sigma_n g(x)\xi_{n+1}\right), \quad n \geq 1, \quad x_0 = \zeta, \tag{III.5}$$

where ξ_{n+1} are independent random variables, $\mathbb{E}\xi_{n+1} = 0$, $\mathbb{E}\xi_{n+1}^2 = 1$, $a_n\ in\mathbb{R}^{\mathbb{1}}$, functions $f, g : \mathbb{R}^{\mathbb{1}} \longrightarrow \mathbb{R}^{\mathbb{1}}$ are continuous and for simlicity that the initial condition $x_0 = \zeta > 0$. Suppose that there exists some $L, L^0 \in (0, \infty)$ such that for all $n \in \mathbb{N}, u \in \mathbb{R}^{\mathbb{1}}$ a.s.

$$-1 < a_n f(u) + \sigma_n g(u)\xi_{n+1} \leq L, \tag{III.6}$$

$$g(u) \neq 0 \quad when \quad u \neq 0, \tag{III.7}$$

$$a_n f(u) \leq L_0 \sigma_n^2 g^2(u) \tag{III.8}$$

$$2L_0(1+L)^2 < 1, \tag{III.9}$$

Theorem III.1.1 *suppose that the conditions $(III.6)$=$(III.9)$ and*

$$\sum_{n=1}^{\infty} \sigma_n^2 = \infty \tag{III.10}$$

are fulfilled. Let x_n be a solution to $III.5$. Then

$$\lim_{n \mapsto \infty} x_n = 0$$

to proof this theorem we need to discuss some necessary results from the theory of Stochastis Processes.
let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$) be a complete filtered probability space. let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of independent random variables with $\mathbb{E}\xi_i = 0$. We supose that filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated, so that $\mathcal{F}_{n+1} = \sigma\{\xi_{i+1} : i = 0, 1, 2, ..., n\}$.
Among all sequences $\{X_n\}_{n\in\mathbb{N}}$ of random variables we distinguish those for which X_n is $\mathcal{F}_n$- measurable for all $n \in \mathbb{N}$.

- A stochastic sequence said $\{\xi_n\}_{n\in\mathbb{N}}$ to be an $\mathcal{F}_n$-martingale-difference, if $\mathbb{E}|\xi_n| < \infty 0$ and $\mathbb{E}(\xi_n|\mathcal{F}_{n-1}) = 0$ a.s. for all $n = 1, 2, ...$ The partial summation of martingale-difference leads at once to a martingale(and conversely):

  $$X_n = \sum_{k=1}^{n} \xi_k$$

  is martingale if only if ξ_k is a martingale-difference.

- A stochastic sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to be increasing if $A_{n+1} - A_n \geq 0$ a.s. for all $n \in \mathbb{N}$.

The following is a variant of the Doob decomposition theorem.

Theorem III.1.2 *suppose that* $\{X_n\}_{n\in\mathbb{N}}$ *is* $\mathcal{F}_n$*-submartingale. Then there exists an* $\mathcal{F}_n$*-martingale* $\{M_n\}_{n\in\mathbb{N}}$ *and an increasing* $\mathcal{F}_{n-1}$*- measurable stochastic sequence* $\{A_n\}_{n\in\mathbb{N}}$ *such that for every* $n = 1, 2, ...$

$$X_n = M_n + A_n, \quad a.s. \tag{III.11}$$

the next lemma can be easily deduced from previous theorem.

Lemma III.1.1 *let* $\{\xi_n\}_{n\in\mathbb{N}}$ *be an* $\mathcal{F}_n$*-martingale-difference, then there exists an* $\mathcal{F}_n$*-martingale-difference* $\{\mu_n\}_{n\in\mathbb{N}}$ *and a positive* $\mathcal{F}_{n-1}$*- measurable stochastic sequence* $\{\eta_n\}_{n\in\mathbb{N}}$ *such that for every* $n = 1, 2, ...$

$$\xi_n^2 = \mu_n + \eta_n, \quad a.s. \tag{III.12}$$

if ξ_n *are independent for all* $n \geq 0$

$$\eta_n = \mathbb{E}(\xi_n^2), \quad \mu_n = \xi_n^2 - \mathbb{E}(\xi_n^2).$$

the following theorem is a variant of martingale convergence theorem.

Theorem III.1.3 *suppose that* $\{X_n\}_{n\in\mathbb{N}}$ *is nonnegative* $\mathcal{F}_n$*-submartingale with Doob's decomposition* $III.1.2$*. Then*

$$\{A_\infty < \infty\} \subseteq \{X_n \longrightarrow\}. \tag{III.13}$$

Here by $\{X_n \longrightarrow\}$ we denote the set of all $\omega \in \Omega$ for which $\lim_{n\mapsto\infty} X_n(\omega)$ exists and is finite.

Lemma III.1.2 *let* $\{Z_n\}_{n\in\mathbb{N}}$ *be nonnegative* $\mathcal{F}_n$*-measurable process,* $\mathbb{E}(Z_n < \infty, \forall n \in \mathbb{N}$

$$Z_n \leq Z_n + u_n - v_n + \nu_{n+1}, \quad n = 0, 1, 2, ..., \tag{III.14}$$

where $\{\nu_n\}_{n\in\mathbb{N}}$ *is an* $\mathcal{F}_n$*-martingale-difference,* $\{u_n\}_{n\in\mathbb{N}}$ $\{v_n\}_{n\in\mathbb{N}}$ *are nonnegative* $\mathcal{F}_n$*-measurable process,* $\mathbb{E}|u_n|, \mathbb{E}|v_n| < \infty$ $\forall n \in \mathbb{N}$*. Then*

$$\left\{\omega : \sum_{n=1}^{\infty} u_n < \infty\right\} \subseteq \left\{\omega : \sum_{n=1}^{\infty} u_n < \infty\right\} \cap \{Z_n \longrightarrow\}.$$

Proof III.1.1 *We observe that the solution* x_n *of* $III.5$ *can be represented in the form*

$$x_{n+1} = x_0 \prod_{i=0}^{n} (1 + a_i f(x_i) + \sigma_i g(x_i)\xi_{i+1}). \tag{III.15}$$

By the assumptions that $x_0 = \zeta$ *and* $(III.6)$ *we see from* $III.15$ *that* $x_n > 0$ *for all* $n \in \mathbb{N}$*. It is also easy to see from* $(III.6)$ *and* $(III.15)$ *that*

$$\mathbb{E}|x_n|^p < \infty \quad n \in \mathbb{N} \text{ and } p > 0.$$

Let $\alpha \in (0, 1)$*. Applying the Taylor expansion of the function* $y = (1 + u)^\alpha$ *up to the third term gives*

$$(1 + u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha - 1)}{2}(1 + \theta)^{(\alpha-2)}u^2, \tag{III.16}$$

where θ lies between 0 and u. Taking into account $III.6$ we can estimate the expression

$$\frac{\alpha(\alpha-1)}{2}(1+\theta)^{(\alpha-2)}$$

when $u = a_n f(x_n) + \sigma_n g(x_n)\xi_{n+1}$ according to

$$1+\theta \le 1+|u| \le 1+L, \quad \frac{\alpha(\alpha-1)}{2(1+\theta)^{(\alpha-2)}} \le \frac{\alpha(2-\alpha)}{2(1+L)^{(2-\alpha)}}. \tag{III.17}$$

Applying $(III.6), (III.16), (III.17)$ we get

$$\begin{aligned} x_{n+1}^{\alpha} &= x_n^{\alpha}\left(1+a_n f(x_n)+\sigma_n g(x_n)\xi_{n+1}\right)^{\alpha} \\ &= x_n^{\alpha}\left[1+\alpha(a_n f(x_n)+\sigma_n g(x_n)\xi_{n+1}) + \frac{\alpha(\alpha-1)}{2(1+\theta)^{(2-\alpha)}}(a_n f(x_n)+\sigma_n g(x_n)\xi_{n+1})^2\right] \\ &\le x_n^{\alpha}\left[1+\alpha(a_n f(x_n)+\sigma_n g(x_n)\xi_{n+1}) + \frac{\alpha(\alpha-1)}{2(1+L)^{(2-\alpha)}}(a_n f(x_n)+\sigma_n g(x_n)\xi_{n+1})^2\right] \\ &= x_n^{\alpha}\left[1+\alpha a_n f(x_n) + \frac{\alpha(\alpha-1)}{2(1+L)^{(2-\alpha)}}\left(a_n^2 f^2(x_n)+\sigma_n^2 g^2(x_n)\right)\right] + \rho_{n+1}, \end{aligned} \tag{III.18}$$

where

$$\begin{aligned} \rho_{n+1} &= \alpha\sigma_n x_n^{\alpha} g(x_n)\xi_{n+1} + \frac{\alpha(\alpha-1)}{(1+L)^{(2-\alpha)}} a_n\sigma_n x_n^{\alpha} f(x_n) g(x_n)\xi_{n+1} \\ &+ \frac{\alpha(\alpha-1)}{2(1+L)^{(2-\alpha)}}\sigma_n^2 x_n^{\alpha} g^2(x_n)\mu_{n+1}, \end{aligned} \tag{III.19}$$

and μ_{n+1} is from $III.12$.We note also that ρ_{n+1} is a martingale–difference. From $III.18$ we get the estimate

$$x_{n+1}^{\alpha} - x_n^{\alpha} \le \alpha x_n^{\alpha}\left[a_n f(x_n) - \frac{(1-\alpha)}{2(1+L)^{(2-\alpha)}} + \sigma_n^2 g^2(x_n)\right] + \rho_{n+1}. \tag{III.20}$$

We substitute condition III.8 into III.20 and get

$$\begin{aligned} x_{n+1}^{\alpha} &\le x_n^{\alpha}\left[1+\alpha L_0\sigma_n^2 g^2(x_n) - \frac{\alpha(1-\alpha)}{2(1+L)^{(2-\alpha)}}\sigma_n^2 g^2(x_n)\right] + \rho_{n+1} \\ &\le x_n^{\alpha} - \alpha x_n^{\alpha}\sigma_n^2 g^2(x_n)\left[\frac{(1-\alpha)}{2(1+L)^{(2-\alpha)}} - L_0\right] + \rho_{n+1}. \end{aligned} \tag{III.21}$$

By $III.9$, $0 < \frac{1}{2} - L_0(1+L)^2 < 1$. Define $\alpha = \frac{1}{2} - L_0(1+L)^2$. Then we have

$$\frac{(1-\alpha)}{2(1+L)^{(2-\alpha)}} - L_0 \ge \frac{(1-\alpha)}{2(1+L)^2} - L_0 = \frac{(\alpha)}{2(1+L)^2} - L_0. \tag{III.22}$$

subsutituting III.22 in III.21 we arrive at

$$x_{n+1}^{\alpha} \leq x_n^{\alpha} - \frac{\alpha^2}{2(1+L)^2} x_n^{\alpha} \sigma_n^2 g^2(x_n) + \rho_{n+1}. \tag{III.23}$$

We can now apply lemma III.1.2 by making the indentification

$$Z_n = x_n^{\alpha}, \quad u_n = 0, \quad v_n = \frac{\alpha^2}{2(1+L)^2} \sum_{n=1}^{n} \sigma_n^2 x_n^{\alpha} g^2(x_n), \quad \nu n = \rho_n, \tag{III.24}$$

to conclude that

$$\lim_{n\to\infty} x_n^{\alpha} \in [0,\infty) exists \ \ a.s., \tag{III.25}$$

and so that

$$\sum_{n=1}^{n} \sigma_n^2 x_n^{\alpha} g^2(x_n) < \infty \ \ a.s. \tag{III.26}$$

We put

$$\Omega_1 = \left\{\omega : \lim_{n\to\infty} x_n(\omega) = 0\right\},$$
$$\Omega_2 = \left\{\omega : \lim_{n\to\infty} x_n(\omega) > 0\right\}.$$

We note that $\mathbb{P}\{\Omega_1 \cup \Omega_2\} = 1$ since $x_n > 0$ for all $n \in \mathbb{N}$. Using III.26 we get for almost every$\omega \in \Omega_2$

$$\sum_{n=1}^{\infty} \sigma_n^2 < c \sum_{n=1}^{\infty} \sigma_n^2 x_n^{\alpha} g^2(x_n) < \infty, \tag{III.27}$$

where $c = c(\omega) > 0$ is some a.s. finite randome variable. This contradicts the assumption III.10 if $\mathbb{P}(\Omega_2) > 0$. In other words, we must have $\mathbb{P}(\Omega_2) = 0$ whence $\mathbb{P}(\Omega_1) = 1$as desired. □

Remark III.1.1 *Suppose that $-1 < a_n f(u) \leq 0$ and$\Sigma_{n=1}^{\infty} |a_n| = \infty$. It was alread mentioned in the Introduction that in this case the solution to III.5 is globally asymptotic stable. Then any noise perturbation to III.5will preserve a.s. global asymptotic stability. This can be readily obtained from the proof of the Theorem and the estimate III.20 in particular.*

Conclusion

Difference equations and stochastic difference equations are highly significant mathematical tools for understanding and predicting the behavior of dynamic systems influenced by deterministic rules and random variations. These equations provide a mathematical framework through which systems evolving in discrete time steps can be analyzed, making them effective tools for representing and studying the complex behavior of various systems, these equations are characterized by their high flexibility and applicability across a wide range of fields, highlighting their importance in both theoretical research and practical applications. In economics, engineering, biology, and finance because they effectively represent and analyze the behavior of these systems.

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