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Dedication

To those who gave me the spirit of life and lit up the darkness of this world...
To those who, after Allah, were the reason for my existence and the driving force behind my determination, always surrounding me with their sincere prayers...
To those who instilled in me a love for knowledge and learning...
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To those who supported me and provided me with every means of happiness and success...
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To my brothers (Khaled, Abd El Djalil) and sisters (Ikram Sarah, Imane), my pillars of strength in life may Allah keep you as a steadfast support for me always.

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Abstract

In this paper, we investigate the existence of weak solutions for various types of fractional differential problems. The study begins by applying fixed point theory to establish the existence and uniqueness of solution for nonlinear fractional elliptic problem. Then, we study the nonlinear fractional elliptic system. In the final part, we present nonlinear elliptic system in both resonance and non-resonance cases, employing the Leray–Schauder topological degree theory to prove the existence of solutions for such system.

Keywords:

Elliptic systems, weak solution, topological degree, homotopy, fractional Laplacian, eigenvalue.

Titre: **Sur la résolution des problèmes elliptiques fractionnaires non linéaires**

Résumé

Dans cet article, nous étudions l'existence de solutions faibles pour divers types de problèmes différentiels fractionnaires. L'étude commence par l'application de la théorie du point fixe pour établir l'existence et l'unicité de la solution d'un problème elliptique fractionnaire non linéaire. Nous étudions ensuite le système elliptique fractionnaire non linéaire. Dans la dernière partie, nous présentons un système elliptique non linéaire dans les cas de résonance et de non-résonance, en utilisant la théorie du degré topologique de Leray-Schauder pour prouver l'existence de solutions pour un tel système.

Mots clés:

Systèmes elliptiques, solution faible, degré topologique, homotopie, Laplacien fractionnaire, valeurs propres.

العنوان حول حل المسائل الإهليلجية الكسرية غير الخطية

ملخص:

في هذا البحث، ندرس وجود حلول ضعيفة لأنواع مختلفة من المشاكل التفاضلية الكسرية. و تبدأ الدراسة بتطبيق نظرية النقطة الثابتة لإثبات وجود و تفرد الحل للمشكلة الإهليلجية الكسرية غير الخطية. ثم ندرس بعد ذلك النظام الإهليلجي الكسري نستخدم نظرية غير الخطي في الجزء الأخير نعرض النظام الإهليلجي غير الخطي في كل من حالتي الرنين وغير الرنين و الدرجة الطوبولوجية ليراي شودر لاثبات وجود حلول لهذا النظام.

الكلمات المفتاحية:

الأنظمة الإهليلجية، الحل الضعيف، الدرجة الطوبولوجية، التحول المتجانس، مؤثر لابلاس الكسري، القيمة الذاتية.

Notations

- $\rightarrow$ designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- $\hookrightarrow$ indicates the continuous embedding.
- ∇ stands for the gradient operator.
- div is the divergence operator.
- Δ^s is the fractional Laplace operator of order s.
- sp denotes the spectrum of an operator.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^n$ is the real space of dimension n.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.
- $< ., . >$ denotes the scalar product.
- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.
- $C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega)$.

- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.

- $L^p_{loc}(\Omega)$ the space of local p-integrable functions on Ω.

- $W^{m,p}(\Omega)$ Sobolev space with norm $\|\cdot\|_{m,p}$.

- $W^{m,p}_{loc}(\Omega)$ the local Sobolev space.

- $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

- $W^{-1,p'}(\Omega)$ is the dual of $W^{1,p}(\Omega)$.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $W^{s,p}(\Omega)$ fractional Sobolev space with norm $\|\cdot\|_{s,p}$.

- $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$.

- $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, $W_0^{s,2}(\mathbb{R}^n) = H_0^s(\mathbb{R}^n)$.

- $U = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ endowed with the norm
$$\|(u,v)\|_U = \|u\|^p_{W_0^{1,p}(\Omega)} + \|v\|^p_{W_0^{1,p}(\Omega)}.$$

- $V = L^p(\Omega) \times L^p(\Omega)$.

- $D^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^n), \text{ such that } u = 0 \text{ in } \mathbb{R}^n \backslash \Omega\}$.

- $\widetilde{U} = D^{s,2}(\Omega) \times D^{s,2}(\Omega)$, with the norm, that we will denote by $\|.\|_U$
$$\|(u,v)\|^2_{\widetilde{U}} = \|u\|^2_{D^{s,2}(\Omega)} + \|v\|^2_{D^{s,2}(\Omega)}$$
.

- $\widetilde{V} = L^2(\Omega) \times L^2(\Omega)$.
- $B(0, R)$ open ball centered at the point 0 with the radius R.
- $\partial B(0, R)$ sphere centered at the point 0 with the radius R.
- Γ is the usual Gamma function.
- $P.V.$ is an abbreviation for "in the principal value sense".

Contents

Introduction

Fractional differential equations have attracted increasing attention in recent years due to their ability to model nonlocal phenomena and memory effects that naturally arise in many scientific and engineering contexts. For details, see [25], [50]. In particular, nonlinear fractional elliptic problems play a crucial role in describing the steady state behavior of complex systems influenced by nonlocal interactions we refer the readers to [4], [41].
Fractional calculus is a natural generalization of classical calculus, extending the concept of differentiation and integration to non-integer (arbitrary) orders. The origins of this idea can be traced back to 1695, when Gottfried Wilhelm Leibniz posed the question of how to define a derivative of fractional order an idea that laid the conceptual foundation for what would later become known as fractional calculus.[37]

Over the years, numerous definitions of fractional derivatives have been proposed. Among the most commonly used and practically significant are the Riemann–Liouville and Caputo definitions, each offering distinct advantages depending on the nature of the problem being studied [47]. In the past two decades, the application of fractional derivatives to model physical phenomena has grown rapidly, driven by their ability to capture memory dependent dynamics, nonlocal interactions, and anomalous diffusion in a wide range of disciplines [46].
This development has led to the formulation of fractional partial differential equations (PDEs), especially of the elliptic type, which generalize classical elliptic models by incorporating operators of fractional order.

A key operator in this context is the fractional Laplacian, which extends the classical Laplace operator to a nonlocal framework. It plays a central role in modeling nonlocal diffusion and has found applications in diverse fields such as mathematical physics, probability theory, and image processing. Although several definitions exist for the fractional Laplacian, one of the most widely used is given by the singular integral:

$$(-\Delta)^s u(x) := C(n,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}} dy \quad \forall x \in \mathbb{R}^n.$$

To tackle nonlinear fractional elliptic problems, various analytical techniques have been developed, including fixed point theorems, semigroup theory, sub-supersolution methods, and topological degree theory. Among these, topological degree methods particularly those developed by Brouwer and Leray–Schauder have proven to be exceptionally powerful. These methods are especially valuable for studying nonlinear partial differential equations, as they

provide conditions under which solutions exist, even in infinite dimensional settings.
The Leray–Schauder degree, in particular, generalizes classical degree theory to Banach spaces, enabling the treatment of compact mappings in infinite dimensions. Its fundamental properties such as homotopy invariance and additivity make it a versatile tool in the study of nonlinear problems [3],[10], [13],[24].

The aim of this thesis is to investigate the existence and uniqueness of solutions to nonlinear fractional elliptic problem and systems involving the fractional Laplacian operator. The analysis relies on topological methods, with particular emphasis on fixed point theory and degree theory, including the Leray–Schauder degrees. Additionally, the thesis explores systems under resonance and nonresonance conditions, and provides a priori estimates for the corresponding solutions.

The thesis is structured into four main chapters:

- In chapter 1, we introduces the necessary mathematical preliminaries, including functional spaces, the fractional Laplacian operator, and topological degree theory.
- Chapter 2 focuses on a nonlinear fractional elliptic problem, we examine the existence of weak solution to problem (1). To establish the existence result, we employ fixed point theory.

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1}$$

- Chapter 3 extends the analysis to system of nonlinear fractional elliptic equations, we use fixed point theory to establish the existence of weak solution for system (2).

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) & \text{in} \quad \Omega, \\ \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) & \text{in} \quad \Omega, \\ \\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in]0, 1[$ and $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying the Caratheodory conditions and also verifying the growth restriction. As far as this result is new and represent fractional version of the classical theorem see [21].

- Chapter 4 presents an application of the Leray-Schauder degree theorem to demonstrate the existence of weak solutions for the nonlinear fractional elliptic systems at resonance and nonresonance of the type (3).

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) + f_1(x) \quad \text{in} \quad \Omega, \\ \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) + f_2(x) \quad \text{in} \quad \Omega, \\ \\ u = v = 0 \quad \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3}$$

  with $s \in (0,1)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, f and g are continuous functions. see [30].

Chapter 1

Preliminaries

This chapter is divided into three sections. First, we review several general results from functional analysis that have been utilized throughout this thesis. In the second section, we introduce the topological degree method. Finally, we define and distinguish between the phenomena of resonance and non-resonance.
For more detailed discussions, the reader is referred to the relevant literature, e.g., [1],[3],[7],[20], [24],[27].

1.1 Functional spaces

Here we define the essential notions on functional spaces, L^p spaces, Sobolev spaces,fractional sobolev spaces and Fractional Laplacians operator.

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$, we note

$$C(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ continuous}\} \ ,$$

and

$$C^m(\Omega) \text{ The space of functions } m \text{ times continuously differentiable on } \Omega,$$

where

$$C^\infty(\Omega) = \underset{m \in \mathbb{N}}{\cap} C^m(\Omega).$$

$$C_c(\Omega)=\{f \in C(\Omega); f(x) = 0 \ , \forall x \in \Omega \backslash K, \text{ where K is compact}\},$$

and D(Ω) the space of functions C^∞ on Ω with compact support in Ω (also called the space of test functions).

1.1.1 The $L^p(\Omega)$ spaces

Let $1 \leqslant p < \infty$ and Ω be an open subset in $\mathbb{R}^n$. We define the standard Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \text{ f is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\},$$

with the norm

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}},$$

if p $= \infty$, we define

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there is a constant } C \text{ s.t } |f(x)| \leqslant C \text{ a.e. on } \Omega \right\}$$

s.t= such that.
with the norm

$$\|f\|_\infty = \text{Inf}\{C; |f(x)| \leqslant C \text{ a.e. on } \Omega\}.$$

Remark 1.1.1. *In particularly, when $p = 2$, $L^2(\Omega)$ is a Hilbert space for the inner product*

$$(f, g) = \int_\Omega f(x)g(x)dx.$$

Recall that $L^p_{loc}(\Omega)$ denotes the set of locally integrable functions on Ω, i.e.

$$L^p_{loc} = \left\{ f : \Omega \to \mathbb{R} \text{ measurable such that} : \forall K compact \subset \Omega, \int_K |f(x)|^p dx < \infty \right\}.$$

In particular

$$L^p(\Omega) \subset L^p_{loc}(\Omega).$$

Proposition 1.1.1. *.[9]*

1. *for $1 \leqslant p \leqslant \infty$, $(L^p(\Omega), \|.\|_p)$ is a Banach space.*

2. *for $1 \leqslant p < \infty, (L^p(\Omega), \|.\|_p)$ is a separable space.*

3. *for $1 < p < \infty$, $(L^p(\Omega), \|.\|_p)$ is a reflexive space.*

Notation: Let $1 \leqslant p \leqslant \infty$; we denote by p' the conjugate exponent,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Some useful inequalities

In this part, we give some inequalities which will be used in the subsequent chapters.

Theorem 1.1.1. ***(Hölder's inequality).*** *[?] Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leqslant p \leqslant \infty$. Then*

$$fg \in L^1(\Omega) \text{ and } \|fg\|_1 \leqslant \|f\|_p \|g\|_{p'} \ .$$

Theorem 1.1.2. ***(Minkowski inequality).*** *[?] Let $f, g \in L^p(\Omega)$ and $p \geqslant 1$, then*

$$f + g \in L^p(\Omega) \text{ and } \|f + g\|_p \leqslant \|f\|_p + \|g\|_p \ .$$

Some results about integration and duality

Theorem 1.1.3. ***(Dominated convergence of lebesgue).*** *[?]*
Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy

1. *$f_n(x) \to f(x)$ a.e. on Ω,*
2. *there is a function $g \in L^1(\Omega)$ such that for all $n \geqslant 1$, $|f_n(x)| \leqslant g(x)$ a.e. on Ω.*

   Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

Lemma 1.1.1. *[?] Let (f_n) be a sequence in $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that $\|f_n - f\|_p \to 0$. Then, there exist a subsequence (f_{n_k}) and a function $h \in L^p(\Omega)$ such that*

1. *$f_{n_k}(x) \to f(x)$ a.e. on Ω,*
2. *$|f_{n_k}(x)| \leqslant h(x)$ $\forall k$, a.e. on Ω.*

Theorem 1.1.4. ***(Riesz representation theorem).*** *[?]*
Let $1 < p < \infty$ and let $\varphi \in (L^p(\Omega))'$. Then there exists a unique function $u \in L^{p'}(\Omega)$ such that

$$\langle \varphi, f \rangle = \int_\Omega u f dx, \forall f \in L^p(\Omega).$$

Moreover,

$$\|u\|_{p'} = \|\varphi\|_{(L^p)'}.$$

Definition 1.1.1. ***(Weak derivative)****. [29] Assume that $f \in L^1_{loc}(\Omega)$ and let $\alpha \in \mathbb{N}^n$ be a multi-index. Then $g \in L^1_{loc}(\Omega)$ is the αth weak partial derivative of f, written $D^\alpha f = g$, if*

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g \varphi dx,$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

Warning: *if such a g exists, it is unique, however it doesn't always exist.*

1.1.2 Sobolev spaces

A Sobolev space is a vector space of functions endowed with a norm that combines the L^p norms of a function and its derivatives up to a specified order. Sobolev spaces form the foundation of the theory of weak (or variational) formulations of partial differential equations.

The concept of Sobolev space is introduced by the Russian mathematician Sergei Sobolev in the 1930s.

Let Ω be an open set of $\mathbb{R}^n$ we define a functional $\|.\|_{m,p}$, where m is a nonnegative integer and $1 \leqslant p \leqslant \infty$, as follows;

$$\|f\|_{m,p} = \left\{ \sum_{0 \leqslant |\alpha| \leqslant m} \|D^\alpha f\|_p^p \right\}^{1/p}, \quad \text{if } 1 \leqslant p < \infty;$$

$$\|f\|_{m,\infty} = \max_{0 \leqslant |\alpha| \leqslant m} \|D^\alpha f\|_\infty;$$

for any functions f that gives meaning to this writing.

We define the space $W^{m,p}(\Omega)$ as being the space of measurable functions $f \in L^p(\Omega)$ such that the derivative in the weak sense $D^\alpha f$,$(0 \leqslant |\alpha| \leqslant m)$ belongs to $L^p(\Omega)$ and the space $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

We associate the space $W^{m,p}(\Omega)$ with the norm $\|.\|_{m,p}$, then have the following proposition

Proposition 1.1.2. . *[9] Let Ω be an open subset of $\mathbb{R}^n$;*

1. *For* $1 \leqslant p \leqslant \infty, W^{m,p}(\Omega)$ *is a Banach space.*

2. *For* $1 \leqslant p < \infty, W^{m,p}(\Omega)$ *is a separable space.*

3. *For* $1 < p < \infty, W^{m,p}(\Omega)$ *is a reflexive space.*

Remark 1.1.2. *If* $p = 2$, *we usually write*

$$H^m(\Omega) = W^{m,2}(\Omega),\ H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Theorem 1.1.5. . *[27] Let us assume that* Ω *is an open subset of* $\mathbb{R}^n$ $(n \geqslant 1)$, $m \in \mathbb{N}, 1 \leqslant p < \infty$ *and* $p^* = \dfrac{np}{n-p}$*. Then*

1. *If* $\dfrac{1}{p} - \dfrac{m}{n} > 0$, *we have* $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, *with* $q \in [p, p^*], \dfrac{1}{p} - \dfrac{1}{p^*} = \dfrac{m}{n}$.

2. *If* $\dfrac{1}{p} - \dfrac{m}{n} = 0$, *we have* $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, +\infty[$.

3. *If* $\dfrac{1}{p} - \dfrac{m}{n} < 0$, *we have* $W_0^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Remark 1.1.3. *The space* $W^{1,p}(\Omega)$ *is equipped with the norm*

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Definition 1.1.2. . *[16] Let* $u : \Omega \to \mathbb{R}$ *be a measurable function with real value. We can consider the map*

$$\begin{aligned} A : \Omega \times \mathbb{R} &\to \mathbb{R} \\ u(x) &\longmapsto f(u)(x); \end{aligned}$$

Where $A(u)$ *is a function with real value defined on* Ω *by :*

$$A(u)(x) = f(x, u(x)),$$

such a map is called the Nemitski operator associated to f.

Theorem 1.1.6. .*[3] Let* $\alpha, \beta \geqslant 1$. *Suppose that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies*

1. *f(x,t) is a measurable with respect to $x \in \Omega$ for all $t \in \mathbb{R}$ and continues with respect $t \in \mathbb{R}$ for a.e. $x \in \Omega$.*

2. *There exists $g \in L^{\beta}(\Omega)$ and $a > 0$ such that*

$$|f(x,u)| \leqslant g(x) + a|u|^{\frac{\alpha}{\beta}}, \ \forall (x,t) \in \Omega \times \mathbb{R} \ , \ (\alpha, \beta \geqslant 1).$$

then the Nemytskii operator A is continuous and compact from $L^{\alpha}(\Omega)$ to $L^{\beta}(\Omega)$.

Remark 1.1.4. *Condition (1) is called the Caratheodory condition and a function f(x,t) satisfying (1) is called a Caratheodory function.*

1.1.3 Fractional Sobolev spaces

Fractional Sobolev spaces have long been a classical subject in functional and harmonic analysis, and are thoroughly addressed in several comprehensive references, such as [31]. In recent years, these spaces and their associated nonlocal equations have found remarkable applications across a wide range of disciplines. Notable examples include the thin obstacle problem [33], optimization [17], mathematical finance [11], elliptic problems involving measure data [26], and gradient potential theory [34], among others.

Let Ω be an open set in $\mathbb{R}^n$. For any real $s > 0$ and for any $p \in [1, \infty)$ we define the fractional Sobolev space $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also referred to as Aronszajn, Gagliardo, or Slobodeckij spaces, named after the researchers who introduced them independently and nearly simultaneously (see [6], [19], [43]).

We begin by fixing a fractional exponent $s \in (0,1)$. For any $p \in [1, \infty)$, the space $W^{s,p}(\Omega)$ is defined as follows:

$$W^{s,p} = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}; \tag{1.1}$$

i.e., an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p})^{\frac{1}{p}},$$

where the term

$$[u]_{s,p} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}},$$

is the so-called Gagliardo (semi) norm of u.

Remark 1.1.5. *The definition in (1.1) can not be plainly extended to the case $s \geqslant 1$. Suppose that Ω is a connected open set in $\mathbb{R}^n$, then any measurable function $u : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy < +\infty,$$

is actually constant (see [8], Proposition 2). This fact is a matter of scaling and it is strictly related to the following result that holds for any u in $W^{s,p}(\Omega)$:

$$\lim_{s \to 1^-} (s-1)^{\frac{1}{p}} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy = C \int_\Omega |\nabla u|^p dx,$$

for a suitable positive constant C depending only on n and p.

When $s > 1$ and it is not an integer we write $s = m + \sigma$, where m is an integer and $\sigma \in (0,1)$.
In this case the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, namely

$$W^{s,p}(\Omega) = \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } |\alpha| = \sigma\},$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}}.$$

Clearly, if $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

Theorem 1.1.7. *[14] For any $s > 0$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$.*

Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$. Note that, in view of Theorem 1.1.7, we have

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n),$$

but in general, for $\Omega \subset \mathbb{R}^n$, $W^{s,p}(\Omega) \neq W_0^{s,p}(\Omega)$, i.e. $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$.

Remark 1.1.6. *For $s < 0$ and $p \in (1, \infty)$, we can define $W^{s,p}(\Omega)$ as the dual space of $W^{-s,q}(\Omega)$ where $1/p + 1/q = 1$. Notice that, in this case, the space $W^{s,p}(\Omega)$ is a actually a space of distributions on Ω, since it is the dual of a space having $C_0^\infty(\Omega)$ as density subset.*

Proposition 1.1.3. *[12] Let Ω be an open set of $\mathbb{R}^n$ and $s \in (0,1)$, then*

1. *For $1 \leqslant p \leqslant \infty$, $W^{s,p}(\Omega)$ is Banach space.*
2. *For $1 \leqslant p < \infty$, $W^{s,p}(\Omega)$ is a separable space.*
3. *For $1 < p < \infty$, $W^{s,p}(\Omega)$ is a reflexive space.*

Corollary 1.1.1. *[12] Let $s \in (0,1)$ and let $p \in]1, \infty[$. Let Ω be a Lipschitz open set of $\mathbb{R}^n$. Then we have :*

1. *If $sp < n$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leqslant np/(n - sp)$.*
2. *If $n = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.*
3. *If $sp > n$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty$ and more precisely,*

$$W^{s,p}(\Omega) \hookrightarrow C^{0,s-n/p}(\Omega).$$

Theorem 1.1.8. *(**Compact embeddings**). [12] Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$. Let $s \in [0, 1[$, let $p > 1$, and let $n \geqslant 1$. Then we have:*

1. *If $sp < n$, then the embedding of $W^{s,p}(\Omega)$ into L^k is compact for every $k < np/(n - sp)$.*
2. *If $sp = n$, then the embedding of $W^{s,p}(\Omega)$ into L^q is compact for every $q < \infty$.*
3. *If $sp > n$, then the embedding of $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}(\Omega)$ is compact for $\lambda < s - n/p$.*

1.1.4 Fractional Laplacian operator

Before giving the definition of fractional laplaction operator we will first giving the definition of gamma function and the Fractional Sobolev spaces H^s.

Gamma Function

Definition 1.1.3. *The function $z \to \Gamma(z)$, $Rez > 0$, defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp^{-t} dt, \; Rez > 0.$$

is called the Gamma function.

Properties of Gamma function

Here the most important properties of the Gamma functions:

1. $\Gamma(1) = \Gamma(2) = 1.$

2. $\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}.$

3. $\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{R}^+.$

Now we will give the definition of **the fractional laplacian operator.**

Definition 1.1.4. *Let $s \in (0,1)$ and for any $u \in S(\mathbb{R}^n)$,*

$$(-\Delta)^s u(x) := C(n,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \backslash B(x,\varepsilon)} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}} dy \quad \forall x \in \mathbb{R}^n, \tag{1.2}$$

Where

$$C(n,s) := \frac{s4^s}{\pi^{\frac{n}{2}}} \frac{\Gamma(s+\frac{n}{2})}{\Gamma(1-s)}. \tag{1.3}$$

Remark 1.1.7. *C(n,s) is following two identities:*

1. $\lim_{s \to 0^+} (-\Delta)^s u = u$ *and* $\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u$

2. *Moreover*

$$C(n,s) \sim s(1-s), as, s \to 0^+ and \quad s \to 1^- \tag{1.4}$$

For more details on the asymptotic behavior of $C(n,s)$, the interested reader is referred to [[14],Section4] (see also [[35],Section1.3.1]).

We recall that by definition:

$$P.V.\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}}dy := \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}}dy$$

P.V. stands for the "Cauchy principal value". Since then, we can write

$$(-\Delta)^s u(x) := C(n,s)P.V.\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}}dy \quad \forall x \in \mathbb{R}^n \tag{1.5}$$

The following theorem (see for instance[[44],Theorem1]) shows that we can do without using the "Cauchy principal value" in (1.5) if $s(0,\frac{1}{2})$.

Theorem 1.1.9. *Let $u \in S(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The integral*

$$\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}}dy$$

is absolutely convergent if and only if $s \in (0,\frac{1}{2})$.
So we have

$$(-\Delta)^s u(x) = C(n,s)\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\|x-y\|^{n+2s}}dy\ ,$$

for $s \in (0,\frac{1}{2})$

We symbolize the space $H_0^s(\Omega)$ by $D^{s,2}(\Omega)$ where $D^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\|_{H^s}}$ ($D^{s,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ compared to the $H^s(\Omega)$ norm), if Ω is a bounded Lipschitz open set (see [16]), then

$$D^{s,2} = \{u \in H^s(\mathbb{R}^n) \text{ such that } u = 0 \text{ in } \mathbb{R}^n\setminus\Omega\},$$

such that

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}}+s} \in L^2(\mathbb{R}^n\times\mathbb{R}^n)\};$$

thus $D^{s,2}(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle u,v\rangle = C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dydx.$$

The norm in $D^{s,2}(\Omega)$ is

$$\|u\|_{D^{s,2}}(\Omega) = \left[\frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dydx \right]^{\frac{1}{2}}.$$

Proposition 1.1.4. *[42] Let $s \in (0,1)$ and Ω be a bounded Lipschitzian subset of $\mathbb{R}^n$ such that $n > 2s$. Let $u : \Omega \to \mathbb{R}$ be a measurable function compactly supported. Then, there exists a positive constant $c_{emb} > 0$ (embedding constant) depending on n and s such that*

$$\|u\|_{L^2(\Omega)} \leqslant c_{emb} \|u\|_{D^{s,2}(\Omega)}.$$

1.2 Topological degree

In this section, we present the concept of the topological degree, a fundamental tool for addressing nonlinear elliptic problems. Originally introduced by L. Brouwer in the context of finite-dimensional spaces, the theory was later extended to infinite-dimensional settings by J. Leray and J. Schauder (see [20] and [32]).

We begin by discussing the existence and uniqueness of the topological degree in finite dimensional spaces.

1.2.1 The Brouwer degree and its properties

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$ (or a bounded open of a banach space E). Let $f \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$ and $y \in \mathbb{R}^n$. The application call topological degree show the existence of solutions of the equation $f(x) = y$ such that $x \in \overline{\Omega}$.

Definition 1.2.1. . *[32] Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$, $f \in C^1(\Omega) \cap C(\overline{\Omega})$, $x_0 \in \Omega$ is called regular point if $J_f(x_0) \neq 0$ (or $J_f(x_0) = detDf(x_0)$ with $Df(x_0) = (\frac{\partial f_i}{\partial x_i})_{i,j}(x_0))$, Otherwise, x_0 is called critical point or singular point.*

Let us designate by :

$$S_f(\Omega) = \{x_0 \in \Omega : J_f(x_0) = 0\},$$

the set of singular points of f on Ω.

Definition 1.2.2. . *(Regular case)* [32] *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open and* $f \in C^1(\Omega) \cap C(\overline{\Omega})$ *a defined function of* Ω *with value in* $\mathbb{R}^n$*, for* $y \notin f(\partial\Omega)$ *a regular value, we define the degree of* f *at the point* y *by*

$$deg(f, \Omega, y) = \sum_{f(x_i)=y; i=\overline{1,n}} \operatorname{sgn}(\det D_x f(x_i))$$

Definition 1.2.3. . [20] *Let* $N \geqslant 1$*. We note by* A *the set of triplets (f,Ω,y) Where* Ω *is an open bounded of* $\mathbb{R}^N$*,* $f \in C(\overline{\Omega}, \mathbb{R}^N)$ *and* $y \in \mathbb{R}^N$ *such that* $y \notin \{f(x); x \in \partial\Omega\}$*.*

Theorem 1.2.1. ***(Brouwer, 1933)*** . [20] *Let* $N \geqslant 1$ *and* A *given by the definition (1.2.3). Then there exists an application* d *of* A *in* $\mathbb{Z}$ *called "topological degree", verifying the following three properties:*

(P1) ***Normalization:*** *d(I,Ω,y)=1 if* $y \in \Omega$*.*

(P2) ***Degree of an union:*** $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ *if* $\Omega_1 \cup \Omega_2 \subset \Omega$*,* $\Omega_1 \cap \Omega_2 = \varnothing$ *and* $y \notin \{f(x), x \in \overline{\Omega} \backslash \Omega_1 \cup \Omega_2\}$*.*

(P3) ***Homotopy invariance:*** *If* $h \in C([0.1] \times \overline{\Omega}, \mathbb{R}^N)$*,* $y \in C([0,1], \mathbb{R}^N)$ *and* $y(t) \notin \{h(t,x), x \in \partial\Omega\}$ *(for all* $t \in [0,1]$*), we have then:*

$$d(h(t,.), \Omega, y(t)) = d(h(0,.), \Omega, y(0)) \text{ for all } t \in [0,1].$$

Brouwer's fixed point theorem, 1912

Brouwer's fixed point is the first consequence of the "topological degree" that we are giving now.

Theorem 1.2.2. . [20] *Let* $N \geqslant 1$*,* $R > 0$ *and* $f \in C(B_R, B_R)$ *with* $B_R = \{x \in \mathbb{R}^N, \|x\| \leqslant R\}$ *(we provided* $\mathbb{R}^N$ *with a norm noted* $\|.\|$*). Then* f *admits a fixed point, that is to say it exists* $x \in B_R$ *such that* $f(x) = x$*.*

Theorem 1.2.1 was generalized (from 1934) in infinite dimension by Leray and Schauder under a hypothesis of compactness that we give now.

1.2.2 The Leray-Schauder degree and its properties

In infinite dimensions it is impossible to define a degree like in finite dimensions. This is why Leray and Schauder have been generalized under compactness assumptions.

Definition 1.2.4. . [20] *Let E be a Banach space (real), B be a part of E and f an application from B to E. We said that f is compact (the terminology of Leray-Schauder is different, they use the expression "completely continuous") if f satisfies the following two properties*

1. *f is continuous.*

2. *$\{f(x), x \in C\}$ is relatively compact in E for any bounded C of B.*

Remark 1.2.1. *We can notice, in the previous definition, that if f is linear (and $B = E$) the second condition leads to the first one. But this is not true for non-linear applications.*

Now we are giving the main results of this part, which states the exictence of the Leray-Schauder degree along with its main properties.

Definition 1.2.5. . [20] *Let E be a Banach space (real). We note A the set of triplets (I - f,y,Ω) where Ω is an open bound of E, f is a compact application from Ω into E (which is equivalent to say that f is continuous and $\{f(x), x \in \Omega\}$ is a relatively compact part of E) and $y \in E$ such that :*

$$y \notin \{x - f(x); x \in \partial\Omega\}.$$

Theorem 1.2.3. ***(Leray-Schauder, 1934)*** [20] *Let E be Banach space (real) and A given by the definition (1.2.5). Then there exists an application d of A in $\mathbb{Z}$ called "topological degree", verifying the following three properties:*

(P1) ***Normalization:*** $d(Id, \Omega, y) = 1$ *if* $y \in \Omega$.

(P2) ***Degree of an union:*** $d(I - f, y, \Omega) = d(I - f, y, \Omega_1) + d(I - f, y, \Omega_2)$ *if* $\Omega_1 \cup \Omega_2 \subset \Omega, \Omega_1 \cap \Omega_2 = \emptyset$ *and* $y \notin \{x - f(x); x \in \overline{\Omega}\backslash\Omega_1 \cup \Omega_2\}$.

(P3) ***Homotopy invariance:*** *If h is a compact application of [0,1]$\times\overline{\Omega}$ in E, $y \in C([0.1], E)$ and $y(t) \notin \{x - h(t, x), x \in \Omega\}$ (for all $t \in [0, 1]$) we have then :*

$$d(I - h(t, .), y, \Omega) = d(I - h(0, .), y(0), \Omega) \text{ for all } t \in [0, 1].$$

Definition 1.2.6. . [20] *An application of the form*

$$f = I - h,$$

where I is the identity application and h is a compact application is called compact perturbation of the identity (or Leray-Schauder application).

Remark 1.2.2. *The essential proprety of the topological degree is:*
If $(I - f,\Omega,y) \in A$ and $d(I - f, \Omega, y) \neq 0$, then there exists $x \in \Omega$ such that $x - f(x) = y$.

Schauder's fixed point theorem

As in finite dimension, a first consequence of the existence of the topological degree is the obtaining of a fixed point theorem that we give now.

Theorem 1.2.4. . *[20] Let E be a Banach space, $R > 0$, $B(0, R) = \{x \in E, \|x\| \leqslant R\}$ and f a compact application from $B(0,R)$ to $B(0,R)$ (that is to say f continuous and $\{f(x), x \in B(0, R)\}$ relatively compact in E). Then f admits a fixed point that is, there exists $x \in B(0, R)$ such that $f(x) = x$.*

1.3 Resonance and non-resonance

Resonance is a phenomenon where some physical systems (Electrical,Mechanical...) are sensitive to certain frequencies.

1.3.1 Mathematical

Let the Dirichlet probleme

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = f(x,u) + h & \text{in} \quad \Omega, \\ \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary, $1 < p < \infty$ and h is given in $W^{-1,p'}(\Omega)$. suppose that

$$F_{\pm} = \lim_{s \to \pm\infty} sup \quad p\frac{F(x,s)}{|s|^p}, \tag{1.6}$$

where $F(x, s) = \int_0^s f(x, t)dt$.

Definition 1.3.1. . *[45] We say that the problem (1.5) is non-resonance under an eigenvalue if $F_+(x) < \lambda_n$ and $F_-(x) < \lambda_n$ a.e. $x \in \Omega$, where F_+ and F_- are defined by (1.6) and λ_n is an eigenvalue of the problem:*

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u & in \quad \Omega, \\ \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{1.7}$$

Definition 1.3.2. . *[45] We will say that the problem (1.5) presents a resonance under an eigenvalue λ_n if $F_+(x) \leqslant \lambda_n$ and $F_-(x) \leqslant \lambda_n$ a.e. $x \in \Omega$ and if at least one of the two inequalities is an equality on a subset of Ω of non-zero measure.*

Remark 1.3.1. . *[45] We note:*

$$f_{\pm} = \lim_{s \to \pm\infty} sup \frac{f(x,s)}{|s|^{p-2}s}.$$

We establish several existence results under different conditions on the report $p\frac{F(x,s)}{|s|^p}$ when $s \to \pm\infty$. Because we always have $F_{\pm} \leqslant f_{\pm}$, the condition we consider are more general than those relating to the report $\frac{f(x,s)}{|s|^{p-2}s}$.

Chapter 2

Existence and uniqueness of solution for a nonlinear fractional elliptic problem

In this chapter, we investigate the existence and uniqueness of weak solutions to a class of nonlocal problems governed by the fractional Laplacian operator. We use the fixed point theory to prove the existence and uniqueness of solutions under suitable assumptions on the non linearitie.

2.1 Position of the problem and the main result

We use the fixed point theory to confirm the existence of a weak solutions to the Dirichlet problem for the fractional Laplacian, given by

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipchitz boundary, $s \in]0,1[$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous function satisfying the Catheodory condition (i.e : $f(., w)$ is measurable for $w \in \mathbb{R}$ and $f(x, .)$ is continuous for almost every $x \in \Omega$),
and also verifying the growth restriction defined below

$$|f(x, \xi)| \leqslant r(x) + a|\xi|^\delta. \tag{2.2}$$

(We are employed the notation that $|.|$ stands for absolute value in $\mathbb{R}$).
Where $\delta \in]0,1[$ and $r \in L^2(\Omega)$ nonnul function; a is nonnegative constant.

We recall that the fractional Laplacian $(-\Delta)^s$ in its nonlocal representation defined as

$$(-\Delta)^s \varphi(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+2s}} dy,$$

along $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $s \in]0,1[$, P.V. denotes the integral in the sense of the principal value and

$$C(n,s) = \frac{4^s\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}}\frac{s}{\Gamma(1-s)}.$$

Throughout the chapter, without futther mention, we always assume that $n > 2s$, and the main result is

Theorem 2.1.1. *Under hypothese (2.2), problem (2.1) has at least one solution $u \in D^{s,2}(\Omega)$.*

2.2 Fixed point formulation of the problem

From the definition of the fractional Laplacian $(-\Delta)^s$, the problem (2.1) is weakly formulated as follows:

$$C(n,s)\int\int_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dydx = \int_\Omega f(x,u(x))\varphi(x)dx,$$

for $\varphi \in D^{s,2}(\Omega)$.

Let us define

$$\hat{L}_u : (\varphi) \longmapsto \hat{L}_u(\varphi),$$

$$\hat{S}_u : (\varphi) \longmapsto \hat{S}_u(\varphi),$$

where

$$\hat{L}_u(\varphi) = C(n,s)\iint\limits_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dydx,$$

and

$$\hat{S}_u(\varphi) = \int_\Omega f(x,u(x))\varphi(x)dx,$$

Lemma 2.2.1. *The operators $\hat{L}$ and $\hat{S}$ are continuous linear functionals on the space $D^{s,2}(\Omega)$.*

Since $D^{s,2}(\Omega)$ is a Hilbert space, by the Riesz representation theorem (see Chapter 1 Theorem 1.1.4) there exists uniquely determined elements $L(u), S(u) \in D^{s,2}(\Omega)$

We have also

$$\hat{L}_u(\varphi) = \langle \hat{L}_u, \varphi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L(u), \varphi \rangle_{\langle D^{s,2}, D^{s,2} \rangle},$$

and

$$\hat{S}_u(\varphi) = \langle \hat{S}, \varphi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S(u), \varphi \rangle_{\langle D^{s,2}, D^{s,2} \rangle} \ ,$$

for all $\varphi \in D^{s,2}(\Omega)$.

The standard norms of $L(u)$ and $S(u)$ are defined by:

$$\begin{cases} \|L_u\|^2_{D^{s,2}(\Omega)} = \|L(u)\|^2_{D^{s,2}}, \\ \\ \|S_u\|^2_{D^{s,2}(\Omega)} = \|S(u)\|^2_{D^{s,2}}, \end{cases}$$

where

$$\|L(u)\|_{D^{s,2}} = \|\hat{L}_u\|_{(D^{s,2})'} = sup_{\|\varphi\| \leqslant 1} |\langle Lu, \varphi \rangle|,$$

and

$$\|S(u)\|_{D^{s,2}} = \|\hat{S}_u\|_{(D^{s,2})'} = sup_{\|\varphi\| \leqslant 1} |\langle S(u), \varphi \rangle| \ ,$$

To prove that the Dirichlet problem (2.1) has at least one weak solution, it is necessary and sufficient to prove that the operator equation

$$L(u) = S(u), \tag{2.3}$$

has at least one solution in the space $D^{s,2}(\Omega)$.

There are several equivalent inner products defined on $D^{s,2}(\Omega)$. If we choose

$$u = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))}{|x-y|^{n+2s}} dydx,$$

then L defined by (2.3) is just an identity on $D^{s,2}(\Omega)$.

Hence (2.3) is equivalent in $D^{s,2}(\Omega)$ to the operator equation

$$u = S(u) \tag{2.4}$$

2.3 Proof of the main result

In this section, we will use the application of the Schauder fixed point theorem to get the existence and the uniqueness of a weak solution of problem (2.1).

Lemma 2.3.1. *The operator S is continuous in $D^{s,2}(\Omega)$.*

Proof. Let $u_n \to u$ in $D^{s,2}(\Omega)$, then we have

$$\|S(u_n) - S(u)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\langle S(u_n) - S(u), \varphi \rangle|,$$

where

$$\|S(u_n) - S(u)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\int_\Omega (f(x, u_n) - f(x, u))\varphi(x) dx|,$$

then

$$\|S(u_n) - S(u)\|_{D^{s,2}} \leqslant sup_{\|\varphi\| \leqslant 1} (\int_\Omega |f(x - u_n) - f(x, u)|^2 dx)^{1/2} (\int_\Omega |\varphi(x)|^2 dx)^{1/2},$$

and

$$\|S(u_n) - S(u)\|_{D^{s,2}} \leqslant sup_{\|\varphi\| \leqslant 1} \|f(x - u_n) - f(x, u)\|_{L^2} \|\varphi\|_{L^2},$$

since

$$\|S(u_n) - S(u)\|_{D^{s,2}} \leqslant sup_{\|\varphi\| \leqslant 1} \|f(x - u_n) - f(x, u)\|_{L^2} c_{emb} \|\varphi\|_{D^{s,2}},$$

thus

$$\|S(u_n) - S(u)\|^2_{D^{s,2}} \leqslant c^2_{emb} \|f(x, u_n) - f(x, u)\|^2_{L^2}.$$

The right-hand side approaches zero as $n \to \infty$ it follows from the continuity of the Nemytski operators from $L^2(\Omega)$ in $L^2(\Omega)$. This proves the continuity of S.

Lemma 2.3.2. *The operator S is compact.*

Proof.
S compact $\Leftrightarrow M \in D^{s,2} \Rightarrow S(M)$ relativement compact.

$S(M)$ relativement compact $\Leftrightarrow \overline{S(M)}$ compact.

$\overline{S(M)}$ compact $\Rightarrow \{w_n\} \subset \overline{S(M)}$, we have $\{w_n\} \to w \in \overline{S(M)}$.

Let $M \subset D^{s,2}$, $\{w_n\} \subset S(M) \subset \overline{S(M)}$.

Let $(u_n) \subset M$, so $S(u_n) = w_n$.

we have, $S(u_n) \to S(u)$, we get

$$w_n \to S(u), \text{ in } D^{s,2}$$

This proves the compactness of $\overline{S(M)}$, i.e., S is a compact operator.

Lemma 2.3.3. *The operator S maps the closure of the ball $B(0; R) \subset D^{s,2}(\Omega)$ into itself.*

Proof. For all $u \in D^{s,2}(\Omega)$, we have from Section 2.2 that

$$\|S(u)\|_{D^{s,2}(\Omega)} = sup_{\|\varphi\| \leqslant 1} |\langle S(u), \varphi \rangle|,$$

using the Cauchy-Schwarz inequality, we obtain

$$\|S(u)\|_{D^{s,2}} \leqslant sup_{\|\varphi\| \leqslant 1} (\int_\Omega |f(x, u(x))|^2 dx)^{\frac{1}{2}} (\int |\varphi(x)|^2 dx)^{\frac{1}{2}},$$

and

$$\|S(u)\|_{D^{s,2}} \leqslant sup_{\|\varphi\| \leqslant 1} (\int_\Omega |f(x, u(x))|^2 dx)^{\frac{1}{2}} c_{emb} \|\varphi\|_{D^{s,2}},$$

and

$$\|S(u)\|_{D^{s,2}} \leqslant c_{emb} (\int_\Omega |f(x, u(x))|^2 dx)^{\frac{1}{2}},$$

From the hypothese (2.2), we get

$$\|S(u)\|_{D^{s,2}} \leqslant c_{emb} (\int_\Omega |r(x) + a|u(x)|^\delta|^2 dx)^{\frac{1}{2}},$$

and

$$\|S(u)\|_{D^{s,2}} \leqslant c_{emb} ((\int_\Omega |r(x)|^2)^{\frac{1}{2}} + (\int_\Omega |a|u(x)|^\delta|^2)^{\frac{1}{2}}),$$

and

$$\|S(u)\|_{D^{s,2}} \leqslant c_{emb}(\|r\|_{L^2} + a(\int_\Omega |u(x)|^{2\delta} dx)^{\frac{1}{2}}), \tag{2.5}$$

where the last estimate is due to the Minkowski inequality for $p = 2$.

Applying the Hölder inequality, we have

$$(\int_\Omega |u(x)|^{2\delta} dx)^{\frac{1}{2}} \leqslant (\int_\Omega |u(x)|^{2\delta \times \frac{1}{\delta}} dx)^{\frac{\delta}{2}} (mes(\Omega))^{\frac{1-\delta}{2}} \ ,$$

and

$$(\int_\Omega |u(x)|^{2\delta} dx)^{\frac{1}{2}} \leqslant (\int_\Omega |u(x)|^{2} dx)^{\frac{\delta}{2}} (mes(\Omega))^{\frac{1-\delta}{2}},$$

and

$$(\int_\Omega |u(x)|^{2\delta} dx)^{\frac{1}{2}} \leqslant (mes(\Omega))^{\frac{1-\delta}{2}} \|u(x)\|^{\delta}_{L^2},$$

and

$$(\int_\Omega |u(x)|^{2\delta} dx)^{\frac{1}{2}} \leqslant c_{emb} (mes(\Omega))^{\frac{1-\delta}{2}} \|u(x)\|^{\delta}_{D^{s,2}}. \tag{2.6}$$

Now, (2.5) and (2.6) yield

$$\|S(u)\|_{D^{s,2}} \leqslant c_{emb}(\|r\|_{L^2} + a c_{emb}^{\delta} (mes(\Omega)^{\frac{1-\delta}{2}}) \|u\|^{\delta}_{D^{s,2}}) \ ,$$

we have

$$\|S(u)\|_{D^{s,2}} \leqslant c_{emb}\|r\|_{L^2} + a c_{emb}^{\delta+1} (mes(\Omega)^{\frac{1-\delta}{2}}) \|u\|^{\delta}_{D^{s,2}} \ .$$

Let

$$k = a c_{emb}^{\delta+1} (mes(\Omega)^{\frac{1-\delta}{2}}) \ ,$$

then

$$\|S(u)\|_{D^{s,2}} \leqslant \underbrace{C_{\mathrm{emb}}\|r\|_{L^2}}_{C} + k\|u\|^{\delta}_{D^{s,2}} \ ,$$

Where upon, we have

$$\|S(u)\|_{D^{s,2}} \leqslant C + k\|u\|^{\delta}_{D^{s,2}}. \tag{2.7}$$

It follows from (2.7) that for any $u \in B(0;R) \subset D^{s,2}(\Omega)$

$$\|S(u)\| \leqslant R; \text{ with } C + kR^{\delta} < R,$$

hence S maps $B(0;R)$ into itself if R is large enough.

Now, we can prove our main result.

Proof. (Theorem 2.1.1) To prove Theorem 2.1.1, we can apply the Schauder fixed point theorem. It follows from lemma 2.3.3 that there is at least one fixed point $u \in D^{s,2}(\Omega)$ of S (which mean the problem (2.1) have a weak solution in $D^{s,2}(\Omega)$).

This completes the proof.

2.4 Particular case

Let us assume that f is Lipschitz continuous functions with respect to the second variable, i.e., there exists constants $c \in \mathbb{R}^+$ for almost every $x \in \Omega$ and for any $w = (w_1, w_2)$, $z = (z_1, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\|f(x, w_1) - f(x, w_2)\|_{L^2(\Omega)} \leqslant c_1\|w_1 - w_2\|_{L^2(\Omega)\times L^2(\Omega)}, \tag{2.8}$$

We can apply the contraction principle to get the following result.

Theorem 2.4.1. *Let the Caratheodory functions f be Lipschitzian continuous with respect to the second variable with constant $c > 0$ such that $|c| < c_{emb}^{-2}$. Then, there is a unique fixed point $u \in D^{s,2}(\Omega)$, i.e., u is a unique weak solution of (2.1).*

2.4.1 Proof of the theorem

To prove Theorem 2.4.1 we need the following contraction principle.

1. **Theorem (Contraction principle)**. [2] Let φ be a contraction mapping from Y to Y. Then φ admits a unique fixed-point in Y.

   So to use the contraction principle we must prove that the operator S is contraction.

Lemma 2.4.1. *The S operator is a contraction.*

Proof. For any $u \in D^{s,2}(\Omega)$ we also have $u \in L^2(\Omega)$, then $f(x,u) \in L^2(\Omega)$.
Then, for all $u_1, u_2 \in D^{s,2}(\Omega)$, we have

$$\|S(u_1) - S(u_2)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\langle S(u_1) - S(u_2), \varphi \rangle|,$$

this means that

$$\|S(u_1) - S(u_2)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\int_\Omega [f(x,u_1) - f(x,u_2)]\varphi(x)dx|,$$

then

$$\|S(u_1) - S(u_2)\|_{D^{s,2}} \leqslant c_{emb} \|f(x,u_1) - f(x,u_2)\|_{L^2},$$

and

$$\|S(u_1) - S(u_2)\|_{D^{s,2}} \leqslant c_{emb} c \|u_1 - u_2\|_{L^2},$$

we use Cauchy-Schwarz inequality, we obtain

$$\|S(u_1) - S(u_2)\|_{D^{s,2}} \leqslant c_{emb} c c_{emb} \|u_1 - u_2\|_{D^{s,2}} \ .$$

We get,

$$\|S(u_1) - S(u_2)\|^2_{D^{s,2}} \leqslant c^2_{emb} c \|u_1 - u_2\|^2_{D^{s,2}},$$

Consequently S is a contraction if $c^2_{emb} c < 1$.

Now, we can prove Theorem 2.4.1.

Proof. According to Lemma 2.4.1 that S is a contraction if $c^2_{emb} c < 1$, Since this condition is satisfied, we can apply the contraction principle to get that there is a unique fixed point $u \in D^{s,2}(\Omega)$ of the operator S. That is, u is the unique weak solution of (2.1).

Chapter 3

Existence and uniqueness of solution for a nonlinear fractional elliptic system

This chapter is devoted to the investigation of the existence of weak solutions for a system of nonlocal equations governed by the fractional Laplacian. Under suitable conditions on the nonlinearities, we apply fixed point theory to demonstrate both the existence and uniqueness of solutions.

3.1 Position of the problem and the main result

Fractional differential equations, characterized by the presence of derivatives of fractional order, have emerged as effective tools for modeling complex phenomena in fields such as image processing, natural sciences, and various branches of engineering and applied mathematics (see [[21],[23]]). Recent research highlights the growing relevance of these equations, with significant developments reported in [[38],[40],[41],[49]].

Among the most studied problems is the Dirichlet problem associated with the fractional Laplacian, which has been approached from probabilistic, potential-theoretic, and analytical perspectives, and has garnered substantial attention (see [[6],[18]]). In this chapter, we focus on establishing the existence of weak solutions to a coupled system involving the fractional Laplacian, utilizing fixed point theory under appropriate conditions.

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) & \text{in} \quad \Omega, \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in]0,1[$ and $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying the Caratheodory conditions (i.e : $f(.,w), g(.,z)$ are measurables for each $w, z \in \mathbb{R}^2$ and $f(x,.), g(y,.)$ are continuous for almost every $x, y \in \Omega$), and also verifying the growth restriction defined below

$$\begin{cases} |f(x,\xi_1,\xi_2)| \leqslant r_1(x) + a|\xi_1|^{\delta_1} + b|\xi_2|^{\delta_1}, \\ \\ |g(x,\eta_1,\eta_2)| \leqslant r_2(x) + c|\eta_1|^{\delta_2} + d|\eta_2|^{\delta_2}. \end{cases} \tag{3.2}$$

(We are employed the notation that $|.|$ stands for absolute value in $\mathbb{R}$).
Where $\delta_1, \delta_2 \in]0,1[$ and $r = (r_1, r_2) \in \tilde{V}$ nonnul functions; a, b, c and d are nonnegative constants.

We recall that the fractional Laplacian $(-\Delta)^s$ in its nonlocal representation defined as

$$(-\Delta)^s \varphi(x) = C(n,s) P.V. \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}} dy,$$

along $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $s \in]0,1[$, P.V. denotes the integral in the sense of the principal value and

$$C(n,s) = \frac{4^s \Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}}} \frac{s}{\Gamma(1-s)}.$$

This study explores a new extension of a well established result by considering it within the framework of fractional calculus (see [16]). While several studies have addressed the classical and linear forms of the problem, the fractional setting remains relatively unexplored.

In the rest of the chapter we will study: In Section 2, we provide the necessary definitions and formulate the main result. Section 3 is devoted to reformulating problem (3.1) within the framework of fixed point theory. Section 4 presents the detailed proof of the main theorem. In Section 5, we examine a specific case as an illustrative example.

Throughout this chapter, unless otherwise specified, we assume that $n > 2s$, Under this standing assumption, we proceed to state the main result

Theorem 3.1.1. *Under hypothesis (3.2), problem (3.1) has at least one solution $(u, v) \in \tilde{U}$.*

3.2 Fixed point formulation of the problem

From the definition of the fractional Laplacian $(-\Delta)^s$, the problem (3.1) is weakly formulated as follows:

$$\begin{cases} C(n,s)\int\int_{\mathbb{R}^{2n}} \dfrac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dydx = \int_\Omega f(x,u(x),v(x))\varphi(x)dx, \\ \\ C(n,s)\int\int_{\mathbb{R}^{2n}} \dfrac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}dydx = \int_\Omega g(x,u(x),v(x))\phi(x)dx, \end{cases}$$

for $(\varphi,\phi)\in\tilde{U}$.

Let us define

$$\hat{L}_{u,v} : (\varphi,\phi) \longmapsto (\hat{L}_u(\varphi), \hat{L}_v(\phi)),$$
$$\hat{S}_{u,v} : (\varphi,\phi) \longmapsto (\hat{S}_1(\varphi), \hat{S}_2(\phi)),$$

where

$$\hat{L}_u(\varphi) = C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dydx,$$
$$\hat{L}_v(\phi) = C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}dydx\ ,$$

and

$$\hat{S}_1(\varphi) = \int_\Omega f(x,u(x),v(x))\varphi(x)dx,$$
$$\hat{S}_2(\phi) = \int_\Omega g(x,u(x),v(x))\phi(x)dx.$$

Lemma 3.2.1. *The operators $\hat{L}$ and $\hat{S}$ are continuous linear functionals on the space $\tilde{U}$.*

Since $\tilde{U}$ is a Hilbert space, by the Riesz representation theorem (see Chapter 1 Theorem 1.1.4) there exists uniquely determined elements $L(u,v), S(u,v) \in \tilde{U}$ such that

$$L(u,v) = (L(u), L(v)) \text{ and } S(u,v) = (S_1(u,v), S_2(u,v)).$$

We have also

$$\begin{cases} \hat{L}_u(\varphi) = \langle \hat{L}_u, \varphi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L(u), \varphi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{L}_v(\phi) = \langle \hat{L}_v, \phi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L(v), \phi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

and

$$\begin{cases} \hat{S}_1(\varphi) = \langle \hat{S}_1, \varphi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_1(u,v), \varphi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{S}_2(\phi) = \langle \hat{S}_2, \phi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_2(u,v), \phi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

for all $(\varphi, \phi) \in \tilde{U}$.

The standard norms of $L(u,v)$ and $S(u,v)$ are defined by:

$$\begin{cases} \|L(u,v)\|^2_{\tilde{U}} = \|L(u)\|^2_{D^{s,2}} + \|L(v)\|^2_{D^{s,2}}, \\ \\ \|S(u,v)\|^2_{\tilde{U}} = \|S_1(u,v)\|^2_{D^{s,2}} + \|S_2(u,v)\|^2_{D^{s,2}}, \end{cases}$$

where

$$\begin{cases} \|L(u)\|_{D^{s,2}} = \|\hat{L}_u\|_{(D^{s,2})'} = sup_{\|\varphi\| \leqslant 1} |\langle Lu, \varphi \rangle|, \\ \\ \|L(v)\|_{D^{s,2}} = \|\hat{L}_v\|_{(D^{s,2})'} = sup_{\|\phi\| \leqslant 1} |\langle Lv, \phi \rangle|, \end{cases}$$

and

$$\begin{cases} \|S_1(u,v)\|_{D^{s,2}} = \|\hat{S}_1\|_{(D^{s,2})'} = sup_{\|\varphi\| \leqslant 1} |\langle S_1(u,v), \varphi \rangle|, \\ \\ \|S_2(u,v)\|_{D^{s,2}} = \|\hat{S}_2\|_{(D^{s,2})'} = sup_{\|\phi\| \leqslant 1} |\langle S_2(u,v), \phi \rangle|. \end{cases}$$

To prove the existence of at least one weak solution to the Dirichlet problem (3.1), it is both necessary and sufficient to prove that the operator equation

$$L(u,v) = S(u,v), \tag{3.3}$$

has at least one solution in the space $\tilde{U}$.

There are several equivalent inner products defined on $D^{s,2}(\Omega)$. If we choose

$$\langle u, v \rangle = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dydx,$$

then L defined by (3.3) is just an identity on $\tilde{U}$.

Hence (3.3) is equivalent in $\tilde{U}$ to the operator equation

$$(u,v) = S(u,v) \tag{3.4}$$

3.3 Proof of the main result

In this section, we establish a series of lemmas to demonstrate the existence and uniqueness of the weak solution to Problem (3.1). The proof relies on the application of the Schauder fixed point theorem.

Lemma 3.3.1. *The operator S is continuous in $\tilde{U}$.*

Proof. Let $(u_n, v_n) \to (u,v)$ in $\tilde{U}$, then we have

$$\begin{cases} \|S_1(u_n,v_n) - S_1(u,v)\|_{D^{s,2}} = sup_{\|\varphi\|\leqslant 1} |\langle S_1(u_n,v_n) - S_1(u,v), \varphi\rangle|, \\ \\ \|S_2(u_n,v_n) - S_2(u,v)\|_{D^{s,2}} = sup_{\|\phi\|\leqslant 1} |\langle S_2(u_n,v_n) - S_2(u,v), \phi\rangle|, \end{cases}$$

then

$$\begin{cases} \sup\limits_{\|\varphi\|\leqslant 1} |\langle S_1(u_n,v_n) - S_1(u,v)\rangle|_{D_{s,2}} \leqslant \sup\limits_{\|\varphi\|\leqslant 1} \left(\int |f(x,u_n,v_n)|^2 dx \right)^{1/2} \left(\int |\varphi(x)|^2 dx \right)^{1/2} \\ \qquad\qquad - \sup\limits_{\|\varphi\|\leqslant 1} \left(\int |f(x,u,v)|^2 dx \right)^{1/2} \left(\int |\varphi(x)|^2 dx \right)^{1/2} \\ \\ \sup\limits_{\|\phi\|\leqslant 1} |\langle S_2(u_n,v_n) - S_2(u,v)\rangle|_{D_{s,2}} \leqslant \sup\limits_{\|\phi\|\leqslant 1} \left(\int |g(x,u_n,v_n)|^2 dx \right)^{1/2} \left(\int |\phi(x)|^2 dx \right)^{1/2} \\ \qquad\qquad - \sup\limits_{\|\phi\|\leqslant 1} \left(\int |g(x,u,v)|^2 dx \right)^{1/2} \left(\int |\phi(x)|^2 dx \right)^{1/2} \end{cases}$$

and

$$\begin{cases} \sup\limits_{\|\varphi\|\leqslant 1} |\langle S_1(u_n,v_n) - S_1(u,v)\rangle|_{D_{s,2}} \leqslant \sup\limits_{\|\varphi\|\leqslant 1} \left(\int |f(x,u_n,v_n)|^2 dx\right)^{1/2} C_{emb}\|\varphi\|_{\tilde{U}} \\ \qquad\qquad - \sup\limits_{\|\varphi\|\leqslant 1} \left(\int |f(x,u,v)|^2 dx\right)^{1/2} C_{emb}\|\varphi\|_{\tilde{U}} \\ \\ \sup\limits_{\|\phi\|\leqslant 1} |\langle S_2(u_n,v_n) - S_2(u,v)\rangle|_{D_{s,2}} \leqslant \sup\limits_{\|\phi\|\leqslant 1} \left(\int |g(x,u_n,v_n)|^2 dx\right)^{1/2} C_{emb}\|\phi\|_{\tilde{U}} \\ \qquad\qquad - \sup\limits_{\|\phi\|\leqslant 1} \left(\int |g(x,u,v)|^2 dx\right)^{1/2} C_{emb}\|\phi\|_{\tilde{U}} \end{cases}$$

since

$$\begin{cases} \sup_{\|\varphi\|\leqslant 1}|\langle S_1(u_n,v_n) - S_1(u,v),\varphi\rangle| \leqslant c_{emb}\|f(x,u_n,v_n) - f(x,u,v)\|_{L^2} \\ \\ \sup_{\|\phi\|\leqslant 1}|\langle S_2(u_n,v_n) - S_2(u,v),\phi\rangle| \leqslant c_{emb}\|g(x,u_n,v_n) - g(x,u,v)\|_{L^2}, \end{cases}$$

thus

$$\|S(u_n,v_n) - S(u,v)\|^2_{\tilde{U}} \leqslant c^2_{emb}\|f(x,u_n,v_n) - f(x,u,v)\|^2_{L^2} + c^2_{emb}\|g(x,u_n,v_n) - g(x,u,v)\|^2_{L^2}.$$

The right-hand side approaches zero as $n \to \infty$ it follows from the continuity of the Nemytski operators from $L^2(\Omega)$ in $L^2(\Omega)$. This proves the continuity of S.

Lemma 3.3.2. *The operator S is compact.*

Proof. Let $M \subset \tilde{U}$ be a bounded set and $\{w_n\}_{n=1}^{\infty} = \{w_{1,n}, w_{2,n}\}_{n=1}^{\infty} \subset S(M)$ be an arbitrary sequence.
Let $\{u_n, v_n\}_{n=1}^{\infty} \subset M$ be such that

$$S(u_n, v_n) = (w_{1n}, w_{2n}).$$

The reflexivity of $\tilde{U}$ implies that $(u_n, v_n) \to (u, v)$ in $\tilde{U}$ at least for a subsequence. As a result of the compact injection of $D^{s,2}(\Omega)$ in $L^2(\Omega)$ (proposition 3.15) that $(u_n, v_n) \to (u, v)$ in $\tilde{V}$.

$$(w_{1n}, w_{2n}) \to S(u,v), \text{ in } \tilde{U}$$

(at least for a subsequence). This proves the compactness of $\overline{S(M)}$, i.e., S is a compact operator.

Lemma 3.3.3. *The operator S maps the closure of the ball $B(0,R) \subset \tilde{U}$ into itself.*

Proof. For all $(u,v) \in \tilde{U}$, we have from Section 3.2 that

$$\begin{cases} \|S_1(u,v)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\langle S_1(u,v), \varphi \rangle|, \\ \\ \|S_2(u,v)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\langle S_2(u,v), \varphi \rangle|, \end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{cases} \|S_1(u,v)\|_{D^{s,2}} \leqslant c_{emb} (\int_\Omega |f(x,u(x),v(x))|^2 dx)^{\frac{1}{2}}, \\ \\ \|S_2(u,v)\|_{D^{s,2}} \leqslant c_{emb} (\int_\Omega |g(x,u(x),v(x))|^2 dx)^{\frac{1}{2}}. \end{cases}$$

From the hypothesis (3.2), we get

$$\begin{cases} \|S_1(u,v)\|_{D^{s,2}} \leqslant c_{emb} (\int_\Omega |r_1(x) + a|u(x)|^{\delta_1} + b|v(x)|^{\delta_1}|^2 dx)^{\frac{1}{2}}, \\ \\ \|S_2(u,v)\|_{D^{s,2}} \leqslant c_{emb} (\int_\Omega |r_2(x) + c|u(x)|^{\delta_2} + d|v(x)|^{\delta_2}|^2 dx)^{\frac{1}{2}}, \end{cases}$$

and

$$\begin{cases} \|S_1(u,v)\|_{D^{s,2}} \leqslant c_{emb} (\|r_1\|_{L^2} + a(\int_\Omega |u(x)|^{2\delta_1} dx)^{\frac{1}{2}} + b(\int_\Omega |v(x)|^{2\delta_1} dx)^{\frac{1}{2}}), \\ \\ \|S_2(u,v)\|_{D^{s,2}} \leqslant c_{emb} (\|r_2\|_{L^2} + c(\int_\Omega |u(x)|^{2\delta_2} dx)^{\frac{1}{2}} + d(\int_\Omega |v(x)|^{2\delta_2} dx)^{\frac{1}{2}}), \end{cases} \tag{3.5}$$

where the last estimate is due to the Minkowski inequality for $p = 2$.

Applying the Hölder inequality, we have

$$\begin{cases} (\int_\Omega |u(x)|^{2\delta_1} dx)^{\frac{1}{2}} \leqslant (\int_\Omega |u(x)|^2 dx)^{\frac{\delta_1}{2}} (mes(\Omega))^{\frac{1-\delta_1}{2}} \leqslant c_{emb}^{\delta_1} (mes(\Omega))^{\frac{1-\delta_1}{2}} \|u\|_{D^{s,2}(\Omega)}^{\delta_1}, \\ \\ (\int_\Omega |v(x)|^{2\delta_2} dx)^{\frac{1}{2}} \leqslant (\int_\Omega |v(x)|^2 dx)^{\frac{\delta_2}{2}} (mes(\Omega))^{\frac{1-\delta_2}{2}} \leqslant c_{emb}^{\delta_2} (mes(\Omega))^{\frac{1-\delta_2}{2}} \|v\|_{D^{s,2}(\Omega)}^{\delta_2}. \end{cases} \tag{3.6}$$

Now, (3.5) and (3.6) yield

$$\begin{cases} \|S_1(u,v)\|^2 \leqslant [c_{emb}\|r_1\| + a c_{emb}^{\delta_1+1} (mes(\Omega))^{\frac{1-\delta_1}{2}} \|u\|^{\delta_1} + b c_{emb}^{\delta_1+1} (mes(\Omega))^{\frac{1-\delta_1}{2}} \|v\|^{\delta_1}]^2, \\ \\ \|S_2(u,v)\|^2 \leqslant [c_{emb}\|r_2\| + c c_{emb}^{\delta_2+1} (mes(\Omega))^{\frac{1-\delta_2}{2}} \|u\|^{\delta_2} + d c_{emb}^{\delta_2+1} (mes(\Omega))^{\frac{1-\delta_2}{2}} \|v\|^{\delta_2}]^2, \end{cases}$$

if we put

$$\begin{cases} k = ac_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ \\ l = bc_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ \\ j = cc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \\ \\ h = dc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \end{cases}$$

then

$$\begin{cases} \|S_1(u,v)\|^2 \leqslant [c_{emb}\|r_1\| + max(k,l)(\|u\|^{\delta_1} + \|v\|^{\delta_1})]^2, \\ \\ \|S_2(u,v)\|^2 \leqslant [c_{emb}\|r_2\| + max(j,h)(\|u\|^{\delta_2} + \|v\|^{\delta_2})]^2, \end{cases}$$

thus

$$\begin{cases} \|S_1(u,v)\|^2 \leqslant 2c_{emb}^2\|r_1\|^2 + 4max^2(k,l)(\|u\|^{2\delta_1} + \|v\|^{2\delta_1}), \\ \\ \|S_2(u,v)\|^2 \leqslant c_{emb}^2\|r_2\|^2 + 4max^2(j,h)(\|u\|^{2\delta_2} + \|v\|^{2\delta_2}). \end{cases}$$

Where-upon, we have

$$\|S(u,v)\|^2 \leqslant \underbrace{2c_{\text{emb}}^2\|r\|^2}_{=C} + \underbrace{4(max^2(k,l) + max^2(j,h))}_{=D} \max(\|(u,v)\|^{2\delta_1}, \|(u,v)\|^{2\delta_2}). \tag{3.7}$$

It follows from (3.7) that for any $(u,v) \in B(0;R) \subset \tilde{U}$

$$\|S(u,v)\| \leqslant R, \text{ with } \sqrt{C + Dmax(R^{2\delta_1}, R^{2\delta_2})} < R,$$

hence S maps $B(0;R)$ into itself if R is large enough.

Now, we can prove our main result.

Proof of Theoram (3.1.1) to establish Theorem (3.1.1), we apply the Schauder fixed point theorem. From Lamma (3.3.3), it follows that the operator S admits at least one fixed point $(u,v) \in \tilde{U}$, which implies that the system (3.1) has a weak solution in $\tilde{U}$.

This completes the proof.

3.4 Particular case

Let us assume that f and g are Lipschitz continuous functions with respect to the second variable, i.e., there exists constants $c_1, c_2 \in \mathbb{R}^+$ for almost every $x \in \Omega$ and for any $w = (w_1, w_2)$, $z = (z_1, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{cases} \|f(x, w_1) - f(x, w_2)\|_{L^2(\Omega)} \leqslant c_1 \|w_1 - w_2\|_{L^2(\Omega)\times L^2(\Omega)}, \\ \\ \|g(x, z_1) - g(x, z_2)\|_{L^2(\Omega)} \leqslant c_2 \|z_1 - z_2\|_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{3.8}$$

We can apply the contraction principle to get the following result.

Theorem 3.4.1. *Let the Caratheodory functions f, g be Lipschitzian continuous with respect to the second variable with constants $c_i > 0$ $(i = 1, 2)$ such that $|c| < c_{emb}^{-2}(c = (c_1, c_2))$. Then, there is a unique fixed point $(u, v) \in \tilde{U}$, i.e., (u, v) is a unique weak solution of (3.1).*

3.4.1 Proof of the theorem

To prove Theorem 3.4.1 we need the following contraction principle.

we need Theorem of Contraction principle [1].

So to use the contraction principle we must prove that the operator S is contraction.

Lemma 3.4.1. *The S operator is a contraction.*

Proof. For any $(u, v) \in \tilde{U}$ we also have $(u, v) \in \tilde{V}$, then $(f(x, u, v), g(x, u, v)) \in \tilde{V}$. Then, for all $(u_1, v_1), (u_2, v_2) \in \tilde{U}$, we have

$$\begin{cases} \|S_1(u_1, v_1) - S_1(u_2, v_2)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\langle S_1(u_1, v_1) - S_1(u_2, v_2), \varphi\rangle|, \\ \\ \|S_2(u_1, v_1) - S_2(u_2, v_2)\|_{D^{s,2}} = sup_{\|\phi\| \leqslant 1} |\langle S_2(u_1, v_1) - S_2(u_2, v_2), \phi\rangle|, \end{cases}$$

this means that

$$\begin{cases} \|S_1(u_1, v_1) - S_1(u_2, v_2)\|_{D^{s,2}} = sup_{\|\varphi\| \leqslant 1} |\int_\Omega [f(x, u_1, v_1) - f(x, u_2, v_2)]\varphi(x) dx|, \\ \\ \|S_2(u_1, v_1) - S_2(u_2, v_2)\|_{D^{s,2}} = sup_{\|\phi\| \leqslant 1} |\int_\Omega [g(x, u_1, v_1) - g(x, u_2, v_2)]\phi(x) dx|. \end{cases}$$

We get,

$$\begin{cases} \|S_1(u_1,v_1) - S_1(u_2,v_2)\|^2_{D^{s,2}} \leqslant c_1^2 c_{emb}^4 \|(u_1,v_1) - (u_2,v_2)\|^2_{\tilde{U}}, \\ \\ \|S_2(u_1,v_1) - S_2(u_2,v_2)\|^2_{D^{s,2}} \leqslant c_2^2 c_{emb}^4 \|(u_1,v_1) - (u_2,v_2)\|^2_{\tilde{U}}. \end{cases}$$

Consequently S is a contraction if $c_{emb}^2|c| < 1$.

Now, we can prove Theorem 3.4.1.

Proof. From Lemma 3.4.1, it follows that the operator S is a contraction if $c_{emb}^2|c| < 1$. Therefore, by applying the contraction principle, we obtain the existence of a unique fixed point $(u,v) \in \tilde{U}$ of the operator S. This implies that (u,v) is the unique weak solution of (3.1).

Chapter 4

Existence of solution for a nonlinear fractional elliptic system at resonance and nonresonance

This chapter is devoted to the study of the existence of solutions for a nonlinear fractional elliptic system under both resonance and nonresonance conditions. Our work contributes to the development of existence results for certain classes of nonlinear systems involving fractional operators. We apply the Leray-Schauder degree theorem to establish the existence of a weak solution to the system under consideration.

4.1 Position of the problem

This study is concerned with the existence of solutions to nonlocal equations involving the fractional Laplacian. We apply the Leray–Schauder degree theorem to establish the existence of a weak solution to the system under consideration.

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) + f_1(x) \quad \text{in} \quad \Omega, \\ \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) + f_2(x) \quad \text{in} \quad \Omega, \\ \\ u = v = 0 \quad \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{4.1}$$

with $s \in (0, 1)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, f and g are continuous functions defined in $\Omega \times \mathbb{R} \times \mathbb{R}$ and $h = (f_1, f_2) \in \tilde{V}$.

This result is novel and is a fractional variant of a traditional theorem derived from the study of Laplcian equations.

We assume that the nonlinear functions f and g are expressed in the following form:

$$\begin{cases} f(x,u,v) = au + f_0(x,v), \\ g(x,u,v) = bv + g_0(x,u), \end{cases}$$

where a, b are real positive constants.In this work we will see 3 class of systems considering the constants a and b.

We begin by recalling the fractional Laplacian $(-\Delta)^s$ is a defined as

$$(-\Delta)^s\varphi(x) = C(n,s)P.V.\int_{\mathbb{R}^n} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}dy,$$

along $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $s \in (0,1)$, P.V. denotes the integral in the sense of the principal value, $C(n,s)$ is a positive constant of normalization and its spectrum in $L^2(\Omega)$ is formed by the sequence $(\lambda_k)_k \in \mathbb{R}^*$ such that $|\lambda_k| \to +\infty$.

We assume that f_0 and g_0 are Caratheodory functions defined on $\Omega \times \mathbb{R} \to \mathbb{R}$. Let λ_1 denote the first eigenvalue of the fractional Laplacian operator $(-\Delta)^s$, and let φ_1 be the corresponding normalized eigenfunction associated with λ_1.

Let $\lambda_1 \in \mathbb{R}^*$ be defined as

$$\lambda_1 = \inf_{\substack{u\in D^{s,2}(\Omega)\\ u\neq 0}} \frac{C(n,s)\displaystyle\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dy\,dx}{\displaystyle\int_\Omega |u|^2\,dx},$$

or equivalently as

$$\lambda_1 = inf\{C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx : \int_\Omega |u|^2dx = 1,\, u \in D^{s,2}(\Omega), u \neq 0\},$$

if

$$C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\varphi_1(x)-\varphi_1(y))(v(x)-v(y))}{|x-y|^{n+2s}}dydx = \lambda_1\int_\Omega \varphi_1(x)v(x)dx, \text{ for all } v \in D^{s,2}(\Omega)\ .$$

The current chapter's goal is to apply the findings of [[36].[41]] to the system (4.1) while keeping in mind the following conditions for the functions f_0 and g_0:

$$\begin{cases} \beta_1^-, \beta_1^+ \in L^2(\Omega) \\ and \\ \beta_1^-(x) \leqslant f_0(x,t) \leqslant \beta_1^+(x) \quad a.e. \quad \Omega \\ where \\ \lim\limits_{t\to-\infty} f_0(.,t) = \beta_1^-(.) \quad a.e. \quad \Omega \\ \lim\limits_{t\to+\infty} f_0(.,t) = \beta_1^+(.) \quad a.e. \quad \Omega, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \beta_2^-, \beta_2^+ \in L^2(\Omega) \\ and \\ \beta_2^-(x) \leqslant g_0(x,t) \leqslant \beta_2^+(x) \quad a.e. \quad \Omega \\ where \\ \lim\limits_{t\to-\infty} g_0(.,t) = \beta_2^-(.) \quad a.e. \quad \Omega \\ \lim\limits_{t\to+\infty} g_0(.,t) = \beta_2^+(.) \quad a.e. \quad \Omega. \end{cases} \tag{4.3}$$

4.2 Preliminaires and the main results

Firsty, we provide a definition of a weak solution.

Definition 4.2.1. *We say $(u,v) \in \tilde{U}$ is a weak solution of the system (4.1) if for any $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \tilde{U}$, we have*

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \dfrac{(u(x)-u(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega au\tilde{w}_1 dx + \int_\Omega f_0(x,v)\tilde{w}_1 dx + \int_\Omega f_1(x)\tilde{w}_1 dx, \\ \\ C(n,s) \iint_{\mathbb{R}^{2n}} \dfrac{(v(x)-v(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega bv\tilde{w}_2 dx + \int_\Omega g_0(x,v)\tilde{w}_2 dx + \int_\Omega f_2(x)\tilde{w}_2 dx. \end{cases}$$

For a fixed $(u,v) \in \tilde{U}$, it is evident that

$$\hat{L}_{u,v} : (\tilde{w}_1, \tilde{w}_2) \longmapsto (\hat{L}_u(\tilde{w}_1), \hat{L}_v(\tilde{w}_2)),$$

$$\hat{S}_{u,v} : (\tilde{w}_1, \tilde{w}_2) \longmapsto (\hat{S}_v(\tilde{w}_1), \hat{S}_u(\tilde{w}_2)),$$

$$\hat{A}_{u,v} : (\tilde{w}_1, \tilde{w}_2) \longmapsto (\hat{A}_u(\tilde{w}_1), \hat{A}_v(\tilde{w}_2)),$$

such that

$$\hat{L}_u(\tilde{w}_1) = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}} dydx,$$

$$\hat{L}_v(\tilde{w}_2) = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}} dydx,$$

$$\hat{S}_v(\tilde{w}_1) = \int_\Omega f_0(x, v(x))\tilde{w}_1(x)dx,$$

$$\hat{S}_u(\tilde{w}_2) = \int_\Omega g_0(x, u(x))\tilde{w}_2(x)dx,$$

$$\hat{A}_u(\tilde{w}_1) = \int_\Omega u\tilde{w}_1(x)dx,$$

$$\hat{A}_v(\tilde{w}_2) = \int_\Omega v\tilde{w}_2(x)dx.$$

Lemma 4.2.1. *$\hat{L}$, $\hat{S}$ and $\hat{A}$ are continuous linear functionals on the space $\tilde{U}$.*

As in the precending chapter,since $\tilde{U}$ is a Hilbert space, the Riesz representation theorem (see to chapter 1, Theorem (1.1.4)) guarantees the existence of uniquely determined elements $L(u,v)$, $S(u,v)$ and $A(u,v) \in \tilde{U}$ such that

$$L(u,v) = (L_1(u), L_2(v)),\ S(u,v) = (S_1(v), S_2(u)) \text{ and } A(u,v) = (A_1(u), A_2(v)),$$

we have also that

$$\begin{cases} \hat{L}_u(\tilde{w}_1) = \langle \hat{L}_u, \tilde{w}_1 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L_1(u), \tilde{w}_1 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{L}_v(\tilde{w}_2) = \langle \hat{L}_v, \tilde{w}_2 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L_2(v), \tilde{w}_2 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

$$\begin{cases} \hat{S}_v(\tilde{w}_1) = \langle \hat{S}_v, \tilde{w}_1 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_1(v), \tilde{w}_1 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{S}_u(\tilde{w}_2) = \langle \hat{S}_u, \tilde{w}_2 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_2(u), \tilde{w}_2 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

and

$$\begin{cases} \hat{A}_u(\tilde{w}_1) = \langle \hat{A}_u, \tilde{w}_1 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle A_1(u), \tilde{w}_1 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{A}_v(\tilde{w}_2) = \langle \hat{A}_v, \tilde{w}_2 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle A_2(v), \tilde{w}_2 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

for all $(\tilde{w}_1, \tilde{w}_2) \in \tilde{U}$.

We can define L as an identity on $\tilde{U}$ and it is evident that S and A are compact and continuous operators.

For $t \in [0,1]$ and $(u,v) \in \tilde{U}$ we define the following homotopy

$$T(t,u,v) = \begin{pmatrix} T_1(t,u,v) \\ T_2(t,u,v) \end{pmatrix} = \begin{pmatrix} u - aA_1(u) - tS_1(v) - (1-t)\varepsilon A_1(u) \\ v - bA_2(v) - tS_2(u) - (1-t)\varepsilon A_2(v) \end{pmatrix}.$$

Equivalently,

$$T(t,u,v) = (u,v) - BA(u,v) - tS(u,v) - (1-t)C(\varepsilon)A(u,v), \quad \forall \varepsilon > 0,$$

where

$$\text{B=} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \textit{and} \quad C(\varepsilon) = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that

$$BA - tS - (1-t)C(\varepsilon)A : [0,1] \times \tilde{U} \to \tilde{U},$$

is a compact and continuous operator, then the existence of at least one solution of the system (4.1) would follow from

$$deg(I - BA - S, B(0,R), 0) \neq 0.$$

Let's now present the main result of this chapter.

Theorem 4.2.1. *Assume that (4.2) and (4.3) are fulfilled. Then (4.1) has at least one solution* $(u,v) \in \tilde{U}$.

4.3 A first class of systems

In this section, the class of systems under consideration is defined by taking functions.

$$\begin{cases} f(x,u,v) = au + f_0(x,v), \\ g(x,u,v) = bv + g_0(x,u), \end{cases}$$

and

$$\begin{cases} f_1(x) = h_1(x), \\ f_2(x) = h_2(x), \end{cases}$$

where $a, b \notin sp((-\Delta)^s)$.

In light of the notation above (4.1) reads as

$$\begin{cases} (-\Delta)^s u(x) = au + f_0(x,v) + h_1(x) \quad in \quad \Omega, \\ (-\Delta)^s v(x) = bv + g_0(x,u) + h_2(x) \quad in \quad \Omega, \\ u = v = 0 \quad on \quad \mathbb{R}^n \backslash \Omega. \end{cases} \tag{4.4}$$

To prove the existence of solution for (4.4), we have to define an admissible homotopy T, the simplest way to do it is to follow Section 4.2. Define

$$T(t,u,v) = (u,v) - BA(u,v) - tS(u,v) - (1-t)C(\varepsilon)A(u,v),$$

$\forall \varepsilon > 0$, $t \in [0,1]$ and $(u,v) \in \tilde{U}$. T is a compact homotopy connecting $I(.) - BA(.) - S(.)$ and $I(.) - BA(.)$.

4.3.1 A priori bounds for solutions

To complete the proof, we must demonstrate the existence $R_1 > 0$ such that, for all $(u,v) \in \tilde{U}$, $\|(u,v)\|_{\tilde{U}} = R_1$ and $t \in [0,1]$, we obtain

$$T(t,u,v) \neq 0. \tag{4.5}$$

We use the indirect proof to establish (4.5).

Lemma 4.3.1. *There exists $R_1 > 0$ such that*

$$\begin{cases} \|(u,v)\|_{\tilde{U}} = R_1, \quad \forall t \in [0,1], \forall (u,v) \in \tilde{U} \\ T(t,u,v) \neq 0. \end{cases}$$

Proof. Let $\varepsilon > 0$ such that $[a, a+\varepsilon] \cap sp((-\Delta)^s) = \varnothing$ and $[b, b+\varepsilon] \cap sp((-\Delta)^s) = \varnothing$. Assume that no such $R_1 > 0$ exists, i.e., we can find sequences $\{(u_n, v_n)\}_{n=1}^{\infty} \subset \tilde{U}$ and $\{t_n\}_{n=1}^{\infty} \subset [0,1]$ such that $\|(u_n, v_n)\|_{\tilde{U}} \to \infty$ (i.e $\|(u_n, v_n)\|_{\tilde{U}} = n$) and

$$(u_n, v_n) - BA(u_n, v_n) - t_n S(u_n, v_n) - (1 - t_n)C(\varepsilon)A(u_n, v_n) = 0. \quad \forall \varepsilon > 0 \tag{4.6}$$

Set

$$w_n = (w_{1,n}, w_{2,n}) = \Big(\frac{u_n}{\|(u_n, v_n)\|_{\tilde{U}}}, \frac{v_n}{\|(u_n, v_n)\|_{\tilde{U}}}\Big),$$

then it follows that $\|w_n\|_{\tilde{U}} = 1$.

Divide (4.6) by $\|(u_n, v_n)\|_{\tilde{U}}$ to obtain

$$w_n - BA(w_n) - (1 - t_n)C(\varepsilon)A(w_n) - t_n \frac{S(u_n, v_n)}{\|(u_n, v_n)\|_{\tilde{U}}} = 0, \forall \varepsilon > 0 \tag{4.7}$$

this is equivalent to

$$C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(w_{1,n}(x) - w_{1,n}(y))(\tilde{w}_1(x) - \tilde{w}_1(y))}{|x-y|^{n+2s}} dydx = \quad [a + (1 - t_n)\varepsilon] \int_\Omega w_{1,n}\tilde{w}_1 dx$$
$$+ \int_\Omega \frac{f_0(x, v_n)\tilde{w}_1}{\|(u_n, v_n)\|} dx + \int_\Omega h_1(x)\tilde{w}_1 dx,$$

$$C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(w_{2,n}(x) - w_{2,n}(y))(\tilde{w}_2(x) - \tilde{w}_2(y))}{|x-y|^{n+2s}} dydx = \quad [b + (1 - t_n)\varepsilon] \int_\Omega w_{2,n}\tilde{w}_2 dx$$
$$+ \int_\Omega \frac{g_0(x, v_n)\tilde{w}_2}{\|(u_n, v_n)\|} dx \quad + \int_\Omega h_2(x)\tilde{w}_2 dx,$$

for any $(\tilde{w}_1, \tilde{w}_2) \in \tilde{U}$.Now, passing to suitable subsequences still denoted by $(t_n, w_{1,n}, w_{2,n})$, we can assume that $t_n \to t \in [0,1]$ and $(w_{1,n}, w_{2,n}) \rightharpoonup (w_1, w_2)$ in $\tilde{U}$.At the same time

$$\int_\Omega \frac{|f_0(x, v_n)|}{\|(u_n, v_n)\|} |\tilde{w}_1| dx \leqslant \int_\Omega \frac{|\beta_1^+(x)|}{\|(u_n, v_n)\|} |\tilde{w}_1| dx \to 0 \quad n \to \infty,$$

and

$$\int_\Omega \frac{|g_0(x,u_n)|}{\|(u_n,v_n)\|}|\tilde{w}_2|dx \leqslant \int_\Omega \frac{|\beta_2^+(x)|}{\|(u_n,v_n)\|}|\tilde{w}_2|dx \to 0 \quad n\to\infty.$$

In summary, we possess

$$t_n \frac{S(u_n,v_n)}{\|(u_n,v_n)\|_{\tilde{U}}} \to 0, \tag{4.8}$$

$$A(w_n) \to A(w). \tag{4.9}$$

(According to the compactness of A, see [[16]] Proposition 2.2.4(iii)).Thus, combining (4.7)-(4.9), we also get that

$$w_n \to w^* \text{ in } \tilde{U}.$$

But $w^* = w$ by virtue of $w_n \rightharpoonup w$. Now, passing to the limit in (4.7), we arrive at

$$w - [B + (1-t)C(\varepsilon)]A(w) = 0, \tag{4.10}$$

and $w \in \tilde{U}$ satisfies $\|w\| = 1$ (it is the strong limit of elements w_n which satisfy $\|w\| = 1$).

Then

$$\begin{aligned}(-\Delta)^s w_1 &= [a + (1-t)\varepsilon]w_1,\\ (-\Delta)^s w_2 &= [b + (1-t)\varepsilon]w_2.\end{aligned}$$

But this goes against what we assumed $[a, a+\varepsilon]\cap sp((-\Delta)^s) = \varnothing$ and $[b, b+\varepsilon]\cap sp((-\Delta)^s) = \varnothing$. It demonstrates that (4.5) holds,i.e. the homotopy T is admissible. This complete the proof.

4.4 A second of systems

In this section, we let

$$\begin{cases} f(x,u,v) = au + f_0(x,v),\\ g(x,u,v) = bv + g_0(x,u),\end{cases}$$

and

$$\begin{cases} f_1(x) = h_1(x),\\ f_2(x) = -h_2(x),\end{cases}$$

where $a = \lambda \in sp((-\Delta)^s)$ and $b = \lambda_1$ which is the first eigenvalue of the fractional Laplace operator.

We obtain the following system

$$\begin{cases} (-\Delta)^s u(x) = \lambda u + f_0(x,v) + h_1(x) & in \quad \Omega, \\ (-\Delta)^s v(x) = \lambda_1 v + g_0(x,v) - h_2(x) & in \quad \Omega, \\ u = v = 0 \quad on \quad \mathbb{R}^n \backslash \Omega. \end{cases} \tag{4.11}$$

Proposition 4.4.1. *Assume that (4.2) and (4.3) are fulfilled. Let $(\varphi, \varphi_1) \in N_\lambda \times N_{\lambda_1}, \lambda_1 > 0$ be the first eigenvalue of the fractional Laplace operator subject to the homogeneous Dirichlet boundary condition and φ_1 the associate eigenfunction normalized by*

$$\int_\Omega \varphi_1 = 1$$

Then the problem (4.11) has at least one weak solution if and only if

1. $-\int_\Omega \beta_1^+ \varphi(x)dx \leqslant \int_\Omega h_1(x)\varphi(x)dx \leqslant -\int_\Omega \beta_1^- \varphi(x)dx.$

2. $\beta_2^- \leqslant \int_\Omega h_2(x)\varphi_1(x)dx \leqslant \beta_2^+.$

with $(h_1, h_2) \in \tilde{V}$.

Proof. We proceed using a method similar to that in Section (4.3). For $\varepsilon > 0$ so small such that $]\lambda, \lambda + \varepsilon] \cap sp((-\Delta)^s) = \varnothing$, $\lambda_1 + \varepsilon < \lambda_2$ we define the homotopy

$$T(t,u,v) = (u,v) - BA(u,v) - tS(u,v) - (1-t)C(\varepsilon)A(u,v) \quad t \in [0.1] \quad (u,v) \in \tilde{U}$$

here

$$\text{B} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_1 \end{pmatrix}.$$

Following the same procedure as in the proof of Section (4.3), we arrive at an analogue of (4.10), namely,

$$w - [B + (1-t)C(\varepsilon)]A(w) = 0, \quad \|w\| = 1 \quad \text{for all} \quad t \in [0,1] \tag{4.12}$$

This is a contradiction if $t \neq 1$ since $\lambda + (1-t)\varepsilon$ and $\lambda_1 + (1-t)\varepsilon$ are not an eigenvalues and $w \neq 0$.

Let us assume $t = 1$, i.e., $t_n \to 1$. Now, we have no contradiction since λ and λ_1 are eigenvalues and

$$w - BA(w) = 0,$$

has a solution with $\|w\| = 1$. Another step is necessary to reach a contradiction and to prove that the homotopy T is admissible. We have to revise the last step when passing to the limit in

$$w_n - BA(w_n) - (1-t_n)C(\varepsilon)A(w_n) - t_n \frac{S(u_n, v_n)}{\|(u_n, v_n)\|_{\tilde{U}}} = 0, \quad t \in [0,1] \quad (u,v) \in \tilde{U}$$

and employ special properties of S. We mean,

$$(u_n, v_n) - BA(u_n, v_n) - t_n S(u_n, v_n) - (1-t_n)C(\varepsilon)A(u_n, v_n) = 0, \quad \forall \varepsilon > 0$$

is equivalent to the integral identity

$$C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u_n(x) - u_n(y)) - (\tilde{w_1}(x) - \tilde{w_1}(y))}{|x-y|^{n+2s}} dydx = [\lambda + (1-t_n)\varepsilon] \int_\Omega u_n \tilde{w_1} dx + \int_\Omega f_0(x, v_n) \tilde{w_1} dx + \int_\Omega h_1(x) \tilde{w_1} dx, \tag{4.13}$$

$$C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(v_n(x) - v_n(y)) - (\tilde{w_2}(x) - \tilde{w_2}(y))}{|x-y|^{n+2s}} dydx = [\lambda_1 + (1-t_n)\varepsilon] \int_\Omega v_n \tilde{w_2} dx + \int_\Omega g_0(x, u_n) \tilde{w_2} dx - \int_\Omega h_2(x) \tilde{w_2} dx, \tag{4.14}$$

for all $(\tilde{w_1}, \tilde{w_2}) \in \tilde{U}$. We take $(\tilde{w_1}, \tilde{w_2}) = (\varphi, \varphi_1)$ in (4.13) - (4.14) and using the fact that

$$C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u_n(x) - u_n(y)) - (\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dydx = \lambda \int_\Omega u_n \varphi dx,$$

and

$$C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(v_n(x) - v_n(y)) - (\varphi_1(x) - \varphi_1(y))}{|x-y|^{n+2s}} dydx = \lambda_1 \int_\Omega v_n \varphi_1 dx,$$

we obtain

$$(1-t_n)\varepsilon\int_\Omega u_n\varphi dx+\int_\Omega f_0(x,v_n)\varphi dx+\int_\Omega h_1(x)\varphi dx=0, \tag{4.15}$$

and

$$(1-t_n)\varepsilon\int_\Omega v_n\varphi_1 dx+\int_\Omega g_0(x,u_n)\varphi_1 dx=\int_\Omega h_2(x)\varphi_1 dx. \tag{4.16}$$

We have, $w_n=(w_{1,n},w_{2,n})=(\frac{u_n}{\|(u_n,v_n)\|_{\tilde U}},\frac{v_n}{\|(u_n,v_n)\|_{\tilde U}})\to w=(w_1,w_2)$ in $\tilde U$ and $w=km$ with a

$$\text{k}=\begin{pmatrix}k_1\\k_2\end{pmatrix},\ \text{m}=\begin{pmatrix}\varphi & 0\\ 0 & \varphi_1\end{pmatrix},$$

such that $k_i\neq 0$. Assume that $k_i>0$, using Theorem 1.2.26 and Remark 1.2.18 in [16], we get (at least for a subsequence) $(u_n,v_n)\to\infty$. Passing to the limit in (4.15) and (4.16) with $t_n\to -1$ and the Lebesgue dominated convergence theorem, we obtain

1. $\displaystyle\int_\Omega h_1(x)\varphi(x)dx\leqslant-\int_\Omega\beta_1^+\varphi(x)dx,$

2. $\displaystyle\beta_2^+\leqslant\int_\Omega h_2(x)\varphi_1(x)dx,$

this contradicts one of the inequalities in (4.4.1). A similar argument applies when $k<0$, leading to a contradiction with the other inequality in (4.4.1).

To prove that conditions (1) and (2) are also necessary, we take (u_0,v_0) as a weak solution of (4.11), i.e. for any $(\tilde w_1,\tilde w_2)\in\tilde U$, we have

$$\begin{cases} C(n,s)\displaystyle\iint_{\mathbb{R}^{2n}}\frac{(u_0(x)-u_0(y))(\tilde w_1(x)-\tilde w_1(y))}{|x-y|^{n+2s}}dydx=\lambda\int_\Omega u_0\tilde w_1dx+\int_\Omega f_0(x,v_0)\tilde w_1dx+\int_\Omega h_1(x)\tilde w_1dx,\\ C(n,s)\displaystyle\iint_{\mathbb{R}^{2n}}\frac{(v_0(x)-v_0(y))(\tilde w_2(x)-\tilde w_2(y))}{|x-y|^{n+2s}}dydx=\lambda_1\int_\Omega v_0\tilde w_2dx+\int_\Omega g_0(x,u_0)\tilde w_2dx-\int_\Omega h_2(x)\tilde w_2dx.\end{cases}$$

Set $(\tilde w_1,\tilde w_2)=(\varphi,\varphi_1)$. Then

$$\begin{cases} \int_\Omega f_0(x,v_0)\varphi dx + \int_\Omega h_1(x)\varphi dx = 0, \\ \\ \int_\Omega g_0(x,u_0)\varphi_1 dx = \int_\Omega h_2(x)\varphi_1 dx. \end{cases}$$

From conditions (4.2) and (4.3) respectively, we obtain

$$-\int_\Omega \beta_1^+\varphi(x)dx \leqslant \int_\Omega h_1(x)\varphi(x)dx \leqslant -\int_\Omega \beta_1^-\varphi(x)dx.$$

$$\beta_2^- \leqslant \int_\Omega h_2(x)\varphi_1(x)dx \leqslant \beta_2^+.$$

4.5 A third class of systems

In the last system, we put $a = \lambda^1$ and $b = \lambda^2$ where $\lambda^1, \lambda^2 \in sp((-\Delta)^s)\backslash\{\lambda_1\}$ and

$$\begin{cases} f_1(x) = h_1(x), \\ f_2(x) = h_2(x), \end{cases}$$

we obtain the following system

$$\begin{cases} (-\Delta)^s u(x) = \lambda^1 u + f_0(x,v) + h_1(x) \quad in \quad \Omega, \\ (-\Delta)^s v(x) = \lambda^2 v + g_0(x,u) + h_2(x) \quad in \quad \Omega, \\ u = v = 0 \quad on \quad \mathbb{R}^n\backslash\Omega. \end{cases} \tag{4.17}$$

Proposition 4.5.1. *Assume that (4.2) and (4.3) are fulfilled. Let $\theta_i \in N_\lambda^i (i = 1,2)$. Then the problem (4.17) has at least one weak solution if and only if*

$$-\int_\Omega \beta_i^+\theta_i(x)dx \leqslant \int_\Omega h_i(x)\theta_i(x)dx \leqslant -\int_\Omega \beta_i^-\theta_i(x)dx,$$

with $h_i \in L^2(\Omega)$ for $i = 1,2$.

Proof. Let (u_0, v_0) be a weak solution of (4.17), i.e. for any $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \tilde{U}$, we have

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}} \dfrac{(u_0(x)-u_0(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}}dydx = \lambda^1\int_\Omega u_0\tilde{w}_1 dx + \int_\Omega f_0(x,v_0)\tilde{w}_1 dx + \int_\Omega h_1(x)\tilde{w}_1 dx, \\ \\ C(n,s)\iint_{\mathbb{R}^{2n}} \dfrac{(v_0(x)-v_0(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}}dydx = \lambda^2\int_\Omega v_0\tilde{w}_2 dx + \int_\Omega g_0(x,u_0)\tilde{w}_2 dx + \int_\Omega h_2(x)\tilde{w}_2 dx. \end{cases}$$

The choice of $(\tilde{w_1}, \tilde{w_2}) = (\theta_1, \theta_2)$, gives

$$\begin{cases} \int_\Omega f_0(x, v_0)\theta_1 dx + \int_\Omega h_1(x)\theta_1 dx = 0, \\ \\ \int_\Omega g_0(x, u_0)\theta_2 dx + \int_\Omega h_2(x)\theta_2 dx = 0. \end{cases}$$

From conditions (4.2) and (4.3) respectively, we obtain

$$-\int_\Omega \beta_i^+ \theta_i(x) dx \leqslant \int_\Omega h_i(x)\theta_i(x) dx \leqslant -\int_\Omega \beta_i^- \theta_i(x) dx.$$

4.5.1 A priori bounds for solutions

Lemma 4.5.1. *There exists $R_2 > 0$ such that*

$$\begin{cases} \|(u, v)\|_{\tilde{U}} = R_2, \quad \forall t \in [0, 1], \forall (u, v) \in \tilde{U} \\ T(t, u, v) \neq 0. \end{cases}$$

Proof. Let $\varepsilon > 0$ so small such that $]\lambda^i, \lambda^i + \varepsilon] \cap sp((-\Delta)^s) = \varnothing$ $(i = 1, 2)$. Recall the proof of the previous lemma with

$$\mathrm{B}= \begin{pmatrix} \lambda^1 & 0 \\ 0 & \lambda^2 \end{pmatrix}.$$

Taking $(\tilde{w_1}, \tilde{w_2}) = (\theta_1, \theta_2)$.

By the same method, we get

$$-\int_\Omega \beta_i^- \theta_i(x) dx \leqslant \int_\Omega h_i(x)\theta_i(x) dx \leqslant -\int_\Omega \beta_i^+ \theta_i(x) dx, \; i = 1, 2$$

This contradicts Proposition 4.5.1.

Now we present the proof of Theorem 4.2.1, which give the existence of solutions for system (4.1).

Proof. (of Theorem 4.2.1). Let

$$B(0, R) = \{(u, v) \in \tilde{U}, \|(u, v)\|_{\tilde{U}} < R\}.$$

We can define the topological degree and by invariance of the homotopy, we have

$$d(T(t,.,.), B(0,R), 0) = d(T(0,.,.), B(0,R), 0),$$

which is equivalent to

$$d(I - BA - tS, B(0,R), 0) = d(I - BA, B(0,R), 0).$$

In particular if $t = 1$:

$$d(I - BA - S, B(0,R), 0) = d(I - BA, B(0,R), 0).$$

On the other hand, for $t = 0$ the system has a unique solution. Then

$$d(I - BA(.) - S(.), B(0,R), 0) = d(I - BA(.), B(0,R), 0) \neq 0,$$

whereupon ther exists $(u,v) \in B(0,R)$ such that :

$$T(1,u,v) = 0 \ ,$$

Hence, the system (4.1) admits a weak solution.

Conclusion

In conclusion, our thesis addressed the existence of solutions for nonlinear elliptic problems involving fractional derivatives. We start our thesis by applying fixed point theory to establish the existence and uniqueness of solution for nonlinear fractional elliptic problem. Then, we study the nonlinear fractional elliptic system. These system were studied using fixed point theory and for nonlinear elliptic systems in resonance and nonresonance, we use the Leray-Schauder topological degree.

Finally, this line of research can be extended to more general boundary value problems involving fractional derivatives, including the development of suitable numerical methods. Additionally, potential applications of these models in image processing may be explored.

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