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Thesis

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Fixed point technique and its applications to functional differential equations with delay

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For long time fixed point theory has been used extensively in nonlinear differential equations to investigate the properties of solutions both quantitatively and qualitatively. One of the most significant qualitative features of differential equations is the determination of their periodicity. The main objective of this thesis is to investigate the existence of bounded positive periodic solutions for a class of first-order nonlinear neutral differential equations with iterative terms and impulsive effects. In the process, we convert the given equation into an equivalent integral equation. Then, we construct a suitable mapping and use a hybrid technique that combines Schauder's fixed point approach, Green's functions method, and some functional analysis tools to achieve our purpose. Easily verifiable sufficient conditions are established. The equation considered is more general and incorporates as special cases various problems widely studied in different literature. In our applications, two examples are analyzed to demonstrate the real power of our results.

Keywords: Fixed-point theorems, impulses, delay equations, integral equations, existence of the solutions, neutral equations, periodic solutions.

Depuis longtemps, la théorie du point fixe est largement utilisée dans l'étude des équations différentielles non linéaires afin d'analyser les propriétés de leurs solutions, aussi bien d'un point de vue quantitatif que qualitatif. Parmi les caractéristiques qualitatives les plus importantes des équations différentielles figure la détermination de leur périodicité. L'objectif principal de cette thèse est d'examiner l'existence de solutions périodiques bornées et positives pour une classe d'équations différentielles neutres non linéaires du premier ordre comportant des termes itératifs et des effets impulsifs. Dans cette optique, nous transformons l'équation considérée en une équation intégrale équivalente. Nous construisons ensuite une application appropriée et mettons en œuvre une technique hybride combinant l'approche du point fixe de Schauder, la méthode des fonctions de Green et certains outils de l'analyse fonctionnelle afin d'atteindre notre objectif. Des conditions suffisantes, aisément vérifiables, sont établies. L'équation étudiée est plus générale et inclut, comme cas particuliers, divers problèmes largement analysés dans la littérature. Enfin, deux exemples sont présentés afin de démontrer la portée et l'efficacité de nos résultats.

Mots-clés: Théorèmes de point fixe, impulsions, équations à retards, équations

tions intégrales, existence de la solutions, équations neutres, solution périodiques.

لقد استُخدمت نظرية النقطة الثابتة منذ زمن طويل بشكل واسع في دراسة المعادلات التفاضلية غير الخطية من أجل تحليل خصائص حلولها، سواء من الناحية الكمية أو النوعية. وتُعدّ من أهم الخصائص النوعية للمعادلات التفاضلية مسألة تحديد دوريتها. يمثّل الهدف الرئيسي من هذه الأطروحة في بحث وجود حلول دورية موجبة ومحدودة لمجموعة من المعادلات التفاضلية المحايدة غير الخطية من الرتبة الأولى، التي تتضمن حدوداً تكرارية وتأثيرات اندفاعية. وفي هذا الإطار، نقوم بتحويل المعادلة المدروسة إلى معادلة تكاملية مكافئة. ثم نبني تطبيقاً مناسباً ونستخدم تقنية هجينة تجمع بين مقارنة النقطة الثابتة لشاوذر، وطريقة دوال غرين، وبعض أدوات التحليل الدالي من أجل بلوغ هدفنا. كما نُثبت شروطاً كافية ويسهل التحقق منها. وتُعدّ المعادلة التي تمت دراستها أكثر شمولية، إذ تضمّ كحالات خاصة عدة مسائل جرى تناولها بشكل واسع في الأدبيات. وأخيراً، تم تقديم مثالين للتطبيق من أجل إبراز مدى قوة وفعالية نتائجنا.

الكلمات المفتاحية: مبرهنات النقطة الثابتة، النبضات، المعادلات ذات التأخير، المعادلات التكاملية، وجود الحلول، المعادلات المحايدة، الحلول الدورية.

I dedicate this work

To my dear husband, who has been my constant support
and motivation through every moment of this journey.
To my beloved parents, whose heartfelt prayers have always lit my
path and given me strength.
To my brothers and sisters, who never stopped believing in me.
To my precious son, the light of my life and my greatest inspiration.
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Acronyms

Abbreviation	Meaning
ODEs	Ordinary differential equations
DDEs	Delayed differential equations
NDEs	Neutral differential equations
SDDEs	State-dependent delay differential equations
IDEs	Iterative differential equations

Sets and numbers

\mathbb{R} : the set of real numbers

\mathbb{R}^* : the set of all non-zero real numbers

\mathbb{R}^n : n -dimensional real Euclidean space

\mathbb{N}^* : the set of all non-zero natural Numbers

$[a, b]$: the interval of numbers between a and b , including a and b

$[a, +\infty)$: left-closed and right-unbounded interval

$\mathcal{C}(E) := \mathcal{C}(E, E)$: is the space of continuous functions from E into itself

$\mathcal{C}([a, b], [0, +\infty))$: is the space of continuous functions from $[a, b]$ into $[0, +\infty)$

τ : a delay

ω : a period

$\{u : Y \subset \mathbb{R} \rightarrow \mathbb{R}, u \in C((t_k, t_{k+1}), \mathbb{R}), (t_k, t_{k+1}) \subset Y,$

$PC(Y, \mathbb{R}) = \{t_k\}_{k=1}^j$, is a sequence of discontinuities,

at each discontinuity point t_k

the one sided limits $u(t_k^+)$ and $u(t_k^-)$ exist, with $u(t_k^-) = u(t_k)$

$P_\omega = \{u \in PC(Y, \mathbb{R}) : u(t + \omega) = u(t)\}$

$P_\omega(n_1, n_2, n_3) = \{u \in P_\omega : n_1 \leq u \leq n_2, |u(\alpha_2) - u(\alpha_1)| \leq n_3|\alpha_2 - \alpha_1|, \forall \alpha_2, \alpha_1 \in \mathbb{R}\}$

Functions

$|\cdot|$: absolute value

$\|\cdot\|_E$: a norm on E

$\|f\|_\infty$: the uniform norm defined by $\sup_{u \in D} |f(u)|$

$u^{[n]}(t)$: the composition of the function $u(t)$ with itself n times or the n^{th} iterate of the function $u(t)$

$\sum_{i=1}^n$: the summation from index $i = 1$ to $i = n$

\approx : approximately equal

$u'(t)$: the first derivative of the function $u(t)$ with respect to t

$\lim_{h \rightarrow 0^+}$: limit as h approaches 0^+

$\Delta u(t_k)$: the impulse at moment t_k

$u(t_k^+)$: the right-hand limits of $u(t)$ at the impulsive moment t_k

$u(t_k^-)$: the left-hand limits of $u(t)$ at the impulsive moment t_k

sup : the supremum

max : the maximum

min : the minimum

$G(t, x)$: the Green's function

When the other notations appear, they are explained

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The development and analysis of mathematical models of dynamic processes have been of great importance in bettering our understanding of nature and the world around us. Since the invention of differential and integral calculus by many authors in past decades, differential equations have aided the investigation of a wide variety of problems in the physical, biological, and social sciences. However, the situation is quite different in many physical phenomena that have a sudden change in their states, which can be used to model many practical problems that arise in the areas of mechanical systems with impact, population dynamics, mathematical economy, chemical technology, electric technology, chemistry, engineering, ecology, control theory, medicine, and so on (see [1], [2], [4], [8], [24], [37], [45], [73], [53]). The theory of impulsive differential equations is a new and important branch of differential equations. The first papers in this theory are related to the names of A. D. Mishkis and V. D. Milman in 1960 and 1963, (see [48]). The last decades have seen major developments in this theory (for example, [32, 70, 71] and references related). This dual focus not only enriches the theoretical study of dynamical systems but also supports the development of effective models for real-world problems.

On the other hand, the realization that differential equations, in general, and indeed impulsive delay differential equations are very important models for describing the true state of several real-life processes/phenomena may have been the tonic. The simultaneous presence of time delays and impulsive effects, as modeled by delay differential equations with impulse effects, introduces significant analytical challenges while also providing a more accurate representation of many real-world systems. These hybrid dynamical systems exhibit complex behaviors that necessitate a thorough investigation of both their qualitative properties and quantitative characteristics—such as stability, boundedness, the existence of periodic or almost periodic solutions, convergence rates, and sensitivity to initial conditions.

The great interest in applications during these and later years has certainly contributed significantly to the rapid development of the theory of impulsive delay differential equations, and recently these theories have been elaborated to a considerable extent. The analysis of these kinds of equations plays a critical role in advancing both the theoretical foundations and applied methodologies of modern dynamical systems theory. Numerous advanced and sophisticated analytical techniques have been developed and employed to explore the qualitative properties of their solutions. Among the most widely adopted analytical tools are Lyapunov functionals, fixed point theory, and comparison principles, which are commonly employed to investigate these properties, particularly in the context of stability, periodicity, and oscillation of solutions, see for instance [35, 60, 61, 72, 69, 73]. Collectively, these methods provide a rigorous framework for analyzing the dynamics of impulsive delay equations and contribute significantly to the broader theory of functional differential equations.

Fixed point theory is essential in mathematics, significantly influencing

theoretical and applied fields. It began with Henri Poincaré in 1886, who introduced the foundational concepts. Maurice Fréchet [30] expanded on this in 1906 by proving a fixed point theorem through the concept of metric spaces. Stefan Banach [7] further advanced the field in 1922 with the Banach contraction principle, establishing a fundamental criterion for the existence of fixed points in complete metric spaces. Since then, researchers have extended these principles to discover new fixed point theorems, broadening their applications. This theory is vital in various scientific disciplines, such as economics for equilibrium models, engineering for control systems, biology for population dynamics, and computer science for algorithm development. Nowadays, fixed point theory stands as one of the most powerful and efficient tools in modern mathematics, with a wide range of applications not only in nonlinear analysis but also across numerous other mathematical disciplines, where it serves as a foundational framework for solving complex problems and establishing the existence and uniqueness of solutions in various contexts, (see [74], [77], [79]). Although a substantial number of definite results have been discovered, a few questions lying at the heart of the remain open. Some of these questions are merely tantalizing while others suggest substantial new directions of research.

In many mathematical settings, one often seeks to find a solution $u^* \in D$ to a given class of equations, where D denotes the domain and the solution is required to satisfy certain desired properties. In many cases these properties might be difficult to express. Sometimes, even given a solution u^* there is no obvious way to verify that this is actually a solution. A common way to overcome these difficulties is to express the solutions as fixed points of an easily described function. More formally we can define a function $f : D \rightarrow D$ such that the solution point $u^* \in D$ satisfies $f(u^*) = u^*$. This way of expressing

solutions is very common in a lot of scientific areas. Because of the importance of such a representation, a lot of interesting and important questions arise. Given a function $f : D \rightarrow D$, is there any fixed point? Is there a procedure that converges to this fixed point? These questions, can be handled from important theorems, some of the most known ones are : Schauder's fixed point Theorem, Banach's fixed point Theorem, Krasnosel'skii's fixed point Theorem and others [63]. Thus, when we deal with the solvability of a certain functional equation (impulsive delay differential equation, fractional differential equation, integral equation, matrix equation, etc), we are reformulating the problem in terms of investigating the existence and uniqueness of a fixed point of a mapping.

This thesis is segregated into four chapters, each chapter is further scattered into different subsections depending on the variety of the subject matter. Examples are discussed in each chapter to illustrate our main results.

The first chapter provides a brief review of some general concepts and fundamental results that serve as essential tools throughout the thesis. Among these are Leibniz's formula and the Ascoli-Arzelà theorem, which play a foundational role in various proofs and arguments presented later. The core of the chapter is dedicated to the exposition of well-known fixed point theorems, specifically those formulated by Banach, Schauder, and Krasnosel'skii, which will be instrumental in establishing the main existence results in subsequent chapters.

Second chapter is divided into two main sections. The first section begins with a general introduction and recalls some basic notions concerning impulsive differential equations, including key definitions and fundamental properties of their solutions. These preliminary concepts lay the groundwork

for the theoretical results developed later. This section also presents several illustrative examples drawn from real-life contexts, especially in biology and medicine, supported by appropriate graphical representations. The second section turns to the application of fixed point theorems in the context of impulsive differential equations with parameter. The focus is on proving the existence of single and multiple positive periodic solutions.

In the **third chapter**, we present fundamental definitions and key properties that form the basis for the study of delay differential equations (DDEs). Several important observations are provided in the form of remarks to enhance clarity and offer deeper theoretical insight. To emphasize the distinguishing features of DDEs, we include illustrative examples that demonstrate their structure and dynamics, along with comparisons to classical ordinary differential equations (ODEs). These comparisons serve to underline the additional complexities introduced particularly by the presence of memory effects. In addition, we introduce the method of steps, a standard technique employed to derive explicit solutions of delay differential equations.

Chapter Four presents the main contributions of this thesis. It is structured into three distinct sections, each serving a specific purpose. In the initial section, we delve into a comprehensive introduction, setting the stage by conducting a thorough literature survey on the key topics related to iterative differential equations. The second section includes a review of the results published in [27], which deal with the following first-order iterative differential equations with impulsive effects.

$$\begin{aligned} u'(t) &= -a(t)f(t, u(t))u(t) + g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\ &\quad + \frac{d}{dt}h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)), t \neq t_k, k \in \mathbb{N}^*, \\ u(t_k^+) - u(t_k^-) &= I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)), t = t_k, k \in \mathbb{N}^*, \end{aligned}$$

where $u^{[0]}(t) = t$, $u^{[1]}(t) = u(t)$, $u^{[2]}(t) = u(u(t))$, $u^{[3]}(t) = u^{[2]}(y(t))$, ...,

$u^{[n]}(t) = u^{[n-1]}(u(t))$ are the iteratives of the state u , $a \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$, and $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^+)$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^+)$, $h \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

The impulsive condition is expressed as

$$\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(t_k, u(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)),$$

where $u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of $u(t)$ at the impulsive moment t_k respectively and $I_k(\cdot, \cdot) \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^+)$, $k \in \mathbb{N}^*$.

By using Schauder's fixed point theorem as well as some properties of a suitable Green function, a set of sufficient conditions for the existence of at least bounded positive periodic solutions is obtained. Some well-known results are improved and generalized. More precisely, our model contains as a particular case the one analyzed in Bouakkaz et al. [12], and Zhao and Liu [79]. In third section, two examples are exhibited to show the efficiency and application of our findings, which are completely new and enrich the existing literature.

CHAPTER 1

Primary concepts

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This chapter presents the essential concepts and basic elements necessary for the development of the thesis. It is structured in two sections. The first provides a brief overview of the key tools of functional analysis, including fundamental definitions, lemmas, and results that will be frequently referenced in subsequent chapters. The second section is devoted to important theorems of fixed-point theory, which will be extensively used in the later stages of this thesis.

1.1 Tools and fundamental concepts

In this section, we review fundamental concepts from functional analysis, including essential tools, definitions, lemmas, and associated results, and examples that are pertinent to our discussion.

1.1.1 Banach space

Let E be an arbitrary nonempty set.

Definition 1.1 [20] Let E be a vector space. A metric, or distance function, on E is an application

$$d : E \times E \rightarrow \mathbb{R},$$

that satisfies:

- i) $d(u, v) \geq 0$ for all $u, v \in E$, and $d(u, v) = 0$ if and only if $u = v$;
- ii) $d(u, v) = d(v, u)$, for all $u, v \in E$, (symmetry);
- ii) $d(u, v) \leq d(u, z) + d(z, v)$, for all $u, v, z \in E$, (triangle inequality).

A metric space (E, d) is a set E equipped with a metric d .

Example 1.1 The set of real numbers \mathbb{R} with the distance defined by $d(u, v) = |u - v|$ is a metric space. The set of complex numbers \mathbb{C} with the distance function $d(z, w) = |z - w|$ is also a metric space.

Definition 1.2 [20] Let (E, d) be a metric space and let $\{u_n\}$ be a sequence of points in E .

- We say that $\{u_n\}$ is a Cauchy sequence if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } d(u_p, u_q) < \varepsilon, \forall p, q \geq n_0.$$

- A sequence $\{u_n\}$ is called a convergent if there exists $u \in E$ such that

$\lim_{n \rightarrow \infty} d(u_n, u) = 0$. This means that

$$\forall \varepsilon > 0, \exists n^* \in \mathbb{N} \text{ such that } d(u_n, u) < \varepsilon, \forall n \geq n^*.$$

- In metric space (E, d) , if sequence $\{u_n\}$ is not convergent, it is termed divergent.

Definition 1.3 [20] A metric space (E, d) is said to be complete if every Cauchy sequence in E converges to a point u in E .

Proposition 1.1 *Every convergent sequence in a metric space is a Cauchy sequence. But the converse is not true in general.*

Definition 1.4 [20] Let E be vector space. A map $\|\cdot\| : E \rightarrow \mathbb{R}$ is called a norm on E if

- a) $\|u\| \geq 0$, for all $u \in E$, (nonnegative);
- b) $\|u\| = 0$ if and only if $u = 0$;
- c) $\|\rho u\| = |\rho| \|u\|$, for all $u \in E$ and $\rho \in \mathbb{k}$, ($\mathbb{k} = \mathbb{R}$ or \mathbb{C}) (homogeneous);
- d) $\|u + v\| \leq \|u\| + \|v\|$, for all $u, v \in E$ (triangle inequality);

The couple $(E, \|\cdot\|)$ is called a normed space.

As an illustration, we consider the following examples:

Example 1.2 The set of real numbers \mathbb{R} with the absolute value norm $\|u\| = |u|$ is a one dimensional real normed linear space. More generally, \mathbb{R}^n , where $n = 1, 2, \dots$, is an n -dimensional linear space. A famous example of norms on \mathbb{R}^n includes:

- 1) $\|u\|_1 = \sum_{i=1}^{i=n} |u_i|$, for all $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$;
- 2) $\|u\|_2 = \left[\sum_{i=1}^n |u_i|^2 \right]^{\frac{1}{2}}$, for all $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$;
- 3) $\|u\|_\infty = \max_{1 \leq i \leq n} |u_i|$, for all $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$. More generally the ℓ_p norm is defined by $\|u\|_p = \left[\sum_{i=1}^n |u_i|^p \right]^{\frac{1}{p}}$ for $p \geq 1$.

Example 1.3 For the space of continuous functions on a bounded interval $[a, b]$, denoted $\mathcal{C}([a, b], \mathbb{R})$, there are famous norms commonly used:

- 1) $\|f\|_1 = \int_a^b |f(t)| dt$;
- 2) $\|f\|_2 = \left(\int_a^b (f(t))^2 dt \right)^{\frac{1}{2}}$;
- 3) $\|f\|_\infty = \sup_{a \leq t \leq b} |f(t)|$.

Among these norms, the supremum norm is particularly important, as it is widely used to define the standard metric on function spaces a concept we will use later.

Theorem 1.1 [20] Let $(E, \|\cdot\|)$ be a normed vector space. The distance associated with the norm $\|\cdot\|$ on E is:

$$\begin{aligned} d & : E \times E \rightarrow \mathbb{R}^+ \\ (u, v) & \longmapsto d(u, v) = \|u - v\|, \end{aligned}$$

which is called the metric induced by the norm. Then, the normed space is a metric space. However, the converse is not always true.

Definition 1.5 [20] Every complete normed space is called a Banach space, i.e. if every Cauchy sequence in E converges to a point u in E .

Example 1.4 For $\omega > 0$, the set

$$C_\omega = \{u \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : u(t + \omega) = u(t)\},$$

of all continuous and periodic functions with period ω endowed with the supremum norm

$$\|u\| = \sup_{t \in \mathbb{R}} |u(t)| = \sup_{t \in [0, \omega]} |u(t)|, \text{ for } u \in C_\omega,$$

is a Banach space.

Corollary 1.1 [22] *Every finite-dimensional normed vector space is complete (Banach space).*

1.1.2 Arzelà Ascoli theorem

The Arzelà-Ascoli Theorem was proven in the late 19th century by Italian mathematicians Cesare Arzelà and Giulio Ascoli. The theorem was first proved as a tool for studying the properties of continuous functions on compact intervals, but it has become one of the fundamental results in modern mathematics. It performs a vital role in understanding the convergence and compactness of function sequences in various areas, including functional analysis, differential equations, and approximation theory.

Proposition 1.2 [53] *Let Ω be a subset of a complete metric space E . Then we have the following:*

- i) Ω is compact if and only if Ω is closed and totally bounded.*
- ii) $\overline{\Omega}$ is compact if and only if Ω is totally bounded.*

Definition 1.6 [49] Let (E, d) is metric space, $\Omega \subseteq E$ is compact if every sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \Omega$ has a convergent subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ and also its limit in Ω . Also Ω is said to be relatively compact if its closure $\overline{\Omega} \subseteq E$ be compact.

Proposition 1.3 *Let Ω be a closed subset of a complete metric space. Then Ω is compact if and only if it is relatively compact.*

Corollary 1.2 (General Heine-Borel Theorem) [49] *Let E be a finite dimensional normed space and let $\Omega \subset E$. Then Ω is compact if and only if Ω is closed and bounded.*

Definition 1.7 [49] The mapping $\mathcal{A}: (E, d_E) \rightarrow (F, d_F)$ is said to be a continuous at $u_0 \in E$ if for $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$d_F(\mathcal{A}(u), \mathcal{A}(u_0)) < \varepsilon, \text{ as } d_E(u, u_0) < \delta.$$

If the function \mathcal{A} is continuous at every point of E , it is said to be continuous on E . Also it is said to be uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall u, w \in E, d_E(u, w) < \delta \text{ yields } d_F(\mathcal{A}(u), \mathcal{A}(w)) < \varepsilon.$$

Definition 1.8 [59] Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed spaces, and let $\mathcal{A}: E \rightarrow F$ be an operator is compact if only if

$$\forall \Omega \subset E, (\Omega \text{ bounded}) \Rightarrow (\overline{\mathcal{A}(\Omega)} \text{ compact}).$$

Equivalently, \mathcal{A} is compact if and only if for every bounded sequence $(u_n)_{n \in \mathbb{N}}$ in E , the sequence $(\mathcal{A}u_n)_{n \in \mathbb{N}}$ has a convergent subsequence in F .

Remark 1.1 If \mathcal{A} is linear, then \mathcal{A} is completely continuous $\Leftrightarrow \mathcal{A}$ compact.

Corollary 1.3 *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed spaces, and let $\mathcal{A}: E \rightarrow F$ be a continuous operator. If $\Omega \subset E$ is a compact set, then $\mathcal{A}(\Omega) \subset F$ is a compact. More specifically, if the image $\mathcal{A}(\Omega)$ is bounded and every sequence in $\mathcal{A}(\Omega)$ has a convergent subsequence, we conclude that:*

$$\mathcal{A}(\Omega) \text{ is compact in } F.$$

Let Ω be a compact subset of a normed vector space $(X, \|\cdot\|)$ and let $\mathcal{C}(\Omega, \mathbb{R})$ be the normed vector space of real valued continuous functions on Ω with the sup-norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|.$$

Let \mathcal{F} be a collection of functions in $\mathcal{C}(\Omega, \mathbb{R})$.

Definition 1.9 [14] The collection \mathcal{F} is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \Omega$ with $\|x - y\| < \delta$, we have $|f(x) - f(y)| < \varepsilon$, for every $f \in \mathcal{F}$.

Definition 1.10 [14] The collection \mathcal{F} is said to be uniformly bounded if there exists $M > 0$ such that

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)| \leq M, \text{ for all } f \in \mathcal{F}.$$

One of the most important results that arises when dealing with theory of operators is the Arzela-Ascoli Theorem. The following theorem is very important and shall use to prove the main results later.

Theorem 1.2 [20] *If \mathcal{F} is a collection of uniformly bounded and equicontinuous functions in $\mathcal{C}(\Omega, \mathbb{R})$, then \mathcal{F} is **relatively compact** in $\mathcal{C}(\Omega, \mathbb{R})$.*

Corollary 1.4 *A collection $\mathcal{F} \subset \mathcal{C}(\Omega, \mathbb{R})$ be **compact** if and only if it be closed, equi-continuous and uniformly bounded.*

Remark 1.2 For $F > 0$, and $\ell > 0$, the set

$$P = \{u \in \mathcal{C}([a, b], \mathbb{R}), \|u\|_\infty < \ell, |u(a_2) - u(a_1)| \leq F |a_2 - a_1|, \forall a_1, a_2 \in \mathbb{R}\},$$

is compact. This result is a direct consequence of Arzelà-Ascoli Theorem.

The next result follows as an immediate consequence of Arzelà-Ascoli Theorem, which offers an alternative means to prove that every collection $\mathcal{F} \subset \mathcal{C}^1([a, b], \mathbb{R})$ is relatively compact, when working with specific sequences in \mathcal{F} as below.

Corollary 1.5 *Let ℓ_1 and ℓ_2 be two strictly positive real numbers. The subset $\mathcal{F} \subset \mathcal{C}^1([a, b], \mathbb{R})$ that satisfy*

$$\forall f_n \in \mathcal{F}, |f_n(t)| \leq \ell_1, \text{ and } |f'_n(t)| \leq \ell_2, \text{ for all } t \in [a, b], n \in \mathbb{N},$$

is relatively compact in $\mathcal{C}([a, b], \mathbb{R})$.

Indeed, the Mean value theorem shows that, for all n and $u, v \in [a, b]$, there exists $c \in (u, v)$, such that

$$|f_n(u) - f_n(v)| \leq |f'_n(c)| \cdot |u - v|.$$

It follows $|f_n(u) - f_n(v)| \leq \ell_2 |u - v|$. Hence for any $\epsilon > 0$, and $\widehat{\delta} = \frac{\epsilon}{\ell_2}$, we have,

$$\forall u, v \in [a, b], |u - v| < \widehat{\delta} \Rightarrow |f_n(u) - f_n(v)| < \epsilon, \quad n = 1, 2, \dots$$

This proves that the family \mathcal{F} is equicontinuous. Moreover, since there exists $\ell_1 > 0$, $|f_n(t)| \leq \ell_1$ for all $t \in [a, b]$, $n \in \mathbb{N}$, which implies that \mathcal{F} uniformly bounded. Finally, since \mathcal{F} is uniformly bounded and equicontinuous, the Arzelà-Ascoli theorem ensures that \mathcal{F} is relatively compact.

1.1.3 Differentiation under the integral sign

First of all, we should point out that if an integral like $\int_a^b h(u, t) dt$ is a function u , which is independent of the integration variable (denoted t), so we can ask about its u -derivative, provided $h(u, t)$ suitable conditions hold. The rule is the u -derivative of the integrals $h(u, t)$ is the integral of the u -derivative of $h(u, t)$:

$$\frac{d}{du} \int_a^b h(u, t) dt = \int_a^b \frac{\partial h}{\partial u}(u, t) dt. \quad (1.1)$$

This operation, called **differentiating under the integral sign**, was first used by **Leibniz**, one of the inventors of calculus. It can be applied as a technique for solving integrals, popularized by Richard Feynman in his book [28]. The rule allows differentiation and integration operations to be interchanged under certain conditions.

Theorem 1.3 (*Fundamental Theorem of Calculus*) [17] Let $\widehat{h} \in \mathcal{C}([a, b], \mathbb{R})$.

Define the function $H : [a, b] \rightarrow \mathbb{R}$ by

$$H(u) = \int_a^u \widehat{h}(t) dt, \quad u \in [a, b],$$

then H is continuous on $[a, b]$, differentiable on (a, b) , and $H'(u) = \widehat{h}(u)$ for all $u \in [a, b]$.

Remark 1.3 Let $\widehat{h} : [a, b] \rightarrow \mathbb{R}$ be a piecewise continuous function. Define the function

$$H(u) = \int_a^u \widehat{h}(t) dt,$$

then the following properties hold:

- 1) H is well defined and continuous on $[a, b]$.
- 2) H is differentiable at all points $u_0 \in [a, b]$, where \widehat{h} is continuous, and $H'(u_0) = \widehat{h}(u_0)$.
- 3) If \widehat{h} is not continuous in u_0 , then H may not be differentiable at u_0 .

Let us now, recall Leibniz's Rule which is used to differentiate an integral with respect to a parameter:

Theorem 1.4 Leibniz integral rule [22] Let $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable functions, and let $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous, and suppose $\frac{\partial h}{\partial u}$ also exists and is continuous. Then the function

$$g(u) = \int_{f_1(u)}^{f_2(u)} h(u, t) dt,$$

is differentiable, and its derivative is given by:

$$\frac{d}{du} \int_{f_1(u)}^{f_2(u)} h(u, t) dt = h(u, f_2(u)) \frac{d}{du} f_2(u) - h(u, f_1(u)) \frac{d}{du} f_1(u) + \int_{f_1(u)}^{f_2(u)} \frac{\partial h}{\partial u}(u, t) dt.$$

In the special case where the functions f_1 and f_2 are constants $f_1(u) = a$ and $f_2(u) = b$ with the values that do not depend on u , this simplifies to equality given by (1.1).

1.2 Basic fixed point theorems

It has been 100 years since Banach published the first metric fixed point theorem. Today, metric fixed point theory is one of the hottest research topics not only in applied mathematics but also in nonlinear functional analysis and topology. In qualitative sciences, almost all questions can be transferred into the metric fixed point problem

$$\mathcal{A}(u) = u,$$

for some suitable function \mathcal{A} . The main goal of the fixed point theory is to understand the existence of a fixed point first. After guaranteeing the existence of a fixed point, they look for its uniqueness. For instance, the existence and uniqueness of certain differential equations, under suitable conditions, can be reduced to the existence and uniqueness of a fixed point. Based on these discussions and observations, we underline the fact that metric fixed point theory is one of the most useful and effective mathematical analysis tools to construct the existence and uniqueness of various problems modeled by nonlinear relations. This is why the metric fixed point theory has been so densely studied in qualitative sciences, in particular, computer science, engineering, physics, etc. Metric fixed points have begun to take shape and sharply develop in different senses. With the invention of several new contraction mappings, various new abstract spaces have been discovered and studied. In connection to the advances in metric fixed point theory, several new results (dealing with fixed point theory) in many disciplines have been recorded. We recommended that anyone interested in fixed point theorems browse through the book [63] by Smart where additional results and many more references can be found.

In this section, we introduce some fixed point theorems, one of which we

will use to prove the intended results. But before that, let us introduce the definition of a fixed point.

Definition 1.11 [63] (**Fixed point**) Let $(E, \|\cdot\|_E)$ be a Banach space, and let $\mathcal{A} : E \rightarrow E$ be a mapping. We call fixed point of \mathcal{A} any point $u \in E$ such that $\mathcal{A}(u) = u$. This is equivalent to saying that the equation $\mathcal{A}(u) - u = 0$ has a solution.

Example 1.5 In mathematics, a fixed point of a function is a value x such that $\mathcal{A}(x) = x$. The existence of fixed points depends on the nature of the function. For instance, the identity function $\mathcal{A}(u) = u$ has every point as a fixed point. In contrast, the fixed point does not exist in translation mapping the form $\mathcal{A}(u) = u + \alpha$, where $\alpha \neq 0$, since the equation $u + \alpha = u$ has no solution.

1.2.1 Banach's fixed point theorem

In 1922, in Banach's PhD thesis, a remarkable fixed point theorem well known as the Banach contraction principle was initiated. The Banach contraction mapping principle is one of the most important theorems in classical functional analysis. Owing to its simplicity, practical utility, and ability to ensure both existence and uniqueness of solutions in a wide range of metric spaces, the Banach Contraction Principle is widely recognized as a central tool in various areas of mathematical analysis and applied sciences. It is perhaps one of the most widely used fixed point theorems in all analysis. The beauty of the Banach contraction principle is that it requires only completeness and the contraction condition on the underlying metric space and mapping, respectively. With these conditions it provides the following assertions:

- The existence and uniqueness of a fixed point.
- The method to compute the approximate fixed points.
- The error estimates for approximate fixed points.

To date, a substantial body of research has been devoted to extensions and generalizations of the Banach Contraction Principle, with numerous papers published in the mathematical literature (see [7, 16, 55]).

Now, we have the definition of Lipschitz mapping.

Definition 1.12 [63] Let (E, d_E) and (F, d_F) be metric spaces. A map $\mathcal{A} : E \rightarrow F$ is said to be Lipschitz continuous if there exists a constant $m \geq 0$, with

$$d_F(\mathcal{A}u, \mathcal{A}v) \leq md_E(u, v), \text{ for all } u, v \in E.$$

In this case, the constant m is referred to as the Lipschitz constant of \mathcal{A} .

\mathcal{A} is said to be a contraction mapping if, in the above inequality, $m < 1$, and it is nonexpansive if $m = 1$. Finally, \mathcal{A} is said to be a contractive mapping if, for all $u, v \in E$ and $u \neq v$,

$$d_F(\mathcal{A}u, \mathcal{A}v) < d_E(u, v).$$

Based on the relationships described above, we can summarize the following implications as follows:

$$\text{contraction} \Rightarrow \text{contractive} \Rightarrow \text{nonexpansive} \Rightarrow \text{Lipschitz},$$

and all such mappings are continuous.

Stefan Banach [7] stated the following theorem in 1922, which is also known as the “**Banach contraction principle**”.

Theorem 1.5 [63] Let (E, d_E) be a complete metric space, and let $\mathcal{A} : E \rightarrow E$ be a contraction mapping with $0 \leq m < 1$. Then, \mathcal{A} has a unique fixed

point $u^* \in E$, i.e., $\mathcal{A}(u^*) = u^*$. Moreover, any recurrent sequence defined by $u_0 \in E$ and $\mathcal{A}(u_{n+1}) = u_n$ converges to u_0 and satisfies the following bound:

$$d_E(u_n, u^*) \leq \frac{m^n}{1-m} d_E(u_0, u_1),$$

where d_E represents the distance in the metric space, and m is the contraction constant with $0 \leq m < 1$.

Theorem 1.6 [63] *If Ω is a closed subset of a Banach space $(E, \|\cdot\|_E)$ and $\mathcal{A} : \Omega \rightarrow \Omega$ is a contraction, then \mathcal{A} admits a unique fixed point in Ω .*

Here's an example that illustrates how contraction mappings in incomplete metric spaces can fail to have fixed points.

Example 1.6 Consider the space (E, d) where $E =]0, 1]$ and $d(u, v) = |u - v|$, $u, v \in E$, the usual Euclidean distance. This is a metric space but it is not complete (because $\frac{1}{n}$ is a Cauchy sequence in (E, d) , but $\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \notin]0, 1]$). Define the map $\mathcal{A} :]0, 1] \rightarrow]0, 1]$ as follows:

$$\mathcal{A}(u) = \frac{u}{5},$$

for all $u \in]0, 1]$. It is easy to verify that \mathcal{A} is a contraction. For any $u, v \in]0, 1]$, we have

$$|\mathcal{A}(u) - \mathcal{A}(v)| = \left| \frac{u}{5} - \frac{v}{5} \right| = \frac{1}{5} |u - v|.$$

This confirms that \mathcal{A} is a contraction mapping with constant $m = \frac{1}{5}$. Looking for fixed points of \mathcal{A} must satisfy $\mathcal{A}(u^*) = u^*$. That is: $u^* = 0$. However, the space (E, d_E) is incomplete, and the fixed point $u^* = 0$ lies outside $E =]0, 1]$. Despite being a contraction, \mathcal{A} cannot have a fixed point within the space (E, d_E) because it is incomplete. This example illustrates how contraction mappings in incomplete metric spaces may not have fixed points.

Banach's fixed point theorem can be generalized in the following way.

Theorem 1.7 [6.3] *Let $(E, \|\cdot\|_E)$ be a Banach space and $\mathcal{A} : E \rightarrow E$ be a mapping such that for some integer p ,*

$$\mathcal{A}^m = \underbrace{\mathcal{A} \circ \mathcal{A} \circ \dots \circ \mathcal{A}}_{p \text{ times}},$$

\mathcal{A}^m is a contraction mapping. Then \mathcal{A} has a unique fixed point.

Proof. Let $u \in E$ be the fixed point of \mathcal{A}^m i.e., $\mathcal{A}^m(u) = u$. Then,

$$\mathcal{A}(u) = \mathcal{A}(\mathcal{A}^m(u)) = \mathcal{A}^{m+1}(u) = \mathcal{A}^m(\mathcal{A}(u)),$$

which implies that $\mathcal{A}(u)$ is a fixed point of \mathcal{A}^m . By uniqueness, we have $u = \mathcal{A}(u)$, that is u is also a fixed point of \mathcal{A} . To prove the uniqueness of the fixed point of \mathcal{A} , let v be some other fixed point of \mathcal{A} . Then, $v = \mathcal{A}(v) = \mathcal{A}(\mathcal{A}(v)) = \dots = \mathcal{A}^m(v)$, that is, v is also a fixed point of \mathcal{A}^m . Hence $v = u$.

■

1.2.2 Schauder's fixed point theorem

Brouwer's Fixed Point Theorem is a fundamental result in topology, originally proven by L.E.J. Brouwer in 1912. The core idea of Brouwer's Fixed Point Theorem is the existence of stationary points under continuous transformations. In the context of finite-dimensional spaces, specifically in \mathbb{R}^n , the theorem guarantees that any continuous mapping of a compact convex set in \mathbb{R}^n to itself has at least one fixed point. Due to its elegance and generality, the Theorem continues to have a profound impact on diverse fields of mathematics, including topology and mathematical analysis. It underscores the concept of invariance in mathematical transformations and has inspired

the development of many other fixed-point theorems and related concepts in mathematics and its applications.

In 1930, Schauder [52], first established an infinite-dimensional generalization of the Brouwer fixed point theorem, now well known as the Schauder fixed point theorem. It asserts that any continuous function from a compact convex subset of a Banach space into itself has at least one fixed point, thereby extending the applicability of Brouwer's original result to broader classes of spaces, including those of infinite dimension, and facilitating its use in more general and complex mathematical contexts. Today, this theorem is one of the most powerful tools in dealing with nonlinear problems in analysis, and in particular, it has played a major role in the development of fixed point theory and the theory of differential equations, underscoring its significance in modern mathematical research.

Definition 1.13 [22] A set Ω in E is said to be convex if, for all $u, v \in \Omega$ and $t \in (0, 1)$, then the line segment $(1 - t)u + tv \in \Omega$.

We start by formulating Brouwer fixed point theorem as follows:

Theorem 1.8 [63] (*Brouwer's fixed point theorem*) Let Ω be a nonempty closed bounded convex subset of a finite-dimensional normed space $(E, \|\cdot\|_E)$ and suppose \mathcal{A} is a continuous function from Ω to itself. Then there exists an element $u^* \in \Omega$ such that $\mathcal{A}(u^*) = u^*$, i.e., u^* is a fixed point of \mathcal{A} .

We now turn to Schauder's fixed point theorem, which generalizes Brouwer's fixed point theorem to infinite-dimensional Banach spaces

Theorem 1.9 [63] Let \mathcal{M} be a non-empty compact convex subset of a Banach space $(E, \|\cdot\|_E)$, and let $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ be a continuous function. Then \mathcal{A} has at least one fixed point in \mathcal{M} .

There is also a version of the Schauder fixed-point where compactness of the image is required (in infinite dimensions) which can be expressed as:

Theorem 1.10 [63] *Let \mathcal{M} be a non-empty, bounded, closed, convex subset of a Banach space $(E, \|\cdot\|_E)$, and let $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ be a compact and continuous mapping. Then \mathcal{A} possesses a fixed point in \mathcal{M} .*

Note that it does not follow from Schauder fixed point theorem that the fixed point is unique.

Example 1.7 The given example shows a case where Schauder's Fixed Point Theorem is applicable but the Contraction Mapping Principle is not. We can use the nonlinear integral equation mentioned below.

$$u(t) = \int_0^1 \frac{e^{\frac{1}{2}(t-s)}}{1+u(s)} ds, \quad t \in [0, 1].$$

Schauder's Fixed Point Theorem is applicable here because the associated operator is compact

$$(\mathcal{A}u)(t) = \int_0^1 \frac{e^{\frac{1}{2}(t-s)}}{1+u(s)} ds, \quad t \in [0, 1],$$

It defines bounded sets in the Banach space $\mathcal{C}([0, 1])$ (the space of continuous functions on $[0, 1]$ is convex and closed). Schauder's theorem ensures the existence of a fixed point for this operator. However, the Contraction Mapping Principle is not applicable because the operator is not a contraction. The kernel $\frac{e^{\frac{1}{2}(t-s)}}{1+u(s)}$ does not satisfy a Lipschitz requirement with a constant lower than 1, which is necessary for contraction.

1.2.3 Krasnoselskii's fixed point theorem

In 1955, Krasnoselskii [39] established the most famous fixed-point solving Equation

$$\mathcal{A}_1 u + \mathcal{A}_2 u = u, \quad u \in \mathcal{M},$$

where \mathcal{M} is a closed convex set (a ball of radius r in practice), of a functional Banach space. His result states that if \mathcal{A}_2 is a contraction, $\mathcal{A}_1(\mathcal{M})$ is relatively compact and \mathcal{M} is left invariant by the operators sum $\mathcal{A}_1 + \mathcal{A}_2$, then the above Equation admits a solution in \mathcal{M} . The theorem was motivated by an observation that the inversion of a perturbed differential operator may yield the sum of a compact and contraction operator. His theorem actually combines both the Banach contraction principle and the Schauder fixed point theorem, and is useful in establishing existence theorems for perturbed operator equations. The importance of this kind of results relies on, among other things, its many applications in nonlinear analysis. For instance, it has a wide range of applications to nonlinear integral equations of mixed type for proving the existence of solutions. Accordingly, Krasnoselskii offered the following fixed point theorem.

Theorem 1.11 [63] *Let \mathcal{M} be a closed convex non-empty subset of a Banach space $(E, \|\cdot\|_E)$ and let $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{M} \rightarrow E$ be two mappings such that*

- i) $\mathcal{A}_1 u + \mathcal{A}_2 v \in \mathcal{M}, \forall u, v \in \mathcal{M},$*
- ii) \mathcal{A}_1 is continuous and compact,*
- iii) \mathcal{A}_2 is a contraction mapping.*

Then there exists $v \in \mathcal{M}$ such that

$$\mathcal{A}_1 v + \mathcal{A}_2 v = v.$$

Proof. Assume that the mapping \mathcal{A}_2 satisfies the hypothesis (iii) of the theorem. Thus, there exists a $c \in [0, 1)$ such that

$$\|\mathcal{A}_2 u - \mathcal{A}_2 v\| \leq c \|u - v\|, \quad u, v \in \mathcal{M}.$$

This yields

$$\begin{aligned}
 \|(I - \mathcal{A}_2)u - (I - \mathcal{A}_2)v\| &= \|(u - v) - (\mathcal{A}_2u - \mathcal{A}_2v)\| \\
 &\leq \|u - v\| + \|\mathcal{A}_2u - \mathcal{A}_2v\| \\
 &\leq \|u - v\| + c\|u - v\| \\
 &= (1 + c)\|u - v\|,
 \end{aligned}$$

and

$$\begin{aligned}
 \|(I - \mathcal{A}_2)u - (I - \mathcal{A}_2)v\| &= \|(u - v) - (\mathcal{A}_2u - \mathcal{A}_2v)\| \\
 &\geq \|u - v\| - \|\mathcal{A}_2u - \mathcal{A}_2v\| \\
 &\geq \|u - v\| - c\|u - v\| \\
 &= (1 - c)\|u - v\|.
 \end{aligned}$$

It follows that

$$(1 - c)\|u - v\| \leq \|(I - \mathcal{A}_2)u - (I - \mathcal{A}_2)v\| \leq (1 + c)\|u - v\|.$$

This inequality shows that $(I - \mathcal{A}_2) : \mathcal{M} \rightarrow (I - \mathcal{A}_2)\mathcal{M}$ is continuous and one to one. Thus, $(I - \mathcal{A}_2)^{-1}$ exist and is continuous. Let us the set $\widehat{F} := (I - \mathcal{A}_2)^{-1}\mathcal{A}_1$. It is clear that \widehat{F} is compact mapping, because \widehat{F} is a composition of a continuous mapping with a compact. Finally under Schauder's fixed point theorem yields the conclusion of the theorem, \widehat{F} has a fixed point, i.e.

$$\exists z^* \in \mathcal{M} \text{ such that } (I - \mathcal{A}_2)^{-1}\mathcal{A}_1z^* = z^*.$$

This is equivalent to $z^* = \mathcal{A}_1z^* + \mathcal{A}_2z^*$. ■

Remark 1.4 It should be noted that when $\mathcal{A}_1 = 0$, the theorem reduces to Banach's fixed-point theorem. Conversely, when $\mathcal{A}_2 = 0$, the theorem coincides with Schauder's fixed-point theorem.

Now, we present an application of Krasnoselskii Theorem. It illustrates the proof of the existence of the solution to a certain type of mixed nonlinear integral functional equations.

Example 1.8 The purpose of the present example is to illustrate the application of the Krasnoselskii Fixed Point Theorem in studying the existence of solutions to the following nonlinear functional integral equations.

$$u(t) = f(t, u(\alpha(t))) + g(t, u(\beta(t))) \int_0^{\varphi(t)} h(t, s, u(\gamma(s))) ds. \quad (1.2)$$

We note that to apply Krasnoselskii's fixed point theorem we need to define a Banach space of continuous functions on the interval $[0, a]$ (noted, $\mathcal{C}([0, a])$), equipped with the supremum norm

$$\|u\| = \sup_{t \in [0, a]} |u(t)|,$$

a closed convex subset $B_{r_0} \subset \mathcal{C}([0, a])$, where B_{r_0} is the closed ball of radius r_0 centered at the origin, and construct two mappings, \mathcal{A}_1 is compact and \mathcal{A}_2 is a contraction (see below). In view of this idea, we write equation (1.2) as follows:

$$(\mathcal{T}u)(t) = (\mathcal{A}_1u)(t) + (\mathcal{A}_2u)(t),$$

where both of \mathcal{A}_1 and \mathcal{A}_2 are mappings from B_{r_0} to $\mathcal{C}([0, a])$, and these mappings are defined as follows:

$$(\mathcal{A}_1u)(t) = g(t, u(\beta(t))) \int_0^{\varphi(t)} h(t, s, u(\gamma(s))) ds,$$

and

$$(\mathcal{A}_2u)(t) = f(t, u(\alpha(t))).$$

We point out that establishing the existence of a fixed point for the operator

\mathcal{T} is equivalent to showing the existence of a solution to the proposed equation (1.2). Before stating our main result related to the existence of solutions of equation (1.2), we suppose the conditions below hold:

S_1) The functions $\alpha, \beta : [0, a] \rightarrow [0, a]$, $\varphi : [0, a] \rightarrow \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous.

S_2) The functions $f, g : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and for all $t \in [0, a]$ and $u, v \in \mathbb{R}$ there exist positive constants d_1, d_2 such that

$$\begin{aligned} |f(t, u) - f(t, v)| &\leq d_1 |u - v|, \\ |g(t, u) - g(t, v)| &\leq d_2 |u - v|. \end{aligned}$$

As a consequence of condition (S_1) there is positive constant T such that

$$\varphi(t) \leq T, \quad \forall t \in [0, a].$$

Under the Lipschitz condition (S_2), for all $t \in [0, a]$, there exist positive constants f^*, g^* such that

$$\begin{aligned} |f(t, 0)| &\leq f^*, \\ |g(t, 0)| &\leq g^*. \end{aligned}$$

S_3) The function $v : [0, a] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Moreover, for all $t \in [0, a]$, $s \in [0, T]$ and $u \in \mathbb{R}$ there exist a function $h^* : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is nondecreasing on \mathbb{R}_+ such that

$$|h(t, s, u)| \leq h^*(|x|).$$

S_4) There exists a positive solution r_0 that satisfies the inequality

$$d_1 r_0 + f^* + T(d_2 r_0 + g^*)h^*(r_0) \leq r_0.$$

For proving the compactness of \mathcal{A}_1 . We need to verify the following two

points: \mathcal{A}_1 is continuous and \mathcal{A}_1 maps any bounded subset of B_{r_0} into a relatively compact subset of B_{r_0} . We first show that \mathcal{A}_1 is continuous in the supremum norm. For $\epsilon > 0$ and any $u, v \in B_{r_0}$ such that $\|u - v\| \leq \epsilon$, we have

$$\begin{aligned}
 & |(\mathcal{A}_1 u)(t) - (\mathcal{A}_1 v)(t)| \\
 \leq & \left| g(t, u(\beta(t))) \int_0^{\varphi(t)} h(t, s, u(\gamma(s))) ds - g(t, v(\beta(t))) \int_0^{\varphi(t)} h(t, s, v(\gamma(s))) ds \right| \\
 & + (|g(t, v(\beta(t))) - g(t, 0)| + |g(t, 0)|) \int_0^{\varphi(t)} |h(t, s, u(\gamma(s))) - h(t, s, v(\gamma(s)))| ds \\
 \leq & d_2 |u(\beta(t)) - v(\beta(t))| Th^*(r_0) + (d_2 |v(\beta(t))| + g^*) T \chi_{v_\epsilon}([0, a], \epsilon) \\
 \leq & T d_2 \|u - v\| h^*(r_0) + T (d_2 \|v\| + g^*) \chi_{v_\epsilon}([0, a], \epsilon) \\
 \leq & T (d_2 h^*(r_0) \epsilon + (d_2 r_0 + g^*) \chi_{v_\epsilon}([0, a], \epsilon)),
 \end{aligned}$$

where

$$\begin{aligned}
 & \chi_{v_\epsilon}([0, a], \epsilon) \\
 = & \sup \{ |h(t, s, u) - h(t, s, v)| : t \in [0, a], s \in [0, T], u, v \in [-r_0, r_0], |u - v| \leq \epsilon \}.
 \end{aligned}$$

Due to the uniform continuity of h on compact sets $[0, a] \times [0, T] \times [-r_0, r_0]$, it follows that $\chi_{v_\epsilon}([0, a], \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Then operator \mathcal{A}_1 is continuous on B_{r_0} under above assumptions.

To show that the map \mathcal{A}_1 is completely continuous, we will show that $\mathcal{A}_1(B_{r_0})$ is relatively compact in $\mathcal{C}([0, a])$. Let $t_1, t_2 \in [0, a]$ with $0 \leq t_1 < t_2$. If $u \in B_{r_0}$, by $(S_1) - (S_4)$, we have

$$\begin{aligned}
 & |(\mathcal{A}_1 u)(t_1) - (\mathcal{A}_1 u)(t_2)| \\
 \leq & |g(t_1, u(\beta(t_1))) - g(t_1, u(\beta(t_2)))| \int_0^{\varphi(t_1)} |h(t_1, s, u(\gamma(s)))| ds \\
 & + |g(t_1, u(\beta(t_2))) - g(t_2, u(\beta(t_2)))| \int_0^{\varphi(t_1)} |h(t_1, s, u(\gamma(s)))| ds
 \end{aligned}$$

$$\begin{aligned}
& + |g(t_2, u(\beta(t_2)))| \left| \int_0^{\varphi(t_2)} h(t_1, s, u(\gamma(s))) ds \right. \\
& + \left. \left| \int_{\varphi(t_2)}^{\varphi(t_1)} h(t_1, s, u(\gamma(s))) ds - \int_{\varphi(t_2)}^{\varphi(t_1)} h(t_2, s, u(\gamma(s))) ds \right| \right. \\
\leq & (d_2 |u(\beta(t_1)) - u(\beta(t_2))| + \chi_g([0, a], \epsilon)) \int_0^{\varphi(t_1)} |h(t_1, s, u(\gamma(s)))| ds \\
& + (|g(t_2, u(\beta(t_2))) - g(t_2, 0)| + |g(t_2, 0)|) \\
& \times \left(\int_0^{\varphi(t_2)} |h(t_1, s, u(\gamma(s))) - h(t_2, s, u(\gamma(s)))| ds \right. \\
& + \left. \left| \int_{\varphi(t_2)}^{\varphi(t_1)} h(t_1, s, u(\gamma(s))) ds \right| \right) \\
\leq & [d_2 \chi(u, \chi(\beta, \epsilon)) + \chi_g([0, a], \epsilon)] d_3 h^*(r_0) + (d_2 r_0 + g^*) \\
& \times [\chi_{v_1}([0, a], \epsilon) + \chi(\varphi, \epsilon) h^*(r_0)],
\end{aligned}$$

where

$$\chi(u_i, \epsilon) = \sup \{|u_i(t_1) - u_i(t_2)| : t_1, t_2 \in [0, a] \text{ and } |t_1 - t_2| \leq \epsilon\},$$

for $i = 1, 2, 3$ such that $u_1 = u$, $u_2 = \beta$, $u_3 = \varphi$. Hence we write $\chi(\beta, \epsilon) \rightarrow 0$, $\chi(u, \epsilon) \rightarrow 0$, $\chi(\varphi, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ the from above estimate by using uniformly continuity of these functions on the set $[0, a]$. Similarly, we get $\chi_g([0, a], \epsilon)$ and $\chi_{v_1}([0, a], \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ since functions g and v are uniformly continuous on the sets $[0, a] \times [-r_0, r_0]$ and $[0, a] \times [0, T] \times [-r_0, r_0]$ respectively. On the other hand for any $u \in \mathcal{A}_1(B_{r_0})$ and all $t \in [0, a]$ we have $|u(t)| \leq r_0$. This means that the set $\mathcal{A}_1(B_{r_0})$ is uniformly bounded and uniformly equicontinuous. So $\mathcal{A}_1(B_{r_0})$ is a relatively compact subset of $\mathcal{C}([0, a])$ from Ascoli-Arzela Theorem.

We now show that \mathcal{A}_2 is a contraction. For any $u, v \in B_{r_0}$, by using (A₂),

we have

$$\begin{aligned}
 |(\mathcal{A}_2 u)(t) - (\mathcal{A}_2 v)(t)| &= |f(t, u(\alpha(t))) - f(t, v(\alpha(t)))| \\
 &\leq d_1 |u(\alpha(t)) - v(\alpha(t))| \\
 &\leq d_1 \|u - v\|,
 \end{aligned}$$

since $d_1 < 1$. So we infer that \mathcal{A}_2 is a contraction on B_{r_0} .

Finally, we check that $T(B_{r_0}) \subset B_{r_0}$. For $u \in B_{r_0}$ and any $t \in [0, a]$, we estimate

$$\begin{aligned}
 &|(\mathcal{A}_1 u)(t) + (\mathcal{A}_2 u)(t)| \\
 = &\left| f(t, u(\alpha(t))) + g(t, u(\beta(t))) \int_0^{\varphi(t)} h(t, s, u(\gamma(s))) ds \right| \\
 \leq &|f(t, u(\alpha(t))) - f(t, 0)| + |f(t, 0)| \\
 &+ (|g(t, u(\beta(t))) - g(t, 0)| + |g(t, 0)|) \\
 &\times \int_0^{\varphi(t)} |h(t, s, u(\gamma(s)))| ds \\
 \leq &d_1 r_0 + f^* + T(d_2 r_0 + g^*) h^*(r_0) \leq r_0.
 \end{aligned}$$

Then $\mathcal{A}_1(B_{r_0}) + \mathcal{A}_2(B_{r_0}) \subset B_{r_0}$. Therefore, on the set B_{r_0} , the conditions of Krasnoselskii's fixed point theorem hold. Hence, we conclude that Eq. (1.2) has at least one fixed point u in B_{r_0} such that $\mathcal{A}_1 u + \mathcal{A}_2 u = u$.

1.2.4 Guo-Krasnoselskii fixed point theorem in cones

The Guo-Krasnoselskii Fixed Point Theorem, independently developed by Mark Krasnoselskii in the 1950s and Yong Zhou Guo in the 1970s, provides a framework for determining the existence of fixed points of compact operators in cones within Banach spaces. Krasnoselskii's pioneering work laid the groundwork for fixed point theory in ordered Banach spaces, while Guo later refined the theorem to specifically address nonlinear differential equations

and boundary value problems. By leveraging the principles of expansion and compression on the boundaries of two open subsets, the theorem is particularly effective in demonstrating the existence of single or multiple positive solutions to such problems. This result extends classical fixed point theory by incorporating the structure of cones, thereby enhancing its applicability in nonlinear analysis. As such, the Guo-Krasnoselskii theorem plays a pivotal role in advancing the study of boundary value problems and other areas of nonlinear functional analysis.

Definition 1.14 (see, [29], [32]) Let E be a Banach space and let Λ be a closed, nonempty subset of E . Λ is said to be a cone if:

- i) $\alpha u + \beta v \in \Lambda$ for all $u, v \in \Lambda$ and all $\alpha, \beta \geq 0$.
- ii) $u, -u \in \Lambda$ imply $u = 0$.

Theorem 1.12 (Guo-Krasnoselskii, [33]) Let E be a Banach space, and let $\Lambda \subset E$ be a cone in E . Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$ and let

$$\mathcal{P} : \Lambda \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow \Lambda,$$

be a completely continuous operator such that either

- i) $\|\mathcal{P}u\| \leq \|u\|$ for $u \in \Lambda \cap \partial\Omega_1$ and $\|\mathcal{P}u\| \geq \|u\|$ for $u \in \Lambda \cap \partial\Omega_2$; or
- ii) $\|\mathcal{P}u\| \geq \|u\|$ for $u \in \Lambda \cap \partial\Omega_1$ and $\|\mathcal{P}u\| \leq \|u\|$ for $u \in \Lambda \cap \partial\Omega_2$.

Then \mathcal{P} has a fixed point in $\Lambda \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

CHAPTER 2

Impulsive differential equations

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In the present chapter, our goal is to introduce the basic concepts concerning this theory and present some fundamental results found in the literature. This chapter is divided in two sections. In the first one, we present an overview of impulsive differential equations, highlighting fundamental definitions and concepts. We conclude this chapter by applying some fixed point theorems to deal with the existence of single and twin periodic solutions of a class of nonlinear differential equations with impulsive effects. Most of the definitions and results from this section are based on the books [6] and [43].

2.1 Review of impulsive differential equations

Impulsive differential equation is a kind of differential equation used to describe the discontinuous development process with jump points (impulses). It has a very wide application background, such as biology, cybernetics, physics and mechanics [23], [31], [41], [47], [58], [65], [76].

2.1.1 Mathematical description of impulsive differential equations

Impulsive Differential Equations model dynamic systems that experience both continuous evolution and sudden, discontinuous state changes. These hybrid systems combine ordinary differential equations, governing smooth behavior between events, with and instantaneous state jumps called impulses. This approach is particularly valuable for modeling real-world phenomena where continuous processes are interrupted by brief but significant events, such as mechanical impacts, electrical pulses, or medical treatments.

The mathematical formulation of an impulsive differential equation typically consists of three main components. Each of these components plays a critical role in describing how the system evolves both continuously and through abrupt changes. Let's break down why each is essential:

1. Continuous Dynamics (Differential Equation): This component governs the evolution of the system's state during intervals between the impulsive events (when no impulses are present). It is expressed as an ordinary or functional differential equation that describes how the system behaves under normal (non-impulsive) conditions. Let $Y \subseteq \mathbb{R}^n$ and $\{t_k\}_{k=1}^{\infty}$ be a sequence such that $t_k < t_{k+1}$. Then, for $t \in \mathbb{R}$, $u \in Y$, we can describe such a

system. Mathematically, this is often written as

$$\frac{du}{dt} = f(t, u(t)), \quad t \neq t_k,$$

where $t \in \mathbb{R}$, $u \in Y \subseteq \mathbb{R}^n$, $f : \mathbb{R}^+ \times Y \rightarrow \mathbb{R}^n$ (Y is an open set).

2. Impulse Moments (Discontinuity Times): These are the specific times $\{t_k\}$, $k \in \mathbb{N}^*$, at which the system experiences sudden changes (impulses). Determines when impulses occur, either at fixed/variable times (time-dependent) or when the state satisfies specific conditions (state-dependent). The impulsive moments t_k , $k = 1, 2, \dots$, are solutions of the previous equation.

3. Impulse Function: The effect of an impulse on the system is captured by an impulsive function, which defines how the state changes at specific moments t_k . Describe the instantaneous change in state at t_k . Mathematically, this is represented as:

$$\Delta u(t_k) = I(t_k, u(t_k^-)), \quad k = 1, 2, \dots,$$

where

$\Delta u(t_k) = u(t_k^+) - u(t_k^-)$: represents the jump in the state at time t_k ,

$u(t_k^+) = \lim_{s \rightarrow t_k^+} u(s)$: is the right-hand limit (state after the impulse),

$u(t_k^-) = \lim_{s \rightarrow t_k^-} u(s)$: is the left-hand limit (state before the impulse),

$I_k \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \mathbb{N}^*$, is called impulsive and it is defined and continuous in the phase space of the considered impulsive system.

2.1.2 Classes of impulsive differential equations

Different types of impulsive differential equations, depending on the way of determining the moments of impulsive effects. There are a variety of types of impulsive differential equations, but we will focus on equations with fixed moments of impulse effect.

Impulsive differential equations can be divided into three classes according to the type of impulse moments [43]:

Class I : An impulsive differential equation with fixed moments and time-dependent dynamics is a system in which the state evolves continuously according to a differential equation, but undergoes instantaneous changes (impulses) at a predefined sequence of time moments $\{t_k\}$. It is written as:

$$\begin{cases} \frac{du}{dt} = f(t, u), & t \neq t_k, k \in \mathbb{N}^*, \\ \Delta u |_{t=t_k} = u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), & t = t_k. \end{cases} \quad (2.1)$$

where

$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defines the continuous dynamics,

$\{t_k\}$ is a strictly increasing sequence of fixed impulse times,

$I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the impulse function at moment t_k ,

$u(t_k^-)$ and $u(t_k^+)$ are the left and right limits of u at t_k .

The following example shows the behavior of solutions is influenced by the impulsive effect.

Example 2.1 Consider, the impulsive differential equation

$$\begin{cases} u'(t) = -5u(t), t \neq k, t > 0, k = 1, 2, \dots, \\ \Delta u = -80u, t = k \\ u(0) = 10. \end{cases} \quad (2.2)$$

Between impulses, that is, $t \in (k, k + 1)$, the solution evolves according to the ODE:

$$u'(t) = -5u(t),$$

whose solution is

$$u(t) = u(k^+) e^{-5(t-k)}, t \in (k, k + 1).$$

At each impulse time $t = k$, the jump condition implies

$$u(k^+) = u(k^-) + \Delta u = u(k^-) - 80u(k^-) = -79u.$$

We define:

$$u(k^+) = (-79)^k u_0 e^{-5k}, \text{ where } u_0 = 10.$$

The full solution, for $t \in (k, k + 1)$ is

$$u(t) = u_k e^{-5(t-k)} = 10 \times (-79)^k e^{-5t}.$$

The final expression is

$$u(t) = u_k e^{-5(t-k)} = 10 \times (-79)^{[t]} e^{-5t}, t > 0.$$

where $[t]$ denotes the greatest integer less than or equal to t . The following figure presents a numerical simulation of the solution $u(t)$ over the interval $t \in [0, 3.5]$, clearly demonstrating its piecewise exponential behavior and the discontinuities occurring at each impulse $t_k, k \in \mathbb{N}$.

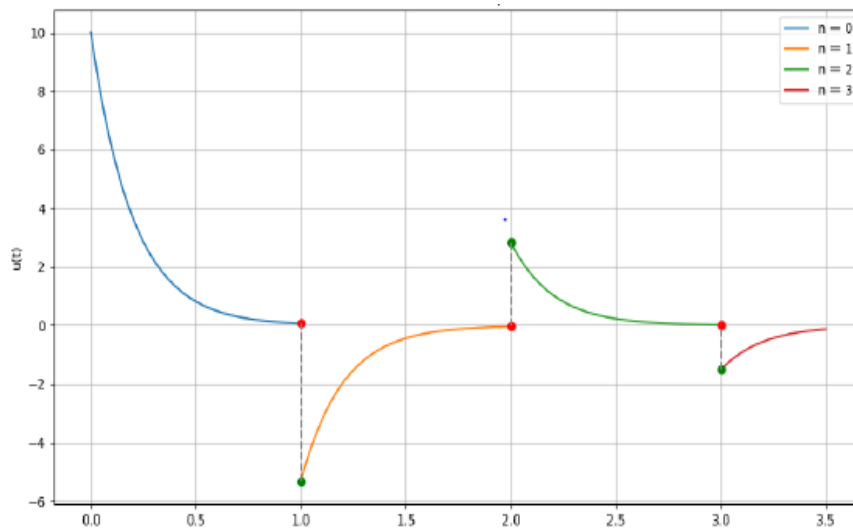


Fig 2.1 Solution of (2.2) at impulse $t = 1, 2, 3, \dots$

Example 2.2 We will consider the following example:

$$\begin{cases} u'(t) = 1 + u^2, & t \neq \frac{k\pi}{4}, k = 1, 2, \dots, \\ \Delta u = u(t^+) - u(t^-) = -1 & t = \frac{k\pi}{4} \\ u(0) = 0. \end{cases} \quad (2.3)$$

This equation describes a system that evolves continuously according to the nonlinear ODE $u'(t) = 1 + u^2$ except at discrete times $t_k = \frac{k\pi}{4}, k \in \mathbb{N}^*$, where the solution experiences a jump of size -1 . The continuous solution between impulses, for each $t \in (t_k, t_{k+1})$ is

$$u(t) = \tan(t + C_k), k \in \mathbb{N}^*$$

The constant C_k is determined by the initial value at the beginning of each interval, which resets after each impulse. Using the initial condition $u(0) = 0$, we construct the solution on successive intervals, for $t \geq 0$,

$$u(t) = \begin{cases} \tan(t), & \text{for } t \in \left[0, \frac{\pi}{4}\right[, \\ \tan\left(t - \frac{\pi}{4}\right), & \text{for } t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right[, \\ \tan\left(t - 2 \times \frac{\pi}{4}\right), & \text{for } t \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right[, \\ \cdot \\ \cdot \end{cases}$$

This is the summary for any $k \in \mathbb{N}^*$,

$$\tan\left(t - k \frac{\pi}{4}\right), \text{ for } t \in \left[\frac{k\pi}{4}, \frac{(k+1)\pi}{4}\right[.$$

This is the corresponding figure of the given model, (see Figure 2.2)

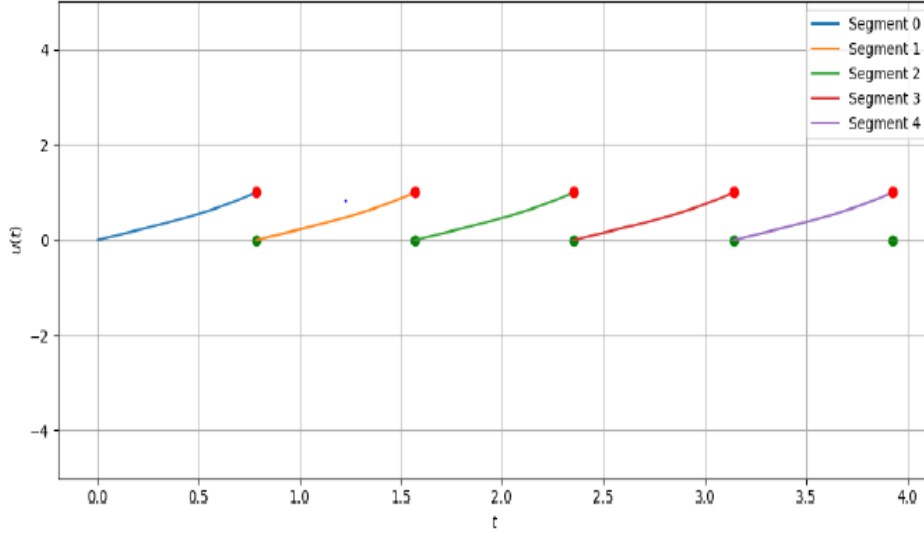


Fig 2.2 Solution of (2.3)

.Class II: System of impulsive differential equations with state-dependent impulse times

$$\begin{cases} \frac{du}{dt} = f(t, u), & t \neq t_k(u), \\ \Delta u = I_k(u), & t = t_k(u), \end{cases} \quad (2.4)$$

where

$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function describing the continuous dynamics.

$t_k(u)$ are impulse moments that depend on the state, i.e., impulses occur when the state satisfies a condition $\phi_k(u(t)) = 0$ or $t = I_k(u)$.

$I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the jump function at impulse time t_k , describing the jump (or discontinuity) in the state at the impulse times.

$$t_k(u) < t_{k+1}(u), \quad (k \in \mathbb{N}^*) \text{ and } \lim_{k \rightarrow \infty} t_k(u) = \infty.$$

This differs from classical impulsive systems where impulse times t_k are fixed. Here, impulses occur when the state hits certain manifolds or thresh-

olds, this introduces a hybrid dynamic that combines continuous flow and discrete jumps, governed by triggering conditions.

The following example illustrates the several situations that arise.

Example 2.3 Consider the following impulsive differential equation with state-dependent moments:

$$\begin{cases} u' = 0, & t \neq t_k(u), \\ \Delta u = u^2 \operatorname{sgn}(u) - u, & t = t_k(u), \quad k \in \mathbb{N}^*, \\ t_k(u) = u + 6k \text{ for } |u| < 3, \text{ where } t \geq 0, u \in \mathbb{R}, \end{cases} \quad (2.5)$$

where $t \geq 0, u \in \mathbb{R}$.

For initial conditions $|u_0| > 3$, the trajectory $u(t)$ evolves linearly without entering the region $|u| < 3$, where impulse effects are defined. Therefore, the solution does not intersect the hypersurfaces $\sigma_k = \{(t, u) : t = u + 6k\}$, and no impulses occur during the motion, (see Figure 2.3).

- For $u_0 = 4$, Time interval: $t \in [0, 50]$, Plot jump lines: $t = u + 6k$, for $k = 1, 2, 3, 4, 5$, $u(t) = u_0$, based on our description, if the blue yellow (representing the trajectory $u(t) = u_0$) doesn't intersect the red dashed lines (representing the hypersurfaces $t = u + 6k$), then your observation is correct: the trajectory never hits the jump region and thus does not trigger any jumps.

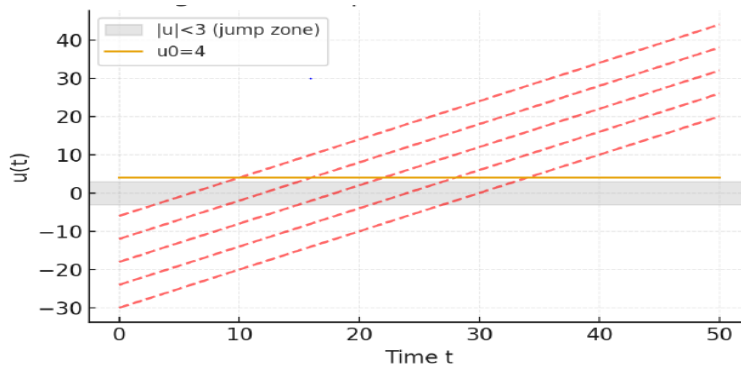


Fig 2.3 The solution of the impulsive differential equation (2.5) has no impulse effect for $|u_0| \geq 3$.

- At each impulse moment, the solution experiences a sudden jump determined by

$$\Delta u = u^2 \operatorname{sgn}(u) - u.$$

Analysis of the Jump Condition

1. For $u > 0$:

$$\Delta u = u(u - 1).$$

- If $0 < u < 1$: the jump $\Delta u < 0$, so u decreases toward zero.
- If $u = 1$: the jump $\Delta u = 0$, hence $u = 1$ is an equilibrium.
- If $u > 1$: the jump $\Delta u > 0$, so u increases further.

2. For $u < 0$:

$$\Delta u = -(u^2 + u).$$

- If $-1 < u < 0$: the jump $\Delta u > 0$, so u increases toward zero. This generates the so-called beating phenomenon.
- If $u = -1$: the jump $\Delta u = 0$, hence $u = -1$ is an equilibrium.
- If $u < -1$: the jump $\Delta u < 0$, so u decreases further (becomes more negative).

3. For $u = 0$:

$\Delta u = 0$, hence $u = 0$ is also an equilibrium.

Behavior of the Solutions

- The solution $u(t)$ that starts from a point $(0, u_0)$ with $1 < |u_0| < 3$ is subject to an impulsive effect only finitely many times. As an illustration, consider the case $u(0) = \sqrt[4]{2}$. The corresponding trajectory intersects the impulsive surface σ_0 three times. At the first intersection, occurring at $t_1 = u_0$ (since $k = 0$), the solution undergoes an impulse

$$\Delta u = u_0^2 \operatorname{sgn}(u_0) - u_0 = \sqrt{2} - \sqrt[4]{2}.$$

The new initial condition immediately after the impulse is therefore

$$u(t_1^+) = u(t_1^-) + \Delta u = \sqrt[4]{2} + \left(\sqrt{2} - \sqrt[4]{2} \right) = \sqrt{2}.$$

When the integral curve intersects the hypersurface a second time, at t_2^- , it is subject to a second impulse,

$$\Delta u = u(t_2^-)^2 \operatorname{sgn}(u(t_2^-)) - u(t_2^-) = 2 - \sqrt{2},$$

so that the solution immediately after the impulse satisfies

$$u(t_2^+) = \sqrt{2} + \left(2 - \sqrt{2} \right) = 2.$$

At the third intersection, t_3 , the solution jumps once more, giving $u(t_3^+) = 4 > 3$. Since this value lies outside the interval $(-3, 3)$, no further impulses occur and the trajectory remains constant thereafter. In this example the solution experiences exactly three impulses before leaving the impulsive region (see Figure 2.4).

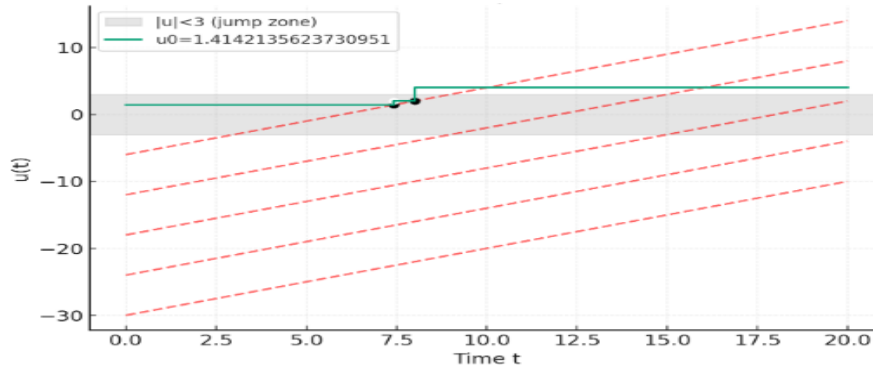


Fig 2.4 Finite number of impulses for $1 < u_0 < 3$. Example $u_0 = \sqrt{2}$: the trajectory hits the hypersurface finitely many times and then settles .

- The solution $u(t)$ that starts from a point $(0, u_0)$ with $0 < u_0 < 1$ is subject to an infinite sequence of impulses. At each impulsive moment the

state decreases but remains positive. Consequently, the solution experiences infinitely many jumps as time progresses. More precisely, if t_k denotes the sequence of impulsive moments, then

$$\lim_{k \rightarrow \infty} t_k \rightarrow \infty, \quad \lim_{k \rightarrow \infty} u(t_k) = 0.$$

Thus, although impulses occur indefinitely, the state converges monotonically to zero as $t \rightarrow \infty$. For example, let $u(0) = 0.5$. At the first impulsive moment we obtain

$$\Delta u = (0.5)^2 - 0.5 = -0.25, \quad u = 0.5 - 0.25 = 0.25.$$

At the next impulse,

$$\Delta u = (0.25)^2 - 0.25 = -0.1875, \quad u = 0.25 - 0.1875 = 0.0625.$$

Hence, the successive impulses reduce the state as

$$0.5 \rightarrow 0.25 \rightarrow 0.0625 \rightarrow \dots,$$

demonstrating concretely how the solution approaches zero through an infinite sequence of impulsive effects (see Figure 2.5).

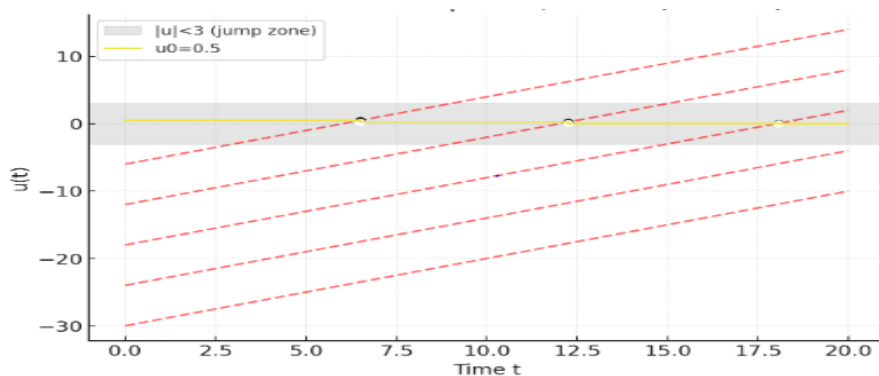


Fig 2.5 Infinite impulses with decay to zero for $0 < u_0 < 1$.

The trajectory experiences infinitely many jumps while $u(t)$ tends to zero.

- Starting at a point $(0, u_0)$ with $-1 < u_0 < 0$, the solution is subject to an impulse effect infinitely many times. At each impulsive moment the state increases while remaining negative. . Consequently, the solution experiences infinitely many jumps as time progresses, but the values of u approach zero from below. More precisely, if t_k denotes the sequence of impulsive moments, then

$$\lim_{k \rightarrow \infty} u(t_k) = 0, \lim_{k \rightarrow \infty} t_k = 6.$$

For example, with $u(0) = -0.5$, the sequence of impulses can be computed by the same method as before, and one finds that the solution increases step by step toward zero while never becoming positive. This behavior is known as the beating phenomenon (see Figure 2.6).

- In addition, the points $(0, 0)$, $(0, 1)$ and $(-1, 0)$ also intersect the impulsive surfaces σ_k infinitely many times, but they remain unchanged under the jump condition $u^2 \operatorname{sgn}(u) - u$. These points are therefore fixed points of the system (see Figure 2.6).

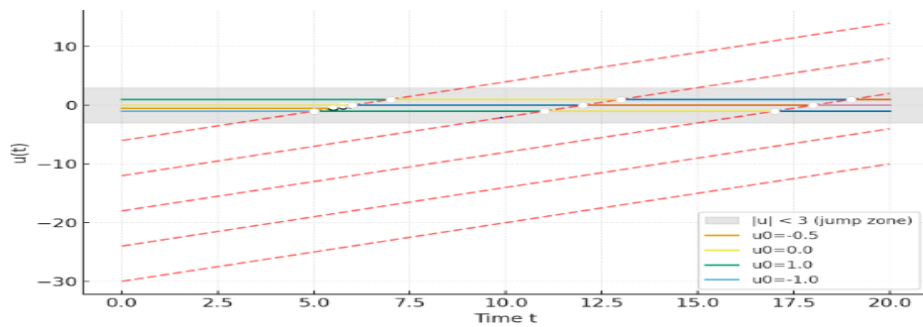


Fig 2.6 Beating and fixed points($-1 < u_0 < 0$ and $u_0 \in \{0, \pm 1\}$).
 Beating for $u_0 = -0.5$: jump times accumulate near $t = 6$. Fixed points $u_0 = 0, \pm 1$: the trajectory meets the hypersurfaces without changing value.

Class III: Autonomous impulsive differential equations describe systems whose state evolves continuously according to a differential equation, but undergoes sudden changes (impulses) when the state reaches a specific region of space, rather than at predefined times. These systems take the form:

$$\begin{aligned}\frac{du}{dt} &= f(u), \quad u \notin \sigma, \\ \Delta u &= I_k(u), \quad u \in \sigma.\end{aligned}\tag{2.6}$$

Here: $u \in \mathbb{R}^n$ is the state vector,

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth vector field describing the continuous evolution of the system,

$\sigma \subset \mathbb{R}^n$ is an $(n - 1)$ -dimensional manifold, often called the impulse surface, which defines the set of states where impulses occur.

$I_k : \sigma \rightarrow \mathbb{R}^n$ is a function that determines the magnitude and direction of the impulse at each point on the manifold.

This system is autonomous, meaning that the dynamics do not explicitly depend on time t , and state-dependent u , as the impulses are triggered by the state entering the manifold σ , not by a fixed sequence of times.

2.1.3 Space of piecewise continuous functions

Since the solutions to an impulsive differential equation are piecewise continuous, the natural functional space for these solutions is the space of piecewise continuous functions, which is defined as follows:

$$\begin{aligned}PC(Y, \mathbb{R}) = \{ & u : Y \subset \mathbb{R} \rightarrow \mathbb{R}, \quad u \in \mathcal{C}((t_k, t_{k+1}), \mathbb{R}), \quad (t_k, t_{k+1}) \subset Y, \\ & \{t_k\}_{k=1}^j, \text{ is a sequence of discontinuities,} \\ & \text{at each discontinuity point } t_k \\ & \text{the one sided limits } u(t_k^+) \text{ and } u(t_k^-) \text{ exist, with } u(t_k^-) = u(t_k)\}.\end{aligned}$$

For convenience, we denote the space of piecewise continuous functions as $PC(Y)$ instead of $PC(Y, \mathbb{R})$, under the assumption that the codomain is understood from the context.

Let us define a piecewise continuous function.

Definition 2.1 [3] Let $t = (t_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of real numbers in $Y = (t_0, \infty)$ such as

$$|t_k| \rightarrow \infty, \text{ when } k \rightarrow \infty, k \in \mathbb{N}^*.$$

We say that the function $f : Y \rightarrow \mathbb{R}$ belongs to the space $PC(Y, \mathbb{R})$ if:

- f is continuous on each interval (t_k, t_{k+1}) .
- For each k , the right hand limit

$$f(t_k^+) = \lim_{t \rightarrow t_k^+} f(t),$$

and the left hand limit

$$f(t_k^-) = \lim_{t \rightarrow t_k^-} f(t),$$

exist, and satisfy

$$f(t_k^-) = f(t_k).$$

Similarly, a function f is said to belong to the space $PC^1(Y, \mathbb{R})$ if both f and its derivative f' belong to $PC(Y, \mathbb{R})$.

Theorem 2.1 [6] Let $Y \subset \mathbb{R}$ be an interval, and let $(PC(Y, \mathbb{R}), \|\cdot\|_{PC})$ be the space of real-valued piecewise continuous functions on J , is a Banach space when equipped with the norm:

$$\|u\|_{PC} = \sup_k \|u_k\|_{\infty},$$

where each $u_k = u|_{(t_k, t_{k+1})}$, and

$$\|u_k\|_{\infty} = \sup_{t \in (t_k, t_{k+1})} |u(t)|.$$

2.1.4 Existence and uniqueness results

The existence and uniqueness of solutions to impulsive differential equations (IDEs) are an extension of the classical results from ordinary differential equations (ODEs), with added considerations due to the presence of impulses, i.e., sudden changes (discontinuities) in the state at specified times. Let $t_0 \in \mathbb{R}$, and let $t_k < t_{k+1}$ ($k \in \mathbb{N}^*$), $\lim_{k \rightarrow \infty} t_k = \infty$. Consider the impulsive differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), & t \neq t_k, \\ \Delta u |_{t=t_k} = u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), & t = t_k, \\ u(t_0^+) = u_0, \end{cases} \quad (2.7)$$

where $f : D \subset \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous (or satisfies a Lipschitz condition).
 - $I_k : \mathbb{R} \rightarrow \mathbb{R}$ is the impulse functions.

To start, we define the solution to the impulsive differential equation, specifying that it is continuous and differentiable on the subintervals between the impulse moments.

Definition 2.2 [43] A function $u : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be a solution of (2.7), if the following conditions are satisfied:

- 1) u is absolutely continuous on each (t_k, t_{k+1}) .
- 2) $\frac{du}{dt} = f(t, u(t))$ hold everywhere in $\mathbb{R} \setminus \{t_k\}$ and the jump condition is satisfied $u(t_k^+) - u(t_k^-) = I(u(t_k^-))$, $t = t_k$.
- 3) For each $k \in \mathbb{N}^*$, $u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k^-) = u(t_k)$, i.e., u is left-continuous at impulse times.

The following theorem guarantees that, for a finite impulse effect, a solution exists after the impulse.

Theorem 2.2 [43] Let $f : \mathbb{R} \times Y \rightarrow \mathbb{R}^n$ be a function that is continuous on each interval $]t_k, t_{k+1}] \times Y$, $k \in \mathbb{N}^*$, ($Y \subset \mathbb{R}^n$). Assume that for every $u \in Y$, the limit

$$\lim_{\substack{(t,v) \rightarrow (t_k^+, u) \\ t > t_k}} f(t, v).$$

exists and finite. Then, for each $(t_0, u_0) \in \mathbb{R} \times Y$, there exists $\theta > t_0$ and a solution

$$\varphi : (t_0, \theta) \rightarrow \mathbb{R}^n$$

to the initial value problem (2.7).

Lemma 2.1 [6] A function $u \in PC(Y)$ is a solution of the problem (2.7) for $t \in Y \subset \mathbb{R}$ if and only if

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s))ds + \sum_{t_0 < t_k < t} I_k(u(t_k)).$$

Proof. Suppose u is a solution of (2.7), $t \neq t_k$. Then, for $t \in [t_0, t_1]$, we get

$$\begin{aligned} u'(t) &= f(t, u(t)) \Rightarrow \int_{t_0}^t u'(s)ds = \int_{t_0}^t f(s, u(s))ds \\ \Rightarrow u(t) &= u_0 + \int_{t_0}^t f(s, u(s))ds. \end{aligned}$$

For $t_1^+ < t \leq t_2^-$, we have

$$\begin{aligned} u'(t) &= f(t, u(t)) \Rightarrow \int_{t_1+\varepsilon}^t u'(s)ds = \int_{t_1+\varepsilon}^t f(s, u(s))ds \\ \Rightarrow u(t) - u(t_1 + \varepsilon) &= \int_{t_1+\varepsilon}^t f(s, u(s))ds \\ \Rightarrow \lim_{\varepsilon \rightarrow 0^+} [u(t) - u(t_1 + \varepsilon)] &= \lim_{\varepsilon \rightarrow 0^+} \int_{t_1+\varepsilon}^t f(s, u(s))ds \\ \Rightarrow \lim_{\varepsilon \rightarrow 0^+} u(t) &= u(t_1^+) + \lim_{\varepsilon \rightarrow 0^+} \int_{t_1+\varepsilon}^t f(s, u(s))ds \end{aligned}$$

After the impulse $(u(t_1^+) - u(t_1^-) = I_1(u(t_1)))$, we get

$$u(t) = u(t_1^-) + I_1(u(t_1^-)) + \int_{t_1}^t f(s, u(s))ds.$$

Since $u(t_1^-) = u(t_1)$, so

$$u(t_1^-) = u_0 + \int_{t_0}^{t_1} f(s, u(s))ds.$$

This yields, If $t_1^+ < t < t_2^-$, the solution is

$$\begin{aligned} u(t) &= u(t_1^+) + \int_{t_1}^t f(s, u(s))ds \\ &= u_0 + \int_{t_0}^{t_1} f(s, u(s))ds + \int_{t_1}^t f(s, u(s))ds + I_1(u(t_1)). \end{aligned}$$

This means that

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s))ds + I_1(u(t_1)).$$

Similarly, for $t \in (t_k, t_{k+1}]$,

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s))ds + \sum_{t_0 < t_k < t} I(u(t_k)).$$

Thus,

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s))ds + \sum_{t_0 < t_k < t} I(u(t_k)), \quad \forall t \in J.$$

■

Theorem 2.3 [43] *Let the assumptions of Theorem 2.2 hold. Moreover, suppose that the function f is locally Lipschitz continuous with respect to u in $\mathbb{R} \times Y$. Then for any $(t, u) \in \mathbb{R} \times Y$ there exists a unique solution of the initial value problem (2.7).*

Example 2.4 We consider the following linear impulsive differential equation

$$\begin{cases} u'(t) = \alpha u(t), & \text{if } t \in]t_k, t_{k+1}], \quad k \in \mathbb{N}, \\ u(t_k^+) = u(t_k) + \beta, & \text{with } \alpha, \beta \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \quad (2.8)$$

To find the explicit solution of the system (2.8), it is necessary to solve the equation on each sub interval $]t_k, t_{k+1}]$. Let's build the solution step by step:

– if $t \in]0, t_1]$, no jump before t_1 , so

$$u(t) = u_0 e^{\alpha t}.$$

Now, we apply the jump at t_1^+ :

$$u(t_1^+) = u(t_1^-) + \beta = u_0 e^{\alpha t_1} + \beta.$$

– if $t \in]t_1, t_2]$, we get the solution of (ODE) with initial value at t_1^+ ,

$$u(t_1^+) = u_0 e^{\alpha t_1} + \beta,$$

so,

$$\begin{aligned} u(t) &= u(t_1^+) e^{\alpha(t-t_1)} \\ &= u_0 e^{\alpha t_1} + \beta e^{\alpha(t-t_1)}, t \in]t_1, t_2]. \end{aligned}$$

We apply the jump at t_2^+ :

$$u(t_2^+) = u(t_2^-) + \beta = u_0 e^{\alpha t_1} + \beta e^{\alpha(t_2-t_1)} + \beta.$$

– if $t \in]t_2, t_3]$, we solve again (EDO) with initial value at t_2^+ ,

$$u(t_2^+) = u_0 e^{\alpha t_1} + \beta e^{\alpha(t_2-t_1)} + \beta,$$

so,

$$\begin{aligned} u(t) &= u(t_2^+) e^{\alpha(t-t_2)} \\ &= u_0 e^{\alpha t_1} + \beta e^{\alpha(t_2-t_1)} + \beta e^{\alpha(t-t_2)}. \end{aligned}$$

This is the summary for any $k \in \mathbb{N}^*$,

$$u(t) = u(t_{k-1}^+) e^{\alpha(t_k-t_{k-1})} + \beta e^{\alpha(t-t_k)}, t \in]t_k, t_{k+1}].$$

In a version of the model in reference (2.8), we consider two parameter values, $\alpha = 0.02$, $\beta = 100$, ($u(0) = 10$ initial condition), and impulse times $t_1 = 0$, $t_2 = 5$, $t_3 = 10$, $t_4 = 15$, $t_5 = 20$. As a result, the solution shows piecewise exponential growth with discrete jumps (impulses) at the specified moments. A clear illustration of the resulting behavior can be found in Figure 2.7, where the discontinuities can be clearly seen at each impulse point.

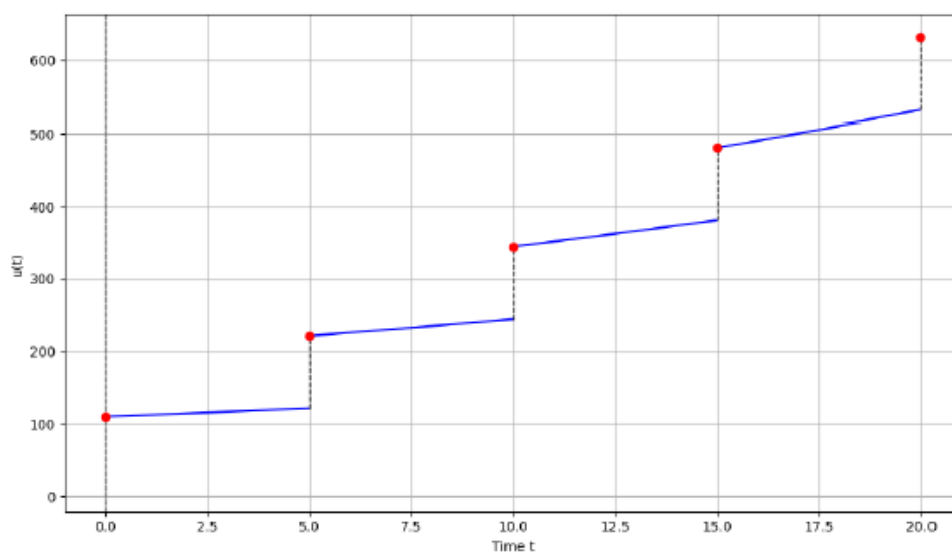


Fig 2.7 Numerical solution with parameters (2.8) for $\alpha = 0.02$, $\beta = 100$, and initial condition $u(0) = 10$.

2.1.5 A survey of examples

Here, two examples are presented to demonstrate the natural occurrence of impulsive differential equations in the study of concrete processes. In general, these equations are utilized to model systems that undergo abrupt and instantaneous changes at specific time intervals, making them a powerful tool for modeling such phenomena. In 1954, Cunningham introduced the foundational concept through a nonlinear differential equation for popula-

tion growth. Since then, IDEs have been utilized in numerous ecological modeling challenges, including fish population dynamics, pest control, and resource harvesting. In recent decades, their use has expanded in biomedical disciplines, especially in the modeling of drug dosing regimens where medication is administered at discrete time intervals. By capturing the sudden pharmacological effects of drug intake, these models are suitable for analyzing treatment strategies and optimizing therapeutic outcomes. Using these models is a good way to evaluate treatment plans and maximize therapeutic results because they accurately depict the sudden pharmacological effects of drug consumption. For appropriate literature, we can refer to [\[15\]](#), [\[19\]](#), [\[46\]](#), [\[49\]](#).

A Fish Population Model with Impulsive Differential Equations

The fish population dynamics are modeled by an impulsive differential equation that combines continuous logistic growth with instantaneous population changes at fixed times due to harvesting. Between the impulsive moments $t \neq t_k$, the population $u(t)$ evolves according to the logistic growth model:

$$\frac{du(t)}{dt} = ru(t) \left(1 - \frac{u(t)}{K}\right),$$

where r is the intrinsic growth rate and K is the environmental carrying capacity.

At the impulsive times $t = t_k$, the population experiences an instantaneous reduction due to harvesting, described by the impulsive condition:

$$u(t_k^+) = (1 - h)u(t_k^-),$$

where h represents the fraction of the population harvested, and $u(t_k^-)$, $u(t_k^+)$ denote the population immediately before and after the impulse, respectively.

These impulsive effects cause discontinuities in the population trajectory, reflected as sharp drops in population size at harvesting times t_k . This modeling approach effectively describes the real-world dynamics of fish populations under human intervention, highlighting the impact of harvesting schedules on population sustainability and recovery.

Here are the values of the parameters used in the simulation: Intrinsic growth rate ($r = 0.5$), Harvesting fraction ($h = 0.3$), ($u(0) = 50$) Initial fish population, Carrying capacity ($K = 100$), Duration of the simulation (30 units).

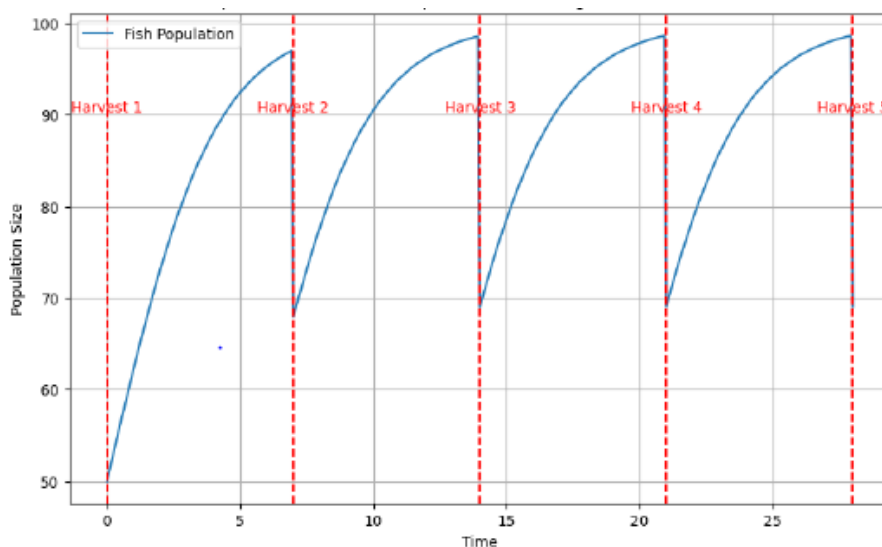


Fig 2.8 Fish population model with impulsive Harvesting.

Over 30 units of time, the above graph displays fish population dynamics combining continuous logistic growth and periodic impulsive harvesting. The population grows smoothly between harvesting times (marked by red dashed lines), approaching the environmental carrying capacity of 100. A sudden decrease in population occurs during each harvesting cycle (discontinuity), which corresponds to the removal of 30% of the population. These sharp ver-

tical drops illustrate the impulsive effect of harvesting on the population. The pattern repeats cyclically: population growth followed by harvesting-induced reductions, emphasizing how periodic interventions impact fish population stability and recovery.

Modeling Drug Dosing with Impulsive Differential Equations

Pharmacokinetic Model with Intermittent Dosing

The impulsive differential equations are essential for modeling the discontinuous nature of drug concentration changes due to discrete dosing events. These equations describe the sharp increase in drug concentration immediately following a dose and the subsequent gradual decline due to absorption and elimination processes. The alterations in drug concentration are contingent on whether a dose is administered, as represented by the following equations:

Let $R(t)$ represent the drug concentration in the body at time t . The dynamics of $R(t)$ are governed by the following system:

$$\begin{aligned} \frac{dR}{dt} &= -d_r R, & t \neq t_k, \\ \Delta R &= \begin{cases} R^i, & \text{if a dose is taken} \\ 0, & \text{if no dose is taken} \end{cases} & t = t_k. \end{aligned}$$

By the definition of an impulsive effect, the solution exhibits a discontinuity at the dosing time t_k , such that:

$$R(t_k^+) = R(t_k^-) + R^i,$$

where

d_r : is the drug elimination rate.

R^i : represents dose size.

$R(t_k^+)$: is the concentration immediately after dosing.

$R(t_k^-)$: is the concentration just before dosing.

We illustrate this with a graph of the drug concentration in the body (or bloodstream) over a 72-hour period, assuming a once - daily dose. The model is based on the following parameters: $R_i = 10$ (dose size), $d_r = 0.2$ (elimination rate), Time resolution $\Delta t = 0.1$ hour and a 24-hour dosing interval, $R(0) = 10$ units Initial dose given at time 0.

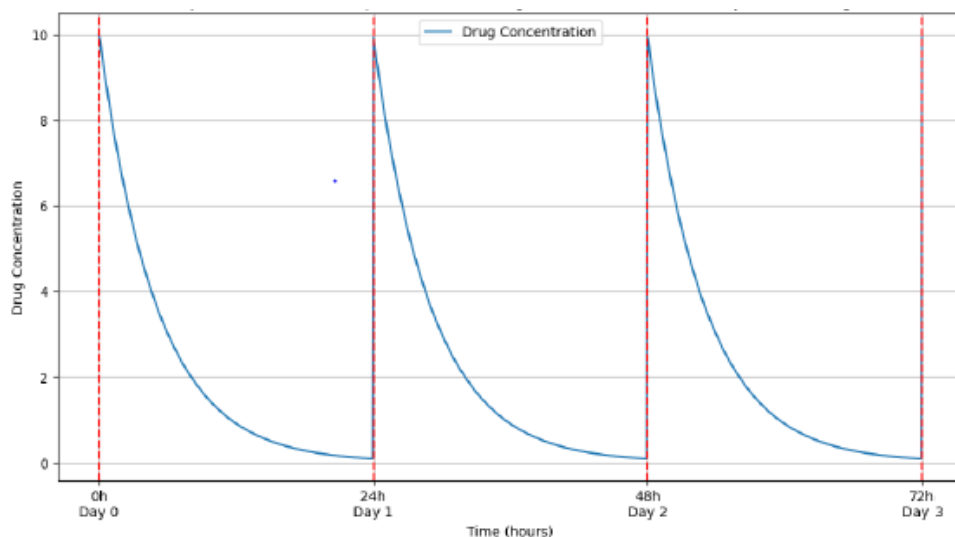


Fig 2.9 Impulsive differential equation model: drug concentration over 3 days with dosing.

From the graph above, we observe the behavior of drug concentration in the body over a 72-hour period. The drug is administered once every 24 hours, resulting in four doses at time points $0h$, $24h$, $48h$, and $72h$. At each dosing time, the dose R^i is added, causing a sudden increase (discontinuity) in the concentration. Between doses, the concentration gradually decreases due to drug elimination. When a dose is missed, no increase occurs, and the

concentration continues to decline. This pattern reflects the impulsive nature of drug administration and highlights the effect of adherence on maintaining therapeutic drug levels.

2.2 Application of fixed point methods

One of the fundamental and initial questions in the study of differential equations, whether ordinary, functional, hybrid, impulsive, or of other forms is establishing the existence and uniqueness of their solutions. This fundamental question forms the basis of any meaningful qualitative or quantitative study, as it ensures that the mathematical model under study is well formulated. Over the years, a variety of fixed point theorems and analytical techniques have been developed to address these issues, providing powerful tools for studying nonlinear phenomena in both theoretical and applied contexts.

The aim of this work is to study the existence of periodic solutions for a nonlinear impulsive differential equations with parameter λ . In the process, we transform the given model into an equivalent integral equation. Then, we construct an appropriate mapping on Banach space. Three distinct theorems are applied to the proposed equation: Schauder's fixed point theorem is employed to establish the existence of at least one periodic solution, while Krasnosel'skii's fixed point theorem index in cones is used to demonstrate the existence of twin positive periodic solutions. To ensure uniqueness, the contraction mapping principle is applied.

2.2.1 Periodic functions

Let $\omega > 0$, and let y be a function defined on a nonempty set Ω .

Definition 2.3 The function y is said to be ω -periodic function if

$$y(t + \omega) = y(t),$$

for all $t \in \Omega$.

Corollary 2.1 *The derivative of ω -periodic function is also a ω -periodic function.*

Proof. Let y be a periodic function with period ω that is differentiable at t . Then

$$\frac{d}{dt}y(t_0) = \lim_{t \rightarrow t_0} \frac{y(t) - y(t_0)}{t - t_0} = \lim_{t \rightarrow t_0} \frac{y(t + \omega) - y(t_0 + \omega)}{t - t_0} = \frac{d}{dt}y(t_0 + \omega).$$

■

Remark 2.1 Let y be ω -periodic function, then

$$\int_t^{t+\omega} y(s)ds = \int_0^\omega y(s)ds,$$

for all $t \in \mathbb{R}$ (This result is derived through a change of variables, where we define $v = s - t$).

A periodic solution in a dynamical system is a solution where the system's state repeats itself after a constant period of time.

Definition 2.4 A solution γ of an equation is called a periodic solution if there exists a positive real number ω such that $\gamma(t + \omega) = \gamma(t)$ for all $t \in \mathbb{R}$. If the solution γ has a period ω , then it also has periods 2ω , 3ω , etc. If ω is the smallest period among all the periods of γ , then this solution is called ω -periodic.

2.2.2 Existence of periodic solutions in impulsive differential equations

A typical result derived from fixed-point theory involves a series of arguments tailored to the specific structure of the equation in question. Each fixed-point theorem produces a distinct result, governed by its own conditions, with each theorem offering particular advantages in different contexts. Many different kinds of problems can be solved by means of fixed point theory. Generally, to solve a problem with fixed point theory is to find:

- 1) a set E consisting of points which would be acceptable solutions;
- 2) a mapping $\mathcal{P} : E \rightarrow E$, with the property that a fixed point solves the problem;
- 3) a fixed point theorem stating that this mapping on this set will have a fixed point.

The most interesting applications of fixed point theorems arise in connection with function spaces. The theorems then yields existence and uniqueness results for differential equations, as we will now see in the following example:

Consider the following functional differential equation with impulsive effects.

$$\begin{aligned} u'(t) &= a(t)u(t) - \lambda g(t, u(t)), t \neq t_k, \\ u(t^+) - u(t^-) &= I_k(t_k, u(t_k)), t = t_k, \end{aligned} \quad (2.9)$$

where the functions $a : \mathbb{R} \rightarrow (0, \infty)$, $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $I : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous function and $\lambda > 0$, and for a positive integer ω , the functions a, g, I are ω -periodic functions in t . We assume that there exists an integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $I_{k+q}(t_{k+q}, u(t_{k+q})) = I_k(t_k, u(t_k))$, ($k \in \mathbb{N}^*$), where $0 < t_1 < t_2 < \dots < t_q < \omega$, ($\omega > 0$ is a given positive constant).

Throughout this section, we assume that

(A₁) Functions a and $g(t, u)$ are periodic in t of period ω :

$$\begin{aligned} a(t + \omega) &= a(t) \text{ with } \int_s^t a(\zeta) d\zeta > 0, \\ g(t + \omega, u) &= g(t, u). \end{aligned}$$

(A₂) The function g is assumed Lipschitz in their components. i.e., there exist positive constant L_g such that for all $u, v \in \mathbb{R}$

$$|g(t, u) - g(t, v)| \leq L_g |u - v|.$$

(A₃) The function I_k is assumed Lipschitz in their components. i.e., there exist positive constants L_k such that for all $u, v \in \mathbb{R}$

$$|I_k(t, u) - I_k(t, v)| \leq L_k |u - v|, k \in \mathbb{N}^*.$$

Before stating the main result, we derive an equivalent integral formulation of equation (2.9). The expression for the equivalent equation of our model is provided below and can be derived as shown in [78]. Hence, we omit the details.

$$u(t) = \lambda \int_t^{t+\omega} \mathcal{G}(t, s) g(s, u(s)) ds + \sum_{t \leq t_k \leq t+\omega} \mathcal{G}(t, t_k) I_k(t_k, u(t_k)), \quad (2.10)$$

where

$$\mathcal{G}(t, s) = \frac{e^{\int_s^t a(\zeta) d\zeta}}{1 - e^{-\int_0^\omega a(\zeta) d\zeta}}.$$

It is straightforward to verify that $\mathcal{G}(t + \omega, s + \omega) = \mathcal{G}(t, s)$ and

$$\mathcal{G}_1 = \frac{e^{-\int_0^\omega a(\zeta) d\zeta}}{1 - e^{-\int_0^\omega a(\zeta) d\zeta}} \leq \mathcal{G}(t, s) \leq \frac{1}{1 - e^{-\int_0^\omega a(\zeta) d\zeta}} = \mathcal{G}_2, \text{ for all } s \in [t, t + \omega]. \quad (2.11)$$

The right hand side of the above equation (2.10) defines a mapping \mathcal{P} on a function space. We then proceed to decide which Banach space $(E, \|\cdot\|_E)$ could be a good candidate for our purposes. Then we restrict \mathcal{P} to the space $(E, \|\cdot\|_E)$ and then we try to make \mathcal{P} map $(E, \|\cdot\|_E)$ to itself. $\mathcal{P} : (E, \|\cdot\|_E)$

$\rightarrow (E, \|\cdot\|_E)$. The mapping in general must not be something as obvious as for example, integrating directly. The mapping will be built to exploit properties useful for us. The solution to the functional equation will be given by the fixed point of the mapping.

Before attempting to prove our desired results, we start by defining

$$PC(\mathbb{R}, \mathbb{R}) = \{u \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : u \in \mathcal{C}((t_k, t_{k+1}), \mathbb{R}) \text{ such that } \\ u(t_k^-), u(t_k^+) \text{ exist and } u(t_k^-) = u(t_k), k \in \mathbb{N}^*\}.$$

For $\omega > 0$, let $(P_\omega, \|\cdot\|)$ a Banach space of ω -periodic continuous functions

$$P_\omega = \{u \in PC(\mathbb{R}, \mathbb{R}), u(t + \omega) = u(t), t \in \mathbb{R}\},$$

equipped with the norm

$$\|u\| = \sup_{t \in \mathbb{R}} |u(t)| = \sup_{t \in [0, \omega]} |u(t)|, \text{ for } u \in P_\omega.$$

Based on equation (2.10), we define an operator $\mathcal{P} : P_\omega \rightarrow P_\omega$ by

$$(\mathcal{P}u)(t) := \lambda \int_t^{t+\omega} \mathcal{G}(t, s)g(s, u(s))ds + \sum_{t \leq t_k \leq t+\omega} \mathcal{G}(t, t_k)I_k(t_k, u(t_k)). \quad (2.12)$$

Let $u \in P_\omega$, from (2.12), it is easy to verify that $(\mathcal{P}u)(t)$ is continuous in (t_k, t_{k+1}) , $(\mathcal{P}u)(t_k^+)$ and $(\mathcal{P}u)(t_k^-)$ exist, and $(\mathcal{P}u)(t_k^-) = (\mathcal{P}u)(t_k)$ for each $k \in \mathbb{N}^*$ and, $(\mathcal{P}u)(t + \omega) = (\mathcal{P}u)(t)$. These conditions shape our approach as we progress through the example.

The first result is based on Banach's fixed point theorem.

Theorem 2.4 *If $(A_1) - (A_3)$ and the following estimate*

$$|\lambda| \mathcal{G}_2 \omega L_g + \mathcal{G}_2 q L_k < 1,$$

hold. Then equation (2.9) has a unique periodic solution.

Proof. We will show that $\mathcal{P} : P_\omega \rightarrow P_\omega$ is a contraction, let $u, v \in P_\omega$

$$\begin{aligned} & |(\mathcal{P}u)(t) - (\mathcal{P}v)(t)| \\ & \leq |\lambda| \mathcal{G}_2 \int_t^{t+\omega} |g(s, u(s)) - g(s, v(s))| ds + \mathcal{G}_2 \sum_{k=1}^q |I_k(t_k, u(t_k)) - I_k(t_k, v(t_k))| \\ & \leq (|\lambda| \mathcal{G}_2 \omega L_g + \mathcal{G}_2 q L_k) \|u - v\|. \end{aligned}$$

According to $|\lambda| \mathcal{G}_2 \omega L_g + \mathcal{G}_2 q L_k < 1$ and the Banach fixed point theorem, \mathcal{P} is a contraction and hence \mathcal{P} has a unique fixed point which is the unique solution of (2.10). ■

The second result relies on the Schauder fixed-point method:

Theorem 2.5 Let $\widehat{L}_g = \sup_{t \in [0, \omega]} |g(t, 0)|$ and $\widehat{L}_k = \sup_{t_k \in [0, \omega]} |I_k(t_k, 0)|$. Suppose $(A_2) - (A_3)$ hold. Let c be a positive constant satisfying the inequality

$$|\lambda| \mathcal{G}_2 \omega (L_g c + \widehat{L}_g) + \mathcal{G}_2 q (L_k c + \widehat{L}_k) \leq c. \quad (2.13)$$

Let $\Omega_\omega(c) = \{u \in P_\omega : \|u\| \leq c\}$. Then equation (2.9) has at least one periodic solution in $\Omega_\omega(c)$.

Proof. First, we show that if $u \in \Omega_\omega(c)$, we have $\|\mathcal{P}u\| \leq c$. Observe that in view of (A_2) and (A_3) we have

$$\begin{aligned} |g(t, u(t))| &= |g(t, u(t)) - g(t, 0) + g(t, 0)| \\ &< |g(t, u(t)) - g(t, 0)| + |g(t, 0)| \\ &< L_g \|u\| + |g(t, 0)|, \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} |I_k(t_k, u(t_k))| &= |I_k(t_k, u(t_k)) - I_k(t_k, 0) + I_k(t_k, 0)| \\ &< |I_k(t_k, u(t_k)) - I_k(t_k, 0)| + |I_k(t_k, 0)| \\ &< L_k \|u\| + |I_k(t_k, 0)|, k \in \mathbb{N}^*, \end{aligned} \quad (2.15)$$

we note $\widehat{L}_g = \sup_{t \in [0, \omega]} |g(t, 0)|$ and $\widehat{L}_k = \sup_{t \in [0, \omega]} |I_k(t_k, 0)|$.

Let $u \in \Omega_\omega(c)$. From, (2.14), (2.15), we get

$$\begin{aligned} |(\mathcal{P}u)(t)| &\leq |\lambda| \int_t^{t+\omega} |\mathcal{G}(t, s)| |g(s, u(s))| ds + \sum_{t \leq t_k \leq t+\omega} |\mathcal{G}(t, t_k)| |I_k(t_k, u(t_k))| \\ &\leq |\lambda| \mathcal{G}_2 \int_t^{t+\omega} (L_g \|u\| + \widehat{L}_g) ds + \mathcal{G}_2 \sum_{t \leq t_k \leq t+\omega} (L_k \|u\| + \widehat{L}_k) \\ &\leq |\lambda| \mathcal{G}_2 \omega (L_g c + \widehat{L}_g) + \mathcal{G}_2 q (L_k c + \widehat{L}_k) \leq c. \end{aligned}$$

Thus, \mathcal{P} maps $\mathcal{P}(\Omega_\omega(c))$ into itself, i.e $\mathcal{P}(\Omega_\omega(c)) \subseteq \Omega_\omega(c)$. Now, we shall prove that \mathcal{P} is continuous. Let the sequence $\{u_n\} \in \Omega_\omega(c)$ such that

$$\lim_{n \rightarrow \infty} \|u_n - u\| = 0.$$

Since $\Omega_\omega(c)$ is closed, we have $u \in \Omega_\omega(c)$. Since the function g, I are globally Lipschitz in u , by definition of \mathcal{P} we have

$$\begin{aligned} &\|\mathcal{P}u_n - \mathcal{P}u\| \\ &\leq \max_{t \in [0, \omega]} \left| \lambda \int_t^{t+\omega} \mathcal{G}(t, s) (g(s, u_n(s)) - g(s, u(s))) ds \right| \\ &\quad + \max_{t \in [0, \omega]} \left| \sum_{t \leq t_k \leq t+\omega} \mathcal{G}(t, t_k) (I_k(t_k, u_n(t_k)) - I_k(t_k, u(t_k))) \right| \\ &\leq \mathcal{G}_2 \lambda \max_{t \in [0, \omega]} \int_t^{t+\omega} |g(s, u_n(s)) - g(s, u(s))| ds \\ &\quad + \mathcal{G}_2 \max_{t \in [0, \omega]} \sum_{t \leq t_k \leq t+\omega} |I_k(t_k, u_n(t_k)) - I_k(t_k, u(t_k))|, \end{aligned}$$

which converge to 0 as $n \rightarrow \infty$ by the Lebesgue convergence theorem. This show that \mathcal{P} is continuous. To show that the map \mathcal{P} is completely continuous, we will show that $\mathcal{P}(\Omega_\omega(c))$ is relatively compact. We know that $\mathcal{P}(\Omega_\omega(c)) \subseteq \Omega_\omega(c)$, which means $\mathcal{P}(\Omega_\omega(c))$ is uniformly bounded because $\Omega_\omega(c)$ is uniformly bounded. Moreover, a direct calculation shows that

$$(\mathcal{P}'u_n)(t) = -a(t)\mathcal{P}(u_n(t)) + \lambda g(t, u_n(t)).$$

Thus, the above expression yields $(\mathcal{P}'u_n)(t) \leq \Theta$, for some positive constant Θ . Therefore the set $\mathcal{P}(\Omega_\omega(c))$ is equicontinuous, and hence by Arzela-Ascoli's theorem, it is relatively compact. By Schauder's fixed point theorem, we conclude that there exists $u \in \Omega_\omega(c)$ such that $u = \mathcal{P}(u)$. ■

The third result is derived using the Guo-Krasnoselskii fixed point theorem in cone, where

$$\forall u \in \mathbb{R}^+, \forall t \in \mathbb{R} : g(t, u(t)) \geq 0, I_k(t_k, u(t_k)) \geq 0, k \in \mathbb{N}^*.$$

Define Λ as a cone in P_ω by

$$\Lambda := \{u \in P_\omega : u(t) \geq \rho \|u\| \text{ for all } t \in [0, \omega]\},$$

where

$$\rho = \frac{\mathcal{G}_1}{\mathcal{G}_2} = e^{-\int_0^\omega a(\zeta) d\zeta}. \quad (2.16)$$

We begin by verifying that the operator \mathcal{P} is well defined on K , i.e., $\mathcal{P}(\Lambda) \subset \Lambda$, where $\lambda > 0$.

From (2.11) and (2.12), we get

$$\begin{aligned} (\mathcal{P}u)(t) &= \lambda \int_t^{t+\omega} \mathcal{G}(t, s) g(s, u(s)) ds + \sum_{t \leq t_k \leq t+\omega} \mathcal{G}(t, t_k) I_k(t_k, u(t_k)) \\ &\geq \mathcal{G}_1 \left[\lambda \int_t^{t+\omega} g(s, u(s)) ds + \sum_{t \leq t_k \leq t+\omega} I_k(t_k, u(t_k)) \right] \\ &\geq \frac{\mathcal{G}_1}{\mathcal{G}_2} \|\mathcal{P}u\| = \rho \|\mathcal{P}u\|, \end{aligned}$$

\mathcal{P} is continuous on $\Omega_\omega(c)$ and the cone Λ is a subset of $\Omega_\omega(c)$. Then \mathcal{P} is continuous on Λ . The proof that \mathcal{P} is completely continuous is similar to the corresponding work in Theorem 2.5, hence we omit it here.

Next, in preparation for the existence theorem, we introduce the following notations:

For a positive constant r and $u \in \Lambda$, we set

$$\begin{aligned} \underline{I}_r &= \inf_{u \in K, \|u\|=r} \frac{\sum_{t \leq t_k \leq t+\omega} I_k(t_k, u(t_k))}{u}, \quad \widehat{I}_r^* = \sup_{u \in K, \|u\|=r} \frac{\sum_{t \leq t_k \leq t+\omega} I_k(t_k, u(t_k))}{u}, \\ \underline{g}_r &= \inf_{u \in K, \|u\|=r} \frac{\int_0^\omega g(s, u(s)) ds}{u}, \quad \widehat{g}_r^* = \sup_{u \in K, \|u\|=r} \frac{\int_0^\omega g(s, u(s)) ds}{u}. \end{aligned} \quad (2.17)$$

Theorem 2.6 *Assume the conditions (2.16), (2.17) hold. Moreover, assume there are positive constants R_1, R_2, R_3 with $R_1 < R_2 < R_3$ such that:*

$$\begin{aligned} \mathcal{G}_2 \left(\lambda \widehat{g}_{R_1}^* + \widehat{I}_{R_1}^* \right) &\leq 1, \\ \mathcal{G}_1 \rho \left(\lambda \underline{g}_{R_2} + \underline{I}_{R_2} \right) &\geq 1, \\ \mathcal{G}_2 \left(\lambda \widehat{g}_{R_3}^* + \widehat{I}_{R_3}^* \right) &\leq 1. \end{aligned}$$

Then, (2.9) possesses two positive ω -periodic solutions u_1, u_2 with $R_1 \leq u_1 \leq R_2 \leq u_2 \leq R_3$.

Proof. Without loss of generality, we assume $R_1 < R_2$. We define $\Omega_{R_1} = \{u \in P_\omega : \|u\| < R_1\}$. Then, for any $u \in \Lambda \cap \Omega_{R_1}$, we have

$$\begin{aligned} (\mathcal{P}u)(t) &\leq \mathcal{G}_2 \lambda \int_t^{t+\omega} \frac{g(s, u(s))}{u(s)} u(s) ds + \mathcal{G}_2 \sum_{t \leq t_k \leq t+\omega} \frac{I_k(t_k, u(t_k))}{u(t_k)} u(t_k) \\ &\leq \mathcal{G}_2 \left(\lambda \widehat{g}_{R_1}^* + \widehat{I}_{R_1}^* \right) \|u\| \leq \|u\|, \end{aligned}$$

this yields

$$\|\mathcal{P}u\| \leq \|u\|, \quad \text{for any } u \in \Lambda \cap \Omega_{R_1}. \quad (2.18)$$

Set $\Omega_{R_3} = \{u \in P_\omega : \|u\| < R_3\}$. Then, for any $u \in \Lambda \cap \Omega_{R_3}$, we obtain the result $\|\mathcal{P}u\| \leq \|u\|$, by applying the same procedure given above. Set

$$\Omega_{R_2} = \{u \in P_\omega : \|u\| < R_2\},$$

for all $u \in K \cap \Omega_{R_2}$, then $\rho R_2 \leq u \leq R_2$. Hence, by virtue of (2.11) and (2.17), we obtain

$$\begin{aligned} (\mathcal{P}u)(t) &\geq \lambda \mathcal{G}_1 \int_t^{t+\omega} \mathcal{G}(t, s) \frac{g(s, u(s))}{u(s)} u(s) ds + \mathcal{G}_1 \sum_{t \leq t_k \leq t+\omega} \frac{I_k(t_k, u(t_k))}{u(t_k)} u(t_k) \\ &\geq \mathcal{G}_1 \rho \left(\lambda \underline{g}_{R_2} + \underline{I}_{R_2} \right) \|u\| \geq \|u\|. \end{aligned}$$

This yields

$$\|(\mathcal{P}u)\| \geq \|u\|, \quad \text{for any } u \in \Lambda \cap \Omega_{R_2}. \quad (2.19)$$

Since $R_1 < R_2 < R_3$, by Guo-Krasnoselski theorem the mapping has at least two fixed points $u_1 \in \Lambda \cap (\overline{\Omega}_{R_3} \setminus \Omega_{R_2})$ and $u_2 \in \Lambda \cap (\overline{\Omega}_{R_2} \setminus \Omega_{R_1})$. It follows that possesses at least two positive periodic solutions of (2.9). That is $R_1 < u_1 < R_2 < u_2 < R_3$. ■

CHAPTER 3

Delayed differential equations

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This chapter presents the essential background material required for the remainder of the thesis. We introduce delay differential equations (DDEs) and the related mathematical concepts in a way that aims to be as accessible and clear as possible. Delay differential equations are a specific type of functional differential equation where the behavior depends on past values. Like ordinary differential equations, they have certain features that make their study more difficult. A comprehensive overview of the theory on delay differential equations is available, for example, in [5], [9], [11], [25], [34], [30], [42], [51], [54], [56], [57], [62], and [68].

3.1 Basic concepts of delay differential equations

3.1.1 A Mathematical viewpoint on the concept of solution

If we open a search engine and type in the research bar "why do we use delay differential equations?", many other questions arise such as: "What are the real life applications of delay differential equations?", "Is it possible to approximate a delay differential equation?" or even "How do we solve a delay differential equation?". These questions are mostly written by graduate students who have started for the first time studying the complex world of infinite-dimensional dynamic systems. When we look at the proposed answers, they are often very long and sometimes unclear. Working with such equations brings new challenges. In particular, questions of existence and uniqueness of solutions must be addressed, as they are essential for understanding the system and ensuring that the model is well-defined. Time delays occur in many real-world systems and are especially common in areas such as control theory, electrodynamics, neutron transportation and population dynamics. Due to their widespread presence, delay differential equations offer a suitable mathematical framework for describing such systems. An example will be presented later to illustrate how these equations can be applied in practice.

There exists a fundamental distinction between initial value problems for ordinary differential equations (ODEs) and those for delay differential equations (DDEs). Consider the delay differential equation of the form

$$\frac{du(t)}{dt} = \frac{\pi}{2}u(t - \tau), \text{ where } \tau \geq 0. \quad (3.1)$$

When the delay parameter is zero ($\tau = 0$), Equation (3.1) reduces to a classical first-order ODE. In such a case, a solution defined on an interval $[0, \gamma]$ is uniquely determined by specifying the initial value $u(0)$. The classical existence and uniqueness theorems are invoked with reference to the function's value at a specified initial time, as defined by the initial condition of the problem.

However, when $\tau > 0$, the equation includes a delay term, and the classical theory is no longer applicable. For $t \in [0, \tau)$, the term involves values of the function before the initial time $t = 0$. Therefore, the equation is not well-defined unless the function is specified on an interval before the origin. This makes it necessary to adjust the initial condition accordingly.

To find a solution for times before $t = 0$, one must provide a value at time $-\tau$. However, assigning a value only at that point is not enough, since for $\tau = 1$, equation (3.1) admits multiple solutions as follows:

$$\begin{aligned} u_1(t) &= \sin \left[\frac{\pi}{2} \left(t + \frac{1}{2} \right) \right], \\ &\text{and} \\ u_2(t) &= \cos \left[\frac{\pi}{2} \left(t + \frac{1}{2} \right) \right], \end{aligned}$$

both satisfying $u_1(0) = u_2(0)$, yet clearly distinct. This indicates that, in delay equations, providing a condition at just one point does not ensure the uniqueness of the solution on $[0, \gamma]$.

Typically, solving a delay differential equation involves defining the initial state of the system over a full interval, not just at a single point. Specifically, to set up a well-defined initial value problem at time t_0 , one must provide an initial function φ belonging to $\mathcal{C}([t_0 - \tau, t_0], \mathbb{R})$, where $\tau > 0$. This initial history function φ supplies the necessary information to evaluate the delayed terms for all $t \geq t_0$. Provided certain regularity and Lipschitz conditions

are met, each initial function corresponds to a unique solution on an interval $[t_0, t_0 + A]$, which may be extended further. When the initial functions are taken to be continuous, the resulting space of solutions has the same infinite dimensionality as $\mathcal{C}([t_0 - \tau, t_0], \mathbb{R})$. This highlights how crucial the initial history is in determining the behavior of solutions. The existence of a solution to an infinite-dimensional equation leads to much more difficulties compared to finite-dimensional problems since there does not exist any on-the-shelf Cauchy theorem.

To clarify more, we give a biological system in which the present rate of change of some unknown function depends upon past values of the same function.

3.1.2 Mathematical modeling with delay differential equations

To introduce the main ideas of delay differential equations (DDEs), we begin with a classical model from population dynamics. This example, originally introduced in [5], illustrates the effect of time delays on system dynamics and helps the reader understand the essential aspects of DDEs. We first consider the classical logistic growth model, also known as the Verhulst model, which describes the evolution of a population $u(t)$ over time $t \geq 0$ in the absence of delay:

$$\frac{du(t)}{dt} = ru(t) \left(1 - \frac{u(t)}{K} \right), \quad (3.2)$$

where

$u(t)$: population size at time t .

$r > 0$: intrinsic (natural) growth rate.

$K > 0$: carrying capacity.

The solution of (3.2), obtained by separation of variables, is given by

$$u(t) = \frac{u_0 K}{u_0 + (u_0 - K) e^{-rt}}, \quad \forall t \geq 0, \quad u(0) = u_0.$$

This explicit formula shows that the positive equilibrium $u^* = K$ of the logistic equation (3.2) is globally stable, that is, $\lim_{t \rightarrow \infty} u(t) = K$ for solution $u(t)$ of (3.2) with any initial value $u(0) = u_0$, and

If $u_0 < K$, the population increases and converges to K as $t \rightarrow \infty$.

If $u_0 > K$, the population decreases and converges to K as $t \rightarrow \infty$.

If $u_0 = K$, the population remains in time at $u^* = K$.

This is illustrated in the following figure.

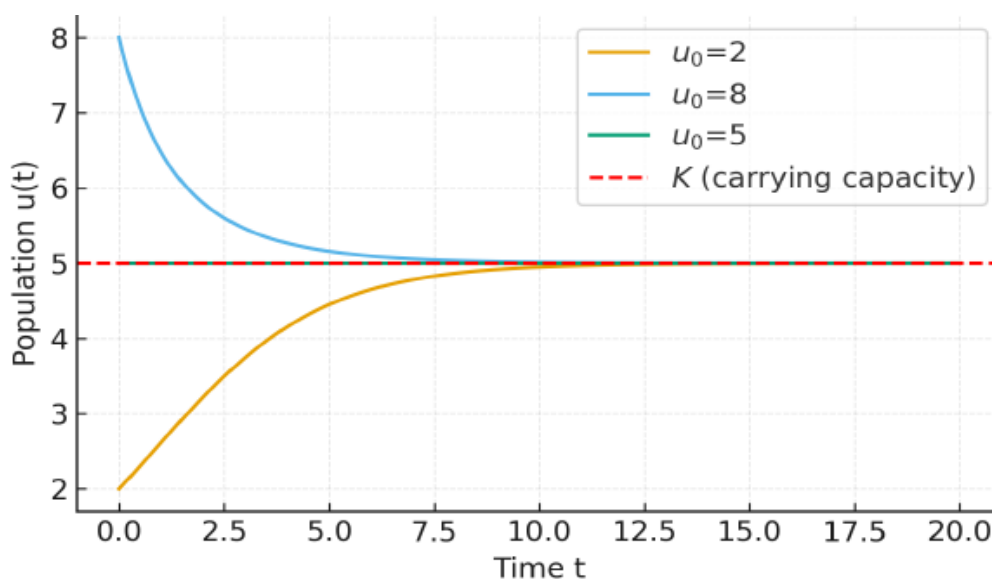


Fig 3.1 Numerical simulation of the logistic model without delay.

While the logistic equation captures the general growth behavior, it does not account for time delays inherent in real biological systems. For example, in certain species such as *Daphnia*, reproduction depends not on the current population size but on the population at an earlier time when resources were

consumed and eggs were formed. Such biological observations motivate the inclusion of delay terms in population models.

The delayed logistic model, also known as the Hutchinson equation (1948) [5], modifies (3.2) by introducing a constant delay $\tau > 0$:

$$\frac{du}{dt}(t) = ru(t) \left(1 - \frac{u(t - \tau)}{K} \right), \quad (3.3)$$

where r and K have the same meaning as in the logistic equation (3.2), and the initial condition is given by a continuous history function $\phi(\theta)$ on $[-\tau, 0]$, i.e., $u(\theta) = \phi(\theta) > 0$.

Unlike the classical model, equation (3.3) accounts for the time required for eggs to develop before hatching, thereby introducing a memory effect into the population dynamics. For small values of τ (e.g., $\tau = 0.5$), the solution remains close to the logistic behavior, converging smoothly to equilibrium. However, for larger delays, the system may exhibit oscillatory dynamics, as illustrated in Figure 3.2.

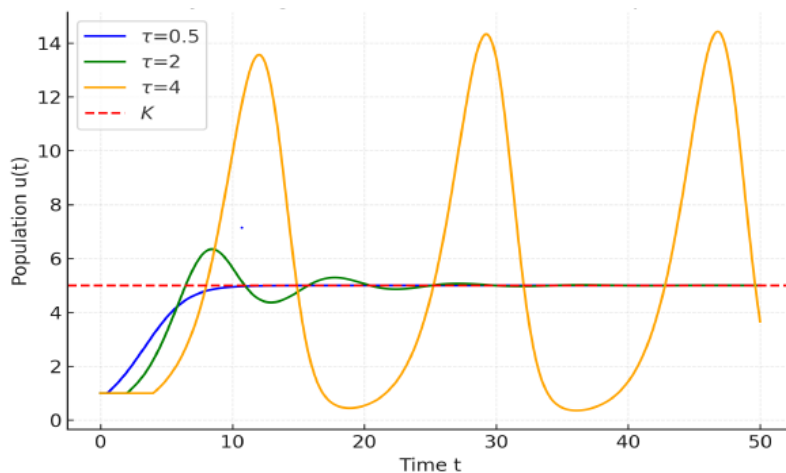


Fig 3.2 Numerical simulation of the solution of equation (3.3) with $r = 0.5$ for different values of τ , and the initial condition $u(t) = 1$ on the interval $[-\tau, 0]$.

The stability of the equilibrium $u^* = K$ depends critically on the product $r\tau$. According to Theorem 1 in [5], the equilibrium is asymptotically stable if $r\tau < \pi/2$ and unstable if $r\tau > \pi/2$. Importantly, the loss of stability at this threshold occurs through a Hopf bifurcation, leading to the emergence of periodic oscillations. This phenomenon is further demonstrated in Figures 3.3 and 3.4, which show numerical solutions of (3.3) both in the time domain and in the phase plane.

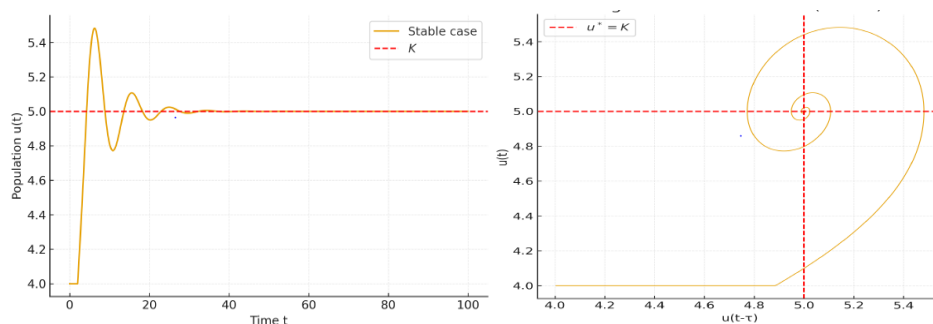


Fig 3.3 Solution of (3.3) when $r\tau < \frac{\pi}{2}$ with $r = 0.5$ and $K = 5$ with the initial $u(t) = 4$ on $[-\tau, 0]$ in the (t, u) plane (left) and in the phase plane (right).

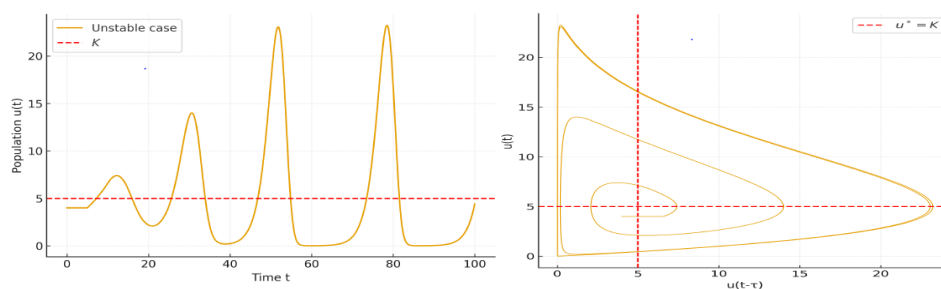


Fig 3.4 Solution of (3.3) when $r\tau > \frac{\pi}{2}$ with $r = 0.5$ and $K = 5$ with the initial $u(t) = 4$ on $[-\tau, 0]$ in the (t, u) plane (left) and in the phase plane (right).

This example highlights two fundamental aspects of DDEs:

1. Infinite-dimensional nature: The state space of a DDE is not \mathbb{R}^n as in ODEs, but a function space $\mathcal{C}([-\tau, 0], \mathbb{R})$, since the system depends on entire history segments.

2. Qualitative behavior: Even simple models may display oscillations, discontinuities in derivatives, and bifurcations phenomena that do not occur in their ODE counterparts.

Thus, the delayed logistic model illustrates how the introduction of delays transforms a simple ODE into a functional differential equation with much richer qualitative behavior. This motivates the systematic study of DDEs and their generalizations, which we develop in the following sections.

3.1.3 Initial value problem

To analyze these systems, we consider the Banach space $\mathcal{C}([a, b], \mathbb{R}^n)$, which consists of all continuous functions from the interval $[a, b]$ into \mathbb{R}^n equipped with the topology of uniform convergence. Let $\tau > 0$ be a fixed real number called the delay. In particular, we define the space $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. For a function $u \in \mathcal{C}$, the continuous norm $\|\cdot\|_\tau$ is given by

$$\|u\|_\tau := \sup_{-\tau \leq \theta \leq 0} |u(\theta)|.$$

Let $\sigma \in \mathbb{R}$, $A > 0$, then for any $u \in \mathcal{C}([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, and for each $t \in [\sigma, \sigma + A]$, we define the history segment of u , denoted by u_t , as:

$$u_t = u(t + \theta) \text{ for } -\tau \leq \theta \leq 0.$$

Since the derivative u' depends on the function's history u_t , it is natural to set an initial condition in terms of a function rather than a single point. Therefore, we consider an initial condition of the form $u_\sigma = \varphi$, where $(\sigma, \varphi) \in \Phi \subset \mathbb{R} \times \mathcal{C}$.

We then consider the initial value problem for delay differential equations.

Definition 3.1 [34] If Φ is a subset of $\mathbb{R} \times \mathcal{C}$. Let $f : \Phi \rightarrow \mathbb{R}^n$ is a given function, and "r" represents the right-hand derivative. We say that the relation

$$\begin{cases} u'(t) = f(t, u_t), & t \geq \sigma, \\ u_\sigma = \varphi, \end{cases} \quad (3.4)$$

is a delayed functional differential equation on Φ and will denote this equation by DDEs.

Definition 3.2 [34] Let $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$. A function

$$u \in \mathcal{C}([\sigma - \tau, \sigma + A], \mathbb{R}^n), \quad A > 0,$$

is called a solution of (3.4) through (σ, φ) if

1. $u'(t) = f(t, u_t)$ for $t \in [\sigma, \sigma + A]$, and
2. $u_\sigma = \varphi$

In such a case we say that u is a solution of (3.4) on $[\sigma, \sigma + A]$. More precisely, for a given $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, a solution of (3.4) through (σ, φ) is a function $u \in \mathcal{C}([\sigma - \tau, \sigma + A], \mathbb{R}^n)$ such that $u'(t) = f(t, u_t)$ for $t \in [\sigma - \tau, \sigma + A]$ and $u_\sigma(\sigma, \varphi) = \varphi$.

3.1.4 Some kind of delay differential equations

When studying functional differential equations, one of the key aspects is the manner in which delays occur. Delays can take different forms, such as a constant, a distributed contribution over a past interval, an explicit function of time, or even a function of the system's state itself. This classification is crucial, since each type of delay leads to distinct mathematical challenges and corresponds to different modeling capabilities. The following section outlines the primary categories of delay differential equations (DDEs).

Differential equations with constant delay

The most straightforward class of DDEs arises when the delay is a positive constant, $\tau > 0$. In their simplest form, it is written as follows:

$$u'(t) = f(t, u(t), u(t - \tau)),$$

where $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function, and the derivative at time t depends on the state at a single, previous time, $t - \tau$. Constant delays are widely used in models with fixed reaction times, such as population dynamics or control systems with communication lags.

Differential equations with distributed delay

In this type, the present rate of change depends on the influence of states over an entire past interval rather than on a single delayed value. A common form is

$$u'(t) = \int_{-\tau}^0 g(\theta) u(t + \theta) d\theta,$$

where $g(\theta)$ specifies how each point in the interval $[-\tau, 0]$ contributes to the present. Such models are particularly relevant in population dynamics (where reproduction depends on the age distribution), in viscoelasticity (where stress depends on the entire history of strain), and in economics (where decisions depend on past averages).

Variable delay differential equations

The delay in this case varies in the time variable or depends on the state.

1) Time-varying delay differential equations:

This class of DDEs is characterized by a delay that is an explicit function of time, $\tau(t)$. In their simplest form, this kind of equation is written as

follows:

$$u'(t) = f(t, u(t), u(t - \tau(t))),$$

where f is a given function. A well-known example is the pantograph equation, given by

$$u'(t) = au(t) + bu(\lambda t), \quad t \geq 0, \quad 0 < \lambda < 1, \quad a, b \in \mathbb{R}.$$

which means the delay is

$$\tau t = t - \lambda t.$$

Here the delay depends proportionally on time. Such models are important in electrodynamics, number theory, and control engineering.

2) State-dependent variable delay differential equations:

A highly complex and significant class of DDEs arises when the time delay itself is governed by the state of the system, rather than being a constant or an explicit function of time. These are known as State-Dependent Delay Equations (SDDEs). Such models are expressed as

$$u'(t) = f(t, u(t), u(t - \tau_1(u(t))), \dots, u(t - \tau_p(u(t))))),$$

where the delays $\tau_i(t)$ satisfy $0 < \tau_i(t) < \tau$ for $i = 1, \dots, p$.

As a specific illustration, if

$$\tau(t) = t - u(t),$$

then the delay term becomes

$$u(t - \tau(t)) = u(u(t)),$$

which corresponds to the second iterate of u , denoted by

$$u^{[2]}(t) = u(u(t)).$$

By repeating this process, one can obtain higher-order iterates, leading to an **iterative differential equation** of the form

$$u'(t) = f(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)),$$

where the iterate $u^{[n]}(t)$ stands for u composed with itself n times, i.e., $u^{[0]}(t) = t$, $u^{[1]}(t) = u(t)$, $u^{[2]}(t) = u(u(t))$, $u^{[3]}(t) = u^{[2]}(u(t))$, and so on.

What distinguishes IDEs from other classes of delay equations is the intrinsic combination of differentiation with functional iteration. This interplay introduces a fundamentally nonlinear and self-referential structure, which often leads to highly nontrivial dynamics such as oscillations, bifurcations, or even chaotic behavior. From a mathematical standpoint, the analysis of IDEs is considerably more demanding, since many of the classical methods developed for standard delay differential equations cannot be applied directly. Questions concerning existence, uniqueness, stability, and long-term asymptotic behavior therefore require new approaches, often drawing simultaneously on techniques from dynamical systems theory and functional iteration. For further details see [26], [39], [50], [64], [66], [74], and [79].

3.1.5 Notion of iterations

To formalize the construction of iterative differential equations, we recall some basic definitions from iteration theory.

Definition 3.3 The composition $u \circ v$ of the function u with the function v is

$$(u \circ v)(t) = u(v(t)),$$

The domain of $u \circ v$ is the set of all t in the domain of v such that $v(t)$ is in the domain of u .

Definition 3.4 For $u : E \rightarrow E$, the n^{th} iterate of function u , denoted by $u^{[n]}$ for some nonnegative integer n , is defined recursively by

$$u^{[0]} = Id_E,$$

and

$$u^{[n+1]} = u \circ u^{[n]},$$

where Id_E is the identity map on E .

We now present and prove a crucial lemma that will serve as a key tool in establishing the main results of this thesis.

Remark 3.1 For $\eta_1 \geq 0$, and $\eta_2, W > 0$, let

$$C_\omega(\eta_1, \eta_2, W) = \{u \in C_\omega, \eta_1 \leq u \leq \eta_2, |u(a_2) - u(a_1)| \leq W |a_2 - a_1|, \forall a_1, a_2 \in \mathbb{R}\}.$$

- 1) $C_\omega(\eta_1, \eta_2, W)$ is a closed, convex and bounded subset of C_ω .
- 2) It follows from conditions $\eta_1 \leq u(t) \leq \eta_2$ and $|u(a_2) - u(a_1)| \leq W |a_2 - a_1|$, in the definition of $C_\omega(\eta_1, \eta_2, W)$, that this subset is equicontinuous and uniformly bounded. Hence, as a consequence of the Arzelà-Ascoli theorem, we conclude that the closed subset $C_\omega(\eta_1, \eta_2, W)$ is compact.

Lemma 3.1 (see, [77]) For any $u, v \in C_\omega(\eta_1, \eta_2, W)$, we have,

$$\|u^{[l]} - v^{[l]}\| \leq \sum_{j=0}^{l-1} W^j \|u - v\|, \quad l = 1, 2, \dots, n.$$

Proof. We will demonstrate this estimate by induction. So, the proof can be given in two steps:

The basis step: for $j = 1$, we have

$$\|u - v\| \leq \|u - v\|,$$

then, the estimate holds for $j = 1$

The inductive step: Now, we suppose that the estimate holds for a given $j = l$ and we want to prove that it also holds for $j = l + 1$. Assume that

$$\|u^{[l]} - v^{[l]}\| \leq \sum_{j=0}^{l-1} W^j \|u - v\|,$$

then

$$\begin{aligned} |u^{[l+1]}(t) - v^{[l+1]}(t)| &\leq |u(u^{[l]}(t)) - u(v^{[l]}(t))| + |u(v^{[l]}(t)) - v(v^{[l]}(t))| \\ &\leq W |u^{[l]}(t) - v^{[l]}(t)| + |u(v^{[l]}(t)) - v(v^{[l]}(t))|, \end{aligned}$$

so

$$\begin{aligned} \|u^{[l+1]} - v^{[l+1]}\| &\leq W \|u^{[l]} - v^{[l]}\| + \|u - v\| \\ &\leq W \sum_{j=0}^{l-1} W^j \|u - v\| + \|u - v\| \\ &\leq \left(\sum_{j=0}^{l-1} W^{j+1} + 1 \right) \|u - v\| \\ &\leq \sum_{j=0}^l W^j \|u - v\|. \end{aligned}$$

By induction we infer that

$$\|u^{[l]} - v^{[l]}\| \leq \sum_{j=0}^{l-1} W^j \|u - v\|, \quad \forall l \in \mathbb{N},$$

which finishes the proof. ■

3.1.6 Existence and uniqueness results

In what follows we present some classic results on the existence and uniqueness of solution of the equation (3.4), which can be found in [5] and [34].

We now state the basic theory of DDEs.

Lemma 3.2 [34] *Let $\sigma \in \mathbb{R}$ and $\varphi \in \mathcal{C}$ be given, and let f be continuous on the product $\mathbb{R} \times \mathcal{C}$. Then, finding a solution of equation (3.4) through (σ, φ) is equivalent to solving the integral equation:*

$$u'(t) = \varphi(\sigma) + \int_{\sigma}^t f(s, u_s) ds \text{ for } t \geq \sigma, \text{ and } x_{\sigma} = \varphi.$$

Lemma 3.3 [5] *If $u \in \mathcal{C}([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then u_t is a continuous function of t for $t \in [\sigma - \tau, \sigma + A]$.*

Proof. Since u is continuous on $[\sigma - \tau, \sigma + A]$, it is uniformly continuous, and thus $\forall \varepsilon > 0, \exists \delta > 0$, such that $|u(t) - u(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently, for t, s in $[\sigma, \sigma + A]$, $|t - s| < \delta$, we have $|u(t + \theta) - u(s + \theta)| < \varepsilon, \forall \theta \in [-\tau, 0]$.

■

The existence and uniqueness of the solutions of DDEs are given by the following theorems.

Theorem 3.1 (Local existence, [34]) *Suppose Φ is an open subset in $\mathbb{R} \times \mathcal{C}$ and $f : \Phi \rightarrow \mathbb{R}^n$ is continuous. For any $(\sigma, \varphi) \in \Phi$, there exists a solution of equation (3.4) through (σ, φ) .*

Definition 3.5 (Lipschitzian, [34]) *We say $f(t, \varphi)$ is Lipschitz in φ in a compact set B of $\mathbb{R} \times \mathcal{C}$ if there is a constant $d > 0$ such that, for any $(t, \varphi_i) \in B, i = 1, 2$,*

$$|f(t, \varphi_1) - f(t, \varphi_2)| < d|\varphi_1 - \varphi_2|.$$

Theorem 3.2 (Existence and uniqueness, [34]) *Suppose Φ is an open set in $\mathbb{R} \times \mathcal{C}$, $f : \Phi \rightarrow \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitzian in φ in each compact set in Φ . If $(t_0, \varphi) \in \Phi$, then there is a unique solution of Eq. (3.4) through (t_0, φ) .*

Proposition 3.1 (*Global existence, [5]*) *If f is at most affine i.e. $f(t, \varphi) \leq a + b|\varphi|$, with $a, b > 0$, then there exists a global solution of the equation (3.4) i.e. $\forall \varphi$, the solution $u(\sigma, \varphi)$ is defined on $[A, \infty[$.*

3.1.7 Primary discontinuities

Among the key features of delay differential equations, primary discontinuities emerge as sudden and non-continuous changes in the solutions or their derivatives. These discontinuities arise due to the system's dependence on past values, which leads to abrupt transitions across time intervals.

Definition 3.6 [10] *If the solution of a DDE and its derivatives of order k are continuous at some point in the time interval, but the derivative of order $k + 1$ is not, then such a point is called a primary discontinuity of the given problem.*

Theorem 3.3 [10] *If ζ_k is a primary discontinuity point where the function $u(t)$ has continuous derivatives up to order $k - 1$, then $u(t)$ is continuously differentiable of all orders.*

Proof. Note that, as t increases, the solution becomes smoother. In fact, at the initial point $t = 0$, the first derivative $u'(t)$ has a primary discontinuity, since the integrable equation

$$u'(t) = f(t, u(t), \psi(t - \tau)), \quad t \in [0, \tau],$$

may satisfy the condition $u(0) = \psi(0)$, but it is unlikely to satisfy the additional condition $u'(0^+) = \psi'(0^-)$. Only for special choices of the initial function $\psi(t)$ is it possible to guarantee continuity of the derivative of the solution at point 0, for such a function must satisfy the condition $\psi'(0^-) = f(0, \psi(0), \psi(-\tau))$. ■

Example 3.1 Consider the equation

$$\begin{cases} u'(t) = u(t-1), & t \geq 0 \\ u(t) = 1, & t < 0. \end{cases} \quad (3.5)$$

Since $u'(0^-) = 0$ and $u'(0^+) = 1$, the derivative function $u'(t)$ has a jump at $t = 0$. At $t = 1$, the first derivative remains continuous, but a discontinuity in the second derivative appears. In general behavior, for each interval $[k, k + 1]$, the solution is a polynomial of degree $k + 1$. At $t = k$, the $(k + 1) - th$ derivative exhibits a jump discontinuity. Specifically:

- At $t = 0$: Discontinuity in u' .
- At $t = 1$: Discontinuity in u'' .
- At $t = 2$: Discontinuity in u''' ,

whose solution is depicted in Fig. 3.5.

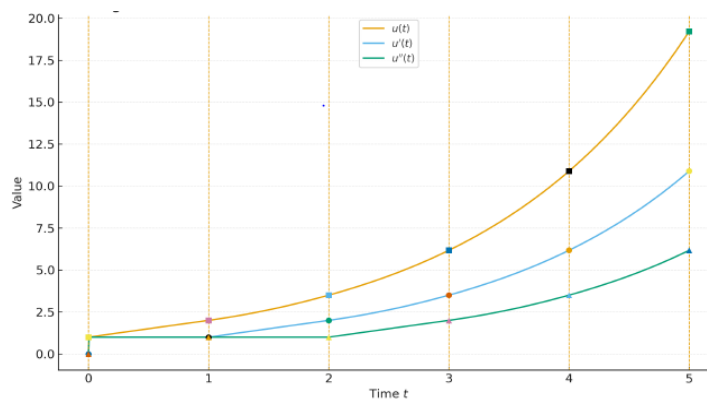


Fig 3.5 Solution of (3.5).

where

- The solution $u(t)$ is represented by a continuous curve with square markers at integer points.
- The derivative $u'(t)$ is shown with circular markers.

- The graph illustrates how discontinuities in higher-order derivatives propagate through the solution, leading to changes in concavity and sharp transitions in slopes at specific points.

3.1.8 Neutral differential equations

Neutral differential equations (NDEs) constitute a special class of functional differential equations in which delayed terms involve not only the state itself but also its highest-order derivatives. This feature makes their analysis considerably more challenging than that of delay differential equations (DDEs) and gives rise to richer dynamical behaviors. Such equations naturally occur in various domains, including mechanical vibrations [18], electrical systems [9], control engineering [40], and biological models [5].

Definition 3.7 [34] Suppose $\Phi \subseteq \mathbb{R} \times \mathcal{C}$ is open with elements (t, φ) . A function $D : \Phi \rightarrow \mathbb{R}^n$ is said to be atomic at δ on Φ if D is continuous together with its first and second Fréchet derivatives with respect to φ , and D_φ , the derivative with respect to φ , is atomic at δ on Φ .

Definition 3.8 [34] Let us consider a Banach space \mathcal{C} and $\Phi \subseteq \mathbb{R} \times \mathcal{C}$. Then a neutral delay differential equation can be written in the form

$$\frac{dD}{dt}(t, u_t) = f(t, u_t), \quad (3.6)$$

where $f : \Phi \rightarrow \mathbb{R}^n$ and $D : \Phi \rightarrow \mathbb{R}^n$ are given continuous functions, with D assumed to be atomic at zero. The function D is called difference operator for NDE (D, f) .

Example 3.2 If $\tau > 0$, u is a scalar, $D(u) = u(t) - e^{u(t-\tau)}$, and $f : \Phi \rightarrow \mathbb{R}^n$ is continuous, then the pair (D, f) defines an NDE,

$$\frac{d}{dt} [u(t) - e^{u(t-\tau)}] = f(t, u_t).$$

Example 3.3 The following equations

$$\begin{aligned}u'(t) &= -5u'(t-2), \\u'(t) &= u(t-1) + \left[u'(t-3) + 1\right]^3,\end{aligned}$$

are neutral delay differential equations. However, this equation

$$u'''(t) = u\left(\frac{t}{2}\right) + u''(t-7) - u'(t-3),$$

doesn't consider as an example of NDEs, since the highest degree of derivative independent of the delay.

3.1.9 Existence and uniqueness

In this part, we consider the questions of existence and uniqueness of solutions of NDEs. This results can be found in [34]

Definition 3.9 [34] A function u is said to be a solution of (3.6) on $[\sigma - \tau, \sigma + A]$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that

$$u \in \mathcal{C}([\sigma - \tau, \sigma + A], \mathbb{R}^n), \quad (t, u_t) \in \Phi, \quad t \in [\sigma, \sigma + A],$$

$D(t, u_t)$ is continuously differentiable and satisfies equation (3.6) on $[\sigma - \tau, \sigma + A]$. For a given $t_0 \in \mathbb{R}$, $\varphi \in \mathcal{C}$, and $(\sigma, \varphi) \in \Phi$, we say $u(t, \sigma, \varphi)$ is a solution of equation (3.6) with initial value φ at σ or simply a solution through (σ, φ) if there is an $A > 0$ such that $u(t, \sigma, \varphi)$ is a solution of equation (3.6) on $[\sigma - \tau, \sigma + A]$ and $u_\sigma(\sigma, \varphi) = \varphi$, we say $u(t, \sigma, \varphi)$ is a solution of (3.6) on $[\sigma - \tau, \infty)$, if for every $A > 0$, $u(t, \sigma, \varphi)$ is a solution of equation (3.6) on $[\sigma - \tau, \sigma + A]$ and $u_\sigma(\sigma, \varphi) = \varphi$.

Theorem 3.4 (*Existence* [34]). *If Φ is an open set in $\mathbb{R} \times \mathcal{C}$ and $(\sigma, \varphi) \in \Phi$, then there exists a solution of the NDE (D, f) through (σ, φ) .*

Theorem 3.5 (*Uniqueness [34]*). *If $\Phi \subseteq \mathbb{R} \times \mathcal{C}$ is open and $f : \Phi \rightarrow \mathbb{R}^n$ is Lipschitz in φ on compact sets of Φ , then, for any $(\sigma, \phi) \in \Phi$, there exists a unique solution of the NDE (D, f) through (σ, φ) .*

3.1.10 Method of steps

The Method of Steps, also referred to as the Method of Successive Integrations, is a classical approach for solving delay differential equations (DDEs) that involves dividing the time domain into successive intervals equal to the delay τ . On the initial interval $[0, \tau]$, the solution is determined using a prescribed history function, which specifies the system state prior to $t = 0$. For each subsequent interval $[k\tau, (k+1)\tau]$, where k is a non-negative integer, the DDE is reduced to an ordinary differential equation (ODE) by treating the delayed terms as known functions derived from solutions on previous intervals. This stepwise procedure effectively transforms the original DDE into a sequence of ODEs, each solvable with standard methods, and the solution on one interval serves as the history function for the next, allowing iterative advancement through the entire time domain. Introduced by Bellman in 1965 for constant delays, the method was subsequently generalized by El'sgol'ts and Norkin in 1973 to accommodate variable delays, under the assumption that the delay remains strictly positive. Although conceptually straightforward and relatively easy to implement, the Method of Steps necessitates careful management of past solutions, including accurate storage and interpolation, particularly in cases involving variable or multiple delays or stiff systems. Nevertheless, it remains a fundamental and widely used technique for analyzing the influence of past states on delayed dynamical systems. To fix the ideas, we explain the steps as follow:

Consider the following delay differential equation

$$\begin{cases} u'(t) = f(t, u(t), u(t - \tau), u'(t - \tau)), & t \geq t_0 \\ u(t) = \varphi_0(t), & \forall t \in [t_0 - \tau, t_0]. \end{cases}$$

For such equations the solution is constructed step by step as follows:

Step 1: Given that a function $\varphi_0(t)$ continuous on $[t_0 - \tau, t_0]$, so one can obtain the solution in the next step interval $[t_0, t_0 + \tau]$ by solving the following equation:

$$u'(t) = f(t, u(t), \varphi_0(t - \tau), \varphi_0'(t - \tau)), \quad \forall t \in [t_0, t_0 + \tau],$$

with the initial condition $u(t_0) = \varphi_0(t_0)$. If we consider $\varphi_1(t)$ is the desired first step solution, which exists by virtue of continuity hypotheses.

Step 2: On the interval $[t_0, t_0 + \tau]$, the function $u(t)$ is the given function $\varphi_1(t)$, therefore one can find the solution $\varphi_2(t)$ to the equation:

$$u'(t) = f(t, u(t), \varphi_1(t - \tau), \varphi_1'(t - \tau)), \quad \forall t \in [t_0 + \tau, t_0 + 2\tau],$$

with initial condition $u(t_0 + \tau) = \varphi_1(t_0 + \tau)$.

These steps may be continued for subsequent intervals.

In general, by assuming that $\varphi_{k-1}(t)$, $\forall (k = 1, 2, \dots)$ is defined on the interval $[t_0 + (k - 2)\tau, t_0 + (k - 1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$u'(t) = f(t, u(t), \varphi_{k-1}(t - \tau), \varphi_{k-1}'(t - \tau)), \quad \forall t \in [t_0 + (k - 1)\tau, t_0 + k\tau],$$

with the initial condition: $u(t_0 + (k - 1)\tau) = \varphi_{k-1}(t_0 + (k - 1)\tau)$. We can continue this process indefinitely, showing that the uniquely defined $u(t)$ exists on $[t_0 - \tau, \infty)$.

We illustrate the method in the following example.

Example 3.4 Let us consider the following simplest delay differential equation:

$$u'(t) = -u(t - \tau), \quad t \geq 0, \quad (3.7)$$

where $\tau > 0$ is called the delay. When $\tau = 0$, this reduces to the simple ordinary differential equation

$$u'(t) = -u(t), \quad (3.8)$$

whose general solution is $u(t) = u(0)e^{-t}$, which decays monotonically to zero.

To solve (3.7), we must prescribe initial data on the entire interval $[-\tau, 0]$. For instance, let

$$u(t) = 1, \quad -\tau \leq t \leq 0, \quad (3.9)$$

as “initial data” for (3.7).

On the first interval $0 \leq t \leq \tau$, we have $t - \tau \leq 0$, so

$$u'(t) = -u(t - \tau) = -1,$$

which integrates to

$$u(t) = u(0) + \int_0^t (-1)ds = 1 - t, \quad 0 \leq t \leq \tau. \quad (3.10)$$

On the next interval $\tau \leq t \leq 2\tau$, substituting the solution from (3.10) gives

$$u'(t) = -u(t - \tau) = -(1 - (t - \tau)),$$

which integrates to

$$\begin{aligned} u(t) &= u(\tau) + \int_{\tau}^t -(1 - (s - \tau)) ds \\ &= 1 - \tau + \left(-s + \frac{1}{2}(s - \tau)^2 \right) \Big|_{s=\tau}^{s=t} \\ &= 1 - t + \frac{(t - \tau)^2}{2}, \quad \tau \leq t \leq 2\tau. \end{aligned} \quad (3.11)$$

In the same way, on the interval $(n - 1)\tau \leq t \leq n\tau$ we obtain a polynomial of degree n :

$$u(t) = 1 + \sum_{k=1}^n (-1)^k \frac{(t - (k - 1)\tau)^k}{k!}, \quad n \geq 1.$$

Thus, $u(t)$ is a polynomial of degree n on each subinterval of the form $[(n - 1)\tau, n\tau)$, It follows that $u(t)$ is a smooth function, except at each $n\tau$, $n \geq 0$.

To illustrate this approach, we compute numerical solutions for various values of the delay parameter τ and examine their behavior for $t > 0$. The corresponding results are presented in Figure 3.6.

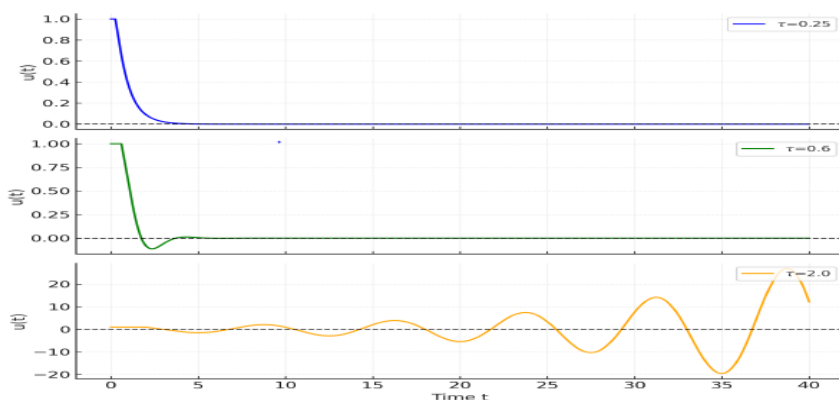


Fig 3.6 Numerical simulation of the solution of equation (3.7) with initial data (3.8) for various τ .

Notice that when $\tau = 0.25$, the solution behaves similarly to the ODE case, decaying smoothly to zero without oscillations. When $\tau = 0.6$, oscillations appear, and in fact the solution repeatedly changes sign. One can rigorously prove that all solutions oscillate whenever $\tau > e^{-1}$. For larger values, such as $\tau = 2$, oscillations persist but remain damped, with the amplitude decreasing over time.

CHAPTER 4

Existence and positivity of periodic solutions in
iterative neutral differential equations with impulses

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The goal of this chapter is to present a very recent work published in [27], namely, Amel Elbahi, Mimia Benhadri, Amel Boulfoul. Positive periodic solutions of a kind of nonlinear iterative differential equations with impulse effects. Rendiconti del Circolo Matematico di Palermo Series 2: 74:62(2025), <https://doi.org/10.1007/s12215-024-01169-2>.

In this chapter, we investigate the existence of bounded positive periodic

solutions for a class of impulsive neutral differential equations with iterative terms. By applying Schauder's fixed point theorem, we derive new sufficient conditions that guarantee the existence of such solutions. We also provide two numerical examples to demonstrate the conditions of the obtained results can be satisfied and applied. A comparison between the obtained results and some existing results is given. The results of this research have new contributions to the qualitative theory of NDEs with impulsive effects, and we improve and extend some results that are available in the literature for the iterative differential equations.

Keywords. Fixed-point theorems, impulses, delay differential equations, existence, neutral equations, positive periodic solutions.

4.1 Introduction

Iterative differential equations are a particular type of functional differential equation involving time and state-dependent delays. This kind of equation arises in the modeling of a wide range of natural phenomena, including cell biology, structured population models, infectious diseases, and position control, see [39], [50], [64], [66]. Despite the appearance of this equation, which dates back to the beginning of the 19th century, the development of their theory has been slow and tedious because of their complicated dynamics, and there are very few works available in the literature that discuss it. Several researchers face challenges when studying these equations, which have recently become more popular and led to several recent contributions (see [13], [26], [38], [50], [66]). Recently, the problem of positive periodic solutions of delay differential equations with or without impulse effects has always been one of the active areas of research and has attracted much attention. However, in

the best of our literature survey there are no published papers have discussed the existence of positive periodic solutions for nonlinear neutral iterative differential equations with impulse effects. To this end, this work makes the first attempt to fill this gap. The results of this work have new extensions and contributions to the theory of the existence of positive periodic solutions to impulsive nonlinear neutral iterative differential equations. The existence of periodic solutions to neutral functional differential equations plays a very important role in solving these practical problems. In recent years, a great deal of work has been performed on the existence of periodic solutions for iterative differential equations. We refer to [\[38\]](#), [\[66\]](#) and the recent studies [\[12\]](#), [\[79\]](#) as inspiring studies on this topic. For example, the nonlinear iterative differential equation of the form

$$u'(t) = f(u(u(t))), \quad (4.1)$$

is handled by [\[38\]](#).

In 1998, Si, Wang [\[64\]](#) and Hou, have discussed the existence of solutions for the following the iterative differential

$$u'(t) = c_1(t)u^{[1]}(t) + c_2(t)u^{[2]}(t) + \dots + c_n(t)u^{[n]}(t). \quad (4.2)$$

In 2017, Zhao and Liu [\[79\]](#), the authors considered the following iterative functional differential equations with variable coefficients:

$$u'(t) = c_0(t)u^{[0]}(t) + c_1(t)u^{[1]}(t) + c_2(t)u^{[2]}(t) + \dots + c_n(t)u^{[n]}(t) + \widehat{F}(t), \quad (4.3)$$

where $c_i \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $i = 0, \dots, n$ are common periodic functions with respect to the time variable. By using fixed point methods, the authors derived some verifiable sufficient conditions for the existence and uniqueness of periodic solutions to [\(4.3\)](#).

In [12], the authors generalized the results of [79] and studied the existence of periodic solutions of the following nonlinear iterative differential equations in the case where no impulses are taken into account.

$$\begin{aligned} \frac{d}{dt}u(t) &= -a(t)u(t) + f(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\ &+ \frac{d}{dt}g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)). \end{aligned} \quad (4.4)$$

By using Schauder's fixed point theorem, sufficient conditions are presented for the existence of periodic solutions of nonlinear iterative differential equations with periodic coefficients.

Motivated by these works, we consider the following class of impulsive neutral differential equation with iterative terms,

$$\begin{aligned} u'(t) &= -a(t)f(t, u(t))u(t) + g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\ &+ \frac{d}{dt}h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)), t \neq t_k, k \in \mathbb{N}^*, \\ u(t_k^+) - u(t_k^-) &= I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)), t = t_k, k \in \mathbb{N}^*, \end{aligned} \quad (4.5)$$

where $u^{[0]}(t) = t$, $u^{[1]}(t) = u(t)$, $u^{[2]}(t) = u(u(t))$, $u^{[3]}(t) = u^{[2]}(u(t))$, ..., $u^{[n]}(t) = u^{[n-1]}(u(t))$ are the iteratives of the state u , $a \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ is a ω -periodic function, and $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^+)$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $h \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

The expression $\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(t_k, u(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k))$, denotes the impulse at moment t_k , $u(t_k^+)$ and $u(t_k^-)$ stand for the right-hand and the left-hand limits of $u(t)$ at the impulsive moment t_k respectively and $I_k(\cdot, \cdot) \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $k \in \mathbb{N}^*$. We assume that there exists an integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $I_{(k+q)}(t_{k+q}, u(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q})) = I_k(t_k, u(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k))$, $k \in \mathbb{N}^*$, where $0 < t_1 < t_2 < \dots < t_q < \omega$, ($\omega > 0$ is a given positive constant).

Since we are searching for the existence of periodic solutions to Eq. (4.5),

it is natural to assume that

$$\begin{aligned}f(t + \omega, u) &= f(t, u), \\h(t + \omega, u_1, u_2, \dots, u_n) &= h(t, u_1, y_2, \dots, u_n), \\g(t + \omega, u_1, u_2, \dots, u_n) &= g(t, u_1, y_2, \dots, u_n).\end{aligned}\tag{4.6}$$

In the study of iterative differential equations there are fundamental questions: is there a periodic solution? And what is it? One of the most elegant ways to prove that an equation has a periodic solution is to pose it as a fixed-point problem. Since many problems in real life, we need to consider the properties of its periodicity. This need serves as a motivation for research in these practical field. It is well known that studying the periodicity of an equation using a fixed point technique involves the construction of a suitable fixed point mapping. Thus, to construct a mapping \mathcal{A} , we begin by transforming (4.5) into a more tractable, but equivalent, equation, which we will then invert to obtain an equivalent integral equation from which we derive a fixed point mapping (see below). After that, we use a suitable Banach space defined below by $(P_\omega, \|\cdot\|)$. The main difficulty of this approach is to define a convex and compact subset of Banach space when the mapping is well defined, which is necessary for using a Schauder's fixed-point theorem.

In 2017, Zhao and Liu have successfully applied Krasnoselskii's fixed point theory to overcome these problems. Their study was the first to discuss the existence of periodic solutions for first order linear iterative differential equations. In addition, by virtue of the Banach contraction principle, the uniqueness and stability of the solution are also analyzed. In that work, they established sufficient conditions for the existence of such solutions. Recently, a vast body of literature has focused on the existence of solutions using fixed-point methods (see [12, 13, 26, 58, 77, 79]). In particular, many authors

have obtained new results for iterative differential equations through the application of these methods. Most of the equations considered in these articles have involved the iterative terms, while impulsive effects have been less frequently addressed. The analysis of iterative differential equations with impulse effects is often linked to the various applications of calculus, which support all mathematical sciences.

The rest of this chapter is organized as follows. Section 2 contains relevant definitions, notation, and an auxiliary lemma. In Section 3, by using the theory of Schauder's fixed point theorem, we obtain sufficient conditions that ensure the existence of bounded positive periodic solutions to the equation (4.5). The chapter ends with illustrative examples to demonstrate the results obtained in the previous sections.

4.2 Preliminaries and equation transformation

Before proceeding to our main objective, we introduce the necessary notations and hypotheses. We define the space of piecewise continuous functions with impulses as

$$PC(\mathbb{R}, \mathbb{R}) = \{u \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : u \in \mathcal{C}((t_k, t_{k+1}), \mathbb{R}) \text{ such that } u(t_k^-), u(t_k^+) \text{ exist and } u(t_k^-) = u(t_k), k \in \mathbb{N}^*\}.$$

Let P_ω denote the Banach space of all ω -periodic continuous functions

$$P_\omega = \{u \in PC(\mathbb{R}, \mathbb{R}) : u(t + \omega) = u(t)\},$$

endowed with the norm

$$\|u\| = \sup_{t \in \mathbb{R}} |u(t)| = \sup_{t \in [0, \omega]} |u(t)|, \text{ for } u \in P_\omega.$$

Moreover, For $n_1 \geq 0$, and $n_2, n_3 > 0$, let

$$P_\omega(n_1, n_2, n_3) = \{u \in P_\omega, n_1 \leq u(t) \leq n_2, |u(a_2) - u(a_1)| \leq n_3 |a_2 - a_1|, \\ \forall a_1, a_2 \in \mathbb{R}\}.$$

Then $P_\omega(n_1, n_2, n_3)$ is a closed convex and bounded subset of P_ω . since the uniform boundedness and equicontinuity of $P_\omega(n_1, n_2, n_3)$ follow from its definition, and it follows from the Arzela-Ascoli theorem that it is compact.

Throughout this chapter, we impose the following hypotheses:

(**H**₁) The function $h(t, u_1, u_2, \dots, u_n)$ is globally Lipschitz in t, u_1, u_2, \dots, u_n . That is, there are ω - periodic functions $L_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ and a positive constant L_0 such that for all $t, s \in \mathbb{R}, u_i, v_i \in \mathbb{R}, i = 1, 2, \dots, n$,

$$|h(t, u_1, u_2, \dots, u_n) - h(s, v_1, v_2, \dots, v_n)| \leq L_0|t - s| + \sum_{i=1}^n L_i(t) \|u_i - v_i\|. \quad (4.7)$$

(**H**₂) There exist ω - periodic functions $C_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ such that for all $t \in \mathbb{R}, u_i, v_i \in \mathbb{R}, i = 1, 2, \dots, n$,

$$|g(t, u_1, u_2, \dots, u_n) - g(t, v_1, v_2, \dots, v_n)| \leq \sum_{i=1}^n C_i(t) \|u_i - v_i\|. \quad (4.8)$$

(**H**₃) There exist positive functions $N_{ik} \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ such that for all $t \in \mathbb{R}, u_i, v_i \in \mathbb{R}, i = 1, 2, \dots, n$,

$$|I_k(t, u_1, u_2, \dots, u_n) - I_k(t, v_1, v_2, \dots, v_n)| \leq \sum_{i=1}^n N_{ik}(t) \|u_i - v_i\|, k \in \mathbb{N}^*. \quad (4.9)$$

For convenience, we also introduce the following notations:

$$\rho = \max_{t \in [0, \omega]} a(t), \sigma = \min_{t \in [0, \omega]} a(t), \quad (4.10)$$

$$L_i^* = \max_{t \in [0, \omega]} L_i(t), C_i^* = \max_{t \in [0, \omega]} C_i(t), i = 1, 2, \dots, n.$$

In order to be able to construct a new fixed mapping, we transform the equation into an equivalent equation.

Lemma 4.1 Suppose (4.6) holds. For $u \in P_\omega \cap C^1(\mathbb{R}, \mathbb{R})$, there is a ω -periodic solution for equation (4.5), if and only if there is a ω -periodic solution of the integral equation

$$\begin{aligned} u(t) = & h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\ & + \int_t^{t+\omega} G(t, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\ & + g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\ & + \sum_{\{k:t_k \in [t, t+\omega]\}} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)), \end{aligned} \quad (4.11)$$

where

$$G(t, x) = \frac{e^{\int_t^x a(s)f(s, u(s))ds}}{e^{\int_0^\omega a(s)f(s, u(s))ds} - 1}, \quad (4.12)$$

and

$$e^{\int_0^\omega a(s)f(s, u(s))ds} \neq 1.$$

Proof. Unlike the procedure carried out in (see [12]), where the authors used the variation of constants formula to rewrite the original equation as an integral equation, we have to proceed in a very different way, which is motivated and justified by the appearance of the impulsive terms in our problem. The technique for constructing a mapping for a fixed point argument comes from an idea in (see [21], [44]).

i) If u is a ω -periodic solution of the equation (4.5). Firstly, from (4.5) for $t \neq t_k, k \in \mathbb{N}^*$, we rewrite

$$\begin{aligned} & \frac{d}{dt} [u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))] \\ = & -a(t)f(t, u(t)) \\ & \times [u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))] \end{aligned}$$

$$\begin{aligned}
 & -a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 & +g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)).
 \end{aligned} \tag{4.13}$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)f(s, u(s))ds}$, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \left[(u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))) e^{\int_0^t a(s)f(s, u(s))ds} \right] \\
 = & e^{\int_0^t a(s)f(s, u(s))ds} \left[-a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \right. \\
 & \left. +g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \right], t \neq t_k, k \in \mathbb{N}^*.
 \end{aligned}$$

Let us denote

$$\begin{aligned}
 \vartheta^*(t, u) = & -a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 & +g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)).
 \end{aligned} \tag{4.14}$$

If $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}^*$, we have

$$\begin{aligned}
 & (u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))) e^{\int_0^t a(s)f(s, u(s))ds} \\
 = & (u(t_k^+) - h(t_k^+, u^{[1]}(t_k^+), u^{[2]}(t_k^+), \dots, u^{[n]}(t_k^+))) e^{\int_0^{t_k} a(s)f(s, u(s))ds} \\
 & + \int_{t_k}^t e^{\int_0^x a(s)f(s, u(s))ds} \vartheta^*(x, u) dx.
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 & (u(t_k^-) - h(t_k^-, u^{[1]}(t_k^-), u^{[2]}(t_k^-), \dots, u^{[n]}(t_k^-))) e^{\int_0^{t_k} a(s)f(s, u(s))ds} \\
 = & (u(t_{k-1}^+) - h(t_{k-1}^+, u^{[1]}(t_{k-1}^+), u^{[2]}(t_{k-1}^+), \dots, u^{[n]}(t_{k-1}^+))) e^{\int_0^{t_{k-1}} a(s)f(s, u(s))ds} \\
 & + \int_{t_{k-1}}^{t_k} e^{\int_0^x a(s)f(s, u(s))ds} \vartheta^*(x, u) dx.
 \end{aligned}$$

Thus, if $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}^*$, we obtain

$$\begin{aligned}
 & (u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))) e^{\int_0^t a(s)f(s, u(s))ds} \\
 = & (u(t_k^-) - h(t_k^-, u^{[1]}(t_k^-), u^{[2]}(t_k^-), \dots, u^{[n]}(t_k^-))) e^{\int_0^{t_k} a(s)f(s, u(s))ds} \\
 & + e^{\int_{t_k}^t a(s)f(s, u(s))ds} \vartheta^*(x, u) dx + e^{\int_0^{t_k} a(s)f(s, u(s))ds} I_k(t_k, u^{[1]}(t_k), \dots, u^{[n]}(t_k))
 \end{aligned}$$

$$\begin{aligned}
&= \left(u(t_{k-1}^+) - h(t_{k-1}^+, u^{[1]}(t_{k-1}^+), u^{[2]}(t_{k-1}^+), \dots, u^{[n]}(t_{k-1}^+)) \right) e^{\int_0^{t_{k-1}^+} a(s)f(s,u(s))ds} \\
&\quad + \int_{t_{k-1}^+}^{t_k} e^{\int_0^x a(s)f(s,u(s))ds} \vartheta^*(x, u) dx + \int_{t_k}^t e^{\int_0^x a(s)f(s,u(s))ds} \vartheta^*(x, u) dx \\
&\quad + e^{\int_0^{t_k} a(s)f(s,u(s))ds} I_k(t_k, u^{[1]}(t_k), \dots, u^{[n]}(t_k)) \\
&= \left(u(t_{k-1}^+) - h(t_{k-1}^+, u^{[1]}(t_{k-1}^+), u^{[2]}(t_{k-1}^+), \dots, u^{[n]}(t_{k-1}^+)) \right) e^{\int_0^{t_{k-1}^+} a(s)f(s,u(s))ds} \\
&\quad + \int_{t_{k-1}^+}^t e^{\int_0^x a(s)f(s,u(s))ds} \vartheta^*(x, u) dx \\
&\quad + e^{\int_0^{t_k} a(s)f(s,u(s))ds} I_k(t_k, u^{[1]}(t_k), \dots, u^{[n]}(t_k)). \tag{4.15}
\end{aligned}$$

For every $t \in \mathbb{R}$, there exists a $k \in \mathbb{N}^*$ such that $t \in (t_k, t_{k+1}]$, then $t + \omega \in (t_k + \omega, t_{k+1} + \omega] = t \in (t_{k+q}, t_{k+1+q}]$. So, from [\(4.15\)](#) we obtain

$$\begin{aligned}
&\left(u(t + \omega) - h(t + \omega, u^{[1]}(t + \omega), u^{[2]}(t + \omega), \dots, u^{[n]}(t + \omega)) \right) e^{\int_0^{t+\omega} a(s)f(s,u(s))ds} \\
&= \left(u(t_{k+q-1}^+) - h(t_{k+q-1}^+, u^{[1]}(t_{k+q-1}^+), u^{[2]}(t_{k+q-1}^+), \dots, u^{[n]}(t_{k+q-1}^+)) \right) \times \\
&\quad e^{\int_0^{t_{k+q-1}^+} a(s)f(s,u(s))ds} + \int_{t_{k+q-1}^+}^{t+\omega} e^{\int_0^x a(s)f(s,u(s))ds} \vartheta^*(x, u) dx \\
&\quad + e^{\int_0^{t_{k+q}} a(s)f(s,u(s))ds} I_{k+q}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q})).
\end{aligned}$$

In consequence,

$$\begin{aligned}
&\left(u(t + \omega) - h(t + \omega, u^{[1]}(t + \omega), u^{[2]}(t + \omega), \dots, u^{[n]}(t + \omega)) \right) e^{\int_0^{t+\omega} a(s)f(s,u(s))ds} \\
&= \left(u(t_{k+1}^+) - h(t_{k+1}^+, u^{[1]}(t_{k+1}^+), u^{[2]}(t_{k+1}^+), \dots, u^{[n]}(t_{k+1}^+)) \right) e^{\int_0^{t_{k+1}^+} a(s)f(s,u(s))ds} \\
&\quad + \int_{t_{k+1}^+}^{t+\omega} e^{\int_0^x a(s)f(s,u(s))ds} \vartheta^*(x, u) dx \\
&\quad + \sum_{\{k:t_k \in [t_{k+1}, t+\omega]\}} e^{\int_0^{t_{k+q}} a(s)f(s,u(s))ds} I_{k+q}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q})) \\
&= \left(u(t_{k+1}^-) - h(t_{k+1}^-, u^{[1]}(t_{k+1}^-), u^{[2]}(t_{k+1}^-), \dots, u^{[n]}(t_{k+1}^-)) \right) e^{\int_0^{t_{k+1}^-} a(s)f(s,u(s))ds} \\
&\quad + \int_{t_{k+1}^-}^{t+\omega} e^{\int_0^x a(s)f(s,u(s))ds} \vartheta^*(x, u) dx \\
&\quad + \sum_{\{k:t_k \in [t, t+\omega]\}} e^{\int_0^{t_{k+q}} a(s)f(s,u(s))ds} I_{k+q}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q}))
\end{aligned}$$

$$\begin{aligned}
 &= \left(u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \right) + q e^{\int_0^t a(s)f(s, u(s))ds} \\
 &\quad + \int_t^{t+\omega} e^{\int_0^x a(s)f(s, u(s))ds} \vartheta^*(x, u) dx + \sum_{\{k: t_k \in [t, t+\omega]\}} e^{\int_0^{t_k} a(s)f(s, u(s))ds} \times \\
 &\quad I_{k+q}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q})). \tag{4.16}
 \end{aligned}$$

Then from (4.16), we get

$$\begin{aligned}
 &\left(u(t+\omega) - h(t+\omega, u^{[1]}(t+\omega), u^{[2]}(t+\omega), \dots, u^{[n]}(t+\omega)) \right) e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} \\
 &\quad - \left(u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \right) e^{\int_0^t a(s)f(s, u(s))ds} \\
 &= \int_t^{t+\omega} e^{\int_0^x a(s)f(s, u(s))ds} \vartheta^*(x, u) dx \\
 &\quad + \sum_{\{k: t_k \in [t, t+\omega]\}} e^{\int_0^{t_k} a(s)f(s, u(s))ds} I_{k+q}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q})).
 \end{aligned}$$

By dividing both sides of the above equation by $e^{\int_0^t a(s)f(s, u(s))ds}$, we obtain

$$\begin{aligned}
 &u(t) - h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &= \frac{1}{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} - 1} \left[\int_t^{t+\omega} e^{\int_t^x a(s)f(s, u(s))ds} \vartheta^*(x, u) dx \right. \\
 &\quad \left. + \sum_{\{k: t_k \in [t, t+\omega]\}} e^{\int_t^{t_k} a(s)f(s, u(s))ds} \right. \\
 &\quad \left. \times I_{k+q}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q}), \dots, u^{[n]}(t_{k+q})) \right]. \tag{4.17}
 \end{aligned}$$

Make substitution of (4.14) into (4.17), we get

$$\begin{aligned}
 u(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &\quad + \int_0^{t+\omega} \frac{e^{\int_t^x a(s)f(s, u(s))ds}}{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} - 1} \\
 &\quad \times \left[-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \right. \\
 &\quad \left. + g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \right] dx \\
 &\quad + \sum_{\{k: t_k \in [t, t+\omega]\}} \frac{e^{-\int_{t_k}^t a(s)f(s, u(s))ds}}{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} - 1} I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)).
 \end{aligned}$$

Thus

$$\begin{aligned}
 u(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &+ \int_0^{t+\omega} G(t, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &+ g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &+ \sum_{\{k:t_k \in [t, t+\omega]\}} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)).
 \end{aligned}$$

It is straightforward to verify that is clear, for all $t, u \in \mathbb{R}^2$, we have

$$\begin{aligned}
 G(t + \omega, t) - G(t, t) &= 1, G(t + \omega, u + \omega) = G(t, u), \\
 \frac{\partial G(t, x)}{\partial t} &= -a(t)f(t, u(t))G(t, x). \tag{4.18}
 \end{aligned}$$

ii) Next, we prove the converse. If u is a ω -periodic solution of equation (4.5), then by (4.11) and (4.12), we obtain

$$\begin{aligned}
 &u(t) e^{\int_0^t a(s)f(s, u(s))ds} \\
 &= e^{\int_0^t a(s)f(s, u(s))ds} h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &+ \int_0^{t+\omega} \frac{e^{\int_0^x a(s)f(s, u(s))ds}}{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} - 1} [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &+ g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &+ \sum_{\{k:t_k \in [t, t+\omega]\}} \frac{e^{\int_0^{t_k} a(s)f(s, u(s))ds}}{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} - 1} I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)). \tag{4.19}
 \end{aligned}$$

For $t_k \neq t$, differentiating (4.19), we have

$$\begin{aligned}
 &\left(u'(t) + a(t)f(t, u(t)) \right) e^{\int_0^t a(s)f(s, u(s))ds} \\
 &= e^{\int_0^t a(s)f(s, u(s))ds} \left[a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \right. \\
 &\quad \left. + \frac{d}{dt} h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \right] \\
 &+ \frac{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds}}{e^{\int_0^{t+\omega} a(s)f(s, u(s))ds} - 1} [-a(t + \omega)f(t + \omega, u(t + \omega))
 \end{aligned}$$

$$\begin{aligned}
& \times h(t + \omega, u^{[1]}(t + \omega), u^{[2]}(t + \omega), \dots, u^{[n]}(t + \omega)) \\
& + g(t + \omega, u^{[1]}(t + \omega), u^{[2]}(t + \omega), \dots, u^{[n]}(t + \omega))] \\
& - e^{\int_0^t a(s)f(s,u(s))ds} [-a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
& + g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))] \\
= & e^{\int_0^t a(s)f(s,u(s))ds} [a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
& + \frac{d}{dt}h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))] \\
& + \frac{e^{\int_0^t a(s)f(s,u(s))ds}}{e^{\int_0^\omega a(s)f(s,u(s))ds} - 1} [-a(t)f(t, u(t))h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
& + g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t))] \left(e^{\int_0^\omega a(s)f(s,u(s))ds} - 1 \right),
\end{aligned}$$

hence

$$\begin{aligned}
u'(t) & = -a(t)f(t, u(t)) + g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
& + \frac{d}{dt}h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)).
\end{aligned}$$

It follows from (4.11) that for $t = t_l$, $l \in \mathbb{N}^*$, we obtain

$$\begin{aligned}
& u(t_l^+) - u(t_l^-) \\
= & \sum_{t_l^+ \leq t_k \leq t_l^+ + \omega} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
& - \sum_{t_l^- \leq t_k \leq t_l^- + \omega} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
= & G(t_l, t_l + \omega) I_l(t_l + \omega, u^{[1]}(t_l + \omega), u^{[2]}(t_l + \omega), \dots, u^{[n]}(t_l + \omega)) \\
& - G(t_l, t_l) I_l(t_l, u^{[1]}(t_l), u^{[2]}(t_l), \dots, u^{[n]}(t_l)) \\
= & I_l(t_l, u^{[1]}(t_l), u^{[2]}(t_l), \dots, u^{[n]}(t_l)).
\end{aligned}$$

Hence, u is a positive ω -periodic solution of (4.5). Thus, the proof of Lemma 4.1 is completed. ■

4.3 Main results

The primary goal of this section is to investigate the existence of positive periodic solutions to equation (4.5) by utilizing Schauder's fixed point theorem. To this end, we prove three main theorems, which form the central results of this study. These theorems provide new and meaningful sufficient conditions ensuring the existence of such positive periodic solutions.

By looking at the three cases $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) > 0$, $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) < 0$, and $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) = 0$, for all $t \in \mathbb{R}$, $u \in P_\omega$, we can prove the existence of positive periodic solutions of the equation (4.5).

To apply Theorem 1.9 we need to define a Banach space $(X, \|\cdot\|)$, a closed convex compact subset \mathcal{M} of X and construct one mapping. Thus, we let $(P_\omega, \|\cdot\|) = (X, \|\cdot\|)$ and $\mathcal{M} = P_\omega(n_1, n_2, n_3)$. From Lemma 4.1, we define the operator $\mathcal{A} : P_\omega(n_1, n_2, n_3) \rightarrow P_\omega$ as follows:

$$\begin{aligned}
 (\mathcal{A}u)(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &+ \int_t^{t+\omega} G(t, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &+ g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &+ \sum_{\{k:t_k \in [t, t+\omega]\}} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)). \quad (4.20)
 \end{aligned}$$

Before presenting the main results, we start by defining the following conditions:

(H₄) There exist nonnegative constants $\hat{\alpha}$ and α such that for all $t \in \mathbb{R}$, $u \in P_\omega(n_1, n_2, n_3)$,

$$\hat{\alpha} \leq f(t, u) \leq \alpha. \quad (4.21)$$

(H₅) There exist nonnegative functions $\hat{\gamma}_k, \gamma_k \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that for all

$t \in \mathbb{R}$, $u \in P_\omega(n_1, n_2, n_3)$,

$$\widehat{\gamma}_k(t) u \leq I_k(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) \leq \gamma_k(t) u, \quad k \in \mathbb{N}^*. \quad (4.22)$$

To simplify our description, we let

$$\lambda_k = \frac{1}{\omega} \sum_{k=0}^q \gamma_k(t_k), \quad \widehat{\lambda}_k = \frac{1}{\omega} \sum_{k=0}^q \widehat{\gamma}_k(t_k).$$

It is easy to see that for $u \in [t, t + \omega]$, thanks to (4.12) and (4.21), for $t \in \mathbb{R}$, we have

$$0 < \frac{1}{e^{\alpha \int_0^\omega a(s) ds} - 1} = m \leq G(t, x) \leq M = \frac{e^{\alpha \int_0^\omega a(s) ds}}{e^{\widehat{\alpha} \int_0^\omega a(s) ds} - 1}. \quad (4.23)$$

Case 1: Now, we start with the case $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) > 0$, $t \in \mathbb{R}$, $u \in P_\omega(n_1, n_2, n_3)$, and we will prove a set of preparatory lemmas in order to use them in the proof of the main existence results. For this reason, we assume that (H₄) and (H₅) hold, and further assume that:

(H₆) There exist nonnegative constants $\widehat{\beta}$ and β such that,

$$\widehat{\beta} u \leq h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) \leq \beta u, \quad \text{for all } t \in [0, \omega], \quad u \in P_\omega(n_1, n_2, n_3). \quad (4.24)$$

(H₇) For all $t \in [0, \omega]$, $u \in P_\omega(n_1, n_2, n_3)$,

$$0 \leq \Upsilon_1 \leq g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \leq \Upsilon_2, \quad (4.25)$$

where

$$\Upsilon_1 = \frac{(1 - \widehat{\beta})}{m\omega} n_1 + \frac{M\alpha\beta\rho}{m} n_2 - \widehat{\lambda}_k n_1, \quad \Upsilon_2 = \frac{(1 - \beta)}{M\omega} n_2 - \lambda_k n_2. \quad (4.26)$$

Lemma 4.2 Suppose that the conditions (4.7)-(4.9) are satisfied. Then the operator \mathcal{A} defined by (4.20) is continuous and compact on $P_\omega(n_1, n_2, n_3)$.

Proof. Since $P_\omega(n_1, n_2, n_3)$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, \omega]$, it follows from the Arzela-Ascoli theorem that $P_\omega(n_1, n_2, n_3)$ is a compact subset of this space. Moreover, since any continuous operator transforms compact sets into compact sets, to show that \mathcal{A} is a compact operator, it suffices to show that it is continuous.

Let us prove that \mathcal{A} is continuous. Take $u, v \in P_\omega(n_1, n_2, n_3)$, $t \in \mathbb{R}$, then we have

$$\begin{aligned}
 & |(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| \\
 \leq & \left| h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) - h(t, v^{[1]}(t), v^{[2]}(t), \dots, v^{[n]}(t)) \right| \\
 & + \int_t^{t+\omega} |G(t, x)| | -a(x) | f(x, u(x)) h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 & - f(x, v(x)) h(x, v^{[1]}(x), v^{[2]}(x), \dots, v^{[n]}(x)) | dx \\
 & + \int_t^{t+\omega} |G(t, x)| |g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 & - g(x, v^{[1]}(x), v^{[2]}(x), \dots, v^{[n]}(x)) | dx + \sum_{\{k:t_k \in [t, t+\omega]\}} |G(t, t_k)| \times \\
 & |I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) - I_k(t_k, v^{[1]}(t_k), v^{[2]}(t_k), \dots, v^{[n]}(t_k))|.
 \end{aligned}$$

In view of conditions (4.7) - (4.9) and (4.21) , (4.23) , we have

$$\begin{aligned}
 & |(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| \\
 \leq & \sum_{i=1}^n L_i(t) \|u^{[i]} - v^{[i]}\| + \omega M \rho \alpha \sum_{i=1}^n L_i(t) \|u^{[i]} - v^{[i]}\| \\
 & + \omega M \sum_{i=1}^n C_i(t) \|u^{[i]} - v^{[i]}\| + M \sum_{0 \leq t_k < \omega} \sum_{i=1}^n N_i(t_k) \|u^{[i]} - v^{[i]}\|.
 \end{aligned}$$

From Lemma 3.1 and notations (4.10), it follows that

$$\begin{aligned} & |(\mathcal{A}u)(t) - (\mathcal{A}v)(t)| \\ & \leq \sum_{i=1}^n \sum_{j=0}^{i-1} L_i^* n_3^j \|u - v\| + \omega M \rho \alpha \sum_{i=1}^n \sum_{j=0}^{i-1} L_i^* n_3^j \|u - v\| \\ & \quad + \omega M \sum_{i=1}^n \sum_{j=0}^{i-1} C_i^* n_3^j \|u - v\| + M \sum_{i=1}^n \sum_{j=0}^{i-1} \sum_{k=1}^q N_i(t_k) n_3^j \|u - v\|, \end{aligned}$$

then

$$\|\mathcal{A}u - \mathcal{A}v\| \leq \Gamma_1 \|u - v\|, \quad (4.27)$$

where

$$\begin{aligned} \Gamma_1 &= (1 + \omega M \rho \alpha) \sum_{i=1}^n \sum_{j=0}^{i-1} L_i^* n_3^j + M \omega \sum_{i=1}^n \sum_{j=0}^{i-1} C_i^* n_3^j \\ & \quad + M \sum_{i=1}^n \sum_{j=0}^{i-1} N_{ik}^* n_3^j, \end{aligned}$$

and

$$N_{ik}^* = \sum_{k=1}^q N_i(t_k), \quad i = 1, 2, \dots, n.$$

This shows that the operator \mathcal{A} is Lipschitz continuous and hence continuous.

Therefore, the compactness of \mathcal{A} follows from its continuity. ■

Lemma 4.3 For all $\alpha_1, \alpha_2 \in \mathbb{R}, t \in [0, \omega]$,

$$\int_{\alpha_1}^{\alpha_1 + \omega} |G(\alpha_2, t) - G(\alpha_1, t)| dt \leq \rho \omega \alpha M |\alpha_2 - \alpha_1|, \quad (4.28)$$

and

$$G(\alpha_2, t_k) - G(\alpha_1, t_k) \leq \rho M \alpha |\alpha_2 - \alpha_1|, \quad k \in \mathbb{N}^*, \quad (4.29)$$

where, ρ, α, M are respectively given by (4.10), (4.21), (4.23).

Proof. For all $\alpha_2, \alpha_1 \in \mathbb{R}$ with $\alpha_1 < \alpha_2$, we have

$$\begin{aligned} |G(\alpha_2, t_k) - G(\alpha_1, t_k)| &= \left| \frac{e^{\int_{\alpha_2}^{t_k} a(s)f(s,u(s))ds}}{e^{\int_0^{\omega} a(s)f(s,u(s))ds} - 1} - \frac{e^{\int_{\alpha_1}^{t_k} a(s)f(s,u(s))ds}}{e^{\int_0^{\omega} a(s)f(s,u(s))ds} - 1} \right| \\ &= \left| \frac{e^{\int_{\alpha_2}^{t_k} a(s)f(s,u(s))ds} (1 - e^{\int_{\alpha_1}^{\alpha_2} a(s)f(s,u(s))ds})}{e^{\int_0^{\omega} a(s)f(s,u(s))ds} - 1} \right|. \end{aligned}$$

By using the Mean value theorem on the function

$$g = e^{\int_{\alpha}^{\alpha_2} a(s)f(s,u(s))ds},$$

in the interval $[\alpha_1, \alpha_2]$, it is easy to see that g is defined and continuous in the interval $[\alpha_1, \alpha_2]$ and differentiable in the interval $] \alpha_1, \alpha_2 [$, hence:

$$g(\alpha_2) - g(\alpha_1) = 1 - e^{\int_{\alpha_1}^{\alpha_2} a(s)f(s,u(s))ds} = a(c)f(c, u(c))e^{\int_c^{\alpha_2} a(s)f(s,u(s))ds}(\alpha_2 - \alpha_1),$$

thus

$$\begin{aligned} &|G(\alpha_2, t_k) - G(\alpha_1, t_k)| \\ &\leq \left| \frac{e^{\int_{\alpha_2}^{t_k} a(s)f(s,u(s))ds} a(c)f(c, u(c))e^{\int_c^{\alpha_2} a(s)f(s,u(s))ds}(\alpha_2 - \alpha_1)}{e^{\int_0^{\omega} a(s)f(s,u(s))ds} - 1} \right| \\ &\leq \rho a \frac{e^{\int_c^{t_k} a(s)f(s,u(s))ds}}{e^{\int_0^{\omega} a(s)f(s,u(s))ds} - 1} |\alpha_2 - \alpha_1| \\ &\leq \rho a \left| \frac{e^{a \int_0^{\omega} a(s)ds}}{e^{\widehat{a} \int_0^{\omega} a(s)ds} - 1} \right| |\alpha_2 - \alpha_1| \\ &\rho a M |\alpha_2 - \alpha_1|. \end{aligned}$$

Similarly, we derive the following result

$$\int_{\alpha_1}^{\alpha_1 + \omega} |G(\alpha_2, t) - G(\alpha_1, t)| dt \leq \omega \rho a M |\alpha_2 - \alpha_1|.$$

■

Lemma 4.4 Suppose (4.6), (4.7) and (4.21)-(4.25) hold. If

$$\begin{aligned} \Gamma_2 = & : L_0 + \sum_{i=1}^n \sum_{j=0}^{i-1} L_i^* n_3^{j+1} + 2M\rho\alpha\beta n_2 + \omega M\rho^2\alpha^2\beta n_2 \\ & + (2 + \omega\rho\alpha) M\Upsilon_2 + 3\omega\lambda_k M\rho\alpha n_2 \leq n_3, \end{aligned} \quad (4.30)$$

then $\mathcal{A}u \in P_\omega(n_1, n_2, n_3)$ for all $u \in P_\omega(n_1, n_2, n_3)$.

Proof. From (4.20), it is easy to verify that $(\mathcal{A}u)(t + \omega) = (\mathcal{A}u)(t)$, and $(\mathcal{A}u)(\cdot)$ is continuous in (t_k, t_{k+1}) , $(\mathcal{A}u)(t_k^+)$ and $(\mathcal{A}u)(t_k^-)$ exist and $(\mathcal{A}u)(t_k^-) = (\mathcal{A}u)(t)$ for $k \in \mathbb{N}^*$.

Let $\alpha_1, \alpha_2 \in [0, \omega]$ with $\alpha_1 < \alpha_2$, we have

$$\begin{aligned} & |(\mathcal{A}(u))(\alpha_2) - (\mathcal{A}(u))(\alpha_1)| \\ \leq & \left| h(\alpha_2, u^{[1]}(\alpha_2), u^{[2]}(\alpha_2), \dots, u^{[n]}(\alpha_2)) - h(\alpha_1, u^{[1]}(\alpha_1), u^{[2]}(\alpha_1), \dots, u^{[n]}(\alpha_1)) \right| \\ + & \left| \int_{\alpha_2}^{\alpha_2+\omega} G(\alpha_2, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \right. \\ & \left. - \int_{\alpha_1}^{\alpha_1+\omega} G(\alpha_1, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \right| \\ + & \left| \int_{\alpha_2}^{\alpha_2+\omega} G(\alpha_2, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \right. \\ & \left. - \int_{\alpha_1}^{\alpha_1+\omega} G(\alpha_1, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ + & \left| \sum_{\{k:t_k \in [\alpha_2, \alpha_2+\omega]\}} G(\alpha_2, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right. \\ & \left. - \sum_{\{k:t_k \in [\alpha_1, \alpha_1+\omega]\}} G(\alpha_1, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right|. \end{aligned}$$

It follows from (4.7) and Lemma 3.1, that

$$\begin{aligned} & \left| h(\alpha_2, u^{[1]}(\alpha_2), u^{[2]}(\alpha_2), \dots, u^{[n]}(\alpha_2)) - h(\alpha_1, u^{[1]}(\alpha_1), u^{[2]}(\alpha_1), \dots, u^{[n]}(\alpha_1)) \right| \\ \leq & L_0|\alpha_2 - \alpha_1| + \sum_{i=1}^n L_i(t) |u^{[n]}(\alpha_2) - u^{[n]}(\alpha_1)| \end{aligned}$$

$$\leq \left(L_0 + \sum_{i=1}^n \sum_{j=0}^{i-1} L_i^* n_3^{j+1} \right) |\alpha_2 - \alpha_1|. \quad (4.31)$$

From (4.21), (4.23), (4.24) and Lemma 4.3, we get

$$\begin{aligned} & \left| \int_{\alpha_2}^{\alpha_2+\omega} G(\alpha_2, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] \right. \\ & \quad \left. - \int_{\alpha_1}^{\alpha_1+\omega} G(\alpha_1, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \right| \\ & \leq \left| \int_{\alpha_2}^{\alpha_1} G(\alpha_2, x) a(x) f(x, u(x)) h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \quad + \left| \int_{\alpha_1+\omega}^{\alpha_2+\omega} G(\alpha_2, x) a(x) f(x, u(x)) h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \quad + \left| \int_{\alpha_1}^{\alpha_1+\omega} (G(\alpha_2, x) - G(\alpha_1, x)) a(x) f(x, u(x)) h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \leq M\rho\alpha\beta n_2 |\alpha_2 - \alpha_1| + M\rho\alpha\beta n_2 |\alpha_2 - \alpha_1| + \omega M\rho^2\alpha^2\beta n_2 |\alpha_2 - \alpha_1| \\ & \leq (2M\rho\alpha\beta n_2 + \omega\rho^2 M\alpha^2\beta n_2) |\alpha_2 - \alpha_1|. \end{aligned} \quad (4.32)$$

In view of (4.22), (4.25) and Lemma 4.3, we obtain

$$\begin{aligned} & \left| \int_{\alpha_2}^{\alpha_2+\omega} G(\alpha_1, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \right. \\ & \quad \left. - \int_{\alpha_1}^{\alpha_1+\omega} G(\alpha_1, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \leq \left| \int_{\alpha_2}^{\alpha_1} G(\alpha_2, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \quad + \left| \int_{\alpha_1+\omega}^{\alpha_2+\omega} G(\alpha_2, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \quad + \left| \int_{\alpha_1}^{\alpha_1+\omega} [G(\alpha_2, x) - G(\alpha_1, x)] g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \right| \\ & \leq M\Upsilon_2 |\alpha_2 - \alpha_1| + M\Upsilon_2 |\alpha_2 - \alpha_1| + \omega M\rho\alpha\Upsilon_2 |\alpha_2 - \alpha_1| \\ & \leq (2M\Upsilon_2 + \omega M\rho\alpha\Upsilon_2) |\alpha_2 - \alpha_1|. \end{aligned} \quad (4.33)$$

In view of (4.22), (4.23) and Lemma 4.3, we obtain

$$\begin{aligned}
 & \left| \sum_{\{k:t_k \in [\alpha_2, \alpha_2 + \omega]\}} G(\alpha_2, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right. \\
 & \quad \left. - \sum_{\{k:t_k \in [\alpha_1, \alpha_1 + \omega]\}} G(\alpha_1, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right| \\
 = & \left| \sum_{\{k:t_k \in [\alpha_2, \alpha_1 + \omega]\}} G(\alpha_2, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right. \\
 & \quad + \sum_{\{k:t_k \in [\alpha_1 + \omega, \alpha_2 + \omega]\}} G(\alpha_2, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
 & \quad - \sum_{\{k:t_k \in [\alpha_1, \alpha_2]\}} G(\alpha_1, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
 & \quad \left. - \sum_{\{k:t_k \in [\alpha_2, \alpha_1 + \omega]\}} G(\alpha_1, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right| \\
 \leq & \left| \sum_{\{k:t_k \in [\alpha_2, \alpha_1 + \omega]\}} (G(\alpha_2, t_k) - G(\alpha_1, t_k)) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right| \\
 & \quad + \left| \sum_{\{k:t_k \in [\alpha_1, \alpha_2]\}} (G(\alpha_2, t_k) - G(\alpha_1, t_k)) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \right| \\
 \leq & 3 \sum_{k=1}^q |(G(\alpha_2, t_k) - G(\alpha_1, t_k))| |I_k(t_k, (t_k)(t_k))| \\
 \leq & 3\omega\rho M\alpha\lambda_k n_2 |\alpha_2 - \alpha_1|. \tag{4.34}
 \end{aligned}$$

Thus, it follows from (4.31)-(4.34) that

$$|(\mathcal{A}(u))(\alpha_2) - (\mathcal{A}(u))(\alpha_1)| \leq \Gamma_2 |\alpha_2 - \alpha_1| \leq n_3 |\alpha_2 - \alpha_1|, \tag{4.35}$$

where Γ_2 is as in (4.30).

Let us show $n_1 \leq \mathcal{A}u \leq n_2$, for all $u \in P_\omega(n_1, n_2, n_3)$.

It follows from (4.21)-(4.25) and (4.23), then for $u \in P_\omega(n_1, n_2, n_3)$, and

$t \in \mathbb{R}$, we get

$$\begin{aligned}
 (\mathcal{A}u)(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &\quad + \int_t^{t+\omega} G(t, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &\quad + g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &\quad + \sum_{\{k:t_k \in [t, t+\omega]\}} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
 &\leq \beta u(t) + M \int_t^{t+\omega} g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \\
 &\quad + M\omega \frac{1}{\omega} \sum_{k=0}^q \gamma_k(t_k) u(t_k) \\
 &\leq \beta u(t) + M \int_t^{t+\omega} g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx + \lambda_k M\omega n_2 \\
 &\leq \beta n_2 + M\omega \left(\frac{(1-\beta)}{M\omega} n_2 - \lambda_k n_2 \right) + M\omega \lambda_k n_2 \\
 &= n_2.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (\mathcal{A}u)(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &\quad - \int_t^{t+\omega} G(t, x) [a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &\quad + \int_t^{t+\omega} G(t, x) g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \\
 &\quad + m \sum_{\{k:t_k \in [t, t+\omega]\}} I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
 &\geq \widehat{\beta} u(t) - M\alpha\beta\rho u(t) + m \int_t^{t+\omega} g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &\quad + m\omega \frac{1}{\omega} \sum_{k=0}^q \widehat{\gamma}_k(t_k) u(t_k) \\
 &\geq \widehat{\beta} n_1 - M\alpha\beta\rho n_2 + m\omega \left[\frac{n_1(1-\widehat{\beta})}{m\omega} + \frac{M\alpha\beta\rho}{m\omega} n_2 - n_1 \widehat{\lambda}_k \right] + m\omega n_1 \widehat{\lambda}_k \\
 &= n_1,
 \end{aligned}$$

which leads to

$$n_1 \leq \mathcal{A}u \leq n_2, \text{ for all } u \in P_\omega(n_1, n_2, n_3). \quad (4.36)$$

Since \mathcal{A} is well defined, i.e. $\mathcal{A}u \in P_\omega$ for all $u \in P_\omega(n_1, n_2, n_3)$, we conclude that $\mathcal{A}(P_\omega(n_1, n_2, n_3)) \subset P_\omega(n_1, n_2, n_3)$. ■

Theorem 4.1 *Suppose that conditions (4.7)-(4.9), (4.21)-(4.25), (4.35) and (4.36) hold. Then equation (4.5) has at least one positive ω - solution u in $P_\omega(n_1, n_2, n_3)$.*

Proof. By virtue of Lemma 4.2 and Lemma 4.4, all the requirements of Schauder's fixed point theorem are fulfilled. So, $\mathcal{A}u = u$ has at least a fixed point u on $P_\omega(n_1, n_2, n_3)$. Finally, via Lemma 4.1, equation (4.5) possesses at least one positive ω -periodic solution satisfying $n_1 \leq u \leq n_2$. ■

Cas 2: To treat the case $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) < 0, t \in \mathbb{R}, u \in P_\omega(n_1, n_2, n_3)$, we replace (H₇), by the following condition.

(H₈) There exist nonpositive constants θ and $\hat{\theta}$ such that for all $t \in [0, \omega]$, $u \in P_\omega(n_1, n_2, n_3)$,

$$\hat{\theta}u \leq h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \leq \theta u. \quad (4.37)$$

Theorem 4.2 *Suppose that (4.7)-(4.9), (4.21), (4.22), (4.37) hold, and further assume that:*

(H₉) For all $t \in [0, \omega]$, $u \in P_\omega(n_1, n_2, n_3)$,

$$\Pi_1 \leq g(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \leq \Pi_2, \quad (4.38)$$

where

$$\Pi_1 = \frac{n_1 - \hat{\theta}n_2}{m\omega} - \hat{\lambda}_k n_1, \quad \Pi_2 = \frac{n_2}{M\omega} + \hat{\theta}\alpha\rho n_2 - \lambda_k n_2.$$

Then the equation (4.5) has at least a solution $u \in P_\omega(n_1, n_2, n_3)$.

Proof. Let us start by proving that $\forall u \in P_\omega(n_1, n_2, n_3)$, $\mathcal{A}u \in P_\omega(n_1, n_2, n_3)$.

It follows from (4.21), (4.22), (4.37), (4.38) and (4.23), then for $u \in P_\omega(n_1, n_2, n_3)$, and $t \in \mathbb{R}$, we get

$$\begin{aligned}
 (\mathcal{A}u)(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &\quad + \int_t^{t+\omega} G(t, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &\quad + g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &\quad + \sum_{\{k:t_k \in [t, t+\omega]\}} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
 &\leq -M\omega\rho\alpha\theta u(t) + M \int_t^{t+\omega} g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \\
 &\quad + M\omega \frac{1}{\omega} \sum_{k=0}^q \gamma_k(t_k) u(t_k) \\
 &\leq -M\omega\rho\alpha\hat{\theta}n_2 + M\omega \left(\frac{n_2}{M\omega} + \rho\alpha\hat{\theta}n_2 - \lambda_k n_2 \right) + \lambda_k M\omega n_2 \\
 &= n_2.
 \end{aligned}$$

Also,

$$\begin{aligned}
 (\mathcal{A}u)(t) &= h(t, u^{[1]}(t), u^{[2]}(t), \dots, u^{[n]}(t)) \\
 &\quad + \int_t^{t+\omega} G(t, x) [-a(x)f(x, u(x))h(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) \\
 &\quad + g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x))] dx \\
 &\quad + \sum_{\{k:t_k \in [t, t+\omega]\}} G(t, t_k) I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k), \dots, u^{[n]}(t_k)) \\
 &\geq \hat{\theta}u(t) + m \int_t^{t+\omega} g(x, u^{[1]}(x), u^{[2]}(x), \dots, u^{[n]}(x)) dx \\
 &\quad + m\omega \frac{1}{\omega} \sum_{k=0}^q \hat{\gamma}_k(t_k) u(t_k) \\
 &\geq \hat{\theta}n_2 + m\omega \left(\frac{n_1 - \hat{\theta}n_2}{m\omega} - \hat{\lambda}_k n_1 \right) + m\omega \hat{\lambda}_k n_1 \\
 &= n_1,
 \end{aligned}$$

which leads to

$$n_1 \leq \mathcal{A}u \leq n_2, \text{ for all } u \in P_\omega(n_1, n_2, n_3). \quad (4.39)$$

The remaining steps of the proof are similar to the corresponding work in Lemma 4.4 hence we omit the details. Also, we conclude from Lemma 4.2 that \mathcal{A} is continuous and compact on $P_\omega(n_1, n_2, n_3)$. We deduce from Schauder's fixed-point theorem (see, [63]) that the operator \mathcal{A} has at least one fixed point $u \in P_\omega(n_1, n_2, n_3)$ which is a solution to (4.7). As a result of Lemma 4.1, u is a solution to problem (4.5). This completes the proof

■

Cas 3: The case $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) = 0$, for $t \in \mathbb{R}$, $u \in P_\omega(n_1, n_2, n_3)$.

Theorem 4.3 Suppose that (4.7)-(4.9), (4.21), (4.22) hold, and further assume that:

(H₁₀) For all $t \in [0, \omega]$, $u \in P_\omega(n_1, n_2, n_3)$,

$$\frac{n_1}{m\omega} - \widehat{\lambda}_k n_1 \leq g(t, u(t), u^{[2]}(t), \dots, u^{[n]}(t)) \leq \frac{n_2}{M\omega} - \lambda_k n_2. \quad (4.40)$$

Then the equation (4.5) has at least a solution $u \in P_\omega(n_1, n_2, n_3)$.

Proof. The proof follows along the lines of the proof of Theorem 4.1 and hence we omit. ■

Remark 4.1 Notice that when $f(t, u) = 1$ and $I_k(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) = 0$, $k \in \mathbb{N}^*$ for all $t \in \mathbb{R}$, $u \in P_\omega(n_1, n_2, n_3)$, then the Eq. (4.5) reduces to equation (4.4), which was studied in [12]. Thus, our results are more general than those obtained in [12]. When $a(t) = 0$ and $h(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) = I_k(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) = 0$, $k \in \mathbb{N}^*$, $g(t, u^{[1]}, u^{[2]}, \dots, u^{[n]}) = \sum_{i=0}^n c_i(t)y^{[i]} + \widehat{F}(t)$, for all $t \in \mathbb{R}$, then the Eq. (4.5) turns into (4.3), which was studied in [79]. Our results are more general than those obtained in [12], [79].

We should point out that some authors [[12], [79]] have imposed hypotheses and employed fixed point methods to some classes of periodic equations of iterative differential equations (without impulses), which however seem to only guarantee that a periodic, rather than positive solution must exist. The research method of this paper is different from the existing research methods (see [[12], [79]]).

4.4 Applications

In this section, we present an analysis of two examples to exemplify two facts. Firstly, we will show how to apply our main result in this paper, Theorem 4.1. More importantly, we highlight the significance of our findings by showing that the existing theory developed by Bouakkaz et al. cannot be applied to Example 4.2.

Example 4.1 We consider the equation

$$\begin{aligned} \frac{d}{dt}u(t) &= -a(t)f(t, u(t))u(t) + g(t, u^{[1]}(t), u^{[2]}(t)) \\ &\quad + \frac{d}{dt}h(t, u^{[1]}(t), u^{[2]}(t)), t \neq t_k = \frac{1}{2}\pi + (k-1)\pi, k \in \mathbb{N}^*, \\ u(t_k^+) - u(t_k^-) &= I_k(t_k, u(t_k), u^{[2]}(t_k)), t = t_k, k \in \mathbb{N}^*. \end{aligned} \quad (4.41)$$

Corresponding to equation (4.5), we let $n = 2$, $\omega = 2\pi$, $a(t) = \frac{0.1}{2\pi}$, $f(t, u(t)) = 0.5|\cos(t)| + 0.5$, $h(t, u^{[1]}(t), u^{[2]}(t)) = \frac{0.0005}{3}|\cos t \sin t| + \frac{0.0005}{6} \times (|\cos t| + 1)u^{[1]}(t) + \frac{0.0005}{3}u^{[2]}(t)$, $I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k)) = \frac{1}{3000}(|\sin(2t_k)| + \pi)u^{[1]}$, $g(t, u^{[1]}(t), u^{[2]}(t)) = \frac{1}{500} + \frac{|\cos t|}{500}|\sin u^{[1]}(t)| + \frac{1}{500}|\sin u^{[2]}(t)|$.

By letting $n_1 = 0.1$, $n_2 = 1$, $n_3 = 10$, i.e., $P_\omega(n_1, n_2, n_3) = P_{2\pi}(0.1, 1, 10)$.

If $u_1, u_2, v_1, v_2 \in P_{2\pi}(0.1, 1, 10)$, $t \in [0, \omega]$, we have

$$\begin{aligned} |h(t, u_1, v_1) - h(s, u_2, v_2)| &\leq L_0 |t - s| + L_1(t) \|u_1 - u_2\| + L_2(t) \|v_1 - v_2\|, \\ |g(t, u_1, v_1) - g(t, u_2, v_2)| &\leq C_1(t) \|u_1 - u_2\| + C_2(t) \|v_1 - v_2\|, \\ |I_k(t, u_1, v_1) - I_k(t, u_2, v_2)| &\leq N_{1k}(t) \|u_1 - u_2\| + N_{2k}(t) \|v_1 - v_2\|, \end{aligned}$$

where

$$\begin{aligned} L_0 &= \frac{0.0005}{3}, L_1(t) = \frac{0.0005}{6} (|\cos t| + 1), L_2(t) = \frac{0.0005}{3}, \\ N_{1k}(t) &= \frac{1}{3000} (|\sin(2t_k)| + \pi), N_{2k}(t) = 0, \\ C_1(t) &= \frac{|\cos t|}{500}, C_2(t) = \frac{1}{500}. \end{aligned}$$

For $\beta = 0.005$ and $\widehat{\beta} = \frac{1}{2} \times 0.00005$, we have

$$\begin{aligned} \widehat{\beta}u &\leq \frac{1}{2} \times 0.00005 \leq h(t, u^{[1]}, u^{[2]}) \leq 0.0005 \leq \beta u, \quad \forall t \in [0, 2\pi], \\ u &\in P_{2\pi}(0.1, 1, 10). \end{aligned}$$

Since $|\sin(2t)| \leq |2t|$ and $|\cos(t)| \leq 1$, for $t \in [0, 2\pi]$, $u \in P_{2\pi}(0.1, 1, 10)$, we obtain

$$0.5 = \widehat{\alpha} \leq f(t, u) \leq \alpha = 1, \quad (4.42)$$

and

$$\frac{\pi}{3000}u = \widehat{\gamma}_k(t_k)u \leq I_k(t_k, u^{[1]}, u^{[2]}) \leq \gamma_k(t_k)u = \frac{(2t_k + \pi)}{3000}u, \quad k \in \mathbb{N}^*.$$

Choosing $q = 2$, we have $t_{k+q} = t_k + \omega$, $I_{(k+q)}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q})) = I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k))$, $k \in \mathbb{N}^*$, and

$$\begin{aligned} \widehat{\lambda}_k &= \frac{1}{\omega} \sum_{0 \leq t_k < \omega} \widehat{\gamma}_k(t_k) = \frac{1}{2\pi} \sum_{k=1}^2 \widehat{\gamma}_k(t_k) = \frac{1}{3000}, \\ \lambda_k &= \frac{1}{\omega} \sum_{0 \leq t_k < \omega} \gamma_k(t_k) = \frac{1}{2\pi} \sum_{k=1}^2 \gamma_k(t_k) = \frac{3}{3000}. \end{aligned}$$

It is easy to see that for $u \in [t, t + 2\pi]$, thanks to (4.12), (4.42), for $t \in \mathbb{R}$, we have $m \leq G(t, x) \leq M$, where

$$m = \frac{1}{e^{0.1} - 1}, \quad M = \frac{e^{0.1}}{e^{0.05} - 1}.$$

For all $t \in [0, \omega]$, $u \in P_{2\pi}(0.1, 1, 10)$, we get

$$0.00182 \simeq \Upsilon_1 \leq g(t, u^{[1]}(t), u^{[2]}(t)) \leq \Upsilon_2 \simeq 0.00634,$$

where Υ_1, Υ_2 are given as in (4.26) by

$$\Upsilon_1 = \frac{(1 - \widehat{\beta})}{m\omega} n_1 + \frac{M\alpha\beta\rho}{m} n_2 - n_1 \widehat{\lambda}_k \simeq 0.00182, \quad \Upsilon_2 = \frac{(1 - \beta)}{M\omega} n_2 - \lambda_k n_2 \simeq 0.00634,$$

and

$$\rho = \max_{t \in [0, 2\pi]} a(t) = \frac{0.1}{2\pi}.$$

On the other hand for $t \in [0, 2\pi]$, $u \in P_{2\pi}(0.1, 1, 10)$,

$$\Upsilon_1 \leq 0.002 \leq g(t, u^{[1]}(t), u^{[2]}(t)) \leq 0.006 \leq \Upsilon_2,$$

and

$$\begin{aligned} \Gamma_2 = & L_0 + n_3 \sum_{i=1}^n \sum_{j=0}^{i-1} L_i^* n_3^j + 2M\rho\alpha\beta n_2 + \omega M\rho^2\alpha^2\beta n_2 \\ & + (2 + \omega\rho\alpha) M\Upsilon_2 + 3\omega\lambda_k M\rho\alpha n_2 \simeq 0.31752 < 10, \end{aligned}$$

Since all the conditions of Theorem 4.1 are satisfied, and hence the equation

(4.41) has at last a positive 2π -periodic solution u in $P_{2\pi}(0.1, 1, 10)$.

Example 4.2 We consider the following neutral differential equation with iterative terms and impulsive effects

$$\begin{aligned} \frac{d}{dt}u(t) &= -a(t)u(t) + g(t, u^{[1]}(t), u^{[2]}(t)) + \frac{d}{dt}h(t, u^{[1]}(t), u(t)), \quad t \neq t_k \\ u(t_k^+) - u(t_k^-) &= I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k)), \quad t_k = 0.4k, \quad k \in \mathbb{N}^*. \end{aligned} \quad (4.43)$$

Corresponding to equation (4.5), we let $n = 2$, $\omega = 0.8$, $a(t) = 1 + 0.02 \sin(10\pi t)$, $f(t, u(t)) = 1$, $h(t, u^{[1]}(t), u^{[2]}(t)) = \frac{0.005}{3} |\sin 3000\pi t| |\cos 3000\pi t| + \frac{0.005}{3} u^{[1]}(t) +$

$$\frac{0.005}{3}u^{[2]}(t), g(t, u^{[1]}(t), u^{[2]}(t)) = 0.33 + 0.01 \sin u^{[1]}(t) + 0.3 \sin u^{[2]}(t),$$

$$I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k)) = 10^{-4} \left(\frac{2|\sin 10\pi t_k|}{\pi} + 8 \right) u^{[1]}(t_k).$$

For some non negative constant $n_3 > 0$, and $n_1 = 0.1$, $n_2 = 1$.

At first, the functions a , f , h , g are 0.8–periodic in t . Furthermore, the Lipschitz conditions given in $(H_1) - (H_3)$ can be easily verified for the functions f , g and I_k , $k \in \mathbb{N}^*$, with the following constants

$$L_0 = 5\pi, L_1^* = \max_{t \in [0, \omega]} L_1(t) = \frac{0.0005}{3}, L_2^* = \max_{t \in [0, \omega]} L_2(t) = \frac{0.0005}{3},$$

$$N_{1k}(t) = 10^{-4} \left(\frac{2|\sin 10\pi t_k|}{\pi} + 8 \right), N_{2k}(t) = 0, k \in \mathbb{N}^*,$$

$$C_1^* = \max_{t \in [0, \omega]} C_1(t) = 0.01, C_2^* = \max_{t \in [0, \omega]} C_2(t) = 0.3.$$

By direct calculation, we obtain

$$m = \frac{1}{e^{0.8} - 1} \simeq 0.8160, M = \frac{e^{0.8}}{e^{0.8} - 1} \simeq 1.8160,$$

$$\rho = \max_{t \in [0, \omega]} a(t) = 1.02.$$

For $\beta = 0.05$ and $\hat{\beta} = \frac{0.005}{3} \times 0.2$, we have

$$\frac{0.005}{3} \times 0.2u \leq \frac{0.005}{3} \times 0.2 \leq h(t, u^{[1]}, u^{[2]}) \leq 0.005 \leq 0.05u, \forall t \in [0, 0.8],$$

$$u \in P_\omega(n_1, n_2, n_3).$$

Since $|\sin(10\pi t_k)| \leq |10\pi t_k|$, for $t \in [0, 0.8]$, $u \in P_\omega(0.1, 1, n_3)$, we obtain

$$\hat{\gamma}_k(t_k)u \leq I_k(t_k, u^{[1]}, u^{[2]}) \leq \gamma_k(t_k)u, k \in \mathbb{N}^*.$$

where

$$\hat{\gamma}_k(t_k) = 8 \times 10^{-4}, \gamma_k(t_k) = 10^{-4}(2 \times 10t_k + 8).$$

Choosing $q = 2$, we have $t_{k+q} = t_k + \omega$, $I_{(k+q)}(t_{k+q}, u^{[1]}(t_{k+q}), u^{[2]}(t_{k+q})) =$

$I_k(t_k, u^{[1]}(t_k), u^{[2]}(t_k)), k \in \mathbb{N}^*$, and

$$\begin{aligned}\widehat{\lambda}_k &= \frac{1}{\omega} \sum_{0 \leq t_k < \omega} \widehat{\gamma}_k(t_k) = \frac{1}{0.8} \sum_{k=1}^2 \widehat{\gamma}_k(t_k) = \frac{2}{1000}, \\ \lambda_k &= \frac{1}{\omega} \sum_{0 \leq t_k < \omega} \gamma_k(t_k) = \frac{1}{0.8} \sum_{k=1}^2 \gamma_k(t_k) = \frac{5}{1000}.\end{aligned}$$

For all $t \in [0, 0.8]$, $u \in P_\omega(0.1, 1, n_3)$, we get

$$0 \leq \Upsilon_1 \leq g(t, u^{[1]}(t), u^{[2]}(t)) \leq \Upsilon_2,$$

where

$$\Upsilon_1 = \frac{(1 - \widehat{\beta})}{m\omega} n_1 + \frac{M\alpha\beta\rho}{m} n_2 - n_1 \widehat{\lambda}_k \simeq 0.2664, \quad \Upsilon_2 = \frac{(1 - \beta)}{M\omega} n_2 - \lambda_k n_2 \simeq 0.6489.$$

On the other hand

$$\Upsilon_1 \leq 0.33 \leq g(t, u^{[1]}(t), u^{[2]}(t)) \leq 0.64 \leq \Upsilon_2.$$

Under condition (4.30) of Lemma 4.4, we get the following second degree inequality:

$$\frac{0.005}{3} n_3^2 - 0.9983 n_3 + 19.2886 \leq 0,$$

which is satisfied for any $n_3 \in [19.9885, 578.9915]$. We see that all the conditions of Theorem 4.1 are satisfied, and hence the equation (4.43) has at last one positive 0.8–periodic solution u in $P_\omega(0.1, 1, n_3)$, for $n_3 \in [19.9885, 578.9915]$.

However, the corresponding inequality (10) in [Theorem 3.3, [12]] used by Bouakkaz et al. in the case no impulses are taken into account (i.e. $I_k = 0$, $k \in \mathbb{N}^*$) is

$$\frac{0.005}{3} n_3^2 + 0.1877 n_3 + 18.1979 \leq 0,$$

which does not hold for any $n_3 > 0$. In this case, the existence of periodic solutions of the equation (4.43) in $P_\omega(0.1, 1, n_3)$ cannot be proved using [

[12], Theorem 3.3], since the crucial condition (10) is not satisfied for any $n_3 > 0$. Thus, in this study, we not only propose an alternative technique to discuss the existence of positive periodic solutions to generalized iterative differential equations but also improve the existing literature.

Conclusion and perspective

Up to now, the study of qualitative and quantitative aspects of neutral impulsive differential equations that include iterative terms has remained a challenging topic for many researchers. These difficulties mainly arise from the complex and layered structure of such equations. The presence of neutral terms, impulsive effects, and iterative terms together creates mathematical difficulties that are not easy to overcome using standard techniques. As a result, this area continues to attract attention and remains an active subject of ongoing investigation within the field of differential equations. Here, the results obtained in this thesis contribute significantly to the fields of the application of Shauther's fixed-point theory to a class of functional impulsive differential equations not addressed before. Our findings have served as a new contribution for further expanding knowledge in the existence of bounded positive periodic solutions. Specifically, the results obtained can be extended in several directions: applying them to different types of equations. Extending our work to other complete metric spaces may uncover new insights and applications, while broadening new conditions could lead to the development of more comprehensive fixed-point methods. Additionally, improving the efficiency of the fixed-point methods could optimize

Conclusion and perspective

practical applications, making the theoretical advancements more applicable to real-world problems. This work emphasizes again the potential for ongoing research to build upon our results, advancing the understanding and practical application of fixed points. We expect that future studies will make use of fixed point methods to explore further qualitative properties of this class of equations damped with stochastic perturbation, especially focusing on the asymptotic and exponential stability of solutions. This remains an open problem for future studies.

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