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Exponential stability of a thermodiffusion Timoshenko beam with distributed delay

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Aknowledgement

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DEDICATION

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Abstract

In this thesis, we are interested in the study of exponontial stability and the existence and uniqueness of a Timoshenko system incorporating distributed delay and thermodiffusion effects. Using the semigroup method, first we establish the well-posedness of the system under suitable assumptions on the weight of the distributed delay, Then by emploing the energy method, we design an appropriate Lyapunov functional to demonstrated exponontial decay of the solution.

Key words: Exponontial stability,Timoshenko system,destibuted delay, thermodiffusion, semigroup method, Energy method, Lyapunov functional.

Résumé

Dans ce memoire, nous nous interressons à l'étude de la stabilité exponontielle et à l'existence et l'unicité d'un système de Timoshenko incorporant un retard distribué et des effets de thermodiffusion. En utilisant la methode des semigroupes,nous etablissons tout d'abord que le système est bien posé sous des hypothèses appropriées sur le poids du retard distribué.Ensuite, en utilisant la méthode de l'énergie, nous concevons une fonctionnelle de Lyapunov appropriée pour demontrer la décroissance exponontielle de la solution.

Mots clés: Stabilité exponontielle, système de Timoshenko, retard distribué, thermodiffusion, méthode des semigroupes,methode de l'énergie, fonction de Lyapunov.

الملخص

في هذا البحث، نحن مهتمون بدراسة الاستقرار الأسي و الوجود و الوحدانية لنظام تيموشينكو الذي يتضمن التأخير الموزع وتأثيرات الانتشار الحراري باستخدام طريقة شبه الزمر حيث نثبث أولا حسن طرح النظام في ظل افتراضات مناسبة على وزن التأخير الموزع ثم باستخدام طريقة الطاقة إذ نقوم بتصميم دالة ليابونوف مناسبة لإثباث الاضمحلال الأسي للحل.

الكلمات المفتاحية :

الاستقرار الأسي ،نظام تيموشينكو، التأخير الموزع ،طريقة أنصاف الزمر، طريقة الطاقة، دالة ليابونوف.

Contents

0.1 Introduction

In last years, the study of Timoshenko-type systems has gained considerable attention in structural mechanics, especially. due to their ability to mode thermomechanical interactions in elastic materials. Unlike the traditional Euler-Bernoulli theory, which neglects shear deformation and rotational effects, the Timoshenko model provides a more comprehensive description of beam dynamics . It does so through a system of coupled partial differential equation that incorporate both mechanical and thermal influences, making it particullary suitable for analyzing long-term behavior and energy dissipation in complex structures.

The governing dynamics are described by

$$\begin{aligned} \rho_1 \mathcal{U}_{tt}(x,t) &= S_x(x,t) \\ \rho_2 \mathcal{V}_{tt}(x,t) &= M_x(x,t) - S(x,t) \end{aligned}$$

where $\mathcal{U}(x,t)$ and $\mathcal{V}(x,t)$ represent transverse displacement and cross-sectional rotation, respectively, defined for $(x,t) \in (0,L) \times \mathbb{R}^+$.

Here , $\rho_1 = \rho A$ and $\rho_2 = \rho I$ incorporate geometric and material properties, with ρ denoting density, A the cross-sectional area, and I the area moment of inertia.

The constitutive relations

$$S(x,t) = k(\mathcal{U}_x(x,t) + \mathcal{V}(x,t)), \quad M(x,t) = b\mathcal{V}_x(x,t),$$

introduce stiffness parameters $b = EI$ (flexural rigidity) and $k = \kappa GA$ (shear stiffness), where E and G are Young's and shear moduli, and κ is the shear correction factor.

Substituting these into the governing equations yields the complete Timoshenko system

$$\rho_1 \mathcal{U}_{tt} - k(\mathcal{U}_x + \mathcal{V})_x = 0$$

$$\rho_2 \mathcal{V}_{tt} - \alpha \mathcal{V}_{xx} + k(\mathcal{U}_x + \mathcal{V}) - \gamma_1 \theta_x - \gamma_2 Px = 0$$

This formulation extends classical beam theory by rigorously addressing shear deformability, making it indispensable for modern applications in composite materials, micro- and nano-structures, and high-frequency vibration analysis.

Regarding the asymptotic behavior of the thermodiffusive Timoshenko beam, several important results have been documented, notably in the work of Aouadi et al. [2].

In their study, the authors developed a novel Timoshenko beam model, which incorporates both temperature and mass diffusion phenomena, and is described by the following system

$$\begin{cases} \rho_1 \mathcal{U}_{tt} - k(\mathcal{U}_x + \mathcal{V})_x + \mu \mathcal{U}_t = 0, \\ \rho_2 \mathcal{V}_{tt} - \alpha \mathcal{V}_{xx} + k(\mathcal{U}_x + \mathcal{V}) - \gamma_1 \theta_x - \gamma_2 Px = 0, \\ c\theta_t + dP_t - \kappa \theta_{xx} - \gamma_1 \mathcal{V}_{tx} = 0, \\ d\theta_t + rP_t - \beta P_{xx} - \gamma_2 \mathcal{V}_{tx} = 0, \end{cases} \tag{1}$$

in $(0, L) \times (0, \infty)$,

where:

$$\alpha = b - \frac{\beta^2}{\varrho}, \qquad c = \rho_3 + \frac{\varpi^2}{\varrho},$$
$$\gamma_1 = \gamma + \frac{\beta\varpi}{\varrho}, \qquad d = \frac{\varpi}{\varrho},$$
$$\gamma_2 = \frac{\beta}{\varrho}, \qquad r = \frac{1}{\varrho},$$

are physical positive constants.

The authors established the well-posedness of the system through semigroup analysis, proving that exponential stability holds precisely when the wave propagation speeds

are matched. In the opposite case, when the mechanical wave speeds vary, the study reveals a deficiency in exponential stability, they demonstrated that exponential

stability is lacking for Neumann boundary conditions when $\mu = 0$, while also establishing exponential stability for (1) without any restrictions on the coefficients if $\mu \neq 0$.

Furthermore, they provided numerical results for both cases of $\mu = 0$ and $\mu \neq 0$.

In their recent work, Ramos et al. [19] investigated a truncated Timoshenko beam model with thermodiffusion effects, replacing the second equation in (1)

with

$$-\rho_2 \mathcal{U}_{ttx} - \alpha \mathcal{V}_{xx} + k(\mathcal{U}_x + \mathcal{V}) - \gamma_1 \theta_x - \gamma_2 Px = 0.$$

They proved the existence and uniqueness of global solutions via the Faedo-Galerkin method while notably establishing exponential stability without equal wave speed assumptions. Recently, Djellali et al. [21] introduced a new Timoshenko beam model with thermal and mass diffusion effects given by

$$\begin{cases} \rho_1\mathcal{U}_{tt} - k_1(\mathcal{U}_x + \mathcal{V})_x - \gamma_1\theta_x - \gamma_2 P_x = 0, & in\ (0,L)\times\mathbb{R}_+, \\ \rho_2\mathcal{V}_{tt} - b\mathcal{V}_{xx} + k_1(\mathcal{U}_x + \mathcal{V}) + \gamma_1\theta + \gamma_2 P = 0, & in\ (0,L)\times\mathbb{R}_+, \\ c\theta_t + dP_t - \kappa\theta_{xx} - \gamma_1(\mathcal{U}_x + \mathcal{V})_t = 0, & in\ (0,L)\times\mathbb{R}_+, \\ d\theta_t + rP_t - hP_{xx} - \gamma_2(\mathcal{U}_x + \mathcal{V})_t = 0, & in\ (0,L)\times\mathbb{R}_+, \end{cases}$$

The authors proved that the system is exponentially stable if and only if the wave speeds of the system are equal. When the speeds of the mechanical waves are different, they showed a lack of exponential stability. For more details about the thermoelastic diffusion problem, see [1, 14].

The significance of the delay term arises in numerous studies because many phenomena depends on their past states Furthermore, its impact on the asymptotic behavior of solutions for the Timoshenko system has been documented in several sources [3, 20, 5, 7, 8]. Motivated by the aforementioned works, in this work, we intend to investigate the question of stabilization for the following one-dimensional thermodiffusion Timoshenko beam with distributed delay

$$\begin{cases} \rho_1\mathcal{U}_{tt} - k_1(\mathcal{U}_x + \mathcal{V})_x - \gamma_1\theta_x - \gamma_2 P_x = 0, & in\ (0,L)\times\mathbb{R}_+, \\ \rho_2\mathcal{V}_{tt} - b\mathcal{V}_{xx} + k_1(\mathcal{U}_x + \mathcal{V}) + \gamma_1\theta + \gamma_2 P + \mu_1\mathcal{V}_t + \int_{\tau_1}^{\tau_2}\mu_2(s)\mathcal{V}_t(t-s)ds = 0, & in\ (0,L)\times\mathbb{R}_+, \\ c\theta_t + dP_t - \kappa\theta_{xx} - \gamma_1(\mathcal{U}_x + \mathcal{V})_t = 0, & in\ (0,L)\times\mathbb{R}_+, \\ d\theta_t + rP_t - hP_{xx} - \gamma_2(\mathcal{U}_x + \mathcal{V})_t = 0, & in\ (0,L)\times\mathbb{R}_+, \end{cases} \tag{2}$$

with initial and boundary conditions

$$\mathcal{U}(x,0)=\mathcal{U}_0(x),\mathcal{U}_t(x,0)=\mathcal{U}_1(x),\mathcal{V}(x,0)=\mathcal{V}_0(x),\quad x\in(0,L),$$

$$\mathcal{V}_t(x,0)=\mathcal{V}_1(x),\theta(x,0)=\theta_0(x),P(x,0)=P_0(x),\quad x\in(0,L), \tag{3}$$

$$\mathcal{U}_x(0,t)=\mathcal{V}(0,t)=\theta(0,t)=P(0,t)=0,\quad t>0,$$

$$\mathcal{U}_x(L,t)=\mathcal{V}(L,t)=\theta(L,t)=P(L,t)=0,\quad t>0. \tag{4}$$

and additional condition:

$$\mathcal{V}_t(x,-t)=f_0(x,t),(x,t)\in(0,1)\times(0,\tau_2), \tag{5}$$

where $\mathcal{U}$ is the transverse displacement of the beam, $\mathcal{V}$ is the rotational angle of a filament, θ is the difference temperature. The coefficients $\rho_1,\rho_2,k_1,b,\gamma_1,\gamma_2,c,d,\kappa,r$ and h are physical positive constants. Working only on rotational angle of a filament, by employing semigroup theory, we first established the well-posedness of the system, then rigorously analyzed the asymptotic decay properties of when

$$\mu_1\geq\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds \text{ and } cr-d^2>0. \tag{6}$$

Unlike previous studies in the literature, such as [2, 1, 7], the presence of both distributed delay and thermodiffusion effects in the system leads to a non-standard energy functional. While we established that the system is dissipative, proving exponential decay stability presented significant challenges. To overcome these difficulties, we carefully selected functionals that provided the necessary estimates for terms arising from the infinite memory and the delay time. These functionals were then used to construct the Lyapunov functional $\mathcal{L}$, which was instrumental in our analysis. Notably, the conditions in (6) played a pivotal role in achieving our result.

This thesis is organized as follows: **chapter1**: is just some fundamental properties definitions , and some theorems useful in the other chapters. **chapter2** establishes the well-posedness of problem (2)-(5) through the Lumer-Phillips theorem in semigroup theory . **chapter3 :** subsequently proves exponential stability under equal wave speed conditions via energy method techniques.

Chapter 1

preliminaries

1.1 Fundamental spaces

In this section we introduce some fundamental definitions, theorems and properties in functional analysis to be used throughout this thesis. see [10].

1.1.1 Complete space

Definition 1 *A normed space* $(E, \|.\|_E)$ *is complete if any Cauchy sequence inside* E *converges to a point of the space .*

1.1.2 Hilbert space

Hilbert spaces arise naturally and frequently in mathematics and physics, typically as function spaces.

Definition 2 *A Hilbert space is a vector space provided with a scalar product* $\langle u, v \rangle$

such that $\sqrt{\langle u, v\rangle}$ *is the norm that allows* E *to be complete.*

1.1.3 Sobolev spaces

Sobolev spaces are particulary functional spaces in mathematical analysis that can be used to solve problems involving partial deferential equations, it consists of the functions of $L^p(\Omega)$.We started with Lebesgue spaces:

The $L^p(\Omega)$ spaces

Definition 3 *Let* Ω *be an open domain in* $\mathbb{R}^n, n \in \mathbb{N}$ *and* $1 \leq p \leq \infty$,*we define the standard Lebesgue space* $L^p(\Omega)$ *as follows :*

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow; f \text{ is mesurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}$$

if $p = \infty$,we obtain :

$$L^p(\Omega) = \{f :\rightarrow \mathbb{R};\ f \text{ is mesurable and there exists a constant } C \text{ such that , } |f(x)| \leq C \text{ in } \Omega\}$$

Notation 4 *Spaces* $L^p(\Omega)$ *supplied with the following norms:*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

and

$$\|f\|_\infty = \inf\{C, |f(x)| \leq C \text{ in } \Omega\}$$

Remark 5 *For $p = 2$, $L^p(\Omega)$ is a Hilbert space equipped with the following inner product:*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

The Sobolev spaces $H^m(\Omega)$:

Definition 6 *[10] The Sobolev spaces $H^m(\Omega)$ and ($m \in \mathbb{N}$) is defined as:*

$$H^m(\Omega) = \begin{cases} u \in L^2(\Omega) : D^m u \in L^2(\Omega) \text{ for all } m \in \mathbb{N}^n \text{ with } |\alpha| \in \sum_{j=1}^{n} m_j \leq m, \text{where the derivative} \\ D^m u \text{ are taken in the weak sense} \end{cases}$$

that's why $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

with the norme:

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \left(\|\partial^\alpha u\|\right)^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

Remark 7 *This is how $H_0^m(\Omega)$ is described*

$$H^m(\Omega) = \left\{ u \in H_0^m(\Omega), v = u' = ...u^{(m-1)} = 0 \text{ on } \partial\Omega \right\}$$

Notation 8 *There is a difference between*

$$H_0^2(\Omega) = \{u \in H^m(\Omega), v = u' = 0 \text{ on } \partial\Omega\}$$

and

$$H^2(\Omega) \cap H_0^1(\Omega) = \left\{u \in H^2(\Omega), u = 0 \text{ on } \partial\Omega\right\}$$

1.2 some useful inequalities

In this section , we shall recall some inequalities which will be used in the subsequent chapters.

1.2.1 Young inequalities

Theorem 9 *[10] Let* $1 \leq p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$,*then:*

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}, a, b > 0$$

Theorem 10 *(Young's inequality with* ε *) Let* $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$,*then:*

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}} \frac{b^q}{q}, a, b > 0$$

The Young inequality has several variants in the following.

Corollary 11 *Let* $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1.$ *Then:*

1. $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.

2. $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q}, \forall \varepsilon > 0.$

3. $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b, 0 < \alpha < 1.$

1.2.2 Holder inequalities

Theorem 12 *[10] Let* $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$,*then if* $f \in L^p(\Omega), g \in L^q(\Omega)$,*we have :*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

Remark 13 *We have the corresponding weighted Holder inequality of the inequality of the integral form.*

Let$1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, f \in L^p(\Omega), g \in L^q(\Omega), \omega(x) > 0$ on Ω.Then:

$$\int_{\Omega} |fg| \, \omega(x) dx \leq \left(\int_{\Omega} |f(x)|^p \, \omega(x) dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^q \, \omega(x) dx \right)^{\frac{1}{q}}$$

.

or

$$\int_{\Omega} |fg| \, dx \leq \|g\|_p \, \|g\|_q \, .$$

1.2.3 Minkowski inequality

Theorem 14 *[10] Assume* $1 < p < \infty, f, g \in L^p(\Omega)$, *then:*

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} .$$

if $0 \neq p < 1$, then:

$$\|f+g\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

.

In the applications, the integral form of the Minkowski inequality is used frequently.

1.2.4 Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality is a special case of the Holder inequality in the case $p = 2$ and $q = 2$.

1.2.5 Poincaré inequality

In this subsection,we shall recall the Poincaré inequality in different forms. see [10]

Let Ω be a bounded domain in $\mathbb{R}^n$ and $f \in H_0^1(\Omega)$.Then there is a positive constant C such that

$$\|f\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}, \forall f \in H_0^1(\Omega$$

1.3 Existence and uniqueness theorems

In this section, we recall some basic knowledge in semigroups,most of which will be used in the subsequent chapters.

1.3.1 C_0-semigroups of Linear Operators

The semigroups is a theory used to establish the existence and uniqueness of the solutions.

Definition 15 *(**semigroups**).*

Let H be a Banach space , the one-parameter family $S(t)$ $0 \leq t \leq \infty$ from H to H is called a semigroup if:

- $S(0) = I$ (I is the identity operator on H)
- $S(t+s) = S(t) \circ S(s)$ for every $t, s \geq 0$ (the semi group property)

Definition 16 *[18] The linear operator $\mathcal{A}$ defined by:*

$$D(\mathcal{A}) = \left\{ x \in H, \lim_{t \to 0^+} (S(t)x - x)/t \text{ ,exists} \right\}$$

and

$$\mathcal{A}x = \lim_{t \to 0^+} (S(t)x - x)/t = \frac{d(S(t)x}{dt}|_{t=0} \text{ for all } x \in D(\mathcal{A})$$

.

Is called the infinitesimal generator of the semigroup $S(t)$, $D(\mathcal{A})$ is called the domain of $\mathcal{A}$.

Definition 17 *(C_0-Semigroups)*

A semi group $S(t)$,$0 \leq t \leq \infty$, from H to H is called a strong continuous semigroup of bounded linear operator if:

$$\lim_{t \to 0^+} (S(t)x) = x, \text{for all } x \in H$$

or

$$lim_{t \to 0^+} \|S(t)x - x\| = 0, \text{for all } x \in H$$

Definition 18 *A semi group $S(t), 0 \leq t \leq \infty$, from H to H is called a semigroup of contraction if there exists a constant $\alpha > 0, (0 < \alpha < 1)$ such that for all $t > 0$,*

$$\|S(t)x - S(t)y\| \leq \alpha \|x - y\|, \text{for all } x, y, \in .H$$

M-dissipative operator

Definition 19 *An unbounded linear operator in X is a pair $(A, D(A)$, where $D(A)$ is a subspace of X and A is a linear mapping from $D(A)$ to X .The subspace $D(A)$ is the domain of A. see [9]*

Definition 20 *An unbounded linear operator $(A, D(A))$ on X , is said to be m-dissipative see[9] if*

- *A is dissipative*
- *The operator $(\lambda I - A)$ is surjective if:*

$$\forall f \in X, \quad \forall \lambda > 0, \quad \exists x \in D(A) \text{ such that } \lambda x - Ax = f.$$

M-dissipative operator in a Hilbert space

In this section ,we assume that X is a Hilbert space

Definition 21 *[13] An unbounded linear operator $(A, D(A))$ on X is said to be dissipative if and only if*

$$(Ax, x)_X \leq 0, \quad \forall x \in D(A).$$

Theorem 22 *[13] If A is m-dissipative ,then $D(A)$ is dense in X.*

1.3.2 Hille-Yoshida Theorem

Definition 23 *[10] An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H^1$ is said to be monotone if it satisfies*

$$\langle \mathcal{A}v, v\rangle \geq 0 \text{ for all } v \in D(\mathcal{A})$$

.

It is called maximal monotone if $R(I + \mathcal{A}) = H$

$$\forall f \in H \ , \exists\ u \in D(\mathcal{A}) \ \text{ such that } u + \mathcal{A}u = f$$

Proposition 24 *Let $\mathcal{A}$ be a maximal monontone operator. Then:*

1. $D(\mathcal{A})$ is dense in H

2. $\mathcal{A}$ is closed operator.

3. For every $\lambda > 0, (I + I\lambda)$ is bijective from $D(\mathcal{A})$ to H, $(I + \lambda\mathcal{A})^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{A})^{-1}\|_{\pounds(H)} \leq 1$.

Theorem 25 *(Hill-Yosida) Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $u_0 \in C^1([0, +\infty), H) \cap C([0, +\infty), D(\mathcal{A}))$.*

satisfying

$$\begin{cases} \frac{du}{dt} + \mathcal{A}u = 0 \text{ on } [0, +\infty) \\ u(0) = u_0 \end{cases}$$

1.3.3 Lumer-Phillips

For more details about it see [17]

Theorem 26 *Let $\mathcal{A} : D(\mathcal{A}) \subseteq H \to H$ a linear operator and $D(\mathcal{A})$ is dense in H.Then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if*

- *$\mathcal{A}$ is dissipative*
- *There exists $\lambda > 0$ such that $\operatorname{Im}(\lambda I - \mathcal{A}) = H$ ($\mathcal{A}$ maximal).*

1.3.4 Lax-Miligram theorem

Ordinary differential equations and linear partial differential equation can both be solved by using the Lax-Miligram theorem, which is a simple and effective method leading to a unique solution to the weak formulation of the problem.

Theorem 27 *[10] **(Lax-Miligram theorem)** Let $a(.,.)$ be a linear form on a Hilbert space H equipped with the norm$\|.\|_H$ and the following properties:*

- $a(.,.)$ is continuous , if

$$\exists \beta_1 > 0 \text{ such that } |a(u,v)| \leq \beta_1 \|u\|_H . \|v\|_H \ , \forall u, v \in H.$$

- $a(.,.)$ coercive (or $H-$elliptic),if

$$\exists \alpha > 0 \text{ such that } a(v,v) \geq \alpha \|v\|_H^2 \ , \forall v \in H.$$

- L is a linear mapping on H (thus L is continuous), if:

$\exists \beta_2 > 0$ such that :

$$|L(v)| \leq \beta_2 \|v\|_H .$$

Then there exists a unique $u \in H$ such that

$$a(u, v) = L(v), \forall v \in H$$

1.4 Lyapunov stability theorems

The investigation of stability for hereditary systems is often related to the construction of Lyapunov functionals.The general method of Lyapunov functionals construction which was proposed by V.Kolmanovski and L.Shaikhet see [11] and successfully used already for functional differential equations , for difference equations with discrete time , for difference equations with continuous time , is used here to investigate the stability of delay evolution equations , in particular differential equations.

1.4.1 Notations and definitions :

Let U and H be two real separable Hilbert space such that $U \subset H = H^* \subset U$ *,where the injection are continuous and dense , Let $\|\|$ and $\|,\|$ and $\|\|_{H^*}$ be the norms in U, H and H respectively, and $((.,.))$, $(.,.)$ be the scalar products in U and H respectively,and $\langle .,. \rangle$ the duality product between U and U *. We assume that:

$$|u| \leq \beta \|u\| , \forall u \in U. \tag{1.1}$$

Let $C(-h,0,H)$ be the Banach space of all continuous functions from $[-h,0]$ to H , $x_t \in C(-h,0,H)$ for each $t \in [0,\infty)$, be the function defined by $x_t(s) = x(t+s)$ for all $s \in [-h,0]$.The space $C(-h,0,H)$ is similarly defined . Let $A(t,.) : U \to U^*, f_{1(t,.)} : C(-h,0,H) \to U^*$ and $f_2(t,.) : C(-h,0,H) \to U^*$ be three families nonlinear operators defined for $t > 0, A(t,0) = 0, f_1(t,0) = 0$, $f_2(t,0) = 0$.

Consider the equation:

$$\frac{du(t)}{dt} = A(t,u(t)) + f_1(t,u_t) + f_2(t,u_t), t > 0 \qquad (1.2)$$

$$u(s) = \psi(s), s \in [-h,0]$$

Let us denote by $u(.,\psi)$ the solution of equation (1.2) corresponding to the initial condition ψ.

Definition 28 *The trivial solution of equation (1.2)is said to be stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that:*

$$|u(t,\psi)| > \varepsilon \text{ for all } t \geq 0 \text{ , if } |\psi(s)|_{C_H} = \sup_{s\in[-h,0]} |\psi(s)| < \delta.$$

Definition 29 *the trivial solution of (* 1.2) *is said to be exponentially stable if it is stable and three exists a positive constant* λ *such that for any* $\psi \in C(-h,0,U)$ *there exists* C *(which may depend on* ψ*)*

Such that

$$|u(t,\psi)| \leq Ce^{-\lambda t} \text{ for } t > 0.$$

1.4.2 Lyapunov type stability theorem

Let us now prove a theorem which will be crucial in our stability investigation.

Theorem 30 *Assume that there exists a functional* $\mathcal{L}(t, u_t)$ *such that the following conditions hold for some positive numbers* c_1, c_2 *and* λ :

$$|u(t; u_t)| \leq c_1 e^{\lambda t} |u(t)|^2, t \geq 0 \tag{1.3}$$

$$|u(0; u_0)| \leq c_2 |\psi|^2_{C_H} \tag{1.4}$$

$$\frac{d\mathcal{L}(t, u_t)}{dt} \leq 0, t \geq 0 \tag{1.5}$$

Then the trivial solution of (1.1)is exponentially stable.

Note that the previous theorem implies that the stability investigation of (1.1)can be reduced to the construction of appropriate Lyapunov functionals.

A formal procedure to construct Lyapunov functionals is described below:

procedure of Lyapunov functionals construction

step1: To transform (1.1) into the form:

$$\frac{dz(t, u_t)}{dt} = A_1(t, u(t)) + A_2(t, u_t) \tag{1.6}$$

where $z(t, .)$ and $A_2(t, .)$ are families of nonlinear operators , $z(t, 0) = 0$, $A_2(t, 0) = 0$, operator $A_1(t, .)$ only depends on t and $u(t)$, but does not depend on the previous values $u(t+s), s < 0$.

step2

Assume that the trivial solution of the auxiliary equation without memory .

$$\frac{dy(t)}{dt} = A_1(t, y(t)) \tag{1.7}$$

is exponentially stable and therefore there exists a Lyapunov function $v(t, y(t))$,which satisfies the conditions of the last theorem.

step3

A Lyapunov functional $\mathcal{L}(t, u_t)$ for eq(1.6) is constructed in the form

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

, where $\mathcal{L}_1(t, u_t) = v(t, z(t, u_t))$.Here the argument y of the functional $v(t, y)$ is replaced on the functional $z(t, x_t)$ from the left-hand part of eq (1.6)

step4

Usually, the functional $\mathcal{L}_1(t, u_t)$ almost satisfies the conditions of the Lyapunov theorem .In order to fully satisfy these conditions, it is necessary to calculate $\frac{d}{dt}\mathcal{L}_1(t, u_{t)}$ and estimate it.Then, the additional functional $\mathcal{L}_2(t, u_t)$ can be chosen in a standard way.

Note that the representation (1.6) or different ways of estimating $\frac{d}{dt}\mathcal{L}_1(t, u_{t)}$, to construct different Lyapunov functionals and ,as a result, to get different sufficient conditions of exponential stability.

Chapter 2

Existence and uniqueness of solution

In this section, we want to establish the existence, uniqueness and smoothness of solutions for problem (2)-(5) by employing the principles of semi-group theory.see[6]

2.1 Assumptions and transformations :

- Firstly, we present the assumptions and transformations required by the nature of our problem.

- Commencing with the new variable:

$$\mathcal{W}(x,\varrho,s,t)=\mathcal{V}_t(x,t-\varrho s), x\in(0,1), \varrho\in(0,1), s\in(\tau_1,\tau_2), t>0,$$

as in [15].

- Through a simple derivation, we verify that $\mathcal{W}$ satisfies:

$$s\mathcal{W}_t(x,\varrho,s,t)+\mathcal{W}_\varrho(x,\varrho,s,t)=0, \varrho\in(0,1), s\in(\tau_1,\tau_2), t>0.$$

- Consequently, system (2)-(5) becomes like this :

$$\begin{cases}
\rho_1\mathcal{U}_{tt}-k_1(\mathcal{U}_x+\mathcal{V})_x-\gamma_1\theta_x-\gamma_2P_x=0, & in\ (0,L)\times\mathbb{R}_+,\\
\rho_2\mathcal{V}_{tt}-b\mathcal{V}_{xx}+k_1(\mathcal{U}_x+\mathcal{V})+\gamma_1\theta & \\
\quad +\gamma_2P+\mu_1\mathcal{V}_t+\int_{\tau_1}^{\tau_2}\mu_2(s)\mathcal{W}(x,1,s,t)ds=0, & in\ (0,L)\times\mathbb{R}_+,\\
s\mathcal{W}_t(x,\varrho,s,t)+\mathcal{W}_\varrho(x,\varrho,s,t)=0, & in\ (0,L)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+,\\
c\theta_t+dP_t-\kappa\theta_{xx}-\gamma_1(\mathcal{U}_x+\mathcal{V})_t=0, & in\ (0,L)\times\mathbb{R}_+,\\
d\theta_t+rP_t-hP_{xx}-\gamma_2(\mathcal{U}_x+\mathcal{V})_t=0, & in\ (0,L)\times\mathbb{R}_+,
\end{cases} \tag{2.1}$$

Under identical boundary and initial conditions, only the additional one (5) should be replaced with:

$$\mathcal{W}(x,\varrho,s,0)=f_0(x,\varrho s), x\in(0,1), \varrho\in(0,1), s\in(0,\tau_2). \tag{2.2}$$

We proceed by introducing the following assumptions:

Following the approach in [12], and in order to guarantee the stability of the system when both thermal and diffusion effects are present, we assume that the physical constants c, r and d achieves the following assumption:

$$cr - d^2 > 0 \tag{2.3}$$

Concerning the weight of the delay, in agreement with [4, 16], we only assume that:

$$\delta := \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \geq 0 \tag{2.4}$$

- Secondly, Let's start by equipping the phase space of system (2.1)-(2.2)

$$H = H_*^1(0,L) \times L_*^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)) \times L^2(0,L) \times L^2(0,L)$$

with the following inner product:

$$\begin{aligned}
\left\langle U, \widetilde{U} \right\rangle_H &= \rho_1 \int_0^L u\widetilde{u}dx + b\int_0^L \mathcal{V}_x\widetilde{\mathcal{V}}_x dx + \varrho_2 \int_0^L v\widetilde{v}dx + c\int_0^L \theta\widetilde{\theta}dx + r\int_0^L P\widetilde{P}dx \\
&\quad + k_1 \int_0^L (\mathcal{U}_x + \mathcal{V})(\tilde{\mathcal{U}}_x + \tilde{\mathcal{V}})dx + d\int_0^L \left(\theta\widetilde{P} + P\widetilde{\theta}\right) dx \\
&\quad + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, \mathcal{W}(x,\varrho,s,t)\tilde{\mathcal{W}}(x,\varrho,s,t)dsd\varrho dx,
\end{aligned}$$

where:

$$\begin{cases} u = \mathcal{U}_t, \;\; v = \mathcal{V}_t, \\ \\ U = (\mathcal{U}, u, \mathcal{V}, v, \mathcal{W}, \theta, P)^T, \tilde{U} = \left(\tilde{\mathcal{U}}, \tilde{u}, \tilde{\mathcal{V}}, \tilde{v}, \tilde{\mathcal{W}}, \tilde{\theta}, \tilde{P}\right)^T \in H, \\ \\ L^2_*(0,L) = \left\{\mathcal{U} \in L^2(0,L) : \int_0^L \mathcal{U}(x)dx = 0\right\} \\ \\ H^1_*(0,L) = H^1(0,L) \cap L^2_*(0,L) \\ \\ H^2_*(0,L) = \{\mathcal{U} \in H^2(0,L) : \mathcal{U}_x(0) = \mathcal{U}_x(L) = 0\}. \end{cases}$$

The spaces $L^2(0,L)$, $H^1(0,L)$ and $H^2(0,L)$ represent the classical Sobolev spaces equipped with the standard inner product. Then, noticing system (2.1)-(2.2) takes the form of a Cauchy problem as follows :

$$\begin{cases} U_t = \mathcal{A}U \\ \\ U(0) = U_0 = (\mathcal{U}_0, \mathcal{U}_1, \mathcal{V}_0, \mathcal{V}_1, \theta_0, P_0)^T, \end{cases} \tag{2.5}$$

where the linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}U(t)=\begin{pmatrix} u \\ \\ \frac{1}{\rho_1}\left[k_1(\mathcal{U}_x+\mathcal{V})_x+\gamma_1\theta_x+\gamma_2 P_x\right] \\ \\ v \\ \\ \frac{1}{\rho_2}\left[b\mathcal{V}_{xx}-k_1(\mathcal{U}_x+\mathcal{V})-\gamma_1\theta-\gamma_2 P-\mu_1 v-\int_{\tau_1}^{\tau_2}\mu_2(s)\mathcal{W}(x,1,s,t)ds\right] \\ \\ -\frac{1}{s}\mathcal{W}_\varrho(x,\varrho,s,t) \\ \\ \frac{1}{\lambda}\left[r\kappa\theta_{xx}-hdP_{xx}+(r\gamma_1-d\gamma_2)(u_x+v)\right] \\ \\ \frac{1}{\lambda}\left[hcP_{xx}-\kappa d\theta_{xx}+(c\gamma_2-d\gamma_1)(u_x+v)\right] \end{pmatrix},$$

Here $\lambda := cr-d^2$. The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{c} U\in H \mid \mathcal{U}\in H_*^2(0,L)\cap H_*^1(0,L)\,;\mathcal{V}\in H^2(0,L)\cap H_0^1(0,L)\,; \\ \\ u\in H_*^1(0,L)\,;v\in H_0^1(0,L)\,;P,\theta\in H^2(0,L)\cap H_0^1(0,L) \\ \\ \mathcal{W}\in L^2((0,L)\times(\tau_1,\tau_2)\,;H^1(0,1)),\mathcal{W}(x,0,s,\cdot)=v \end{array}\right\}. \quad (2.6)$$

2.2 Well-posedness

In this part of the chapter which is the most important part in it ,we use the Lumer-Philips theorem and Lax-Miligram theorem (see [10] for mor details) to prove the existence and uniqueness of solution .

This theorem constitutes the first main objective of this work.:

Theorem 31 *Assume that μ_1, μ_2 are subject to (2.4), then for any initial datum $U_0 \in H$ there exists a unique solution $U \in C([0,\infty), H)$ for problem (2.5). Furthermore, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), H).$$

Proof. First, we will prove that operator $\mathcal{A}$ generates a C_0-semigroup on H.

We begin by showing that is dissipative.

Take $U = (\mathcal{U}, u, \mathcal{V}, v, \mathcal{W}, \theta, P)^T$, then

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_H =& \int_0^L \left[k_1(\mathcal{U}_x + \mathcal{V})_x + \gamma_1\theta_x + \gamma_2 P_x\right] u dx + b\int_0^L \mathcal{V}_x v_x dx \\
&+ \int_0^L \left[b\mathcal{V}_{xx} - k_1(\mathcal{U}_x + \mathcal{V}) - \gamma_1\theta - \gamma_2 P - \mu_1 v - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{W}(x,1,s,t)ds\right] v dx \\
&+ c\int_0^L \frac{1}{\lambda}\left[r\kappa\theta_{xx} - hdP_{xx} + (r\gamma_1 - d\gamma_2)(u_x + v)\right]\theta dx \\
&+ r\int_0^L \frac{1}{\lambda}\left[hcP_{xx} - \kappa d\theta_{xx} + (c\gamma_2 - d\gamma_1)(u_x + v)\right] P dx \\
&+ d\int_0^L \frac{1}{\lambda}\left[r\kappa\theta_{xx} - hdP_{xx} + (r\gamma_1 - d\gamma_2)(u_x + v)\right] P dx \\
&+ d\int_0^L \frac{1}{\lambda}\left[hcP_{xx} - \kappa d\theta_{xx} + (c\gamma_2 - d\gamma_1)(u_x + v)\right]\theta dx \\
&- \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}_\varrho(x,\varrho,s,t)\mathcal{W}(x,\varrho,s,t) ds d\varrho dx.
\end{aligned}$$

By applying integration by parts followed some simplifications, we get:

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_H = &-\mu_1 \int_0^L v^2 dx - \kappa \int_0^L \theta_x^2 dx - h \int_0^L P_x^2 dx \\ &- \int_0^L v \int_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{W}(x,1,s,t)dsdx \\ &- \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}_\varrho(x,\varrho,s,t)\mathcal{W}(x,\varrho,s,t)dsd\varrho dx.\end{aligned} \tag{2.7}$$

Regarding the final term in the right-hand side of equation (2.7), we have

$$\begin{aligned}-\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}_\varrho(x,\varrho,s,t)\mathcal{W} dsd\varrho dx &= -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(s)| \frac{\partial}{\partial \varrho}\mathcal{W}^2(x,\varrho,s,t)d\varrho dsdx \\ &= -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}^2(x,1,s,t)dsdx \\ &+ \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}^2(x,0,s,t)dsdx \\ &= -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}^2(x,1,s,t)dsdx \\ &+ \frac{1}{2}\int_0^L v^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)| dsdx \end{aligned} \tag{2.8}$$

while using Young's inequality on the penultimate term, we get

$$-\int_0^L v \int_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{W}(x,1,s,t)dsdx \leq \frac{1}{2}\int_0^L v^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)| dsdx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \mathcal{W}^2(x,1,s,t)dsdx. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), we can conclude that

$$\langle \mathcal{A}U, U\rangle_H \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right)\int_0^L v^2 dx - \kappa \int_0^L \theta_x^2 dx - h \int_0^L P_x^2 dx.$$

Observe that from (2.4), $\langle \mathcal{A}U, U\rangle \leq 0$, which means that the operator $\mathcal{A}$ is dissipative.

Now, we will demonstrate that the operator $\mathcal{A}$ is maximal.To establish this, it suffices to prove that the operator $I - \mathcal{A}$ is surjective.

For this purpose, let $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in H$, we prove that there exists $U=(\mathcal{U},u,\mathcal{V},v,\mathcal{W},\theta,P)^T\in D(\mathcal{A})$ satisfying

$$(I-\mathcal{A})U=F \tag{2.10}$$

which it is straightforward equivalent to following system:

$$\begin{cases} \mathcal{U}-u=f_1\in H_*^1(0,L) \\ \rho_1 u-k_1(\mathcal{U}_x+\mathcal{V})_x-\gamma_1\theta_x-\gamma_2 P_x=\rho_1 f_2\in L_*^2(0,L) \\ \mathcal{V}-v=f_3\in H_0^1(0,L) \\ \rho_2 v-b\mathcal{V}_{xx}+k_1(\mathcal{U}_x+\mathcal{V})+\gamma_1\theta+\gamma_2 P \\ \qquad +\mu_1 v+\int_{\tau_1}^{\tau_2}\mu_2(s)\mathcal{W}(x,1,s,t)ds=\rho_2 f_4\in L^2(0,L) \\ s\mathcal{W}+\mathcal{W}_\varrho(x,\varrho,s,t)=sf_5\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)) \\ \lambda\theta-r\kappa\theta_{xx}+hdP_{xx}-(r\gamma_1-d\gamma_2)(u_x+v)=\lambda f_6\in L^2(0,L) \\ \lambda P-hcP_{xx}+\kappa d\theta_{xx}-(c\gamma_2-d\gamma_1)(u_x+v)=\lambda f_7\in L^2(0,L). \end{cases} \tag{2.11}$$

Assume we have identified U with the requisite regularity. Then, we can determine

$\mathcal{W}$. In fact, according to (2.6), (2.11), $\mathcal{W}$ is the unique solution of

$$\begin{cases} s\mathcal{W}+\mathcal{W}_{\varrho}(x,\varrho,s,t)=sf_5 \\ \mathcal{W}(x,0,s,t)=v, \end{cases}$$

which is

$$\mathcal{W}\left(x,\varrho\right)=\mathcal{V}e^{-s\varrho}-f_3e^{-s\varrho}+se^{-s\varrho}\int_0^{\varrho}f_5\left(x,\sigma\right)e^{s\sigma}d\sigma. \tag{2.12}$$

Substituting $u=\mathcal{U}-f_1$, $v=\mathcal{V}-f_3$ and $\mathcal{W}$ into $(2.11)_2$ and $(2.11)_4$, we get

$$\begin{cases} \rho_1\mathcal{U}-k_1(\mathcal{U}_x+\mathcal{V})_x-\gamma_1\theta_x-\gamma_2P_x=g_1\in L^2_*(0,L) \\ \eta\mathcal{V}-b\mathcal{V}_{xx}+k_1(\mathcal{U}_x+\mathcal{V})+\gamma_1\theta+\gamma_2P=g_2\in L^2(0,L) \\ \lambda\theta-r\kappa\theta_{xx}+hdP_{xx}-(r\gamma_1-d\gamma_2)(\mathcal{U}_x+\mathcal{V})=g_3\in L^2(0,L) \\ \lambda P-hcP_{xx}+\kappa d\theta_{xx}-(c\gamma_2-d\gamma_1)(\mathcal{U}_x+\mathcal{V})=g_4\in L^2(0,L) \end{cases} \tag{2.13}$$

where

$$\begin{cases} \eta = \rho_2 + \mu_1 + \int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}ds \\ g_1 = \rho_1 \left(f_1 + f_2\right) \\ g_2 = \eta f_3 + \rho_2 f_4 - \int_{\tau_1}^{\tau_2} se^{-s}\mu_2(s) \int_0^1 f_5\left(x,\sigma,s\right)e^{s\sigma}d\sigma ds \\ g_3 = \lambda f_6 - \left(r\gamma_1 - d\gamma_2\right)\left(f_3 - \left(f_1\right)_x\right) \\ g_4 = \lambda f_7 - \left(c\gamma_2 - d\gamma_1\right)\left(f_3 - \left(f_1\right)_x\right). \end{cases}$$

To address the solution of the problem (2.13), we multiply $(2.13)_1$ by $\tilde{\mathcal{U}}$, $(2.13)_2$ by $\tilde{\mathcal{V}}$, $(2.13)_3$ by $\dfrac{c}{\lambda}\tilde{\theta}$, $(2.13)_4$ by $\dfrac{r}{\lambda}\tilde{P}$, $(2.13)_3$ by $\dfrac{d}{\lambda}\tilde{P}$ and $(2.13)_4$ by $\dfrac{d}{\lambda}\tilde{\theta}$, we then integrate the sum of these products over $(0,L)$ to derive the following variational formulation:

$$B((\mathcal{U},\mathcal{V},\theta,P),(\tilde{\mathcal{U}},\tilde{\mathcal{V}},\tilde{\theta},\tilde{P})) = F((\tilde{\mathcal{U}},\tilde{\mathcal{V}},\tilde{\theta},\tilde{P})) \tag{2.14}$$

where $B : \left[H_*^1\left(0,L\right) \times H_0^1\left(0,L\right) \times H_0^1\left(0,L\right) \times H_0^1\left(0,L\right)\right]^2 \to \mathbb{R}$ is the bilinear form given by

$$B((\mathcal{U},\mathcal{V},\theta,P),(\tilde{\mathcal{U}},\tilde{\mathcal{V}},\tilde{\theta},\tilde{P}))=\rho_1\int_0^L \mathcal{U}\tilde{\mathcal{U}}dx+k_1\int_0^L(\mathcal{U}_x+\mathcal{V})\left(\tilde{\mathcal{U}}_x+\tilde{\mathcal{V}}\right)dx+\eta\int_0^L\mathcal{V}\tilde{\mathcal{V}}dx$$

$$+b\int_0^L\mathcal{V}_x\tilde{\mathcal{V}}_xdx+c\int_0^L\theta\tilde{\theta}dx+r\int_0^L P\tilde{P}dx$$

$$+d\int_0^L\left(\theta\tilde{P}+P\tilde{\theta}\right)dx+\kappa\int_0^L\theta_x\tilde{\theta}_xdx+h\int_0^L P_x\tilde{P}_xdx$$

$$+\int_0^L(\gamma_1\theta+\gamma_2P)\left(\tilde{\mathcal{U}}_x+\tilde{\mathcal{V}}\right)dx-\int_0^L\left(\gamma_1\tilde{\theta}+\gamma_2\tilde{P}\right)(\mathcal{U}_x+\mathcal{V})dx$$

and $F:[H_*^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)]\to\mathbb{R}$ is the linear functional defined by :

$$F\left(\tilde{\mathcal{U}},\tilde{\mathcal{V}},\tilde{\theta},\tilde{P}\right)^T=\int_0^1 g_1\tilde{\mathcal{U}}dx+\int_0^1 g_2\tilde{\mathcal{V}}dx+\frac{c}{\lambda}\int_0^1 g_3\tilde{\theta}dx$$

$$+\frac{r}{\lambda}\int_0^1 g_4\tilde{P}dx+\frac{d}{\lambda}\int_0^1 g_3\tilde{P}dx+\frac{d}{\lambda}\int_0^1 g_4\tilde{\theta}dx.$$

Now, for $V=H_*^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)$ equipped with the norm

$$\|(\mathcal{U},\mathcal{V},\theta,P)\|_V=\|\mathcal{U}\|_2^2+\|(\mathcal{U}_x+\mathcal{V})\|_2^2+\|\mathcal{V}_x\|_2^2+\|\theta_x\|_2^2+\|P_x\|_2^2,$$

applying the Cauchy-Schwarz and Poincaré inequalities, it follows directly that both

B and F are continuous operators. Furthermore, we obtain

$$B((\mathcal{U},\mathcal{V},\theta,P),(\mathcal{U},\mathcal{V},\theta,P)) \geq \rho_1 \int_0^L \mathcal{U}^2 dx + k_1 \int_0^L (\mathcal{U}_x+\mathcal{V})^2 dx + b \int_0^L \mathcal{V}_x^2 dx$$

$$+ \kappa \int_0^L \theta_x^2 dx + h \int_0^L P_x^2 dx + \left(c - \frac{d^2}{r}\right) \int_0^L \theta^2 dx$$

$$+ \left(r - \frac{d^2}{c}\right) \int_0^L P^2 dx,$$

by (2.3), it follows that

$$B((\mathcal{U},\mathcal{V},\theta,P),(\mathcal{U},\mathcal{V},\theta,P)) \geq C \left\|(\mathcal{U},\mathcal{V},\theta,P)\right\|_V,$$

for some positive constant C. Thus B is coercive. As a consequence, the Lax-Milgram lemma ensures the existence of a unique solution

$$\mathcal{U} \in H_*^1(0,L), \mathcal{V} \in H_0^1(0,L), \theta \in H_0^1(0,L), P \in H_0^1(0,L),$$

for the system (2.13). To establish that the solution obtained belongs to $D(\mathcal{A})$, we will proceed with the following steps. Substituting $\mathcal{U}$ and $\mathcal{V}$ in $(2.11)_1$, $(2.11)_3$, it remains clear that

$$u \in H_*^1(0,L), v \in H_0^1(0,L).$$

Similarly, inserting v in (2.12) and bearing in mind $(2.11)_5$, we get

$$\mathcal{W} \in L^2((0,L) \times (\tau_1,\tau_2); H^1(0,1)).$$

Now, if $\left(\tilde{\mathcal{V}},\tilde{\theta},\tilde{P}\right) \equiv (0,0,0) \in \left(H_0^1(0,L)\right)^3$, then (2.14) reduces to :

$$\rho_1 \int_0^L \mathcal{U}\tilde{\mathcal{U}} dx + k_1 \int_0^L (\mathcal{U}_x+\mathcal{V})\tilde{\mathcal{U}}_x dx = \int_0^L g_1 \tilde{\mathcal{U}} dx, \forall \tilde{\mathcal{U}} \in H_*^1(0,L), \tag{2.15}$$

which implies

$$-k_1\mathcal{U}_{xx} = k_1\mathcal{V}_x - \rho_1\mathcal{U} + g_1 \in L^2(0, L).$$

Applying standard regularity theory for linear elliptic equations, we consequently obtain

$$\mathcal{U} \in H^2(0, L) \cap H_*^1(0, L).$$

Furthermore, the result (2.15) remains valid for all $\vartheta \in C^1([0, L]) \subset H_\star^1(0, L)$. Hence, by using integration by parts we obtain

$$\mathcal{U}_x(L)\,\vartheta(L) - \mathcal{U}_x(0)\,\vartheta(0) = 0.$$

Therefore, $\mathcal{U}_x(0) = \mathcal{U}_x(L) = 0$, and so we get

$$\mathcal{U} \in H_*^2(0, L) \cap H_*^1(0, L).$$

Also, by setting $\left(\tilde{\mathcal{U}}, \tilde{\theta}, \tilde{P}\right) \equiv (0, 0, 0)$, $\left(\tilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{P}\right) \equiv (0, 0, 0)$ and $\left(\tilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{\theta}\right) \equiv (0, 0, 0)$ in (2.14) we can respectively derive the following equations

$$-b\mathcal{V}_{xx} = -k_1\mathcal{U}_x - (k_1 + \eta)\,\mathcal{V} - \gamma_1\theta - \gamma_2 P + g_2 \in L^2(0, L),$$

$$-\kappa\theta_{xx} = -c\theta - dP + \gamma_1(\mathcal{U}_x + \mathcal{V}) + \frac{c}{\lambda}g_3 + \frac{d}{\lambda}g_4 \in L^2(0, L)$$

and

$$-hP_{xx} = -d\theta - rP + \gamma_2(\mathcal{U}_x + \mathcal{V}) + \frac{d}{\lambda}g_3 + \frac{r}{\lambda}g_4 \in L^2(0, L).$$

Arguing analogously to the previous case, we can show that

$$\mathcal{V}, \theta, P \in H^2(0, L) \cap H_0^1(0, L),$$

Thus we establish the existence of a unique solution $U \in D(\mathcal{A})$ satisfying (2.10). This result confirms that $\mathcal{A}$ is maximal dissipative, and consequently, follows directly from the Lumer-Phillips theorem (see [18]). ■

Chapter 3

Exponential Stability

In this section, our aim is to prove an exponential stability result for the problem (2.1)-(2.2). for more details see([6]) by employing the energy method on the basis of the multiplier thechnique ,we design an appropriate Lyapunov functional to demonstrate exponontial decay of the solution.

3.1 The associated energy of the problem

We assume (2.3) and (2.4), the associated energy of this problem is defined by:

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_1 \mathcal{U}_t^2 + k_1(\mathcal{U}_x + \mathcal{V})^2 + \rho_2 \mathcal{V}_t^2 + b\mathcal{V}_x^2 + c\theta^2 + rP^2 + 2d\theta P\right] dx$$

$$+ \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| \mathcal{W}^2(x, \varrho, s, t) ds d\varrho dx. \tag{3.1}$$

Without a doubt, this energy is greater than the standard :

$$\frac{1}{2}\int_0^L \left[\rho_1 \mathcal{U}_t^2 + k_1(\mathcal{U}_x + \mathcal{V})^2 + \rho_2 \mathcal{V}_t^2 + b\mathcal{V}_x^2 + c\theta^2 + rP^2 + 2d\theta P\right] dx$$

and it contains supplementary term arising from the distributed delay. Nevertheless, it is evident from the next lemma that the system is dissipative, meaning that the energy functional is decreasing. Although this result does not imply exponential decay towards the equilibrium. It is necessary to construct an equivalent functional $\mathcal{L}$ that satisfies

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t)$$

and such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -CE(t)$$

for all $t > 0$ and some positive constants α_1, α_2, C.

In this section, we articulate and demonstrate our stability theorem concerning the energy of the solution to system (2.1)-(2.2), employing the multiplier technique.

To attain our objective, we rely on the following lemmas.

Lemma 32 *Let $(\mathcal{U}, \mathcal{V}, \mathcal{W}, \theta, P)^T$ be the solution of (2.1)-(2.2), then the energy functional defined by (3.1) satisfies*

$$E'(t) \leq -\delta \int_0^L \mathcal{V}_t^2 dx - \kappa \int_0^L \theta_x^2 dx - h \int_0^L P_x^2 dx \tag{3.2}$$

Proof. We multiply equations $(2.1)_1$, $(2.1)_2$, $(2.1)_4$, and $(2.1)_5$ by $\mathcal{U}_t$, $\mathcal{V}_t$, θ, and P, respectively,

then integrate over the domain $(0, L)$. Applying integration by parts while incorporating the boundary conditions (4), and summing the results yields:

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left[\rho_1 \mathcal{U}_t^2 + k_1(\mathcal{U}_x + \mathcal{V})^2 + \rho_2 \mathcal{V}_t^2 + b\mathcal{V}_x^2 + c\theta^2 + rP^2 + 2d\theta P\right] dx$$
$$= -\kappa \int_0^L \theta_x^2 dx - h \int_0^L P_x^2 dx - \mu_1 \int_0^L \mathcal{V}_t^2 dx$$
$$- \int_0^L \mathcal{V}_t \int_{\tau_1}^{\tau_2} \mu_2(s) \mathcal{W}(x,1,s,t) ds dx \tag{3.3}$$

- By multiplying $(2.1)_3$ with $|\mu_2(s)|\,\mathcal{W}$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, while considering (2.8), yield

$$\frac{1}{2}\frac{d}{dt}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,\mathcal{W}^2 ds d\varrho dx = -\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,\mathcal{W}_\varrho(x,\varrho,s,t)\mathcal{W}(x,\varrho,s,t) ds d\varrho dx$$
$$= -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,\mathcal{W}^2(x,1,s,t) ds dx$$
$$+ \frac{1}{2}\int_0^L \mathcal{V}_t^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds dx. \tag{3.4}$$

- A combination of (3.3) and (3.4) gives

$$E'(t) = -\kappa \int_0^L \theta_x^2 dx - h \int_0^L P_x^2 dx - \mu_1 \int_0^L \mathcal{V}_t^2 dx - \int_0^L \mathcal{V}_t \int_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{W}(x,1,s,t) ds dx$$

$$-\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,\mathcal{W}^2(x,1,s,t) ds dx + \frac{1}{2}\int_0^L \mathcal{V}_t^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds dx. \tag{3.5}$$

- By substituting (2.9) with $v = \mathcal{V}_t$ into (3.5), and employing the assumption given in (2.4), we deduce (3.2), thereby concluding the proof.

■

Remark 33 *The energy functional $E(t)$, as defined in (3.1) is non-negative. Indeed,*

- *by using (2.3) we can easily show that :*

$$c\theta^2 + rP^2 + 2d\theta P > c_1\theta^2 + r_1P^2$$

where $c_1 = \frac{1}{2}\left(c - \frac{d^2}{r}\right)$ *and* $c_2 = \frac{1}{2}\left(r - \frac{d^2}{c}\right)$.

- Consequently

$$E(t) > \frac{1}{2}\int_0^L \left[\rho_1\mathcal{U}_t^2 + k_1(\mathcal{U}_x + \mathcal{V})^2 + \rho_2\mathcal{V}_t^2 + b\mathcal{V}_x^2 + c_1\theta^2 + r_1P^2\right] dx$$

$$+\frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| \mathcal{W}^2(x, \varrho, s, t) ds d\varrho dx > 0.$$

Remark 34 • *So,* $E(t)$ *is non-negative.*

3.2 Construction of Lyapunov functional:

3.2.1 Functional I_1

Lemma 35 *Let* $(\mathcal{U}, \mathcal{V}, \mathcal{W}, \theta, P)^T$ *be the solution of (2.1)-(2.2), then the functional*

$$I_1(t) = \rho_2 \int_0^L \mathcal{V}_t\mathcal{V}dx - \rho_1 \int_0^L \mathcal{U}_t \int_0^x \mathcal{V}(y, t)dydx,$$

satisfies the following estimate

$$\frac{d}{dt}I_1(t) \leq -\frac{b}{2}\int_0^L \mathcal{V}_x^2dx + \frac{\rho_1}{2}\int_0^L \mathcal{U}_t^2dx + (\rho_2 + C_1)\int_0^L \mathcal{V}_t^2dx$$

$$+\frac{L^2}{2b}\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| \mathcal{W}^2(x, 1, s, t)dsdx \qquad (3.6)$$

for some positive constant C_1.

Proof. A simple differentiation of I_1 gives

$$\frac{d}{dt}I_1(t) = \rho_2 \int_0^L \mathcal{V}_{tt}\mathcal{V}dx + \rho_2 \int_0^L \mathcal{V}_t^2 dx - \rho_1 \int_0^L \mathcal{U}_t \int_0^x \mathcal{V}_t(y,t)dydx$$

$$- \rho_1 \int_0^L \mathcal{U}_{tt} \int_0^x \mathcal{V}(y,t)dydx.$$

From the first two equations of system (2.1), we establish

$$\frac{d}{dt}I_1(t) = \int_0^L \left[b\mathcal{V}_{xx} - k_1(\mathcal{U}_x + \mathcal{V}) - \gamma_1\theta - \gamma_2 P - \mu_1\mathcal{V}_t - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{W}(x,1,s,t)ds \right] \mathcal{V}dx$$

$$+ \rho_2 \int_0^L \mathcal{V}_t^2 dx - \rho_1 \int_0^L \mathcal{U}_t \left(\int_0^x \mathcal{V}_t(y,t)dy \right) dx$$

$$+ \int_0^L [-k_1(\mathcal{U}_x + \mathcal{V})_x - \gamma_1\theta_x - \gamma_2 P_x] \left(\int_0^x \mathcal{V}(y,t)dy \right) dx,$$

integrating by parts and considering the boundary conditions (4), we arrive at

$$\frac{d}{dt}I_1(t) = -b \int_0^L \mathcal{V}_x^2 dx + \rho_2 \int_0^L \mathcal{V}_t^2 dx - \rho_1 \int_0^L \mathcal{U}_t \left(\int_0^x \mathcal{V}_t(y,t)dy \right) dx$$

$$- \int_0^L \mathcal{V} \int_{\tau_1}^{\tau_2} \mu_2(s)\mathcal{W}(x,1,s,t)dsdx. \qquad (3.7)$$

Now, by employing Young's, Poincaré, and Cauchy-Schwarz inequalities to estimate the terms on the right-hand side of (3.7), we find for some positive constant C_1

$$-\rho_1 \int_0^L \mathcal{U}_t \left(\int_0^x \mathcal{V}_t(y,t)dy \right) dx \le \frac{\rho_1}{2} \int_0^L \mathcal{U}_t^2 dx + C_1 \int_0^L \mathcal{V}_t^2 dx \qquad (3.8)$$

and

$$-\int_0^L \mathcal{V}\int_{\tau_1}^{\tau_2}\mu_2(s)\mathcal{W}(x,1,s,t)dsdx \leq \frac{b}{2L^2}\int_0^L \mathcal{V}^2dx + \frac{L^2}{b}\int_0^L\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,\mathcal{W}(x,1,s,t)ds\right)^2dx \tag{3.9}$$

$$\leq \frac{b}{2}\int_0^L \mathcal{V}_x^2dx + \frac{L^2}{2b}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,\mathcal{W}^2(x,1,s,t)dsdx.$$

The estimate (3.6) is obtained by substituting (3.8)–(3.9) into (3.7). ■

3.2.2 Functional I_2

Lemma 36 *Let $(\mathcal{U},\mathcal{V},\mathcal{W},\theta,P)^T$ be the solution of (2.1)-(2.2), then the functional*

$$I_2(t) = -\rho_1\int_0^L \mathcal{U}_t\mathcal{U}dx$$

satisfies, for any $\epsilon_2 > 0$, the estimate

$$\frac{d}{dt}I_2(t) \leq -\rho_1\int_0^L \mathcal{U}_t^2dx + \epsilon_2\int_0^L \mathcal{V}_x^2dx + \left(\epsilon_2 + \frac{k_1^2}{\epsilon_2}\right)\int_0^L(\mathcal{U}_x+\mathcal{V})^2dx$$

$$+\frac{L^2\gamma_1^2}{\epsilon_2}\int_0^L\theta_x^2dx + \frac{L^2\gamma_2^2}{\epsilon_2}\int_0^L P_x^2dx \tag{3.10}$$

Proof. A simple differentiation of I_2, using $(2.1)_2$, then integrating by parts, yields

$$\frac{d}{dt}I_2(t) = -\rho_1\int_0^L \mathcal{U}_t^2dx - \rho_1\int_0^L \mathcal{U}_{tt}\mathcal{U}dx$$

$$= -\rho_1\int_0^L \mathcal{U}_t^2dx + k_1\int_0^L(\mathcal{U}_x+\mathcal{V})\mathcal{U}_xdx + \gamma_1\int_0^L\theta\mathcal{U}_xdx + \gamma_2\int_0^L P\mathcal{U}_xdx$$

Using Young's with appropriate constants and Poincaré inequalities, we get the estimates

$$k_1\int_0^L(\mathcal{U}_x+\mathcal{V})\mathcal{U}_x\le\frac{\epsilon_2}{4}\int_0^L\mathcal{U}_x^2dx+\frac{k_1^2}{\epsilon_2}\int_0^L(\mathcal{U}_x+\mathcal{V})^2dx,$$

$$\gamma_1\int_0^L\theta\mathcal{U}_x\le\frac{\epsilon_2}{8}\int_0^L\mathcal{U}_x^2dx+\frac{L^2\gamma_1^2}{\epsilon_2}\int_0^L\theta_x^2dx,$$

$$\gamma_2\int_0^L P\mathcal{U}_x\le\frac{\epsilon_2}{8}\int_0^L\mathcal{U}_x^2dx+\frac{L^2\gamma_2^2}{\epsilon_2}\int_0^L P_x^2dx$$

true for any $\epsilon_2>0$. Finally, by using the following inequality

$$\frac{1}{2}(\mathcal{U}_x+\mathcal{V}-\mathcal{V})^2\le(\mathcal{U}_x+\mathcal{V})^2+\mathcal{V}^2$$

the estimate (3.10) is established. ■

3.2.3 Functional I_3

Lemma 37 *Let $(\mathcal{U},\mathcal{V},\mathcal{W},\theta,P)^T$ be the solution of (2.1)-(2.2), then the functional*

$$I_3(t)=-\frac{\rho_1}{\gamma_1}\int_0^L\left[c\theta+dp-\gamma_1\left(\mathcal{U}_x+\mathcal{V}\right)\right]\left(\int_0^x\mathcal{U}_t(y,t)dy\right)dx$$

satisfies, for any $\epsilon_3>0$, the estimate

$$\frac{d}{dt}I_3(t)\le-\frac{k_1}{2}\int_0^L(\mathcal{U}_x+\mathcal{V})^2dx+\epsilon_3\int_0^L\mathcal{U}_t^2dx$$

$$+\left(\frac{L^2}{2}\xi_1+\frac{\xi_2}{\epsilon_3}\right)\int_0^L\theta_x^2dx+\frac{L^2}{2}\xi_3\int_0^L P_x^2dx \qquad (3.11)$$

where $\xi_1=\frac{1}{k_1}\left(\frac{ck_1}{\gamma_1}-\gamma_1\right)^2+\frac{c\gamma_2}{\gamma_1}+d+c,\xi_2=\frac{\kappa^2\rho_1^2}{4\gamma_1^2}$ and $\xi_3=\frac{1}{k_1}\left(\frac{dk_1}{\gamma_1}-\gamma_2\right)^2+\frac{c\gamma_2}{2\gamma_1}+\frac{d}{2}+\frac{d\gamma_2}{\gamma_1}$.

Proof. Through a direct differentiation of I_3 with respect to t, using $(2.1)_3$, applying integration by parts, considering the boundary conditions stated in (4), and since $\int_0^L \mathcal{U}(x)dx = 0$ it follows that:

$$\frac{d}{dt}I_3(t) = -\frac{1}{\gamma_1}\int_0^L \left[c\theta + dp - \gamma_1(\mathcal{U}_x + \mathcal{V})\right]\left[k_1(\mathcal{U}_x + \mathcal{V}) + \gamma_1\theta + \gamma_2 P\right]dx$$

$$+ \frac{\kappa\rho_1}{\gamma_1}\int_0^L \theta_{xx}\left(\int_0^x \mathcal{U}_t(y,t)dy\right)dx$$

$$= -k_1\int_0^L (\mathcal{U}_x + \mathcal{V})^2 dx + c\int_0^L \theta^2 dx + \frac{d\gamma_2}{\gamma_1}\int_0^L P^2 dx - \frac{\kappa\rho_1}{\gamma_1}\int_0^L \theta_x\mathcal{U}_t dx$$

$$+ \left(\frac{ck_1}{\gamma_1} - \gamma_1\right)\int_0^L \theta(\mathcal{U}_x + \mathcal{V})dx + \left(\frac{dk_1}{\gamma_1} - \gamma_2\right)\int_0^L P(\mathcal{U}_x + \mathcal{V})dx$$

$$+ \left(\frac{c\gamma_2}{\gamma_1} + d\right)\int_0^L \theta P dx \tag{3.12}$$

The estimation of the last four terms on the right-hand side of (3.12) proceeds as follows:

$$-\frac{\kappa\rho_1}{\gamma_1}\int_0^L \theta_x\mathcal{U}_t dx \leq \epsilon_3\int_0^L \mathcal{U}_t^2 dx + \left(\frac{\kappa\rho_1}{2\gamma_1}\right)^2 \frac{1}{\epsilon_3}\int_0^L \theta_x^2 dx, \tag{3.13}$$

$$\left(\frac{ck_1}{\gamma_1} - \gamma_1\right)\int_0^L \theta(\mathcal{U}_x + \mathcal{V})dx \leq \frac{k_1}{4}\int_0^L (\mathcal{U}_x + \mathcal{V})^2 dx + \left(\frac{ck_1}{\gamma_1} - \gamma_1\right)^2 \frac{L^2}{2k_1}\int_0^L \theta_x^2 dx, \tag{3.14}$$

$$\left(\frac{dk_1}{\gamma_1}-\gamma_2\right)\int_0^L P\left(\mathcal{U}_x+\mathcal{V}\right)dx \leq \frac{k_1}{4}\int_0^L \left(\mathcal{U}_x+\mathcal{V}\right)^2 dx + \left(\frac{dk_1}{\gamma_1}-\gamma_2\right)^2 \frac{L^2}{2k_1}\int_0^L P_x^2 dx, \tag{3.15}$$

and

$$\left(\frac{c\gamma_2}{\gamma_1}+d\right)\int_0^L \theta P dx \leq \frac{L^2}{4}\left(\frac{c\gamma_2}{\gamma_1}+d\right)\int_0^L \theta_x^2 dx + \frac{L^2}{4}\left(\frac{c\gamma_2}{\gamma_1}+d\right)\int_0^L P_x^2 dx. \tag{3.16}$$

Estimate (3.11) follows by substituting (3.13)-(3.16) into (3.12). ■

3.2.4 Functional I_4

Lemma 38 *Let $(\mathcal{U},\mathcal{V},\mathcal{W},\theta,P)^T$ be the solution of (2.1)-(2.2), then the functional*

$$I_4(t)=\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\varrho}\left|\mu_2(s)\right|\mathcal{W}^2(x,\varrho,s,t)dsd\varrho dx$$

satisfies, for some positive constant C_4, the following estimate

$$\frac{dI_4(t)}{dt} \leq -C_4\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|\mathcal{W}^2(x,\varrho,s,t)dsd\varrho dx$$

$$-C_4\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\mathcal{W}^2(x,\varrho,s,t)dsd\varrho dx+\mu_1\int_0^L \mathcal{V}_t^2 \tag{3.17}$$

Proof. See [4] ■

3.3 The Lyapunov functional $\mathcal{L}$

The next step involves defining a Lyapunov functional $\mathcal{L}$ and proving its equivalence to the energy functional E.

Lemma 39 *Let* $(\mathcal{U},\mathcal{V},\mathcal{W},\theta,P)^T$ *be the solution of (2.1)-(2.2), then for* N, N_3 *and* N_4 *sufficiently large, the functional defined by :*

$$\mathcal{L} := NE + I_1 + I_2 + N_3 I_3 + N_4 I_4 \tag{3.18}$$

satisfies

$$aE(t) \leq \mathcal{L}(t) \leq bE(t), \forall t \geq 0, \tag{3.19}$$

for some positive constants a *and* b.

Proof. We have

$$|\mathcal{L}(t) - NE(t)| \leq \rho_2 \int_0^L |\mathcal{V}_t\mathcal{V}|\, dx + \rho_1 \int_0^L \left|\mathcal{U}_t \int_0^x \mathcal{V}(y,t)dy\right| dx + \rho_1 \int_0^L |\mathcal{U}_t\mathcal{U}|\, dx$$

$$+ \frac{\rho_1 c}{\gamma_1} N_3 \int_0^L \left|\theta \int_0^x \mathcal{U}_t(y,t)dy\right| dx + \frac{\rho_1 d}{\gamma_1} N_3 \int_0^L \left|p \int_0^x \mathcal{U}_t(y,t)dy\right| dx$$

$$+ \rho_1\gamma_1 N_3 \int_0^L \left|(\mathcal{U}_x + \mathcal{V}) \int_0^x \mathcal{U}_t(y,t)dy\right| dx$$

$$+ N_4 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\varrho} |\mu_2(s)| \mathcal{W}^2(x,\varrho,s,t) ds d\varrho dx$$

Arguing as in previous proofs and using the same inequalities, (3.1), the fact that $e^{-s\varrho} \leq 1, \forall \varrho \in [0,1]$, it is not difficult to show that

$$|\mathcal{L}(t) - NE(t)| \leq CE(t).$$

Consequently,

$$(N - C)\, E(t) \leq \mathcal{L}(t) \leq (N + C)\, E(t)$$

By choosing a sufficiently large N, we achieve the estimate (3.19) ■

Having rigorously developed and verified all the essential techical lemmas required for our analysis , we are now in a position to advance to the core contribution of this section , namely the precise formulation and detailed proof of the main theorem that encapsulate the fundamental result of our study.

Theorem 40 *Assume that (2.3), (2.4) hold and let* $(\mathcal{U}, \mathcal{V}, \mathcal{W}, \theta, P)^T$ *be the solution of (2.1)-(2.2). Then, for any* $U_0 \in D(\mathcal{A})$*, there exist two positive constants,* α *and* β*, such that the energy functional defined by (3.1) satisfies*

$$E(t) \leq \alpha e^{-\beta t}, \forall t \geq 0, \tag{3.20}$$

Proof. By differentiating (3.18) and taking in account (3.2), (3.6), (3.10), (3.11), (3.17), we get

$$\mathcal{L}'(t) \leq -\left[\frac{b}{2} - \epsilon_2\right] \mathcal{V}_x^2 - \left[\rho_1 - \frac{\rho_1}{2} - N_3\epsilon_3\right] \mathcal{U}_t^2$$

$$- \left[N\delta - (\rho_2 + C_1) - N_4\mu_1\right] \mathcal{V}_t^2$$

$$- \left[N_3\frac{k_1}{2} - \left(\epsilon_2 + \frac{k_1^2}{\epsilon_2}\right)\right] (\mathcal{U}_x + \mathcal{V})^2$$

$$- \left[N\kappa - \frac{L^2\gamma_1^2}{\epsilon_2} - N_3\left(\frac{L^2}{2}\xi_1 + \frac{\xi_2}{\epsilon_3}\right)\right] \theta_x^2$$

$$- \left[Nh - \frac{L^2\gamma_2^2}{\epsilon_2} - N_3\frac{L^2}{2}\xi_3\right] P_x^2$$

$$- \left[N_4C_4 - \frac{L^2}{2b}\right] \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, \mathcal{W}^2(x, \varrho, s, t)dsd\varrho dx$$

$$- N_4C_4 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, \mathcal{W}^2(x, \varrho, s, t)dsd\varrho dx$$

Now, we proceed to carefully choose all the constants. Primarily, we pick ϵ_2 so small that

$$\epsilon_2 < \frac{b}{2},$$

then we choose N_3 large enough so that

$$N_3 > \frac{2}{k_1}\left(\epsilon_2 + \frac{k_1^2}{\epsilon_2}\right)$$

and we select ϵ_3 so small that

$$\epsilon_3 < \frac{\rho_1}{2N_3}.$$

Finally, when N_4 and N are large enough such that

$$N_4 > \frac{L^2}{2bC_4},$$

$$N > \max\left\{\frac{1}{\delta}\left(\rho_2 + C_1 + N_4\mu_1\right), \frac{1}{\kappa}\left(\frac{L^2\gamma_1^2}{\epsilon_2} + N_3\left(\frac{L^2}{2}\xi_1 + \frac{\xi_2}{\epsilon_3}\right)\right), \frac{1}{h}\left(\frac{L^2\gamma_2^2}{\epsilon_2} + N_3\frac{L^2}{2}\xi_3\right)\right\},$$

it is straightforward to verify that for some positive constant β_0 the functional $\mathcal{L}$ satisfies:

$$\mathcal{L}'(t) \leq -\beta_0 E(t), \quad \forall t > 0. \tag{3.21}$$

A combination of (3.19) and (3.21) gives

$$\mathcal{L}'(t) \leq -\beta\mathcal{L}(t), \quad \forall t > 0,$$

where $\beta = \frac{\beta_0}{b}$. Thus, a direct integration over $(0, t)$ yields

$$\mathcal{L}'(t) \leq \mathcal{L}(0)e^{-\beta t}, \quad \forall t > 0. \tag{3.22}$$

Finally, by combining (3.19) and (3.22), we arrive at the result given in (3.20) with $\alpha = \frac{b}{a}E(0)$, which completes the proof. ■

conclusion

In this work, we have studied the exponotial stability of one-dimensional thermo-diffusion Timoshenko beam model with distributed delay, passing by proving the well-posedness (Existence and uniqueness of solution) of our problem; by using some essential and important theorems such as: Lumer-Philips and Semigroup theory.

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