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Multidimensional fractional Fourier transform: Properties and applications

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CONTENTS

Dedication

To the ones who gave meaning to every step of this journey…

*To my **father and mother**, whose love is the quiet strength behind all I do. Your prayers, your patience, and your endless support carried me further than words can ever say.*

*To my beloved **siblings**, whose presence is a comfort and whose encouragement lifted me in moments of doubt.*

*To the gentle memory of my **brother-in-law**, forever missed, forever cherished. May this work be a whisper of love to his soul.*

*To my **nieces and nephews**, the little sparks of joy who remind me why the future is always worth building.*

*To my dearest **friends**, who stood beside me with kindness, laughter, and unwavering belief.*

This work is for all of you – with all my heart.

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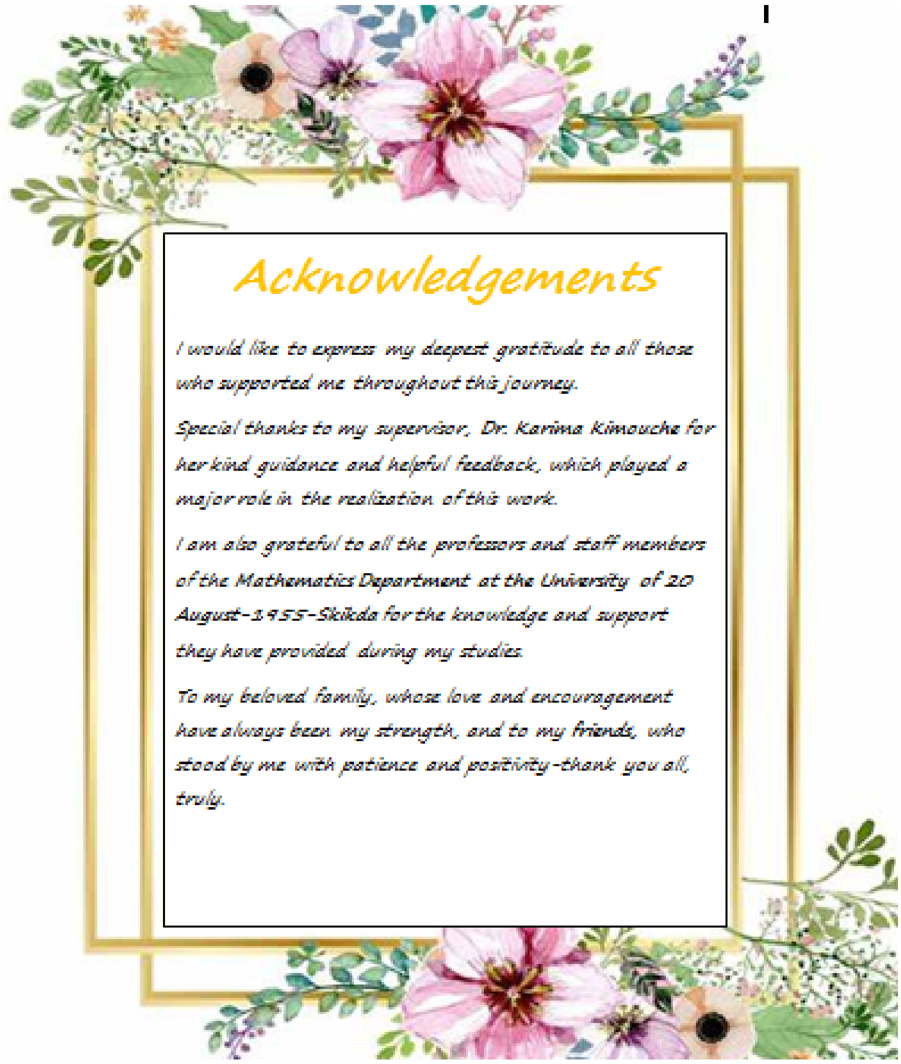

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Abstract

This master's thesis focuses on the study of the multidimensional fractional Fourier transform in both its classical and windowed forms as a powerful extension of the conventional Fourier transform with enhanced capability for analyzing non-stationary signals. The work presents formal definitions of both transforms and explores their main mathematical properties, including linearity, time-shift, convolution, scaling, and energy preservation. Special attention is given to the role of window function in improving time-frequency localization in the n-dimensional window fractional Fourier transform framework. A comparative analysis between these transforms is also provided. Finally, selected applications are presented to highlight the relevance of these transforms in various fields.

Keywords:

- Fourier transform:
- Multidimensional fractional Fourier transform
- Windowed fractional Fourier transform
- Kernel function
- Window function

Résumé

Ce mémoire de master porte sur l'étude de la transformée de Fourier fractionnaire multidimensionelle, tant dans ses formes classique que fenêtrée. Il s'agit d'une extension puissante de la transformée de Fourier conventionnelle, offrant une capacité accrue d'analyse des signaux non stationnaires. Ce travail présente les définitions formelles des deux transformées et explore leurs principales propriétés mathématiques, notamment la linéarité, le décalage temporel, la convolution, la mise à l'échelle et la préservation de l'énergie. Une attention particulière est portée au rôle de la fonction de fenêtre dans l'amélioration de la localisation temps-fréquence dans le cadre de la transformée de Fourier fractionnaire fenêtrée à n dimensions. Une analyse comparative de ces transformées est également fournie. Enfin, une sélection d'applications est présentée pour souligner l'intérêt de ces transformées dans divers domaines.

Mots clés:

- Transformée de Fourier
- Transformée de Fourier fractionnaire multidimensionnelle
- Transformée de Fourier fractionnaire fenêtrée
- Fonction noyau
- Fonction Fenêtrée.

Abstract in arabic

تركز أطروحة الماستر هذه على دراسة تحويل فورييه الكسري متعدد الأبعاد، بصيغتيه الكلاسيكية والنافذية، كامتداد قوي لتحويل فورييه التقليدي، مع قدرة مُحسّنة على تحليل الإشارات غير الثابتة. يقدم العمل تعريفات لكلا التحويلين، ويستكشف خصائصهما الرياضية الرئيسية، بما في ذلك الخطية، والانزياح الزمني، والالتواء، والقياس، وحفظ الطاقة. ويولي العمل اهتمامًا خاصًا لدور دالة النافذة في تحسين تحديد موضع التردد الزمني في إطار تحويل فورييه الكسري متعدد الأبعاد. كما يقدم تحليلًا مقارنًا بين هذه التحويلات. وأخيرًا، يعرض تطبيقات مختارة لتسليط الضوء على أهمية هذه التحويلات في مجالات مختلفة.

الكلمات المفتاحية:

- تحويل فورييه الكسري متعدد الأبعاد

- تحويل فورييه الكسري النافذة

- دالة النواة

- دالة النافذة

List of symbols

$L^2(R^n)$: The space of square integrable functions in $\mathbb{R}^n$

$\mathcal{F}[f(\mathbf{x})](\xi)$ or $\widehat{f}(\boldsymbol{\xi})$: The $n-$dimensional Fourier transform of function $f(\mathbf{x})$

$\mathcal{F}^{-1}(\mathcal{F}[f(\mathbf{x})])$: The $n-$dimensional inversion formula of Fourier transform

$\overline{f(\mathbf{x})}$: The conjugate of $f(\mathbf{x})$

$f * g$: The convolution of the function $f(\mathbf{x})$ and $g(\mathbf{x})$

$\nabla_{\mathbf{t}} f(x_1, x_2, ..., x_n) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_v}\right]^T$: The gradient vector of the function $f(x_1, x_2, ..., x_n)$

$\|f\|_{L^2}$: The L^2-norm of the function f

$R^{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi})$ or $\mathcal{F}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$: The $n-$dimensional fractional Fourier transform

$K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})$: The $n-$dimensional kernel of fractional Fourier transform

$(W_{\psi} f)(\mathbf{u}, \boldsymbol{\xi})$: The $n-$dimensional window Fourier transform

$\psi(\mathbf{x})$: The $n-$dimensional window function in $L^2(\mathbb{R}^n)$.

General introduction

The Fourier transform (FT) plays a central role in applied mathematics, engineering and signal processing due its powerful ability to convert signals from the time or spatial domain into the frequency domain. However, this transformation lacks time-frequency localization, making it less effective for analyzing non-stationary signals. This limitation led to the development of the fractional Fourier transform (FrFT) (also sometimes called angular Fourier transform), which generalizes the Fourier transform by introducing a rotation parameter that follows a smooth transition between time and frequency representations.

Wiener first introduced the FrFT in his 1929 work [19]. Namias proposed the FrFT in [10] using its eigenvalue equation. Since Namias' work in 1980, the FrFT has attracted a lot of interest. For example, a rigorous mathematical framework of the properties of fractional Fourier transform on the Schwartz space S of rapidly decreasing functions was given by McBride and Kerr [9]. Compared with FT, the FrFT contains one extra free parameter and is more suited to process non-stationary signals, especially chirp signals. As a result, FrFT can achieve effects that the classical Fourier transform or time-frequency analysis cannot.

In recent decades, the FrFT has become an attractive tool and has various applications in many fields of applied sciences, such as signal processing [3], [5], [20], [15], image processing [4], [14], [7], [16], optics [2], [11], [12], [13], communications [6], [21], quantum mechanics [10] and so on. For example, in 1997; Luis B. Almeida, developed the product and convolution theorems for the one dimensional fractional Fourier transform and obtained the same to more general forms in [8].

In 1994; the kernel of fractional Fourier transform was extended to n-dimensions by T.Alieva, F. Agullo-Lopez and L.B.Almeida [1] and using this kernel developed n-dimensional fractional, opening up new directions in signal and image processing, optics and time-frequency analysis. The multidimensional FrFT (MFrFT) retains the essential features of the classical FT while offering greater flexibility in analyzing signals with multiple variables or spatial dimensions.

Furthermore, The windowed fractional Fourier transform (WFrFT) was introduced as an improvement over the FrFT, allowing localized analysis in both time and fractional frequency through the use of window functions. This is particularly beneficial for nonstationary signal analysis, where precise time-frequency representation is required.

This master's thesis is devoted to the study of the Multidimensional Fractional Fourier Transform and its applications. It begins with the formal definitions and extends to a detailed examination of their mathematical properties, such as linearity, time-shifting, convolution, scaling,and energy preservations work also includes an overview of window function and their roles in WFrFT. A comparison between FrFT and WFrFT is conducted to highlight their respective advantages and limitations. Finally,the master's thesis explores practical application in signal analysis, biomedical signals, optics and time-frequency filtering.

The work is organized as follows: In Chapter 1, we introduced the definition of the Fourier transform and we give some examples and basic properties from a mathematical point of view, in addition we give some application of FT in many fields. Chapter 2, studied the multidimensional fractional Fourier transform ($n-$FrFT), and its properties with some applications. Finally, we introduced the notion of $n-$dimensional windowed fractional Fourier transform which is the improvement of the fractional Fourier transform and we give some comparison.

CHAPTER 1

Fourier transform

Contents

1.1 Definitions

Joseph Fourier in 1822 published first work about Fourier transform, which is an integral transform that converts a mathematical function from the time domain to the frequency domain. The Fourier transform has evolved into a widely recognized discipline of harmonic analysis and has been successfully applied scientific and engineering pursuits.

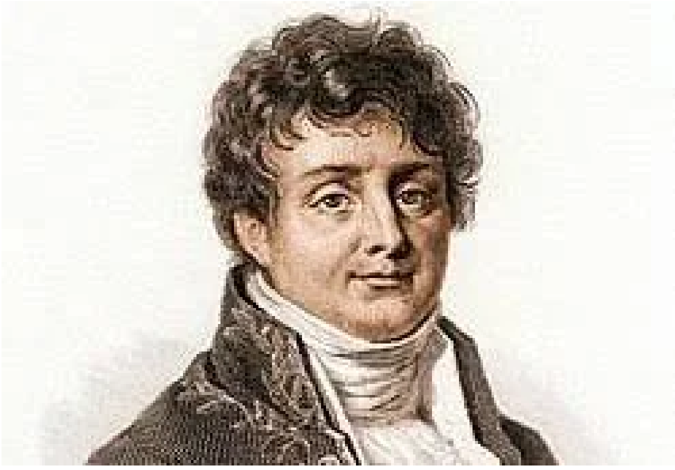

Figure 1.1: Joseph Fourier (1768-1830).

Let us begin with definition of the classical Fourier transform.

Definition 1.1 The Fourier transform (FT) of any function $f(t) \in L^2(\mathbb{R})$ is defined as:

$$\mathcal{F}[f(t)](\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} dt, \tag{1.1}$$

and corresponding inversion Formula is given by

$$\mathcal{F}^{-1}(\mathcal{F}[f(t)](\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}[f(t)](\xi) e^{i\xi t} d\xi, \tag{1.2}$$

Example 1.1 Consider a function

$$f(t) = \begin{cases} e^{-\alpha t} & \text{for } \ t \succeq, \alpha \succ 0. \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Then, the Fourier transform of $f(t)$ is obtained as:

$$\begin{aligned}
\mathcal{F}[f(t)](\xi) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\xi t} e^{-\alpha t} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty (\cos \xi t - i \sin \xi t) e^{-\alpha t} dt \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_0^\infty \cos \xi t e^{-\alpha t} dt - i \int_0^\infty \sin \xi t e^{-\alpha t} dt \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \frac{\alpha}{\alpha^2 + \xi^2} - \frac{i\xi}{\alpha^2 + \xi^2} \right\}.
\end{aligned}$$

1.2 Properties

1. **Linearity:** The Fourier transform (FT) is a linear operation this means that the Fourier transform of sum of signal is the sum of their individual Fourier transform

$$\mathcal{F}\{a.f(t) + b.g(t)\} = a.\mathcal{F}\{f(t)\} + b.\mathcal{F}\{g(t)\}, \tag{1.3}$$

where a and b are constant and $f(t)$ and $g(t)$ are two function.

2. **Time shifting:** If a signal $f(t)$ is shifted by t_0 in time. Its Fourier transform will experience a phase shift in the frequency domain

$$\mathcal{F}\{f(t - t_0) = e^{-i\omega_0 t_0}.\mathcal{F}\{f(t)\}, \tag{1.4}$$

where t_0 is the time shift, and ω is the angular frequency.

3. **Frequency shifting:** If a signal $f(t)$ is multiplied by complex exponential $e^{i\omega_0 t}$, this results in a shift in the frequency domain

$$\mathcal{F}\{f(t).e^{i\omega_0 t}\} = \mathcal{F}\{f(t)\}|_{\omega-\omega_0}. \tag{1.5}$$

4. **Scaling:** If the time domain is scaled by a factor of a, the frequency domain is scaled by$\frac{1}{a}$, and the amplitude is scaled by

$$|a|^{-1}.\mathcal{F}\{f(at)\} = \frac{1}{|a|}.\mathcal{F}\{f(t)\}\bigg|_{\frac{\omega}{q}}, \tag{1.6}$$

where a is a scaling factor.

5. **Convolution Theorem:** The Fourier transform of the convolution of two functions is equal to the point wise product of their Fourier transform.

$$\mathcal{F}\{f(t) * g(t)\} = \mathcal{F}\{f(t)\}.\mathcal{F}\{g(t)\}, \tag{1.7}$$

where $*$ represent the convolution operation.

6. **Parseval Theorem:** Parseval theorem states that the total energy of a signal in the time is equal to the total energy of that signal in the frequency domain.

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |F(\omega)|^2 \, d\omega. \tag{1.8}$$

This shows that the total energy (or power) is conserved between the time and frequency domain.

1.3 Applications

1. **Signal processing:** Fourier transform is used to analyse the frequency components of signals.

2. **Image processing :** Its helps in tasks like image filtering or compressions by working in the frequency domain.

3. **Audio processing:** It is used to analyse the frequencies in audio signals.

1.4 Limitation of Fourier transform

- **No time localization:** Fourier transform (FT) shows which frequencies but not accurate.

- **Fixed resolution:** It uses the same resolution for all parts of the signal,which is not suitable for non stationary signals.

- **Poor time-frequency analysis:** FT can not track haw frequencies change over time.

- **Limited flexibility:** There is not control over the balance between time and frequency resolution.

- **Not ideal for short or finite signals:** Applying FT limited duration signals can lead to spectral leakage and inaccurate results.

- **Sensitive to noise:** Fourier Transform (FT) is easily affected by noise, which can distort the frequency signals.

- **Limited phase information in some application:** FT may not provide clear or sufficient phase representation, which is critical in certain systems.

1.5 Multidimensional Fourier transform

The n-dimensional Fourier transform is a mathematical process used to convert a function from the spatial or time domain into frequency domain in $n-$dimension.

Using vector notations $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$. We have

Definition 1.2 The $n-$dimensional Fourier transform ($n-$FT) of any function $f(x_1, x_2, ..., x_n) \in L^2(\mathbb{R}^n)$ is defined and denoted as:

$$\begin{aligned}\mathcal{F}[f(x_1, x_2, ..., x_n)](\xi_1, \xi_2, ..., \xi_n) &= \widehat{f}(\xi_1, \xi_2, ..., \xi_n) \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{\mathbb{R}^n} f(x_1, x_2, ..., x_n) e^{-i\sum_{i=1}^n \xi_i . x_i} dx_1 dx_2 ... dx_n,\end{aligned}$$

or simply

$$\mathcal{F}[f(\mathbf{x})](\xi) = \widehat{f}(\boldsymbol{\xi}) = \frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\boldsymbol{\xi}.\mathbf{x}} d\mathbf{x}.$$

where $\boldsymbol{\xi}.\mathbf{x} = \sum_{i=1}^n \xi_i . x_i$, $d\mathbf{x} = dx_1 dx_2 ... dx_n$ and corresponding inversion Formula (denoted $f(\mathbf{x})$) is given by

$$\begin{aligned}&\mathcal{F}^{-1}(\mathcal{F}[f(x_1, x_2, ..., x_n)](\xi_1, \xi_2, ..., \xi_n))(x_1, x_2, ..., x_n) \\ = &\frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{\mathbb{R}^n} \mathcal{F}[f(x_1, x_2, ..., x_n)](\xi_1, \xi_2, ..., \xi_n) e^{i\sum_{i=1}^n \xi_i . x_i} d\xi_1 d\xi_2 ... d\xi_n.\end{aligned}$$

or simply

$$f(\mathbf{x}) = \frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}.\mathbf{x}} d\boldsymbol{\xi}.$$

Example 1.2 $n-$dimensional Fourier transform of some functions are given in the following table

$f(x_1, x_2, ...x_n)$	$\mathcal{F}[f(x_1, x_2, ..., x_n)](\xi_1, \xi_2, ..., \xi_n)$
$rect(x_1, x_2, ..., x_n)$	$\frac{\sin(\pi\zeta_1)}{\pi\zeta_1} \frac{\sin(\pi\zeta_2)}{\pi\zeta_2} ... \frac{\sin(\pi\zeta_n)}{\pi\zeta_n}$
$rect(a_1 x_1, a_2 x_2, ..., a_n x_n)$	$a_1 \sin c(a_1\zeta_1)\, a_2 \sin c(a_2\zeta_2) ... a_n \sin c(a_n\zeta_n)$
$\exp\{-\pi \sum_{i=1}^n x_i^2\}$	$\exp\left\{-\pi\left(\sum_{i=1}^n \zeta_i^2\right)\right\}$
$\cos(\pi(x_1^2 + x_2^2 + ... + x_n^2))$	$\sin\left(\pi\left(\zeta_1^2 + \zeta_2^2 + ... + \zeta_n^2\right)\right)$
$\exp\{i\pi(x_1^2 + x_2^2 + ... + x_n^2)\}$	$i\exp\left\{-i\pi\left(\zeta_1^2 + \zeta_2^2 + ... + \zeta_n^2\right)\right\}$
$\delta(x_1, x_2, ..., x_n)$	1

Remark 1.1 The properties of the $n-$dimensional Fourier transform are analogous to the one-dimensional transform.

CHAPTER 2

Multidimensional fractional Fourier transform

The fractional Fourier Transform (FrFT) is generalization of the classical Fourier transform (FT). While the FT maps a time-domain signal to its frequency-domain representation. The FrFT provide a transformation into intermediate domain between time and frequency controlled by a fractional order parameter. This feature makes it particularly useful in signal processing, optics and communication systems.

2.1 Definitions

Analogous to the one dimensional case, the multidimensional kernel is defined as follows. We have

Definition 2.1 Let $\mathbf{x} = (x_1, x_2, ...x_n) \in \mathbb{R}^n, \boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$; with all components of α are constants. The $n-$dimensional kernel $K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})$ is defined as follows

$$K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) = \prod_{i=1}^{n} K_{\alpha_i}(x_i, \xi_i). \tag{2.1}$$

where

$$K_{\alpha_i}(x_i, \xi_i) = \begin{cases} \frac{c(\alpha_i)}{\sqrt{2\pi}} \exp\left\{ i \left[a(\alpha_i)(x_i^2 + \xi_i^2) - x_i \xi_i b(\alpha_i) \right] \right\}, \alpha_i \neq k\pi, \\ \delta(x_i - \xi_i), \alpha_i = 2\pi k, \\ \delta(x_i + \xi_i), \alpha_i = (2k+1)\pi, \end{cases} \tag{2.2}$$

and $a(\alpha_i) = \frac{1}{2}\cot\alpha_i \left(i.e, \cot\alpha = \frac{1}{\tan(\alpha)}\right)$, $b(\alpha_i) = \csc\alpha_i \left(i.e, \csc\alpha = \frac{1}{\sin(\alpha)}\right)$ and $c(\alpha_i) = \sqrt{1 - i\cot\alpha_i}$ for all $i = 1, 2, ..., n$.

Remark 2.1 Note that this kernel is continuous, in the generalized function sense, even when α is a multiple of π.

Similarly to one dimensional, the kernel has the following properties.

Properties

1. The Kernel is symmetric:
$$K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) = K_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \mathbf{x}).$$

2.
$$K^*_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) = K_{-\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}),$$
where $K^*_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})$ is a conjugate of $K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})$.

3.
$$K_{\boldsymbol{\alpha}}(-\mathbf{x}, \boldsymbol{\xi}) = K_{\boldsymbol{\alpha}}(\mathbf{x}, -\boldsymbol{\xi}).$$

4.
$$K_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}, \boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}')K_{\boldsymbol{\beta}}(\boldsymbol{\xi}', \boldsymbol{\xi})\, d\boldsymbol{\xi}'.$$

Proof. (1.), (2.) and (3.) are trivial. For (3.) we have by definition

$$\begin{aligned}
K_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}, \boldsymbol{\xi}) &= \prod_{i=1}^{n} K_{\alpha_i+\beta_i}(x_i, \xi_i) \\
&= K_{\alpha_1+\beta_1}(x_1, \xi_1)K_{\alpha_2+\beta_2}(x_2, \xi_2)...K_{\alpha_n+\beta_n}(x_n, \xi_n) \\
&= \left(\int_{\mathbb{R}} K_{\alpha_1}(x_1, \xi_1')K_{\beta_1}(\xi_1', \xi_1)\, d\xi_1'\right)\left(\int_{\mathbb{R}} K_{\alpha_2}(x_2, \xi_2')K_{\beta_2}(\xi_2', \xi_2)\, d\xi_2'\right)... \\
&\quad \left(\int_{\mathbb{R}} K_{\alpha_n}(x_n, \xi_n')K_{\beta_n}(\xi_n', \xi_n)\, d\xi_n'\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}...\int_{\mathbb{R}} \left(\prod_{i=1}^{n} K_{\alpha_i}(x_i, \xi_i')K_{\beta_i}(\xi_i', \xi_i)\, d\xi_i'\right) \\
&= \int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} K_{\alpha_i}(x_i, \xi_i')\right)\left(\prod_{i=1}^{n} K_{\beta_i}(\xi_i', \xi_i)\right) d\xi' \\
&= \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}')K_{\boldsymbol{\beta}}(\boldsymbol{\xi}', \boldsymbol{\xi})\, d\boldsymbol{\xi}'.
\end{aligned}$$

■

Analogous to the classical definition of Fourier transform, the multidimensional fractional Fourier transform on $L^2(\mathbb{R}^n)$ is defined as follows. We have

Definition 2.2 Let $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$.The $n-$dimensional fractional Fourier transform with parameter $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ (i.e, the vector of fractional orders) of $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ denoted by $R^{\alpha}[f(\mathbf{x})](\boldsymbol{\xi})$ or $\mathcal{F}_{\alpha}(\xi)$
(i.e, $\mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n}(\xi_1, \xi_2, ..., \xi_n)$) and is given by the following integral transform

$$\begin{aligned} R^{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi}) &= \mathcal{F}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) d\mathbf{x}, \end{aligned} \tag{2.3}$$

where all components of $\boldsymbol{\alpha}$ are constants, and

$$\begin{aligned} f(\mathbf{x}) &= f(x_1, x_2, ..., x_n), \\ d\mathbf{x} &= dx_1 dx_2 ... dx_n, \\ K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) &= \prod_{i=1}^{n} K_{\alpha_i}(x_i, \xi_i). \end{aligned}$$

if $\alpha_i \neq \pi k$,for all $k = 0, 1, 2, ...$and $i = 1, 2, ..., n$.

Remark 2.2

1. If $n = 1$, then the equation (2.3) reduces to one dimensional fractional Fourier transform.
2. If $\boldsymbol{\alpha} = \left(\frac{\pi}{2}, \frac{\pi}{2}, ..., \frac{\pi}{2}\right)$, then the equation (2.3) reduces to $n-$dimensional Fourier transform.

Definition 2.3 Let $f \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. If $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, where all the components of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ are constants, $R^{\alpha}[f(\mathbf{x})](\boldsymbol{\xi})$ or $\mathcal{F}_{\alpha}(\boldsymbol{\xi})$ is $n-$dimensional fractional Fourier transform of $f(\mathbf{x})$, then

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} K_{-\boldsymbol{\alpha}}(\boldsymbol{\xi}, \mathbf{x}) \mathcal{F}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{2.4}$$

where

$$\overline{K}_{\alpha_i}(x_i, \xi_i) = \prod_{i=1}^{n} \left[\frac{\overline{c\,(\alpha_i)}}{\sqrt{2\pi}} \exp\left\{-i\left[a(\alpha_i)(x_i^2 + \xi_i^2) - 2x_i\xi_i b(\alpha_i)\right]\right\}, \alpha_i \neq k\pi \right]. \tag{2.5}$$

and $d\boldsymbol{\xi} = d\xi_1 d\xi_2 ... d\xi_n$ (for more detail see [17]).

Remark 2.3 1. The corresponding inversion formula is also a FrFT with angle $-\boldsymbol{\alpha}$ and is given by

$$f(\mathbf{x}) = \mathcal{F}_{-\boldsymbol{\alpha}}\{\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi})\}(\mathbf{x}) = \int_{\mathbb{R}^n} K_{-\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})\mathcal{F}_{\alpha}(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.6}$$

2. It is easy to see that, when $\alpha_i = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$, for all $i = 1, ..., n$, the FrFT is reduced to the identity operation, the FT, time- reverse operation, and the IFT, respectively.

2.2 Properties

The $n-$dimensional fractional Fourier transform ($n-$FrFT) extends the classical FrFT into multiple dimensions while preserving its essential structure and proprieties. Due to its separable kernel, the $n-$FrFT can be decomposed into a product of unidimensional transforms along each axis. This structure allows for a natural generalization of key properties from the 1D case. In this section, we present and prove the most fundamental properties of the n-FrFT, these properties are crucial for theoretical analysis and practical application in multidimensional signal processing and related fields.

2.2.1 Linearity

Theorem 2.1 *Let* $f(\mathbf{x}), g(\mathbf{x}) \in L^2(\mathbb{R}^n)$, *with* $\mathbf{x} = (x_1, x_2, ...x_n) \in \mathbb{R}^n$ *and* $\lambda, \mu \in \mathbb{C}$. *Then the* $n-$*dimensional fractional Fourier transform satisfies:*

$$\mathcal{F}_{\boldsymbol{\alpha}}(\lambda f(\mathbf{x}) + \mu g(\mathbf{x})) = \lambda\mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi}) + \mu\mathcal{F}_{\boldsymbol{\alpha}}\{g(\mathbf{x})\}(\boldsymbol{\xi}). \tag{2.7}$$

for all $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$.

Proof. Using the definition of the $n-$dimensional FrFT

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{\alpha}}(\lambda f(\mathbf{x}) + \mu g(\mathbf{x}))(\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} (\lambda f(\mathbf{x}) + \mu g(\mathbf{x}))K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \lambda f(\mathbf{x})K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})d\mathbf{x} + \int_{\mathbb{R}^n} \mu g(\mathbf{x})K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})d\mathbf{x} \\
&= \lambda \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})f(\mathbf{x})d\mathbf{x} + \mu \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})g(\mathbf{x})d\mathbf{x} \\
&= \lambda\mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi}) + \mu\mathcal{F}_{\boldsymbol{\alpha}}\{g(\mathbf{x})\}(\boldsymbol{\xi}).
\end{aligned}$$

where $K_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{u})$ is the separable kernel associated with the $n-$dimensional FrFT, and the integral taken over all components $(x_1, x_2, ..., x_n)$. ■

This completes the proof.

2.2.2 Translation

For simplicity, we put $\alpha_i = \alpha$ for all $i = 1, n$.

Theorem 2.2 *Let $f(\mathbf{x}) \in L^2(\mathbb{R}^n)$ where $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and let $\mathbf{h} = (h_1, h_2, ..h_n) \in \mathbb{R}^n$ be as constant constant shift vector. Then, the $n-$dimensional fractional Fourier transform satisfies the following translation property*

$$\mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x}-\mathbf{h})\}(\boldsymbol{\xi}) = e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{h})}\mathcal{F}_{\boldsymbol{\alpha}}\left\{f(\mathbf{x})\right\}(\boldsymbol{\xi} - \mathbf{h}\cos\alpha). \tag{2.8}$$

for all $\boldsymbol{\xi}$ =$(\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, where

$$f(\mathbf{x}-\mathbf{h}) = f(x_1 - h_1, x_2 - h_2, ..., x_n - h_n),$$

and $\phi_{\boldsymbol{\alpha}}\,(\boldsymbol{\xi},\mathbf{h})$ is a phase term depending on the fractional order and kernel.

Proof. From the definition of the $n-$dimensional FrFT, we have

$$\mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x}-\mathbf{h})\}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})f(\mathbf{x}-\mathbf{h})d\mathbf{x}.$$

we perform the change of variables: $\mathbf{u} = \mathbf{x} - \mathbf{h}$, thus $\mathbf{x} = \mathbf{u} + \mathbf{h}$, and $d\mathbf{x} = d\mathbf{u}$. Then

$$\mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x}-\mathbf{h})\}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{u}+\mathbf{h}, \boldsymbol{\xi})f(\mathbf{u})d\mathbf{u}.$$

Using the property of the kernel

$$K_{\boldsymbol{\alpha}}(\mathbf{u}+\mathbf{h}, \boldsymbol{\xi}) = e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{h})}K_{\boldsymbol{\alpha}}(\mathbf{u}, \boldsymbol{\xi} - \mathbf{h}\cos\alpha),$$

where $\phi_{\boldsymbol{\alpha}}\,(\boldsymbol{\xi},\mathbf{h}) = \left(\frac{1}{2}(\|\mathbf{h}\|^2)\cot(\alpha) - \mathbf{h}.\boldsymbol{\xi}\csc(\alpha)\right)$, we get

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x}-\mathbf{h})\}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{h})}K_{\boldsymbol{\alpha}}(\mathbf{u}, \boldsymbol{\xi} - \mathbf{h}\cos\alpha)f(\mathbf{u})d\mathbf{u} \\
&= e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{h})}\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{u}, \boldsymbol{\xi} - \mathbf{h}\cos\alpha)f(\mathbf{u})d\mathbf{u} \\
&= e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{h})}\mathcal{F}_{\boldsymbol{\alpha}}\left\{f(\mathbf{u})\right\}(\boldsymbol{\xi} - \mathbf{h}\cos\alpha).
\end{aligned}$$

This completes the proof. ■

2.2.3 Orthogonality

Theorem 2.3 *Let* $f(\mathbf{x}), g(\mathbf{y}) \in L^2(\mathbb{R}^n)$, *where* $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$,*and let* $\alpha \in \mathbb{R}^n$. *Then the following orthogonality relation holds:*

$$\langle \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})], \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{y})] \rangle = \langle f(\mathbf{x}), g(\mathbf{y}) \rangle . \tag{2.9}$$

where $\langle ., . \rangle$ *denoted the inner product in* $L^2(\mathbb{R}^n)$.

Proof. By definition we have

$$\langle \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})], \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})] \rangle = \int_{\mathbb{R}^n} \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi}) \overline{\mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})](\boldsymbol{\xi})} d\boldsymbol{\xi},$$

subsisting the expression for $\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})]$ and $\mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})]$, we obtain

$$\begin{aligned} \langle \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})], \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})] \rangle &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) f(\mathbf{x}) d\mathbf{x} \right) \left(\overline{\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{y}, \boldsymbol{\xi}) g(\mathbf{y}) d\mathbf{y}} \right) d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{g(\mathbf{y})} \left(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) \overline{K_{\boldsymbol{\alpha}}(\mathbf{y}, \boldsymbol{\xi})} d\boldsymbol{\xi} \right) d\mathbf{x} d\mathbf{y}, \end{aligned}$$

Using the orthogonality property of the kernel

$$\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) \overline{K_{\boldsymbol{\alpha}}(\mathbf{y}, \boldsymbol{\xi})} d\boldsymbol{\xi} = \delta(\mathbf{x} - \mathbf{y}).$$

Where $\delta(\mathbf{x} - \mathbf{y})$ is the Dirac delta function,we get:

$$\begin{aligned} \langle \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})], \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})] \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{g(\mathbf{y})} \left(\delta(\mathbf{x} - \mathbf{y}) \right) d\mathbf{x} d\mathbf{y} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{g(\mathbf{y})} d\mathbf{x} d\mathbf{y} \\ &= \langle f(\mathbf{x}), g(\mathbf{y}) \rangle . \end{aligned}$$

This completes the proof. ■

Remark 2.4 If we set $f = g$ in Parseval's relation (orthogonality), we obtain the following energy conservation property

$$\|\mathcal{F}_{\boldsymbol{\alpha}}[f]\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} .$$

This means that L^2-norm (or the total energy) of the function f is preserved under the multidimensional fractional Fourier transform.

2.2.4 Convolution

Theorem 2.4 *Let* $f(\mathbf{x}), g(\mathbf{x}) \in L^2(\mathbb{R}^n)$, *where* $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. *Then*

$$\mathcal{F}_{\boldsymbol{\alpha}}[(f * g)(\mathbf{x})](\boldsymbol{\xi}) = \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi})\mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})](\boldsymbol{\xi}). \tag{2.10}$$

Where $*$ *denotes the convolution operation defined by*

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{v})g(\mathbf{x}-\mathbf{v})d\mathbf{v}.$$

Proof. By applying the definition of the fractional Fourier transform to the convolution of the function $f(\mathbf{x})$ and $g(\mathbf{x})$, we have

$$\mathcal{F}_{\boldsymbol{\alpha}}[(f * g)(\mathbf{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi})(f * g)(\mathbf{x})d\mathbf{x}.$$

Substituting the definition of convolution

$$\mathcal{F}_{\boldsymbol{\alpha}}[(f * g)(\mathbf{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) \left(\int_{\mathbb{R}^n} f(\mathbf{v})g(\mathbf{x}-\mathbf{v})d\mathbf{v} \right) d\mathbf{x}.$$

Changing the order of integration (Fubini' Theorem)

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\alpha}}[(f * g)(\mathbf{x})](\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) f(\mathbf{v}) g(\mathbf{x} - v) d\mathbf{v} d\mathbf{x} \\ &= \int_{\mathbb{R}^n} f(\mathbf{v}) \left(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) g(\mathbf{x} - \mathbf{v}) d\mathbf{x} \right) d\mathbf{v}, \end{aligned}$$

perform the change of variables $\mathbf{w} = \mathbf{x} - \mathbf{v}$ thus $\mathbf{x} = \mathbf{w} + \mathbf{v}$ and $d\mathbf{x} = d\mathbf{w}$. Thus

$$\mathcal{F}_{\boldsymbol{\alpha}}[(f * g)(\mathbf{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(v)(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{w} + \mathbf{v}, \boldsymbol{\xi}) g(\mathbf{w}) d\mathbf{w}) dv.$$

Using the property of the kernel

$$K_{\boldsymbol{\alpha}}(\mathbf{w} + \mathbf{v}, \boldsymbol{\xi}) = K_{\boldsymbol{\alpha}}(\mathbf{v}, \boldsymbol{\xi}) K_{\boldsymbol{\alpha}}(\mathbf{w}, \boldsymbol{\xi}).$$

Thus:

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\alpha}}[(f * g)(\mathbf{x})](\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} f(\mathbf{v}) K_{\boldsymbol{\alpha}}(\mathbf{v}, \boldsymbol{\xi}) \left(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{w}, \boldsymbol{\xi}) g(\mathbf{w}) d\mathbf{w} \right) d\mathbf{v} \\ &= \left(\int_{\mathbb{R}^n} f(\mathbf{v}) K_{\boldsymbol{\alpha}}(\mathbf{v}, \boldsymbol{\xi}) d\mathbf{v} \right) \left(\int_{\mathbb{R}^n} g(\mathbf{w}) K_{\boldsymbol{\alpha}}(\mathbf{w}, \boldsymbol{\xi}) d\mathbf{w} \right) \\ &= \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi}) \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})](\boldsymbol{\xi}). \end{aligned}$$

This completes the proof. ■

2.2.5 Multiplication

Theorem 2.5 *Let $f(\mathbf{x}), g(\mathbf{x}) \in L^2(\mathbb{R}^n)$. Then:*

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})g(\mathbf{x})](\boldsymbol{\xi}) = \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi}) * \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})](\boldsymbol{\xi}), \tag{2.11}$$

where $*$ denotes convolution over $\mathbb{R}^n$.

Proof. we have

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})g(\mathbf{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}.$$

Since

$$g(\mathbf{x}) = \int_{\mathbb{R}^n} K_{-\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\zeta}) \mathcal{F}_{\boldsymbol{\alpha}}[g](\boldsymbol{\zeta}) d\boldsymbol{\zeta},$$

thus

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})g(\mathbf{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) f(\mathbf{x}) \left(\int_{\mathbb{R}^n} K_{-\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\zeta}) \mathcal{F}_{\boldsymbol{\alpha}}[g](\boldsymbol{\zeta}) d\boldsymbol{\zeta} \right) d\mathbf{x}.$$

Applying Fubini's Theorem

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})g(\mathbf{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} \mathcal{F}_{\boldsymbol{\alpha}}[g](\boldsymbol{\zeta}) \left(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) K_{-\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\zeta}) f(\mathbf{x}) d\mathbf{x} \right) d\boldsymbol{\zeta}.$$

Using the property

$$K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) K_{-\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\zeta}) = K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi} - \boldsymbol{\zeta}),$$

we get

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})g(\mathbf{x})](\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} \mathcal{F}_{\boldsymbol{\alpha}}[g](\boldsymbol{\zeta}) \left(\int_{\mathbb{R}^n} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi} - \boldsymbol{\zeta}) f(\mathbf{x}) d\mathbf{x} \right) d\boldsymbol{\zeta} \\ &= \int_{\mathbb{R}^n} \mathcal{F}_{\boldsymbol{\alpha}}[g](\boldsymbol{\zeta}) \mathcal{F}_{\boldsymbol{\alpha}}[f](\boldsymbol{\xi} - \boldsymbol{\zeta}) d\boldsymbol{\zeta}, \end{aligned}$$

which gives

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})g(\mathbf{x})](\boldsymbol{\xi}) = \mathcal{F}_{\boldsymbol{\alpha}}[f(\mathbf{x})](\boldsymbol{\xi}) * \mathcal{F}_{\boldsymbol{\alpha}}[g(\mathbf{x})](\boldsymbol{\xi}).$$

This completes the proof. ■

The following table defined some properties of $n-$dimensional fractional Fourier transform:

Description	function	$n-$FrFT wit angle α
Modulation	$e^{i2\pi \mathbf{F}.\mathbf{x}} f(\mathbf{x})$	$e^{-i\frac{\Vert\mathbf{v}\Vert^2 \sin\alpha\cos\alpha}{2} + i\mathbf{h}.\boldsymbol{\xi}\csc(\alpha)} \mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi} - \mathbf{v}\sin\alpha)$.
Derivative	$\frac{d}{d\mathbf{x}} f(\mathbf{x})$	$\cos\alpha \frac{d}{d\mathbf{x}} \mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi}) + i\boldsymbol{\xi}\sin\alpha \mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi})$
Integration	$\int_{-\infty}^{\mathbf{x}} f(\mathbf{x}) d\mathbf{x}$	$\sec\alpha . e^{-\frac{i}{2}\Vert\boldsymbol{\xi}\Vert^2 \tan\alpha} \int_{-\infty}^{\mathbf{x}} e^{-\frac{i}{2}\Vert\mathbf{v}\Vert^2 \tan\alpha} \mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\mathbf{v}) d\mathbf{v}$
time multiplication	$\Vert\mathbf{t}\Vert f(\mathbf{x})$	$\Vert\boldsymbol{\xi}\Vert \cos\alpha \mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi}) + i\sin\alpha \frac{d}{d\boldsymbol{\xi}} \mathcal{F}_{\boldsymbol{\alpha}}\{f(\mathbf{x})\}(\boldsymbol{\xi})$

2.3 Applications

The Fractional Fourier Transform (FrFT) has found numerous application in both theoretical and practical domains. Bellow are some of its most significant uses

1. **Optical propagation:**

   FrFT provides an elegant framework for modeling wave propagation in free space. Classical diffraction patterns, such as the Fraunhofer and Fresnel approximations, appear as special cases of the FrFT.

2. **Graded-index (GRIN) optical fibers:**

   The FrFT simplifies the analysis of light propagation in GRIN fibers, where the refractive index varies quadratically. It helps describe periodic beam focusing and imaging behavior.

3. **Gaussian beam analysis:**

   Since the FrFT of the Gaussian function is still a Gaussian, it is particularly suitable for modeling laser beam evolution and focusing characteristics.

4. **Quantum mechanics:**

   The FrFT is mathematically related to the Green's function of the quantum harmonic oscillator, making it applicable in quantum optics and wave mechanics.

5. **Efficient computation:**

   The FrFT can be implemented using fast numerical algorithms,such as FFT-based methods,making it attractive for real-time signal processing.

These applications show the FrFT's versatility as a bridge between classical Fourier analysis and modern wave-based systems, during transient events like epileptic spikes or cardiac anmalies.

CHAPTER 3

Multidimenstional windowed fractional Fourier transform

The windowed fractional Fourier transform (WFrFT) is an advanced tool that extends the classical fractional Fourier transform (FrFT) by incorporating time-localized analyzing non-stationary signals where both time and frequency localization are essential.

In this chapter, we define the $n-$dimensional WFrFT formally, explore its mathematical proprieties, compare it to the standard FrFT, and discuss its practical application.

3.1 Definitions

Definition 3.1 The windowed fractional Fourier transform (WFrFT) is an extension of the classical fractional Fourier transform (FrFT) that provide localized analysis of non-stationary signals, It is obtained by multiplying the function $f(x) \in L^2(\mathbb{R})$ with a window function $\psi(x) \in L^2(\mathbb{R})$ centered at a time $u \in \mathbb{R}$, and applying the FrFT. The WFrFT is defined as

$$\left(W_\psi f\right)(u, \xi) = \int_{\mathbb{R}} f(x)\overline{\psi(x-u)}K_\alpha(x, \xi)dx, \tag{3.1}$$

Where

- $f(x)$: the function (input signal), assumed to be in the space of square-integrable function $L^2(\mathbb{R})$.

- $\psi(x)$: the window function, also in $L^2(\mathbb{R})$ used to localize the signal around a specific time.

- $u \in \mathbb{R}$: the time shift parameter, which determines the centre of the window.

$K_\alpha(x,\xi)$: the kernel of the fractional Fourier transform of order α, which depends on both t and ξ.

- ξ is denotes the fractional frequency variable.

- $(W_\psi x)(u,\xi)$: the result of the windowed fractional Fourier transform, representing the time-frequency representation the time- frequency of the signal.

The windowed fractional Fourier transform (WFrFT) in $\mathbb{R}^n$ is an extension of the classical FrFT to higher dimension, enabling localized time frequency analysis for multidimensional non-stationary signals.

Definition 3.2 Let $f(\mathbf{x}) \in L^2(\mathbb{R}^n)$ and $\psi(\mathbf{x}) \in L^2(\mathbb{R}^n)$ a window function centred at $\mathbf{u} \in \mathbb{R}^n$. The $n-$dimensional WFrFT is defined as

$$(W_\psi f)(\mathbf{u},\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}, \tag{3.2}$$

where

- $\mathbf{x},\mathbf{u},\boldsymbol{\xi} \in \mathbb{R}^n$, are vectors in n-dimensional space.

- $K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})$ is the multidimensional FrFT kernel

$$K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi}) = \prod_{i=1}^{n} K_{\alpha_i}(x_i,\xi_i) \tag{3.3}$$

- $\psi(\mathbf{x}-\mathbf{u})$ is window function centered at $\mathbf{u}$.

3.2 Properties

3.2.1 Linearity

Theorem 3.1 *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, ..., x_n)$ be two functions in $L^2(\mathbb{R}^n)$, and let $a, b \in \mathbb{C}$ be a constants. The $n-$dimensional windowed fractional Fourier* *transform is linear* $(W_\psi f)(\mathbf{u},\boldsymbol{\xi})$

$$W_\psi\left[af(\mathbf{x})+bg(\mathbf{x})\right](\mathbf{u},\boldsymbol{\xi})=aW_\psi\left[f(\mathbf{x})\right](\mathbf{u},\boldsymbol{\xi})+bW_\psi\left[g(\mathbf{x})\right](\mathbf{u},\boldsymbol{\xi}). \tag{3.4}$$

Proof. Using the definition of the $n-$dimensional WFrFT, we have

$$W_\psi[af(\mathbf{x})+g(\mathbf{x})](\mathbf{u},\boldsymbol{\xi})=\int_{\mathbb{R}^n}[af(\mathbf{x})+bg(\mathbf{x})]\,\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x},$$

by linearity of the integral

$$\begin{aligned}&W_\psi[af(\mathbf{x})+bg(\mathbf{x})](\mathbf{u},\boldsymbol{\xi})\\=\;&a\int_{\mathbb{R}^n}f(\mathbf{x})\overline{\psi(\mathbf{x}-\mathbf{u})}K_\alpha(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}+\mathbf{b}\int_{\mathbb{R}^n}g(\mathbf{x})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}\\=\;&aW_\psi\left[f(\mathbf{x})\right](\mathbf{u},\boldsymbol{\xi})+bW_\psi\left[g(\mathbf{x})\right](\mathbf{u},\boldsymbol{\xi}).\end{aligned}$$

This completes the proof. ■

3.2.2 Time-Shift

Theorem 3.2 *Let* $f(\mathbf{x})\in L^2(\mathbb{R}^n)$ *and let* $\mathbf{a}\in\mathbb{R}^n,\psi\in L^2(\mathbb{R}^n)$ *be a window function. The* $n-$*dimensional WFrFT of the shifted function* $f(\mathbf{x}-\mathbf{a})$ *is given by*

$$W_\psi\left[f(\mathbf{x}-\mathbf{a})\right](\mathbf{u},\boldsymbol{\xi})=e^{-i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a})}W_\psi[f(\mathbf{x})](\mathbf{u}-\mathbf{a},\boldsymbol{\xi}-\mathbf{a}\cos\alpha), \tag{3.5}$$

where $\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a})$ *is a phase term depending on the fractional order and kernel.*

Proof. Start from the definition of $n-$dimensional WFrFT

$$W_\psi[f(\mathbf{x})](\mathbf{u},\boldsymbol{\xi})=\int_{\mathbb{R}^n}f(\mathbf{x})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}.$$

Now, apply it to the shifted function $f(\mathbf{x}-\mathbf{a})$

$$W_\psi[f(\mathbf{x}-\mathbf{a})](\mathbf{u},\boldsymbol{\xi})=\int_{\mathbb{R}^n}f(\mathbf{x}-\mathbf{a})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}.$$

Let change variables: $\mathbf{y}=\mathbf{x}-\mathbf{a}$, thus $\mathbf{x}=\mathbf{y}+\mathbf{a},d\mathbf{x}=d\mathbf{y}$, we get

$$W_\psi[f(\mathbf{x}-\mathbf{a})](\mathbf{u},\boldsymbol{\xi})=\int_{\mathbb{R}^n}f(\mathbf{y})\overline{\psi(\mathbf{y}+\mathbf{a}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{y}+\mathbf{a},\boldsymbol{\xi})d\mathbf{y}.$$

Now use the kernel property of the fractional Fourier transform

$$K_{\boldsymbol{\alpha}}(\mathbf{y}+\mathbf{a},\boldsymbol{\xi})=e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a})}K_{\boldsymbol{\alpha}}(\mathbf{y},\boldsymbol{\xi}-\mathbf{a}\cos\alpha),$$

where $\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a}) = \left(\frac{1}{2}(\|\mathbf{a}\|^2)\cot(\alpha) - \mathbf{a}.\boldsymbol{\xi}\csc(\alpha)\right)$, so the expression becomes

$$\begin{aligned} W_\psi[f(\mathbf{x}-\mathbf{a})](\mathbf{u},\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} f(\mathbf{y})\overline{\psi(\mathbf{y}+\mathbf{a}-\mathbf{u})}e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a})}K_{\boldsymbol{\alpha}}(\mathbf{y},\boldsymbol{\xi}-\mathbf{a}\cos\alpha)d\mathbf{y} \\ &= e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a})}\int_{\mathbb{R}^n} f(\mathbf{y})\overline{\psi(\mathbf{y}-\mathbf{z})}K_{\boldsymbol{\alpha}}(\mathbf{y},\boldsymbol{\xi}-\mathbf{a}\cos\alpha)d\mathbf{y}. \end{aligned}$$

Now set: $\mathbf{z} = \mathbf{u} - \mathbf{a}$, so $\mathbf{y} + \mathbf{a} - \mathbf{u} = \mathbf{y} - \mathbf{z}$. Then

$$W_\psi[f(\mathbf{x}-\mathbf{a})](\mathbf{u},\boldsymbol{\xi}) = e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi},\mathbf{a})}W_\psi[f(\mathbf{x})](\mathbf{u}-\mathbf{a},\boldsymbol{\xi}-\mathbf{a}\cos\alpha).$$

This completes the proof. ■

3.2.3 Convolution

Theorem 3.3 *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be two functions in $L^2(\mathbb{R}^n)$, and let $\psi(\mathbf{x})$ be a window function. Then, the $n-$dimensional windowed fractional Fourier transform ($n-$WFrFT) satisfies the following convolution property.*

$$W_\psi[f*g](\mathbf{u},\boldsymbol{\xi}) = (W_\psi f)(\mathbf{u},\boldsymbol{\xi}).\mathcal{F}_\alpha[g](\boldsymbol{\xi}). \tag{3.6}$$

Proof. We start of the definition of WFrFT in $\mathbb{R}^n$

$$(W_\psi f)(\mathbf{u},\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}.$$

Let apply it to $f*g$

$$W_\psi[f*g](\mathbf{u},\boldsymbol{\xi}) = \int_{\mathbb{R}^n} (f*g)(\mathbf{x})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}.$$

Recall that in $\mathbb{R}^n$, the convolution is

$$(f*g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y}.$$

Substitute into $n-$WFrFT

$$W_\psi[f*g](\mathbf{u},\boldsymbol{\xi}) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}.$$

Apply Fubini's theorem to change the order of integration

$$W_\psi[f*g](\mathbf{u},\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{y})\left(\int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y})\overline{\psi(\mathbf{x}-\mathbf{u})}K_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\xi})d\mathbf{x}\right)d\mathbf{y}.$$

Now perform the change of variable $\mathbf{z} = \mathbf{x} - \mathbf{y}$,thus $\mathbf{x} = \mathbf{z} + \mathbf{y}$, so $d\mathbf{x} = d\mathbf{z}$

$$W_{\psi}[f * g](\mathbf{u}, \boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\mathbb{R}^n} g(\mathbf{z}) \overline{\psi(\mathbf{z} + \mathbf{y} - \mathbf{u})} K_{\boldsymbol{\alpha}}(\mathbf{z} + \mathbf{y}, \boldsymbol{\xi}) d\mathbf{z} \right) d\mathbf{y},$$

$$K_{\boldsymbol{\alpha}}(\mathbf{z} + \mathbf{y}, \boldsymbol{\xi}) = e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \mathbf{y})} K_{\boldsymbol{\alpha}}(\mathbf{z}, \boldsymbol{\xi} - \mathbf{y} \cos \alpha),$$

where $\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \mathbf{y}) = \left((\|\mathbf{y}\|^2) \cot(\alpha) + -\mathbf{y}.\boldsymbol{\xi} \csc(\alpha) \right)$

$$\begin{aligned} W_{\psi}[f * g](\mathbf{u}, \boldsymbol{\xi}) &= \int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\mathbb{R}^n} g(\mathbf{z}) \overline{\psi(\mathbf{z} + \mathbf{y} - \mathbf{u})} e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \mathbf{y})} K_{\boldsymbol{\alpha}}(\mathbf{z}, \boldsymbol{\xi} - \mathbf{y} \cos \alpha) d\mathbf{z} \right) d\mathbf{y} \\ &= (W_{\psi} f)(\mathbf{u}, \boldsymbol{\xi}).\mathcal{F}_{\alpha}[g](\boldsymbol{\xi}). \end{aligned}$$

This complete the proof. ■

3.2.4 Scaling

Theorem 3.4 *Let $f_a(\mathbf{x}) = f(a\mathbf{x})$,where $a \in \mathbb{R} \backslash \{0\}$ is a scaling factor. Then the $n-$ WFrFT satisfies the following*

$$W_{\psi}[f(a\mathbf{x})](\mathbf{u}, \boldsymbol{\xi}) = \frac{1}{|a|^n} W_{\psi}[f] \left(\frac{\mathbf{u}}{a}, \frac{\boldsymbol{\xi}}{a} \right) e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, a, \boldsymbol{\alpha})}. \tag{3.7}$$

Proof. Start with the WFrFT definition in $\mathbb{R}^n$:

$$W_{\psi}[f](\mathbf{u}, \boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{\psi(\mathbf{x} - \mathbf{u})} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) d\mathbf{x}.$$

Apply it to the scaled function $f(a\mathbf{x})$, we get

$$W_{\psi}[f(a\mathbf{x})](\mathbf{u}, \boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{ax}) \overline{\psi(\mathbf{ax} - u)} K_{\boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\xi}) d\mathbf{x}.$$

Now perform the change the variable $\mathbf{y} = a\mathbf{x}$,so $\mathbf{x} = \frac{\mathbf{y}}{a}$, $d\mathbf{x} = \frac{1}{|a|^n} d\mathbf{y}$

$$W_{\psi}[f(\mathbf{ax})](\mathbf{u}, \boldsymbol{\xi}) = \frac{1}{|a|^n} \int_{\mathbb{R}^n} f(\mathbf{y}) \overline{\psi\left(\frac{\mathbf{y}}{a} - \mathbf{u} \right)} K_{\boldsymbol{\alpha}} \left(\frac{\mathbf{y}}{a}, \boldsymbol{\xi} \right) d\mathbf{y},$$

we have

$$K_{\boldsymbol{\alpha}} \left(\frac{\mathbf{y}}{a}, \boldsymbol{\xi} \right) = e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \mathbf{y})} K_{\boldsymbol{\alpha}}(\mathbf{y}, \boldsymbol{\xi}),$$

where $\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, a, \boldsymbol{\alpha}) = (\|\mathbf{y}\|^2) \left(1 - \frac{1}{a^2}\right) \cot(\alpha)$. This give:

$$W_{\psi}[f(a\mathbf{x})](\mathbf{u}, \boldsymbol{\xi}) = \frac{1}{|a|^n} W_{\psi}[f] \left(\frac{\mathbf{u}}{a}, \frac{\boldsymbol{\xi}}{a} \right) e^{i\phi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, a, \boldsymbol{\alpha})}.$$

This complete the proof. ■

3.2.5 Parseval's Theorem

Theorem 3.5 *Let $f \in L^2(\mathbb{R}^n)$ be a square-integrable function, and let $\psi \in L^2(\mathbb{R}^n)$ be a $n-$dimensional window function. Then the $n-$dimensional windowed fractional Fourier transform ($n-$WFrFT) satisfies the following energy-preserving identity:*

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} |W_\psi[f](\mathbf{u},\boldsymbol{\xi})|^2 \, d\mathbf{u}d\boldsymbol{\xi} = \|\psi\|_2^2 \cdot \|f\|_2^2 . \tag{3.8}$$

This means that the total energy of the $n-$WFrFT of f, weighted by the window ψ, equals the energy of the original scaled by the energy of the window.

3.3 Window Function in the WFrFT

In the windowed Fractional Fourier Transform (WFrFT), The window function ψ plays a crucial role. It localizes the analysis of the signal in both time and frequency, allowing for a time-frequency representation. The window must belong to $L^2(\mathbb{R}^n)$,i.e., it should be square-integrable and $\psi(\mathbf{x}) = \prod_{i=1}^n \psi(x_i)$.

3.3.1 Examples of Window functions

1. **Gaussian window:**

$$\psi(x) = e^{\frac{-x^2}{2}}.$$

This window is:

(a) Optimal for time frequency resolution.

(b) Minimizes the uncertainty (best joint localization in time and frequency).

(c) Remaina invariant under the fractional Fourier transform (self-FrFT).

2. **Hamming window:**

$$\psi(x) = 0.54 - 0.46\cos\left(\frac{2\pi x}{N-1}\right), x \in [0, N-1].$$

This window is

(a) Reduces spectral leakage

(b) Common in digital signal processing (DSP).

3. **Hanning window:**

$$\psi(x) = 0.5\left(1 - \cos\left(\frac{2\pi x}{N-1}\right)\right).$$

This window is

(a) Smooths the edges of the window more than the Hamming windows

4. **Rectangular Window:**

$$\psi(x) = \begin{cases} 1, & x \in [-L/2, L/2] \\ 0 & elsewhere. \end{cases}$$

This window is

(a) Simplest window,but causes high spectral leakage.

(b) Not smooth;not ideal for frequency domain analysis.

5. **Other Windows: Blackman,Kaiser,Tukey,**etc.

These windows are

1. (a) Each offer of trade-of between main-lobe width and side-lobe attenuation.

Remark 3.1 - The choice of window function effects time frequency resolution directly.

- In the case of the WFrFT ,it is preferable to choose windows that are well matched to the fractional domain.

- Gaussian windows are often ideal due to their smoothness and transformation proprieties.

3.4 Comparison between FrFT and WFrFT

Feature	FrFT	WFrFT
Time frequency localization	Weak	Good,due to windrowing
Window function	Not used	Applied, chosen based on signal type
Nature of Transform	Global	Localized
Suitable for	Stationary or longue-duration signals	Not stationary or Short-duration signals
Special case	-	Reduces to FrFT when $\psi(x) = 1$

This comparison highlights the main conceptual and practical differences between the classical Fractional Fourier Transform (FrFT) and its windowed version (WFrFT). While FrFT performs a global transformation, WFrFT enables localized time-frequency analysis by applying a window function. As a result, WFrFT is more suitable for analyzing non stationary or short duration signals, where the global nature of FrFT my fail to capture local variations. This makes WFrFT a powerful tool in modern signals processing applications.

3.5 Applications

The Windowed Fractional Fourier Transform (WFrFT) extends the classical FrFT by incorporating a window function that enables local analysis in the time-frequency domain. This makes the WFrFT particularly well-suited for analyzing non-stationary and time-varying signals, where the standard FrFT may fail to provide sufficient resolution. Some notable applications of the WFrFT include:

1. **Speech and audio signal processing:**

   WFrFT is used for analyzing short-time speech signals,detecting time-varying pitch and formant structures with improved resolution.

2. **Biomedical signal analysis:**

   It is employed to study localized features in signals such as EEG or ECG ,especially during transient events like epileptic spikes or cardiac anmalies.

3. **Time-frequency filtering:**

   By applying a windowed transformation,WFrFT allows selective filtering in both time and fractional frequency domains,enhancing signal denoising and compression.

4. **Radar and sonar systems:**

WFrFT enables better detection of moving targets by adapting the analysis window to the signal's behavior over time

Compared to traditional FrFT ,the WFrFT offers enlaced flexibility and precision in application requiring simultaneous time and frequency localization.

CHAPTER 4 Conclusion

In this Master's thesis, we studied the multidimensional fractional Fourier transform (MFrFT) and its windowed version (WFrFT) and advanced extensions of the classical Fourier transform. So, we presented the formal definitions of both transforms and analyzed their main mathematical properties, including linearity,time-shift, convolution, scaling, and energy preservation. We also examined the role of window functions in enhancing time-frequency localization within the WFrFT framework. A comparative analysis between FrFT and WFrFT was included to highlight their differences in terms of performance and analytical precision. The theoretical findings were supported by selected applications demonstrating the usefulness of these transforms in fields such as optics, biomedical signal analysis, and time-frequency filtering.

We hope this work provides a modest contribution to the understanding of this advanced topic and serves as a foundation for future studies, whether theoretical or applied.

Appendix

The following properties of one dimensional kernel that useful in properties in n-dimensional kernel. For $i = 1, ..., n$, we have

1. The Kernel is symmetric:

$$K_{\alpha_i}\left(x_i, \xi_i\right) = K_{\alpha_i}\left(\xi_i, x_i\right).$$

2.

$$K^*_{\alpha_i}\left(x_i, \xi_i\right) = K_{-\alpha_i}\left(x_i, \xi_i\right),$$

where $K^*_{\alpha_i}\left(x_i, \xi_i\right)$ is a conjugate of $K_{\alpha_i}\left(x_i, \xi_i\right)$.

3.

$$K_{\alpha_i}\left(-x_i, \xi_i\right) = K_{\alpha_i}\left(x_i, -\xi_i\right).$$

4.

$$K_{\alpha_i+\beta_i}\left(x_i, \zeta_i\right) = \int_{\mathbb{R}} K_{\alpha_i}(x_i, \xi_i) K_{\beta_i}\left(\xi_i, \zeta_i\right) d\xi_i.$$

5.

$$\int_{\mathbb{R}} K_{\alpha_i}(x_i, \xi_i) K^*_{\alpha_i}\left(x_i, \xi'_i\right) dx_i = \delta\left(\xi_i - \xi'_i\right).$$

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