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Mémoire

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Well-posedness of solution for a nonlinear wave equation

Option : AFA

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Bouzitouna Ahlem

Abstract

In this memory, first, we defined the classi¢ cation of partial differential equation and canonical forms, we also exposed same theorems and de¢ nitions which has importants rolein the sequel. Finally we are going to solve a class of hyperbolic equation, the essential but is study the existence and uniquess of the solution, also, we study the behavionr of the solution use the energy method.

Key words:

* Hyperbolic equation
* Method energy

Résumé

Dans ce mémoire, premiérement, on fait la classiffication de l'équation partielles differentielle et la forme canonique, ainsi que nous avous rapplé quelques théorémes et quelques déffinition important dans la suite. Finalement nous allons résoudre une classe d'équation hyperbolique, le but essentile est d'étudier l'existence, l'unicité de solution, de plus la comportement de le solution par la méthode de l'energé.

Mots clés:

1. * Équation hyperbolique
2. * Méthode de l'enragé

ملخص

ندرس في هذه المذكرة:

في الفصل الأول من المذكرة نقوم بعرض جملة من أنواع المعادلات التفاضلية الجزئية المهمة والتي تمثل في المعادلات:

التكافئية الزائدية والناقصية و كتابة الشكل القانوني لكل صنف.

في الفصل الثاني نقوم بعرض بعض النظريات والتعريفات المهمة.

وفي الفصل الثالث نقوم بدراسة السلوك التقاربي لنوع من المعادلات التفاضلية الجزئية الزائدية بطريقة الطاقة وفق شروط حدية لديركلي متجانسة وشروط إبتدائية. ثم البرهنة على أن الحل ينفجر في وقت زمني محدد.

الكلمات المفتاحية:

* معادلة زائدية.

* طريقة الطاقة.

LIST OF SYMBOLS

Sets and numbers

❁ $\mathbb{R}^n$	The euclidean space.
❁ Ω	a bounded regular domain in $\mathbb{R}^n$.
❁ $[a, b]$	a closed interval of $\mathbb{R}^n$.
❁ $H_0^1(\Omega)$	$u \in H^1(\Omega), u(0) = u(1) = 0$.
❁ $L^p(\Omega)$	space of p-th order integrable functions on Ω.
❁ $L^{p'}(\Omega)$	Dual space of $L^p(\Omega)$.
❁ $L^\infty(\Omega)$	L infinity space.
❁ $W^{1,p}(I)$	Sobolev space.
❁ B^*	Dual space of B.
❁ $C_c^1(I)$	Space of all first derivative function has a complect support.
❁ $\partial\Omega$	The boundary of Ω.

Functions

❊ $\vert . \vert$	Absolute value.
❊ $\Vert . \Vert$	Norm.
❊ $\Vert f \Vert_2$	Norm of $L^2(\Omega)$.
❊ $\Vert u \Vert_{L^p([a,b],X)}$	Norm of $L^2([a, b] : X)$ defined by : $\int_a^b (\Vert u(t) \Vert_X^p)^{1/p}$.
❊ sup	Superior value.
❊ inf	Inferior value.
❊ Δ	Laplacien.
❊ ∇u	Devergence.

Abbreviation Meaning

● PDE	Partial differential equation.
● ODE	Ordinary differential equation.

CONTENTS

INTRODUCTION

The study of partial differential equations (PDE) started in a century eighteen (18) action Euler, Alembert, Lagrange and Laplace. This equations are used to mathematically formulate, and thus aid the solution of, physical. The importance of a differential eqution as a thecnique for determining a function is that if we know the function and possibly some of its derivatives at a particular point.

In chapter 01:

* classification of partial differential equation
* canonical forms In this chapter werite general forms the partial differential equation and used (Δ) for prove la classiffication the this equation rite canonical forme for all class the equation.

In chapter 02: we exposed same therems and deffintions has importants role in the sequel. It used for prove our main results.

In chapter 03: we are going to solve a class of parabolic equation, We prove the existence and uniquess of the solution, also, we study the behavion of the solution also used the Feado-Galerkin method.

CHAPTER 1

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATION AND CANONICAL FORMS

1.1 second-order partial differential equation

The most general case of second-order linear partial differential equation (PDE) in two independent variables is given by

$$A\frac{\partial^{2u}}{\partial^2 x} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \tag{1.1}$$

Where the coefficients A, B and C are functions of x and y and do not vanish simultaneously, because in that case the second-order PDE degenerates to one of first order. Further the coefficients D, E and F are also assumed to be functions of x and y. We shall assume that the function $u(x, y)$ and the coefficients are twice continuously differentialble in some domain Ω. The classification of second-order PDE depends on the form of the leading part of the equation consisting of the second order terms. So, for simplicity of notation, we combine the lower ordre terms and rewrite the above aquation in the folowing form

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = \phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \tag{1.2}$$

or using the short-hand notations for partial derivatives,

$$A(x, y)u_{x\,x} + B(x, y)u_{xy} + C(x, y)u_{yy} = \phi(x, y, u, u_x, u_y). \tag{1.3}$$

As we shall see, there are fundanmentally theree types of $PDE_{s-}hyperbolic$,and elliptic $PDE_{S.}$From the physical point of view, these PDE_S respectively represents the wave propagation, the time-dipendent diffusion processes and the steady state or equilibrium processes. Thus, hyperbolic equations model the transport of some physical quantity, such as fluids or waves. Parabolic problems descrde evolutionary phenomena that lead to a steady state described by an elleptic equation.and elleptic equations are associated to a special state of a system, in principle corresponding to the minimum of the invergy. Mathematically, these classification of second-order PDE_S is based upon the possibility of redesing equation (1.3) by coordinate transformation to canonical or standard from at a point. It may be noted that, for the purposes of classification, it is not necessary to restrict considuration to linear equation. It is applicable to quasilinear second-order PDE as well. Aquasilinear second-ordre PDE is linear in the second derivatives only. The type of second-ordre PDE (1.2) at a point $(x_{0,}y_0)$ depents on the sign of the discriminant defined as

$$\Delta(x_{0,}y_0) = \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B\left(x_{0,}y_0\right)^2 \text{-} 4A\left(x_0, y_0\right)C\left(x_{0,}y_0\right) \ . \tag{1.4}$$

The classification of second-order linear PDE_s is given by the following.
If $\Delta\left(x_0 y_0\right) \succ 0$, the equation is hyperbolic.
$\Delta\left(x_0, y_0\right) = 0$ the equation is parabolic.
$\Delta\left(x_0, y_0\right) \prec 0$ the equation is elliptic.
In order to illustrate the significance of the discriminant Δ and thus the classification of the PDE (1.3). We try to reduce the given eqution (1.3) to a canonical form. To do this, we transform the independent variables x and y to the new independent variables ξ and η through the change of variables

$$\xi = \xi\left(x, y\right), \qquad\qquad \eta = \eta\left(x, y\right). \tag{1.5}$$

Where both ξ and η are twice continuously differntiable and that the Jacobian

$$J = \frac{\partial\left(\xi, \eta\right)}{\partial\left(x, y\right)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0, \tag{1.6}$$

in the region under consideration. The nonvanishing of the Jacobian of the transformation ensure that a one to one transformation exists betwrrn the new and old variables. This simply means that the new independent variables can serve as new coordinate variables without any apply the chain rule to compute the terms of the equation (1.3), in terms of ξ and η as follows:

$$u_x = \omega_\xi \xi_x + \omega_\eta \eta_x, \tag{1.7}$$

$$u_y = \omega_\xi \xi_y + \omega_\eta \eta_y,$$

$$u_{xx} = \omega_{\xi\xi}\xi_x^2 + 2\omega_{\xi\eta}\xi_x\eta_x + \omega_{\eta\eta}\eta_x^2 + \omega_\xi\xi_{xx} + \omega_\eta\eta_{xx},$$

$$u_{yy} = \omega_{\xi\xi}\xi_y^2 + 2\omega_{\xi\eta}\xi_y\eta_y + \omega_{yy}\eta_y^2 + \omega_\xi\xi_{yy} + \omega\eta_{yy},$$

$$u_{xy} = \omega_{\xi\xi}\xi_x\xi_y + \omega_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + \omega_{\eta\eta}\eta_x\eta_y + \omega_\xi\xi_{xy} + \omega_\eta\eta_{xy},$$

substituting these expressions into equation (1.3) we obtain the transformed PDE as

$$a\omega_{\zeta\zeta} + b\omega_{\zeta\eta} + c\omega_{\eta\eta} = \phi\left(\zeta, \eta, \omega, \omega_{\zeta,}\omega_\eta\right). \tag{1.8}$$

Where Φ becomes ϕ and the new coefficients of the higher order terms a, b and c are expressed via the original coefficients and the change of variables formulas as follows :

$$\begin{aligned} a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\ b &= 2A\xi_x\eta_y + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y, \\ c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned} \tag{1.9}$$

As this stage the form of the EDP (1.8) is no simpler than that of the original EDP (1.2) , but this is to be expected because so far the choice of the new vaeiable ξ and has been that equation (1.9) can be written in matrix form as

$$\begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} + \begin{vmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{vmatrix} J^2,$$

where J is Jacobian of the change of variables given by (1.6). Expanding the determinant and multiplying by the factor, -4 to obtain

$$b^2 - 4ac = J^2\left(B^2 - 4AC\right) \Rightarrow \delta = J^2\Delta. \tag{1.10}$$

Where $\delta = b^2 - 4ac$ is the discriminant of the equation (1.8). This shows that the discriminant of (1.2) has the same sign as the discriminant of the transformed equation (1.8) and therfore it is clear that any real nonsingular ($J \neq 0$) transformation does not change the type of PDE. Note that the discriminant involoves only the coefficients of second-order derivatives of the corresponding PDE.

1.2 Canonical forms

Let use now try to contruct transformation, which will make one, or possibly two of the coefficients of the leading second-order terms of equation (1.8) vanish, thus reducing the equation to asimpler form called canonical form. For convenience, we reproduce below the original PDE

$$A(x;y)\,u_{xx}+B(x,y)\,u_{xy}+C(x,y)\,u_{yy}=\Phi(x,y,u,u_x,u_y)\,, \tag{1.11}$$

and the corresponding transformrd PDE

$$a(\xi,\eta)\,\omega_{\xi\xi}+b(\xi,\eta)\,\omega_{\eta\eta}+c(\xi,\eta)\,\omega_{\eta\eta}=\phi(\xi,\eta,\omega,\omega_\zeta,\omega_\eta)\,. \tag{1.12}$$

We again mention here that for the (1.2) or (1.8) to remain a second-order PDE, the coefficients A,B and C (or a , b and c) do not vanish simultaneously. By definition, a PDE is hyperbolic if the discriminant $\Delta = b^2-4ac \succ 0$. Since the sign of dicriminant is invariant under the change of coordinater see equatio (1.10), it follows that for a hyperboulic PDE,we should have $b^2-4ac \succ 0$. The simplest case of satisfying this condition is $a=c=0$. So, if we try to chose the new variables ξ and η such that the cofficients a and c vanish, we get the following canonical form of hyperbolic euqation :

$$\omega_{\xi\eta}=\psi(\xi,\eta,\omega,\omega_\xi,\omega_\eta)\,. \tag{1.13}$$

Where $\psi=\frac{\phi}{b}$, this form is called the first canonical form of the hyperbolic eqution, we also have another simple case for which $b^2-4ac \succ 0$ condition is satified. This the case when $b=0$ and $c=-a$. Is this case (1.10) reduces to

$$\omega_{\alpha\alpha}-\omega_{\beta\beta}=\psi(\alpha,\beta,\omega,\omega_\alpha,\omega_\beta)\,, \tag{1.14}$$

which is the second canonical form of the hyperbolicequatoin. By definition, a PDE is parabolic if the discriminant $\Delta=b^2-4ac=0$. It follows that for a parabolic PDE, we should have $b^2-4ac=0$. Thesimplest case of satisfying this condition is a (or c) = 0. In this case another necessary requirement $b=0$ will follow automatically (sinec b^2-4ac). So if we try to chose the new variables ξ and η such that the coefficients a and b vanish, we get the following canonical form of parabolic eqution :

$$\omega_{\eta\eta}=\psi(\xi,\eta,\omega,\omega_\zeta,\omega_\eta)\,, \tag{1.15}$$

where $\psi=\frac{\phi}{c}$. By definition, a PDE is elliptic if the discriminant $\Delta=b^2-4ac \prec 0.It$ follows that for a elliptic PDE, we should have $b^2-4ac \prec 0$. The simplesr case of satisfying this condition is $b=0$ and $c=a$. Soif we tey to chose the new variables ξ and η such that b vanishes and $c=a$, we get the following canonical form of elliptic eqution :

$$\omega_{\xi\xi}+\omega_{\eta\eta}=\psi(\xi,\eta,\omega,\omega_\xi,\omega_\eta)\,, \tag{1.16}$$

where $\psi = \frac{\phi}{a}$.
In summary eqution (2.8) can be reduced to a canonical form if the coordinate fransformation $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ can be selected such that :
.$a = c = 0$ corresponds to the first canonical form of hyperbolic PDE given by (1.15) .$b = 0, c = -a$ corresponds to the second canonical foem of hyperbolic PDE given by (1.14) .$a = b = 0$ corresponds to the canonical form of parabolic PDE given by (1.15). $b = 0, c = a$ corresponds to the canonical form of elliptic PDE given by (1.16)

1.2.1 Hyperbolic equtions

For a hyperbolic PDE the discriminant $(\Delta = b^2 - 4ac) \succ 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variablrs ξ and η such that the coefficients a and c vanish in (1.8) form (1.9) we have

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0, \tag{1.17}$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0. \tag{1.18}$$

Divising eqution (1.7) and (1.8) throughout by ξ_y^2 and η_y^2 respectively to obtain

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_x}\right) + C = 0, \tag{1.19}$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0. \tag{1.20}$$

Equation (1.10) is a quadratic euqation for $\left(\frac{\xi_x}{\xi_y}\right)$ whose roots are given by

$$\mu_1(x, y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

$$\mu_2(x, y) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

The roots of the equation (1.20) can also be found in an identical manner, so as only two distinct roots are possible between the two equation (1.19) and (1.20). Here we may consider μ_1 as the root of (1.19) and μ_2 as that of (1.20) That is :

$$\mu_1(x, y) = \frac{\xi_x}{\xi_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \tag{1.21}$$

$$\mu_2(x,y) = \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{1.22}$$

The above equations lead to the following two first-order diferential aquation

$$\xi_x - \mu_1(x,y)\,\xi_Y = 0, \tag{1.23}$$

$$\eta_x - \mu_2(x,y)\,\eta_y = 0. \tag{1.24}$$

These are the equations that define the new coordinate variables ξ and η that are necessary to make $a = c = 0$ in (1.8).
As the total derivative of ξ along the coordinate $\xi(x,y) =$ constant, $d\xi = 0$. It follows that

$$d\xi = \xi_x dx = \xi_y dy = 0,$$

and hence the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

We also have a similar result along coordinate line $\eta(x,y) =$ constant ,i.e,

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}.$$

Using these results equation (1.19) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.25}$$

This is called the characteristic polynomial of the PDE (1.2) and its roots are given by

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \lambda_1(x,y), \tag{1.26}$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \lambda_2(x,y). \tag{1.27}$$

The required variables ξ and η are determined by the respective solutions of the two ordinary differential equations (1.26) and (1.27), known as the characteristic equation of the PDE (1.2) They are ordinary differential equation for families of curves in the xy-plane along wich $\xi =$ constant and $\eta =$constant.Clearly these families of curves depend on the coefficients A,B and C in the original PDE (1.2).

1.2. Canonical forms

Integration of equation (1.26) leads to the family of curvilinear coordinates $\xi(x,y) = c_1$ while the of (1.27) gives another family of curvilinear coordinates $\eta(x,y) = c_2$, where c_1

and c_2 are arbitrary constants of integration. These two families of curvilinear coordinates $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are called characteristic curves of the hyperbolic equation (1.3) or simply the characteristics of the equation. Hence second-order hyperbolic eqution have twe families of characteristic curves. The fact that $\Delta \succ 0$ means that the characteristic are real curves in xy-plane.
If the coefficients A, B and C are constants it is easy to integrate euations (1.26) and (1.27) to obtain the expressions for change of variables formulas for reducing a hyperbolic PDE to the canonical from . Thus integration of (1.26) produces

$$y = \frac{B+\sqrt{B^2-4AC}}{2A}x + c_1 \qquad and \qquad y = \frac{B-\sqrt{B^2-4AC}}{2A}x + c_2. \tag{1.28}$$

Or

$$y = \frac{B+\sqrt{B^2-4AC}}{2A}x = c_1 \qquad and \qquad y = \frac{B-\sqrt{B^2-4AC}}{2A}x = c_2 \quad . \tag{1.29}$$

Thus when the coefficients A, B and C are two constants the two families of characteristic curves associated with PDE reduces to two distinct families of parallel straight lines.
Since the families of curves ξ =constant and η =constant are the characteristic curves, the change of variables are given by the following equations :

$$\xi = y - \frac{B-\sqrt{B^2-4AC}}{2A}x = y - \lambda_1 x, \tag{1.30}$$

$$\eta = y - \frac{B+\sqrt{B^2-4AC}}{2A}x = y - \lambda_2 x. \tag{1.31}$$

The first canonical form of the hyperbolic is :

$$\omega_{\xi\eta} = \psi(\xi, \eta, \omega, \omega_\xi, \omega_\eta), \tag{1.32}$$

where $\psi = \frac{\phi}{b}$ and b is calculated from (1.9)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B(\xi_x\eta_x + \xi_y\eta_y) + 2C\xi_y\eta_y, \\ &= 2A\left(\frac{B^2-(B^2-4AC)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C, \\ &= 4C - \frac{B^2}{A} = \frac{\Delta}{A}. \end{aligned} \tag{1.33}$$

Each of the familiies $\xi(x, y) =$ constant and $\eta(x, y) =$ constant forms an envelop of the domain of the xy-plane in which th PDE is hyperbolic.
The transformation $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ can be regarded as a mapping from the xy-plane to the $\xi\eta - plane$. ξ and the curves along which ξ and η are constant in the xy-plane brcome coordinates lines in the $\xi\eta - plane$. Sinec these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other worde, charactreristic curves of a hyperbolic PDE are those curves for the PDE. Must be referred as coordinate curves in order that it take on canonical form. We now determine the Jacobian of transformation definded by (1.30) and (1.31). We have

$$J = \begin{vmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 1 \end{vmatrix} = \lambda_2 - \lambda_1.$$

We know that $\lambda_1 = \lambda_2$ only if $B^2 - 4AC = 0$. However, for an hyperbolic PDE, $B^2 - 4AC \neq 0$. Hence Jacobian is nonsingular for the given transformation. A consequence of $\lambda_1 \neq \lambda_2$ is that at no point can the particular curves from each family share a common trangent line. It is easy to show that the hyperbolic PDE has a second canonical form. The following linear change of variables

$$\alpha = \xi + \eta \qquad\qquad \beta = \xi - \eta,$$

converts (1.32) in to

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi(\alpha, \beta, \omega, \omega_\alpha, \omega_\beta), \tag{1.34}$$

which is the seconde canonical form of the hyperbolic equtions

1.2.2 Parabolic equtions

For a parabolic PDE the discriminant $\Delta = B^2 - 4AC = 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ξ and η such that the coefficients a and b vanish in (1.8). Thus, from (1.9) we have

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2.$$

Dividing the above eqution throughhout by ξ_y^2 to obtain

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C + 0. \tag{1.35}$$

As the total derivative of ξ along the coordinate line $\xi(x, y)$ =constant, $d\xi = 0$.It follows that

$$d\xi = \xi_x dx + \xi_y dy = 0,$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

Using this result, equation (1.35) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.36}$$

This is called the characteristic polynomial of the PDE (1.2) Since $B^2 - 4AC = 0$ in this case the characteristi (1.35) has only root, given by

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda(x, y). \tag{1.37}$$

Hence we see that for a parabolic PDE there is only one family of real characteristic curves.

The required varibles ξ is determined by the ordinary differential equation (1.37), known as the characteristic equations of thr PDF (1.2). This is an ordinary differential equation for families of curves in the xy-plane along which $\xi = const$. To determine the second transformation variablr η, we set $b = 0$ in (1.9) so that

$$\begin{aligned}
2A\xi_x\eta_x + B(\xi_x\eta_x + \xi_y\eta_y) + 2C\xi_y\eta_y &= 0,\\
2A\frac{\xi_x}{\xi_y}\eta_x + B\left(\frac{\xi_x}{\xi_y}\eta_y + \eta_x\right) + 2C\eta_y &= 0,\\
2A\left(\frac{-B}{2A}\right)\eta_x + B\left[\left(\frac{-B}{2A}\right)\eta_y + \eta_x\right] + 2C\eta_y &= 0,\\
-B\eta_x - \frac{B^2}{2A}\eta_y + B\eta_x + 2C\eta_y &= 0,\\
\left(B^2 - 4AC\right)\eta_y &= 0.
\end{aligned}$$

Since $B^2 - 4AC$ for a parabolic PDF, η_y could be an arbitrary function of (x, y) and consequently the transformation variable η can be chosen arbitrary, as long as the change of coordinates formulas define a non-degenerate transfoemation. lf the corfficients A,B and C are constants, it is easy to integrate equation (1.37) to obtain the expressions for change of variables formulas for reducing a parabolic PDE to the canonical foem. Thus integration of (1.37) produces

$$y = \frac{B}{2A}x + c_1, \tag{1.38}$$

or

$$y - \frac{B}{2A}x = c_1, \tag{1.39}$$

sincr the families of curves $\xi =$ const are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{B}{2A}x, \tag{1.40}$$

$$\eta = x, \tag{1.41}$$

where we have set $\eta = x$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -\frac{B}{2A} & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$$

Naw, we have from (1.9)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y, \\ &= 2A\left(-\frac{B}{2A}\right) + B + 0 = 0. \end{aligned}$$

In these new coordinate variables given by (1.39) and (1.41) equation (1.8) reduces to following canonical form:

$$\omega_{\eta\eta} = \psi(\xi, \eta, \omega, \omega_\xi, \omega_{\cdot\eta}) \tag{1.42}$$

Where$\psi = \frac{\varphi}{\xi}$. As the choice of η is arbitrary the form taken by ψ will depend on the choice of η. we have from (1.9)

$$c = A\eta_x^2 + B\eta_\chi\eta_\gamma + C\eta_\gamma^2 = A. \tag{1.43}$$

Equation (1.8) may also assume the form

$$\omega_{\xi\xi} = \psi(\xi, \eta, \omega_\xi, \omega_\eta), \tag{1.44}$$

if we choose $c = 0$ instead of $a = 0$.

1.2.3 Elliptic equations

For an elliptic PDE the discriminant $\Delta = B^2 - 4AC \prec 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ξ and η to produce $b = 0$ and $a = 0$, or $b = 0$ and $a - c = 0$. Then, from (1.9) we abtain the following equation :

$$A(\xi_\chi^2 - \eta_\chi^2) + B(\xi_\chi\xi_\gamma - \eta_\chi\eta_\gamma) + C(\xi_\gamma^2 6 - \eta_\gamma^2) = 0, \tag{1.45}$$

$$2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = 0. \tag{1.46}$$

For hyperbolic and parabolic PDE_S, ξ and η are satisfied by equations that are not coupled each other (see (1.17) and (1.35)). However,equation (1.46) are coupled since both unknowns ξ and η appear in both equations. In an attempt to separate them, we add the first of these equation to complex number i times the second to give

$$A\left(\xi_x + i\eta_x^2\right) + B\left(\xi_x + i\eta_y\right)\left(\xi_x + i\eta_y\right) + C\left(\xi_y + i\eta_y\right)^2 = 0.$$

Dividing the above equation throughout by $\left(\xi_y + i\eta_y\right)^2$to obtain

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi^x + i\eta_y}{\xi^x + i\eta_y}\right) + C = 0. \tag{1.47}$$

This equation can be solved for two possible values of the ratio

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{-B \pm \sqrt{B - 4AC}}{2A} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{1.48}$$

Clearly, these twe roots are complex conjugates and are given by

$$\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}, \tag{1.49}$$

$$\frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A}. \tag{1.50}$$

Where $\beta\left(x, y\right)$ is the complex conjugate of $\alpha\left(x, y\right)$. They are given by:

$$\alpha\left(x, y\right) = \xi\left(x, y\right) - i\eta\left(x, y\right), \tag{1.51}$$

$$\beta\left(x, y\right) = \xi\left(x, y\right) - i\eta\left(x, y\right). \tag{1.52}$$

We will now proceed in a purely formal fashion. As the total derivative of α along the coordinate line $\alpha(x, y)$ =constant, $d\alpha = 0$, it follows that

$$d\alpha = \alpha_x dx + \alpha_y dy = 0,$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = \frac{\alpha_x}{\alpha_y},$$

we also have a similar result along coordinate line $\beta(x, y)$ =constant,i.e,

$$\frac{dy}{dx} = -\frac{\beta_x}{\beta_y}.$$

From the forgoing discussion it follows that :

$$\frac{dy}{dx} = \lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A}, \tag{1.53}$$

$$\frac{dy}{dx} = \lambda_2 = \frac{B + i\sqrt{4AC - B^2}}{2A}. \tag{1.54}$$

Equations (1.53) and (1.54) are called the characteristic equation of the PDE (1.3). Clearly, the solution of this differential eqution are necessarily complex-valued and as consequence there are no real characteristic axist for an elliptic EDP.The complex variables α and β are determined by the respective solution of the two ordinary differential equtions (1.53) and (1.54). Integration of equation (1.53) leads to the family of curvilinear coordinates $\alpha(x, y) = c_1$, where the integration of (1.54) gives another family of curvilinear coordinates $\beta(x, y) = c_1$ where $c_1$and c_2 are complex constants of integration. Since α and β are complex function the characteristic curves of the elliptic equation (1.3) are not real. Now the real and imaginary parts of α and β give the required trans formation variables ξ and η we have

$$\xi = \frac{\alpha + \beta}{2}, \qquad \eta = \frac{\alpha - \beta}{2}. \tag{1.55}$$

With the choice of coordinate variables (1.55), equation (1.8) reduces to following canonical form.

$$\omega_{\xi\xi} + \omega_{\eta\eta} = \psi(\xi, \eta, \omega, \omega_\xi, \omega_\eta), \tag{1.56}$$

where $\psi = \frac{\phi}{\alpha}$.

NOTE : It may be noted that the quasilinear second-order equation in two independent variables can also be classified in a similar way according to rule analogous to those developed above for semilinear equations. However, since A, B, and C are now functions of u_x, u_y and u its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.

1.2. Canonical forms

CHAPTER 2

DEFINITIONS AND THEOREMS IMPORTANTS

Definition 2.1 We denote by $L^P([a,b];X)$, the space of L^P functions from $[a,b]$ into X.It is a Banach space for the norm

$$\|u\|_{L^P([a,b];X)} = \left(\int_a^b \|u(t)\|_X^p \ dt \right)^{\frac{1}{p}}.$$

Definition 2.2 $L^\infty([a,b];X)$ is the space of measurable functions from $[a,b]$ into X being essentially bounded. It is a banach spacr of the norm

$$\|u\|_{L^\infty([a,b];X)} = \sup_{t\in[a,b]} \|u(t)\|_X.$$

(I) We recall some notion : Let B be a Banach space. Usually we will encounter the following three different concepts of convergence .

(i) Strong convergence : let $u_n \in B, u \in B$ such that as $n \to +\infty$,

$$\|u_n - u\|_B \to 0. \tag{2.1}$$

Then u_n is said to strongly converge to u.

(ii)] Weak convergence: let $u_n, u \in B$ such that for any $f \in B'$ as $n \to +\infty$

$$f(u_n) \to f(u). \tag{2.2}$$

Then u_n is said to weakly converge to u.

(iii) Weakly star convergence: let $u_n, u \in B$,and let B be the dual space of another banach space $B^*, i.e, B = (B^*)'$. If for any $f \in B^*$, as $n \to +\infty$

$$u_n(f) \to u(f). \tag{2.3}$$

Then u_n is said to weakly star converge to u.

Remark 2.1 It is well known that strong convergence implies weak convergence, and weak convergence implies weakly star convergence. When B is a reflixive banach space, weak convergence is equivalent to weakly star convergence.

Lemma 2.1 *Let S be a number with $2 \leq S \leq M \prec +\infty$ then there is a constant C depending on Ω and S such that :*

$$\left\| u \right\|_s \leq C \left\| \nabla u \right\|_m , \qquad\qquad u \in H_0^1 \left(\Omega \right) .$$

Lemma 2.2 *Any bounded set in reflexive banach space is weakly compact i.e any sequence in a bounded set has a weakly converging subsequence.*

Example 2.1 Since for $1 < p < \infty, L^p(\Omega)$ is a reflixive Banach space, any bounded set in L^p is weakly compact.

Lemma 2.3 *Let B be a Banach space such that $B = (B^*)'$ where B^* is reflixive Banach space. Then any bounded set in B is weakly star compact, i.e., any sequence in a bounded set in B has a weakly star converging subsequence.*

Example 2.2 Since for $1 \prec P \prec \infty, L^p \left(\Omega \right)$ is a reflixive banach space and
$L^\infty \left(\Omega \right) = \left(L^1 \left(\Omega \right) \right)$, an Y bounded set in $L^p \left(\Omega \right)$ with $1 \prec P \leq \infty$ is weakly star compact. In particular, any bounded set in $L^\infty \left(\Omega \right)$ is weakly star compact.

Lemma 2.4 *Let B_0, B, B_1 be three banach space. Suppose that B_0 is continuously imbedded into B, wich is also continuously imbedded into B_1, and imbedding from B_0 into B is compat. Then for any $\eta \succ 0$, there is a positive constant C_η depending only on η such that for any $v \in B_0$, the following holds.*

$$\left\| v \right\|_B \leq \eta \left\| v \right\|_{B_0} + C_\eta \left\| v \right\|_{B_1} .$$

Theorem 2.1 *(**Local existence**) Suppose that $u_0 \in H_0^1 \left(\Omega \right), u_1 \in L^2 \left(\Omega \right)$, then there exists $T \succ 0$ such that problem (2.1), (2.2) has unique solution u stisfiying:*

$$\begin{cases} u \in L^\infty \left([0, T], H_1^0 \left(\Omega \right) \right) \cap L^2 \left([0, T], L^2 \left(\Omega \right) \right) \\ u' \in L^\infty \left([0, T], L^2 \left(\Omega \right) \right) . \end{cases}$$

Theorem 2.2 *Let B_0, B, B_1 be three banach space B_0, B_1 are reflexive. Suppose that $B_0$is continuously imbedded into B , which is also continuously imbedded into B_1, and imbedding from B_0 into B is compact .For any given P_0, P_1 with $1 \prec p_0, p_1 \prec \infty$, let*

$$W = \left\{ v/v \in L^{P_0} \left([O, T], B_0 \right), v_t \in L^{p_1} [0, T], B_1 \right\}.$$

Then the imbedding from W into $L^{P_0} \left([0, T], B \right)$ is compact.

Remark 2.2 It can be seen from the proof that if the assumption of reflexivity of $B_0, B_1$is replaced by the assumption that $B_0, B_1$are the dual space of reflexive banach spaces B_0^*, B_1^*,respectively, then the conclusion of Theorem 2.2 still holds.

Lemma 2.5 *Suppose that Ω is a bounded domain in $\mathbb{R}^n$. Let $u_n(x), u(x)$ be real function in $L^P(\Omega), (1 \leq p \prec \infty)$ such that u_n strongly converges to u in $L^p(\Omega)\,(1 \leq p < \infty)$. such that u_n almost everywhere converges yo u. Thet u also belongs $L^p(\Omega)$ and u_n weakly converges in $L^p(\Omega)$ to u.*

Lemma 2.6 *Suppose that Ω is a bounded domain in $\mathbb{R}^n$. Let $u_n(x)$ be bounded sequence in $L^p(\Omega)$, and u_n weakly converges in $L^p(\Omega)$ to u .*

Remark 2.3 When $p = \infty$, then the conclusion becomes that u_n weakly star converges to u.

Lemma 2.7 *Let B be a banach space, and $B = (B^*)'$ with B^* being another banach space. Suppose that for $1 \prec p \leq \infty$,*

$$\begin{cases} u_n \to u & \text{weakly star in } L^p\left([0, T], B\right), \\ u_n' \to u' & \text{weakly star in } L^p\left([0, T], B\right). \end{cases}$$

Then

$$u_n(0) \to u(0) \qquad \text{weakly star in } B.$$

Theorem 2.3 ***(Hölder's inequality)*** *Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $f, g \in L^1$*

$$\int |fg| \leq \|f\|_p \|g\|_{p'}.$$

Definition 2.3 The Sobolev space $W^{1,p}(I)$ is defined to be , and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$

$$W^{1,p}(I) = \left\{ u \in L^p(I) ; \exists g \in L^p(I) \text{ such that } \int u\varphi' = -\int g\varphi \ \forall \varphi \in C_c^1(I) \right\}.$$

We set

$$H^1(I) = W^{1,2}(I).$$

For $u \in W^{1,p}(I)$ we denote $u' = g$.

Theorem 2.4 *L^p is a vector space and $\|\|_p$ is a norm for any p,$1 \leq p \leq \infty$.*

Example 2.3 $L^2(\Omega)$ equipped with the scalar product?

$$(u, v) = \int_\Omega u(x)\, v(x)\, d\mu,$$

is a Hilbert space. In particular, L^2 is a Hilbert space.

Definition 2.4 *A* hilbert space is a vector space H equipped with a scalar product such that H is complete for the norm$\|\|$.

CHAPTER 3

BLOW UP OF SOLUTIONS OF WAVE EQUATION

In the section we will state and prove our result concerning the following nonlinear wave equation

$$u_{tt} - \Delta u + a u_t \left| u_t \right|^{m-2} = b u \left| u \right|^{p-2}, \quad x \in \Omega \ , \quad t \succ 0,$$

$$u(x,t) = 0, \qquad x \in \partial\Omega \ , \qquad t \geq 0, \tag{3.1}$$

$$u(x,0) = u_0(x) \, , \qquad u_t(x,0) = u_1(0) \, , \quad x \in \Omega.$$

Theorem 3.1 *Suppose that* $m \succ 2, P \succ 2, and$

$$p \leq 2\frac{n-1}{n-1}, \qquad n \geq 3. \tag{3.2}$$

Assume further that

$$(u_{0,} u_1) \in H_0^1(\Omega) \times L^2(\Omega) \, . \tag{3.3}$$

Then the problem (3.1) *has a unique local solition*

$$u \in C\left([0,T) \, ; H_0^1(\Omega)\right), \quad u_t \in C\left([0,T) \, ; L^2(\Omega)\right) \cap L^m(\Omega \times (0,T)) \, , \tag{3.4}$$

T *is small.*

Remark 3.1 The condition on p, in (3.2) is needed to establish the local existence result(see [22]). In fact under this condition, the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$.

We introduce the energy functional related of equation (3.4), and we used the energy method $E(0) \prec 0$. Where

$$E(t) := \frac{1}{2}\int_{\Omega}[u_t^2 + |\nabla u|^2](x,t)dx - \frac{b}{p}\int_{\Omega}|u(x,t)|^p\, dx. \tag{3.5}$$

Lemma 3.1 *suppose that* (3.2) *holds. Then there exists a positive constant* $C \succ 1$ *depending on* Ω *only such that*

$$\|u\|_p^s \leq C\left(\|u\|_2^2 + \|u\|_p^p\right), \tag{3.6}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq S \leq P$.

Proof. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2$ by Sobolev embedding theorems. If$\|u\|_p \succ 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (3.6) follows.
We set

$$H(t) := -E(t),$$

and ues, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (3.5), (3.6) we have

Corollary 3.1 *Let the assumption of the lemma hold. Then we have*

$$\|u\|_p^s \leq C\left(|H(t)| + \|u_t\|_2^2 + \|u\|_p^p\right), \tag{3.7}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq S \leq P$.

Theorem 3.2 *Let the conditions of the theorem 3.1 be fulfilled. Assume further that* $p \succ m$ *and*

$$E(0) \prec 0. \tag{3.8}$$

Then the solution (3.4) *blows up in finite time.*

■

Remark 3.2 Note that contrary to [22], no condition on the size of the initial data has been done. The blow up takes place for any initial data satisfing (3.8).

Proof. We multiply Equation (3.1) by u_t and integrate over Ω to get

$$E'(t) = -a\int_{\Omega}|u_t(x,t)|^m\, dx, \tag{3.9}$$

for almost evry t in $[0,T)$ since $E'(t)$ is absolntely continuous (see [22]) hance $H'(t) \geq 0$. So we have

$$0 \prec H(0) \leq H(t) \leq \frac{b}{p} \|u\|_p^p, \tag{3.10}$$

for every t in $[0,T)$, by virtue of (3.9). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)\,dx, \tag{3.11}$$

for ε small to be chosen later and

$$0 \prec \alpha \leq \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{3.12}$$

By taking a derivative of (3.11) and using equation (3.1). We obtain

$$\begin{aligned} L'(t) \quad &: \; = (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx \\ &+\varepsilon \int_\Omega |u(x,t)|^p\,dx - a\varepsilon \int_\Omega |u_t|^{m-2} u_t u(x,t)\,dx. \end{aligned} \tag{3.13}$$

We then exploit Young's inequality $XY \leq \frac{\delta^R}{R} X^r + \frac{\delta^q}{q} Y^q, X, Y \geq 0$, for all $\delta \succ 0, \frac{1}{r} + \frac{1}{q} = 1$. With $r = m$ and $q = \frac{m}{(m-1)}$ to estimate the last term in (3.13) as follows

$$\int_\Omega |u_t|^{m-1} |u|\,dx \leq \frac{\delta^m}{m} \|u\|_m^m + \frac{m-1}{m} \delta^{\frac{-m}{m-1}} \|u\|_m^m,$$

which yields, by substitution in (3.13)

$$\begin{aligned} L'(t) \geq &\left[(1-\alpha) H^{-\alpha}(t) - \frac{m-1}{m} \varepsilon \delta^{-\frac{m}{(m-1)}}\right] H'(t) \\ &+ \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx + \varepsilon \left[pH(t) + \frac{p}{2} \int_\Omega \left[u_t^2 + |\nabla u|^2\right](x,t)\,dx\right] \\ &- \varepsilon a \frac{\delta^m}{m} \|u\|_m^m, \quad \text{for all } \delta \succ 0. \end{aligned} \tag{3.14}$$

Of course (3.14) remain valid even if δ is time dependant sincethe integralis taken over that x variable. Therefor by taking δ so that $\delta^{\frac{-m}{(m-1)}} = kH^{-\alpha}(t)$, for large k ■
to be specified later, and subtituting in (3.14). We arrive at

$$\begin{aligned} L'(t) \quad &\geq \quad \left[(1-\alpha) - \frac{m-1}{m} \varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p}{2} + 1\right) \int_\Omega u_t^2(x,t)\,dx \\ &+\varepsilon \left(\frac{p}{2} - 1\right) \int_\Omega |\nabla u|^2(x,t)\,dx + \varepsilon \left[pH(t) - \frac{k^{1-m}}{m} a H^{\alpha(m-1)}(t) \|u\|_m^m\right]. \end{aligned} \tag{3.15}$$

By exploiting (3.10) and the inequality $\|u\|_m^m \leq C \|u\|_p^m$. We obtain

$$H^{\alpha(m-1)}(t) \|u\|_m^m \leq \left(\frac{b}{p}\right)^{\alpha(m-1)} C \|u\|_p^{m+\alpha p(m-1)},$$

hence (3.15) yields

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx \quad (3.16)$$
$$+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)\,dx$$
$$+\varepsilon\left[pH(t) - \frac{k^{1-m}}{m} a\left(\frac{b}{p}\right)^{\alpha(m-1)} C\,\|u\|_p^{m+\alpha p(m-1)}\right].$$

Remark 3.3 We then use Corolary 3.1 and (3.12), for $s = m + \alpha p(m-1) \leq P$, to deduce from (3.16)

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx \quad (3.17)$$
$$+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)\,dx$$
$$+\varepsilon\left[pH(t) - C_1 k^{1-m}\left\{H(t) + \|u_t\|_2^2 + \|u\|_p^p\right\}\right],$$

where $C_1 = a\left(\frac{b}{p}\right)^{\alpha(m-1)} C\diagup m$. By noting that

$$H(t) = \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2,$$

and writing $p = (p+2)\diagup 2 + (p-2)\diagup 2$, (3.14) yields

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \frac{p-2}{4}\|\nabla u\|_2^2 \quad (3.18)$$
$$+\varepsilon\left[\begin{array}{c}\left(\frac{p+2}{2}C_1 k^{1-m}\right) H(t) + \left(\frac{p-2}{2p}b - C_1 k^{1-m}\right)\|u\|_p^p \\ +\left(\frac{p+6}{4} - C_1 k^{1-m}\right)\|u_t\|_2^2\end{array}\right].$$

At this point, we choose k large enough so that the coefficients of $H(t)$, $\|u_t\|_2^2$, and $\|u\|_p^p$, in (3.18) are strictly positive, hernce we get

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right], \quad (3.19)$$

when $\gamma \succ 0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha) - \varepsilon k(m-1)\diagup m \geq 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0 u_1(x)\,dx \succ 0.$$

Therefore (3.19) takes the from

$$L'(t) \geq \gamma\varepsilon \left[H(t) + \|u\|_2^2 + \|u\|_p^p \right]. \tag{3.20}$$

Consequently we have

$$L'(t) \geq L(0) \succ 0, \text{ for all } \ t \geq 0.$$

Next we would like to show that

$$L'(t) \geq \Gamma L^{1/(1-\alpha)}(t), \quad \text{for all} \quad t \geq 0, \tag{3.21}$$

where Γ is a positive constant depending on $\varepsilon\gamma$ and C (the constant of Lemma 3.1) Once (3.21) is established, we obtain in a standard way the finite time blow up of $L(t)$, hence of u (see [21] for instance).

To prove (3.21), we first estime

$$\left| \int_\Omega u u_t(x,t)\, dx \right| \leq \|u\|_2 \|u_t\| \leq C \|u\|_p \|u_t\|_2,$$

which implies

$$\left| \int_\Omega u u_t(x,t)\, dx \right|^{1/(1-\alpha)} \leq C \|u\|_p^{1/(1-\alpha)} \|u_t\|_2^{1/(1-\alpha)}.$$

Again Young' s inequality gives

$$\left| \int_\Omega u u_t(x,t)\, dx \right|^{1/(1-\alpha)} \leq C \left[\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)} \right], \tag{3.22}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, *to get* $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by [12].

Therefore (3.22) becomes

$$\left| \int_\Omega u u_t(x,t)\, dx \right|^{1/(1-\alpha)} \leq C \left[\|u\|_p^s + \|u_t\|_2^2 \right],$$

where $s = 2/(1-2\alpha) \leq p$. By using Corollary 3.1 we obtain

$$\left| \int_\Omega u u_t(x,t)\, dx \right|^{1/(1-\alpha)} \leq C \left[H(t) + \|u\|_p^p + \|u_t\|_2^2 \right], \quad \text{for all} \qquad t \geq 0 \quad . \tag{3.23}$$

Finally by noting that

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)\, dt \right)^{1/(1-\alpha)} \\ &\leq 2^{1/(1-\alpha)} \left(H(t) + \left| \int_\Omega u u_t(x,t)\, dx \right|^{1/(1-\alpha)} \right), \end{aligned}$$

and combining it with (3.20) and (3.23), the inequality (3.17) is established. This completes the proof.

CONCLUSION

In this work, we notice that the last result related to the behavior of the solution resulted from the fact that the primary energy of the equation is negative. The question is what happens if the primary energy is positive and this requires future research.

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