Algerian Democratic and Popular Republic

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

20 August 1955 University of Skikda

Faculty of Sciences

Department of Mathematics

Ref:………………

جامعة 20 أوت 1955 –سكيكدة

كلية العــلوم

قسم الرياضيات

المرجع:………..

Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: ***Applied Functional Analysis***

Study of a nonlinear fractional PDEs system

Presented by:

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Publicly discussed:

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University year : 2024/2025

Dedication

I dedicate this modest work to my dearest mother and father. I thank my parents Ahcen and Akila whose work could not have been completed without their tireless support and encouragement.
To my perfect wife for unwavering support and to my beautiful daughter Noursine.
To my family, professors, and friends.
With all my heart, thank you all.

Acknowledgement

First and foremost, i would like to thank my Allah who gave me the strength and courage to do this work.
I am deeply grateful to my supervisor professor **Hakim Lakhal** for giving me the opportunity to work with him during all years of my thesis, he was always available for guidance. I would also like to thank my co-supervisor, Dr. **Kamel Slimani**, for his considerable time, his scientific expertise, and kindness. Their vast experiences encouraged me all the time.
A special thanks goes to the members of my PhD thesis committee for their valuable feedback and suggestions.
I would also like also to express my deepest gratitude to **my parents**, **my wife**, and the rest of **my family** for their encouragement and support throughout my research pursuit.
Finally, I would like to express my sincere gratitude and respect to all **my professors** and **teachers** who gave me priceless lessons. My gratitude to them known by no bounds.

Abstract

Over the past two decades, nonlinear systems of elliptic type involving fractional operators have been extensively studied by numerous researchers in various contexts because they can serve as models for several physical phenomena, and many results on the solvability of these systems have been established. The most interesting aspect has been proving the existence and multiplicity of nontrivial solutions in appropriate fractional Sobolev spaces, using variational methods and critical point theorems.

In this regard, the main focus of this thesis is to investigate the concept of existence and multiplicity of nontrivial solutions for classes of nonlinear fractional Schrödinger-Poisson systems and fractional Kirchhoff-Schrödinger-Poisson systems driven by two kinds of fractional operators in appropriate fractional frameworks. In other words, we analyze different classes of these fractional systems under various types of assumptions imposed on the potentials and nonlinearities. To achieve these results, the main techniques employed for the proofs are variational methods based on the mountain pass theorem, the symmetric mountain pass theorem and the fountain theorem.

Key-words: Fractional Kirchhoff-Schrödinger-Poisson systems; Fractional Schrödinger-Poisson systems; Fractional Sobolev spaces; Calculus of variations; Variational methods; Symmetric mountain pass theorem; Fountain theorem; Mountain pass theorem; Multiplicity of solutions; Nontrivial solution; Concave-convex nonlineairities; Superlinear terms; Palais-Smale condition; Cerami condition.

Résumé

Au cours des deux dernières décennies, les systèmes non linéaires de type elliptique impliquant des opérateurs fractionnaires ont été largement étudiés par de nombreux chercheurs dans divers contextes, en raison de leur capacité à modéliser plusieurs phénomènes physiques. De nombreux résultats sur la résolubilité de ces systèmes ont été établis. L'un des aspects les plus intéressant de ces études est la démonstration de l'existence et la multiplicité de solutions non triviales dans des espaces de Sobolev fractionnaires appropriés, en utilisant des méthodes variationnelles et des théorèmes de points critiques.

À cet égard, l'objectif principal de cette thèse est d'étude du concept de l'existence et de la multiplicité de solutions non triviales pour des classes de systèmes de Schrödinger-Poisson non linéaires fractionnaires et de systèmes de Kirchhoff-Schrödinger-Poisson non linéaires fractionnaires, régis par deux types d'opérateurs fractionnaires dans des cadres fractionnaires adaptés. Plus précisément, nous analysons différentes classes de ces systèmes non linéaires fractionnaires sous différentes types d'hypothèses imposées sur les fonctions de potentiels et de non-linéarités. Pour établir ces résultats, nous employons principalement pour les preuves des méthodes variationnelles, notamment le théorème du col de montagne, le théorème du col de montagne symétrique et le théorème de la fontaine.

Mots clés: Systèmes fractionnaires de Kirchhoff-Schrödinger-Poisson ; Systèmes fractionnaires de Schrödinger-Poisson; Espaces de Sobolev fractionnaires ; Calcul des variations; Méthodes variationnelles; Théorème du col de montagne symétrique; Théorème du col de montagne; Théorème de la fontaine; Multiplicité des solutions; Solution non triviale; Non-linéairités concaves-convexes ; Termes superlinéaires; Condition de Palais-Smale; Condition de Cerami.

الملخص

خلال العقدين الماضيين، تمت دراسة الأنظمة الكسرية غير الخطية من النوع الإهليلجي بشكل مكثف من قبل العديد من الباحثين في سياقات مختلفة، و ذلك لقدرتها على نمذجة العديد من الظواهر الفيزيائية. وقد أسفرت هذه الدراسات نتائج هامة حول قابلية حل هذه الأنظمة. أحد الجوانب الأكثر إثارة للاهتمام يكمن في إثبات وجود وتعدد الحلول غير التافهة في فضاءات سوبولاف الكسرية المناسبة باستخدام مناهج التباين ونظريات النقاط الحرجة.

في هذا الإطار، يتمثل الهدف الرئيسي لهذه الأطروحة في دراسة مفهومي الوجود وتعدد الحلول غير التافهة لفئات من أنظمة شرودينغر بواسون الكسرية غير الخطية وأنظمة كيرشوف شرودينغر بواسون الكسرية غير الخطية التي يقودها نوعان من المؤثرات الكسرية ضمن أطر كسرية مناسبة. بعبارة أخرى، نقوم بتحليل فئات مختلفة من هذه الأنظمة الكسرية غير الخطية تحت فرضيات متنوعة تتعلق بدوال الكمون و الدوال غير خطية. لتحقيق هذه النتائج ، نستخدم بشكل أساسي مناهج التباين المرتكزة على نظرية ممر الجبل، نظرية ممر الجبل المتماثل و نظرية النافورة.

الكلمات المفتاحية: أنظمة شرودينجر بواسون الكسرية؛ أنظمة كيرشوف شرودينجربواسون الكسرية؛ فضاءات سوبولاف الكسرية؛ الحساب التبايني؛ مناهج التباين؛ نظرية ممر الجبل؛ نظرية النافورة؛ نظرية ممر الجبل المتماثل؛ تعدد الحلول؛ حل غير تافه؛ لاخطية مقعرة محدبة؛ حدود فوق خطية؛ شرط بالي سمول؛ شرط سيرامي .

List of Symbols

Notations

$\forall$: For all.
$\exists$: There exists.
$\equiv$: Equivalent.
$\sum$: Summation.
$a.e.$: Almost everywhere.
$\to$: Strong convergence.
$\rightharpoonup$: Weak convergence.
$o_n(1)$: Denotes $o_n(1) \to 0$ as $n \to \infty$.
$\langle \cdot, \cdot \rangle$: Scalar product of $\mathbb{R}^d$, duality between X and X'.
supp(f): Support of the function f.
$P.V$: Denotes the principal value.
meas(Ω): Measure of the set Ω.

Constants and special functions

C and C_i ($i \in \mathbb{N}$): Generic constants whose exact value is inessential.
Γ: Gamma function.
$C_{d,\sigma}$: Constant of the distributional Riesz fractional gradient.
$c_{d,\sigma}$: Normalisation constant.
$\zeta_{d,\sigma}$: Constant of the Riesz potential.
p^*_σ: Fractional critical Sobolev exponent.

Kernels and operators

∇: The classical gradient.
$-\Delta$: Laplacian operator.
$(-\Delta)^\sigma$: Fractional Laplace operator.
D^σ: Distributional Riesz fractional gradient.
div^σ: Distributional Riesz fractional divergence.
$-div^\sigma D^\sigma$: Distributional Riesz fractional derivative.
I_σ: Riesz kernel of order σ.
G_σ: Bessel kernel with paramater σ.
$[u]_{\sigma,p}$: Gagliardo seminorm for $W^{\sigma,p}(\mathbb{R}^d)$.

Sets and spaces

$\mathbb{N}$: Set of natural numbers.
$\mathbb{R}$: Set of real numbers.
$\mathbb{R}^+$: Set of positive real numbers.
$\mathbb{R}^d$: Euclidean space of d-dimensional vectors.
X: Arbitrary Banach space.
X^*: Dual space of the Banach space X.
Ω: Open bounded subset of $\mathbb{R}^d$.
$\partial\Omega$: Boundary of Ω.
$B_\varepsilon(x)$: The open ball of $\mathbb{R}^d$ centered at x with radius $\varepsilon > 0$.
$C_0^\infty(\mathbb{R}^d)$: Infinitely differentiable functions with compact support on $\mathbb{R}^d$.
$L^p(\mathbb{R}^d)$: Lebesgue spaces.

$L^{\sigma,p}(\mathbb{R}^d)$: Bessel potential space.
$W^{\sigma,p}(\mathbb{R}^d)$: Fractional Sobolev spaces.
$H^{\sigma}(\mathbb{R}^d)$: Fractional Sobolev space.
$D^{\sigma}(\mathbb{R}^d)$: Fractional homogeneous Sobolev space.
$S^{\sigma,p}(\mathbb{R}^d)$: Spaces of fractionally differentiable functions.
$\mathcal{S}(\mathbb{R}^d)$: Schwartz space.
$S_0^{\sigma,p}(\mathbb{R}^d)$: Space of functions $u \in L^p(\mathbb{R}^d)$ with $D^\sigma u \in L^p(\mathbb{R}^d;\mathbb{R}^d)$.

Contents

General introduction

Context and motivations

The theory of fractional derivatives involving real (complex) order derivative is a generalization of the integer order of differential calculus. This notion originated in 1695 when Leibniz introduced the notation $\frac{d^n x}{d^n y}$. L'Hôpital wrote to Leibniz asking: "What would the result be if $n = \frac{1}{2}$". Leibniz replied: "It will lead to a paradox, from which one day, useful consequences will be drawn". In these words, the concept of the theory of fractional derivatives grew, and this idea motivated several well-known mathematicians including Lacroix, Fourier, Liouville, Riemann, Caputo, and others to develop this notion.

Since its inception, great efforts have been devoted by many mathematicians and physicists to the study of this theory which is motivated both from a purely mathematical viewpoint and by the description and analysis of many nonlinear complex phenomena appearing in several fields of applied sciences, such as physics involving diffusion phenomena and phase transition [27], electromagnetism [85], finance [99], image processing [57], quantum mechanics [72], Lévy processes [17], population dynamics [62], game theory [28] and optimal control [53]. More precisely, fractional derivative operators capture the memory of all past physical processes and complex events. For instance, fractional models of dynamical systems retain the memory of their earlier states [9]. This improved accuracy is important for scientific and engineering applications that need exact modeling. This considerable attention has led many researchers to significant advancements in the theory of fractional derivatives, as well as to develop various definitions of fractional derivative operators that describe nonlinear models of real-world problems. Among them, we can list the Riemann-Liouville fractional derivative, Hilfer fractional derivative, Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Hadamard fractional derivative, and others. Particularly, the prototypical fractional derivatives operators, namely the fractional

Laplacian denoted by $(-\Delta)^{\sigma}$ and defined for a smooth function φ by

$$(-\Delta)^{\sigma}\varphi(x) = C_{d,\sigma} P.V. \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+\sigma p}} dy, \tag{1}$$

where $P.V.$ is the Cauchy principal value and $C_{d,\sigma}$ is a normalization constant. This nonlocal operator can be seen as a fractional generalization of the ordinary Laplacian operator and the fundamental difference between the two operators above is that the classical Laplacian describes the local behavior of the function, whereas the fractional Laplacian considers nonlocal interactions between different parts of the function. The fractional Laplacian operators have piqued researchers interest due to their intriguing analytical structure (symmetry, positivity and continuity) and because of several applications in different subjects including finance, physics, fluid dynamics, image processing, various fields including stochastic processes of Lévy type, anomalous diffusion, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, geophysical fluid dynamics, drifts and game theory, phase transitions, population dynamics, optimal control, and game theory, for more details and applications, we refer interested readers to [17, 19, 27, 28, 44, 45, 57, 78] and the references therein. Notably, in the context of fractional quantum mechanics, a nonlinear fractional Schrödinger equation was obtained [72] as a result of expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths.

After the construction of these different kinds of fractional derivatives, a natural question arises: what the most adequate framework to study (nonlinear) problems related to the partial differential equations with a fractional operator (Laplacian)?. The answer is the well-known fractional Sobolev spaces $W^{\sigma,p}(\mathbb{R}^d)$ $\sigma \in (0,1)$. These functional spaces play a central role in harmonic analysis, partial differential equations and calculus of variations. Furthermore, fractional Sobolev spaces are the cornerstone and provide an important functional framework for studying boundary value problems related to fractional partial differential equations. For a nice overview of fractional Sobolev spaces with applications to the partial differential equations driven by fractional Laplacian operator, we refer to [19, 45] and their references.

Partial differential equations have experienced a significant development in various directions, especially in modeling many phenomena in several fields of science. They have a strong dependence on the ordinary gradient operator (derivative) which gives them their local nature, usually imposes some regularity and prevents them from capturing some nonlocal information. In this context, locality means that the behavior of an object depends only on its immediate neighborhood. However, nonlocal models and corresponding nonlocal operators are gathering increasing attention more recently and being considered as alternative models to local ones, because they require less smoothness and they can capture discontinuity effects that the previous ones cannot describe and also long-range interactions are included and lead to refined predictions. These effects make nonlocal models suitable for several practical applications including nonlocal diffusion [27], image processing [57], phase transitions [35], continuum mechanics [49], and machine learning [87]. See also [46] for an introduction to nonlocal modeling. As a direct consequence, this huge interest has been accompanied by significant mathematical progress of the new operators involved. Especially, the so-called distributional Riesz fractional gradient, which for sufficiently regular function φ is defined as

$$D^\sigma \varphi(x) = C_{d,\sigma} \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+\sigma}} \frac{x-y}{|x-y|} dy, \quad \text{for } x \in \mathbb{R}^d, \tag{2}$$

where $C_{d,\sigma}$ is a suitable constant. It was Shieh and Spector in a series of two interesting papers [91, 92] who brought it to the attention of the partial differential equations community. They developed a general comprehensive theory for partial differential equations based on distributional Riesz fractional gradient as well as discussed properties of the associated fractional Sobolev spaces which are equivalent to the well-known Bessel potential spaces $L^{\sigma,p}(\mathbb{R}^d)$ in some sense, this naturally leads the authors to pose new types of fractional problems. This fractional operator has an intriguing structure and properties analogous to the ordinary gradient. Moreover, it is the only fractional operator (up to a constant) that satisfies translational and rotational invariance, homogeneity under dilations, and a weak requirement of continuity as recently shown by the author in [93]. These three basic natural requirements show that the above definition in (2) is well posed not only from a mathematical perspective but also from a physical perspective and establishes the distributional Riesz fractional gradient in some sense as the canonical choice of fractional derivative among several operators with infinite interaction range. Furthermore, this operator and therefore the associated fractional problems have a wide range of applications in analysis [41, 91], quasiconvexity [70] and some physical background and significance, especially in the theory of electromagnetic fields [8], fractal media [85], nonlocal elasticity models [14], and multidimensional processes [54].

It is worth mentioning that most of the work on the theory of fractional derivatives is devoted to the solvability (existence and multiplicity of nontrivial solutions) of nonlinear boundary value problems generated by fractional partial differential equations. The variational methods based on the critical point theory are one of the most powerful analytic techniques for studying these problems that can be understood and solved in terms of the minimization of a functional usually related to the energy, in a suitable functional space, and most cases, comparable results have not been achieved with others methods. Moreover, it was observed that the solutions to a great number of problems are in effect critical points of functionals. In other words, variational methods based on critical point theory are indispensable as a tool dedicated to problems related to finding the minimum or the maximum values of a special mathematical object called functional. These functionals can represent, for example, a surface area, an action, path length, and energy or cost. Mathematically, this object is a mapping from the space of functions to the real (or complex) numbers. For more details we refer the reader to consult [3, 19, 61, 90, 105] and the references therein.

In the following, we outline the thesis organization, which consists of six chapters defining the contributed work. Below, we briefly summarize the content of each chapter.

Guide to the thesis

Chapter 1: This chapter introduces the topic of this thesis and we present some notations, theorems, and technical results that will be needed for this study. The first section provides the definitions of fractional Sobolev spaces and the fractional Laplacian operator. In the second section, some auxiliary results are presented that will be used throughout the thesis. We finish the chapter in the last section by giving an agile exposition of the calculus of variations and we give some variational methods and critical point theorems that are used throughout the thesis.

Chapter 2: This chapter concerns a class of Kirchhoff-Schrödinger-Poisson systems with superlinear terms involving fractional Laplacian operator. The existence of infinitely many nontrivial solutions to the given problem has been obtained using variational methods combined with the fountain theorem. More specifically, here we deal with the following fractional Kirchhoff-Schrödinger-Poisson system

$$\begin{cases} M([u]_\sigma^2)(-\Delta)^\sigma u + V(x)u + K(x)\phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where

$$[u]_\sigma^2 = \int_{\mathbb{R}^3} \left|(-\Delta^{\frac{\sigma}{2}})u\right|^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy,$$

$\sigma \in (\frac{3}{4}, 1)$, $\gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, λ is a real positive parameter, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous and function. The nonlinearity $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that obeys certain assumptions and $(-\Delta)^\sigma$ is the fractional Laplacian.

Chapter 3: In this chapter, we extend our study to another class of fractional Kirchhoff-Schrödinger-Poisson in the case of combined effect of concave-convex nonlinearities, and we prove the existence and multiplicity of nontrivial solutions. More precisely, we consider the fractional Kirchhoff-Schrödinger-Poisson system including concave-convex nonlinearities given by

$$\begin{cases} M([u]_\sigma^2)(-\Delta)^\sigma u + V(x)u + K(x)\phi u = f(x,u) + \lambda g(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\sigma \in (\frac{3}{4}, 1)$, $\gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, λ is a real parameter, $1 < q < 2$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$, $V : \mathbb{R}^3 \to (0, \infty)$ are continuous functions, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies some conditions and $(-\Delta)^\sigma$ denotes the fractional Laplacian. In particular, we do not use the well-known Ambrosetti-Rabinowitz condition in the convex term. Under certain different assumptions from the previous chapter, we show that at least one nontrivial solutions exists provided that

$\lambda \in (0, \Lambda_*)$ using the mountain pass theorem-Cerami version. Then, infinitely many nontrivial solutions for all $\lambda > 0$ are derived by exploiting the fountain theorem as the main tool.

Chapter 4: This chapter seeks to establish the existence of a sequence of infinitely many large energy solutions for a general class of fractional Schrödinger-Poisson systems, which has not been studied before of the following form

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + K(x)\phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\sigma, \gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, $\lambda \in \mathbb{R}_+$, and $(-\Delta)^\sigma$ stands for the fractional Laplacian and the nonlinearity $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that fulfills some suitable assumptions. To achieve this result, we utilize variational techniques in combination with the fountain theorem under the Cerami condition.

Chapter 5: In this chapter, we shift our focus to another fractional operator called distributional Riesz fractional gradient which is currently undergoing a great development. More precisely, this chapter is divided into two sections. We first establish a comprehensive general theory for partial differential equations based on the distributional Riesz fractional gradient, as well as we introduce some abstract results on the perspective of the new functional spaces (Bessel potential space) essential for dealing with this fractional operator. As an application, in section 5.2 we apply the theory developed to investigate the multiplicity property of nontrivial solutions to the following superlinear Schrödinger-Poisson system

$$\begin{cases} -div^\sigma(D^\sigma u) + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -div^\gamma(D^\gamma \phi) = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\sigma, \gamma \in (0,1)$, $4\sigma + 2\gamma > 3$, and $-div^\sigma(D^\sigma)$ denotes the distributional Riesz fractional derivative. Using variational methods, based on the symmetric mountain pass theorem under some suitable assumptions imposed on f, K and V, we get the existence of multiple nontrivial solutions.

Chapter 6: This chapter deals with the multiplicity property of nontrivial solutions in Bessel potential space to a class of fractional Schrödinger-Poisson systems involving distributional Riesz fractional derivative with combined effect of nonlinearities. The main result is obtained by applying the symmetric mountain pass theorem as the key tool. More specifically, we consider a new class of fractional Schrödinger-Poisson systems with concave-convex nonlinearities given by

$$\begin{cases} -div^{\sigma}(D^{\sigma}u) + V(x)u + \phi u = f(x,u) + \lambda g(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -div^{\gamma}(D^{\gamma}\phi) = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\sigma, \gamma \in (0;1]$ with $\sigma > \frac{3}{4}$, $\lambda > 0$ is a parameter, $q \in (1,2)$, $2\gamma + 4\sigma > 3$, and $-div^{\sigma}(D^{\sigma})$ is the distributional Riesz fractional derivative operator. Under different assumptions from the previous chapter imposed on f, g and V, we discuss the multiplicity result of nontrivial solutions in Bessel potential space under a specific parameter interval by using the symmetric mountain pass theorem as the main tool under the Palais-Smale condition.

CHAPTER 1

Preliminaries

This chapter is devoted to recalling some basic definitions and classical results of functional analysis and calculus of variations that will play a crucial role in this thesis. Specifically, we first collect the definitions of fractional Sobolev spaces and related fractional operator with some of their elementary properties. We refer the interested reader to [45] for further references and for some of the proofs of these results. Secondly, we provide an agile exposition of the calculus of variations and we cite some variational methods and critical point theorems.
First, let us recall that the space of continuous functions $C(\mathbb{R}^d)$ is defined as

$$C(\mathbb{R}^d) = \Big\{ u : \mathbb{R}^d \to \mathbb{R} \text{ with } u \text{ continuous} \Big\}.$$

$C^k(\mathbb{R}^d)$ the space of real functions k times continuously differentiable on $\mathbb{R}^d$.

$$C^k(\mathbb{R}^d) = \Big\{ u \in C(\mathbb{R}^d) : D^\alpha u \in C(\mathbb{R}^d) \text{ for all } \alpha \text{ such that } |\alpha| \leq k \Big\}.$$

Let K be a compact subset of $\mathbb{R}^d$, we introduce the following space that contains continuous functions with compact support.

$$C_c(\mathbb{R}^d) = \Big\{ u \in C(\mathbb{R}^d) \text{ such that } u(x) = 0 \text{ for all } x \in \mathbb{R}^d \setminus K \Big\}.$$

We note $C_c^\infty(\mathbb{R}^d)$ the space of infinitely differentiable (smooth) functions with compact support on $\mathbb{R}^d$. It is worth mentioning that in this thesis we use the notation $C_0^\infty(\mathbb{R}^d)$ instead of $C_c^\infty(\mathbb{R}^d)$ for the space of smooth function with compact support.

1.1 Fractional Sobolev spaces

Based on recent developments in the theory of fractional Sobolev spaces $W^{\sigma,p}(\mathbb{R}^d)$ with $\sigma \in (0,1)$, the study of nonlinear problems involving fractional derivative operators of elliptic type has been emerged in the literature in order to find nontrivial weak solutions. It has become a very active and interesting research area, both for pure mathematical standpoint and for the concrete applications. Indeed, these types of fractional Sobolev spaces $W^{\sigma,p}(\mathbb{R}^d)$ with $\sigma \in (0,1)$ and corresponding fractional operators, which are the focus of this thesis, are nowadays of central importance in the analysis of partial differential equations. The prototype example of fractional operator is the fractional Laplacian that will be defined below.

1.1.1 Lebesgue spaces $L^p(\mathbb{R}^d)$

First, we briefly recall the definitions and some elementary properties of the Lebesgue spaces. We refer the reader to [24] for further references and for some of the proofs of the results in this section. Throughout of this chapter, we have $d \geq 1$.

Definition 1.1.1. [24] For any $p \in [1,\infty)$ we define the Lebesgue spaces $L^p(\mathbb{R}^d)$ as follows :

$$L^p(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R}^d \text{ measurable} : \int_{\mathbb{R}^d} |u(x)|^p dx < \infty \right\}.$$

This is a normed space thanks to the Minkowski inequality. The norm, which is denoted by $\|\cdot\|_p$ is defined by

$$\|u\|_p = \left(\int_{\mathbb{R}^d} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we denote

$$L^\infty(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R}^d \text{ measurable} : |u(x)| \leq C \text{ a.e in } \mathbb{R}^d \text{ for some } C > 0 \right\},$$

with the norm

$$\|u\|_\infty = \inf \left\{ C > 0 \ : \ |u(x)| \leq C \ \text{ a.e. in } \ \mathbb{R}^d \right\}.$$

Particularly, if $p = 2$ the space $L^2(\mathbb{R}^d)$ is a Hilbert space.

Notation 1.1.1. *: Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent of p, that is,* $\dfrac{1}{p} + \dfrac{1}{p'} = 1.$

Theorem 1.1.1. ***(Hölder inequality)*** *[24] Let $u \in L^p(\mathbb{R}^d)$ and $v \in L^{p'}(\mathbb{R}^d)$ with $1 \leq p \leq \infty$. Then, $uv \in L^1(\mathbb{R}^d)$ and*

$$\left| \int_{\mathbb{R}^d} u(x)v(x)dx \right| \leq \|u\|_p \|v\|_{p'}.$$

Remark 1.1.1. In the space $L^2(\mathbb{R}^d)$, the Cauchy-Schwartz inequality is given as follows:

$$\left|\int_{\mathbb{R}^d} u(x)v(x)dx\right| \leq \|u\|_2\|v\|_2.$$

Theorem 1.1.2. ***(Minkowski inequality)****[24] Let $u, v \in L^p(\mathbb{R}^d)$ with $p \geq 1$. Then*

$$u + v \in L^p(\mathbb{R}^d) \ \text{ and } \ \|u+v\|_p \leq \|u\|_p + \|v\|_p.$$

Theorem 1.1.3. *[24]*

1. *$L^p(\mathbb{R}^d)$ is a Banach space for all $p \in [1, \infty]$.*
2. *If $1 \leq p < \infty$ then $L^p(\mathbb{R}^d)$ is separable space.*
3. *If $1 < p < \infty$ then $L^p(\mathbb{R}^d)$ is reflexive space.*
4. *If $1 < p < \infty$ then $L^p(\mathbb{R}^d)$ is uniformly convex space.*

Theorem 1.1.4. *[24] Let $u \in L^1(\mathbb{R}^d)$ and $v \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Thus, for a.e. $x \in \mathbb{R}^d$, the function $y \mapsto u(x-y)v(y)$ is integrable on $\mathbb{R}^d$. Set*

$$(u * v)(x) = \int_{\mathbb{R}^d} u(x-y)v(y)dy.$$

Then, $u * v \in L^p(\mathbb{R}^d)$ and

$$\|u * v\|_p \leq \|u\|_1\|v\|_p$$

1.1.2 Fractional Sobolev spaces $W^{\sigma,p}(\mathbb{R}^d)$

In this subsection, we investigated to the definition of fractional Sobolev spaces, and we recall some results of regarding continuous and compact embedding of these spaces. We first begin by fixing the fractional exponent $\sigma \in (0, 1)$.

Definition 1.1.2. ([45]) For $p \in [1, \infty)$, recalling that the fractional Sobolev spaces $W^{\sigma,p}(\mathbb{R}^d)$ is defined as follows

$$W^{\sigma,p}(\mathbb{R}^d) = \left\{ u \in L^p(\mathbb{R}^d) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{d}{p}+\sigma}} \in L^p(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

endowed with the natural norm

$$\|u\|_{\sigma,p} = \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p\sigma}} dxdy + \int_{\mathbb{R}^d} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

where the term

$$[u]_{\sigma,p}=\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\sigma}}dxdy\right)^{\frac{1}{p}},$$

is the Gagliardo seminorm of u.

Theorem 1.1.5. *([45]) For $p\in[1,\infty)$, then we have*

1. *$W^{\sigma,p}(\mathbb{R}^d)$ is a Banach space.*
2. *$W^{\sigma,p}(\mathbb{R}^d)$ is separable space.*
3. *If $1<p<\infty$ then $W^{\sigma,p}(\mathbb{R}^d)$ is reflexive space.*
4. *If $1<p<\infty$ then $W^{\sigma,p}(\mathbb{R}^d)$ is uniformly convex space.*

Theorem 1.1.6. *([45]) Let $\sigma\in(0,1)$ and $p\in(1,\infty)$ be such that $\sigma p<d$. Then, there exists a constant $C:=C(d,\sigma,p)>0$ such that*

$$\|u\|_{p^*_\sigma}\leq C\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\sigma}}dxdy\right)^{\frac{1}{p}}.$$

Here, p^*_σ is the fractional critical Sobolev exponent, that is,

$$p^*_\sigma:=\begin{cases}\frac{dp}{d-\sigma p} & \text{if } d>\sigma p\\ \infty & \text{if } d\leq\sigma p.\end{cases}$$

$W^{\sigma,p}(\mathbb{R}^d)$ is continuously embedded into $L^q\left(\mathbb{R}^d\right)$ for all $q\in[p,p^*_\sigma]$.

Theorem 1.1.7. *[45] For any $\sigma>0$, the space $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{\sigma,p}(\mathbb{R}^d)$.*

1.1.3 Fractional Sobolev space $H^\sigma(\mathbb{R}^d)$

Here, we focus our interest on the Hilbert case $p=2$. They are strictly related to the fractional Laplacian operator and simply denoted by $H^\sigma(\mathbb{R}^d)$. Thus, for $\sigma\in(0,1)$, the fractional Sobolev space $H^\sigma(\mathbb{R}^d)$ is defined as

$$H^\sigma(\mathbb{R}^d)=\left\{u\in L^2(\mathbb{R}^d):\frac{|u(x)-u(y)|}{|x-y|^{\frac{d}{2}+\sigma}}\in L^2(\mathbb{R}^d\times\mathbb{R}^d)\right\},$$

endowed with the inner product

$$\langle u,v\rangle_{H^\sigma}=\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\sigma}}dxdy+\int\limits_{\mathbb{R}^d}|u(x)v(x)|dx,$$

and the norm

$$\|u\|_{H^\sigma} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2\sigma}} dxdy + \int_{\mathbb{R}^d} |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

We write the seminorm as $[u]_{\sigma,2} = [u]_\sigma$.

Theorem 1.1.8. *For any $\sigma \in (0,1)$ and $2 \le q \le 2^*_\sigma$, then $H^\sigma(\mathbb{R}^d)$ is continuously embedded in $L^q(\mathbb{R}^d)$.*

1.1.4 Fractional Laplace operator $(-\Delta)^\sigma$

As far as we known, there are ten equivalent definitions of the fractional Laplacian in the whole space ($\mathbb{R}^d$) (see e.g. [71]). In this thesis we will present only two definitions (the most frequently used) including the following, as an integral in the form of the Cauchy principle value, and in the form of a well defined integral.

Definition 1.1.3. [45] The fractional Laplace operator $(-\Delta)^\sigma$ for $\sigma \in (0,1)$ can be defined along any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ as

$$\begin{aligned}(-\Delta)^\sigma \varphi(x) &= c_{d,\sigma} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+2\sigma}} dy \\ &= c_{d,\sigma} P.V. \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+2\sigma}} dy, \quad \forall x \in \mathbb{R}^d\end{aligned}$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space, $P.V.$ is the Cauchy principal value and

$$c_{d,\sigma} := \pi^{-\frac{d}{2}} \frac{\sigma 4^\sigma \Gamma(\sigma + \frac{d}{2})}{\Gamma(1-\sigma)}. \tag{1.1}$$

is a normalisation constant (see Proposition 3.3 in [45]) and the Gamma function Γ given by

$$\Gamma(t) := \int_0^\infty s^{t-1} e^{-s} ds, \quad t > 0.$$

For more detailed information, the interested reader is referred to [19, 27, 45] where the calculations are carried out.

Definition 1.1.4. (See [45]) For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\sigma \in (0,1)$, the fractional Laplace operator $(-\Delta)^\sigma$ can be defined as

$$(-\Delta)^\sigma \varphi(x) = -\frac{1}{2} c_{d,\sigma} \int_{\mathbb{R}^d} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{d+\sigma}} dy, \quad \forall x \in \mathbb{R}^d$$

where $c_{d,\sigma}$ is the constant defined by (1.1).

We will conclude this subsection with the following nice proposition from [45].

Proposition 1.1.1. *For any $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have the following statements.*

1. $(-\Delta)^\sigma \varphi \to -\Delta\varphi$ *as* $\sigma \to 1^-$.
2. $(-\Delta)^\sigma \varphi \to \varphi$ *as* $\sigma \to 0^+$.

Here $(-\Delta)$ is the classical Laplacian operator.

1.1.5 Homogeneous fractional Sobolev space $D^{\sigma,2}(\mathbb{R}^d)$

In this subsection, we will present a particular class of fractional Sobolev spaces. For $\sigma \in (0,1)$, we define the homogeneous fractional Sobolev space $D^{\sigma,2}(\mathbb{R}^d)$ as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{D^{\sigma,2}} = \left(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\sigma}} dxdy \right)^{\frac{1}{2}}. \tag{1.2}$$

It is well known that $D^{\sigma,2}(\mathbb{R}^d)$ is continuously embedded into $L^{2^*_\sigma}(\mathbb{R}^d)$ (see Theorem 6.5 in [45]), and there exists a best constant $C_\sigma > 0$ such that

$$C_\sigma = \inf_{u \in D^{\sigma,2}(\mathbb{R}^d)\setminus\{0\}} \frac{\int\limits_{\mathbb{R}^d} |(-\Delta)^{\frac{\sigma}{2}} u|^2 dx}{\left(\int\limits_{\mathbb{R}^d} |u^{2^*_\sigma}| dx\right)^{\frac{2}{2^*_\sigma}}}, \tag{1.3}$$

where

$$\int\limits_{\mathbb{R}^d} |(-\Delta)^{\frac{\sigma}{2}} u|^2 dx = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\sigma}} dxdy.$$

Using (1.3), we can give a concrete characterization of the completion $D^{\sigma,2}(\mathbb{R}^d)$ as a functional space (see [23]).

Theorem 1.1.9. *For $\sigma \in (0,1)$, thus we can define $D^{\sigma,2}(\mathbb{R}^d)$ as*

$$D^{\sigma,2}(\mathbb{R}^d) = \left\{ u \in L^{2^*_\sigma}(\mathbb{R}^d) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{d}{2}+\sigma}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

equipped with the norm (1.2).

Theorem 1.1.10. *(See [23]) Let $\sigma \in (0,1)$ such that $2\sigma < d$. Then, $D^{\sigma,2}(\mathbb{R}^d)$ is a Banach space.*

1.1.6 Bessel potential spaces $L^{\sigma,p}(\mathbb{R}^d)$

There are two important potentials in harmonic analysis and potential theory, namely the Riesz potential and the Bessel potential. Here, we will introduce the two potentials although the first potential will only used in the chapter 5.

Definition 1.1.5. [29, 30] Let $0 < \sigma < d$, $x \in \mathbb{R}^d$ and $u \in L^p(\mathbb{R}^d)$. The Riesz potential of order σ, $\mathcal{I}_\sigma$, is given by

$$\mathcal{I}_\sigma u\left(x\right) = \left(I_\sigma * u\right)\left(x\right) = \zeta_{d,\sigma} \int\limits_{\mathbb{R}^d} \frac{u\left(y\right)}{\left|x-y\right|^{d-\sigma}} dy,$$

where

$$I_\sigma := \frac{\zeta_{d,\sigma}}{\left|x\right|^{d-\sigma}}.$$

is the Riesz kernel, and

$$\zeta_{d,\sigma} := \pi^{-\frac{d}{2}} 2^{-\sigma} \frac{\Gamma(\frac{d-\sigma}{2})}{\Gamma(\frac{\sigma}{2})}.$$

Definition 1.1.6. Let $\sigma \in \mathbb{R}$ and $x \in \mathbb{R}^d$. The Bessel potential of order σ, $\mathcal{G}_\sigma$, is defined as

$$\mathcal{G}_\sigma u\left(x\right) = \left(G_\sigma * u\right)\left(x\right),$$

where G_σ is the Bessel kernel satisfies the following statements (see [29, 83] for the proof).

Theorem 1.1.11. *For $\sigma \in \mathbb{R}_+$. Then,*

1. $G_\sigma(x) := \dfrac{1}{(4\pi)^{\frac{\sigma}{2}}\Gamma(\frac{\sigma}{2})} \int\limits_0^\infty e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{\sigma-d}{2}-1} \frac{dt}{t}.$
2. $G_\sigma(x) \in L^1(\mathbb{R}^d)$.
3. $\|G_\sigma\|_1 = 1$

Remark 1.1.2. This result, together with Theorem 1.1.4 gives

$$\|\mathcal{G}_\sigma u\|_p = \|G_\sigma * u\|_p \leq \|G_\sigma\|_1 \|u\|_p = \|u\|_p. \tag{1.4}$$

Now for $\sigma \in \mathbb{R}_+$, we define the Bessel potential spaces $L^{\sigma,p}(\mathbb{R}^d)$ by [29, 30]

$$L^{\sigma,p}(\mathbb{R}^d) = \{u : u = G_\sigma * f \quad \text{for some} \quad f \in L^p(\mathbb{R}^d)\},$$

in the sense that we define $L^{\sigma,p}(\mathbb{R}^d)$ to be the image of $L^p(\mathbb{R}^d)$ under $\mathcal{G}_\sigma$.
The norm $\|\cdot\|_{L^{\sigma,p}}$ of $u \in L^{\sigma,p}(\mathbb{R}^d)$ is $\|u\|_{L^{\sigma,p}} = \|f\|_p$ if $G_\sigma * f$ (see (1.4)).
The following theorem, summarize the main properties of $L^{\sigma,p}(\mathbb{R}^d)$.

Theorem 1.1.12. *[1]*

1. *Let $\sigma \in \mathbb{R}$ and $1 \leq p \leq \infty$. Then, $L^{\sigma,p}(\mathbb{R}^d)$ is a Banach space.*
2. *Let $\sigma \geq 0$ and $1 < p < \infty$. Then, $C_0^\infty(\mathbb{R}^d)$ is dense in $L^{\sigma,p}(\mathbb{R}^d)$.*
3. *For $\sigma \in (0,1)$ and and $1 < p < \infty$, Then, $L^{\sigma,p}(\mathbb{R}^d)$ is reflexive space.*
4. *If σ is a non-negative integer and $1 < p < \infty$, the space $L^{\sigma,p}(\mathbb{R}^d)$ is coincides with the space $W^{\sigma,p}(\mathbb{R}^d)$, where the norms in the two spaces are equivalent. In particular, for any real $\sigma \in (0,1)$ we have the following identification $W^{\sigma,2}(\mathbb{R}^d) = L^{\sigma,2}(\mathbb{R}^d)$.*
5. *If $1 < p < \infty$, $\sigma \in (0,1)$ and $\varepsilon > 0$, for any σ we have*

$$L^{\sigma+\varepsilon,p}(\mathbb{R}^d) \hookrightarrow W^{\sigma,p}(\mathbb{R}^d) \hookrightarrow L^{\sigma-\varepsilon,p}(\mathbb{R}^d).$$

Thanks to the Theorem 1.1.12, we conclude this subsection by the following interesting properties about $L^{\sigma,2}(\mathbb{R}^d)$.

Theorem 1.1.13. *[50, 97]*

1. *For $\sigma \in (0,1)$, we have $W^{\sigma,2}(\mathbb{R}^d) = H^\sigma(\mathbb{R}^d) = L^{\sigma,2}(\mathbb{R}^d)$ with equivalence of norms.*
2. *If $\sigma \geq 0$ and $2 \leq q \leq 2_\sigma^*$, then $L^{\sigma,2}(\mathbb{R}^d)$ is continuously embedded in $L^q(\mathbb{R}^d)$.*

1.2 Some important results

Lemma 1.2.1. ***(Fatou's Lemma)***[24] *Let $\{u_n\}$ be a sequence of functions of $L^1(\Omega)$ such that*

1. *$u_n(x) \geq 0$ a.e. on Ω for all n.*
2. *$\sup\limits_n \int\limits_\Omega u_n < \infty$.*

For all $x \in \Omega$, we set $u(x) = \liminf_{n\to\infty} u_n(x)$. Thus, $u \in L^1(\Omega)$ and we have,

$$\int\limits_\Omega \liminf_{n\to\infty} u_n(x)dx \leq \liminf_{n\to\infty} \int\limits_\Omega u_n(x)dx.$$

Theorem 1.2.1. ***(Dominated convergence theorem [24])*** *Let $\{u_n\}$ be a sequence of functions of $L^1(\Omega)$. Suppose that*

1. *$u_n(x) \to u(x)$ a.e. on Ω.*
2. *There exist a function $h \in L^1(\Omega)$ such for all $n \in \mathbb{N}$ and for almost everywhere in Ω*

$$|u_n(x)| \leq h(x).$$

Then, $u \in L^1(\Omega)$ *and* $\lim_{n\to\infty}\|u_n - u\|_1 = 0$.

The following theorem is the Lebesgue theorem from [24].

Theorem 1.2.2. *Let* $\{u_n\}$ *be a sequence of* $L^p(\Omega)$ *and* $u \in L^p(\Omega)$ *such that* $\lim_{n\to\infty}\|u_n - u\|_p = 0$. *Then, there exists a subsequence* $\{u_{n_k}\}$ *such that*
a) $u_{n_k}(x) \to u(x)$ *a.e in* Ω.
b) $|u_{n_k}(x)| \leq h(x)$ *for all* k *and a.e in* Ω, *with* $h \in L^p(\Omega)$.

Theorem 1.2.3. *[24] Let* $\{u_n\}$ *be a sequence of a Banach space* X. *Then*

1. $\{u_n \rightharpoonup u \;\text{ in } X\} \Leftrightarrow \{\langle u_n, v\rangle \to \langle u, v\rangle \;\; \forall v \in X^*\}$.

2. *If* $u_n \rightharpoonup u$ *in* X *and* $v_n \to v$ *in* X^*, *thus* $\langle u_n, v_n\rangle \to \langle u, v\rangle$.

Theorem 1.2.4. ***(Lax-Milgram theorem)****[24] Let* $\mathcal{L}$ *be a continuous linear form on Hilbert space* H *and* μ *is a continuous and coercive bilinear form, then there exists one and only one function* $u \in H$ *such that:*

$$\mu(u, v) = \mathcal{L}(v), \quad \forall v \in H.$$

Moreover, if the bilinear form μ *is symmetric, then* u *is the only element of* H *that minimizes the functional* $\Phi : H \to \mathbb{R}$ *defined by*

$$\Phi(v) = \frac{1}{2}\,\mu(v, v) - \mathcal{L}(v), \quad \forall v \in H,$$

i.e.

$$\Phi(u) = \min_{v\in H} \Phi(v) \;\; \textit{and} \;\; \Phi(u) < \Phi(v) \;\; \textit{if } u \neq v.$$

Definition 1.2.1. We say that the following function

$$\begin{aligned} f : \mathbb{R}^d \times \mathbb{R} &\to \mathbb{R} \\ (x, u) &\to f(x, u) \end{aligned}$$

satisfies the Carathéodory conditions if $u \mapsto f(x, u)$ is continuous for almost every $x \in \mathbb{R}^d$ and $x \mapsto f(x, u)$ is measurable for every $u \in \mathbb{R}$.

Given any f satisfying the Carathéodory conditions and a function $u : \mathbb{R}^d \to \mathbb{R}$, we can define another function by composition

$$\mathcal{F}(u)(x) := f(x, u(x)).$$

The composition operator $\mathcal{F}$ is called a Nemytskii operator.

Definition 1.2.2. [105] Let $\Phi : U \to \mathbb{R}$ where U is an open subset of a Banach space X. The functional Φ has a Gateaux derivative $f \in X^*$ at $u \in U$ if, for every $h \in X$,

$$\lim_{t\to 0} \frac{1}{t}\,[\Phi(u + th) - \Phi(u) - < f, th >] = 0.$$

The Gateaux derivative at u is denoted by $\Phi'(u)$.
The functional Φ has a Fréchet derivative $f \in X^*$ at $u \in U$ if

$$\lim_{h \to 0} \frac{1}{\|h\|} \left[\Phi(u+h) - \Phi(u) - < f, h > \right] = 0.$$

The functional Φ belongs to $C^1(U, \mathbb{R})$ if the Fréchet derivative of Φ exists and is continuous on U.

Remark 1.2.1. [105] a) The Gateaux derivative is given by

$$< \Phi'(u), h > := \lim_{t \to 0} \frac{1}{t} \left[\Phi(u+th) - \Phi(u) \right].$$

b) Any Fréchet derivative is a Gateaux derivative but the converse is not true.

Before moving on to the next section, by using the mean value theorem, it is easy to prove the following result.

Proposition 1.2.1. *[105] If Φ has a continuous Gateaux derivative on U then $\Phi \in C^1(U, \mathbb{R})$.*

Definition 1.2.3. Let $\Phi \in C^1(X, \mathbb{R})$ be a functional.

- We say that $u \in X$ is a critical point of Φ if $\Phi'(u) = 0$ in X^*.
- We say that $c \in \mathbb{R}$ is a critical value of Φ, if there exists a critical point $u \in X$ of Φ such that $\Phi(u) = c$.

1.3 Calculus of variations

One of the most powerful tools in mathematical analysis and modern physics is the calculus of variations. It has been recognized that when they apply, variational methods can yield better results that most other methods. In simple sense, the calculus of variations is concerned with the study of problems related to finding the minimum or the maximum values of functionals, which are usually written as integrals depending on one or several functions and their derivatives.
Variational methods applied in the following chapters play a great role in many other disciplines of mathematics including optimization, geometry, and the theory of partial differential equations. A particular class of differential equations is the following class of Euler-Lagrange equations

$$E'(u) = 0 \tag{1.5}$$

where E is a functional defined on an appropriate function space, which is Fréchet-differentiable. Differential problems that can be written as (1.5) are problems that have a variational structure. The Euler-Lagrange equation is often taken as the starting point to solve any variational problem. For solving partial differential equations of variational type, an extensive methods has been developed, methods that are concerned with the minimization of functional are called the direct

method of calculus of variations. It is called “direct” since we demonstrate the existence of solutions to minimization problems without the detour through a differential equation. On the other hand, another forceful tool in this topic is the critical point theory, that consists of trying to find solutions by looking for stationary points of the (unbounded) functional. We notice that there are two kinds of critical points of a functional, the local extrema (maxima and minima) and the saddle points.
The most celebrated to approaches to critical point theory are the mountain pass theorem of Ambrosetti and Rabinowitz [3], the symmetric mountain pass theorem [90], the fountain theorem of Bartsch [11] and Ljusternik-Schnirelman theory [61].
In this section, we present some of these tools, but firstly we review a basic method of proving the existence of solution (a minimizer) of a variational problem provided the following two conditions are fulfilled.

Definition 1.3.1. [56] A functional $\Phi : X \to \mathbb{R}$ is called coercive if:

$$\lim_{\|u\| \to \infty} \Phi(u) = +\infty.$$

One of the main ingredient in the direct method of calculus of variations for determining the existence of minimizers is (sequential) weakly lower semicontinuous.

Definition 1.3.2. [24, 56] A functional $\Phi : X \to \mathbb{R}$ is called (sequential) weakly lower semicontinuous on X if for every $u_n \rightharpoonup u$ in X, we have

$$\Phi(u) \leq \liminf_{n \to \infty} \Phi(u_n).$$

The direct method of calculus of variations can be phrased as follows (Theorem 1.2 in [98]).

Theorem 1.3.1. *Assume that X is a reflexive Banach space, and let $Y \subset X$ be a weakly closed subset of X. It is assumed that $\Phi : Y \to \mathbb{R} \cup \{+\infty\}$ is coercive and (sequential) weakly lower semi-continuous on Y with respect to X. Then, Φ is bounded from below on Y and attains its infimum in Y.*

We recall that a subset $Y \subset X$ is called weakly closed if $u \in Y$ whenever there is a sequence $\{u_n\} \subset Y$ converging weakly to u in X.

Remark 1.3.1.
- A good example of a (sequential) weakly lower semi-continous functional is the norm in a Banach space.
- An important examples of weakly closed sets are closed convex subsets of Banach spaces.

In general it is not clear that a bounded and lower semi-continuous functional Φ actually attains its infimum. For example, the function $f(x) = \arctan(x)$, neither attains its supremum nor its infimum on the real line. However, in many nonlinear problems, the associated "energy" functional may be unbounded from above and below, and so that it has no maximum or minimum. This forces us to look for results where other kinds of critical points (saddle points) may be

identified. Note that to demonstrate the existence of saddle points which are obtained by minimax arguments, we will strengthen the regularity hypothesis on functional and in general require, the functional to be of class C^1, that is continuously Fréchet differentiable.
A particularly interesting models for the minimax results is the so-called mountain pass theorem of Ambrosetti and Rabinowitz [3].
Now, we proceed to introduce definitions of compactness conditions. We first start by defining what is meant by the Palais-Smale condition ((PS)-condition for short). The original idea of the (PS)-condition was established by Palais [86] and Smale [94].

Definition 1.3.3. Let X be a real Banach space and assume that $\Phi \in C^1(X, \mathbb{R})$. The functional Φ fulfills the (PS)-condition, if any sequence $\{u_n\} \subset X$ satisfying

$$(\Phi(u_n))_n \text{ is bounded} \quad \text{and} \quad \Phi'(u_n) \to 0 \text{ as } n \to \infty, \tag{1.6}$$

admits a convergent subsequence.

Any sequence satisfying (1.6) is called a Palais-Smale sequence.
In general, when dealing with critical point theory, we need a weaker condition denoted by (PS)$_c$, which was provided by Brézis-Coron-Nirenberg in [25].

Definition 1.3.4. Let X be a real Banach space and assume that $\Phi \in C^1(X, \mathbb{R})$. The functional Φ fulfills the (PS)-condition at the level $c \in \mathbb{R}$, if any sequence $\{u_n\} \subset X$ satisfying

$$\Phi(u_n) \to c \quad \text{and} \quad \Phi'(u_n) \to 0 \text{ as } n \to \infty, \tag{1.7}$$

possesses a convergent subsequence.

Remark 1.3.2. In the sequel, an alternative definition of (PS)-condition is taken as follows:
The functional Φ is said to fulfill the (PS)-condition if and only if it fulfills the (PS)-condition at any level $c \in \mathbb{R}$ ((PS)$_c$-condition for short).

Next, we present one of the weak version of (PS)$_c$-condition called the Cerami condition, which was initially introduced by Cerami [32].

Definition 1.3.5. Let X be a real Banach space and assume that $\Phi \in C^1(X, \mathbb{R})$. The functional Φ fulfills the Cerami condition at level $c \in \mathbb{R}$ ((C)$_c$-condition for short), if any $\{u_n\} \subset X$ satisfying

$$\Phi(u_n) \to c \quad \text{and} \quad \|\Phi'(u_n)\|_*(1 + \|u_n\|) \to 0 \text{ as } n \to \infty.$$

possesses a convergent subsequence.

The following theorem is the standard mountain pass theorem.

Theorem 1.3.2. ***(Mountain pass theorem** [3]) Let X be a real Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional satisfying the (PS)$_c$-condition with $\Phi(0) = 0$. Assuming that the following geometric conditions hold, that is,*

(i) There exist $\rho > 0$ and $\delta > 0$ such that $\Phi(u) \geq \delta$ for $\|u\| = \rho$.
(ii) There exists $e \in X$ with $\|e\| > \rho$ such that $\Phi(e) \leq 0$.
Let $\Gamma = \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}$. Then

$$c := \inf_{\gamma \in \Gamma, t \in [0,1]} \max \Phi(\gamma(t)) > \delta$$

is a critical value of Φ.

We mention that the mountain pass theorem can be expressed as follows.

Lemma 1.3.1. *Let X be a real Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional satisfying the (PS)$_c$-condition with $\Phi(0) = 0$. Assuming that the following geometric conditions hold, that is,*

1. *There exist $\delta, \rho > 0$ such that $\Phi(u) \geq \delta > 0$ for all $u \in X$ with $\|u\| = \rho$.*
2. *There exists $\varphi > 0$ in X, such that $\Phi(t\varphi) \to -\infty$ as $t \to \infty$.*
3. *There exists $\psi > 0$ in X, such that $\Phi(t\psi) < 0$ for all $t \to 0^+$.*

The preceding theorems help us to get the existence of at least one nontrivial solution. The symmetric mountain pass theorem [90] may be considered as an extension of the Ljusternik-Schnirelman theorem [61] to Banach spaces and unbounded (from below or from above) functional. It is powerful tool to get a multiplicity result asserting the existence of multiple critical points. We choose $\{e_i\}_i$ an orthonormal basis of space X and define $X_i = \mathbb{R}e_i$,

$$Y_k = \oplus_{i=1}^{k} X_i \quad Z_k = \overline{\oplus_{i=k+1}^{\infty} X_i} \quad k \in \mathbb{Z}.$$

Evidently, we have $X = Y_k \oplus Z_k$.
Then, its exact statement is the following.

Theorem 1.3.3. ***(Symmetric mountain pass theorem [90])*** *Assume that $X = Y_k \oplus Z_k$ be a Banach space where Y is finite dimensional, let $\Phi \in C^1(X, \mathbb{R})$ be even and $\Phi(0) = 0$, satisfies the (PS)$_c$-condition, if*
(i) there exist constants $\rho, \delta > 0$ satisfying $\Phi\big|_{\partial B_\rho \bigcap Z_k} \geq \delta$, where B_ρ is the open ball in X of radius ρ about 0 and ∂B_ρ is its boundary,
(ii) for any finite dimensional subspace $\tilde{X} \subset X$, there is $C = C(\tilde{X}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{X} \setminus B_C$,
then, Φ has an unbounded sequence of critical points.

The next theorem is the fountain theorem developed by Bartsch [11] which can be seen as a generalization of multiplicity result of the symmetric mountain pass theorem. This theorem is a way to study the existence of infinitely many nontrivial solutions. Note that a decomposition of the Banach space plays an important role in proving these theorems. To do this, we refer to the following Lemma.

Lemma 1.3.2. *[116] Let X be a reflexive and separable Banach space and X^* its dual space. Then, there exist $\{e_n\} \subseteq X$ and $\{e_n^*\} \subseteq X^*$ such that*

$$X = \overline{\text{span}\,\{e_n : n = 1, 2, ...\}}, \quad X^* = \overline{\text{span}\,\{e_n^* : n = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us define $X_n = \text{span}\,\{e_n\}$, $Y_k = \oplus_{n=1}^{k} X_n$ and $Z_k = \overline{\oplus_{n=k}^{\infty} X_n}$ for $k \in \mathbb{Z}$. Now, we recall the fountain theorem under the $(\text{PS})_c$-condition.

Lemma 1.3.3. ***(Fountain theorem [11])*** *Let X be a real reflexive Banach space, consider an even functional $\Phi \in C^1(X, \mathbb{R})$ satisfies the $(PS)_c$-condition for any $c > 0$. If for each sufficiently large $k \in \mathbb{N}$, there exist $\alpha_k > \beta_k > 0$ such that*

(i) $\rho_k := \inf\limits_{u \in Z_k, \|u\| = \beta_k} \Phi(u) \to \infty,$ *as $k \to \infty$,*

(ii) $\delta_k := \max\limits_{u \in Y_k, \|u\| = \alpha_k} \Phi(u) \leq 0,$

then the functional Φ has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset X$ such that $\Phi'(u_n) = 0$ and $\Phi(u_n) \to \infty$ as $n \to \infty$.

We finishes this chapter by the remark as below.

Remark 1.3.3. It was shown that the preceding critical point theorems hold true also under the $(\text{C})_c$-condition, see for instance [10].

CHAPTER **2**

Multiplicity of solutions for a large class of fractional Kirchhoff-Schrödinger-Poisson systems

This chapter includes the results of the following research article:
• H.Boutebba, H.Lakhal, and K.Slimani. Multiplicity of solutions for a class of fractional Kirchhoff-Schrödinger-Poisson systems with superlinear terms. (submitted to an international journal).

This chapter seeks to establish the existence of a sequence of infinitely many solutions with high energies to a new class of superlinear fractional Kirchhoff-Schrödinger-Poisson systems under certain suitable assumptions on f, M, V and K. The main result also provide a blow-up property of the associated energy levels. To reach our goal, we use a variational approach combined with the fountain theorem.
Note that from now on, we limit the work space in dimension $d = 3$.

2.1 Introduction

In recent years, the study of elliptic problems involving nonlocal fractional operators has received significant attention from researchers in many different contexts due to the practical applications to several phenomena. These kinds of operators occur in a quite natural way to the description of some phenomena in finance, quantum mechanics, probability, water waves, and physics; see, for instance, [17, 27, 72, 99] and the references therein. On the other hand, the literature on the fractional Laplacian operator and their application is quite large and interesting, we refer the reader to [19, 45] for the elementary properties of fractional Sobolev spaces and the study of fractional Laplacian based on variational methods.
Herein, we intend to study the following fractional Schrödinger-Kirchhoff-Poisson type system

$$\begin{cases} M([u]_\sigma^2)(-\Delta)^\sigma u + V(x)u + K(x)\phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{2.1}$$

where

$$[u]_\sigma^2 = \int\limits_{\mathbb{R}^3} |(-\Delta)^{\frac{\sigma}{2}} u|^2 dx = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy,$$

$\sigma, \gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, λ is a real positive parameter, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function. The nonlinearity $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that obeys certain assumptions and $(-\Delta)^\sigma$ is the fractional Laplacian which can be defined along all function $\varphi \in C_0^\infty(\mathbb{R}^3)$ as

$$(-\Delta)^\sigma \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{3+2\sigma}} dy, \quad x \in \mathbb{R}^3,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^3 : |x-y| < \varepsilon\}$.
Throughout the paper, the potential functions V, K and Kirchhoff-type function M are assumed that there hold following:
(V): $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$ where V_0 is a constant, and for any $M > 0$ meas $\{x \in \mathbb{R}^3 : \ V(x) \leq M\} < \infty$.
(K) : $K(x) \in L^q(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $q \in [\frac{6}{4\sigma+2\gamma-3}, \infty)$, and $K(x) \geq 0$ for all $x \in \mathbb{R}^3$.
(M_1) : $M(s) \geq m_0 > 0$ for all $s \in \mathbb{R}_0^+$ where m_0 is a constant.
(M_2): there exists $\tau \in [2, \dfrac{3}{3-2\sigma})$ such that $\tau\mathcal{M}(s) := \tau\int_0^s M(\theta)d\theta \geq M(s)s$ for all $s \in \mathbb{R}_0^+$.

Remark 2.1.1. 1. • The assumption (M_1) implies that $M(s) > 0$ for all $s > 0$. Thus, by assumption (M_2) for all $0 < s \leq 1$ we obtain $\frac{M(s)}{\mathcal{M}(s)} \leq \frac{\tau}{s}$. Hence, integrating on $[s, 1]$ with $0 < s < 1$, we deduce

$$\mathcal{M}(s) \geq \mathcal{M}(1)s^\tau \quad \text{for all} \quad 0 \leq s \leq 1.$$

- While, by integrating (M_2) once again we infer

$$\mathcal{M}(s) \leq \frac{\mathcal{M}(s_0)}{s_0^\tau} s^\tau \text{ for all } \quad s \geq s_0 > 0.$$

Choosing $s_0 = 1$, we arrive at

$$\mathcal{M}(s) \leq \mathcal{M}(1) s^\tau \quad \text{for all} \quad s \geq 1.$$

As a direct result, if $\|u\| > 1$, we have

$$\mathcal{M}([u]_\sigma^2) \leq \mathcal{M}(\|u\|^2) \leq \mathcal{M}(1) \|u\|^{2\tau}. \tag{2.2}$$

2. The assumption (M_2) implies to impose the lower bound $\sigma > \frac{3}{4}$ in the fractional Laplacian operator and this leads us to $2\gamma + 4\sigma > 3$.

The typical function for M is given by $M(s) = a + bs^{\tau-1}$ for $s \in \mathbb{R}_0^+$, where $\tau \in [2, \dfrac{3}{3-2\sigma})$, $a \in \mathbb{R}_0^+$, $b \in \mathbb{R}_0^+$ and $a + b > 0$. In particular, when $M(0) > 0$, the Kirchhoff problem is called nondegenerate, while it is called degenerate if $M(0) = 0$. In this paper, the assumption (M_1) covers the nondegenerate case.

This work is motivated by the large interest in the recent literature applying variational methods combined with the nonlocal analysis as well as by the fact that in nature, nonlocal operators of elliptic type appear in many physical phenomena, in finance, in probability, and also in water waves, see for example [17, 27, 72, 99] and their references.

Note that the presence of the nonlocal term M in (2.1) implies that this system is no longer pointwise, which introduces certain mathematical difficulties. We would like to point out that such nonlocal problems can model several physical and biological systems including population density [39].

Another important reason for studying system (2.1) is the significant feature of Kirchhoff-type equations. To be more precise, Kirchhoff [69] in 1883 considered the following model

$$\rho h \frac{\partial^2 u}{\partial t^2} - \left(p_0 + \frac{Eh}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = f(x, u), \tag{2.3}$$

as an extension of the classical D'alembert's wave equation for free vibrations of elastic strings. Here ρ, p_0, h, E, L are constants which represent some physical meanings.

In the last few years, due to their importance in mathematical purposes such as calculus of variations, probability theory, and approximation theory, there have been many interesting generalizations of the Kirchhoff model to the fractional case by many scholars. Among them, we pay special attention to [52] where the authors considered the following stationary Kirchhoff-type equation

$$\begin{cases} M\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2\sigma}}dxdy\right)(-\Delta)^\sigma u = \lambda f(x,u)+|u|^{2^*_\sigma-2}u & \text{in } \Omega, \\ u=0 & \text{in } \mathbb{R}^d\setminus\Omega, \end{cases} \tag{2.4}$$

where $2^*_\sigma = \frac{6}{3-2\sigma}$ is the critical Sobolev exponent and M is non-degenerate. By employing an appropriate truncation argument in combination with the mountain pass theorem, they proved that (2.4) admits at least one nontrivial solution when λ is large enough. After that, the existence and multiplicity of solutions for fractional Kirchhoff-type elliptic equations or systems are given using suitable variational methods. For more recent advances related to the nondegenerate case we refer the reader to [66, 75, 80, 88, 103] while the degenerate case is treated in [80] once again and [51]. Here, we would like to mention that the degenerate case of (2.4) is studied in [6] by introducing a new technical method based on the asymptotic property of the critical mountain pass level. See also the notable work of Kim-Kim-Park [64] for the case of concave-convex nonlinearities with the fractional $p(\cdot)$-Laplacian operator.
For similar systems to (2.1) involving fractional operators without the presence of the Kirchhoff's term of the following form

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + K(x)\phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.5}$$

It is well known that numerous authors have devoted to the existence and multiplicity of solutions for (2.5) under various assumptions on the potential functions V, K and the nonlinearity f. In our recent work [22], we showed the existence of infinitely many high energy solutions for (2.5) by means of the fountain theorem. When $\lambda = 1$ in (2.5), He and Lei [60] established a multiplicity result of solutions by using the symmetric mountain pass theorem. Jin and Yang [63] considered the existence of infinitely many solutions when $K(x) \equiv 1$ and $\lambda = 1$ in (2.5). Kim and Bae [65] considered the existence of infinitely many solutions when $K(x) \equiv 1$ in (2.5). Meng-Zhang-He [81], Ye-Teng [110], Yu-Zhao-Zhao [112] and Teng [100] investigated the existence of solutions for (2.5) with critical nonlinearity term. Teng and Agarwal [101] obtained the existence and nonexistence of ground state and bounded state solution for (2.5) with the subcritical choquard nonlinearity. For elliptic problems driven by another fractional operator instead of the well-known fractional Laplacian we refer the interested reader to [21, 58] when we prove the existence and multiplicity of nontrivial solutions by using the mountain pass theorem and the symmetric mountain pass theorem, respectively, in an appropriate functional space.
Motivated by all the work just described above, particularly by [66, 75], the main purpose of this chapter is to obtain the existence of infinitely many high energy solutions by using the fountain theorem.
Next, we consider appropriate superlinear growth assumptions for f. Let us denote $F(x,s) =$

$\int_0^s f(x,t)dt$ and let $\tau \in \mathbb{R}$ be given in (M_2).
(F_1) : There exist nonnegative functions $\varrho \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\xi \in L^\infty(\mathbb{R}^3)$ such that, for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$

$$|f(x,s)| \leq \varrho(x) + \xi(x)|s|^{r-1}, \quad r \in (2\tau, 2^*_\sigma).$$

(F_2) : $f(x,-s) = -f(x,s)$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$.
(F_3) : $\lim\limits_{|s|\to\infty} \frac{F(x,s)}{|s|^{2\tau}} = \infty$ uniformly for almost all $x \in \mathbb{R}^3$.
(F_4) : There exist $\mu > 2\tau$ and $\varsigma > 0$ such that

$$\mu F(x,s) \leq sf(x,s) + \varsigma |s|^2, \text{ for all } (x,s) \in \mathbb{R}^3 \times \mathbb{R}.$$

Our main result can be given as follows.

Theorem 2.1.1. *Let $\sigma, \gamma \in (0,1)$ with $\sigma > \frac{3}{4}$. Suppose that (F_1)-(F_4), (M_1)-(M_2), (K) and (V) hold. Then, the problem* (2.1) *admits a sequence of nontrivial solutions $\{u_n\}$ in E such that $\Phi_\lambda(u_n) \to \infty$ as $n \to \infty$ for all $\lambda > 0$.*

The rest of this chapter is designed as follows. In the next section, we present some auxiliary results and we set up the functional framework necessary to our problem. In section 2.3, we prove the existence of infinitely many large energy solutions for the given problem by using the fountain theorem as the main tool.

2.2 Preliminaries and variational settings

This section is devoted to recall some useful results of fractional Sobolev spaces, embeddings and variational formulations.
Recalling that for $\sigma \in (0,1)$, the fractional Sobolev space $H^\sigma(\mathbb{R}^3)$ is defined as

$$H^\sigma(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy \right)^{\frac{1}{2}} < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^\sigma} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy + \int_{\mathbb{R}^3} |u|^2 dx \right)^{\frac{1}{2}}.$$

Moreover, we define the fractional Sobolev space $D^{\sigma,2}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_{D^{\sigma,2}} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy \right)^{\frac{1}{2}}.$$

It follows from Theorem 6.5 in [45] that the space $D^{\sigma,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\sigma}(\mathbb{R}^3)$, and there exists a best constant $C_\sigma > 0$ such that

$$C_\sigma = \inf_{u \in D^{\sigma,2}(\mathbb{R}^3)\setminus\{0\}} \frac{[u]_\sigma^2}{\|u\|_{2^*_\sigma}^2} \tag{2.6}$$

In view of the potential $V(x)$, let us define the natural solution space E for (2.1) by

$$E = \left\{ u \in H^\sigma\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty \right\},$$

Thus, E is a Hilbert space with the following inner product

$$\langle u, w\rangle = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{\sigma}{2}} u(-\Delta)^{\frac{\sigma}{2}} w + V(x)uw)dx,$$

and corresponding norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{\sigma}{2}} u|^2 + V(x)|u|^2)dx.$$

In particular, we have the following result.

Lemma 2.2.1. $(E, \|\cdot\|)$ *is a uniformly convex Banach space.*

Proof. The proof is similar to that of [[88], Lemma 10] and hence omitted. □

Because of the assumption (V), we get the following embedding theorem.

Lemma 2.2.2. *([88], Lemma 1) Let the assumption (V) hold. Then, E is continuously embedded into $H^\sigma(\mathbb{R}^3)$. Further, E is continuously embedded into $L^r(\mathbb{R}^3)$ for all $r \in [2, 2^*_\sigma]$.*

E is continuously embedded in $L^r(\mathbb{R}^3)$ for all $r \in [2, 2^*_\sigma]$, i.e., there exists a positive constant C_r such that

$$|u|_r \le C_r\|u\|. \tag{2.7}$$

Thanks to the assumption (V), Lemma 2.2.2 and (2.6) (see Lemma 2.3 in [80] for the proof), we obtain the following compactness result.

Lemma 2.2.3. *[80] The space E is compactly embedded in $L^r(\mathbb{R}^3)$ for all $r \in [2, 2^*_\sigma)$.*

Now, we are going to reduce system (5.11) to a single equation. For every $u \in E$, the linear operator $\mathcal{L}_u : D^{\gamma,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{L}_u(w) = \int_{\mathbb{R}^3} K(x)u^2 w dx,$$

is well defined on $D^{\gamma,2}(\mathbb{R}^3)$ and continuous. Indeed, by Hölder inequality, (K) and (2.6)-(2.7), we have

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|K\|_q \|u\|^2_{\frac{12q}{(q(3+2\gamma)-6)}} \|w\|_{2^*_\gamma} \\ &\leq C\|u\|^2 \|w\|_{D^{\gamma,2}}. \end{aligned} \tag{2.8}$$

Here we have used the fact $2 < \frac{12}{3+2\gamma} < \frac{12q}{(q(3+2\gamma)-6)} \leq 2^*_\sigma$ and $q \in [\frac{6}{4\sigma+2\gamma-3}, \infty)$.
Set

$$\eta(u,w) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{\gamma}{2}} u (-\Delta)^{\frac{\gamma}{2}} w dx, \quad \forall u, w \in D^{\gamma,2}(\mathbb{R}^3).$$

For every $u \in E$, one can use the Lax-Milgram theorem, and then there exists a unique $\phi_u^\gamma \in D^{\gamma,2}(\mathbb{R}^3)$ such that $\mathcal{L}_u(w) = \eta(\phi_u^\gamma, w)$, that is

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{\gamma}{2}} \phi_u^\gamma (-\Delta)^{\frac{\gamma}{2}} w dx = \int_{\mathbb{R}^3} K(x) u^2 w dx, \quad \forall w \in D^{\gamma,2}\left(\mathbb{R}^3\right). \tag{2.9}$$

As a result, ϕ_u^γ fulfills the Poisson equation

$$(-\Delta)^\gamma \phi_u^\gamma = K(x)u^2, \quad x \in \mathbb{R}^3,$$

and it can be represented by the so-called γ-Riesz potential (see [97])

$$\phi_u^\gamma(x) = c_\gamma \int_{\mathbb{R}^3} \frac{K(x)u^2(y)}{|x-y|^{3-2\gamma}} dy, \tag{2.10}$$

where

$$c_\gamma = 2^{-2\gamma} \frac{\Gamma\left(\frac{3-2\gamma}{2}\right)}{\pi^{\frac{3}{2}}\Gamma(\gamma)}.$$

Then, $\phi_u^\gamma > 0$ for all $u \neq 0$. Taking $w = \phi_u^\gamma$ in (2.8) and (2.9), we derive

$$\|\phi_u^\gamma\|_{D^{\gamma,2}} \leq C\|u\|^2. \tag{2.11}$$

If we take $w = \phi_u^\gamma$ in (2.8) and (2.9) again, and by (2.11) we get

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx &\leq C_1 \|u\|^2 \|\phi_u^\gamma\|_{D^{\gamma,2}} \\ &\leq C\|u\|^4. \end{aligned} \tag{2.12}$$

When substituting ϕ_u^γ in (2.1), we get the following single equation

$$M([u]^2_\sigma)(-\Delta)^\sigma u + V(x)u + K(x)\phi_u^\gamma u = \lambda f(x,u), \quad x \in \mathbb{R}^3. \tag{2.13}$$

Therefore, the energy functional $\Phi_\lambda : E \to \mathbb{R}$ associated to (2.13) is given by

$$\Phi_\lambda(u) = \frac{1}{2}\left(\mathcal{M}([u]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u|^2 dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.14}$$

is well defined and $\Phi_\lambda \in C^1(E,\mathbb{R})$ (see for instance[88] for the proof), and any critical points of Φ_λ are a weak solutions (2.13). Moreover, its Fréchet derivative for any $u, w \in E$ is

$$< \Phi_\lambda'(u), w >= \int_{\mathbb{R}^3} (M([u]_\sigma^2)(-\Delta)^{\frac{\sigma}{2}}u(-\Delta)^{\frac{\sigma}{2}}w + V(x)uw + K(x)\phi_u^\gamma uw - \lambda f(x,u)w)dx.$$

The following assertion is crucial to establish our main result. The fundamental idea of proofs of this consequences follows similar arguments as in [88, 96, 102].

Lemma 2.2.4. *[88, 96, 102] Φ_λ' is of type (S_+), that is, if for every sequence $\{u_n\} \subset E$ satisfying $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty}\langle \Phi_\lambda'(u_n) - \Phi_\lambda'(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in E as $n \to \infty$.*

Now, consider the operator $\Lambda : H^\sigma(\mathbb{R}^3) \to D^{\gamma,2}(\mathbb{R}^3)$ as follows $\Lambda[u] = \phi_u^\gamma$. The following Lemma summarize the main properties of Λ, which is particularly helpful for the study of the problem (2.1). The proof can be derived from the same lines of [81, 101, 109]. Here we omit it.

Lemma 2.2.5. *If $2\gamma + 4\sigma > 3$, for each $u \in H^\sigma(\mathbb{R}^3)$, we have*

1. *If $u_n \rightharpoonup u$ in $H^\sigma(\mathbb{R}^3)$, then $\Lambda[u_n] \rightharpoonup \Lambda[u]$ in $D^{\gamma,2}(\mathbb{R}^3)$.*
2. *$\Lambda[\tau u] = \tau^2\Lambda[u]$ for all $\tau \in \mathbb{R}$.*
3. *If $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 2_\gamma^*$, then*

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma(x)u_n w dx \to \int_{\mathbb{R}^3} K(x)\phi_u^\gamma(x)uwdx \qquad \text{for all} \quad w \in E$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma(x)u_n^2 dx \to \int_{\mathbb{R}^3} K(x)\phi_u^\gamma(x)u^2 dx.$$

2.3 Proof of main result

In this section, we first prove that the functional Φ_λ fulfills the (C)$_c$-condition which plays a decisive role in showing the existence of nontrivial solutions for our problem. Then, we prove that the functional Φ_λ has the remaining geometric conditions of the fountain theorem.

Lemma 2.3.1. *Let $\sigma, \gamma \in (0,1)$ with $2\gamma + 4\sigma > 3$. Assume that (F_1),(F_3)-(F_4), (M_1)-(M_2) and (V) hold. Then, Φ_λ fulfills the (C)$_c$-condition for all $\lambda > 0$.*

Proof. Let $\{u_n\}$ be a (C)$_c$-sequence, that is,

$$\Phi_\lambda(u_n) \to c \quad \text{and} \quad \|\Phi'_\lambda(u_n)\|_*(1+\|u_n\|) \to 0 \text{ as } n \to \infty, \tag{2.15}$$

which implies that

$$c = \Phi_\lambda(u_n) + o_n(1) \text{ and } \langle \Phi'_\lambda(u_n), u_n\rangle = o_n(1). \tag{2.16}$$

From Lemma 2.2.4, we have that Φ'_λ verifies the (S_+) condition, that is, for any sequence $\{u_n\} \subset E$ such that $u_n \rightharpoonup u$ in E and

$$\limsup_{n\to\infty}\langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle \leq 0,$$

we have, passing to a subsequence if necessary, $u_n \to u$ in E as $n \to \infty$. Hence, it suffices to assure that the sequence $\{u_n\}$ is bounded in E since E is reflexive. To do this, suppose on the contrary that the sequence $\{u_n\}$ is unbounded in E. Then, we may suppose that

$$\|u_n\| \to \infty \text{ as } n \to \infty. \tag{2.17}$$

Define a sequence $\{v_n\}$ by $v_n = \frac{u_n}{\|u_n\|}$. Then, it is clear that $\{v_n\} \subset E$ and $\|v_n\| = 1$. Then, up to a subsequence $\{v_n\}$, still denoted by $\{v_n\}$, we obtain $v_n \rightharpoonup v$ in E as $n \to \infty$.
Due to Lemma 2.2.3,

$$\begin{aligned} v_n(x) &\to v(x) \quad \text{a.e. in } \mathbb{R}^3 \\ v_n &\to v \quad \text{in } L^p(\mathbb{R}^3) \text{ for } 2 \leq p < 2^*_\sigma \end{aligned} \tag{2.18}$$

as $n \to \infty$. Owing to the relation (2.16), we have

$$\begin{aligned} c &= \Phi_\lambda(u_n) + o_n(1) \\ &= \frac{1}{2}\left(\mathcal{M}([u_n]^2_\sigma) + \int_{\mathbb{R}^3} V(x)|u_n|^2dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi^\gamma_{u_n}u_n^2dx - \lambda\int_{\mathbb{R}^3} F(x,u_n)dx + o_n(1) \end{aligned}$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, from (M_1)-(M_2) we arrive at

$$\begin{aligned} \int_{\mathbb{R}^3} F(x,u_n)dx =& \frac{1}{2\lambda}\left(\mathcal{M}([u_n]^2_\sigma) + \int_{\mathbb{R}^3} V(x)|u_n|^2dx\right) + \frac{1}{4\lambda}\int_{\mathbb{R}^3} K(x)\phi^\gamma_{u_n}u_n^2dx - \frac{c}{\lambda} + \frac{o_n(1)}{\lambda} \\ \geq& \frac{M([u_n]^2_\sigma)[u_n]^2_\sigma}{2\lambda\tau} + \frac{1}{2\lambda}\int_{\mathbb{R}^3} V(x)|u_n|^2dx - \frac{c}{\lambda} + \frac{o_n(1)}{\lambda} \\ \geq& \frac{\min\left\{\frac{m_0}{\tau},1\right\}}{2\lambda}\|u_n\|^2 - \frac{c}{\lambda} + \frac{o_n(1)}{\lambda} \\ \to& \infty \text{ as } n \to \infty. \end{aligned} \tag{2.19}$$

Here we utilized the inequality

$$\begin{aligned}\mathcal{M}([u]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u|^2dx &\geq \frac{M([u]_\sigma^2)[u]_\sigma^2}{\tau} + \int\limits_{\mathbb{R}^3} V(x)|u|^2dx \\ &\geq \min\left\{\frac{m_0}{\tau}, 1\right\}\|u\|^2, \end{aligned} \tag{2.20}$$

which is easily deduced from definition of the Kirchhoff function and the norm $\|\cdot\|$.
Thus, due to the relation (2.19), we have

$$0 < \frac{1}{2\lambda} \leq \int\limits_{\mathbb{R}^3} \limsup_{n\to\infty} \frac{|F(x,u_n)|}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} dx. \tag{2.21}$$

Owing to assumption (F_3), there exists $u_0 > 1$ such that $F(x,u) > |u|^{2\tau}$ for all $x \in \mathbb{R}^3$ and $|u| > u_0$. Since f fulfills the Carathéodory condition and by means of (F_1), there exists a constant $C > 0$ such that for all $|F(x,u)| < C, \ \forall\ (x,u) \in \mathbb{R}^3 \times [-u_0, u_0]$. Therefore, we conclude that $F(x,u) \geq C_0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ and for some $C_0 \in \mathbb{R}$. Thus,

$$\frac{F(x,u_n) - C_0}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} \geq 0, \tag{2.22}$$

for all $x \in \mathbb{R}^3$ and for all $n \in \mathbb{N}$.
Denote $\Sigma_0 = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, then meas$(\Sigma_0) > 0$. Using (F_3), (M_1)-(M_2) and (2.12), (2.17), (2.19) and for all $x \in \Sigma_0$, we infer

$$\begin{aligned}&\lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} \\ \geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}(1)\|u_n\|^{2\tau} + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}C\|u_n\|^4} \\ \geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|^{2\tau} + \frac{1}{2}C\|u_n\|^{2\tau}} \\ =\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1+\frac{1}{2}C)\|u_n\|^{2\tau}} \\ =\ & \lim_{n\to\infty} \frac{1}{\mathcal{M}(1)+1+\frac{1}{2}C} \cdot \frac{F(x,u_n)}{|u_n(x)|^{2\tau}} \cdot |v_n(x)|^{2\tau} \\ =\ & \infty. \end{aligned} \tag{2.23}$$

Taking into account (2.12), (2.19)-(2.23) and the Fatou lemma, we infer

$$
\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int\limits_{\mathbb{R}^3} F(x,u_n)dx}{\lambda \int\limits_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1)} \\
&= \liminf_{n\to\infty} \frac{\int\limits_{\mathbb{R}^3} F(x,u_n)dx}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \\
&\geq \liminf_{n\to\infty} \frac{\int\limits_{\Sigma_0} F(x,u_n)dx}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \\
&\quad - \limsup_{n\to\infty} \int\limits_{\Sigma_0} \frac{C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \\
&= \liminf_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n) - C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\|u_n\|^4} dx \\
&\geq \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n) - C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\|u_n\|^4} dx \\
&= \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\|u_n\|^4} dx \\
&\quad - \int\limits_{\Sigma_0} \limsup_{n\to\infty} \frac{C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\|u_n\|^4} dx \\
&= \infty,
\end{aligned} \tag{2.24}
$$

which is a contradiction. Thus, $v(x) = 0$ for almost all $x \in \mathbb{R}^3$. In addition, by the convergence (2.18) we obtain for $n \to \infty$

$$
\begin{aligned}
v_n &\to 0 \quad \text{in} \quad L^p(\mathbb{R}^3) \quad \text{for} \quad 2 \leq p < 2_\sigma^*. \\
v_n(x) &\to 0 \quad \text{a.e. in} \quad \mathbb{R}^3
\end{aligned}
$$

Now we consider the case that $v(x) = 0$ in $\mathbb{R}^3$.

It follows from (M_1)-(M_2) and (F_4) that for n large enough

$$
\begin{aligned}
c+1 \geq{}& \Phi_\lambda(u_n) - \frac{1}{\mu}\langle \Phi'_\lambda(u_n), u_n\rangle \\
={}& \frac{1}{2}\mathcal{M}([u_n]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \lambda\int_{\mathbb{R}^3} F(x,u_n)dx \\
&- \frac{1}{\mu}M([u_n]_\sigma^2)[u_n]_\sigma^2 - \frac{1}{\mu}\int_{\mathbb{R}^3} V(x)|u_n|^2dx - \frac{1}{\mu}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx + \frac{\lambda}{\mu}\int_{\mathbb{R}^3} f(x,u_n)u_n dx \\
={}& \frac{1}{2}\mathcal{M}([u_n]_\sigma^2) - \frac{1}{\mu}M([u_n]_\sigma^2)[u_n]_\sigma^2 + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx \\
&+ \left(\frac{1}{4}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx + \lambda\int_{\mathbb{R}^3}\left(\frac{f(x,u_n)u_n}{\mu} - F(x,u_n)\right)dx \\
\geq{}& \left(\frac{1}{2\tau}-\frac{1}{\mu}\right)M([u_n]_\sigma^2)[u_n]_\sigma^2 + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx \\
&+ \left(\frac{1}{4}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx + \lambda\int_{\mathbb{R}^3}\left(\frac{f(x,u_n)u_n}{\mu} - F(x,u_n)\right)dx \\
\geq{}& \left(\frac{1}{2\tau}-\frac{1}{\mu}\right)\min\{m_0,1\}\|u_n\|^2 - \frac{\lambda\varsigma}{\mu}\int_{\mathbb{R}^3}|u_n|^2dx.
\end{aligned}
$$

Since $v_n = \frac{u_n}{\|u_n\|}$, we assert that

$$
c+1 \geq \left(\frac{1}{2\tau}-\frac{1}{\mu}\right)\min\{m_0,1\}\|u_n\|^2 - \frac{\lambda\varsigma}{\mu}\|v_n\|_2^2\|u_n\|^2.
$$

By dividing the both sides by $\|u_n\|^2$ and using the fact that $\|u_n\| \to \infty$ as $n\to\infty$, we obtain

$$
\begin{aligned}
\|v_n\|_2^2 &\geq \frac{\mu}{\lambda\varsigma}\left(\frac{1}{2\tau}-\frac{1}{\mu}\right)\min\{m_0,1\} \\
&= \frac{(\mu-2\tau)}{\lambda\varsigma 2\tau}\min\{m_0,1\}.
\end{aligned}
\tag{2.25}
$$

Dividing the both sides of (2.25) by $\frac{(\mu-2\tau)}{\lambda\varsigma 2\tau}\min\{m_0,1\}$ and then taking the limit of supremum of this inequality as $n\to\infty$, we find the following

$$
1 \leq \frac{\lambda\varsigma 2\tau}{\min\{m_0,1\}(\mu-2\tau)}\limsup_{n\to\infty}\|v_n\|_2^2 = \frac{\lambda\varsigma 2\tau}{\min\{m_0,1\}(\mu-2\tau)}\|v\|_2^2,
$$

which shows that $v(x)\neq 0$. In any case we get contradiction. Hence, $\{u_n\}$ is bounded in E. The proof of Lemma 2.3.1 is completed. □

In order To prove our main result, we need to apply the fountain theorem of Bartsch [11]. To do this, we refer to the following Lemma.

Lemma 2.3.2. *Let H be a reflexive and separable Banach space. Then, there exist $\{e_n\} \subseteq H$ and $\{e_n^*\} \subseteq H^*$ such that*

$$H = \overline{\textit{span}\,\{e_n : n = 1, 2, ...\}}, \quad H^* = \overline{\textit{span}\,\{e_n^* : n = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Let us define $H_n = \text{span}\,\{e_n\}$, $Y_k = \oplus_{n=1}^{k} H_n$ and $X_k = \overline{\oplus_{n=k}^{\infty} H_n}$ for $k \in \mathbb{Z}$.
Now, we recall the fountain theorem under the $(C)_c$-condition.

Theorem 2.3.1. *Assume that H is a real reflexive Banach space, consider an even functional $J \in C^1(H, \mathbb{R})$ fulfills the $(C)_c$-condition. If for each large enough $k \in \mathbb{N}$, there exist $\alpha_k > \beta_k > 0$ such that*
(i) $\rho_k := \inf_{u \in X_k, \|u\| = \beta_k} J(u) \to \infty, \text{ as } k \to \infty.$
(ii) $\delta_k := \max_{u \in Y_k, \|u\| = \alpha_k} J(u) \leq 0.$
Then, J possesses an unbounded sequence of critical values.

Note that E is a separable and reflexive Banach space and define X_k and Y_k as in Lemma 2.3.2. Plainly, due to the assumption (F_2), Lemma 2.3.1, Φ_λ is an even functional fulfills the $(C)_c$-condition. Consequently, it remains to shows that there exist $\alpha_k > \beta_k > 0$ such that (i) and (ii) in Theorem 2.3.1 are verified.

Lemma 2.3.3. *Let $2\tau < r < 2_\sigma^*$ and we denote*

$$\psi_k := \sup_{u \in X_k, \|u\| = 1} \left(\int_{\mathbb{R}^3} |u(x)|^r dx \right).$$

Then, $\psi_k \to 0$ as $k \to \infty$.

Lemma 2.3.4. *Under the assumptions (M_2) and (F_1)-(F_3), the geometry condition in the fountain theorem hold.*

Proof. For any $u \in X_k$, it follows from the assumptions (M_2), (F_1) and the Hölder inequality that

$$\begin{aligned}
\Phi_\lambda(u) =& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u)\,dx \\
\geq& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx \\
&- \lambda \int_{\mathbb{R}^3} |\varrho(x)|u(x)|dx - \lambda \int_{\mathbb{R}^3} \frac{\xi(x)}{r}|u(x)|^r dx \\
\geq& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx \\
&- \lambda \|\varrho\|_2 \|u\|_2 - \frac{\lambda}{r}\|\xi\|_\infty \int_{\mathbb{R}^3} |u(x)|^r dx \\
\geq& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx \\
&- \lambda C_2 \|u\| - \frac{\lambda C_3}{r}\int_{\mathbb{R}^3} |u(x)|^r dx. \\
\geq& \frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\}\|u_n\|^2 - \lambda C_2\|u\| - \frac{\lambda C_3}{r}\psi_k^r \|u\|^r.
\end{aligned}$$

Since $2 < r$ and $\psi_k \to 0$ as $k \to \infty$, we get

$$\beta_k = \left(\frac{\lambda C_3 \psi_k^r}{\min\left\{\frac{m_0}{\tau}, 1\right\}}\right)^{\frac{1}{2-r}} \to \infty \text{ as } k \to \infty.$$

Thus, if $u \in X_k$ and $\|u\| = \beta_k$, then we infer that

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\}\|u\|^2 - \lambda C_2\|u\| - \frac{\lambda C_3}{r}\psi_k^r\|u\|^r \\
&= \left(\frac{1}{2} - \frac{1}{r}\right)\min\left\{\frac{m_0}{\tau}, 1\right\}\beta_k^2 - \lambda C_2 \beta_k.
\end{aligned}$$

Hence, we arrive at

$$\inf_{u \in X_k, \|u\| = \beta_k} \Phi_\lambda(u) \to \infty \text{ as } k \to \infty,$$

which implies (i) in Theorem 2.3.1.

Next, suppose that (ii) in Theorem 2.3.1 is not fulfilled for some k. Then, there exists a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\| > 1 \text{ and } \|u_n\| \to \infty \text{ as } n \to \infty \text{ and } \Phi_\lambda(u_n) > 0. \tag{2.26}$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then clearly $\|v_n\| = 1$. Since dim$Y_k < \infty$, there exists $v \in Y_k \setminus \{0\}$ such that up to subsequence

$$\|v_n - v\| = 0 \quad \text{and} \quad v_n(x) \to v(x) \text{ for a.e. } x \in \mathbb{R}^3 \text{ as } n \to \infty.$$

For all $x \in \Sigma_0 := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Thus, we obtain that $|u_n(x)| \to \infty$ as $n \to \infty$. According to the assumptions (M_1)-(M_2) and (F_3), we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \\
\geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}(1)\|u_n\|^{2\tau} + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}C\|u_n\|^4} \\
\geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|^{2\tau} + \frac{1}{2}C\|u_n\|^{2\tau}} \\
=\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1+\frac{1}{2}C)\|u_n\|^{2\tau}} \\
=\ & \lim_{n\to\infty} \frac{1}{\mathcal{M}(1)+1+\frac{1}{2}C}.\frac{F(x,u_n)}{|u_n(x)|^{2\tau}}.|v_n(x)|^{2\tau} \\
=\ & \infty.
\end{aligned}$$

Since meas$(\Sigma_0) \neq 0$, we obtain

$$\lim_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} dx \geq \infty. \tag{2.27}$$

Consequently, from (2.27) we deduce

$$\begin{aligned}
\Phi_\lambda(u_n) =& \frac{1}{2}\mathcal{M}([u_n]_\sigma^2) + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx - \lambda\int\limits_{\Sigma_0} F(x,u_n)\,dx \\
\leq& \frac{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx}{2}. \\
& \left(1 - 2\lambda\int\limits_{\Sigma_0} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dxdx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} dx\right) \\
& \to -\infty \text{ as } n \to \infty,
\end{aligned}$$

which contradicts (2.26). This complete the proof of Lemma 2.3.4.
With the aid of Theorem 2.3.1, we are ready to demonstrate the existence of a sequence of nontrivial solutions with high energies to the given problem.

Proof of Theorem 2.1.1. From the assumption (F_2), one can easily check that Φ_λ is an even functional. Based on Lemmas 2.3.1, 2.3.4 all conditions of the fountain theorem are verified. Consequently, we assert that the problem (2.1) admits a sequence of nontrivial solutions $\{u_n\}$ in E satisfying $\Phi_\lambda(u_n) \to \infty$ as $n \to \infty$ for all $\lambda > 0$. $\square$

CHAPTER **3**

Existence and multiplicity of solutions for fractional Kirchhoff-Schrödinger-Poisson system with concave-convex nonlinearities

This chapter includes the results of the following research article:
• H.Boutebba, H.Lakhal, and K.Slimani. Nontrivial and infinitely many large energy solutions for fractional Kirchhoff-Schrödinger-Poisson systems with concave-convex nonlinearities. (submitted to an international journal).

This chapter investigates the existence and multiplicity of nontrivial solutions for a class of fractional Kirchhoff-Schrödinger-Poisson systems involving combined effect of concave-convex nonlinearities, in particular when the convex term does not require the well-known Ambrosetti-Rabinowitz condition. To achieve these results, we employ variational tools based on the mountain pass theorem for the existence of at least one nontrivial solution, and the fountain theorem for the existence of a sequence of infinitely many large energy solutions under different assumptions imposed on f and certain suitable assumptions on g, M, V and K.

3.1 Introduction

In the last decades, the investigation into problems of elliptic type involving fractional operators has gained a great deal of attention from authors in many different contexts because they can be substantiated as a model for several physical phenomena arising from optimization, finance, image process, quantum mechanics, game theory, anomalous diffusion in plasma, and probability. We refer the reader to [17, 27, 28, 57, 72, 99] and their references. On the other hand, an increasing amount of interest has been devoted to the study of fractional Sobolev spaces and the corresponding fractional operators, we refer the interested reader to [12, 19, 21, 45, 91] for fundamental properties of fractional Sobolev spaces and the study of fractional operator based on variational approaches and critical point theory.
At the same time, the variational problems of Kirchhoff-type have also been extensively investigated by many researchers motivated by its significant feature and the powerful background in diverse applications in physics and biology, see for instance [5, 39, 69]. With the widespread applications of Kirchhoff-type problems, abundant results on the solvability of these problems have been brought out after the seminal work of Lions [77]. Their most interesting was to investigate the existence and multiplicity of solutions using variational approaches, we refer to [40, 59, 95, 104] and the references therein.
In this context, the present paper discusses the results regarding the existence and multiplicity of nontrivial solutions to a new class of fractional Schrödinger-Kirchhoff-Poisson systems involving the concave-convex nonlinearities of the following form

$$\begin{cases} M([u]_\sigma^2)(-\Delta)^\sigma u + V(x)u + K(x)\phi u = f(x,u) + \lambda g(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

where

$$[u]_\sigma^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\sigma}{2}} u|^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy,$$

$\sigma \in (\frac{3}{4}, 1)$, $\gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, λ is a real parameter, $1 < q < 2$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$, $V : \mathbb{R}^3 \to (0,\infty)$ are continuous functions, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies some conditions and $(-\Delta)^\sigma$ denotes the fractional Laplacian.
In the following of the chapter, we shall assume that the potential functions V, K and Kirchhoff-type function M verify that:
(V): V is a continuous function verifying $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$ where V_0 is a constant, and we have

$$\lim_{|x|\to\infty} V(x) = +\infty.$$

(K) : $K(x) \in L^\infty(\mathbb{R}^3)$ and $K(x) \geq 0$ for all $x \in \mathbb{R}^3$.
(M_1) : The function M verifies $M(s) \geq m_0 > 0$ for all $s \in \mathbb{R}_0^+$ where m_0 is a constant.
(M_2): there exists a positive constant $\tau \in [2, \dfrac{3}{3-2\sigma})$ such that $sM(s) \leq \tau\mathcal{M}(s)$ for any $s \geq 0$,

where $\mathcal{M}(s) = \int\limits_0^s M(\theta)d\theta$.
A typical example for M, due to Kirchhoff, is given for all $s \in \mathbb{R}_0^+$ by

$$M(s) = a + bs^{\tau-1}, \;\; a, b \in \mathbb{R}_0^+, \;\; a + b > 0, \text{ and } \tau \in [2, \frac{3}{3-2\sigma}).$$

If $M(0) > 0$, Kirchhoff problems are said to be nondegenerate, while Kirchhoff problems are called degenerate if $M(0) = 0$. From a physical interpretation, the degenerate case means that the base tension of the string is zero. It is worth mentioning that the assumption (M_1) means that M is nondegenerate, more precisely the condition $M(s) \geq m_0 > 0$ for all $s \in \mathbb{R}_0^+$ is important to assure that the related energy functional of problem fulfills both the geometry structure and the boundedness of a Palais-Smale and the Cerami sequences.
In this recent years, a tremendous amount of attention has been devoted to the generalization of the Kirchhoff model to the fractional case by many researchers. Fiscella and Valdinocci [52] first considered a stationary Kirchhoff equation involving a critical exponent

$$M\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\sigma}}dxdy\right)(-\Delta)^\sigma u = \lambda f(x,u) + |u|^{2_\sigma^*-2}u, \tag{3.2}$$

where M is nondegenerate, by using the mountain pass theorem combined with a suitable truncation argument, the authors obtained the existence of nonnegative solutions for problem (3.2) as sufficiently large λ. From then on, many authors have dedicated themselves to the study of the existence and multiplicity of solutions for several kinds of fractional Kirchhoff elliptic problems using some different approaches under various assumptions on potential and nonlinearity. Li-Rădulescu-Zhang [75] showed the existence of infinitely many solutions for possibly degenerate Kirchhoff-type problem by employing the fountain theorem for the subcritical case and the symmetric mountain pass theorem for the critical case. Indeed, the literature on the study of the existence and multiplicity results related to the nondegenerate setting is quite large and interesting, see for instance [64, 66, 80, 88, 107, 117], otherwise the degenerate setting is treated in [6, 51, 80, 89, 108].
We should mention that the study of a system similar to (3.1) has been motivated by the following Schrödinger-Poisson system

$$\begin{cases}(-\Delta)^\sigma u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3,\\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{3.3}$$

which was widely investigated by numerous researchers using different methods under various assumptions on the potential functions V, K and the nonlinearity f. He and Lei [60] established a multiplicity of property of solutions for (3.3). In our recent work [22], we obtained the existence of infinitely many solutions for (3.3) with $f(x,u) = \lambda f(x,u)$. When $K(x) \equiv 1$ in (3.3), Jin and Yang [63] studied the existence of infinitely many solutions under several assumptions

on f. Kim and Bae [65] considered the existence of infinitely many solutions when $K(x) \equiv 1$ in (3.3). The existence of solutions for (3.3) with critical nonlinearity term is still proved by Meng-Zhang-He [81] and Yu-Zhao-Zhao [112] when the potential satisfies $V(x) \equiv 1$.
On the other hand, The concave-convex-type elliptic problems have been extensively considered since the pioneering work of Ambrosetti-Brezis-Cerami [2] for the following problem

$$\begin{cases} (-\Delta)u = \lambda|u|^{q-2}u + |u|^{r-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < 2 < r < 2^*_\sigma$.
For more elliptic problems involving concave-convex nonlinearities, for instance, we refer the reader to [31, 43, 95, 106] and their references. On the subject of the existence and multiplicity of nontrivial solutions for concave-convex-type elliptic problems driven by fractional non-local operators via vartional methods combined with critical point theorems, the authors in [108] first introduced a multiplicity property of solutions for nonhomogeneous fractional p-Laplacian Kirchhoff type problem. The existence of infinitely many small energy solutions to a class of Schrödinger-Kirchhoff Type Problems involving $r(\cdot)$-Laplacian have been obtained by Kim in [68]. A remarkable related works on fractional problems with concave-convex nonlinearities can be found in [21, 38, 40, 64, 67, 73].
Before stating our main results, let us consider some suitable assumptions on f and g where $F(x,s) = \int\limits_0^s f(x,t)dt$, and let $\tau \in \mathbb{R}$ be given in (M_2).
(F_1) : $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ fulfills the Carathéodory condition.
(F_2) : There exists a nonnegative function $\xi \in L^\infty(\mathbb{R}^3)$ such that for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$

$$|f(x,s)| \le \xi(x)|s|^{p-1}, \quad p \in (2\tau, 2^*_\sigma).$$

(F_3) : $\lim\limits_{|s|\to\infty} \frac{F(x,s)}{|s|^{2\tau}} = \infty$ uniformly for almost all $x \in \mathbb{R}^3$.
(F_4) : There exist $\kappa \ge 2\tau$ and $T > 0$ such that

$$f(x,s)s - \kappa F(x,s) \ge -\varsigma|s|^2 - \eta(x), \quad \forall x \in \mathbb{R}^3, \ \ |s| \ge T,$$

where $\varsigma \ge 0$ and $\eta \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with $\eta(x) \ge 0$.
(F_5) : $F(x,s) \ge 0$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$.
(G) : $g : \mathbb{R}^3 \to \mathbb{R}^+$ and $g(x) \in L^{\frac{2}{2-q}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.
Our main results are the following theorems.

Theorem 3.1.1. *Let $\sigma, \gamma \in (0,1)$ with $\sigma > \frac{3}{4}$. Suppose that (F_1)-(F_5), (M_1)-(M_2), (G), (K) and (V) hold. Then, there exists a positive constant Λ_* such that the problem* (3.1) *admits at least one nontrivial solution in E, provided that $\lambda \in (0, \Lambda_*)$.*

Theorem 3.1.2. *Let $\sigma, \gamma \in (0,1)$ with $\sigma > \frac{3}{4}$. Suppose that (F_1)-(F_4), (M_1)-(M_2), (G), (K) and (V) hold. If $f(x,-s) = -f(x,s)$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$, then the problem* (3.1) *admits a sequence of nontrivial solutions $\{u_n\}$ in E such that $\Phi_\lambda(u_n) \to \infty$ as $n \to \infty$ for all $\lambda > 0$.*

The rest of this chapter is designed as follows. In the forthcoming section, we present some auxiliary results and we set up the functional framework necessary to our problem. In section 3.3, by exploiting the mountain pass theorem-Cerami version, we establish the existence of at least one nontrivial solution for the problem (3.1). In section 3.4, we prove the existence of infinitely many large energy solutions for the problem (3.1) by using the fountain theorem as the main tool.

3.2 Variational framework and preliminary lemmas

In this section, we briefly introduce some useful definitions and technical results which are particularly helpful for the proof of our main results and we state the variational framework of problem (3.1).
For $\sigma \in (0,1)$ and $p \in [1,\infty)$, we define the fractional Sobolev space $W^{\sigma,p}(\mathbb{R}^3)$ as

$$W^{\sigma,p}(\mathbb{R}^3) = \left\{ u \in L^p(\mathbb{R}^3) : u \text{ is measurable and } [u]_{\sigma,p} < \infty \right\},$$

where $[u]_{\sigma,p}$ is the Gagliardo seminorm defined by

$$[u]_{\sigma,p} = \left(\int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|u(x)-u(y)|^p}{|x-y|^{3+p\sigma}} dxdy \right)^{\frac{1}{p}}.$$

The norm of $W^{\sigma,p}(\mathbb{R}^3)$ is

$$\|u\|_{\sigma,p}^p = [u]_{\sigma,p}^p + |u|_p^p,$$

where $|u|_p = \left(\int_{\mathbb{R}^3} |u|^p dx \right)^{\frac{1}{p}}$ is the norm of u in the space $L^p(\mathbb{R}^3)$.
For $p = 2$, the space $W^{\sigma,2}(\mathbb{R}^3)$ is simply denoted $H^\sigma(\mathbb{R}^3)$ with the norm $\|u\|_\sigma^2 = [u]_\sigma^2 + |u|_2^2$.
Furthermore, the fractional homogeneous Sobolev space $D^{\sigma,2}(\mathbb{R}^3)$ is defined as the closure of $C_0^\infty(\mathbb{R}^3)$ under the seminorm

$$\|u\|_{D^{\sigma,2}}^2 = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy = [u]_\sigma^2.$$

It follows from Theorem 6.5 in [45] that the space $D^{\sigma,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2_\sigma^*}(\mathbb{R}^3)$, and there exists a best constant $C_\sigma > 0$ such that

$$C_\sigma = \inf_{u \in D^{\sigma,2}(\mathbb{R}^3) \setminus \{0\}} \frac{[u]_\sigma^2}{\|u\|_{2_\sigma^*}^2} \tag{3.4}$$

On the other hand, on account of the coercive potential $V(x)$, our working space is defined as follows

$$E=\left\{u\in H^{\sigma}\left(\mathbb{R}^3\right):\int\limits_{\mathbb{R}^3}V(x)|u|^2dx<\infty\right\},$$

endowed with the following inner product

$$\langle u,w\rangle=\int\limits_{\mathbb{R}^3}((-\Delta)^{\frac{\sigma}{2}}u(-\Delta)^{\frac{\sigma}{2}}w+V(x)uw)dx,$$

and corresponding norm

$$\|u\|^2=[u]_{\sigma}^2+\int\limits_{\mathbb{R}^3}V(x)|u|^2dx.$$

Consequently, we have $(E,\|\cdot\|)$ is a uniformly convex Banach space (i.e. reflexive). Considering the assumption (V) and (3.4), it yields the following embedding theorem.

Lemma 3.2.1. *([88], Lemma 1) Let the assumption (V) hold. Then, E is continuously embedded into $H^{\sigma}(\mathbb{R}^3)$. Further, E is continuously embedded into $L^r(\mathbb{R}^3)$ for all $r\in[2,2_{\sigma}^*]$.*

Therefore, E is continuously embedded in $L^r(\mathbb{R}^3)$ for all $r\in[2,2_{\sigma}^*]$, i.e., there exists a positive constant C_r such that

$$|u|_r\leq C_r\|u\|. \tag{3.5}$$

Thanks to the coercivity of $V(x)$, Lemma 3.2.1 and (3.4) (see Lemma 2.3 in [80] for the proof), we obtain the following compactness result.

Lemma 3.2.2. *[80] The space E is compactly embedded in $L^r(\mathbb{R}^3)$ for all $r\in[2,2_{\sigma}^*)$.*

Since $2\gamma+4\sigma>3$, there holds $\frac{12}{3+2\gamma}<\frac{6}{3-2\sigma}$. Thus, E is continuously embedded in $L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)$. For any $u\in E$ and $w\in D^{\gamma,2}(\mathbb{R}^3)$, using the Hölder inequality, (K) and (3.4)-(3.5), one has

$$\begin{aligned}\int\limits_{\mathbb{R}^3}K(x)u^2wdx &\leq \|K\|_{\infty}\|u\|^2_{\frac{12}{3+2\gamma}}\|w\|_{2_{\gamma}^*}\\ &\leq C_1\|u\|^2\|w\|_{D^{\gamma,2}}.\end{aligned} \tag{3.6}$$

For any $u\in E$, one can use the Lax-Milgram theorem, and then there exists a unique $\phi_u^{\gamma}\in D^{\gamma,2}(\mathbb{R}^3)$ such that

$$\int\limits_{\mathbb{R}^3}(-\Delta)^{\frac{\gamma}{2}}\phi_u^{\gamma}(-\Delta)^{\frac{\gamma}{2}}wdx=\int\limits_{\mathbb{R}^3}K(x)u^2wdx,\quad\forall w\in D^{\gamma,2}\left(\mathbb{R}^3\right). \tag{3.7}$$

Therefore, ϕ_u^γ is a weak solution of

$$(-\Delta)^\gamma \phi_u^\gamma = K(x)u^2, \quad x \in \mathbb{R}^3,$$

and it can be expressed by

$$\phi_u^\gamma(x) = c_\gamma \int\limits_{\mathbb{R}^3} \frac{K(x)u^2(y)}{|x-y|^{3-2\gamma}} dy \geq 0, \quad \forall x \in \mathbb{R}^3, \tag{3.8}$$

which is called γ-Riesz potential, where

$$c_\gamma = \frac{\Gamma\left(\frac{3-2\gamma}{2}\right)}{2^{2\gamma}\pi^{\frac{3}{2}}\Gamma(\gamma)}.$$

Taking $w = \phi_u^\gamma$ in (3.6) and (3.7), we obtain

$$\|\phi_u^\gamma\|_{D^{\gamma,2}} \leq C_2 \|u\|^2. \tag{3.9}$$

Consequently, we have that

$$\int\limits_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx \leq C \|u\|^4. \tag{3.10}$$

By replacing ϕ_u^γ in (3.1), the problem (3.1) reduces to the following equivalent form

$$M([u]_\sigma^2)(-\Delta)^\sigma u + V(x)u + K(x)\phi_u^\gamma u = f(x,u) + \lambda g(x)|u|^{q-2}u, \quad x \in \mathbb{R}^3. \tag{3.11}$$

The associated energy functional $\Phi_\lambda : E \to \mathbb{R}$ is given by

$$\Phi_\lambda(u) = \frac{1}{2}(\mathcal{M}([u]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u|^2 dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \int\limits_{\mathbb{R}^3} F(x,u)dx - \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} g(x)|u|^q dx \tag{3.12}$$

By Lemmas 2 and 3 in [88] (see also [64] Lemmas 3.2, 3.3), we have that $\Phi_\lambda \in C^1(E,\mathbb{R})$. Moreover, any critical points of Φ_λ are exactly a weak solutions of (3.11), and its Fréchet derivative for any $u, w \in E$ is given by

$$< \Phi_\lambda'(u), w > = \int\limits_{\mathbb{R}^3} M \quad ([u]_\sigma^2)(-\Delta)^{\frac{\sigma}{2}}u(-\Delta)^{\frac{\sigma}{2}}w dx + \int\limits_{\mathbb{R}^3} V(x)uw dx + \int\limits_{\mathbb{R}^3} K(x)\phi_u^\gamma uw dx$$
$$- \int\limits_{\mathbb{R}^3} f(x,u)w dx - \lambda \int\limits_{\mathbb{R}^3} g(x)|u|^{q-2}uw dx,$$

Lemma 3.2.3. *[88, 96, 102] Φ_λ' is of type (S_+), that is, if for every sequence $\{u_n\} \subset E$ satisfying $u_n \rightharpoonup u$ in E and $\limsup\limits_{n\to\infty}\langle \Phi_\lambda'(u_n) - \Phi_\lambda'(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in E as $n \to \infty$.*

Concerning the function ϕ_u^γ, its main properties are listed as follows.

Lemma 3.2.4. *[81, 101, 109] For any $u \in H^\sigma(\mathbb{R}^3)$ with $2\gamma + 4\sigma > 3$, we have*

1. *$\phi_{\pi u}^\gamma = \pi^2 \phi_u^\gamma$ for all $\pi \in \mathbb{R}$.*
2. *If $u_n \rightharpoonup u$ in $H^\sigma(\mathbb{R}^3)$, then $\phi_{u_n}^\gamma \rightharpoonup \phi_u^\gamma$ in $D^{\gamma,2}(\mathbb{R}^3)$.*
3. *If $u_n \rightharpoonup u$ in E, then*
$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n w dx \to \int_{\mathbb{R}^3} K(x)\phi_u^\gamma u w dx \qquad \forall w \in E,$$
and
$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx \to \int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx.$$

3.3 Proof of Theorem 3.1.1.

In this section, we prove that the problem (3.1) has at least one nontrivial solution. We first start by demonstrating that the functional Φ_λ possesses the mountain pass geometry. Then, we prove that Φ_λ verifies the (C)$_c$-condition which is a key lemma for the proof of our main results.

Lemma 3.3.1. *Assume that the assumptions (F_1)-(F_3), (F_5), (G), (M_1)-(M_2), (V) and (K) hold. Then, we have the following:*

1. *There exists a positive constant $\Lambda_* > 0$ such that for any $\lambda \in (0, \Lambda_*)$, we can choose $\alpha > 0$ and $0 < \delta < 1$ such that $\Phi_\lambda(u) \geq \alpha > 0$ for all $u \in E$ with $\|u\| = \delta$.*
2. *There exists $\varphi > 0$ in E, such that $\Phi_\lambda(t\varphi) \to -\infty$ as $t \to \infty$.*
3. *There exists $\psi > 0$ in E, such that $\Phi_\lambda(t\psi) < 0$ for all $t \to 0^+$.*

Proof. We first prove statement 1. Suppose that $\|u\| < 1$, then it follows from the assumptions (G), (F_2) and (3.5) that

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\left(\mathcal{M}([u]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u|^2 dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\
&\geq \frac{1}{2}\left(\mathcal{M}([u]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u|^2 dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \frac{\|\xi\|_\infty}{p}\|u\|_p^p - \frac{\lambda}{q}\|g\|_{\frac{2}{2-q}}\|u\|_2^q \\
&\geq \frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\}\|u\|^2 - \frac{C_p}{p}\|\xi\|_\infty\|u\|^p - \frac{\lambda C_2}{q}\|g\|_{\frac{2}{2-q}}\|u\|^q \\
&\geq \left(\frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\} - \frac{C_3}{p}\|u\|^{p-2} - \frac{\lambda C_4}{q}\|u\|^{q-2}\right)\|u\|^2. \qquad (3.13)
\end{aligned}$$

Let us define $\Psi_\lambda(s) : (0,\infty) \to \mathbb{R}$ as follows

$$\Psi_\lambda(s) = \frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\} - \frac{C_3}{p}s^{p-2} - \frac{\lambda C_4}{q}s^{q-2}. \tag{3.14}$$

Thus,

$$\Psi_\lambda^{'}(s) = -\frac{C_3}{p}(p-2)s^{p-3} - \frac{\lambda C_4}{q}(q-2)s^{q-3}.$$

$\Psi_\lambda^{'}(s) = 0$, implies that

$$-\frac{C_3}{p}(p-2)s^{p-3} = \frac{\lambda C_4}{q}(q-2)s^{q-3}.$$

By dividing the both side by s^{q-3} we derive

$$s^{p-q} = \frac{\lambda p C_4(2-q)}{qC_3(p-2)}.$$

Since $1 < q < 2 < p$, we get

$$s_0 = \left(\frac{\lambda p C_4(2-q)}{qC_3(p-2)}\right)^{\frac{1}{p-q}}.$$

On the other hand, we have

$$\Psi_\lambda^{''}(s) = -\frac{C_3}{p}(p-2)(p-3)s^{p-4} - \frac{\lambda C_4}{q}(q-2)(q-3)s^{q-4}, \quad s > 0,$$

Thus,

$$\Psi_\lambda^{''}(s_0) = -\frac{C_3}{p}(p-2)(p-3)s_0^{p-4} - \frac{\lambda C_4}{q}(q-2)(q-3)s_0^{q-4} < 0.$$

Hence, we deduce that $\max\limits_{s\in\mathbb{R}^+} \Psi_\lambda(s) = \Psi_\lambda(s_0)$.
Thus, there exists a constant $\Lambda_* > 0$ such that for any $\lambda \in (0, \Lambda_*)$, we can choose $\alpha > 0$ and sufficiently small $\delta > 0$, such that $\Phi_\lambda(u) \geq \alpha > 0$ for all $u \in E$ with $\|u\| = \delta$.
In order to prove the statement 2, we choose $\varphi \in E$ with $\varphi > 0$. From the Fatou lemma and the assumption (F_3), it follows that

$$\lim_{t\to\infty}\int\limits_{\mathbb{R}^3} \frac{F(x,t\varphi)}{|t|^{2\tau}}dx \geq \int\limits_{\mathbb{R}^3} \lim_{t\to\infty}\frac{F(x,t\varphi)}{|t\varphi|^{2\tau}}|\varphi|^{2\tau}dx = \infty. \tag{3.15}$$

Therefore, owing to the assumption (M_2) and the relations (3.10), (3.15)

$$\begin{aligned}\Phi_\lambda(t\varphi) =& \frac{1}{2}\mathcal{M}([t\varphi]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|t\varphi|^2dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{t\varphi}^\gamma(t\varphi)^2dx \\ &- \int_{\mathbb{R}^3} F(x,t\varphi)dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|t\varphi|^q dx \\ \leq& \frac{1}{2}\mathcal{M}(1)t^{2\tau}[\varphi]_\sigma^{2\tau} + \frac{1}{2}t^2\int_{\mathbb{R}^3} V(x)|\varphi|^2dx + \frac{1}{4}t^4C\|\varphi\|^4 - \int_{\mathbb{R}^3} F(x,t\varphi)dx \\ \leq& t^{2\tau}\left(\frac{1}{2}\mathcal{M}(1)[\varphi]_\sigma^{2\tau} + \frac{1}{2}t^{2-2\tau}\int_{\mathbb{R}^3} V(x)|\varphi|^2dx + \frac{1}{4}t^{4-2\tau}C\|\varphi\|^4 - \int_{\mathbb{R}^3}\frac{F(x,t\varphi)}{|t|^{2\tau}}dx\right) \\ \to& -\infty \ \text{ as } \ t\to\infty,\end{aligned}$$

because $\tau \in [2, \frac{3}{3-2\sigma})$. Thus, Φ_λ is unbounded from below.
Finally, it remains to prove the statement 3. Taking $\psi \in E$ with $\psi > 0$, for t small enough, by the assumptions (F_5), (M_1) and (G) we derive

$$\begin{aligned}\Phi_\lambda(t\psi) =& \frac{1}{2}\mathcal{M}([t\psi]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|t\psi|^2dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{t\psi}^\gamma(t\psi)^2dx \\ &- \int_{\mathbb{R}^3} F(x,t\psi)dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|t\psi|^q dx \\ \leq& \frac{\max\{\sup_{\mu\in[0,1]} M(\mu), 1\}}{2}t^{2\tau}\|\psi\|^2 + \frac{1}{4}t^4C\|\psi\|^4 - \frac{\lambda t^q C_4}{q}\|\psi\|^q \\ <& 0,\end{aligned}$$

thanks to the fact that $q < 2 < 4 \leq 2\tau$. This completes the proof of Lemma 3.3.1. □

Lemma 3.3.2. *Assume that $\sigma, \gamma \in (0,1)$ and $2\gamma + 4\sigma > 3$. If the assumptions (F_1)-(F_4), (M_1)-(M_2) and (V) are satisfied. Then, the functional Φ_λ fulfills the (C)$_c$-condition for all $\lambda > 0$.*

Proof. Let $\{u_n\} \subset E$ be a (C)$_c$-sequence of functional Φ_λ, i.e.

$$\Phi_\lambda(u_n) \to c \quad \text{and} \quad \|\Phi_\lambda'(u_n)\|_*(1+\|u_n\|) \to 0 \ \text{ as } \ n\to\infty, \tag{3.16}$$

which means

$$c = \Phi_\lambda(u_n) + o_n(1) \ \text{ and } \ \langle \Phi_\lambda'(u_n), u_n\rangle = o_n(1). \tag{3.17}$$

If a sequence $\{u_n\} \subset E$ is bounded in E, it follows that $u_n \to u$ in E (since the operator Φ_λ' verifies the (S_+) condition and E is reflexive). Hence, it suffices to ensure that the sequence $\{u_n\}$ is bounded in E, arguing by contradiction. Suppose that the sequence $\{u_n\}$ is unbounded in E.

Then, we may assume that

$$\|u_n\| > 1 \text{ and } \|u_n\| \to \infty \text{ as } n \to \infty. \tag{3.18}$$

Define a sequence $\{v_n\}$ by $v_n = \frac{u_n}{\|u_n\|}$. Then, obviously we have $\{v_n\} \subset E$ and $\|v_n\| = 1$. Then, going to a subsequence $\{v_n\}$, still denoted by $\{v_n\}$, we obtain $v_n \rightharpoonup v$ in E as $n \to \infty$. From Lemma 3.2.2, we deduce

$$\begin{aligned} v_n(x) &\to v(x) \quad \text{a.e. in } \mathbb{R}^3 \\ v_n &\to v \quad \text{in } L^p(\mathbb{R}^3) \text{ for } 2 \leq p < 2^*_\sigma \end{aligned} \tag{3.19}$$

as $n \to \infty$.
Denote $\Sigma_0 = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, then meas$(\Sigma_0) > 0$. Due to the relations (3.12) and (3.17), one has

$$\begin{aligned} c &= \Phi_\lambda(u_n) + o_n(1) \\ &= \frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \int_{\mathbb{R}^3} F(x,u_n)dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u_n|^q dx + o_n(1) \end{aligned} \tag{3.20}$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, from (2.20) we infer that

$$\begin{aligned} \int_{\mathbb{R}^3} F(x,u_n)dx =& \frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx\right) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx \\ & - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u_n|^q dx - c + o_n(1) \\ \geq& \frac{M([u_n]_\sigma^2)[u_n]_\sigma^2}{2\tau} + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{4}C\|u_n\|^4 - \frac{\lambda}{q}\|g\|_{\frac{2}{2-q}}\|u_n\|_2^q - c + o_n(1) \\ \geq& \frac{\min\left\{\frac{m_0}{\tau}, 1\right\}}{2}\|u_n\|^2 + \frac{1}{4}C\|u_n\|^4 - \frac{\lambda C_2}{q}\|g\|_{\frac{2}{2-q}}\|u_n\|^q - c + o_n(1) \\ \to& \infty \text{ as } n \to \infty, \end{aligned} \tag{3.21}$$

thanks to $q < 2$.

Furthermore, by assumption (G) we obtain

$$\begin{aligned}\Phi_\lambda(u_n) =&\frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2 dx\right) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \int\limits_{\mathbb{R}^3} F(x,u_n)dx \\ &- \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} g(x)|u_n|^q dx \\ \leq&\frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2 dx\right) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \int\limits_{\mathbb{R}^3} F(x,u_n)dx.\end{aligned}$$

Hence, one has

$$\frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2 dx\right) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx \geq \int\limits_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1). \tag{3.22}$$

In accordance with assumption (F_3), there is $u_0 > 1$ such that $F(x,u) > |u|^{2\tau}$ for all $x \in \mathbb{R}^3$ and $|u| > u_0$. By means of the assumptions (F_1) and (F_2), there exists a constant $C > 0$ such that for all $|F(x,u)| < C$, $\forall\ (x,u) \in \mathbb{R}^3 \times [-u_0, u_0]$.
Therefore, we have $F(x,u) \geq C_0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ and for some $C_0 \in \mathbb{R}$, and thus

$$\frac{F(x,u_n) - C_0}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \geq 0, \tag{3.23}$$

for all $x \in \mathbb{R}^3$ and for all $n \in \mathbb{N}$.
By the convergence (3.19) we have $|u_n(x)| = |v_n(x)|.\,\|u_n\| \to \infty$ as $n \to \infty$ for all $x \in \Sigma_0$. It follows from (M_1)-(M_2), (F_3) and equations (3.10), (3.18), (3.21) that for all $x \in \Sigma_0$

$$\begin{aligned}&\lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \\ \geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}(1)\|u_n\|^{2\tau} + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}C\|u_n\|^4} \\ \geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|^{2\tau} + \frac{1}{2}C\|u_n\|^{2\tau}} \\ =\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1+\frac{1}{2}C)\|u_n\|^{2\tau}} \\ =\ & \lim_{n\to\infty} \frac{1}{\mathcal{M}(1)+1+\frac{1}{2}C}.\frac{F(x,u_n)}{|u_n(x)|^{2\tau}}.|v_n(x)|^{2\tau} \\ =\ & \infty.\end{aligned} \tag{3.24}$$

According to (3.10), (3.21)-(3.24) and the Fatou lemma, we deduce that

$$
\begin{aligned}
1 &= \liminf_{n\to\infty} \frac{\int\limits_{\mathbb{R}^3} F(x,u_n)dx}{\int\limits_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1)} \\
&\geq \liminf_{n\to\infty} \frac{\int\limits_{\mathbb{R}^3} F(x,u_n)dx}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} \\
&\geq \liminf_{n\to\infty} \frac{\int\limits_{\Sigma_0} F(x,u_n)dx}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} \\
&\quad - \limsup_{n\to\infty} \int\limits_{\Sigma_0} \frac{C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx} \\
&= \liminf_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n) - C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\left\|u_n\right\|^4}dx \\
&\geq \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n) - C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\left\|u_n\right\|^4}dx \\
&= \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\left\|u_n\right\|^4}dx \\
&\quad - \int\limits_{\Sigma_0} \limsup_{n\to\infty} \frac{C_0}{\frac{1}{2}(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx) + \frac{1}{4}C\left\|u_n\right\|^4}dx \\
&= \infty,
\end{aligned} \tag{3.25}
$$

which is a contradiction. Thus, $v(x) = 0$ for almost all $x \in \mathbb{R}^3$.
Now, we consider the case that $v(x) = 0$ in $\mathbb{R}^3$.
Since $V(x) \to \infty$ as $|x| \to \infty$, by similar proof as in [73] we achieve

$$
\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{\kappa}\right)\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx\right) - C_6 \int\limits_{|u_n|\leq T} (|u_n|^2 + \xi(x)|u_n|^p)dx \\
&\geq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\kappa}\right)\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx\right) - C_7,
\end{aligned}
$$

where C_6 and C_7 are positive constants.

Thus, it follows from the assumptions (M_1)-(M_2) and (F_2)-(F_4) that for n large enough

$$
\begin{aligned}
c+1 \geq\ & \Phi_\lambda(u_n) - \frac{1}{\kappa}\left\langle \Phi'_\lambda(u_n), u_n \right\rangle \\
=& \frac{1}{2}\mathcal{M}([u_n]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \int_{\mathbb{R}^3} F(x,u_n)dx \\
& - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u_n|^q dx - \frac{1}{\kappa}M([u_n]_\sigma^2)[u_n]_\sigma^2 - \frac{1}{\kappa}\int_{\mathbb{R}^3} V(x)|u_n|^2dx - \frac{1}{\kappa}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx \\
& + \frac{1}{\kappa}\int_{\mathbb{R}^3} f(x,u_n)u_n dx + \frac{\lambda}{\kappa}\int_{\mathbb{R}^3} g(x)|u_n|^q dx \\
=& \frac{1}{2}\mathcal{M}([u_n]_\sigma^2) - \frac{1}{\kappa}M([u_n]_\sigma^2)[u_n]_\sigma^2 + \left(\frac{1}{2}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx \\
& + \left(\frac{1}{4}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx + \int_{\mathbb{R}^3}\left(\frac{f(x,u_n)u_n}{\kappa} - F(x,u_n)\right)dx \\
& - \lambda\left(\frac{1}{q}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} g(x)|u_n|^q dx \\
\geq & \left(\frac{1}{2\tau}-\frac{1}{\kappa}\right)M([u_n]_\sigma^2)[u_n]_\sigma^2 + \left(\frac{1}{2}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx \\
& + \int_{|u_n|>T}\left(\frac{f(x,u_n)u_n}{\kappa} - F(x,u_n)\right)dx - C_6\int_{|u_n|\leq T}(|u_n|^2+\xi(x)|u_n|^p)dx \\
& - \lambda\left(\frac{1}{q}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} g(x)|u_n|^q dx \\
\geq & \left(\frac{1}{2\tau}-\frac{1}{\kappa}\right)m_0[u_n]_\sigma^2 + \left(\frac{1}{2\tau}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx \\
& - \frac{1}{\kappa}\int_{\mathbb{R}^3}\left(\varsigma|u_n|^2+\eta(x)\right)dx - C_7 - \lambda\left(\frac{1}{q}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} g(x)|u_n|^q dx \\
\geq & \frac{1}{2}\left(\frac{1}{2\tau}-\frac{1}{\kappa}\right)\left(m_0[u_n]_\sigma^2 + \int_{\mathbb{R}^3} V(x)|u_n|^2dx\right) \\
& - \frac{1}{\kappa}\int_{\mathbb{R}^3}\left(\varsigma|u_n|^2+\eta(x)\right)dx - C_7 - \lambda\left(\frac{1}{q}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3} g(x)|u_n|^q dx \\
\geq & \frac{1}{2}\left(\frac{1}{2\tau}-\frac{1}{\kappa}\right)\min\{m_0,1\}\|u_n\|^2 - \frac{\varsigma}{\kappa}\int_{\mathbb{R}^3}|u_n|_2^2dx - \frac{1}{\kappa}\|\eta\|_1 \\
& - \lambda\left(\frac{1}{q}-\frac{1}{\kappa}\right)\|g\|_{\frac{2}{2-q}}\|u_n\|_2^q - C_7.
\end{aligned}
$$

Since $\|u_n\| \to \infty$ as $n \to \infty$ and $v_n = \frac{u_n}{\|u_n\|}$, we deduce that

$$\|v_n\|_2^2 \geq \frac{\kappa - 2\tau}{4\tau\varsigma} \min\{m_0, 1\}.$$

Dividing the both sides by $\frac{(\kappa-2\tau)}{4\varsigma\tau}\min\{m_0, 1\}$ and then taking the limit of supremum of this inequality as $n \to \infty$, we find the following

$$1 \leq \frac{4\tau\varsigma}{\min\{m_0, 1\}(\kappa - 2\tau)} \limsup_{n\to\infty} \|v_n\|_2^2 = \frac{4\tau\varsigma}{\min\{m_0, 1\}(\kappa - 2\tau)} \|v\|_2^2,$$

which means that $v(x) \neq 0$. Hence, $\{u_n\}$ is bounded in E. In conclusion, the proof of Lemma 3.3.2 is complete.
Now, using Lemmas 3.3.1, 3.3.2 to obtain our first main results in this chapter.
Proof of Theorem 3.1.1. First, note that $\Phi_\lambda(0) = 0$. From Lemma 3.3.1, there exists a positive number Λ_* such that the mountain pass geometric condition is verified for any $\lambda \in (0, \Lambda_*)$. Furthermore, by Lemma 3.3.2, the functional Φ_λ fulfills the $(C)_c$-condition. Subsequently, thanks to the mountain pass theorem, we derive that Φ_λ possesses a critical point $u_0 \in E$ with $\Phi_\lambda(u_0) = \bar{\jmath}$ and $\Phi'_\lambda(u_0) = 0$, which in turn proves the existence of nontrivial solution for problem (3.1) with positive energy. □

3.4 Proof of Theorem 3.1.2

In this section, we will give the proof of Theorem 3.1.2 which is mainly based on the fountain theorem.

Lemma 3.4.1. *Let H be a separable and reflexive Banach space. Then, there exist $\{e_n\} \subseteq H$ and $\{e_n^*\} \subseteq H^*$ such that*

$$H = \overline{span\{e_n : n = 1, 2, ...\}}, \quad H^* = \overline{span\{e_n^* : n = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $H_n = \text{span}\{e_n\}$, $Y_k = \oplus_{n=1}^k H_n$ and $X_k = \overline{\oplus_{n=k}^\infty H_n}$ for $k \in \mathbb{Z}$.

Theorem 3.4.1. *([11]) Assume that H is a real reflexive Banach space, consider an even functional $J \in C^1(H, \mathbb{R})$ fulfills the $(C)_c$-condition. If for each sufficiently large $k \in \mathbb{N}$, there exist $\alpha_k > \beta_k > 0$ such that*
(i) $\rho_k := \inf_{u\in X_k, \|u\|=\beta_k} J(u) \to \infty$, *as* $k \to \infty$.
(ii) $\delta_k := \max_{u\in Y_k, \|u\|=\alpha_k} J(u) \leq 0$.
Then, J has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset H$ such that $J'(u_n) = 0$ and $J(u_n) \to \infty$ as $n \to \infty$.

Note that E is a separable and reflexive Banach space and define X_k and Y_k as in Lemma 3.4.1. Plainly, by $f(x,-s) = -f(x,s)$ holds for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$ and Lemma 3.3.2, the functional Φ_λ is an even functional fulfills the $(\mathrm{C})_c$-condition for all $\lambda > 0$. Consequently, it remains to demonstrate that there exist $\alpha_k > \beta_k > 0$ such that the statements (i) and (ii) in Theorem 3.4.1 are verified.

Lemma 3.4.2. *Let $2\tau < p < 2^*_\sigma$ and we denote*

$$\psi_k := \sup_{u \in X_k, \|u\|=1} \|u\|_p.$$

Then, we assert $\psi_k \to 0$ as $k \to \infty$.

Lemma 3.4.3. *Under the assumptions (F_1)-(F_3), (M_2) and (G), the geometry condition in the fountain theorem hold.*

Proof. For any $u \in X_k$, it follows from the assumptions (F_1), (M_2), (G), the Hölder inequality and Lemma 3.4.2 that

$$\begin{aligned}
\Phi_\lambda(u) =& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\
\geq& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} \frac{|\xi(x)|}{p}|u(x)|^p dx \\
&- \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u(x)|^q dx \\
\geq& \frac{1}{2}\mathcal{M}([u]_\sigma^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \frac{\|\xi\|_\infty}{p}\int_{\mathbb{R}^3} |u(x)|^p dx \\
&- \frac{\lambda}{q}\|g\|_{\frac{2}{2-q}} \|u\|_2^q \\
\geq& \frac{1}{2\tau} m_0 [u]_\sigma^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \frac{C_8}{p}\psi_k^p \|u\|^p - \lambda C_9 \|u\|^q \\
\geq& \frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\} \|u\|^2 - \frac{C_8}{p}\psi_k^p \|u\|^p - \lambda C_9 \|u\|^q,
\end{aligned}$$

where C_8 and C_9 are positive constants. Since $2 < 2\tau < p$ and $\psi_k \to 0$ as $k \to \infty$, we derive that

$$\beta_k = \left(\frac{C_8 \psi_k^p}{\min\left\{\frac{m_0}{\tau}, 1\right\}}\right)^{\frac{1}{2-p}} \to \infty \text{ as } k \to \infty.$$

Consequently, for any $u \in X_k$ with $\|u\| = \beta_k$ we obtain

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\min\left\{\frac{m_0}{\tau}, 1\right\}\|u\|^2 - \frac{C_8}{p}\psi_k^p\|u\|^p - \lambda C_9\|u\|^q \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{m_0}{\tau}, 1\right\}\beta_k^2 - \lambda C_9 \beta_k^q.\end{aligned}$$

Note that $p > 2\tau > 2$, then we get

$$\inf_{u \in X_k, \|u\| = \beta_k} \Phi_\lambda(u) \to \infty \text{ as } k \to \infty,$$

which implies (i) in Theorem 3.4.1.
Now, arguing by contradiction to the statement (ii) in Theorem 3.4.1. Then, let us suppose that this is false for some k. Then, we can find a sequence $\{u_n\}$ in Y_k such that

$$\|u_n\| > 1 \text{ and } \|u_n\| \to \infty \text{ as } n \to \infty \text{ and } \Phi_\lambda(u_n) > 0. \tag{3.26}$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then it is clear that $\|v_n\| = 1$. Since, dim$Y_k < \infty$, there is $v \in Y_k \setminus \{0\}$ such that up to subsequence

$$\|v_n - v\| = 0 \quad \text{and} \quad v_n(x) \to v(x) \text{ for a.e. } x \in \mathbb{R}^3 \text{ as } n \to \infty.$$

For all $x \in \Sigma_0 := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, we get $|u_n(x)| \to \infty$ as $n \to \infty$. According to the assumptions (M_1)-(M_2) and (F_3), we deduce that

$$\begin{aligned}&\lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \\ \geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}(1)\|u_n\|^{2\tau} + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}C\|u_n\|^4} \\ \geq\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|^{2\tau} + \frac{1}{2}C\|u_n\|^{2\tau}} \\ =\ & \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1+\frac{1}{2}C)\|u_n\|^{2\tau}} \\ =\ & \lim_{n\to\infty} \frac{1}{\mathcal{M}(1)+1+\frac{1}{2}C}.\frac{F(x,u_n)}{|u_n(x)|^{2\tau}}.|v_n(x)|^{2\tau} = \infty.\end{aligned}$$

Since meas$(\Sigma_0) \neq 0$, we obtain

$$\lim_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} dx \geq \infty. \tag{3.27}$$

Combining this with assumption (G), we get

$$\begin{aligned}
\Phi_\lambda(u_n) =& \frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx\right) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx - \int\limits_{\Sigma_0} F(x,u_n)dx \\
&- \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} g(x)|u_n|^q dx \\
\leq& \frac{1}{2}\left(\mathcal{M}([u_n]_\sigma^2) + \int\limits_{\mathbb{R}^3} V(x)|u_n|^2dx\right) + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx - \int\limits_{\Sigma_0} F(x,u_n)dx \\
\leq& \frac{\mathcal{M}([u_n]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx}{2} \cdot \\
&\left(1 - 2\int\limits_{\Sigma_0} \frac{F(x,u_n)}{\mathcal{M}([u_n]_\sigma^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2dxdx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2dx}dx\right) \\
&\to -\infty \text{ as } n\to\infty,
\end{aligned}$$

which is a contradiction with (3.26). This shows that Φ_λ fulfills the condition (ii) in Theorem 3.4.1. This finishes the proof.

With the aid of Theorem 3.4.1, we are ready to demonstrate the existence of a sequence of nontrivial solutions with large energy to the given problem.

Proof of Theorem 3.1.2. From $f(x,-s) = -f(x,s)$ holds for all $(x,s)\in\mathbb{R}^3\times\mathbb{R}$, one can easily check that Φ_λ is an even functional. Further, in view of Lemma 3.3.2 and Lemma 3.4.3, the functional Φ_λ fulfills all conditions of Theorem 3.4.1. As a direct consequence, we assert that the problem (3.1) admits a sequence of nontrivial solutions $\{u_n\}$ in E satisfying $\Phi_\lambda(u_n)\to\infty$ as $n\to\infty$ for all $\lambda>0$. □

CHAPTER 4

Infinitely many high energy solutions to a class of nonlinear fractional Schrödinger-Poisson systems

This chapter includes the results of the following research article:

• H.Boutebba, H.Lakhal, and K.Slimani. (2024). Infinitely many distributional solutions to a general kind of nonlinear fractional Schrödinger-Poisson systems. The Journal of Analysis, 32(2), 1079-1091.

In this chapter, we consider a general class of nonlinear fractional Schrödinger-Poisson systems. Under certain appropriate assumptions imposed on nonlinearity function f, and potentials functions V and K, we prove the existence of infinitely many high energy solutions for any $\lambda > 0$. The main achievement of the present chapter consists in proving the Cerami condition and the conditions of the fountain theorem.

4.1 Introduction

This chapter focus on the following general class of nonlinear fractional Schrödinger-Poisson systems

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + K(x)\phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.1}$$

where $\sigma, \gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, $\lambda \in \mathbb{R}_+$, and $(-\Delta)^\sigma$ stands for the fractional Laplacian and the function $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. In [45], Di Nezza-Palatucci-Valdinoci have showed that $(-\Delta)^\sigma$ becomes the standard Laplacian $(-\Delta)$ as $\sigma \to 1$. In fact, this fractional operator naturally arises in different contexts, such as the description of several phenomena in finance, quantum mechanic, probability and in physics. For further details about this fractional operator, see for instance [19, 44, 45, 78].
When $\sigma = \gamma = \lambda = 1$ in (4.1), Benci and Fortunato [16] firstly proposed the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.2}$$

to describe the interaction of the charged particle for nonlinear Schrödinger equations with an electrostatic field. There are a plenty of works deal with system (4.2), we refer the readers to [4, 7, 33, 76, 113, 114] for more details about the physical backgrounds and the existence results related to systems like (4.2). Among them, we pay special attention to [114] where Zhang-Nui-Han studied system (4.2) with $V(x) \equiv 1$ and $f(x,u) = f(x)|u|^{p-1}u + h(x)$ under suitable assumptions on f, h and k where $3 < p < 5$. At least four positive solutions obtained using variational tools combined with Lusternik–Schnirelmann category.
On the other side, the study of the existence of distributional solutions, multiple solutions, and infinitely many solutions for nonlinear Schrödinger-Poisson systems driven by fractional operators has been vigorously investigated in the past few years, see for instance [22, 75]. In particular, Li-Rădulescu-Zhang in [75] studied the existence of infinitely many solutions for the following fractional Kirchhoff-Schrödinger-Poisson type system

$$\begin{cases} M([u]_\sigma^2)(-\Delta)^\sigma u + V(x)u + \phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

via the fountain theorem in the subcritical case and the symmetric mountain pass theorem established by Kajikiya in the critical case where $[u]_\sigma$ is the Gagliardo norm of u and M is the Kirchhoff-type function. When $\lambda = 1$, system (4.1) becomes in the following form

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.3}$$

He and Lei in [60] showed that (4.3) possesses multiple solutions when g is a superlinear term fulfills some condition by using the symmetric mountain pass theorem. When f is asymptotically linear at infinity and $\sigma = \gamma$, $\sigma \in (\frac{1}{2}, 1)$, Chen and Liu [37] proved the existence of ground state solutions of (4.3) by variational methods. However, similar systems to (4.1) has been widely studied by many scholars. Meng-Zhang-He in [79] considered the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^{\sigma} u + V(x)u + K(x)\phi|u|^{q-2}u = h(x)g(u) + |u|^{2_{\sigma}^{*}-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\gamma}\phi = K(x)|u|^{q} & \text{in } \mathbb{R}^3, \end{cases} \tag{4.4}$$

where $\sigma, \gamma \in (0, 1)$, $3 < 4\sigma < 3 + 2\gamma$, $2_{\sigma}^{*} = \frac{6}{3-2\sigma}$ is the fractional critical Sobolev exponent, and $q \in (1, \dfrac{2_{\sigma}^{*}}{2})$. Under some appropriate conditions on f, K and V they obtained ground state solutions for (4.4) via variational methods. We notice that when $f(u) = |u|^{p-2}u$ and $q = 2$, $\gamma \in (0, 1)$, $\sigma \in (\dfrac{3}{4}, 1)$, $2\sigma + 2\gamma > 3$, the ground state and sign changing solutions are obtained by Ye and Teng [111] under different conditions on f, K and V. Yu-Zhao-Zhao [112] proved the existence of positive and sign changing solutions for system (4.4) when $f(u) = |u|^{p-2}u$ and $q = 2$, $\gamma \in (0, 1)$, and $\sigma \in (\dfrac{3}{4}, 1)$ via variational methods.
To the best of our knowledge, there is no similar result for system (4.1) even in the local case, the main aim of this Chapter is to fill this gap by showing the existence of infinitely many solutions for any real $\lambda > 0$ using the fountain theorem as the main tool.
Before state our main result, we introduce the following assumptions on the nonlinearity function f, and the potentials functions K and V.
(F_1) : There exist nonnegative functions $\varrho \in L^2(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$, $\xi \in L^{\infty}(\mathbb{R}^3)$ such that, for all $(x, s) \in \mathbb{R}^3 \times \mathbb{R}$

$$|f(x, s)| \le \varrho(x) + \xi(x)|s|^{p-1}, \quad p \in (2, 2_{\sigma}^{*}),$$

where $2_{\sigma}^{*} = \frac{6}{3-2\sigma}$ is the fractional critical Sobolev exponent.
(F_2) : $\lim\limits_{|s| \to \infty} \frac{F(x,s)}{|s|^4} = \infty$, uniformly for a.e. $x \in \mathbb{R}^3$ $F(x, s) = \int\limits_0^s f(x, t)dt$.
(F_3) : There exists $\varsigma > 0$ such that

$$4F(x, s) \le sf(x, s) + \varsigma|s|^2, \quad \text{for all} \quad (x, s) \in \mathbb{R}^3 \times \mathbb{R}.$$

(F_4) : $f(x, -s) = -f(x, s)$, $x \in \mathbb{R}^3$, $s \in \mathbb{R}$.
(K) : $K(x) \in L^q(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$ for some $q \in [\frac{6}{4\sigma+2\gamma-3}, \infty)$ and $K(x) \ge 0$ for all $x \in \mathbb{R}^3$.
(V): $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x \in \mathbb{R}^3} V(x) \ge V_0 > 0$ where V_0 is a constant, and for any $M > 0$ meas $\{x \in \mathbb{R}^3 : \ V(x) \le M\} < \infty$.
The main result of this chapter is the following theorem.

Theorem 4.1.1. *Let $\sigma, \gamma \in (0, 1)$ with $4\sigma + 2\gamma > 3$. Suppose that the assumptions (F_1)-(F_4), (K) and (V) hold. Then, the problem (4.1) possesses a sequence of nontrivial solutions $\{u_n\} \subset E$ such* *that* $\Phi_\lambda(u_n) \to \infty$ *as* $n \to \infty$ *for any* $\lambda > 0$.

Remark 4.1.1.
- In this chapter, we do not impose the Ambrosetti-Rabinowitz's 4-superlinearity condition:

$$\exists \mu > 4 \text{ such that } 0 < \mu F(x,s) \leq sf(x,s) \text{ for all } x \in \mathbb{R}^3 \text{ and } s \in \mathbb{R}\backslash\{0\}, \tag{4.5}$$

  which was first introduced by Ambrosetti and Rabinowitz in [3]. This condition is important to ensure that the corresponding functional Φ has the mountain pass geometry and to guarantee that the (PS), or (C) sequence of Φ is bounded.

- The assumption (F_3), is weaker than the (4.5)-condition.

The rest of this chapter is organized as follows. In the forthcoming section, we present some auxiliary results and we set up the functional framework necessary to our problem. In the last section, we prove the existence of infinitely many large energy solutions for problem (5.11) by means of the fountain critical point theorem.

4.2 Preliminaries and variational settings

In this section we introduce the functional framework needed for studying system (4.1), and the complete introduction on fractional Sobolev spaces can be found in [93].
For $\sigma \in (0,1)$, the homogeneous fractional Sobolev space $D^{\sigma,2}(\mathbb{R}^3)$ can be characterized as follows

$$D^{\sigma,2}(\mathbb{R}^3) = \left\{ u \in L^{2^*_\sigma}(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2\sigma}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

which is also the completion of $C_0^\infty(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_{D^{\sigma,2}} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy \right)^{\frac{1}{2}} = \Big(\int_{\mathbb{R}^3} \left|(-\Delta^{\frac{\sigma}{2}})u\right|^2 dx\Big)^{\frac{1}{2}}.$$

Moreover, we recall that the well-known fractional Sobolev space $H^\sigma(\mathbb{R}^3)$ can be given as follows

$$H^\sigma(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2\sigma}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

equipped with the norm

$$\|u\|_{H^\sigma(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\sigma}} dxdy + \int_{\mathbb{R}^3} |u|^2 dx \right)^{\frac{1}{2}}.$$

Next, we define the natural solution space for (4.1) as follows

$$E = \left\{ u \in H^{\sigma}\left(\mathbb{R}^3\right) : \int\limits_{\mathbb{R}^3} V(x)|u|^2 dx < \infty \right\}.$$

E is equipped with the following inner product

$$\langle u, w \rangle = \int\limits_{\mathbb{R}^3} ((-\Delta)^{\frac{\sigma}{2}} u (-\Delta)^{\frac{\sigma}{2}} w + V(x) u w) dx,$$

and its corresponding norm given by

$$\|u\|^2 = \int\limits_{\mathbb{R}^3} (|(-\Delta)^{\frac{\sigma}{2}} u|^2 + V(x)|u|^2) dx.$$

Consequently, we have $(E, \|\cdot\|)$ is a uniformly convex Banach space (i.e. reflexive). Furthermore, in view of the assumption (V), we have the following lemma which discuss the embedding properties of E.

Lemma 4.2.1. *[88] E is compactly embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2^*_\sigma)$, and continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2^*_\sigma]$.*

Lemma 4.2.2. *[45] For any $\sigma \in (0,1)$, $D^{\sigma,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\sigma}(\mathbb{R}^3)$, that is, there is $C_\sigma > 0$ such that*

$$\left(\int\limits_{\mathbb{R}^3} |u|^{2^*_\sigma} dx \right)^{\frac{2}{2^*_\sigma}} \leq C_\sigma \int\limits_{\mathbb{R}^3} \left|(-\Delta)^{\frac{\sigma}{2}} u\right|^2 dx, \; u \in D^{\sigma,2}\left(\mathbb{R}^3\right).$$

It is easy to reduce the system (4.1) to a single fractional Schrödinger equation. In fact, for every $u \in E$, obviously the linear operator $\mathcal{L}_u : D^{\gamma,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{L}_u(w) = \int\limits_{\mathbb{R}^3} K(x) u^2 w dx,$$

is well defined on $D^{\gamma,2}(\mathbb{R}^3)$ and continuous. Indeed, using the Hölder inequality, (K) and Lemmas 4.2.1, 4.2.2 we get

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|K\|_q \|u\|^2_{\frac{12q}{(q(3+2\gamma)-6)}} \|w\|_{2^*_\gamma} \\ &\leq C\|u\|^2 \|w\|_{D^{\gamma,2}}. \end{aligned} \tag{4.6}$$

Here we have used the fact $2 < \frac{12}{3+2\gamma} < \frac{12q}{(q(3+2\gamma)-6)} \leq 2^*_\sigma$ where $\sigma, \gamma \in (0,1)$ satisfy $2\gamma + 4\sigma > 3$, and $q \in [\frac{6}{4\sigma+2\gamma-3}, \infty)$.

Set

$$\eta(u,w)=\int\limits_{\mathbb{R}^3}(-\Delta)^{\frac{\gamma}{2}}u(-\Delta)^{\frac{\gamma}{2}}wdx,\ \ \forall u,w\in D^{\gamma,2}(\mathbb{R}^3).$$

Then, in light of Lax-Milgram theorem, there exists a unique $\phi_u^\gamma\in D^{\gamma,2}(\mathbb{R}^3)$ such that $\mathcal{L}_u(w)=\eta(\phi_u^\gamma,w)$ for every $w\in D^{\gamma,2}(\mathbb{R}^3)$, that is

$$\int\limits_{\mathbb{R}^3}(-\Delta)^{\frac{\gamma}{2}}\phi_u^\gamma(-\Delta)^{\frac{\gamma}{2}}wdx=\int\limits_{\mathbb{R}^3}K(x)u^2wdx. \tag{4.7}$$

Therefore, ϕ_u^γ is a weak solution of the fractional Poisson equation

$$(-\Delta)^\gamma\phi_u^\gamma=K(x)u^2,\quad x\in\mathbb{R}^3,$$

and the representation formula holds, i.e.,

$$\phi_u^\gamma(x)=c_\gamma\int\limits_{\mathbb{R}^3}\frac{K(x)u^2(y)}{|x-y|^{3-2\gamma}}dy\quad x\in\mathbb{R}^3,$$

which is called the γ-Riesz potential (see [97]) where

$$c_\gamma=2^{-2\gamma}\frac{\Gamma\left(\frac{3-2\gamma}{2}\right)}{\pi^{\frac{3}{2}}\Gamma(\gamma)}.$$

Thus, $\phi_u^\gamma(x)\geq 0$ for all $x\in\mathbb{R}^3$. Taking $w=\phi_u^\gamma$ in (4.6) and (4.7), we infer

$$\|\phi_u^\gamma\|_{D^{\gamma,2}}\leq C\|u\|^2. \tag{4.8}$$

Substituting ϕ_u^γ in (4.1), we get the fractional Schrödinger equation

$$(-\Delta)^\sigma u+V(x)u+K(x)\phi_u^\gamma u=\lambda f(x,u),\ \ x\in\mathbb{R}^3. \tag{4.9}$$

Whose solutions are then the critical points of the energy functional $\Phi_\lambda:E\to\mathbb{R}$ defined by

$$\Phi_\lambda(u)=\frac{1}{2}\int\limits_{\mathbb{R}^3}\left(\left|(-\Delta)^{\frac{\sigma}{2}}u\right|^2+V(x)|u|^2\right)dx+\frac{1}{4}\int\limits_{\mathbb{R}^3}K(x)\phi_u^\gamma u^2dx-\lambda\int\limits_{\mathbb{R}^3}F(x,u)dx.$$

Furthermore, if we take $w=\phi_u^\gamma$ in (4.6) and (4.7) again, and by (4.8) we deduce

$$\begin{aligned}\int\limits_{\mathbb{R}^3}K(x)\phi_u^\gamma u^2dx&\leq C\|u\|^2\|\phi_u^\gamma\|_{D^{\gamma,2}}\\&\leq C\|u\|^4\end{aligned} \tag{4.10}$$

Hence, the functional Φ_λ is well defined in E and $\Phi_\lambda \in C^1(E,\mathbb{R})$. Moreover, for all $u, w \in E$, its derivative is

$$< \Phi'_\lambda(u), w >= \int_{\mathbb{R}^3} ((-\Delta)^{\frac{\sigma}{2}} u.(-\Delta)^{\frac{\sigma}{2}} w + V(x)uw + K(x)\phi_u^\gamma uw - \lambda f(x,u)w)dx.$$

Clearly, the pair $(u, \phi_u^\gamma) \in E \times D^{\gamma,2}(\mathbb{R}^3)$ is a weak solution of (4.1) if and only if u is a critical point of $\Phi_\lambda(u)$.

The following assertion is crucial to establish our main result. The fundamental idea of proofs of this consequences follows similar arguments as in [88, 96, 102].

Lemma 4.2.3. *[88, 96, 102] Φ'_λ is of type (S_+), that is, if for every sequence $\{u_n\} \subset E$ satisfying $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty} \langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u \rangle \leq 0$, then $u_n \to u$ in E as $n \to \infty$.*

4.3 Proof of main result

In this section, we first start by proving the following Lemma which plays a crucial role in establishing the existence of a nontrivial weak solution to the given problem. Then, we demonstrate that the functional Φ_λ possesses the remaining geometric conditions in the fountain theorem.

Lemma 4.3.1. *Let $\sigma, \gamma \in (0,1)$ satisfies $2\gamma + 4\sigma > 3$. Assume that (F_1)-(F_3), and (V) hold. Then the functional Φ_λ satisfies the (C)$_c$-condition for any $\lambda > 0$.*

Proof. Let $\{u_n\} \subset E$ be a (C)$_c$ sequence of Φ_λ, that is,

$$\Phi_\lambda(u_n) \to c \quad \text{and} \quad \|\Phi'_\lambda(u_n)\|_*(1 + \|u_n\|) \to 0 \;\text{ as }\; n \to \infty, \tag{4.11}$$

which implies that

$$c = \Phi_\lambda(u_n) + o_n(1) \;\text{ and }\; \langle \Phi'_\lambda(u_n), u_n \rangle = o_n(1). \tag{4.12}$$

Indeed, by (4.11) we deduce

$$\begin{aligned} |\langle \Phi'_\lambda(u_n), u_n \rangle| &\leq \|\Phi'_\lambda(u_n)\|_* \|u_n\| \\ &\leq \|\Phi'_\lambda(u_n)\|_*(1 + \|u_n\|) \to 0 \;\text{ as }\; n \to \infty. \end{aligned}$$

By $\Phi_\lambda(u_n) \to c$ for n large enough, we get

$$\Phi_\lambda(u_n) \leq c + 1$$

From the same arguments as in for instance, Lemma 6 in [88], we know that Φ'_λ is a mapping of type (S_+). Thus, if a sequence $\{u_n\} \subset E$ is bounded in E, we conclude that $u_n \to u$ in E since E is reflexive. Hence, it suffices to ensure that the sequence $\{u_n\}$ is bounded in E. To do this,

arguing by contradiction.
Assume that the sequence $\{u_n\}$ is unbounded in E. Then we may suppose that

$$\|u_n\| \geq 1 \ \text{ and } \ \|u_n\| \to \infty \ \text{ as } \ n \to \infty \tag{4.13}$$

Let $\{v_n\} \subset E$ such that $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Hence, up to a subsequence $\{v_n\}$, we still write it as $\{v_n\}$, and by Lemma 4.2.1 we have that

$$\begin{aligned} v_n &\rightharpoonup v \ \text{ in } \ E, \\ v_n &\to v \ \text{ a.e. in } \ \mathbb{R}^3, \\ v_n &\to v \ \text{ in } \ L^p\left(\mathbb{R}^3\right) \ \text{ for } \ 2 \leq p < 2^*_\sigma, \end{aligned} \tag{4.14}$$

as $n \to \infty$. There are two possible cases.
First, we consider the case that $v(x) \neq 0$ in $\mathbb{R}^3$. Set $\Sigma_0 = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Then meas$(\Sigma_0) > 0$. Due to (4.12), we have that

$$c = \Phi_\lambda(u_n) + o_n(1) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi^\gamma_{u_n} u_n^2 dx - \lambda \int_{\mathbb{R}^3} F(x, u_n)dx + o_n(1) \tag{4.15}$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, we assert that

$$\int_{\mathbb{R}^3} F(x, u_n)dx \geq \frac{1}{2\lambda}\|u_n\|^2 - \frac{c}{\lambda} + o_n(1) \to \infty \ \text{ as } n \ \to \infty. \tag{4.16}$$

Combining the relations (4.15) and (4.10), we obtain for n large enough that the following holds

$$\int_{\mathbb{R}^3} F(x, u_n)dx + \frac{c}{\lambda} - o_n(1) = \frac{1}{2\lambda}\|u_n\|^2 + \frac{1}{4\lambda}\int_{\mathbb{R}^3} K(x)\phi^\gamma_{u_n} u_n^2 dx \leq \frac{3}{4\lambda}\|u_n\|^4. \tag{4.17}$$

Taking account into (F_2) we can choose $u_0 > 1$ such that $F(x, u) > |u|^4$ for all $x \in \mathbb{R}^3$ and $|u| > u_0$. Since f is a Carathéodory function and by means of (F_1), we assert that there exists $C > 0$ such that $|F(x, u)| < C, \ \forall\, (x, u) \in \mathbb{R}^3 \times [-u_0, u_0]$. Hence, there is a $C_0 \in \mathbb{R}$ such that $F(x, u) \geq C_0$ for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$, and thus

$$\frac{F(x, u_n) - C_0}{\|u_n\|^4} \geq 0, \tag{4.18}$$

for all $x \in \mathbb{R}^3$ and for all $n \in \mathbb{N}$.
It follows from the convergence (4.14) that $|u_n| = |v_n|\|u_n\| = \infty$ as $n \to \infty$ for all $x \in \Sigma_0$. Hence, the assumption (F_2) implies that

$$\lim_{n\to\infty} \frac{F(x, u_n)}{\|u_n\|^4} = \lim_{n\to\infty} \frac{F(x, u_n)}{|u_n(x)|^4}|v_n(x)|^4 = \infty, \tag{4.19}$$

for all $x \in \Sigma_0$.
It follows from the relations (4.16)-(4.19) and the Fatou lemma that

$$\begin{aligned}
\frac{3}{4\lambda} &= \liminf_{n\to\infty} \frac{\frac{3}{4}\int\limits_{\mathbb{R}^3} F(x,u_n)dx}{\lambda \int\limits_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1)} \\
&\geq \liminf_{n\to\infty} \int\limits_{\mathbb{R}^3} \frac{\frac{3}{4}F(x,u_n)}{\frac{3}{4}\|u_n\|^4}dx \\
&\geq \liminf_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n)}{\|u_n\|^4}dx - \limsup_{n\to\infty} \int\limits_{\Sigma_0} \frac{C_0}{\|u_n\|^4}dx \\
&= \liminf_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n) - C_0}{\|u_n\|^4}dx. \\
&\geq \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n) - C_0}{\|u_n\|^4}dx. \\
&= \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n)}{\|u_n\|^4}dx - \int\limits_{\Sigma_0} \limsup_{n\to\infty} \frac{C_0}{\|u_n\|^4}dx \\
&= \infty,
\end{aligned} \tag{4.20}$$

which is a contradiction. Thus, we get that $\text{meas}(\Sigma_0) = 0$ and $v(x) = 0$ for almost all $x \in \mathbb{R}^3$.
Next, we only need to consider the case that $v(x) = 0$ in $\mathbb{R}^3$.
It follows from the assumption (F_3) that for sufficiently n large

$$\begin{aligned}
c + 1 \geq{}& \Phi_\lambda(u_n) - \frac{1}{4}\langle \Phi_\lambda'(u_n), u_n\rangle \\
={}& \frac{1}{2}\int\limits_{\mathbb{R}^3} (|(-\Delta)^{\frac{\sigma}{2}} u_n|^2 + V(x)|u_n|^2)dx + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \lambda \int\limits_{\mathbb{R}^3} F(x,u_n)dx \\
&- \frac{1}{4}\int\limits_{\mathbb{R}^3} (|(-\Delta)^{\frac{\sigma}{2}} u_n|^2 + V(x)|u_n|^2)dx - \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} f(x,u_n)u_n dx \\
={}& \left(\frac{1}{2} - \frac{1}{4}\right)\|u_n\|^2 + \lambda \int\limits_{\mathbb{R}^3} \left(\frac{f(x,u_n)u_n}{4} - F(x,u_n)\right)dx \\
\geq{}& \frac{1}{4}\|u_n\|^2 - \frac{\lambda\varsigma}{4}\int\limits_{\mathbb{R}^3} |u_n|^2 dx \\
\geq{}& \frac{1}{4}\|u_n\|^2 - \frac{\lambda\varsigma}{4}\|v_n\|_2^2\|u_n\|^2,
\end{aligned}$$

which implies

$$\frac{c+1}{\|u_n\|^2} \geq \frac{1}{4} - \frac{\lambda\varsigma}{4} \|v_n\|_2^2 .$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, then

$$\|v_n\|_2^2 \geq \frac{1}{\lambda\varsigma}.$$

Dividing the both sides by $\frac{1}{\lambda\varsigma}$ and then taking the limit of supremum of this inequality as $n \to \infty$, we find the following

$$1 \leq \lambda\varsigma \limsup_{n\to\infty} \|v_n\|_2^2 = \lambda\varsigma \|v\|_2^2 ,$$

which shows that $v(x) \neq 0$. In any case, we arrive at contradiction.
Hence, $\{u_n\}$ is bounded in E. This complete the proof. □

Next, applying the fountain theorem, we indicate infinitely many high energy solutions for (4.1). To do this, let H be a separable and reflexive Banach space. Then, there exist $\{e_n\} \subseteq H$ and $\{e_n^*\} \subseteq H^*$ such that

$$H = \overline{\text{span}\{e_n : n = 1, 2, ...\}}, \quad H^* = \overline{\text{span}\{e_n^* : n = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases}$$

Let us denote $H_n = \text{span}\{e_n\}$, $Y_k = \oplus_{n=1}^k H_n$ and $X_k = \overline{\oplus_{n=k}^\infty H_n}$, $k \in \mathbb{Z}$. Then, we recall the fountain theorem under the (C)$_c$-condition.

Theorem 4.3.1. *Assume that H is a real reflexive Banach space, and consider an even functional $J \in C^1(H, \mathbb{R})$ satisfies the (C)$_c$-condition. If for each sufficiently large $k \in \mathbb{N}$, there exist $\alpha_k > \beta_k > 0$ such that*
(i) $\rho_k := \inf_{u\in X_k, \|u\|=\beta_k} J(u) \to \infty, \ k \to \infty.$
(ii) $\delta_k := \max_{u\in Y_k, \|u\|=\alpha_k} J(u) \leq 0.$
Then the functional J has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset H$ such that $J'(u_n) = 0$ and $J(u_n) \to \infty$ as $n \to \infty$.

According to the assumption (F_4) and Lemma 4.3.1, Φ_λ is an even functional satisfies the (C)$_c$-condition for all $\lambda > 0$. Note that E is a separable and reflexive Banach space. Thanks to Lemma 4.3.1, it suffices to prove that there exist $\alpha_k > \beta_k > 0$ such that (i) and (ii) in Theorem 4.3.1 are satisfied for a sufficiently large k. But first, we need to the following assertion which will be needed later.

Lemma 4.3.2. *Let $2 < q < 2^*_\sigma$ and we denote*

$$\varpi_k := \sup_{u \in X_k, \|u\|=1} \left(\int_{\mathbb{R}^3} |u(x)|^q dx \right).$$

Then, $\varpi_k \to 0$ as $k \to \infty$.

Proof. Suppose the contrary that there exists $\varepsilon_0, k_0 > 0$ and a sequence $\{u_k\} \subset X_k$ such that

$$\|u_k\| = 1, \int_{\mathbb{R}^3} |u_k(x)|^q dx \geq \varepsilon_0 \text{ for all } k \geq k_0.$$

Since the sequence $\{u_k\}$ is bounded in E, there exists $u \in E$ such that $u_k \rightharpoonup u$ in E as $k \to \infty$, and

$$\langle e_i^*, u \rangle = \lim_{k \to \infty} \langle e_i^*, u_k \rangle = 0 \text{ for } i = 1, 2, \ldots.$$

Hence, we get that $u = 0$ since $e_i^* \neq 0$. However, we obtain

$$0 < \varepsilon_0 \leq \lim_{k \to \infty} \int_{\mathbb{R}^3} |u_k(x)|^q dx = \int_{\mathbb{R}^3} |u(x)|^q dx = 0,$$

which is a contradiction. This finishes the proof. □

Now, we are in position to prove the statements (i) and (ii) in Theorem 4.3.1.

Lemma 4.3.3. *Under the assumptions (F_1), (F_2) and (F_4), the geometry condition in the fountain theorem hold.*

Proof. For every $u \in X_k$, using (F_1) and the Hölder inequality, we derive

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\frac{\sigma}{2}} u\right|^2 + V(x)|u|^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^\gamma u^2 dx - \lambda \int_{\mathbb{R}^3} F(x, u)\, dx \\
&\geq \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^\gamma u^2 dx - \lambda \int_{\mathbb{R}^3} |\varrho(x)| |u(x)| dx - \lambda \int_{\mathbb{R}^3} \frac{\xi(x)}{p} |u(x)|^p dx \\
&\geq \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^\gamma u^2 dx - \lambda \|\varrho\|_2 \|u\|_2 - \frac{\lambda}{p} \|\xi\|_\infty \int_{\mathbb{R}^3} |u(x)|^p dx \\
&\geq \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^\gamma u^2 dx - \lambda C_2 \|u\| - \frac{\lambda C_3}{p} \int_{\mathbb{R}^3} |u(x)|^p dx. \\
&\geq \frac{1}{2} \|u\|^2 - \lambda C_2 \|u\| - \frac{\lambda C_3}{p} \varpi_k^p \|u\|^p,
\end{aligned}$$

where C_2 and C_3 are positive constants. Choose $\beta_k = (\lambda C_3 \varpi_k^p)^{\frac{1}{2-p}}$.

Since $2 < p$ and $\varpi_k \to 0$ as $k \to \infty$, we get $\beta_k \to \infty$ as $k \to \infty$. Hence, if $u \in X_k$ and $\|u\| = \beta_k$, then we deduce that

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \lambda C_2\|u\| - \frac{\lambda C_3}{p}\varpi_k^p\|u\|^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\beta_k^2 - \lambda C_2\beta_k \\ &\to \infty \text{ as } k \to \infty.\end{aligned}$$

Thanks to $2 < p$, we assert that

$$\inf_{u\in X_k, \|u\|=\beta_k} \Phi_\lambda(u) \to \infty \text{ as } k \to \infty,$$

which implies (i) in Theorem 4.3.1.
Next, suppose that condition (ii) in Theorem 4.3.1 is not satisfied for some k. Then, there exists a sequence $\{u_n\}$ in Y_k such that

$$\|u_n\| > 1 \text{ and } \|u_n\| \to \infty \text{ as } n \to \infty \text{ and } \Phi_\lambda(u_n) > 0. \tag{4.21}$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then clearly $\|v_n\| = 1$. Since, dim$Y_k < \infty$, there exists $v \in Y_k \setminus \{0\}$ such that up to subsequence

$$\|v_n - v\| = 0 \quad \text{and} \quad v_n(x) \to v(x) \text{ for a.e. } x \in \mathbb{R}^3 \text{ as } n \to \infty.$$

For all $x \in \Sigma_0 := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, we get $|u_n(x)| \to \infty$ as $n \to \infty$. Hence, by assumption (F_2) we obtain

$$\lim_{n\to\infty} \frac{F(x,u_n)}{\|u_n\|^4} = \lim_{n\to\infty} \frac{F(x,u_n)}{|u_n(x)|^4}|v_n(x)|^4 = \infty.$$

As a consequence, we get

$$\lim_{n\to\infty} \frac{F(x,u_n)}{\int\limits_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{\sigma}{2}}u_n|^2 + V(x)|u_n|^2\right)dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} \geq \infty.$$

Since meas$(\Sigma_0) \neq 0$, we derive

$$\lim_{n\to\infty} \frac{\int\limits_{\Sigma_0} F(x,u_n)}{\int\limits_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{\sigma}{2}}u_n|^2 + V(x)|u_n|^2\right)dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx} dx \geq \infty \tag{4.22}$$

Therefore, using (4.22), we get

$$\begin{aligned}\Phi_\lambda\left(u_n\right) =&\frac{1}{2}\int\limits_{\mathbb{R}^3}\left(\left|(-\Delta)^{\frac{\sigma}{2}}u_n\right|^2+V(x)|u_n|^2\right)dx+\frac{1}{4}\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}^{\gamma}u_n^2dx-\lambda\int\limits_{\Sigma_0}F\left(x,u_n\right)dx\\ \leq&\frac{\int\limits_{\mathbb{R}^3}\left(\left|(-\Delta)^{\frac{\sigma}{2}}u_n\right|^2+V(x)|u_n|^2\right)dx+\frac{1}{2}\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}^{\gamma}u_n^2dx}{2}.\\ &\left(1-2\lambda\int\limits_{\Sigma_0}\frac{F(x,u_n)}{\int\limits_{\mathbb{R}^3}\left(\left|(-\Delta)^{\frac{\sigma}{2}}u_n\right|^2+V(x)|u_n|^2\right)dx+\frac{1}{2}\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}^{\gamma}u_n^2dx}dx\right)\\ &\to-\infty\ \text{ as }\ n\to\infty,\end{aligned}$$

which contradicts (4.21).
The proof of Lemma 4.3.3 is completed.
Proof of Theorem 4.1.1. Combining the assumption (F_1) and Lemma 4.3.1, one can easily check that Φ_λ is an even functional satisfies the $(C)_c$-condition for any $\lambda > 0$. In addition, by Lemma 4.3.3 the functional Φ_λ meets all fountain theorem requirements. Consequently, we assert that the problem (4.1) possesses a sequence of nontrivial solutions $\{u_n\}$ in E satisfying $\Phi_\lambda(u_n) \to \infty$ as $n \to \infty$ for all $\lambda > 0$. □

CHAPTER 5

Existence theory of solution for partial differential equations related to distributional Riesz fractional gradient

This chapter includes the results of the following research article:

• H.Boutebba, H.Lakhal, and K.Slimani. (accepted paper 2024). The multiplicity of solutions to a new class of superlinear fractional Schrödinger-Poisson systems. Novi Sad Journal of Mathematics.

The main goal of chapter 5 is to present the existence theory for the solutions to the fractional systems related to the distributional Riesz fractional gradient which is the basis of the rest of the thesis.

We do not aim to be exhaustive and all the proofs will be omitted, we will focus on definitions and results that will be relevant for the sequel. We begin by presenting several results regarding the distributional Riesz fractional gradient and corresponding fractional framework. As an application, in section 5.2 we apply variational methods together with the symmetric mountain pass theorem to solve fractional systems related to the distributional Riesz fractional gradient.

5.1 Distributional Riesz fractional gradient

5.1.1 Introduction

In recent years, great attention has been given to the search for a good notion of fractional derivatives operators. This considerable interest led many researchers to develop a variety of suitable definitions for fractional derivatives operators. In particular, there has been a renewed interest in the so-called distributional Riesz fractional gradient D^σ, which may be defined in finite integral form for a sufficiently smooth function u (see [91]) as

$$D^\sigma u(x) = C_{d,\sigma} \int\limits_{\mathbb{R}^d} \frac{[u(x) - u(y)]}{|x-y|^{d+\sigma}} \frac{x-y}{|x-y|} dy, \quad x \in \mathbb{R}^d, \tag{5.1}$$

where $C_{d,\sigma}$ is a suitable constant. This fractional operator has been recently studied in a series of two interesting papers [91, 92] by Shieh and Spector who developed a general theory for fractional partial differential equations as well as considered typical calculus of variations problems with standard growth conditions based on this operator. This naturally leads the authors to pose a new class of fractional partial differential equations and to allow us to provide an interesting characterization of the Bessel potential spaces defined in Chapter 1 similar to the classical characterization of the Sobolev spaces. Starting from these papers, such an operator has been the subject of extensive research by an increasing number of researchers in various directions, and many interesting works have emerged [13, 42, 45, 91].
Furthermore, as recently shown by the author in [93], the fundamental aspect of the operator in (5.1) is that it has the advantage of satisfying three natural physical requirements. Indeed, the definition in (5.1) is an alternative to the well-known fractional Laplacian and shares many similarities with the classical gradient (for example, being vector valued contrasting with the scalar fractional Laplacian), and makes it suitable for anisotropic and inhomogeneous problems. Indeed, it is the unique translationally and rotationally invariant α-homogeneous operator, that establishes it as a canonical choice for a fractional derivative, and that object deserves more attention in the literature. On the other side, another strong motivation for considering this operator and therefore the associated fractional systems is that they can be used in a multi-dimensional situation (for example a thin material that is usually viewed as 2D and a normal 3D crystalline solid), peridynamics [15] and nonlocal diffusion [47].
In this regard, in this chapter, we establish some abstract results on the distributional Riesz fractional gradient defined above and corresponding functional space and state some of their elementary properties. As an application, using variational methods combined with the symmetric mountain pass theorem, we obtain the existence of multiple non-trivial solutions for a class of fractional Schrödinger-Poisson system driven by distributional Riesz fractional gradient.

5.1.2 Distributional Riesz fractional gradient $D^\sigma, 0<\sigma<1$

The starting point of research pursued in [45] for the development of a general theory for fractional partial differential equations depending on distributional Riesz fractional derivative, is the distributional Riesz fractional gradient D^σ. In what follows for $0<\sigma<1$, consider $u \in L^p\left(\mathbb{R}^d\right)$ with $1<p<\infty$ such that $I_{1-\sigma} * u$ is well defined, we recall that the Riesz fractional partial derivatives $\dfrac{\partial u}{\partial x_j}$ of order σ, can be characterized in distributionally as

$$\langle \frac{\partial^\sigma u}{\partial x_j^\sigma}, w\rangle = -\langle I_{1-\sigma} * u, \frac{\partial w}{\partial x_j}\rangle = -\int_{\mathbb{R}^d} (I_{1-\sigma} * u)\frac{\partial w}{\partial x_j} dx, \quad \forall w \in C_0^\infty(\mathbb{R}^d)$$

where I_σ is the Riesz kernel of order σ.
By definition of distributional Riesz partial derivatives, these operators are nonlocal, in the sense that in order to be computed they require information on the entire $\mathbb{R}^d$.
As a consequence, similarly to the classical derivatives, we can express the distributional Riesz fractional gradient D^σ of order σ as the vector, in terms of its distributional Riesz fractional partial derivatives

$$(D^\sigma u)_j = \frac{\partial^\sigma u}{\partial x_j^\sigma} = \frac{\partial}{\partial x_j} I_{1-\sigma} * u, \quad j=1,...,d,$$

and the distributional Riesz fractional divergence div^σ of order σ for $\varphi \in L^p \in (\mathbb{R}^d;\mathbb{R}^d)$ is given by

$$D^\sigma.\varphi = div^\sigma \varphi = \sum_{j=1}^{d} \frac{\partial^\sigma \varphi_j}{\partial x_j^\sigma}.$$

Additionally, the distributional Riesz fractional gradient D^σ and the distributional Riesz fractional divergence div^σ can be written in finite integral form for sufficiently regular functions u and vectors φ, respectively, by (see [91, 93])

$$D^\sigma u(x) = C_{d,\sigma} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+\sigma}} \frac{x-y}{|x-y|} dy,$$

and

$$div^\sigma \varphi(x) = C_{d,\sigma} \int_{\mathbb{R}^d} \frac{\varphi(x)-\varphi(y)}{|x-y|^{d+\sigma}} . \frac{x-y}{|x-y|} dy,$$

for the constant

$$C_{d,\sigma} = \pi^{-\frac{d}{2}} 2^\sigma \frac{\Gamma(\frac{d+\sigma+1}{2})}{\Gamma(\frac{1-\sigma}{2})}.$$

The above fractional operators enjoy a duality property, which is a fractional integration by parts whose proof follows the same line of Theorem 1.4 in [82]. In particular, for smooth functions we have the following Corollary.

Corollary 5.1.1. *Let* $\sigma \in (0,1)$*, for all* $u \in C_0^\infty(\mathbb{R}^d)$ *and* $\varphi \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$*, we have*

$$\int_{\mathbb{R}^d} D^\sigma u(x).\varphi(x)dx = -\int_{\mathbb{R}^d} u(x)div^\sigma \varphi(x)dx. \tag{5.2}$$

It was observed in [91] that in the distributional sense, the composition of div^σ and D^σ it coincides with the fractional Laplacian for $u \in C_0^\infty(\mathbb{R}^d)$ via the identity

$$\begin{aligned}(-\Delta)^\sigma u &= -\sum_{j=1}^{d} \frac{\partial^\sigma}{\partial x_j^\sigma}\frac{\partial^\sigma}{\partial x_j^\sigma} u \\ &= -div^\sigma D^\sigma u, \end{aligned} \tag{5.3}$$

where for $0 < \sigma < 1$, we have

$$(-\Delta)^\sigma u(x) = -\frac{1}{2}C_{d,\sigma}^2 \int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2\sigma}}dy, \quad \forall x \in \mathbb{R}^d. \tag{5.4}$$

Furthermore, we have the convergence of D^σ to the classical gradient ∇ as $\sigma \nearrow 1$, that is,

$$D^\sigma u \to \nabla u, \ \text{ in } L^p(\mathbb{R}^d) \quad \text{for } u \in W^{1,p}(\mathbb{R}^d).$$

We refer the reader to [13, 42, 79] for more detailed information and the proof.
We notice that for $u, w \in C_0^\infty(\mathbb{R}^d)$, it follows from Corollary 5.1.1, relation (5.4), Lebesgue and Fubini theorems, that

$$\begin{aligned}\int_{\mathbb{R}^d} D^\sigma u.D^\sigma w dx =& \int_{\mathbb{R}^d} w(-\Delta)^\sigma u dx \\ &= \frac{1}{2}C_{d,\sigma}^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{d+2\sigma}}dxdy. \end{aligned} \tag{5.5}$$

As a result, equation (5.3) is to be understood in the following sense

$$\int_{\mathbb{R}^d} D^\sigma u.D^\sigma w dx = \int_{\mathbb{R}^d} (-\Delta)^\sigma u.w dx = \int_{\mathbb{R}^d} (-\Delta)^{\frac{\sigma}{2}} u.(-\Delta)^{\frac{\sigma}{2}} w dx, \tag{5.6}$$

which is very important formulas that are going to be particularly helpful for the variational formulation of partial differential equations involving fractional derivatives operators.

Furthermore, due to (5.5), we have the following equality of norms

$$\|D^\sigma u\|_2^2 = \frac{C_{d,\sigma}^2}{2}[u]_\sigma^2 = \frac{C_{d,\sigma}^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2\sigma}}dxdy.$$

We finishes this subsection by an alternative representation of the distributional Riesz fractional gradient D^σ in terms of the classical gradient ∇ with the Riesz kernel, which is extremely helpful.

Theorem 5.1.1. *Let $\sigma \in (0,1)$ and $u \in C_0^\infty(\mathbb{R}^d)$. Then,*

$$D^\sigma u = I_{1-\sigma} * \nabla u \quad \textit{for } u \in C_0^\infty(\mathbb{R}^d).$$

5.1.3 Spaces of fractionally differentiable functions $S^{\sigma,p}(\mathbb{R}^d)$

Now that we have introduced the notion of distributional Riesz fractional gradient, we proceed to present the functional spaces essential for dealing with problems in which the Riesz fractional gradient is present, that will serve as a foundation for our existence results in the calculus of variations and partial differential equations. Thus, we define the space of fractionally differentiable functions $S^{\sigma,p}(\mathbb{R}^d)$ as follows.

Definition 5.1.1. ([91]) Set $\sigma \in (0,1)$ and $p \in [1,\infty)$. For $u \in C_0^\infty(\mathbb{R}^d)$, we can thus define the space of fractionally differentiable functions $S^{\sigma,p}(\mathbb{R}^d)$ as the closure of $C_0^\infty(\mathbb{R}^d)$ under the following norm

$$\|u\|_{S^{\sigma,p}}^p = \|u\|_p^p + \|D^\sigma u\|_p^p. \tag{5.7}$$

Moreover, we define the fractional spaces $S_0^{\sigma,p}(\mathbb{R}^d)$ as the space of function $u \in L^p(\mathbb{R}^d)$ with distributional Riesz fractional gradient also in $L^p(\mathbb{R}^d;\mathbb{R}^d)$, i.e.

$$S_0^{\sigma,p}(\mathbb{R}^d) := \left\{u \in L^p(\mathbb{R}^d) : D^\sigma u \in L^p(\mathbb{R}^d;\mathbb{R}^d)\right\}. \tag{5.8}$$

To this space we endow the norm given by (5.7).
By Theorem A.1 in [26], if $p \in [1,\infty)$, then $C_0^\infty(\mathbb{R}^d)$ is dense in $S_0^{\sigma,p}(\mathbb{R}^d)$, that is, $S^{\sigma,p}(\mathbb{R}^d) = S_0^{\sigma,p}(\mathbb{R}^d)$. Furthermore, we have $S_0^{0,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ with the equivalent norms when $p \in (1,\infty)$.
For the reader convenience, we will denote the vector space of fractional differentiable functions by $S^{\sigma,p}(\mathbb{R}^d)$.
However, the $S^{\sigma,p}(\mathbb{R}^d)$ with the distributional Riesz fractional gradient as a mathematical object, was studied in [91, 92]. The first remarkable fact is the identification of $S^{\sigma,p}(\mathbb{R}^d)$ with the Bessel potential spaces $L^{\sigma,p}(\mathbb{R}^d)$ addressed in Theorem 1.7 of [91].

Theorem 5.1.2. *If $p \in (1,\infty)$ and $\sigma \in (0,1)$, we have*

$$L^{\sigma,p}(\mathbb{R}^d) = S^{\sigma,p}(\mathbb{R}^d),$$

where $L^{\sigma,p}(\mathbb{R}^d)$ is defined in Chapter 1, and the norms in the two spaces being equivalent.

As a consequence of Theorem 5.1.2, we will able to transport many properties that hold on $L^{\sigma,p}(\mathbb{R}^d)$ to $S^{\sigma,p}(\mathbb{R}^d)$ from the existing literature that allows us to understand the structure of this space.

Proposition 5.1.1. *For $\sigma \in (0,1)$ and $p \in (1,\infty)$. We have*

1. *$C_0^\infty(\mathbb{R}^d)$ is dense in $S^{\sigma,p}(\mathbb{R}^d)$.*
2. *$S^{\sigma,p}(\mathbb{R}^d)$ is reflexive.*
3. *If $\gamma < \sigma$, then $S^{\sigma,p}(\mathbb{R}^d) \hookrightarrow S^{\gamma,p}(\mathbb{R}^d)$.*
4. *If $0 < \sigma_1 < \sigma < \sigma_2 < 1$, then $S^{\sigma_2,p}(\mathbb{R}^d) \hookrightarrow W^{\sigma,p}(\mathbb{R}^d) \hookrightarrow S^{\sigma_1,p}(\mathbb{R}^d)$.*
5. *If $p = 2$, then $S^{\sigma,2}(\mathbb{R}^d) = W^{\sigma,2}(\mathbb{R}^d)$ with equivalent norms.*

Recalling that $W^{\sigma,p}(\mathbb{R}^d)$ is the classical fractional Sobolev spaces and $\hookrightarrow$ denotes a continuous inclusion with an inequality of the norms.

Due to the last Proposition combined with Theorem 5.1.2, we obtain

$$S^{\sigma,2}(\mathbb{R}^d) = L^{\sigma,2}(\mathbb{R}^d) = W^{\sigma,2}(\mathbb{R}^d) = H^\sigma(\mathbb{R}^d).$$

Remark 5.1.1.

1. In light of the Theorem 5.1.2, we will simply denote the space $S^{\sigma,p}(\mathbb{R}^d)$ if $p \in (1,\infty)$ and $\sigma \in (0,1)$ as $L^{\sigma,p}(\mathbb{R}^d)$.
2. A great difference between $L^{\sigma,p}(\mathbb{R}^d)$ and $W^{\sigma,p}(\mathbb{R}^d)$ is that in the latter space there is no suitable notion of distributional Riesz fractional gradient, even though it possibly is the most adequate space to define the fractional Laplacian.
3. As mentioned in Theorem 5.1.2 that the norms of $L^{\sigma,2}(\mathbb{R}^d)$ and $S^{\sigma,2}(\mathbb{R}^d)$ are equivalent. Thus, $S^{\sigma,2}(\mathbb{R}^d)$ is a Banach space for every $\sigma \in (0,1)$ and $1 < p < \infty$.
4. $S^{\sigma,2}(\mathbb{R}^d)$ is a Hilbert space for the inner product

$$\langle u, w\rangle_{S^{\sigma,2}} = \int_{\mathbb{R}^d} (D^\sigma u . D^\sigma w + uw)dx.$$

5. Another possibility for the definition of fractional Sobolev space $H^\sigma(\mathbb{R}^d)$, inspired by the classical construction of Sobolev spaces, would be to consider the space defined as the class of $L^2(\mathbb{R}^d)$ functions whose distributional Riesz fractional gradient is also in $L^2(\mathbb{R}^d)$. For $u \in L^2(\mathbb{R}^d)$, we have

$$H^\sigma(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : D^\sigma u \in L^2(\mathbb{R}^d;\mathbb{R}^d)\right\}.$$

We completes this short introduction with an application of the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ to the theory of (fractional) partial differential equations that is based on a result proved in ([91], Theorem 1.13), and is a simple consequence of the Lax-Milgram theorem, by the boundedness and ellipticity of the matrix A.

Theorem 5.1.3. *([45]) Let $0 < \sigma < 1$. Assume that $h \in L^2(\mathbb{R}^d)$ and consider $A : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ measurable with coefficients bounded such that*

$$\lambda|z|^2 \leq A(x)z.z \ \text{ and } \ A(x)z.z \leq \Lambda|z|^2$$

For all $x, z \in \mathbb{R}^d$ and some $\lambda, \Lambda > 0$. Then, there exists a unique $u \in L^{\sigma,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} A(x)D^\sigma u.D^\sigma w dx = \int_{\mathbb{R}^d} hw dx$$

for every $w \in C_0^\infty(\mathbb{R}^d)$.

In the following, the matrix A being the identity.

5.2 The multiplicity of solutions to a new class of fractional Schrödinger-Poisson systems with superlinear terms

5.2.1 Introduction

In last years, non local systems involving fractional Laplacian operators have recieved the attention of several scholars, mainly the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.9}$$

which used to study the solitary waves solutions for the fractional Schrödinger equations. Recently, there have been many papers devoted to the study on the existence of solutions to (5.9) by means of variational approaches and critical point theory under different assumptions imposed on V, K and f. We call attention to the work of He and Lei [60], where it was proved the multiplicity of nontrivial solutions for (5.9) under some assumptions on V and K with $\sigma, \gamma \in (0,1)$, $2\gamma + 4\sigma > 3$, by exploiting the symmetric mountain pass theorem. We notice that when $K(x) \equiv 1$, Zhang [115] considered the following system

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.10}$$

where $\sigma, \gamma \in (0,1)$, and established the multiplicity of solutions via the fountain theorem. In recent years, similar system like (5.10) has been concentrically investigated by many authors, for instance, Kim et al [65] for infinitely many solutions when $f(x,u) = \lambda f(x,u)$, Li in [74] and Boutebba et al in [58] for the nontrivial solutions when $V(x) = 1$ and with distributional Riesz fractional gradient instead the fractional Laplacian respectively, and Gao et al [55] for ground state nontrivial solutions when $f(x,u) = f(u)$.
In this viewpoint, the present chapter aims to improve the previous results related to the (5.9) as well as we would like to improve the assumptions required and to extend the result in [60] to our case considering the presence of distributional Riesz fractional gradient instead of fractional Laplacian operator.
Motivated by the above discussion, mainly by the growing interest in the current literature around distributional Riesz fractional gradient D^σ, in this section we study in Bessel potential space the following class of fractional Schrödinger-Poisson systems

$$\begin{cases} -div^\sigma(D^\sigma u) + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -div^\gamma(D^\gamma \phi) = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.11}$$

where $\sigma, \gamma \in (0,1)$, $4\sigma + 2\gamma > 3$ and $-div^\sigma(D^\sigma)$ is distributional Riesz fractional derivative.
As stated by Shieh and Spector in [91], the distributional Riesz fractional derivative operator

$-div^{\sigma}(D^{\sigma})$ is consistent with the fractional Laplacian operator $(-\Delta)^{\sigma}$. They proved that $(-\Delta)^{\sigma}$ coincides with the composition of distributional Riesz fractional divergence $-div^{\sigma}$ and distributional Riesz fractional gradient D^{σ}. Furthermore, distributional Riesz fractional derivative $-div^{\sigma}(D^{\sigma})$ and fractional Laplacian $(-\Delta)^{\sigma}$ in contrast to the usual Laplacian $(-\Delta)$, which is local operator, they are paradigms of the vast family of nonlocal nonlinear operators, and this has immediate consequences in the formulation of several basic questions such as the Dirichlet problem. For this, for example the Dirichlet datum can be given in $\mathbb{R}^d \setminus \Omega$ (which is different from the classical case of the Laplacian) and not simply on $\partial\Omega$. In addition, the value of $-div^{\sigma}(D^{\sigma})$ and $(-\Delta)^{\sigma}$ at any point x depends not only on the values of u on Ω, but actually on the whole space $\mathbb{R}^d$, which implies that the problem (5.11) is no longer a pointwise equations. Thus, it is often called nonlocal problem. This causes some mathematical difficulties which make the study of such a problem particularly interesting.
Throughout this section, we require the following assumptions on f, K and V.
(F_1) : $f \in C(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R})$ for all $(x, s) \in \mathbb{R}^3 \times \mathbb{R}$, and there exist constant $C_1 > 0$, and $p \in (2; 2^*_{\sigma})$ such that

$$|f(x, s)| \leq C_1(|s| + |s|^{p-1}),$$

where $2^*_{\sigma} = \frac{6}{3-2\sigma}$ is the fractional critical Sobolev exponent.
(F_2) : $f(x, -s) = -f(x, s)$, $x \in \mathbb{R}^3$, $s \in \mathbb{R}$.
(F_3) : $\lim\limits_{|s|\to\infty} \frac{F(x,s)}{|s|^4} = \infty$ uniformly in $x \in \mathbb{R}^3$ and $F(x, s) \geq 0$ for all $(x, s) \in \mathbb{R}^3 \times \mathbb{R}$ where $F(x, s) = \int\limits_0^s f(x, t)dt$.
(F_4) : There exist $\lambda > 0$ such that

$$4F(x, s) \leq sf(x, s) + \lambda|s|^2 \text{ for all } (x, s) \in \mathbb{R}^3 \times \mathbb{R}.$$

(K) : $K(x) \in L^{\frac{6}{4\sigma+2\gamma-3}}(\mathbb{R}^3) \cup L^{\infty}(\mathbb{R}^3)$, $K(x) \geq 0 \; \forall x \in \mathbb{R}^3$.
(V): $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$ where V_0 is a constant, and for any $M > 0$ meas $\{x \in \mathbb{R}^3 : \; V(x) \leq M\} < \infty$.
The main result of this section is as follows.

Theorem 5.2.1. *Let $\sigma, \gamma \in (0, 1)$ with $4\sigma + 2\gamma > 3$. Assume that system* (5.11) *satisfies (F_1)-(F_4), (K) and (V). Then,* (5.11) *has infinitely many nontrivial solutions in E.*

The rest of this section is organized as follows. In the next subsection, we introduce some work space and key results that will be needed later. In the last subsection, we use variational methods in combination with the symmetric mountain pass theorem to prove our main result.

5.2.2 Preliminaries and variational settings

Firstly, we introduce the variational framework for studying system (5.11).
As mentioned in section 5.1, space of fractionally differentiable functions $S^{\sigma,2}(\mathbb{R}^d)$ is the conve-

nient setting for our problem, so we will give some sketches of the vector space of fractional differentiable functions $S^{\sigma,2}(\mathbb{R}^d)$. For $\sigma \in (0,1)$ and $u \in C_0^\infty(\mathbb{R}^d)$, the space $S^{\sigma,2}(\mathbb{R}^d)$ is defined as the closure of $C_0^\infty(\mathbb{R}^d)$ under the norm

$$\|u\|_{S^{\sigma,2}}^2 = \|u\|_2^2 + \|D^\sigma u\|_2^2. \tag{5.12}$$

It is exactly the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ defined for $\sigma \in \mathbb{R}_+$ by

$$L^{\sigma,2}(\mathbb{R}^d) = \{u : u = G_\sigma * f \quad \text{for some} \quad f \in L^2(\mathbb{R}^d)\},$$

where the Bessel potential G_σ is defined in Chapter 1.
The norm of $L^{\sigma,2}(\mathbb{R}^d)$ if $G_\sigma * f$, is $\|u\|_{L^{\sigma,2}} = \|f\|_2$.
The following result summarize the main properties of the space $L^{\sigma,2}(\mathbb{R}^d)$.

Theorem 5.2.2. *1. For $\sigma \in (0,1)$, we have $H^\sigma(\mathbb{R}^d) = W^{\sigma,2}(\mathbb{R}^d) = L^{\sigma,2}(\mathbb{R}^d) = S^{\sigma,2}(\mathbb{R}^d)$.*

2. If $\sigma \geq 0$ and $2 \leq q \leq 2_\sigma^*$, then $L^{\sigma,2}(\mathbb{R}^d)$ is continuously embedded in $L^q(\mathbb{R}^d)$.

Remark 5.2.1.
- Though our working space involves $\|D^\sigma u\|_2$, we will not separate the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ from the fractional Sobolev space $H^\sigma(\mathbb{R}^d)$ despite the fact that $L^{\sigma,2}(\mathbb{R}^d)$ is topologically compatible with $H^\sigma(\mathbb{R}^d)$, and the norm in the two spaces being equivalent given by (5.12).
- An alternative definition of Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$, inspired by the classical construction of Sobolev spaces, would be to consider the space defined as the class of $L^2(\mathbb{R}^d)$ functions whose distributional Riesz fractional gradient is also in $L^2(\mathbb{R}^d)$, that is

$$L^{\sigma,2}(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : D^\sigma u \in L^2(\mathbb{R}^d;\mathbb{R}^d)\right\}.$$

The fractional homogeneous Sobolev space $D^{\sigma,2}(\mathbb{R}^d)$ can be defined also as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{D^{\sigma,2}} = (\int_{\mathbb{R}^d} |D^\sigma u|^2 dx)^{\frac{1}{2}}.$$

Now, we introduce the work space of (5.11) as

$$E = \left\{ u \in L^{\sigma,2}\left(\mathbb{R}^d\right) : \int_{\mathbb{R}^d} |D^\sigma u|^2 + V(x)|u|^2) dx < \infty \right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^d} (|D^\sigma u|^2 + V(x)|u|^2) dx,$$

and the inner product

$$\langle u, w\rangle = \int_{\mathbb{R}^d} (D^\sigma u.D^\sigma w + V(x)\, uw)dx.$$

By assumption (V), we have the following embedding property of E.

Lemma 5.2.1. *[100] E is compactly embedded in $L^q\left(\mathbb{R}^d\right)$ for $q \in [2, 2^*_\sigma)$, and continuously embedded in $L^q\left(\mathbb{R}^d\right)$ for $q \in [2, 2^*_\sigma]$.*

Remark 5.2.2. The previous definitions of $D^{\sigma,2}(\mathbb{R}^d)$ and E, coincides with any standard definitions found in the literature.

Lemma 5.2.2. *[91] For any $\sigma \in (0, 1)$, $D^{\sigma,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\sigma}(\mathbb{R}^3)$, that is, there is $C_\sigma > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2^*_\sigma} dx\right)^{\frac{2}{2^*_\sigma}} \leq C_\sigma \int_{\mathbb{R}^3} |D^\sigma u|^2 dx,\ u \in D^{\sigma,2}\left(\mathbb{R}^3\right).$$

Now, we are going to reduce system (5.11) to a single equation. For any $u \in E$, the linear operator $\mathcal{L}_u : D^{\gamma,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{L}_u(w) = \int_{\mathbb{R}^3} K(x)u^2 w dx,$$

is well defined on $D^{\gamma,2}(\mathbb{R}^3)$ and continuous. In fact, for $K \in L^{\frac{6}{4\sigma+2\gamma-3}}(\mathbb{R}^3)$ using the Hölder inequality, (K) and Lemmas 5.2.1, 5.2.2, we obtain

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|K\|_{\frac{6}{4\sigma+2\gamma-3}} \|u\|^2_{\frac{6}{3-2\sigma}} \|w\|_{2^*_\gamma} \\ &\leq C\|u\|^2 \|w\|_{D^{\gamma,2}}. \end{aligned} \tag{5.13}$$

While for $K \in L^\infty(\mathbb{R}^3)$, we derive that

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|K\|_\infty \|u\|^2_{\frac{12}{3+2\gamma}} \|w\|_{2^*_\gamma} \\ &\leq C\|u\|^2 \|w\|_{D^{\gamma,2}}. \end{aligned} \tag{5.14}$$

Therefore, set

$$\eta(u, w) = \int_{\mathbb{R}^3} D^\gamma u.D^\gamma w dx, \quad \forall u, w \in D^{\gamma,2}(\mathbb{R}^3).$$

Thus, the Lax-Milgram theorem implies that there is a unique $\phi^\gamma_u \in D^{\gamma,2}\left(\mathbb{R}^3\right)$ such that $\mathcal{L}_u(w) =$

$\eta\left(\phi_u^\gamma, w\right)$ for every $w \in D^{\gamma,2}\left(\mathbb{R}^3\right)$, that is,

$$\int_{\mathbb{R}^3} D^\gamma \phi_u^\gamma . D^\gamma w dx = \int_{\mathbb{R}^3} K(x) u^2 w dx. \tag{5.15}$$

In other words, ϕ_u^γ is a weak solution of

$$-div^\gamma\left(D^\gamma \phi_u^\gamma\right) = K(x)u^2.$$

Moreover, the representation formula holds

$$\phi_u^\gamma(x) = c_\gamma \int_{\mathbb{R}^3} \frac{K(x)u^2(y)}{|x-y|^{3-2\gamma}} dy, \quad x \in \mathbb{R}^3, \tag{5.16}$$

which is called γ-Riesz potential (see [97]), where

$$c_\gamma = 2^{-2\gamma} \frac{\Gamma\left(\frac{3-2\gamma}{2}\right)}{\pi^{\frac{3}{2}} \Gamma\left(\gamma\right)}.$$

Taking $w = \phi_u^\gamma$ in (5.13), (5.14) and (5.15), we get

$$\|\phi_u^\gamma\|_{D^{\gamma,2}} \leq C \left\|u\right\|^2. \tag{5.17}$$

Consequently, we deduce that

$$\int_{\mathbb{R}^3} K(x) \phi_u^\gamma u^2 dx \leq C \left\|u\right\|^4. \tag{5.18}$$

Substituting ϕ_u^γ in (5.11), then (5.11) can be reduced to the following single fractional Schrödinger equation

$$-div^\sigma(D^\sigma u) + V(x)u + K(x)\phi_u^\gamma u = f(x,u), \quad x \in \mathbb{R}^3. \tag{5.19}$$

Whose solutions can be obtained by seeking critical values of the energy functional $\Phi : E \to \mathbb{R}$

$$\Phi\left(u\right) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|D^\sigma u|^2 + V(x)u^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} F\left(x,u\right) dx. \tag{5.20}$$

Hence, Φ is well defined in E and $\Phi \in C^1(E, \mathbb{R})$, and for all $u, w \in E$, its derivative

$$< \Phi'\left(u\right), w >= \int_{\mathbb{R}^3} (D^\sigma u . D^\sigma w + V(x)uw + K(x)\phi_u^\gamma uw - f\left(x,u\right)w) dx. \tag{5.21}$$

Definition 5.2.1. 1. If $u \in E$ is a weak solution of (5.19), then $\left(u, \phi_u^\gamma\right) \in E \times D^{\gamma,2}\left(\mathbb{R}^3\right)$ is a weak solution of 5.11.

2. $u \in E$ is a weak solution of (5.19) if

$$\int_{\mathbb{R}^3} (D^\sigma u.D^\sigma w + V(x)uw + K(x)\phi_u^\gamma uw - f(x,u)\, w)dx = 0.$$

5.2.3 Proof of main result

In this section, the existence result of infinitely many solutions to (5.11) is provided by taking into account the symmetric mountain pass theorem. Before going to prove our main result, we need to shows that the functional Φ_λ satisfies the the (C)$_c$-condition.

Lemma 5.2.3. *Φ satisfies the (C)$_c$-condition on E, if the assumptions (F_1)-(F_4), (K) and (V) hold.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be the (C)$_c$ sequence in E, That is

$$\Phi(u_n) \to c \quad \text{and} \quad \|\Phi'(u_n)\|_*(1 + \|u_n\|) \to 0 \;\text{ as }\; n \to \infty,$$

which means

$$c = \Phi(u_n) + o_n(1) \;\text{ and }\; \langle \Phi'(u_n), u_n\rangle = o_n(1). \tag{5.22}$$

First, we argue by contradiction to verify that $\{u_n\}$ is bounded in E. Assume that $\|u_n\| \to \infty$ as $n \to \infty$. Consider a sequence $\{v_n\} \subset E$ such that $v_n = \frac{u_n}{\|u_n\|}$. Then it is obvious that $\|v_n\| = 1$. Hence, there is a subsequence $\{v_n\}$ (we still write it as v_n) such that $v_n \rightharpoonup v$ in E as $n \to \infty$. We have by Lemma 5.2.1 that,

$$\begin{aligned} v_n &\to v \;\text{ a.e. in }\; \mathbb{R}^3 \\ v_n &\to v \;\text{ in }\; L^p\left(\mathbb{R}^3\right) \;\text{ for }\; 2 \le p < 2_\gamma^*, \end{aligned} \tag{5.23}$$

as $n \to \infty$. There are two possible cases.
Case $v(x) \neq 0$ in $\mathbb{R}^3$. Set $\Sigma_0 = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, thus meas$(\Sigma_0) > 0$. Since $\|u_n\| \to \infty$ as $n \to \infty$ and by (5.20) and (5.22), we get

$$c = \Phi(u_n) + o_n(1) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx - \int_{\mathbb{R}^3} F(x,u_n)dx + o_n(1). \tag{5.24}$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, we derive

$$\int_{\mathbb{R}^3} F(x,u_n)dx \ge \frac{1}{2}\|u_n\|^2 - c + o_n(1) \to \infty \;\text{ as } n \;\to \infty. \tag{5.25}$$

When combining (5.18) and (5.24), we infer for n large enough that the following holds

$$\int_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n^2 dx \le \frac{3}{4}\|u_n\|^4. \tag{5.26}$$

Moreover, the assumption (F_3) implies that there is $z_0 > 1$ such that $F(x,u) > |u|^4$ for all $x \in \mathbb{R}^3$ and $|u| > z_0$. By means of (F_1), we derive that there is $C > 0$ such that $|F(x,u)| < C$ for all $(x,u) \in \mathbb{R}^3 \times [-z_0, z_0]$. Hence, there is $C_0 \in \mathbb{R}$ such that $F(x,u) \geq C_0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$. Thus, we have

$$\frac{F(x,u_n) - C_0}{\|u_n\|^4} \geq 0. \tag{5.27}$$

By the convergence (5.23), we have $|u_n| = |v_n|\|u_n\| = \infty$ as $n \to \infty$ for all $x \in \Sigma_0$. Hence, the assumption (F_3) implies that

$$\lim_{n\to\infty} \frac{F(x,u_n)}{\|u_n\|^4} = \lim_{n\to\infty} \frac{F(x,u_n)}{|u_n|^4}|v_n|^4 = \infty, \tag{5.28}$$

for all $x \in \Sigma_0$.

Thus, we see that meas$(\Sigma_0) = 0$. Indeed, if meas$(\Sigma_0) \neq 0$, combining with (5.26)-(5.28) and the Fatou lemma we infer that

$$\begin{aligned}
\frac{3}{4} &= \liminf_{n\to\infty} \frac{\frac{3}{4}\int\limits_{\mathbb{R}^3} F(x,u_n)dx}{\int\limits_{\mathbb{R}^3} F(x,u_n)dx + c - o_n(1)} \\
&\geq \liminf_{n\to\infty} \int\limits_{\mathbb{R}^3} \frac{\frac{3}{4}F(x,u_n)}{\frac{3}{4}\|u_n\|^4}dx \\
&\geq \liminf_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n)}{\|u_n\|^4}dx - \limsup_{n\to\infty} \int\limits_{\Sigma_0} \frac{C_0}{\|u_n\|^4}dx \\
&= \liminf_{n\to\infty} \int\limits_{\Sigma_0} \frac{F(x,u_n) - C_0}{\|u_n\|^4}dx. \\
&\geq \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n) - C_0}{\|u_n\|^4}dx \\
&= \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{F(x,u_n)}{\|u_n\|^4}dx - \int\limits_{\Sigma_0} \limsup_{n\to\infty} \frac{C_0}{\|u_n\|^4}dx \\
&= \infty,
\end{aligned} \tag{5.29}$$

which is a contradiction. This means that $v(x) = 0$ for a.e. $x \in \mathbb{R}^3$.

Case $v(x) = 0$. From (F_4), for sufficiently n large one has

$$\begin{aligned} c+1 &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{4}\right)\|u_n\|^2 + \int_{\mathbb{R}^3}\left(\frac{f(x,u_n)u_n}{4} - F(x,u_n)\right)dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3}|u_n|^2 dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{\lambda}{4}\|v_n\|_2^2\|u_n\|^2, \end{aligned}$$

which implies

$$\frac{c+1}{\|u_n\|^2} \geq \frac{1}{4} - \frac{\lambda}{4}\|v_n\|_2^2.$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, then

$$\|v_n\|_2^2 \geq \frac{1}{\lambda}.$$

This implies that

$$1 \leq \lambda \limsup_{n\to\infty}\|v_n\|_2^2 = \lambda\|v\|_2^2,$$

which shows that $v(x) \neq 0$, then we arrive at contradiction in any cases. Hence, $\{u_n\}$ is bounded in E. Up to a subsequence, we have that $u_n \rightharpoonup u$ in E, from Lemma 5.2.1 we conclude that $u_n \to u$ in $L^p(\mathbb{R}^3)$, for all $2 \leq p < 2_\sigma^*$. Clearly, we have $\|u_n - u\|_2^2 \to 0$ as $n \to \infty$.
Because $u_n \rightharpoonup u$ in E and $\Phi'_\lambda(u_n) \to 0$ in E^* as $n \to \infty$, we have

$$\langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle \to 0.$$

For $K \in L^\infty(\mathbb{R}^3)$, from (K) and the Hölder inequality we infer

$$\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n(u_n-u)\,dx\right| \leq \|K\|_\infty \left\|\phi_{u_n}^\gamma\right\|_{\frac{6}{3-2\gamma}}\|u_n\|_{\frac{12}{3+2\gamma}}\|u_n-u\|_{\frac{12}{3+2\gamma}} \to 0 \text{ as } n\to\infty.$$

For $K \in L^{\frac{6}{4\sigma+2\gamma-3}}(\mathbb{R}^3)$, by Lemma 2.2.5 (see [81], Proposition 2.4 for the proof), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^\gamma u_n(u_n-u)\,dx = \int_{\mathbb{R}^3} K(x)\phi_u^\gamma u(u_n-u)\,dx.$$

Thus, we deduce that

$$\left|\int_{\mathbb{R}^3} K(x)\left(\phi_{u_n}^{\gamma} u_n-\phi_u^{\gamma} u\right)(u_n-u)\,dx\right| \to 0 \text{ as } n\to\infty.$$

According to (F_1) and the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)(u_n-u)\,dx\right| \\
&\leq C_1\int_{\mathbb{R}^3}(|u_n|+|u|)|u_n-u|\,dx+C_1\int_{\mathbb{R}^3}\left(|u_n|^{p-1}+|u|^{p-1}\right)|u_n-u|\,dx \\
&\leq C_1\left(\|u_n\|_2+\|u\|_2\right)\|u_n-u\|_2+C_1\left(\|u_n\|_p^{p-1}+\|u\|_p^{p-1}\right)\|u_n-u\|_p \\
&\leq C\left(\|u_n\|+\|u\|\right)\|u_n-u\|_2+C\left(\|u_n\|^{p-1}+\|u\|^{p-1}\right)\|u_n-u\|_p\to 0
\end{aligned}$$

as $n\to\infty$. Thus, we get

$$\begin{aligned}
\|u_n-u\|^2 &= \langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle-\int_{\mathbb{R}^3}K(x)\left(\phi_{u_n}^{\gamma}u_n-\phi_u^{\gamma}u\right)(u_n-u)\,dx \\
&\quad+\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)(u_n-u)dx\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore, $\{u_n\}$ converges strongly in E. □

To prove Theorem 6.2.1, we need to apply the symmetric mountain pass theorem under the (C)$_c$-condition. To do this, we choose $\{e_i\}_i$ an orthonormal basis of E and define $X_i=\mathbb{R}e_i$,

$$Y_k=\oplus_{i=1}^{k}X_i,\quad X_k=\overline{\oplus_{i=k+1}^{\infty}X_i},\quad k\in\mathbb{Z}.$$

Clearly, we have $E=Y_k\oplus X_k$.
Thus, its exact statement is the following.

Theorem 5.2.3. *Let $E=Y_k\oplus X_k$ be a Banach space where Y is finite dimensional. Suppose that $\Phi\in C^1(X,\mathbb{R})$ an even functional such that $\Phi(0)=0$, satisfies the (C)$_c$-condition, if*
(i) There exist constants $\rho,\delta>0$ satisfying $\Phi\big|_{\partial B_\rho\bigcap X_k}\geq\delta$, where B_ρ is the open ball in X of radius ρ about 0 and ∂B_ρ is its boundary.
(ii) for any finite dimensional subspace $\tilde{X}\subset X$, there is $C=C(\tilde{X})>0$ such that $\Phi(u)\leq 0$ on $\tilde{X}\setminus B_C$.
Then, Φ has an unbounded sequence of critical points.

Lemma 5.2.4. *Suppose that the assumptions (F_1), (F_3) and (V) are satisfied. Then, for any finite dimensional subspace $\tilde{E} \subset E$, we have*

$$\Phi(u) \to -\infty, \qquad \|u\| \to \infty, \quad u \in \tilde{E}.$$

Proof. We argue indirectly, suppose that there is $C > 0$ for some $\{u_n\} \subset \tilde{E}$ and all $n \in \mathbb{N}$, such that $\Phi(u_n) \geq -C$ with $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to a subsequence, we may suppose that $v_n \rightharpoonup v$ in E. Since $\dim(\tilde{E}) < \infty$, then $v_n \to v \in \tilde{E}$ in E, $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^3$ and so $\|v\| = 1$. Set $\Sigma_0 = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, thus meas$(\Sigma_0) > 0$, and we have $|u_n(x)| \to \infty$ for almost every $x \in \Sigma_0$.
Dividing by $\|u_n\|^4$ in both sides of (5.20) and by (5.18), we get

$$\lim_{n\to\infty} \frac{4\int\limits_{\mathbb{R}^3} F(x,u_n)}{\|u_n\|^4} dx = \lim_{n\to\infty} \frac{2\|u_n\|^2 + \int\limits_{\mathbb{R}^3} K(x)\phi_{u_n}^{\gamma} u_n^2 - 4\Phi(u_n)}{\|u_n\|^4} dx \leq C, \tag{5.30}$$

for $x \in \Sigma_0$. Since $|u_n(x)| \to \infty$, it follows from similar arguments as in (5.27)-(5.29) with the assumption (F_3) that, for n large

$$\begin{aligned}\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \frac{4F(x,u_n)}{\|u_n\|^4} dx &\geq \lim_{n\to\infty} \int\limits_{\Sigma_0} \frac{4F(x,u_n)}{\|u_n\|^4} dx \\ &\geq \int\limits_{\Sigma_0} \liminf_{n\to\infty} \frac{4F(x,u_n)}{|u_n|^4} |v_n|^4 dx = \infty,\end{aligned}$$

which is a contradiction with (5.30). □

Corollary 5.2.1. *Assume that (F_1), (F_3) and (V) are satisfied. Then, there exists a constant $C = C(\tilde{E}) > 0$ for every finite dimensional subspace $\tilde{E} \subset E$ such that*

$$\Phi(u) \leq 0 \quad \textit{for all} \quad u \in \tilde{E} \quad \textit{with} \quad \|u\| \geq C.$$

Lemma 5.2.5. *For $2 \leq p < 2_\gamma^*$ and $k \in \mathbb{N}$, we have that*

$$\Gamma_k := \sup_{u \in X_k, \|u\|=1} \|u\|_p \to 0 \quad \textit{as} \quad k \to \infty$$

We can choose by Lemma 5.2.5 an integer $k \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2C_1}\|u\|^2, \quad \|u\|_p^p \leq \frac{p}{4C_1}\|u\|^p \quad \forall u \in X_k. \tag{5.31}$$

Lemma 5.2.6. *If (F_1) and (V) hold. Then, there exist constants $\rho, \delta > 0$ satisfying $\Phi\Big|_{\partial B_\rho \bigcap X_k} \geq \delta > 0$.*

Proof. According to (F_1), we have

$$|F(x,u)| \leq \frac{C_1}{2}|u|^2 + \frac{C_1}{p}|u|^p \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{5.32}$$

For $u \in X_k$, by (5.31) and (5.32), choosing $\rho := \|u\| = \frac{1}{2}$, we derive

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int\limits_{\mathbb{R}^3}\left(|D^\sigma u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_u^\gamma u^2 dx - \int\limits_{\mathbb{R}^3} F(x,u)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C_1}{2}\|u\|_2^2 - \frac{C_1}{p}\|u\|_p^p \\ &\geq \frac{1}{4}(\|u\|^2 - \|u\|^p) \\ &= \frac{2^{p-2}-1}{2^{p+2}} := \delta > 0.\end{aligned}$$

This is a desired result.

Proof of Theorem 5.2.1. Let $Y = Y_k$ and $X = X_k$. Due to the assumption (F_2), it is obvious that $\Phi(u)$ is an even functional. Based on Lemmas 5.2.3, 5.2.5 and Corollary 5.2.1, the functional $\Phi(u)$ satisfies all conditions of the symmetric mountain pass theorem. Consequently, the system (5.11) possesses infinitely many nontrivial solutions. □

CHAPTER 6

Multiplicity of solutions for nonlinear fractional Schrödinger-Poisson system with (PS) condition

This chapter includes the results of the following research article:

- H.Boutebba, H.Lakhal, and K.Slimani. (2024). Existence of multiple solutions for nonlinear fractional Schrödinger-Poisson system involving new fractional operator. International Journal of Nonlinear Analysis and Applications, 15(11), 83-92.

In this chapter, we continue to focus our attention on study of the existence and multiplicity of nontrivial solutions in Bessel potential space for a class of fractional systems with concave-convex nonlinearities driven by the so-called distributional Riesz fractional gradient operator. The main achievement of the present chapter consists in proving the Palais-Smale condition and the conditions of the symmetric mountain pass theorem.

6.1 Introduction

In the last few years, nonlinear systems involving fractional and nonlocal differential operators of elliptic type, have been studied extensively by many scholars due to their ability to capture effects that local operators cannot describe. These effects include numerous complex phenomena, such as electrical circuits, diffusion, phase transitions, finance, quantum mechanics and obstacle problem. For more related results, we refer the reader to [17, 18, 27, 48, 72, 79, 91].
Recently, due to their real physical meaning, the fractional Schrödinger-Poisson system has been extensively studied by many researchers. Benci and Fortunato in [16] proposed the following classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{6.1}$$

to describe quantum particles governed by nonlinear Schrödinger equations interacting with an unknown electrostatic field. It also appears in plasma physics, semiconductor theory, and other related applications. The nonlinearity f represents the particles interacting with each other, and the nonlocal term ϕu concerns the interaction with the electric field. We refer the reader interested in further details to [7, 84, 113, 114] and their references to get more physical background to the system (6.1).
In the last decade, there have been many interesting works about the existence of infinitely many solutions, positive solutions, concentration of solutions, ground state solutions, and multiplicity of solutions for Schrödinger-Poisson systems with fractional operators via variational methods and critical point theory, see for instance [20, 22, 58, 60, 63, 65, 81, 110] and the references therein. Among these works, we highlight [34] where Che and Chen studied the following system

$$\begin{cases} (-\Delta)^{\sigma} u + V_{\lambda}(x)u + \lambda\phi u = f(x,u) + g(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\gamma} \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{6.2}$$

where $\sigma, \gamma \in (0,1]$, $\lambda > 0$, $2\gamma + 2\sigma > 3$, $1 < q < 2$, $V_{\lambda}(x)$ is allowed to be sign-changing potential, and $(-\Delta)^{\sigma}$ is the fractional Laplacian operator, under some assumptions on f and V_{λ}, multiplicity and concentration of solutions are obtained. In [36], similar system to (6.2) was studied by Chen-Li-Wei. More precisely, they proved the existence of multiple solutions for the system of the following form

$$\begin{cases} (-\Delta)^{\sigma} u + V(x)u + \phi u = f(x,u) + \lambda g(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\gamma} \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{6.3}$$

where $\sigma, \gamma \in (0, 1]$, $\lambda > 0$, $1 < q < 2$, $2\gamma + 4\sigma > 3$. Notice that when $\lambda = 0$, system (6.3) reduces to the system as below

$$\begin{cases} (-\Delta)^\sigma u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\gamma \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{6.4}$$

In recent years, a system similar to (6.4) has been extensively studied by many scholars. For example, Gao et al [55] studied ground state solutions when $f(x,u) = f(u)$. Li [74] analyzed a nontrivial solution when $V(x) = 1$. Zhang [115] investigated the existence and multiplicity of solutions.

As a consequence of the pioneering works of Shieh and Spector [91, 92] concerning the study of a new class of fractional partial differential equations depending on the distributional Riesz fractional gradient, an increasing number of authors have been interested in studying its theoretical structure see for instance [26, 48, 79, 82, 91, 92, 93], and in understanding its applications in the theory of electromagnetic fields, multidimensional processes, and in fractal media see e.g [8, 54, 85] and their works. Such an operator is an intrinsic object for the study of problems in the calculus of variations and fractional partial differential equations and it seems to be the natural notion for a fractional differential object. This operator, which is the focus of this chapter, the unique object satisfying some basic physical and mathematical requirements as proved in [93] on fractional gradient analysis.

In the present section, we build upon all the works just described, by using the distributional Riesz fractional derivative instead of the usual fractional Laplacian, we study a new class of fractional Schrödinger-Poisson systems involving concave-convex nonlinearities, which take the following form

$$\begin{cases} -div^\sigma(D^\sigma u) + V(x)u + \phi u = f(x,u) + \lambda g(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -div^\gamma(D^\gamma \phi) = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{6.5}$$

where $\sigma, \gamma \in (0, 1]$ with $\sigma > \frac{3}{4}$ satisfies $2\gamma + 4\sigma > 3$, $\lambda > 0$ is a parameter, $q \in (1, 2)$, and $div^\sigma(D^\sigma)$ is the distributional Riesz fractional derivative.

In this works, we impose the following assumptions on f, g and V.

(F_1) : $f \in C(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R})$ for every $x \in \mathbb{R}^3$ and $s \in \mathbb{R}$, there exists constant $C_1 > 0$, and $p \in (4; 2^*_\sigma)$ such that

$$|f(x,s)| \le C_1(|s| + |s|^{p-1})$$

where $2^*_\sigma = \frac{6}{3-2\sigma}$ is the fractional critical Sobolev exponent.

(F_2) : $f(x, -s) = -f(x, s)$, $x \in \mathbb{R}^3$, $s \in \mathbb{R}$.

(F_3) : There exist $\mu > 4$ and $l > 0$ such that

$$0 < \mu F(x,s) \le s f(x,s)$$

holds $\forall x \in \mathbb{R}^3$ and $|s| \geq l$ where $F(x,s) = \int\limits_0^s f(x,t)dt$.

(F_4) $\inf_{x\in\mathbb{R}^3,|s|=1} F(x,s) > 0$.
(G) : $g : \mathbb{R}^3 \to \mathbb{R}^+$ and $g(x) \in L^{\frac{2}{2-q}}(\mathbb{R}^3)$.
(V) : $V \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$, where V_0 is a constant, and for every $M > 0$ meas $\{x \in \mathbb{R}^3 : \ V(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure.
We next fix the following notation.

For $1 < q < 2$, $L^q\left(\mathbb{R}^d\right)$ is the weighted Lebesgue space with the norm $\|u\|_q = (\int\limits_{\mathbb{R}^d} g(x)\,|u|^q dx)^{\frac{1}{q}}$.

Under the above assumptions, our main result is the following theorem.

Theorem 6.1.1. *Let $\sigma, \gamma \in (0,1]$ with $\sigma > \frac{3}{4}$. Assume that (F_1)-(F_4), (G) and (V) are satisfied. Then, there exists $\Lambda_* > 0$ such that system* (6.5) *possesses multiple nontrivial solutions in Bessel potential space with negative energy for any $\lambda \in (0, \Lambda_*)$.*

Remark 6.1.1.

1. The assumptions (F_3) and (F_4) implies that the range of p in the assumption (F_1) should be $(4; 2^*_\sigma)$ not $(2; 2^*_\sigma)$. Indeed, if $p \leq 4$, by assumption (F_1), one has

$$|F(x,s)| \leq \int\limits_0^s |f(x,t)|dt \leq C_1 \int\limits_0^s (|t| + |t|^{p-1})dt \leq \frac{C_1}{2}|s|^2 + \frac{C_1}{p}|s|^p$$

for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$, then we deduce that

$$\limsup_{|s|\to+\infty} \frac{F(x,s)}{s^4} \leq C \ \text{ uniformly in } \ x \in \mathbb{R}^3. \tag{6.6}$$

For any $x \in \mathbb{R}^3$, $l \in \mathbb{R}$, define

$$h(s) := F(x, s^{-1}l)s^\mu, \quad s \geq 1.$$

Then, by (F_3), one has

$$h'(s) = s^{\mu-1}[\mu F(x, s^{-1}l) - s^{-1}l f(x, s^{-1}l)] \leq 0$$

for $|l| \geq 1$, $s \in [1, |l|]$. That is, $h(1) \geq h(|l|)$. Therefore, by (F_4), we derive

$$F(x,l) \geq F\left(x, \frac{l}{|l|}\right)|l|^\mu \geq \inf_{x\in\mathbb{R}^3,|s|=1} F(x,s)|l|^\mu$$

for $x \in \mathbb{R}^3$ and $|l| \geq 1$, which contradicts (6.6). That is (F_3) implies that the range of p in (F_1) should be $(4, 2^*_\sigma)$.

2. The choice of $\sigma > \frac{3}{4}$ ensures that the interval $(4; 2^*_\sigma)$ in (F_1) is nondegenerated.

The rest of this chapter is structured as follows. In the next section, we introduce some work space and key results that will be needed later. In the last section, we use variational methods together with the symmetric mountain pass theorem to prove our main result.

6.2 Preliminaries and variational settings

Firstly, let us recall briefly some definitions and basic properties for Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ and fractional Sobolev space $D^{\sigma,2}(\mathbb{R}^d)$ that are necessary for the study of the given problem. Fixing $\sigma \in (0,1)$. We define the vector space of fractional differentiable functions $S^{\sigma,2}(\mathbb{R}^d)$ for sufficiently smooth functions as the closure of $C_0^\infty(\mathbb{R}^d)$ with the norm

$$\|u\|^2_{S^{\sigma,2}} = \|u\|_2^2 + \|D^\sigma u\|_2^2. \tag{6.7}$$

By Theorem 5.1.2, it is exactly the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ defined for $\sigma \in \mathbb{R}_+$ by

$$L^{\sigma,2}(\mathbb{R}^d) = \{u : u = G_\sigma * f \quad \text{for some} \quad f \in L^2(\mathbb{R}^d)\},$$

where the Bessel potential G_σ is defined in Chapter 1.
The norm of the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ if $G_\sigma * f$, is $\|u\|_{L^{\sigma,2}} = \|f\|_2$.
The following theorem recall the key properties of the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$.

Theorem 6.2.1. *1. For $\sigma \in (0,1)$, we have $H^\sigma(\mathbb{R}^d) = W^{\sigma,2}(\mathbb{R}^d) = L^{\sigma,2}(\mathbb{R}^d) = S^{\sigma,2}(\mathbb{R}^d)$.*

2. If $\sigma \geq 0$ and $2 \leq q \leq 2^*_\sigma$, then $L^{\sigma,2}(\mathbb{R}^d)$ is continuously embedded in $L^q(\mathbb{R}^d)$.

We define the homogeneous fractional Sobolev space $D^{\sigma,2}(\mathbb{R}^d)$ in term of the distributional Riesz fractional gradient as follows

$$D^{\sigma,2}(\mathbb{R}^d) = \left\{u \in L^{2^*_\sigma}(\mathbb{R}^d) : D^\sigma u \in L^2(\mathbb{R}^d)\right\}.$$

The fractional homogeneous Sobolev space $D^{\sigma,2}(\mathbb{R}^d)$ can be defined also as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{D^{\sigma,2}} = \Big(\int_{\mathbb{R}^d} |D^\sigma u|^2 dx\Big)^{\frac{1}{2}}.$$

Now, we introduce the work space of our problem as follows

$$E = \left\{u \in L^{\sigma,2}\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} (|D^\sigma u|^2 + V(x)|u|^2) dx < \infty\right\},$$

with the norm

$$\|u\|^2 = \int\limits_{\mathbb{R}^3} (|D^\sigma u|^2 + V(x)|u|^2)dx.$$

Lemma 6.2.1. *[88] Assume that (V) hold. Then, E is compactly embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2^*_\sigma)$, and continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2^*_\sigma]$.*

By the previous Lemma, one obtains that there exists a positive constant C_q such that

$$|u|_q \leq C_q \|u\|. \tag{6.8}$$

Lemma 6.2.2. *([45]) For any $\sigma \in (0,1)$, $D^{\sigma,2}(\mathbb{R}^d)$ is continuously embedded in $L^{2^*_\sigma}(\mathbb{R}^d)$, i.e there exists $C_\sigma > 0$ such that :*

$$\left(\int\limits_{\mathbb{R}^d} |u|^{2^*_\sigma} dx\right)^{\frac{2}{2^*_\sigma}} \leq C_\sigma \int\limits_{\mathbb{R}^d} |D^\sigma u|^2 dx,\ u \in D^{\sigma,2}\left(\mathbb{R}^d\right).$$

If $2\gamma + 4\sigma > 3$, thus E is continuously embedded into $L^{\frac{12}{3+2\gamma}}(\mathbb{R}^3)$. For any $u \in E$, the linear operator $\mathcal{L}_u : D^{\gamma,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined as:

$$\mathcal{L}_u(w) = \int\limits_{\mathbb{R}^3} u^2 w dx.$$

By Hölder inequality and Lemmas 6.2.1, 6.2.2, we obtain

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|u\|^2_{\frac{12}{3+2\gamma}} \|w\|_{2^*_\gamma} \\ &\leq C\|u\|^2 \|w\|_{D^{\gamma,2}}. \end{aligned} \tag{6.9}$$

Hence, the Lax-Milgram theorem implies that there exists a unique $\phi_u^\gamma \in D^{\gamma,2}(\mathbb{R}^3)$ such that

$$\int\limits_{\mathbb{R}^3} D^\gamma \phi_u^\gamma . D^\gamma w dx = \int\limits_{\mathbb{R}^3} u^2 w dx \quad \forall w \in D^{\gamma,2}\left(\mathbb{R}^3\right). \tag{6.10}$$

i.e. ϕ_u^γ is a weak solution of

$$-div^\gamma (D^\gamma \phi_u^\gamma) = u^2.$$

From Lemma 6.2.2 and relations (6.9)-(6.10), we derive

$$\begin{aligned} \|\phi_u^\gamma\|^2_{D^{\gamma,2}} = \int\limits_{\mathbb{R}^3} |D^\gamma \phi_u^\gamma|^2 dx &\leq \|u\|^2_{\frac{12}{3+2\gamma}} \|\phi_u^\gamma\|_{2^*_\gamma} \\ &\leq C\|u\|^2_{\frac{12}{3+2\gamma}} \|\phi_u^\gamma\|_{D^{\gamma,2}}. \end{aligned} \tag{6.11}$$

Then,

$$\|\phi_u^\gamma\|_{D^{\gamma,2}} \leq C \|u\|^2_{\frac{12}{3+2\gamma}} . \tag{6.12}$$

For $x \in \mathbb{R}^3$, we have

$$\phi_u^\gamma(x) = c_\gamma \int\limits_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2\gamma}} dy, \tag{6.13}$$

which is called γ-Riesz potential (see [97]), where

$$c_\gamma = \pi^{-\frac{3}{2}} 2^{-2\gamma} \frac{\Gamma\left(\frac{3-2\gamma}{2}\right)}{\Gamma(\gamma)}.$$

Substituting ϕ_u^γ in (6.5), it leads to the following single equation

$$-div^\sigma(D^\sigma u) + V(x)u + \phi_u^\gamma u = f(x,u) + \lambda g(x)|u|^{q-2}u, \quad x \in \mathbb{R}^3 . \tag{6.14}$$

We define the energy functional $\Phi_\lambda : E \to \mathbb{R}$ associated to (6.5) by

$$\Phi_\lambda(u) = \frac{1}{2} \int\limits_{\mathbb{R}^3} \left(|D^\sigma u|^2 + V(x)|u|^2\right) dx + \frac{1}{4} \int\limits_{\mathbb{R}^3} \phi_u^\gamma u^2 dx - \int\limits_{\mathbb{R}^3} F(x,u)\, dx - \frac{\lambda}{q} \int\limits_{\mathbb{R}^3} g(x)|u|^q dx.$$

Hence, Φ_λ is well defined in E and $\Phi_\lambda \in C^1(E,\mathbb{R})$. Moreover, its Fréchet derivative for all $w \in E$, is

$$< \Phi'_\lambda(u), w >= \int\limits_{\mathbb{R}^3} (D^\sigma u.D^\sigma w + V(x)uw + \phi_u^\gamma uw - f(x,u)w - \lambda g(x)|u|^{q-2}uw) dx. \tag{6.15}$$

Definition 6.2.1. 1. If $u \in E$ is a weak solution of (6.14), then the pair $(u, \phi_u^\gamma) \in E \times D^{\gamma,2}(\mathbb{R}^3)$ is a weak solution of (6.5) .

2. $u \in E$ is a weak solution of (6.14) if

$$\int\limits_{\mathbb{R}^3} \left(D^\sigma u.D^\sigma w + V(x)uw + \phi_u^\gamma uw - f(x,u)\, w - \lambda g(x)|u|^{q-2}uw\right) dx = 0.$$

6.3 Proof of main result

In this section, we first start by proving the following Lemma which plays a central role in establishing the existence of a nontrivial weak solution to the given problem. Then, we demonstrate that the functional Φ_λ possesses the remaining geometric conditions in the symmetric mountain pass theorem.

Lemma 6.3.1. *The functional Φ_λ satisfies the (PS)$_c$-condition on E, if (F_1)-(F_3), (G) and (V) hold.*

Proof. We will prove that $\{u_n\}$ bounded in E using arguing by contradiction.
Let $\{u_n\}$ be a (PS)$_c$ sequence in E such that

$$\Phi_\lambda(u_n) \to c \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0 \;\text{ as }\; n \to \infty.$$

This implies that for sufficiently large n

$$\Phi_\lambda(u_n) \leq c+1, \quad \text{and} \quad \|\Phi'_\lambda(u_n)\|_* \to 0.$$

Hence, for $\varepsilon = \mu > 0$ and for n large enough

$$\|\Phi'_\lambda(u_n)\|_* \leq \mu. \tag{6.16}$$

Observe that by (6.16) for n large enough

$$\begin{aligned} -\langle \Phi'_\lambda(u_n), u_n\rangle &\leq \|\Phi'_\lambda(u_n)\|_* \|u_n\| \\ &\leq \mu \|u_n\|. \end{aligned}$$

As a result, we obtain for n large enough

$$c+1+\|u_n\| \geq \Phi_\lambda(u_n) - \frac{1}{\mu}\langle \Phi'_\lambda(u_n), (u_n)\rangle \tag{6.17}$$

Combining (6.8) and (6.17) with the assumptions (F_3), (G), we get

$$\begin{aligned} c+1+\|u_n\| &\geq \Phi_\lambda(u_n) - \frac{1}{\mu}\langle \Phi'_\lambda(u_n), (u_n)\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2 + \left(\frac{1}{4}-\frac{1}{\mu}\right)\int\limits_{\mathbb{R}^3} \phi^{\gamma}_{u_n} u_n^2 dx + \int\limits_{\mathbb{R}^3}\left(\frac{u_n f(x,u_n)}{\mu} - F(x,u_n)\right)dx \\ &\quad + \lambda\left(\frac{1}{\mu}-\frac{1}{q}\right)\int\limits_{\mathbb{R}^3} g(x)|u_n|^q dx \\ &\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2 + \lambda\left(\frac{1}{\mu}-\frac{1}{q}\right)\|g\|_{\frac{2}{2-q}} C_q \|u_n\|^q . \end{aligned} \tag{6.18}$$

Which implies that $\{u_n\}$ is bounded in E. Up to a subsequence, we suppose that $u_n \rightharpoonup u$ in E. Then, by Lemma 6.2.1 we have

$$u_n \to u \text{ in } L^p\left(\mathbb{R}^3\right) \;\text{ when }\; 2 \leq p < 2^*_\sigma. \tag{6.19}$$

By (6.19) and Theorem 1.2.2, there exist a subsequence $\{u_n\}$, still denoted by $\{u_n\}$, and $h_1 \in L^p(\mathbb{R}^3)$ and $h_2 \in L^p(\mathbb{R}^3)$ such that

$$|u_n(x)| \leq h_1(x) \quad |u(x)| \leq h_2(x) \quad \text{a.e. in } \mathbb{R}^3,\ \forall n \in \mathbb{N}.$$

On the other hand, obviously we have $\|u_n - u\|_2^2 \to 0$ as $n \to \infty$, and because $u_n \rightharpoonup u$ in E and $\Phi'_\lambda(u_n) \to 0$ in E^* as $n \to \infty$, we have

$$\langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u \rangle \to 0. \tag{6.20}$$

Combining the Hölder inequality, Lemma 6.2.2 and (6.12), we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \phi_{u_n}^\gamma u_n (u_n - u)\, dx \right| &\le \left\| \phi_{u_n}^\gamma \right\|_{2_\gamma^*} \|u_n\|_{\frac{12}{3+2\gamma}} \|u_n - u\|_{\frac{12}{3+2\gamma}} \\
&\le C \left\| \phi_{u_n}^\gamma \right\|_{D^{\gamma,2}} \|u_n\|_{\frac{12}{3+2\gamma}} \|u_n - u\|_{\frac{12}{3+2\gamma}} \\
&\le C \|u_n\|^3_{\frac{12}{3+2\gamma}} \|u_n - u\|_{\frac{12}{3+2\gamma}} \\
&\le C \|u_n\|^3 \|u_n - u\|_{\frac{12}{3+2\gamma}} \\
&\to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Similarly, we can prove that

$$\left| \int_{\mathbb{R}^3} \phi_u^\gamma u (u_n - u)\, dx \right| \le C \|u\|^3 \|u_n - u\|_{\frac{12}{3+2\gamma}} \to 0 \ \text{ as } \ n \to \infty.$$

Hence, we have,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \left(\phi_{u_n}^\gamma u_n - \phi_u^\gamma u \right) (u_n - u)\, dx \right| \le \left| \int_{\mathbb{R}^3} \phi_{u_n}^\gamma u_n (u_n - u)\, dx \right| + \left| \int_{\mathbb{R}^3} \phi_u^\gamma u (u_n - u)\, dx \right| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Using (F_1) and the Hölder inequality, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u)) (u_n - u)\, dx \right| \\
&\le C_1 \int_{\mathbb{R}^3} (|u_n| + |u|) |u_n - u|\, dx + C_1 \int_{\mathbb{R}^3} \left(|u_n|^{p-1} + |u|^{p-1} \right) |u_n - u|\, dx \\
&\le C_1 (\|u_n\|_2 + \|u\|_2) \|u_n - u\|_2 + C_1 \left(\|u_n\|_p^{p-1} + \|u\|_p^{p-1} \right) \|u_n - u\|_p \\
&\le C (\|u_n\| + \|u\|) \|u_n - u\|_2 + C \left(\|u_n\|^{p-1} + \|u\|^{p-1} \right) \|u_n - u\|_p \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

By assumption (G) and the Hölder inequality, we derive

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^3}(g(x)|u_n|^{q-2}u_n-g(x)|u|^{q-2}u)(u_n-u)dx=0.$$

In fact, Using (6.19) and the Hölder inequality, we get

$$\begin{aligned}
&\left|\int\limits_{\mathbb{R}^3}(g(x)|u_n|^{q-2}u_n-g(x)|u|^{q-2}u)(u_n-u)dx\right|\\
&\leq\int\limits_{\mathbb{R}^3}g(x)|u_n|^{q-1}\left|u_n-u\right|dx+\int\limits_{\mathbb{R}^3}g(x)|u|^{q-1}\left|u_n-u\right|dx\\
&\leq\int\limits_{\mathbb{R}^3}g(x)|h_1|^{q-1}\left|u_n-u\right|dx+\int\limits_{\mathbb{R}^3}g(x)|h_2|^{q-1}\left|u_n-u\right|dx\\
&\leq\|g\|_{\frac{2}{2-q}}\left(\int\limits_{\mathbb{R}^3}(|h_1|^{q-1}\left|u_n-u\right|)^{\frac{2}{q}}dx\right)^{\frac{q}{2}}+\|g\|_{\frac{2}{2-q}}\left(\int\limits_{\mathbb{R}^3}(|h_2|^{q-1}\left|u_n-u\right|)^{\frac{2}{q}}dx\right)^{\frac{q}{2}}\\
&\leq\|g\|_{\frac{2}{2-q}}\|h_1\|_2^{q-1}\|u_n-u\|_2+\|g\|_{\frac{2}{2-q}}\|h_2\|_2^{q-1}\|u_n-u\|_2\\
&\to 0\ \text{ as }\ n\to\infty.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
\|u_n-u\|^2\ &=\langle\Phi_\lambda'(u_n)-\Phi_\lambda'(u),u_n-u\rangle-\int\limits_{\mathbb{R}^3}\left(\phi_{u_n}^\gamma u_n-\phi_u^\gamma u\right)(u_n-u)\,dx\\
&+\int\limits_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)dx+\lambda\int\limits_{\mathbb{R}^3}\left(g(x)(|u_n|^{q-2}u_n-|u|^{q-2}u)(u_n-u)dx\right)dx\\
&\to 0\ \ \text{as}\ \ n\to\infty.
\end{aligned}$$

It follows that $\{u_n\}$ converges strongly in E. □

In order to demonstrate our main result, we need to use the symmetric mountain pass theorem. To do this, we choose $\{e_i\}_i$ an orthonormal basis of E and define $X_i=\mathbb{R}e_i$,

$$Y_k=\oplus_{i=1}^k X_i,\quad X_k=\overline{\oplus_{i=k+1}^\infty X_i},\quad k\in\mathbb{Z}. \tag{6.21}$$

Evidently, we have $E=Y_k\oplus X_k$.
Thus, its exact statement is the following.

Theorem 6.3.1. *Let $E=Y_k\oplus X_k$ be a Banach space where Y is finite dimensional. Suppose that $\Phi\in C^1(X,\mathbb{R})$ an even functional such that $\Phi(0)=0$, satisfies the (PS)$_c$-condition, if*
(i) There exist constants $\rho,\delta>0$ satisfying $\Phi\Big|_{\partial B_\rho\bigcap X_k}\geq\delta$, where B_ρ is the open ball in X of radius

ρ about 0 and ∂B_ρ is its boundary.
(ii) for any finite dimensional subspace $\tilde{X} \subset X$, there is $C = C(\tilde{X}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{X} \setminus B_C$.
Then, Φ has an unbounded sequence of critical points.

Corollary 6.3.1. *Under the assumptions (F_1) and (F_3), for every finite dimensional subspace $\tilde{E} \subset E$, there exists a constant $C = C(\tilde{E}) > 0$ such that*

$$\Phi_\lambda(u) \leq 0 \quad \text{for all} \quad u \in \tilde{E} \quad \text{with } \|u\| \geq C.$$

Proof. By (F_1) and (F_3), there exist $C_2, C_3 > 0$ such that

$$F(x,u) \geq C_2|u|^\mu - C_3|u|^2, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{6.22}$$

Indeed, by (F_3) there exist $\mu > 4$ and $l > 0$ such that

$$0 < \mu F(x,u) \leq u f(x,u), \quad \forall x \in \mathbb{R}^3 \quad \text{and} \quad |u| \geq l$$

where

$$F(x,u) = \int_0^u f(x,t)dt.$$

Thus,

$$\frac{\mu}{u} \leq \frac{f(x,u)}{F(x,u)}$$

Hence, for $u > u_0 > 0$ we derive

$$\int_{u_0}^{u} \frac{\mu}{u} du \leq \int_{u_0}^{u} \frac{f(x,u)}{F(x,u)} du.$$

Which implies

$$[\mu log|u|]_{u_0}^u \leq [log|F(x,u)|]_{u_0}^u.$$

This fact gives

$$log\frac{|u|^\mu}{|u_0|^\mu} \leq log\frac{F(x,u)}{F(x,u_0)},$$

hence, we infer

$$\frac{F(x,u_0)}{|u_0|^\mu}|u|^\mu \leq F(x,u).$$

Consequently, we infer that there exist $l > 0, C_2 > 0$ such that

$$C_2|u|^\mu \leq F(x,u), \quad \forall |u| \geq l,\ x \in \mathbb{R}^3. \tag{6.23}$$

In other side from (F_1), one can has

$$|F(x,u)| \leq \frac{C_1}{2}|u|^2 + \frac{C_1}{p}|u|^p, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Thus if $|u| \leq l$ and $x \in \mathbb{R}^3$, there exists C_3 such that

$$\begin{aligned}|F(x,u)| &\leq |u|^2\left(\frac{C_1}{2} + \frac{C_1}{p}|u|^{p-2}\right)\\ &\leq C_3|u|^2.\end{aligned} \tag{6.24}$$

Combining (6.23) and (6.24) we get (6.22).
Since dim$E < \infty$ and all norms are equivalent in the finite dimensional space, thus for all $u \in \tilde{E}$, by (6.22) we derive

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|D^\sigma u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx\\ &\leq \frac{1}{2}\|u\|^2 + C\|u\|^4 - C_2\|u\|_\mu^\mu + C_3\|u\|_2^2 - \frac{\lambda}{q}\|u\|_q^q.\end{aligned}$$

Thanks to $q < 2 < 4 < \mu$, we deduce that $\Phi_\lambda(u) \to -\infty$ as $\|u\| \to \infty$.
Hence, there exists a constant $C > 0$ such that

$$\Phi_\lambda(u) \leq 0 \quad \text{for all} \quad u \in \tilde{E} \quad \text{with} \quad \|u\| \geq C.$$

This completes the proof of Corollary 6.3.1. □

Lemma 6.3.2. *For $2 \leq r < 2_\sigma^*$, $k \in \mathbb{N}$ and we denote*

$$\psi_k := \sup_{u \in X_k, \|u\|=1} \|u\|_r.$$

Then, we assert $\psi_k \to 0$ as $k \to \infty$.

According to the previous lemma, there exists a constant $k \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2C_1}\|u\|^2, \quad \|u\|_p^p \leq \frac{p}{4C_1}\|u\|^p, \quad \forall u \in X_k, \tag{6.25}$$

where $p \in (4, 2_\sigma^*)$.

Indeed, we know that

$$u = \|u\| \frac{u}{\|u\|} \text{ with } \|u\| = 1. \tag{6.26}$$

According to the Lemma 6.3.2, there exists a k such that $\psi_k < \varepsilon$ for any $\varepsilon > 0$. This implies that for all $u \in X_k$ with $\|u\| = 1$

$$\|u\|_r < \varepsilon \quad \text{ for } \quad 2 \leq r < 2_\sigma^*. \tag{6.27}$$

By (6.26), we infer

$$\|u\|_r = \|u\| \left\| \frac{u}{\|u\|} \right\|_r .$$

This fact with (6.27) give

$$\|u\|_r < \varepsilon \|u\| ,$$

as desired.

Lemma 6.3.3. *Suppose that (F_1), (G) and (V) are satisfied. Then, there exist constants $\rho, \delta > 0$ satisfying $\Phi_\lambda \Big|_{\partial B_\rho \bigcap X_k} \geq \delta > 0$.*

Proof. According to (F_1), we have

$$|F(x,u)| \leq \frac{C_1}{2}|u|^2 + \frac{C_1}{p}|u|^p, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{6.28}$$

For $u \in X_k$, combining (6.25)-(6.28) and Lemma 6.2.1, we infer

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^3} \left(|D^\sigma u|^2 + V(x)u^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^\gamma u^2 dx - \int_{\mathbb{R}^3} F(x,u)\, dx - \frac{\lambda}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx \\
&\geq \frac{1}{2} \|u\|^2 - \frac{C_1}{2} \|u\|_2^2 - \frac{C_1}{p} \|u\|_p^p - \lambda \|g\|_{\frac{2}{2-q}} \|u\|_2^q \\
&\geq \frac{1}{2} \|u\|^2 - \frac{1}{4} \|u\|^2 - \frac{1}{4} \|u\|^p - \lambda \|g\|_{\frac{2}{2-q}} C_q \|u\|^q \\
&\geq \frac{1}{4} (\|u\|^2 - \|u\|^p) - \lambda \|g\|_{\frac{2}{2-q}} C_q \|u\|^q \\
&\geq \|u\|^2 \left(\frac{1}{4} - \frac{1}{4} \|u\|^{p-2} - \lambda \|g\|_{\frac{2}{2-q}} C_q \|u\|^{q-2} \right).
\end{aligned}$$

Let us define $\Psi(s) : (0, \infty) \to \mathbb{R}$ as follows

$$\Psi(s) = \frac{1}{4} - \frac{1}{4} s^{p-2} - \lambda \|g\|_{\frac{2}{2-q}} C_q s^{q-2}, \quad s > 0.$$

Thus,

$$\Psi'(s) = -\frac{1}{4}(p-2)s^{p-3} - \lambda\|g\|_{\frac{2}{2-q}} C_q(q-2)s^{q-3}, \quad s>0.$$

$\Psi'(s) = 0$, implies that

$$-\frac{1}{4}(p-2)s^{p-3} = \lambda\|g\|_{\frac{2}{2-q}} C_q(q-2)s^{q-3}, \quad s>0.$$

By dividing the both side by s^{q-3} we derive

$$s^{p-q} = \frac{4\lambda(2-q)C_q\|g\|_{\frac{2}{2-q}}}{(p-2)}.$$

Since $1<q<2<p$, there exists $\rho_\lambda > 0$ such that

$$\rho_\lambda = \left(\frac{4\lambda(2-q)C_q\|g\|_{\frac{2}{2-q}}}{(p-2)}\right)^{\frac{1}{p-q}}.$$

On the other hand, we have

$$\Psi''(s) = -\frac{1}{4}(p-2)(p-2)s^{p-4} - \lambda\|g\|_{\frac{2}{2-q}} C_q(q-2)(q-3)s^{q-4}, \quad s>0,$$

Thus,

$$\Psi''(\rho_\lambda) = -\frac{1}{4}(p-2)(p-3)\rho_\lambda^{p-4} - \lambda\|g\|_{\frac{2}{2-q}} C_q(q-2)(q-3)\rho_\lambda^{q-4} < 0.$$

Hence, we deduce that $\max_{s\in\mathbb{R}^+}\Psi(s) = \Psi(\rho_\lambda)$.
Thus, there exists Λ_* such that for any $\lambda < \Lambda_*$ we have

$$\Phi_\lambda(u) \geq \|\rho_\lambda\|^2\,\Psi(\rho_\lambda) := \delta > 0.$$

Letting $\rho = \|\rho_\lambda\| > 0$. Thus, we deduce that $\Phi_\lambda\Big|_{\partial B_\rho \bigcap X_k} \geq \delta > 0$.
Proof of Theorem 6.1.1. Let $Y = Y_k$ and $X = X_k$. Obviously, $\Phi_\lambda(u)$ is even due to (F_2). Based on Lemma 6.3.1, Lemma 6.3.3 and Corollary 6.3.1, the functional $\Phi_\lambda(u)$ satisfies all conditions of the symmetric mountain pass theorem. Thus, there exists $\Lambda_* > 0$ such that system (6.5) admits multiple nontrivial solutions with negative energy for any $\lambda \in (0, \Lambda_*)$. □

Conclusion and Perspective

In the present thesis, our main scientific contributions focused on the existence and multiplicity results of nontrivial solutions for several classes of nonlinear fractional systems of partial differential equations involving two different kinds of fractional derivatives operators. More specifically, this thesis aims to study the doubling properties of nontrivial solutions that is, the existence and multiplicity of nontrivial solutions for various classes of nonlinear fractional Schrödinger-Poisson systems and Kirchhoff-Schrödinger-Poisson systems under different types of assumptions on potential functions and nonlinearities functions. The main results also establish a blow-up property for the associated energy levels. The main ingredients are variational and crucial roles are played by the critical point theorems, namely, the mountain pass theorem, the symmetric mountain pass theorem, and the fountain theorem.

We have shown interest in a new fractional derivative operator, the so-called distributional Riesz fractional gradient and we have given its consistency with the well-known fractional Laplace operator. Furthermore, we introduce a new research direction which has a strong relationship with several applications in these circumstances.

For the perspective and the possible generalization, it would be interesting to extend the results of the present thesis by considering any fractional integro-differentiable operators that generalize the fractional Laplacian operator such that, fractional magnetic Laplacian operator, fractional $a(\cdot)$-Laplacian operator and fractional double phase operator. In addition, since the concept of distributional Riesz fractional gradient is very recent, it will be useful to expand the results related to the distributional Riesz fractional gradient to the results driven by distributional Riesz fractional gradient with external magnetic potential. These suggestions will be treated in the future.

APPENDIX **A**

List of publications

The publications that constitute the basis of this PhD thesis can be found in:
B.HAMZA, H.LAKHAL, S.KAMEL, and B.TAHAR. (2023). The nontrivial solutions for nonlinear fractional Schrödinger-Poisson system involving new fractional operator. Advances in the Theory of Nonlinear Analysis and its Application, 7(1), 121-132. doi: 10.31197/atnaa.1141136.
H.Boutebba, H.Lakhal, and K.Slimani. (2024). Infinitely many distributional solutions to a general kind of nonlinear fractional Schrödinger-Poisson systems. The Journal of Analysis, 32(2), 1079-1091. doi: 10.1007/s41478-023-00674-4.
H.Boutebba, H.Lakhal, and K.Slimani. (2024). Existence of multiple solutions for nonlinear fractional Schrödinger-Poisson system involving new fractional operator. International Journal of Nonlinear Analysis and Applications, 15(11), 83-92. doi: 10.22075/ijnaa.2023.28823.3999.
H.Boutebba, H.Lakhal, and K.Slimani. (accepted paper 2024). The multiplicity of solutions to a new class of superlinear fractional Schrödinger-Poisson systems. Novi Sad Journal of Mathematics. doi: 10.30755/NSJOM.16284.
H.Boutebba, H.Lakhal, and K.Slimani. (accepted paper 2024). Multiple nontrivial solutions to a class of fractional Schrödinger-Maxwell systems in Bessel potential space. International Journal of Nonlinear Analysis and Applications.
H.Boutebba, H.Lakhal, and K.Slimani. (accepted paper 2025). Infinitely many high energy solutions to a new class of superlinear fractional Schrödinger-Poisson systems. boletim da sociedade paranaense de matemática.

APPENDIX B

List of submitted works

H.Boutebba, H.Lakhal, and K.Slimani. Nontrivial and infinitely many large energy solutions for fractional Kirchhoff-Schrödinger-Poisson systems with concave-convex nonlinearities. (submitted to an international journal).

H.Boutebba, H.Lakhal, and K.Slimani. Multiplicity of solutions for a class of fractional Kirchhoff-Schrödinger-Poisson systems with superlinear terms. (submitted to an international journal).

H.Boutebba, H.Lakhal, and K.Slimani. On a new class of nondegenerate fractional Kirchhoff-Schrödinger-Poisson type systems: Existence and multiplicity. (submitted to an international journal).

H.Boutebba, H.Lakhal, and K.Slimani. Existence results for nonlinear fractional Schrödinger-Poisson system in bounded domain. (submitted to an international journal).

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