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Mémoire

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Existence des solutions périodiques positives pour certaine classe d'équations différences a retards.

Option :AFA

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Abstract in English

Fixed point theory has a long history of being used in nonlinear difference and differential equations, in order to prove existence, uniqueness, or other qualitative properties of solutions. However, using the fixed point theorem for the periodicity of solutions has a more recent appearance. In this dissertation, we focus on the study of the quantitative and qualitative properties of some nonlinear delay differences and differential equations. We start by giving some fixed point theorems. After we introduce results for delay differential equations and necessary relevant definitions. Finally, we derive some sufficient conditions for the existence of positive periodic solutions for the system of neutral difference equations, by using the Krasnoselskii fixed-point theorem. An example is also given to illustrate the claims established.

Keywords: Fixed points theorem ; Krasnoselskii's theorem; Periodicity; Positivity; Delay differential equations; neutral difference equation; Integral equation; Variable delays.

Abstract in French

La théorie du point fixe est utilisée depuis longtemps dans les équations différent non linéaires, afin de prouver l'existence, l'unicité ou d'autres propriétés qualitatives. Cependant, l'utilisation du théorème du point fixe pour la stabilité et la périodicité des solutions a une apparition plus récente. Nous nous concentrons sur l'étude des propriétés quantitatives et qualitatives de certains types d'équations différentielles et différences de retard non linéaires. Nous commençons par donner quelques théorèmes de point fixe. Ensuite, nous donnons des notions préliminaires sur les équations différentielle à retard qui sont necéssaires par la suite. A la fin, en utilisant le théorème du point fixe nous étudions l'existence des solutions périodiques positives pour le système des equations différence de type neutre. De plus nous étudions l'unicité de la solution periodic par l'application du contraction de Banach. Finalement, un exemple est également donné pour illustrer l'éfficacité des résults établies.

Mots clés: Points fixe; Theorème de Krasnoselskii; Solutions périodiques positives; Equations à retard; Equations différence neutre; Equation intégrale; Retards.

Abstract in Arabic

Dedication

I dedicate my humble work
to my family, a special feeling
of gratitude to my loving
parents Warda and Rabah whose words
of encouragement and push for tenacity ring
in my dear sister Soumaya and her son Abdelillah.
To my brothers Ayyoub, Mouhamed and Abdessalem,
My Allah protect them .
Without forgetting all the teachers and all my friends
Thank you for everything

Bendiaf Loubna

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We thank God for helping us to accomplish this work.

In the name of Allah,the Beneficent, we thank Allah (SWT) the Most High, the Almighty, who bestowed upon us the grace of reason and religion.

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After praising God Almighty and thanking him for ending this dissertation, we extend our sincere thanks and great gratitude to the distinguished **Dr. Benhadri Mimia** for the continuous knowledge and guidance, she gave us and for her continuous effort, advice and directions from the beginning of the research phase until the completion of this dissertation, and whatever phrases and sentences. We wrote, the words of thanks we remain unable to fulfill her right, so may God reward her with the best reward for us and make that in the balance of her good deeds.

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Acronyms

Abbreviation	**Meaning**
ODEs	Ordinary Differential Equations
FDEs	Functional Differential Equations
DDEs	Delay Differential Equations
NDDEs	Neutral Delay Differential Equations

List of Symbols

Here we state some conventions regarding mathematical notation that we will use in this dissertation.

$\mathbb{R}^n$	Euclidean space of n- dimensions;
$\mathbb{R}^+ = [0, \infty)$	set of positive real numbers;
(X, d)	this is called metric space;
(a, b)	open interval from a to b;
$\mathbb{N}$	set of natural numbers;
$\frac{d}{dt}$	first derivative with respect to t;

$[a, b]$	closed interval from a to b;
$\mathbb{C}\left([a, b], \mathbb{R}\right)$	space of continuous functions mapping from the interval $[a, b]$ to $\mathbb{R}$;
ω	period;

$\mathbb{C}([-\tau, 0]; \mathbb{R}^n)$	the space of all continuous $\mathbb{R}^n -$ valued functions φ defined on $[-\tau, 0]$; with a norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \left|\varphi\left(\theta\right)\right|$;
$f : A \to B$	the mapping f from A to B;
$d(x, y)$	this is called the distance between x and y;

$\|.\|_\infty$ uniform norm;

$\mathbb{C}_\omega$ set of periodic functions;

$|x|$ the Euclidean norm of a vector x;

$\sum_{i=0}^{n}$ summation from index $i = 0$ to $i = n$.

Other notations will be explained when they first appear.

0.1 Introduction

The fixed point theory is at the heart of the nonlinear analysis because it provides the necessary tools to have theorems of existence for many different non-linear problems. It uses its tools of analysis and topology. Advancements in fixed point theory enrich many scientific fields such as biology, chemistry, computer science, economics and game theory. Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point theory and metric theory. Or, fixed point and topological theory, (see [2], [55]).

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known the contraction mapping theorem or the contraction mapping principle). It was first stated by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space, and provides a constructive method to find those fixed points.

Concerning the topological branch, results are obtained using topological properties of the set X. The main result is Schauder fixed point theorem which stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem.

In 1955, Krasnoselskii studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated a hybrid theorem known under its name. The reader is referred to the classical textbook on fixed point [57]. This is a captivating result and it has a number of interesting applications. The proof of this result combines the Banach contraction principle and Schauder fixed point theorem and thus it is a blend of the two branches.

Delay difference and differential equations are a class of differential equations where the derivatives at the current time depend on the solution at previous times. Strictly speaking, delay differential equations are a specific example of functional differential equations, in which the functional part of the differential equations is the evaluation of a functional on the past of the process. Mathematical modeling involving delay differential equations is widely used for analysis and predictions in various areas of the life sciences, for example, population dynamics, epidemiology, immunology, physiology, neural networks, (see for example, [6, 11]), and the references cited therein. The delays can represent gestation times, incubation periods, or transport delays. In many cases time delays can be substantial such as gestation, forestation, deforestation and maturation or can represent little lags such as acceleration and deceleration in physical processes. Due to their importance in numerous applications have been devoted to the existence of periodic solutions of several different types of delay differential equations. For some specific work concerning the existence of periodic solutions to periodic population models which were carried out using the fixed point theory, the reader is referred to see ([37], [24], [27], [45], [56]), amongst others, and the references therein.

The difference equations fascinate their readers as they occur widely in many applications such as classical electrodynamics, population models, mechanical models, the dynamics of economic systems, and the transmission of infectious diseases, refer to ([5],[39], [53]).

Now, intensive scientific work has been carried out on various dynamical aspects of difference and differential equations and has been reported. In particular, a number of existence results are investigated by many authors, (see, [52] , [53]). In this dissertation, we have been interested in the use of fixed point theory to solve the problems of periodicity and positivity for a system neutral delay difference equations. In this work, we present a collection of results to some problems of neutral difference equations using fixed point theory.

This dissertation consists of three chapters.

Chapter 1 : The first chapter is devoted to point out the tools which are needed in the next chapters. The aim of this chapter is to introduce the basic concepts, notations, and elementary results which will be used throughout this work. We recall some classical results from functional analysis such as the Ascoli-Arzela theorem. The main part of the chapter is dedicated to the presentation of fixed point theorems, the Schauder's and Banach fixed point theorem and that of Krasnoselskii. Moreover, we analyze some examples to illustrate how to apply these theorems to some specific differential equations.

Chaptre 2 : This chapter is devoted to DDEs analysis, we present some basic preliminaries and discuss the existence and uniqueness theorems for the solution and properties. The authors provide the appropriate mathematical tools which will be needed to understand the concepts that will be developed in this dissertation for the study of the periodicity of delay differential equations (DDEs). A significant and interesting model of delay equations emanating from biology is given in the beginning of this chapter.

Chaptre 3 : This chapter exposes results published in [53], we will investigate the existence of positive periodic solutions for neutral delay difference equations. In the process, we convert the given system into an equivalent integral equation. Then we construct appropriate mappings and use Krasnoselskii's fixed point theorem in Banach space to show the existence of a positive periodic solution of the equation. Easily verifiable sufficient conditions are established. In particular, the results improve some previous ones in the literature. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

CHAPTER 1

Fixed point Theorems

Contents

This chapter, we will mention (and complete) some important theorems in the theory of the fixed point, as well as some tools of functional analysis necessary for the rest of this dissertation. In particular, the Ascoli-Arzela theorem is an element that is very useful in this work to prove compactness in function spaces challenged on compact sets and not necessarily compact sets. The theorems of Banach, Schauder and Krasnoselskii will be presented in this chapter. These elements of the analysis have been presented. Taken from a few books chosen as (see [14], [55], [60]).

1.1 Notation and preliminaries

This section contains an elementary set of definitions, theorems, and examples which were motivated by the examples in the last section and were formulated to aid us in deciding which fixed point theorem to use and which stability properties to prove.

1.1.1 Normed and Banach space

Definition 1.1 *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ a function. Then d is called.a metric on X if the following properties hold.*

1) $d(x, y) = 0$ *if and only if* $x = y$ *for some* $x, y \in X$;
2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called metric space.

Definition 1.2 The metric space (X, d) is complete if every Cauchy sequence in (X, d) has a limit in that space. Asequence $\{x_n\} \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $d(x_n, x_m) < \varepsilon$.

Definition 1.3 *A* vector space $(\mathcal{V}, +, \cdot)$ is a normed space if for each $x, y \in \mathcal{V}$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(i) $\|x\| = 0$ *if and only if* $x = 0$,
(ii) $\|\alpha x\| = |\alpha| \|x\|$ *for each* $\alpha \in \mathbb{R}$, *and*
(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Remark 1.1 A normed space is a vector space and it is a metric space with $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition 1.4 A Banach space is a complete normed space.

We list two examples, (see Burton, [14]).

Example 1.1 a) The space $\mathbb{C}([a, b], \mathbb{R}^n)$ consisting of all continuous functions $f : [a, b] \to \mathbb{R}^n$ is a vector space over the reals.

b) If $\|f\| = \max_{a\leq t\leq b} |f(t)|$, where $|.|$ is a norm in $\mathbb{R}^n$, then $(\mathbb{C}, \|.\|)$ is a Banach space.

Example 1.2 Let $X = \mathbb{R}^n, n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:

i) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$;

ii) $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $p \in (1, \infty)$;

iii) $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Definition 1.5 *A* sequence $\{x_n\}$ in a normed space X is said to be Cauchy if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$, *i.e.*, for $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \geq n_0$.

Definition 1.6 *A* normed space $(X, \|.\|)$ is said to be complete if it is complete as a metric space (X, d), i.e., every Cauchy sequence is convergent in X.

Definition 1.7 *A* complete normed space is called a Banach space.

Example 1.3 The linear space $\mathbb{C}([a,b])$ of continuous functions on the closed and bounded interval $[a,b]$ is a Banach space with the uniform convergence norm $\|f\|_{\infty} = \sup_{t\in[a,b]} |f(t)|$.

Definition 1.8 Every finite- dimensional normed space is a Banach space.

Definition 1.9 *A* closed subspace of a Banach space is a Banach space.

Proposition 1.1 *Let X be a metric space. Then the following are equivalent*

i) X is a compact.

ii) Every sequence in X has a convergent subsequence.

iii) X is compete and totally bounded.

Proposition 1.2 *Let X be a subset of a complete metric space X. Then we have the following:*

a) $\mathcal{M}$ is compact if and only if $\mathcal{M}$ is closed and totally bounded.

b) $\overline{\mathcal{M}}$ is compact if and only if $\mathcal{M}$ is totally bounded.

A subset $\mathcal{M}$ of a topological space is said to be relatively compact if its closure is compact, i.e., $\overline{\mathcal{M}}$ is compact. In particular, we have an interesting result.

Proposition 1.3 *Let $\mathcal{M}$ be a closed subset of a complete metric space. Then $\mathcal{M}$ is compact if and only if it is relatively compact.*

Definition 1.10 ([14]) Let $\{f_n\}$ be a sequence of real valued functions with $f_n : [a,b] \to \mathbb{R}$.

a) $\{f_n\}$ is uniformly bounded on $[a,b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a,b]$.

b) $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply

$$|f_n(t_1) - f_n(t_2)| < \epsilon$$

for all n.

The following results give the main method of proving compactness in the spaces in which we are interested.

Definition 1.11 (Ascoli-Arzela, [14]) If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.

Remark 1.2 Let k_1 and k_2 be two strictly positive real numbers. The subset F of continuous real functions on $[a, b]$, differentiable on $]a, b[$ which satisfy

$$|f(t)| \leq k_1 \text{ and } \sup_{c \in]a,b[} |f'(c)| \leq k_2$$

for all $t \in [a, b]$,is relatively compact in $C([a, b], \mathbb{R})$.

Indeed, for all $f \in F$; the finite increment theorem proves that for all $t_0, t \in [a, b]$ there exists $c \in]t_0, t[$ such that

$$|f(t) - f(t_0)| = |f'(c)| |t - t_0|,$$

so $|f(t) - f(t_0)| \leq k_2 |t - t_0|$ let's fixe $t_0 \in [a, b]$. Let $\varepsilon > 0$ and by taking $\eta = \dfrac{\varepsilon}{k_2}$, we have

$$\forall t \in [a, b], |t - t_0| \leq \eta \Rightarrow |f(t) - f(t_0)| \leq \varepsilon.$$

F is said to be "uniformly bounded" if and only if

$$\forall x \in [a, b], \{f(x), f \in F\} \text{ is bounded,}$$

which exactly is the equicontinuity of F at t_0. As we can take for t_0 any point of $[a;b]$, we deduce that F is equicontinuous. We have $t \in [a,b], |f(t)| \leq k_1$ for all $f \in F$ wich implies that $\|f\|_\infty \leq k_1$, so we conclude that

$$\forall f \in F, B'(0,k_1),$$

i,e

$$F \subset B'(0,k_1),$$

hence the boundedness of F. Finally, as F is bounded and equicontinuous, then the Ascoli-Arzela's theorem ensures that F is relatively compact

But if we manipulate function spaces defined on infinite t-intervals. So, for compactness we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([14], Theorem 1.2.2 p. 20) and is as follows.

Theorem 1.1 *([14]) Let $\mathbb{R}^+ = [0,\infty)$ and let $q : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\phi_k(t)\}$ is an equicontinuous sequence of $\mathbb{R}^d$-valued functions on $\mathbb{R}^+$ with $|\phi_k(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^d$.*

Proof. It is clear that set of functions $\{\phi_k(t)\}$ is uniformly bounded on $\mathbb{R}^+$. Thus, considering intervals $[0,n]$, n a positive integer, and using a diagonalization process there is a subsequence, say $\{\phi_k(t)\}$ again, converging uniformly on any compact subset of $\mathbb{R}^+$ to some continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$. Because $\phi(t) \to 0$ as $t \to \infty$, it will now be possible to show that $\|\phi_k - \phi\| \to 0$ as $k \to \infty$, where $\|.\|$ denotes the supremum metric on $\mathbb{R}^+$. From the definition of $q(t)$, for any $\varepsilon > 0$ there is a $T > 0$ with $q(t) < \varepsilon/2$ if $t \geq T$, which yields

$$|\phi_k(t) - \phi(t)| \leq 2q(t) < \varepsilon \text{ if } k \in \mathbb{N} \text{ and } t \geq T, \qquad (**)$$

where $\mathbb{N}$ denote the set of positive integers. On the other hand, since $\{\phi_k(t)\}$ converges to $\phi(t)$ uniformly on $[0, T]$ as $k \to \infty$, for the ε there is a $\kappa \in \mathbb{N}$ with

$$|\phi_k(t) - \phi(t)| < \varepsilon \text{ if } k \geq \kappa \text{ and } 0 \leq t \leq T,$$

which together with (**), implies that $\|\phi_k - \phi\| < \varepsilon$ if $k \geq \kappa$. This shows that

$$\|\phi_k - \phi\| \to 0 \text{ as } k \to \infty.$$

Let us give an example for compact set: (see Burton, [14]). ■

1.2 Fixed point theorems

Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point and metric theory. Or, fixed point and topological theory.

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known as the contraction mapping theorem or the contraction mapping principle). It was first stated by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space, and provides a constructive method to find those fixed points. Concerning the topological branch, results are obtained using topological properties of the set X. The main result is Schauder fixed point theorem which was stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem. Although historically the two branches of the fixed point theory had separate development. In 1958, Krasnoselskii established that the sum of two operators $\mathcal{A}+\mathcal{B}$ has a fixed point in a nonempty closed convex subset $\mathcal{M}$ of a Banach space X, where one of them is a contraction and the other one is compact (see below).

1.2.1 Banach fixed point

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem, the famous contraction principle, which is one of the most important results of analysis. It is the most widely applied fixed point result in different areas of mathematics and applications. It requires the structure of a complete metric space with a contractive condition on the map which is easy to test in many situations. It has been generalized in many different directions. Moreover, the proof of the Banach contraction principle gives a sequence of approximate solutions and useful information as regards the rate of convergence toward the fixed point. This is very important since a fundamental principle both in mathematics and computer science is iteration. Particularly, fixed point iterations and monotone iterative techniques are the core methods when solving a large class of abstract and applied mathematical problems and play an important role in many algorithms.

Let describe this theorem.

Definition 1.12 Let f be a mapping in the set M. We call fixed point of f any point x satisfying $f(x) = x$. If there exists such x, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.

Theorem 1.2 *(Contraction Mapping Principle, Smart* [55] *). Let* $(\mathcal{S}, \rho)$ *be a complete metric space and let* $\mathcal{P} : \mathcal{S} \to \mathcal{S}$*. If there is a constant* $0 \leq \alpha < 1$ *such that for each pair* $\phi_1, \phi_2 \in \mathcal{S}$*, we have*

$$\rho(\mathcal{P}\phi_1, \mathcal{P}\phi_2) \leq \alpha\rho(\phi_1, \phi_2),$$

then there is one and only one point $\phi \in \mathcal{S}$ with $\mathcal{P}\phi = \phi$.

Example 1.4 Let $f(t, x)$ be a continuous real-valued function defined for t in the interval $[0, T]$, and x in $\mathbb{R}$. The Cauchy initial value problem is the problem of finding a continuously differentiable function x on $[0, T]$ satisfying the differential equation

$$\begin{cases} x'(t) = f(t, x(t)), \; t \in [0, T] \\ x(0) = \zeta. \end{cases} \tag{1.1}$$

Proof. Consider the space $C([0, T])$ of continuous real-valued functions with standard supremum norm and f is $L-$lipschitzian with respect to x. Integrating both sides of (1.1) we obtain

$$x(t) = \zeta + \int_0^t f(s, x, (s))ds.$$

We denote the function defined by the right side of the above by $\mathcal{P}x$. Precisely,

$$(\mathcal{P}x)(t) = \zeta + \int_0^t f(s, x(s))ds.$$

Thus $\mathcal{P} : \mathbb{C}([0, T] \to \mathbb{C}([0, T])$, and a solution to (1.1) corresponds to a fixed point x of $\mathcal{P}$. Observe that for any $x, y \in [0; T]$,

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| &= \left| \int_0^t f(s, x(s))ds - \int_0^t f(s, y(s))ds \right| \\ &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds \\ &\leq L \int_0^t |x(s) - y(s)| \, ds \\ &= Lt \|x - y\| . \end{aligned}$$

It follows that

$$\|\mathcal{P}x - \mathcal{P}y\| \leq LT \|x - y\| .$$

If $LT < 1$, then the result is immediate via the Banach Contraction Principle.

■

1.2.2 Schauder's fixed point theorem

Definition 1.13 Let (X, d) be a generalized metric space. A subset C of X is called compact if every open cover of C has a finite subcover A subset C of X is sequentially compact if every sequence in C contains a convergent subsequence with a limit in C.

Definition 1.14 A set C of topological space is said relatively compact if its closure is compact, i.e., $\overline{C}$ is compact. The set C is sequentially relatively compact if every sequence in C contains a convergent subsequence (the limit need not be an element of C), i.e., $\overline{C}$ is sequentially compact.

Definition 1.15 Let X ,Y be two generalized metrics spaces $K \subset X$ and $f : K \to Y$ be a an open operator. Then f is called:

(i) compact, if for any bounded subset $A \subset K$ we have $f(A)$ is relatively compact or $\overline{f(A)}$ is compact;

(ii) Complete continuous, if f is continuous and compact;

(iii) with relatively compact range, if f is continuous and $f(K)$ is relatively compact or $\overline{f(K)}$ is compact.

Theorem 1.3 *(Schauder's first point theorem, Burton* [14] *) Let $\mathcal{M}$ be a nonempty compact convex subset of a Banach space and let $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ be continuous. Then $\mathcal{P}$ has a fixed point in $\mathcal{M}$.*

Theorem 1.4 *(Schauder's second fixed point theorem, Burton* [14] *) Let $\mathcal{M}$ be a nonempty convex subset of a normed space and let $\mathcal{P} : \mathcal{M} \to \mathcal{K}$ where $\mathcal{K}$ is a compact subset of $\mathcal{M}$. Then $\mathcal{P}$ has a fixed point in $\mathcal{K}$.*

1.2.3 Krasnoselskii fixed point

The fixed point theorem of Krasnoselskii combined the two main fixed point theorems, Banach contraction mapping principle and Schauder fixed point theorem into the following result. Firstly, we recall the theorem of Schauder. Schauder's theorems which require compactness instead of completeness and which yield a, possibly non unique, fixed point. More precisely, Schauder's fixed point theorem shows us that a continuous map on a compact convex subset of a Banach space has a fixed point. The next two results are found in Smart [55] and Burton [14].

Krasnoselskii's theorem may be combined with Banach and Schauder's fixed point theorems. In a certain sense, we can interpret this as follows: if a compact operator has the fixed point property, under a small perturbation, then this property can be inherited. The theorem is useful in establishing the existence results for perturbed operator equations. It also has a wide range of applications to nonlinear integral equations of mixed type for proving the existence of periodic solutions. Thus the existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. (see [55] Smart, 1980;p.31).

Two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result:

Theorem 1.5 *(**Krasnoselskii,** see Burton* [12] *). Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $(X, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathcal{M}$ into X such that the following conditions hold*

(*i*) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$, $\forall x, y \in \mathcal{M}$;

(*ii*) *$\mathcal{A}$ is continuous and $\mathcal{A}\mathcal{M}$ is contained in a compact set;*

(iii) $\mathcal{B}$ *is a contraction with* $\alpha < 1$;

then there is a $z \in \mathcal{M}$, *with* $z = \mathcal{A}z + \mathcal{B}z$.

Proof. According to the condition (iii) we have

$$\begin{aligned} \|(I-\mathcal{B})x-(I-\mathcal{B})y\| &= \|(x-y)-(\mathcal{B}x-\mathcal{B}y)\| \\ &\leq \|x-y\|+\|\mathcal{B}x-\mathcal{B}y\| \\ &\leq \|x-y\|+\alpha\|x-y\| \\ &= (1+\alpha)\|x-y\|, \end{aligned}$$

and

$$\begin{aligned} \|(I-\mathcal{B})x-(I-\mathcal{B})y\| &= \|(x-y)-(\mathcal{B}x-\mathcal{B}y)\| \\ &\geq \|x-y\|-\|\mathcal{B}x-\mathcal{B}y\| \\ &\geq \|x-y\|-\alpha\|x-y\| \\ &= (1-\alpha)\|x-y\|. \end{aligned}$$

In short

$$(1-\alpha)\|x-y\| \leq \|(I-\mathcal{B})x-(I-\mathcal{B})y\| \leq (1+\alpha)\|x-y\|.$$

This inequality shows that $(I-\mathcal{B}) : \mathcal{M} \to (I-\mathcal{B})\mathcal{M}$ is continuous and one to one. Thus, $(I-\mathcal{B})^{-1}$ exists and is continuous. Let us pose $U := (I-\mathcal{B})^{-1}\mathcal{A}$ It is clear that U is a compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in \mathcal{M} \text{ such that } (I-\mathcal{B})^{-1}\mathcal{A}z{=}z.$$

This is equivalent to $z{=}\ \mathcal{A}z{+}\mathcal{B}z$. ■

Remark 1.3 If $\mathcal{A} = 0$, the theorem can be summed up in Banach's theorem and if $\mathcal{B} = 0$ then the theorem is none other than Schauder's theorem.

In addition, an example is eventually analyzed to illustrate the effectiveness of the proved results of Krasnoselskii theorem.

Example 1.5 For better understanding this observation, now, we analyze an example to illustrate the application of Krasnoselskii fixed point theorem for proving the existence of $\omega-$periodic solutions of the following differential equation,

$$x'(t) = -a(t)x(t) - g(t,x), \tag{1.2}$$

where $a(t) = a(t+\omega)$, and the function $g(t,x)$ is periodic in t of period ω.

Proof. We can transform this equation in another form while writing, formally

$$x'(t)e^{-\int_0^t a(s)ds} = -a(t)e^{-\int_0^t a(s)ds}x(t) - g(t,x)e^{-\int_0^t a(s)ds},$$

thus

$$x'(t)e^{-\int_0^t a(s)ds} + a(t)e^{-\int_0^t a(s)ds}x(t) = -g(t,x)e^{-\int_0^t a(s)ds},$$

or

$$\left(x(t)e^{-\int_0^t a(s)ds}\right)' = -g(t,x)e^{-\int_0^t a(s)ds},$$

then integrating from $t-\omega$ to t, we obtain

$$\int_{t-\omega}^t \left(x(u)e^{-\int_0^u a(s)ds}\right)' du = -\int_{t-\omega}^t g(u,x)e^{-\int_0^u a(s)ds}du,$$

which gives

$$x(t)e^{-\int_0^t a(s)ds} - x(\omega - t)e^{-\int_0^{\omega-t} a(s)ds} = -\int_{t-\omega}^t g(u,x)e^{-\int_0^u a(s)ds}du,$$

or

$$x(t) = x(\omega - t)e^{-\int_{\omega-t}^t a(s)ds} - \int_{t-\omega}^t g(u,x)e^{-\int_t^u a(s)ds}du. \tag{1.3}$$

If we suppose that $e^{-\int_{\omega-t}^t a(s)ds} := \alpha$ and if $(X, \|.\|)$ is a Banach space of functions $\varphi : \mathbb{R} \to X$ continuous and $\omega-$ periodic, then the equation (1.3) can be written as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (\mathcal{H}\varphi)(t),$$

where $\mathcal{B}$ is contraction provides that the constant $\alpha < 1$ and $\mathcal{A}$ is compact mapping. ■

This example shows the birth of the mapping $\mathcal{H}\varphi := \mathcal{B}\varphi + \mathcal{A}\varphi$ which is identified with a sum of a contraction and a compact mapping.

Finally, let us indicate a very useful Theorem 1.6 when trying to prove that the existence and positivity of a solution to a periodic problem.

Definition 1.16 *(see,* [31] , [27] *) Let X be a Banach space and let Ω be a closed, nonempty subset of X. Ω is a cone if*

i) $\alpha u + \beta v \in \Omega$ for all $u, v \in \Omega$ and all $\alpha, \beta \geq 0$;

ii) $u, -u \in \Omega$ imply $u = 0$.

The proof of Krasnoselskii's fixed point theorem stated below can be found in [29]

Theorem 1.6 *(Guo-Krasnoselskii,* [29] *) Let X be a Banach space, and let $\Omega \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\mathcal{P} : \Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \Omega,$$

be a completely continuous operator such that either

i) $\|\mathcal{P}u\| \leq \|u\|$ for $u \in \Omega \cap \partial\Omega_1$ and $\|\mathcal{P}u\| \geq \|u\|$ for $u \in \Omega \cap \partial\Omega_2$; or

ii) $\|\mathcal{P}u\| \geq \|u\|$ for $u \in \Omega \cap \partial\Omega_1$ and $\|\mathcal{P}u\| \leq \|u\|$ for $u \in \Omega \cap \partial\Omega_2$.

Then $\mathcal{P}$ has a fixed point in $\Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

CHAPTER 2

Retarded differential equations with applications

Contents

This chapter provides background material necessary for the rest of the dissertation. Some preliminaries and basic definitions are given for delay differential equations. Strictly speaking a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process. Like ordinary differential equations, delay differential equations have several features which make their analysis more complicated. The survey of the theory related to delay differential equations can be found for example in the books, [6], [14], [7], [21], [24], [34], [39], [41].

2.1 Basic concepts of delay differential equations

Motivation

The questions have been asked by many researchers "Why study this subject?" Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier? The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. To clarify more, we give a biological system in which the present rate of change of some unknown function depends upon past values of the same function.

Real example of delay differential equation

To have a better understanding and reading of this section, we will focus on a simple real-life example. The goal is to help the reader to understand the most relevant aspects of delay differential equations. The following is an example presented in [6]. Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $t = \alpha\rho$. Then the dynamics of N can be described by the linear delay differential equation

$$\frac{d}{dt}N(t) = -\mu N(t) - rN(t-h), \tag{2.1}$$

which involves a nonlocal term, $rN(t-h)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N. Equation (2.1) describes the changes in N.

With deeper study and understanding of population dynamics, people started to consider introducing state-dependent delay into population models, as was pointed out in Arino et al. [6].

In the context of population dynamics, the delay arises frequently as the maturation time from birth to adulthood and this time is in some cases a function of the total population.

Mathematical point of view:

To determine a solution past time t_0, we need to prescribe the value of $N(t_0 - h)$. Suppose we have the initial value $N(t_0 - h)$. Once we advance, say to $N(\varepsilon)$, with $t_0 < \varepsilon < t_0 + h$ small, notice that to calculate the derivative at $t = \varepsilon$ so that we can advance the next step, we need to know

$$\frac{d}{dt}N(\varepsilon) = -\mu N(\varepsilon) - rN(\varepsilon - h),$$

where $\varepsilon - h \in (t_0 - h, t_0)$. In this manner, we realize that we need to know the values of $N(.)$ on the whole interval $[t_0 - h, t_0]$. If we do not specify these values, we obtain an unsatisfactory notion of uniqueness, as in the following example

$$x'(t) = -\frac{\pi}{2}x(t-1), x(0) = \frac{1}{\sqrt{2}}.$$

Here

$$x_1(t) = \sin\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right] \text{ and } x_2(t) = \cos\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right],$$

are both solutions to the above equation at $t_0 = 0$. But if we specify the initial behavior on the interval $[-1, 0]$, we obtain that only one solution exists to each delay differential equations, by the existence-uniqueness result in Theorem 2.1 that we give below.

Clearly, to begin with, an initial value problem requires more information than an analogous problem for a system without delays. For an ordinary differential system, a unique solution is determined by an initial point in Euclidean space at an initial time t_0. For a delay differential system, one requires information on the entire interval $[t_0 - h, t_0]$. Each such initial function determines a unique solution to the delay differential equation. If we require that initial functions be continuous, then the space of solutions has the same dimensionality as $C([t_0 - h, t_0], \mathbb{R})$.

In the next section, with the previous discussion as a guide, let us now define the DDEs problem for a given initial function.

2.1.1 A general initial value problem

Suppose $\tau > 0$ is a given real number $\tau > 0$, denote $C([a, b], \mathbb{R}^n)$, the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We will denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ as $|x|$ from now on in order to avoid confusion with another norm we shall use. If $[a, b] = [-\tau, 0]$, we let C=$C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element φ in C by

$$\|\varphi\|_\tau := \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|.$$

Let $\sigma \in \mathbb{R}, A > 0$ and $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$, be defined by

$$x_t = x(t + \theta) \text{ for } -\tau \leq \theta \leq 0.$$

Definition 2.1 ([34]) *If Ω is a subset of $\mathbb{R} \times C$, Let $f : \mathbb{R} \times C \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation*

$$\begin{cases} x'(t) = f(t, x_t), t \geq \sigma, \\ \textit{and } x_\sigma = \varphi, \end{cases} \tag{2.2}$$

is a retarded functional differential equation and we will denote this equation by DDEs. The quantitie $\tau \geq 0$, is called the delay.

Definition 2.2 ([34]) *A function x is said to be a solution of (2.2) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, and x satisfies (2.2) for $t \in [\sigma, \sigma + A]$. In such a case we say that x is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ for a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say that $x = x(\sigma, \varphi)$, is a solution of (2.2) with an initial value at σ or simply a solution of (2.2) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.*

Equation (2.2) is a very general type of equation and includes ordinary differential equations ($\tau = 0$). Although the structure of these equations is similar to ordinary differential equations, the crucial difference is that a delay differential equation (or a system of equations) is an infinite dimensional problem and the corresponding phase space is a functional space usually the space of continuous functions is considered.

Definition 2.3 *Equation (2.2) is called:*

i) *linear if $f(t, \varphi) = L(t, \varphi)$, where L is linear in φ.*

ii) nonhomogeneous if $f(t,\varphi) = L(t,\varphi) + h(t)$*, where* $h(t) \neq 0$*, it is called homogeneous if* $h = 0$.

iii) autonomous if $f(t,\varphi) = g(\varphi)$*, where* g *does not depend on* t.

Equation (2.2) is a very general type of equation and includes differential-difference equations. To be more explicit we give some classes of equations that can be expressed by (2.2), we have equations with a fixed delay (the simplest possible case) such as

$$x'(t) = f(t, x(t), x(t-\tau)),$$

or nonlinear nonautonomous differential equations with multiple time varying delays on the same state x

$$x'(t) = f(t, x(t), x(t-\tau_1(t)), ..., x(t-\tau_p(t))),$$

with $0 \leq \tau_i(t) \leq \tau$ for all $i = 1, ..., p$.
We also have integrodifferential equations with a distributed delay

$$x'(t) = \int_{-\tau}^{0} g(t, x(t+\theta))\, d\theta,$$

where we see how in the integration process we need to know the values of x in $[t-\tau, t]$ for each t where the vector field is defined.

The delay also may be functions $\tau_i(t, x(t)) > 0, (i = 1, 2, ..., p)$ (state-dependent delays), the following class of functional differential equations with state and time-varying delays,

$$x'(t) = f(t, x(t), x(t-\tau_1(t, x(t))), ..., x(t-\tau_p(t, x(t)))),$$

Iterative differential equations have distinctive characteristics as a particular type of state-dependent delay-differential equation

$$x'(t) = f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}, ..., x^{[n]}\right),$$

where the iterate $x^{[n]}(t)$ stands for x composed with itself n times, i.e. $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t))$, $x^{[3]}(t) = x^{[2]}(x(t))$,..., $x^{[n]}(t)$ $= x^{[n-1]}(x(t))$ are the iterates of the state x. In recent years, the iterative differential equations have been the subject of several investigations (see, [58] , [38] and the references cited therein).

2.1.2 Existence and uniqueness theory

The existence and uniqueness theory for delay equations can be derived from the more general theory of functional differential equations. Since we intend to consider only equations of the form (2.2) we will not make use of the full generality available. Nevertheless, the more general theory leads to a presentation that is simpler and also benefits from an analogy with similar results in the theory of ordinary differential equations.

We now state the basic theory of DDEs.

Lemma 2.1 *([34]) Let $\sigma \in \mathbb{R}$ and $\varphi \in C$ be given and f be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of equation (2.2) through (σ, φ) is equivalent to solving the integral equation:*

$$x(t) = \varphi(\sigma) + \int_{\sigma}^{t} f(s, x_s)\, ds \text{ for } t \geq \sigma, \text{ and } x_\sigma = \varphi.$$

Lemma 2.2 *([34]) If $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [\sigma - \tau, \sigma + A]$.*

Proof. Since x is continuous on $[\sigma - \tau, \sigma + A]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently for t, s in $[\sigma, \sigma + A]$, $|t - s| < \delta$, we have $|x(t + \theta) - x(s + \theta)| < \varepsilon, \forall \theta \in [-\tau, 0]$. ■

The existence and uniqueness of the solutions of DDEs are given by the following Theorems.

Theorem 2.1 *(Local existence, [34]) Suppose $\overline{\Omega}$ is an open subset in $R \times C$ and $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous. For any $(\sigma, \varphi) \in \overline{\Omega}$, there exists a solution of equation (2.2) through (σ, φ).*

Definition 2.4 (Lipschitzian, [34]). *We say $f(t, \varphi)$ is Lipschitz in φ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \varphi_i) \in K, i = 1, 2,$*

$$|f(t, \varphi_1) - f(t, \varphi_2)| < k\, |\varphi_1 - \varphi_2| \, .$$

Theorem 2.2 *(Existence and uniqueness, [44]) Suppose $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$, $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitzian in φ in each compact set in $\overline{\Omega}$. If $(t_0, \varphi) \in \overline{\Omega}$, then there is a unique solution of Eq. (2.2) through (t_0, φ).*

Proposition 2.1 . *If f is at most affine i.e. $f(t, \varphi) \leq a + b\,|\varphi|$, with $a, b > 0$, then there exists a global solution of the equation (2.2) i.e. $\forall\varphi$, the solution $x(\sigma, \varphi)$ is defined on $[A, \infty[$.*

In the following we also require continuous dependence of solutions on initial conditions, for which the following theorem gives a result analogous to that for ordinary differential equations.

Theorem 2.3 . *Suppose x is a solution through (t_0, φ) of the equation (2.2) and that it is unique on $[t_0 - \tau, \beta]$. If $\{(t_n, \varphi_n)\} \subset \mathbb{R} \times C$ is a sequence such that $(t_n, \varphi_n) \to (t_0, \varphi)$ as $n \to \infty$, then for all sufficiently large n every solution x_n through φ_n exists on $[t_n - \tau, \beta]$, and $x_n \to x$ uniformly on $[t_0 - \tau, \beta]$.*

2.2 Neutral delay differential equations

Now are ready to give the definition of an other class of delay differential equations so-called the Neutral delay differential equations (NDDEs).

Definition 2.5 [34] *Suppose that* $\mathbb{R} \times C$ *is open with elements* (t, φ)*. A function* $D : \overline{\Omega} \to \mathbb{R}^n$ *is said to be atomic at* β *on* $\overline{\Omega}$ *if* D *is continuous together with its first and second Fréchet derivatives with respect to* φ*; and* D_φ*, the derivative with respect to* φ*, is atomic at* β *on* $\overline{\Omega}$*.*

Definition 2.6 [34] *Suppose that* $\overline{\Omega} \subseteq \mathbb{R} \times C$ *is open,* $f : \overline{\Omega} \to \mathbb{R}^n$*,* $D : \overline{\Omega} \to \mathbb{R}^n$ *are given continuous functions with* D *atomic at zero. The equation*

$$\frac{dD}{dt}(t, x_t) = f(t, x_t), \tag{2.3}$$

is called the neutral delay differential equation NDDE (D, f)*.*

If the delayed argument occurs in the highest order derivative of the state we call it neutral functional differential equation.

The following equations are some examples of neutral differential equations

Example 2.1 [34] *If* $\tau > 0$*,* B *is an* $n \times n$ *constant matrix,* $D(\varphi) = \varphi(0) - B\varphi(-\tau)$*, and* $f : \overline{\Omega} \to \mathbb{R}^n$ *is continuous, then the pair* (D, f) *defines an NDDE,*

$$\frac{d}{dt}[x(t) - Bx(t - \tau)] = f(t, x_t).$$

Example 2.2 [34] *If* $\tau > 0$*,* x *is a scalar,* $D(\varphi) = \varphi(0) - \sin(-\tau)$*, and* $f : \overline{\Omega} \to \mathbb{R}^n$ *is continuous, then the pair* (D, f) *defines an NDDE,*

$$\frac{d}{dt}[x(t) - \sin x(t - \tau)] = f(t, x_t). \tag{2.4}$$

Remark 2.1 *Note that when x is continuously differentiable, (2.4) is equivalent to*

$$x^{'}(t) - (\cos x(t-\tau))\, x^{'}(t-\tau) = f(t, x_t).$$

Definition 2.7 [34] *A function x is said to be a solution of (2.3) on $[\sigma - \tau, \sigma + A]$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that*

$$x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n), \ \ (t, x_t) \in \overline{\Omega}, \ \ t \in [\sigma, \sigma + A],$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (2.3) on $[\sigma, \sigma + A]$. For a given $t_0 \in \mathbb{R}, \varphi \in C$, and $(\sigma, \varphi) \in \overline{\Omega}$, we say $x(t, \sigma, \varphi)$ is a solution of equation (2.3) with initial value φ at σ or simply a solution through (σ, φ) if there is an $A > 0$ such that $x(t, \sigma, \varphi)$ is a solution of equation (2.3) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$; we say $x(t, \sigma, \varphi)$ is a solution of (2.3) on $[\sigma - \tau, \infty)$, if for every $A > 0$, $x(t, \sigma, \varphi)$ is a solution of equation (2.3) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.

Theorem 2.4 *(Existence,* [44] *) If $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$ and $(t_0, \varphi) \in \overline{\Omega}$, then there exists a solution of the NDDE (L, f) through (t_0, φ).*

2.3 Method of steps

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solving the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solutions (e.g. the variation of constants method, the separation of variables method and others). But these methods are inapplicable to dealing with delay differential equations. Hence qualitative and numerical analysis of these equations gather great importance. The method of steps was first proposed by Bellman and Cooke [7]. This approach, provides a method for finding explicit solutions. The desired solution is found at successive intervals by solving ordinary differential equations without delays in each interval. As an illustration of this approach, consider the DDE:

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau)), t \geq t_0 \\ x(t) = \varphi_0(t), \ t_0 - \tau \leq t \leq t_0. \end{cases} \tag{2.5}$$

For such equations the solution is constructed step by step as follows:

Given that a function $\varphi_0(t)$ continuous on $[t_0 - \tau, t_0]$, therefore one can obtain the solution in the next step interval $[t_0, t_0 + \tau]$ by solving the following ordinary differential equation:

$$x'(t) = f(t, x(t), \varphi_0(t-\tau)) = g_0(t, x(t), \text{ for } t_0 \leq t \leq t_0 + \tau.$$

Under suitable hypotheses on g_0, existence and uniqueness of a solution of this equation (hence a solution of (2.5) on $[t_0 - \tau, t_0]$ can be established. Denoting this solution by $\varphi_1(t)$ and restricting equation (2.5) to the interval $[t_0 + \tau, t_0 + 2\tau]$, we find the ordinary differential equations

$$x'(t) = f(t, x(t), \varphi_1(t-\tau)) = g_1(t, x(t) \text{ for } t_0 + \tau \leq t \leq t_0 + 2\tau,$$

with the initial condition $x(t_0+\tau) = \varphi_1(t_0+\tau)$, for which we can again establish existence and uniqueness of a solution φ_2. Thus we have now extended the solution x to the interval $[t_0 + \tau, t_0 + 2\tau]$, and we now have a formula for $x(t)$ when $t \in [t_0 - \tau, t_0 + 2\tau]$.

In general, by assuming that $\varphi_{k-1}(t), \forall (k = 1, 2, ...)$ is defined on the interval $[t_0 + (k-2)\tau, t_0 + (k-1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$x'(t) = f(t, x(t), \varphi_{k-1}(t-\tau)), \text{ for } t_0 + (k-1)\tau \leq t \leq t_0 + k\tau,$$

with the initial condition $x(t_0 + (k-1)\tau) = \varphi_{k-1}(t_0 + (k-1)\tau)$. We can continue this process indefinitely, showing that the uniquely defined $x(t)$ exists on $[t_0 - \tau, \infty)$.

Example 2.3 For an exemple of this method we look first at a very simple DDE

$$y'(t) = -y(t-1),$$

with

$$y(t) = H(t) = 2, \text{ for } -5 \le t \le 0.$$

The solution in the interval $0 \le t \le 5$ is given by:

$$y(t) = \int_0^t -H(x-1)\, dx + y(0) = 1 - t.$$

Now we can solve for the solution in the interval $5 \le t \le 10$. This solution is given by:

$$\begin{aligned} y(t) &= \int_5^t -H(x-1)\, dx + y(5) \\ &= \int_5^t -(2-x)\, dx + y(5) \\ &= \frac{t^2}{2} - 2t - \frac{25}{2} + 10 - 3 \\ &= \frac{t^2}{2} - 2t - \frac{11}{2}. \end{aligned}$$

2.4 Primary discontinuity of delay differential equation

Definition 2.8 [24] *If the solution of a DDE and its derivatives of order μ are continuous at some point in the time interval, but the derivative of order $\mu + 1$ is not, then such a point is called a primary discontinuity of the given problem.*

Theorem 2.5 [24] *The points* $\xi_\mu := \mu\tau$ *the primary discontinuities of problem (2.5). More precisely,* $x^{(\mu)}$ *is continuous at* ξ_μ *but* $x^{(\mu+1)}$ *is, in general, not, even if the functions* φ *and* f *have continuous derivatives of all orders.*

Proof. See [24]. Note that, as t increases, the solution becomes smoother. In fact, at the initial point $t = 0$, the first derivative $x'(t)$ has a primary discontinuity, since the integrable equation

$$x'(t) = f(t, x(t), \varphi(t-\tau)), t \in [0, \tau],$$

may satisfy the condition $x(0) = \varphi(0)$, but it is unlikely to satisfy the additional condition $x'(0^+) = \varphi'(0^-)$. Only for special choices of the initial function $\varphi(t)$ is it possible to guarantee continuity of the derivative of the solution at point 0, for such a function must satisfy the condition $\varphi'(0^-) = f(0, \varphi(0), \varphi(-\tau))$. ■

Example 2.4 *We illustrale this method by using the special cases of equation (2.5), the following is an example presented in (*[34]*), canada 1998, Let*

$$\begin{aligned} x'(t) &= ax(t-\tau), t \in [0, +\infty), \\ x(t) &= 1, t \in [-\tau, 0], \end{aligned}$$

where a *is positive constant. Using the method of steps, it is easy to see that the solution* $x(t)$ *is a piecewise polynomial. On each subinterval* $[i\tau, (i+1)\tau]$, $x(t)$ *is an* $(i+1)$*-th. order polynomial, i.e.,*

$$x(t) = \sum_{j=1}^{i+1} \frac{a^j}{j!}(t-(j-1)\tau)^j, i \in \mathbb{N}.$$

It is also clear that integer multiples of τ *are primary discontinuities for this particular problem.*

As a generalization of (2.5), we consider

$$x'(t) = f(t, x(t), x(t - \tau(t)), x'(t - \tau(t))), t \in [0, \overline{a}],$$

where $t - \tau(t)$ is a strictly increasing function and

$$0 < \tau(t) \leq t, \ \overline{a} = \inf_{t \geq 0} \{t - \tau(t)\}.$$

Remark 2.2 *The method of steps can be extended to delay differential equations with additive noise term.*

CHAPTER 3

Positive periodic solutions of neutral difference equations

Contents

The goal of this chapter is to present a work published in [59], namely, **E. Yankson**, Positive solutions for a system of periodic neutral delay difference equations, **African Diaspora Journal of Mathematics** , *11 (2) (2011), 90-97.*

The basic theory of linear difference equations was developed in the eighteenth century by de Moivre, Euler, Lagrange, Laplace, and others. Generating functions first used by the Moivre to solve the Fibonacci equation, were exploited by Laplace as part of his work in probability theory.

The development of the study of periodic solutions of difference equations is relatively rapid. There have been many approaches to studying periodic solutions of difference equations, such as critical point theory,fixed point theorems in Banach spaces. The study of the existence of periodic solutions for difference equations has gained the attention of many researchers in recent times see for example [3] [5], and [20]. The study of positive periodic solutions of differential and differece equations has gained the attention of a lot of researchers in recent years: see [15] − [17] , [52] , [53] , [59] and references therein.

By using the fixed point method, some sufficient conditions are obtained for the existence of positive periodic solutions for a system of neutral delay difference equations. In the process, we invert the given equation into an equivalent integral equation. Then, we construct an appropriate mapping and use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions of the given system. Easily verifiable sufficient conditions are established. Also, by using the contraction mapping principle, the uniqueness of the solution is obtained. For this type of equation, we combine different techniques to prove new results.

3.1 Short biographies about difference equations

A difference equation is a mathematical equation that defines a sequence of values by relating the value of a current term to previous terms in the sequence. In other words, it describes how a quantity changes from one time step to the next in discrete time. Difference equations are commonly used in various fields such as mathematics, physics, engineering, economics, and biology to model dynamic systems.

Difference equations arise in almost every field of scientific inquiry, from population dynamics (the study of a single species) to economics (the study of a single commodity) to physics (the study of the motion of a single body). Such equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$ st generation $x(n+1)$ is a function of the nth generation $x(n)$ .This relation expresses itself in the difference equation

$$x(n+1) = f(x(n)). \tag{3.1}$$

By using x_0 as an initial point, the following sequence can be generated

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...,$$

$f(x_0) = x_1$ is called the first iterate of x_0 under f, $f^2(x_0) = f(f(x_0)) = x_2$ is called the second iterate of x_0 under f. In general $f^n(x_0) = x_n$ is the n^{th} iterate of x_0 under f. This iterative procedure is an example of a discret dynamical system. That is, systems that can be described by a set of n variables $x_1, x_2, ..., x_n$ which are specified by the following difference equation,

$$x(n+1) = f(x(n)).$$

The difference equations and discret dynamical equations are equivalent in the sense that they represent the same physical model. For instance, when we discuss difference equations we often refer to the analytic theory of subject, and when we discuss discrete dynamical system, we refer to the topological and geometrical aspects.

Difference equation (3.1) is called time-invariante or autonomus, if the function f in (3.1) is replaced by a function g of two variables $g : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}^+$ is the set of nonnegative integers, then we have

$$x(n+1) = g(n, (x(n))).$$

The basic theory of linear difference equations was developed in the eighteenth century by de Moivre, Euler, Lagrange, Laplace, and others. Actually, asymptotic analysis entered the subject somewhat earlier when Maclaurin (1742), and independently Euler (1755), stymied by the dearth of closed formulas for sums, discovered recipes for approximate summation The foundation for a thorough study of the asymptotic properties of solutions of linear difference equations was laid in the 1880's by poincaré,who formalized the concept of asymptotic series and also showed that under favorable conditions the ratio of consecutive values of a solution must approach a characteristic root. In 1909, Perron gave a significant extension of this result, and the asymptotic theory was brought to a certain level of completeness by Birkhoff and his students in 1930.

The efficient application of linear difference equations to the computation of special functions orginated in 1952 with Miller's algorithm for Bessel functions.

During the 1950's, several ecologists used simple nonlinear difference equations, including the logistic equation to study the change in populations from one year (or season) to the next with the emphasis on the stability of the iteration.However, in the early 1970's May investigated the variety of complex behavior exhibited by the logistic equation and pondered the possible relationship of this behavior to observed fluctuations in real populations. Additional discoveries about the logistic and related equations were soon made by York, Sarkovskii, Feigenbaum, and others, and the remarkably intricate properties of these equations led to their becoming a focus in the developing area of chaotic dynamical systems. The excitement of these discoveries attracted the attention of researchers who attempted to apply the results to fields from economics to medicine.

3.2 Statement of the problem and preliminaries

Throughout this paper, a vector $x(n) = [x_1(n), x_2(n), ..., x_k(n)]^t \in \mathbb{R}^k$ is said to be positive if $x_i \geq 0, 1 \leq i \leq k$.

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_-$ the negative integers, $\mathbb{Z}^+$ the non-negative integers, and $T \geq 1$ is an integer. In this chapter we consider the system of neutral difference equations with the highest-order differences of the state occurring with delays,

$$x(n+1) = A(n)x(n) + C(n) \triangle x(n - \tau(n)) + g(n, x(n - \tau(n))), \quad (3.2)$$

where $x(n) = [x_1(n), x_2(n), ..., x_k(n)]^t \in \mathbb{R}^k$, $A(n) = diag[a_1(n), a_2(n), ..., a_k(n)]$, and $C(n) = diag[c_1(n), c_2(n), ..., c_k(n)]$, a_j, c_j, τ are T-periodic, $g : \mathbb{Z} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is continuous in x and $g(n, x)$ is T-periodic in n and x, whenever x is T-periodic. Let P_T be the set of all real T-periodic sequences $x : \mathbb{Z} \rightarrow \mathbb{R}^k$. Endowed with the maximun norm

$$\|x\| = \max_{\theta \in \mathbb{Z}} \sum_{j=1}^{k} |x_j(\theta)|,$$

where $x = (x_1, x_2, ..., x_k)^t$, P_T is a Banach space. Here t stands for the transpose.

In the process we will require the following:

a) For any sequence

$$\sum_{n=a}^{b} x(n) = 0 \text{ and } \prod_{n=a}^{b} x(n) = 1 \text{ whenever } a > b.$$

b)

$$\sum (Ex(n)\,\Delta z(n)) = x(n)\,z(n) - \sum z(n)\,\Delta x(n),$$

where E is defined as $Ex(n) = x(n+1)$.

In 2010, by means of Krasnoselskii's fixed point theorem, Raffoul and Yankson [53] gave the sufficient conditions that guarantee the existence of positive periodic solutions of the following neutral scalar difference equation,

$$x(n+1) = a(n)\,x(n) + c\Delta x(n-\tau) + g(n, x(n-\tau)), \tag{3.3}$$

with a constant delay τ was considered.

In this chapter, by using Krasnosel'skii's fixed point theorem for the sum of a completely continuous operator and a contraction to obtain sufficient conditions for the existence of positive periodic solutions x for the generalized system given by (3.2).

Before giving our main theorem related to the existence of positive periodic soltions of the system (3.2), we suppose the conditions below hold.

(H1) There exist a constant $\sigma_j > 0$ such that $\sigma_j < c_j(n), j = 1, 2, ..., k,$ for all $n \in [0, T-1]$.

(H2) $0 < a_j(n) < 1$ for all $n \in [0, T-1], j = 1, 2, ..., k.$

(H3) There exist constants α_j, such that $\|c_j\| \leq \alpha_j \leq 1, j = 1, 2, ..., k.$

To simplify our description, we start this section by introducing some notations. Let

$$G_j(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a_j(s)}{1 - \prod_{s=n}^{n+T-1} a_j(s)}, u \in [n, n+T-1]. \tag{3.4}$$

Note that the denominator in $G_j(n,u)$ is not zero since $0 < a_j(n) < 1$ for $n \in [0, T-1]$.

Define

$$G(n,u) = diag\left[G_1(n,u), G_2(n,u), ..., G_k(n,u)\right]. \tag{3.5}$$

It is evident that $G(n,u) = G(n+T, u+T)$ for all $(n,u) \in \mathbb{Z}^2$. Also, let

$$q_j := \min\left\{G_j(n,u) : n \geq 0, u \leq T\right\} = G_j(n,n) > 0, j = 1, ..., k, \tag{3.6}$$

and

$$\begin{aligned} Q_j &: = \max\left\{G_j(n,u) : n \geq 0, u \leq T\right\} = G_j(n, n+T-1) \\ &= G_j(0, T-1) > 0, j = 1, ..., k. \end{aligned} \tag{3.7}$$

We also set the notations $q = \min_{1 \leq j \leq k} q_j$ and $Q = \max_{1 \leq j \leq k} Q_j$.

3.3 Inversion of the system (3.2)

Notice that problem (3.2) can be written as follows:

$$x_j(n+1) = a_j(n) x_j(n) + c_j(n) \Delta x_j(n - \tau(n)) + g_j(n, x_j(n - \tau(n))), j = \overline{1,k}. \tag{3.8}$$

Lemma 3.1 *Suppose (H2) holds. Then $x_j(n) \in P_T$ is a solution of if and only if*

$$\begin{aligned} x_j(n) &= c_j(n-1) x_j(n - \tau(n)) \\ &+ \sum_{u=n}^{n+T-1} G_j(n,u) g_j(u, x_j(u - \tau(u))) x_j(u - \tau(u)) \phi_j(u) a_j(u), j = \overline{1,k}, \end{aligned} \tag{3.9}$$

where

$$\phi_j(u) = c_j(u) - c_j(u-1).$$

Proof. *We have*

$$x_j(n+1) = a_j(n)\,x_j(n) + c_j(n)\,\Delta x_j(n-\tau(n)) + g_j(n, x(n-\tau(n))),\, j = \overline{1,k}. \tag{3.10}$$

Since

$$\Delta Z(n) = Z(n+1) - Z(n). \tag{3.11}$$

We rewite (3.10) *as*

$$x_j(n+1) - a_j(n)\,x_j(n) = c_j(n)\,\Delta x_j(n-\tau(n)) + g_j(n, x(n-\tau(n))). \tag{3.12}$$

By dividing the both sides of the above equation, we get

$$x_j(n+1)\,a_j^{-1}(n) - x_j(n) = [c_j(n)\,\Delta x_j(n-\tau(n)) + g(n, x(n-\tau(n)))]\,a_j^{-1}(n). \tag{3.13}$$

Multipling the both sides of the above equation by $\prod_{s=0}^{n-1} a_j^{-1}(s)$, *we get*

$$\begin{aligned} & \left[x_j(n+1)\,a_j^{-1}(n) - x_j(n)\right] \prod_{s=0}^{n-1} a_j^{-1}(s) \\ = & \ [c_j(n)\,\Delta x_j(n-\tau(n)) + g(n, x(n-\tau(n)))] \prod_{s=0}^{n} a_j^{-1}(s). \end{aligned} \tag{3.14}$$

Since

$$\begin{aligned} \prod_{s=0}^{n} a_j^{-1}(s) &= a_j^{-1}(0) \times a_j^{-1}(1) \times a_j^{-1}(2) \times \ldots \times a_j^{-1}(n) \\ &= \prod_{s=0}^{n-1} a_j^{-1}(s) \times a_j^{-1}(n) = \prod_{s=0}^{n} a_j^{-1}(s). \end{aligned} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\Delta\left[x_j(n)\prod_{s=0}^{n-1}a_j^{-1}(s)\right] &= x_j(n+1)\prod_{s=0}^{n}a_j^{-1}(s)-x_j(n)\prod_{s=0}^{n-1}a_j^{-1}(s)\\ &= x_j(n+1)a_j^{-1}(n)\prod_{s=0}^{n-1}a_j^{-1}(s)-x_j(n)\prod_{s=0}^{n-1}a_j^{-1}(s)\\ &= \left[x_j(n+1)a_j^{-1}(n)-x_j(n)\right]\prod_{s=0}^{n-1}a_j^{-1}(s). \qquad (3.16)\end{aligned}$$

Substituting (3.15) into (3.16) gives

$$\begin{aligned}&\Delta\left[x_j(n)\prod_{S=0}^{n-1}a_j^{-1}(s)\right]\\ = &\left[c_j(n)\Delta x_j(n-\tau(n))+g_j(n,x_j(n-\tau(n)))\right]\prod_{s=0}^{n}a_j^{-1}(s). \qquad (3.17)\end{aligned}$$

Summing equation (3.10) from n *to* $n+T-1$*, we obtian*

$$\begin{aligned}&\sum_{u=n}^{n+T-1}\Delta\left[x_j(u)\prod_{s=0}^{u-1}a_j^{-1}(s)\right]\\ = &\sum_{u=n}^{n+T-1}\left[c_j(u)\Delta x_j(u-\tau(u))+g_j(u,x_j(u-\tau(u)))\right]\prod_{s=0}^{u}a_j^{-1}(s)(3.18)\end{aligned}$$

It follows:

$$
\begin{aligned}
& \sum_{u=n}^{n+T-1} \Delta\left[x_j(n) \prod_{s=0}^{n-1} a_j^{-1}(s)\right] \\
= & \sum_{u=n}^{n+T-1}\left[x_j(n+1) \prod_{s=0}^{n} a_j^{-1}(s)-x_j(n) \prod_{s=0}^{n} a_j^{-1}(s)\right] \\
= & \sum_{s=0}^{n+T-1}\left[x_j(n+1) \prod_{s=t_0}^{n-1} a_j^{-1}(s) \times a_j^{-1}(n)-x_j(n) \prod_{s=t_0}^{n-1} a_j^{-1}(s)\right] \\
= & \sum_{s=0}^{n+T-1}\left[x_j(n+1) \prod_{s=t_0}^{n-1} a_j^{-1}(s) \times a_j^{-1}(n)-x_j(n) \prod_{s=t_0}^{n-1} a_j^{-1}(s)\right] \\
= & x_j(t_0+1) \prod_{s=t_0}^{t_0-1} a_j^{-1}(s) \times a_j^{-1}(t_0)-x_j(t_0) \prod_{s=t_0}^{t_0-1} a_j^{-1}(s) \\
& +x_j(t_0+2) \prod_{s=t_0+1}^{t_0} a_j^{-1}(s) \times a_j^{-1}(t_0+1) \\
& -x_j(t_0+1) \prod_{s=t_0+1}^{t_0} a_j^{-1}(s) \\
& +x_j(t_0+3) \prod_{s=t_0+2}^{t_0+1} a_j^{-1}(s) \\
& \times a_j^{-1}(t_0+2)-x_j(t_0+2) \prod_{s=t_0+2}^{t_0+1} a_j^{-1}(s) \\
& +\ldots+x_j(t_0+n) \prod_{s=t_0}^{t_0+n-1} a_j^{-1}(s) \\
= & x_j(t_0+n) \prod_{s=t_0}^{t_0+n-1} a_j^{-1}(s) \\
& -x_j(t_0) \prod_{s=0}^{t_0-1} a_j^{-1}(s) .
\end{aligned} \tag{3.19}
$$

Consecontly

$$\begin{aligned} & x(n+T)\prod_{s=0}^{n+T-1} a_j^{-1}(s) - x(n)\prod_{s=0}^{n-1} a_j^{-1}(s) \\ = & \sum_{u=n}^{n+T-1} \left[c_j(u)\Delta x_j(u-\tau(n)) \right. \\ & \left. + g_j(u, x_j(u-\tau(u)))\right] \prod_{s=0}^{u} a_j^{-1}(s). \end{aligned} \tag{3.20}$$

Since $x(n+T) = x(n)$, we obtain

$$\begin{aligned} & x(n)\left[\prod_{s=0}^{n+T-1} a_j^{-1}(s) \prod_{s=0}^{n-1} a_j^{-1}(s)\right] \\ = & \sum_{u=n}^{n+T-1} \left[c_j(u)\Delta x_j(u-\tau(u)) \right. \\ & \left. + g_j(u, x_j(u-\tau(u)))\right] \prod_{s=0}^{u} a_j^{-1}(s). \end{aligned} \tag{3.21}$$

But

$$\begin{aligned} & \sum_{u=n}^{n+T-1} c_j(u)\Delta x_j(u-\tau(u)) \prod_{s=0}^{u} a_j^{-1}(s) \\ = & c_j(n-1)x_j(n-\tau(u))\left[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\right] \\ & - \sum_{u=n}^{n+T-1} x_j(u-\tau(u))\Delta\left[c_j(u-1)\prod_{s=0}^{u-1} a_j^{-1}(s)\right] \\ = & c_j(n-1)x_j(n-\tau(u))\left[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\right] \\ & - \sum_{u=n}^{n+T-1} x_j(u-\tau(u))\left[c_j(u) - c_j(u-1)a_j(u)\right]\prod_{s=0}^{u} a_j^{-1}(s). \end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21) gives

$$\begin{aligned} & x(n)\left[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\right] \\ = \; & c_j(n-1)\, x_j(n-\tau(u))\left[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\right] \\ & - \sum_{u=n}^{n+T-1} x_j(u-\tau(u))\left[c_j(u) - c_j(u-1)\, a_j(u)\right] \\ & \prod_{s=0}^{u} a_j^{-1}(s) + \left[g_j(u, x_j(u-\tau(u)))\right] \prod_{s=0}^{u} a_j^{-1}(s). \qquad (3.23) \end{aligned}$$

Dividing throughgh by $\left[\prod_{s=0}^{n+T-1} a_j^{-1}(s) - \prod_{s=0}^{n-1} a_j^{-1}(s)\right]$ *gives the desired result.* ■

3.4 Existence of positive periodic solutions

In this section, we obtain sufficient conditions for the existence of positive periotic solutions for (3.2). For some nonnegative constant L and J, we define the set

$$\mathbb{M} = \left\{ \phi \in P_T : L \leq \|\phi\| \leq J, \text{ with } \frac{L}{k} \leq \phi_j \leq \frac{J}{k}, j = 1, 2, ..., \right\}, \qquad (3.24)$$

which is a closed convex and bounded subset of the Banach space P_T. Further, we assume that for all $u \in \mathbb{Z}$ and $\rho \in \mathbb{M}$,

$$\frac{(1-\sigma_j)\, L}{T q_j k} \leq g_j\left(u, \rho_j, \rho_j\right) - \rho_j \phi_j(u)\, a_j(u) \leq \frac{(1-\alpha_j)\, J}{T Q_j k}. \qquad (3.25)$$

Define a mapping $H : \mathbb{M} \rightarrow P_T$ by

$$\begin{aligned} (Hx)(n) \;=\; & C(n-1)\, x(n-\tau(n)) \\ & + \sum_{u=n}^{n+T-1} G(n,u)\left[g(u, x(u), x(u-\tau(u))) - \Phi(u)\, A(u)\, x(u-\tau(u))\right], \end{aligned}$$

where

$$\Phi(u) = diag[\phi_1(u), ..., \phi_k(u)].$$

We denote

$$(Hx) = (H_1x_1, H_2x_2, ..., H_kx_k)^t. \tag{3.26}$$

It is clear that $(Hx)(n+T) = (Hx)(n)$. We note that to apply Krasnoselskii's fixed point theorem we need to define a Banach space $X = P_T$, a closed convex subset $\mathbb{M}$ of X, and to construct two mappings; one is contraction and the other is compact. In view of these idea, we write equation (3.9) in the form

$$(H\varphi)(n) = (D\varphi)(n) + (F\varphi)(n),$$

where both of the mappings D and F are defined as follows: $D : \mathbb{M} \rightarrow P_T$ by

$$(D\varphi)(n) = C(n-1)\varphi(n-\tau(n)), \tag{3.27}$$

and $F : \mathbb{M} \rightarrow P_T$ by

$$(F\varphi)(n) = \sum_{u=n}^{n+T-1} G(n,u)[g(u,\varphi(u),\varphi(u-\tau(u))) - \Phi(u)A(u)\varphi(u-\tau(u))]. \tag{3.28}$$

Lemma 3.2 *Suppose (H3) hold. Then the operator D defined by (3.27) is a contraction.*

Proof. Let $\varphi, \psi \in \mathbb{M}$ and $\alpha = \max_{1\leq j\leq k}\alpha_j$. Then

$$\|(D\varphi) - (D\psi)\| = \max_{n\in[0,T-1]} \sum_{j=1}^{k} \left|(D_j\varphi_j)(n) - (D_j\psi_j)(n)\right|.$$

But,

$$\left\|(D_j\varphi_j)(n) - (D_j\psi_j)(n)\right\| = \left|c_j(n-1)\varphi_j(n) - c_j(n-1)\psi_j(n)\right| \leq \alpha_j \left\|\varphi_j - \psi_j\right\|.$$

Thus,

$$\|(D\varphi)-(D\psi)\| \leq \sum_{j=1}^{k} \alpha_j \left\|\varphi_j-\psi_j\right\| \leq \alpha \|\varphi-\psi\| .$$

This completes the proof of lemma 3.3. ■

Lemma 3.3 . *Suppose that (H1), (H2), (H3) and (3.25) hold. Then the operator F defined by (3.28) is completely continuous on $\mathbb{M}$.*

Proof. For $n \in [0, T-1]$ and for $\varphi \in \mathbb{M}$, we have by (3.25) hold that

$$\begin{aligned}\left|\left(F_j\varphi_j\right)(n)\right| &\leq \left|\sum_{u=n}^{n+T-1} G_j(n,u)\left[g_j\left(u,\varphi_j(u-\tau(u))\right)-\varphi_j(u-g(u))\,\phi_j(u)\,a_j(u)\right]\right| \\ &\leq Q_j T\frac{(1-\alpha_j)J}{TQ_jk} \\ &\leq \frac{(1-\alpha_j)J}{k}.\end{aligned}$$

Thus,

$$\begin{aligned}\|(F\varphi)\| &\leq \sum_{j=1}^{k}\frac{(1-\alpha_j)J}{k} \\ &\leq (1-\alpha^*)J,\end{aligned}$$

where $\alpha^* = \min_{1\leq j\leq k} \alpha_j$. It so follows that

$$\|(F\varphi)\| < J.$$

This showns that $F(M)$ is uniformly bounded. Given to the continuity of every terms, we have that F is continuous. ■

Next we shown that F maps bounded subsets into compact sets. Let $S = \{\varphi \in P_T : \|\varphi\| \leq \mu\}$ and $Q = \{(F\varphi)(n) : \varphi \in S\}$, then S is a subset of $\mathbb{R}^k$ which is closed and bounded and thus compact. As F is continuous in φ, it maps compact sets into compact sets. Hence $Q = F(S)$ is compact. This completes the proof.

Theorem 3.1 *Suppose that (H1), (H2), (H3), and (3.25) hold. Also impose that the hypothesis of Lemma 3.4 also hold. Then equation (3.2) has a positive periodic solution.*

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then we have that

$$\begin{aligned}
& \left(D_j\varphi_j\right)(n) + \left(F_j\psi_j\right)(n) \\
= \ & c_j(n-1)\varphi_j(n-\tau(n)) \\
& + \sum_{u=n}^{n+T-1} G_j(n,u)\left[g_j\left(u,\psi_j(u),\psi_j(u-\tau(u))\right) - \psi_j(u-\tau(u))\phi_j(u)a_j(u)\right] \\
\leq \ & \frac{\alpha_j J}{k} + Q_j \sum_{u=n}^{n+T-1}\left[g_j\left(u,\psi_j(u),\psi_j(u-\tau(u))\right) - \psi_j(u-\tau(u))\phi_j(u)a_j(u)\right] \\
\leq \ & \frac{\alpha_j J}{k} + \frac{Q_j T(1-\alpha_j)J}{TQ_j k} = \frac{J}{k}.
\end{aligned}$$

This,

$$\|(D\varphi)(n) + (F\psi)(n)\| \leq \sum_{j=1}^{k}\frac{J}{k} = J.$$

On the other hand,

$$\begin{aligned}
& \left(D_j\varphi_j\right)(n) + \left(F_j\psi_j\right)(n) \\
= \ & c_j(n-1)\varphi_j(n-\tau(n)) \\
& + \sum_{u=n}^{n+T-1} G_j(n,u)\left[g_j\left(u,\psi_j(u),\psi_j(u-\tau(u))\right) - \psi_j(u-\tau(u))\phi_j(u)a_j(u)\right] \\
\geq \ & \frac{\sigma_j L}{k} + q_j \sum_{u=n}^{n+T-1}\left[g_j\left(u,\psi_j(u),\psi_j(u-\tau(u))\right) - \psi_j(u-\tau(u))\phi_j(u)a_j(u)\right] \\
\geq \ & \frac{\sigma_j L}{k} + \frac{q_j T(1-\sigma_j)L}{Tq_j k} = \frac{L}{k}.
\end{aligned}$$

Thus,

$$\|(D\varphi)(n) + (F\psi)(n)\| \geq \sum_{j=1}^{k}\frac{L}{k} = L.$$

This shows that $(D\varphi)(n) + (F\psi)(n) \in \mathbb{M}$. Thus by Theorem 2.1 equation (3.2) has a positive periodic solution in $\mathbb{M}$. ■

Example 3.1 The neutral difference system:

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{12} \end{pmatrix} \begin{pmatrix} \Delta x_1(n-6) \\ \Delta x_2(n-6) \end{pmatrix}$$
$$+ \begin{pmatrix} \dfrac{1}{x_1^2(n-6)+100} + \frac{7}{80}x_1(n-6) + \frac{1}{20} \\ \dfrac{1}{x_2^2(n-6)+100} + \frac{7}{80}x_2(n-6) + \frac{1}{20} \end{pmatrix}, \qquad (3.29)$$

has a positive periodic solutions $x = (x_1, x_2)$, of period 6 satisfying $\frac{12}{7} \leq \|x\| \leq 3$. To see this, we have

$$\begin{pmatrix} g_1(u,\rho_1) \\ g_2(u,\rho_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho_1^2+100} + \frac{7}{80}\rho_1 + \frac{1}{20} \\ \frac{1}{\rho_2^2+100} + \frac{7}{80}\rho_2 + \frac{1}{20} \end{pmatrix}.$$

and

$$a_1(n) = \frac{1}{8}, a_2(n) = \frac{1}{6}.$$

$$c_2(n) = \frac{1}{10}, c_2(n) = \frac{1}{12} \text{ and, } T = 6.$$

Let

$$\begin{aligned} q_1 &= \min\{G_1(n,u) : n \geq 0, u \leq T\} = G_1(n,n) > 0, j = 1,2. \\ q_2 &= \min\{G_2(n,u) : n \geq 0, u \leq T\} = G_2(n,n) > 0. \end{aligned}$$

A simple calculation yields

$$q_1 = G_1(n,n) = \frac{\prod_{s=n+1}^{n+6-1} \frac{1}{8}}{1 - \prod_{s=n}^{n+6-1} \frac{1}{8}} = \frac{\prod_{s=1}^{5} \frac{1}{8}}{1 - \prod_{s=n}^{5} \frac{1}{8}} = \frac{8}{262143},$$

and

$$q_2 = G_2(n,n) = \frac{\prod_{s=1}^{5} \frac{1}{6}}{1 - \prod_{s=n}^{5} \frac{1}{6}} = \frac{6}{46655}.$$

Let

$$Q_1 = \max\left\{G_1(n,n), n \geq 0, u \leq T\right\} = G_1(n, n+T-1) = G_1(0, T-1) > 0,$$

consequently

$$Q_1 = G_1(n, n+T-1) = \frac{\prod_{s=n+6-1+1}^{n+6-1} \frac{1}{8}}{1 - \prod_{s=n}^{n+6-1} \frac{1}{8}} = \frac{\prod_{6}^{5} \frac{1}{8}}{1 - \prod_{n}^{5} \frac{1}{8}} = \frac{262144}{262143}.$$

On the other hand,

$$Q_2 = G_2(n, n+T-1) = \frac{\prod_{6}^{5} \frac{1}{6}}{1 - \prod_{n}^{5} \frac{1}{6}} = \frac{1}{1 - \prod_{n}^{5} \frac{1}{6}} = \frac{46656}{46655}.$$

Let $L = \frac{12}{7}, J = 3$ and, define the set:

$$\mathbb{M} = \left\{\phi \in P_6 : \frac{12}{7} \leq \|\phi\| \leq 3 \text{ with } \frac{12}{14} \leq \phi_j \leq \frac{3}{2}, j = 1, 2\right\}.$$

Then for $\rho_1, \rho_2 \in \left[\frac{12}{7}, 3\right]$, we have

$$\begin{aligned} g_1(u, \rho_1) - c_1\left[1 - a_1(u)\right]\rho_1 &= \frac{1}{\rho_1^2 + 100} + \frac{7}{80}\rho_1 + \frac{1}{20} - \frac{1}{10}\left[1 - \frac{1}{8}\right]\rho_1 \\ &\leq \frac{1}{100} + \frac{1}{20} = 0.06 < \frac{(1-c_1)\,J}{TQ_1 2}, \end{aligned}$$

and,

$$\begin{aligned} g_1(u, \rho_1) - c_1\left[1 - a_1(u)\right]\rho_1 &= \frac{1}{\rho_1^2 + 100} + \frac{7}{80}\rho_1 + \frac{1}{20} - \frac{1}{10}\left[1 - \frac{1}{8}\right]\rho_1 \\ &> \frac{7}{80}\rho_1 + \frac{1}{20} - \frac{7}{80}\rho_1 \\ &> \frac{7}{80}\frac{12}{7} + \frac{1}{20} - \frac{7}{20}3 > \frac{(1-c_1)\,L}{Tq_1 2}. \end{aligned}$$

On the other hand:

$$\begin{aligned} g_2(u,\rho_2) - c_2[1-a_2(u)]\rho_2 &= \frac{1}{\rho_2^2+100} + \frac{7}{80}\rho + \frac{1}{20} + \frac{1}{12}\left[1-\frac{1}{6}\right]\rho_2 \\ &< \frac{1}{100} + \frac{1}{20} = 0,06 < \frac{(1-c_2)\,J}{TQ_22}, \end{aligned}$$

and

$$\begin{aligned} g_2(u,\rho_2) - c_2[1-a_2(u)]\rho_2 &= \frac{1}{\rho_2^2+100} + \frac{7}{80}\rho_2 + \frac{1}{20} + \frac{1}{12}\left[1-\frac{1}{6}\right]\rho_2 \\ &> \frac{7}{80}\rho + \frac{1}{20} - \frac{5}{72}\rho_2 \\ &> \frac{7}{80}\frac{12}{7} + \frac{1}{20} - \frac{5}{72}3 > \frac{(1-c_2)\,J}{Tq_22}. \end{aligned}$$

By Theorem 3.3 equation (3.29) has a positive periodic solution $x = (x_1, x_2)$ with period $T = 6$ such that $\frac{12}{7} \leq \|x\| \leq 3$.

3.5 Conclusion

This dissertation studies the positive periotic solutions for a system of nonlinear neutral delay difference. The approach used in our project is based on the fixed point technique. This method relies mainly on necessary arguments: an elementary variation of parameter formula, a non-empty closed convex and bounded subset $\mathbb{M}$ of a Banach space and a fixed point application. The benefit of this approach is that the fixed point arguments can yield existence, and at least one positive periodic solutions for such an equation in one step. The main difficulty of this approach is to define a subset $\mathbb{M}$ of a Banach space and a suitable mapping, when the mapping F is compact on $\mathbb{M}$. The results in this work extend and improve some exist results in the literature in some ways. Recently, Raffoul et al. [**53**] have addressed this technique to investigate the periodicity special cases of delay difference equations. However, there are many problems to be solved for impulsive difference equations and other variants, persistence, and so on. We leave these for our -future work.

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