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# Thesis

A view to obtaining the diploma of

## Doctorate of 3° cycle (LMD) in Mathematics

**Option: Numerical analysis, Partial differential equations and applications**

# Study of a quasi-linear boundary problem with non-regular boundary conditions and applications

Presented by:

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# ABSTRACT

In this dissertation, we study some quasilinear hyperbolic equations with source terms in three parts.
In the first part, we prove the global existence of solutions by using the potential and Nihari's functional for a quasilinear hyperbolic problem involving the weighted Laplacian and $p-$Laplacian operator, after that, by using the Nakao's inequality we study the decay of solutions and finally, we establish the blow up of solutions. In the second part, we prove the global existence results for a hyperbolic problem involving the fractional Laplacian operator and we study the blow up of solutions by using the concavity method. In the third part, we consider the boundary value problem related to the hyperbolic wave equation with non-regular boundary condition, we show the local existence theorem and we prove the finite time blow up result.
**Keywords**: Quasilinear equation, global existence, exponential stability, energy function, decay, hyperbolic problem, fractional laplacian, Potential functional, Blow up of solution, Nakao's inequality, source term, non-regular boundary conditions, concavity method.

# RÉSUMÉ

L'objectif de cette thèse est l'étude des certaines types des équations hyperboliques quasi-linéaires avec terme source en trois parties.

Dans la première patie, nous prouvons l'existence globale des solutions en utilisant le potentiel et la fonction de Nihari pour une équation hyperbolique de type quasi-linéaire avec l'opérateur laplacien et $p-$laplacien pondéré, puis en utilisant l'inégalité de Nakao nous étudions la décroissance des solutions et puis, nous obtenons des résultats d'explosion. Dans la deuxième partie, nous étudions l'existence globale pour une équation hyperbolique d'ordre fractionnaire et nous obtenons l' explosion des solutions en utilisant la méthode de concavité. Dans la troisième partie, nous considèrons le problème aux limites lié au l'équation hyperbolique d'onde avec des conditions aux limites non régulières. On montre le théorème d'existence local des solutions et on trouve les résultats d'explosion.

**Mots clés**: Équation quasi-linéaire, existence globale, stabilité exponentielle, fonction énergétique, désintégration, problème hyperbolique, laplacien fractionnaire, Fonctionnelle potentielle, Explosion de la solution, inégalité de Nakao, terme source, conditions aux limites non régulières, la méthode de concavité.

# ملخص

خلال هذه الأطروحة نقوم بدراسة حول بعض المعادلات التفاضلية الجزئية الزائدية الشبه الخطية ذات منبع في ثلاثة أجزاء.

في الجزء الأول، نقوم بدراسة وجود الحل الكلي باستعمال دالة نهاري-Nihari للمعادلات التفاضلية ذات المشتقات الجزئية الزائدية من النوع الشبه الخطي للمؤثر اللابلاسي الموزون و p-اللابلاسي الموزون ثم ندرس الاضمحلال للحلول باستخدام متباينة ناكاو وبعد ذلك نقوم بدراسة تفجير الحلول. في الجزء الثاني، نعمل على دراسة وجود الحل الكلي لمسألة زائدية ذات مؤثر من رتبة كسرية بعدئذ نطبق طريقة التقعرمن اجل دراسة تفجير الحلول. وفي الجزء الاخير، نقدم دراسة حول وجود الحل المحلي لمسألة زائدية متعلقة بمعادلة الموجة ذات شروط حدية غير منتظمة كما نقوم بدراسة تفجير الحلول.

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At the end of this work, I would like to express my deep gratitude to my supervisor Professor M. Maouni, for his advice, for his encouragement, for his great patience during the period of preparation and writing of this thesis. I greatly appreciated his support which allowed me to complete this work.

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# DEDICACE

With the help and success of God Almighty, this research was completed. I dedicate this humble work to those to whom I have never been able to express my sincere love. To the woman whose blood I carry with pride and whom I miss so much that I tremble whenever I find myself facing difficulties without her, to the woman whom life took from me early on, to "my mother, may God have mercy on her soul". To my strength and support at all times, to the flame that shines. My life and guides me to the right path, to my guardian angel, to the source of strength and strictness, "my dear father". To my five angles "my sisters" and their husbands and children. To my husband and my other half. To all my family and relatives. To all my friends and colleagues. To all those present in the heart and absent on the line.

# LIST OF SYMBOLS

•$\nabla u$: stands for the gradient operator.

•$div$: is the divergence operator.

•$\frac{\partial}{\partial x}$: partial derivative.

•$\mathbb{N}$: the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, ...\}$.

•$\mathbb{R}$: the set of real numbers.

•$\mathbb{R}^n$: is the real space of dimension $n$.

•$\Omega$: a bounded domain in $\mathbb{R}^n$.

•$\partial\Omega$: denote the boundary of domain $\Omega$.

•$< .,. >$: denotes the scalar product.

•$C^m(\Omega)$: space of $m$ times continuously differentiable functions on $\Omega$, $m \in \mathbb{N}$.

•$C^\infty(\Omega) = \bigcap_{m\in\mathbb{N}} C^m(\Omega)$.

•$C_0^\infty(\Omega)$: the space of $C^\infty(\Omega)$ functions with compact support in $\Omega$.

•$L^2(\Omega)$: Lebesgue space with norm $||.||_2$.

•$L^p(\Omega)$: Lebesgue space with norm $||.||_p$.

•$W^{2,s}(\Omega)$: fractional Sobolev space with norm $||.||_{2,s}$.

•$W_0^{2,s}(\Omega)$: denote the closure of $C_0^\infty(\Omega)$ in the norm $||.||_{W_0^{2,s}(\Omega)}$.

•$\mathbb{W}_0^{1,2}(\Omega, \alpha)$: the weighted Sobolev space, defined as the closure of $C_0^\infty(\Omega)$ in the norm $||.||_{\mathbb{W}_0^{1,2}}$.

•$\mathbb{W}_0^{1,p}(\Omega, \alpha)$: the $p-$weighted Sobolev space, defined as the closure of $C_0^\infty(\Omega)$ in the norm $||.||_{\mathbb{W}_0^{1,p}}$.

# INTRODUCTION

The partial differential equations that control the behavior of many wave events that we may observe or encounter in our environment can be derived. By studying these partial differential equations, it is possible to establish some common form for various wave events, even while the fundamental physical conditions that give rise to them differ. Having said that, there are many other kinds of what we refer to as wave phenomena, and they are also highly diverse. However, examining the structure of the partial differential equations governing the events makes it easy to understand why the characteristic of a wave is unquestionably shared by these phenomena.

The nonlinear partial differential equations that have garnered interest over the past two decades are complex and have a broad range of historical applications. In many application domains, including physics, engineering, and information theory which is crucial to entropy and the applied sciences they play a basic role. Also, they have long been employed to explain a wide range of natural events, including population growth, alterations in the climate, earthquakes, changes in atomic structure, and more. Analytical studies conducted utilizing a variety of techniques can help us comprehend these natural events much better.

In the set of all partial differential equations there exists a class of partial differential equations called equations of hyperbolic type. This type is an effective tool to engineering problems and model physical systems. Moreover, it has a wide range of applications, such as opening channel hydraulics, description traffic flow, gas flow networks and air traffic management. For more details, see ([1, 10, 17] and [49]).

The main focus of this thesis is the study of a class of quasilinear partial differential equation with respectively a regular and nonregular boundary conditions, furthermore the study of fractional equation, we prove the existence result, decay and blow up of solutions of this problems.
The study of the quasilinear hyperbolic PDEs is an active research area and has attracted great interest in recent years due to its applications in various fields. It's began in the 1960$s$ and the fields has grown steadily since then. The main challenge is to understand the qualitative properties of the solutions of these equations, such as the existence, decay and blow up of the solutions, Research in this area has continued to grow, and have become a major area of study in modern mathematics. For results of these studies, we refer the reader to ([27], [33], [40], [45] and [50]).
Over the past four decades, The subject of fractional differential equations is gaining much attention and importance. The so-called fractional differential equations are specified by generalizing the standard integer order derivative to arbitrary order. Due to the effective memory function of fractional derivative, fractional differential equations have been widely used to describe many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more details see([41] and [28]).
The difference between standard and fractional Laplacian can be explained from the probabilistic interpretation. The standard Laplace operator represents the infinitesimal generator of a Brownian motion with continuous sampling trajectories, therefore for a particle in a domain; it must leave the domain via the boundary point on $\partial\Omega$: On the other hand, the fractional Laplacian is the infinitesimal generator of an s-stable symmetric Levy process with discontinuous sample paths, the particles can 'jump' out of the domain without touch no boundary point on $\partial\Omega$.
Overall, it is evident that fractional differential equations containing the fractional Laplacian provide an exciting and diverse field of study that is significant from a theoretical and practical standpoint. These classes of equations had various applications in gas dynamics, materials science, and finance. For example, in gas dynamics, quasilinear equations are used to indicate the possible direction of the process of rapid variation of the state of the gas. In physics, they are represent the longitudinal motion of a visco-elastic configuration which obeys a nonlinear Voight model.

Nonregular boundary problems are widely applied in many mathematical models, such difusion phenomena, hydrologic fitration process, thermo-elasticity and hydrodynamics. This kind of problems has been introduced by a group of physicists to underline the fact that the kinetics of the process. The way the system's parts interact with the walls defines this kind of choice. Evolution equations with dynamical boundary conditions, or first order equations in time, have been thoroughly investigated since the first work was introduced in 1969. Afterwards, it was extensively researched by mathematicians and physicists, who were successful in their creative endeavors.

**The main results of this thesis:**

This thesis contains five chapters and it is mainly explores the study of a quasilinear hyperbolic equation and hyperbolic equation involving the fractional Laplacian. We use the potential and Nihari's functionals to prove global existence of solution, decay of energy being based on Nakao's inequality and we use the concavity method to obtain the blow up results.

**In chapter 1:** We recall some notations and we give some mathematical concepts, some lemmas and assumptions which will be used throughout this thesis.

**In chapter 2:** We consider the following value problem related to the quasilinear hyperbolic equation with nonlinear source terms:

$$\begin{cases} u_{tt} - |\nabla u| div\big(\alpha(x)\dfrac{\nabla u}{|\nabla u|}\big) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{1}$$

We start by proving the global existence of nontrivial solutions, thus, we obtain the decay of the energy by using Nakao's inequality and we get the finite time blow up of solutions.

**In chapter 3:** In this chapter, we study the following quasilinear hyperbolic equation involving the weighted $p-$Laplacian operator with source terms:

$$\begin{cases} u_{tt} - div\big(\alpha(x)|\nabla u|^{p-2}\nabla u\big) - \Delta u_t + |u_t|^{q-1}u_t = \xi|u|^{m-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{2}$$

We start by showing the local existence theorem, following that, we prove the global existence of solutions and decay of energy, later, we obtain the finite time blow up of solutions with non-positive initial energy.

**In chapter 4:** We consider the following hyperbolic equation involving the fractional Laplacian:

$$\begin{cases} u_{tt} + (-\Delta)^s u - (-\Delta)^s u_t + |u_t|^{q-1} u_t = \alpha |u|^{p-1} u, & x \in \Omega, t > 0 \\ u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{3}$$

We prove the global existence results and we establish the blow up of solutions by using the concavity method.

**In chapter 5:** We study the following value problem related to the hyperbolic wave equation with non-regular boundary conditions:

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2} u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \Gamma_0, t > 0, \\ u(t,x) = -|u_t|^{q-2} u_t, & x \in \Gamma_1, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{4}$$

We show the local existence theorem and we obtain the blow up results.

# CHAPTER 1

# PRELIMINARIES

This chapter collects several basic tools that will be required throughout this work. The common link between all of the results in this chapter is that they are preparatory for the main results, which are contained in the following chapters.

## 1.1 The $L^p(X;E)$ spaces

In all that follows, $(X, A, \mu)$ is a measured space and $E$ is a Banach space (on $\mathbb{R}$) where the norm note by $||.||$.

**Definition 1.1** [21]

1. For $1 \leq p < +\infty$, we define:

$$L^p(X;E) = \{f : X \to E \quad \mu - \text{mesurable and} \int_X ||f||^p d\mu < +\infty\},$$

with

$$||f||_{L^p(X;E)} = \Big(\int_X ||f||^p d\mu\Big)^{\frac{1}{p}}.$$

2. If $p = \infty$, let

$$L^\infty(X;E) = \{f : X \to E \quad \mu - \text{mesurable and} \quad \exists C > 0, \quad ||f|| \leq C \quad \text{for}$$

$$\mu - \text{a.e.} \quad x\},$$

with

$$||f||_{L^{\infty}(X;E)} = \inf\{C > 0, \quad ||f(x)|| \leq C \quad \text{for} \quad \mu-\text{a.e.} \quad x\}.$$

3. In particularly, if $p = 2$, $L^2(X;E)$ equipped with the inner product

$$< f, g >= \int_X f(x)g(x)d\mu.$$

## 1.2 The $L^{\infty}(0,T;L^2(\Omega))$

The $L^{\infty}(0,T;L^2(\Omega))$ is the set of functions $u$ such that for all $t \in (0,T)$ and $u(t)$ in $L^2(\Omega)$ for the norm

$$||u||_{L^{\infty}(0,T;L^2(\Omega))} = \sup_{0<t<T} ||u(t)||^2_{L^2(\Omega)}.$$

## 1.3 Weighted Sobolev spaces

**Definition 1.2** [14] We introduce the weighted Sobolev space $\mathbb{W}_0^{1,2}(\Omega,\alpha)$ defined as the closure of $C_0^{\infty}(\Omega)$ in the norm

$$||u||_{\mathbb{W}_0^{1,2}(\Omega,\alpha)} = (\int_{\Omega} \alpha(x)|\nabla u|^2 dx)^{\frac{1}{2}}.$$

$\alpha(x)$ is a weight function, which is strictly positive and continuously differentiable function in $\Omega$. It is a Hilbert space with scalar product

$$< u, v >_{\mathbb{W}_0^{1,2}(\Omega,\alpha)}= \int_{\Omega} \alpha(x)\nabla u \nabla v dx.$$

**Definition 1.3** [3] We introduce the $p-$weighted sobolev space $\mathbb{W}_0^{1,p}(\Omega,\alpha)$ defined as the closure of $C_0^{\infty}(\Omega)$ in the norm

$$||u||_{\mathbb{W}_0^{1,p}(\Omega,\alpha))} = \Big( \int_{\Omega} \alpha(x)|\nabla u|^p dx\Big)^{\frac{1}{p}},$$

and denote by $\mathbb{W}^{-1,p}(\Omega,\alpha)$ the dual space of $\mathbb{W}_0^{1,p}(\Omega,\alpha)$, with $\alpha(x)$ is a weight function, which is strictly positive and continuously differentiable function in $\Omega$ and $2 \leq p < \infty$.

**Lemma 1.1** *[3] Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then the following embedding hold:*

1. *$\mathbb{W}_0^{1,p}(\Omega,\alpha) \hookrightarrow W_0^{1,\beta}(\Omega)$ continuously if $1 \leq \beta \leq \dfrac{np}{n+\gamma}$ with $\gamma \geq 0$.*
2. *$\mathbb{W}_0^{1,p}(\Omega,\alpha) \hookrightarrow L^r(\Omega)$ compactly if $1 \leq r \leq p_{\gamma}^* = \dfrac{pn}{n-p+\gamma}$.*

## 1.4 Fractional Sobolev space

The fractional Laplacian $(-\Delta)^s u$ of the function $u$ is given by

$$(-\Delta)^s u(x) = C\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \quad \forall x \in \mathbb{R}^n,$$

where $C = C(n,s)$ be a constant of normalisation depends on $n$ and $s$.

### 1.4.1 Fractional Sobolev space

Before we define the fractional Sobolev space mathematically, we have in literature, the fractional Sobolev spaces are called also Aronszajin, Gagliardo or Slobodeckij spaces by the name of the ones who introduced them.

**Definition 1.4** Let be $\Omega$ a bounded set in $\mathbb{R}^n$ and for any $s \in (0,1)$ and $p \in [1,\infty)$. We define $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega\times\Omega)\},$$

this is a Banach space with respect to the norm

$$||u||_{W^{s,p}(\Omega)} = \big(||u||^p_{L^p(\Omega)} + [u]^p_{s,p}\big)^{\frac{1}{p}},$$

with

$$[u]^p_{s,p} = \Big(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\Big)^{\frac{1}{p}},$$

is the so-called Gagliardo (semi) norm of $u$.

### 1.4.2 The space $H^s(\Omega)$

In this part, we focus on the case $p = 2$, this is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ and $W^{s,2}_0(\mathbb{R}^n)$ out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^n)$ and $H^s_0(\mathbb{R}^n)$:

**Definition 1.5** [46] Let be $\Omega$ a bounded set of $\mathbb{R}^n$, then we define

$$H^s(\Omega) = W^{s,2}(\Omega), \quad \text{with} \quad 0 < s < 1,$$

$$H^s(\Omega) = \{u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega \times \Omega)\},$$

the sobolev space $H^s(\Omega)$ equipped with the following inner product and norm

$$< u, v >_{H^s(\Omega)} = \int_\Omega u(x)v(x)dx + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dxdy.$$

$$||u||_{H^s(\Omega)} = (\int_\Omega |u|^2 + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy)^{\frac{1}{2}}.$$

Let $W_0^{s,2}(\Omega)$ be closed linear subspace of $W^{s,2}(\Omega)$ (see [5]), and it's norm is given by

$$||u||_{W_0^{s,2}(\Omega)} = (\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy)^{\frac{1}{2}}.$$

**Lemma 1.2** *[19] Let $\Omega$ be a bounded domain. Then*

1. *The embedding $W_0^{s,2}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $p \in [1, 2_s^*)$, where $2_s^* = \dfrac{2n}{n-2s}$, $n > 2s$.*
2. *The embedding $W_0^{s,2}(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ is continuous.*

**Theorem 1.1** *(Poincare Inequality)[46] Let $\Omega$ be an open set of $\mathbb{R}^n$, let $s \in ]0; 1[$ and $p \in [1; +\infty[$ so there exists $C = C(n; s; p)$ such that*

$$||u||_{L^p(\Omega)} \leq C||u||_{W_0^{s,p}(\Omega)}, \quad \forall u \in W_0^{s,p}(\Omega).$$

*Therefore, if $\Omega$ is bounded then $||.||_{W_0^{s,p}(\Omega)}$ is a norm of $W_0^{s,p}(\Omega)$ equivalent to $||.||_{W^{s,p}(\Omega)}$ with*

$$||u||_{W_0^{s,p}(\Omega)} = (\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy)^{\frac{1}{p}}.$$

## 1.5 Some useful inequalities and lemmas

In this part, we will review several inequalities that will be used in later chapters.

**Lemma 1.3** *[2] Let $p$ be a number with $2 \leq p < \infty (n = 1, 2)$ or $2 \leq p \leq \frac{2n}{n-2} (n \geq 3)$, then there is a constant $C_* = C_*(\Omega, p)$ such that*

$$||u||_p \leq C_* ||\nabla u||_2, \quad \textit{for} \quad u \in H_0^1(\Omega).$$

**Theorem 1.2** *[12](The Young inequality) Let $a, b \in \mathbb{R}_+$ and $p, q \in ]1, +\infty[$ with $\frac{1}{p}+\frac{1}{q} = 1$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Theorem 1.3** *[12](The generalized Young inequality) Let $a, b \in \mathbb{R}_+$ and for all $\epsilon > 0$, we have*

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}.$$

**Theorem 1.4** *[12](The Hölder inequality) Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq \infty$ and with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$fg \in L^1(\Omega) \quad \textit{and} \quad ||fg||_{L^1(\Omega)} \leq ||f||_{L^p(\Omega)}||g||_{L^q(\Omega)}.$$

**Lemma 1.4** *(Nakao's inequality) [42] Let $\phi(t)$ be a non-increasing and non-negative function defined on $[0, T]$, $T > 1$, satisfying*

$$\phi^{1+\nu}(t) \leq w_0(\phi(t) - \phi(t+1)), \quad t \in [0, T].$$

*For $w_0$ is a positive constant and $\nu$ is a non-negative constant. Then we have, for each $t \in [0, T]$,*

$$\begin{cases} \phi(t) \leq \phi(0)e^{-w_1[t-1]^+}, & \nu = 0, \\ \phi(t) \leq (\phi(0)^{-\nu} + w_0^{-1}\nu[t-1]^+)^{-\frac{1}{\nu}}, & \nu > 0, \end{cases}$$

*where $[t-1]^+ = \max\{t-1, 0\}$ and $w_1 = \ln(\frac{w_0}{w_0-1})$.*

**Lemma 1.5** *[35] Let us have $\delta > 0$ and let $B(t) \in C^2(0, \infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{1.1}$$

*If*

$$B'(0) > r_2 B(0) + K_0, \tag{1.2}$$

*with $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$, then $B'(t) > K_0$ for $t > 0$, where $K_0$ is a constant.*

**Lemma 1.6** *[35] If $H(t)$ is a nonincreasing function on $[t_0, \infty)$ and satisfies the differential inequality*

$$[H'(t)]^2 \geq a + b[H(t)]^{2+\frac{1}{b}} \quad \textit{for } t \geq t_0, \tag{1.3}$$

*where $a > 0, b \in R$, then there exists a finite time $T^*$ such that*

$$\lim_{t \to T^{*-}} H(t) = 0.$$

*Upper bounds for $T^*$ are estimated as follows:*

1. *If $b < 0$ and $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$ then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}.$$

2. *If $b = 0$, then*

$$T^* \leq t_0 + \frac{H(t_0)}{H'(t_0)}.$$

3. *If $b > 0$, then*

$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \quad \text{or} \quad T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left[1 - (1 + cH(t_0))^{-\frac{1}{2\delta}}\right],$$

   *where $c = (\frac{a}{b})^{2+\frac{1}{\delta}}$.*

CHAPTER 2

# GLOBAL EXISTENCE, DECAY AND BLOW-UP OF SOLUTIONS FOR A QUASILINEAR HYPERBOLIC EQUATION WITH SOURCE TERMS

The results presented in this chapter have been published in [7].

In this chapter, we consider the following value problem related to the quasilinear hyperbolic equation involving the weighted laplacian operator with source terms:

$$\begin{cases} u_{tt} - |\nabla u| div\big(\alpha(x)\dfrac{\nabla u}{|\nabla u|}\big) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Where $\Omega$ is bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n (n \geq 1)$, $p, q \geq 1$ and $\alpha(x)$ is a weight function, which is strictly positive and continuously differentiable function in $\Omega$.
Many authors studied the following equation

$$u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u. \tag{2.2}$$

In the absence of the damping mechanism $|u_t|^{m-2}u_t$, the source term $|u|^{p-2}u$ causes finite time blow-up of solutions with the negative-initial-energy in [4, 27].

In [24, 29], the damping assures global existence for arbitrary initial data in the absence of the source terms.

In [32, 31], Levine was showed the interaction between the damping term $|u_t|^{m-2}u_t$ and the source terms $|u|^{p-2}u$, and he studied the global non-existence of solutions in the case of the initial energy is negative. Then, in [26] Georgiev and Todorova developed the result of Levine when nonlinear damping ($m > 2$).

Many researches have introduced a new method related between $m$ and $p$ in their work, for which there is the global existence and the other relations between $m$ and $p$ for which there is finite time blow-up. In particularly, they showed that if $m \geq p$ then, the solutions with negative-energy continue to exist globally and blow-up finite-time if $p > m$ and the initial energy is sufficiently negative, this result has been generalized in [33, 34]to unbounded domains and to an abstract setting. In these works, the authors showed that no-solution can be extended on $[0, \infty)$ with negative-initial-energy if $p > m$ and they proved with non-continuation theorems. This generalization allowed them to apply their non-continuation results to the quasilinear situations in the particular case of which the result of [26].

In the presence of the strong damping term $\Delta u_t$ the following semilinear wave equations

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{q-1} u_t = |u|^{p-1} u, \tag{2.3}$$

studied by Yu in [53].

In the case of ($q = 1$), Gazzola and Squassina studied the global existence and blow up of solutions in [25] . Then, when $\omega = 1$ and $\mu = 1$, Chen and Liu was studied the global existence, decay and exponential growth of solutions in [16].

Many researchers have investigated to prove the existence, blow up and asymptotic behavior of solutions of the (2.3) (for more details see [39, 47, 51, 45, 50]).

In [39], Messaoudi extended the blow up of solutions results of [26] with the negative-initial-energy. Then, he studies decay of solutions by using the techniques combination of the perturbed energy and potential well methods in [40].

In [50], Wu and Xue studied the uniform energy decay rates of solutions by utilizing the multiplier method.

The authors obtained the blow up of solutions under some restriction for a system of semilinear wave equation and they gave an estimate for the blow up time $T$ in [35], this result allowed them also to apply their theorem to quasilinear situations, of which problem (2.1) is a particular case. So, we prove the same result of [35] in the case of a quasilinear hyperbolic equation with strong damping and source terms.

## 2.1 The main of results

In this section, we show the local existence theorem and we prove the global existence of solution for problem, then we obtain the decay of energy. Finally, we get the blow up of solutions for (2.1).

### 2.1.1 Global existence of solutions

Now, we show the local existence theorem which can be established by [26] and [52].

**Theorem 2.1 (Local existence)** *We suppose that:* $2 < p+1 < \dfrac{2n}{n-2}$, $u_0 \in \mathbb{W}_0^{1,2}(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *then, the problem* (2.1) *has a unique local solution* $u(t)$ *satisfying*

$$u \in C\big([0,T); \mathbb{W}_0^{1,2}(\Omega)\big),$$

$$u_t \in C\big([0,T); L^2(\Omega)\big) \cap L^{q+1}\big([0,T) \times \Omega\big).$$

In order to state our main results, we first introduce the following Nihari's functionals

$$J(t) = \frac{1}{2}||u||^2_{\mathbb{W}_0^{1,2}} - \frac{1}{p+1}||u||^{p+1}_{p+1}, \tag{2.4}$$

$$I(t) = ||u||^2_{\mathbb{W}_0^{1,2}} - ||u||^{p+1}_{p+1}. \tag{2.5}$$

And the energy functional

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||u||^2_{\mathbb{W}_0^{1,2}} - \frac{1}{p+1}||u||^{p+1}_{p+1}, \tag{2.6}$$

then, we denote the Nihari space

$$W = \{u : u \in \mathbb{W}_0^{1,2}(\Omega, \alpha), \quad I(u) > 0\} \cup \{0\}. \tag{2.7}$$

Next, we get the energy function (2.6) which is a nonincreasing function along the solution of (2.1) by the next lemma:

**Lemma 2.1** *Let $u$ be a solution to problem (2.1), then, $E(t)$ is a non-increasing function for $t > 0$ and*

$$E^{'}(t) = -(||u_t||_{q+1}^{q+1} + ||\nabla u_t||^2) \leq 0. \tag{2.8}$$

**Proof.** By multiplying the first equation of (2.1) by $u_t$ and integrating over $\Omega$, by using integrating by parts, we obtain

$$E(t) - E(0) = -\int_0^t ||u_\tau||_{q+1}^{q+1} d\tau - \int_0^t ||\nabla u_\tau||_2^2 d\tau \quad \text{for} \quad t \geq 0. \tag{2.9}$$

■

**Lemma 2.2** *Assume that $u_0 \in W$, $u_1 \in L^2(\Omega)$ and $(p > 1)$ satisfying*

$$I(0) > 0 \quad \text{and} \quad \beta = C_* \Big(\frac{2(p+1)}{(p-1)} E(0)\Big)^{\frac{p-1}{2}} < 1. \tag{2.10}$$

*Then, $I(t) > 0$, for any $t \geq 0$.*

**Proof.** Since $I(0) > 0$ and by the continuity of $u(t)$, there exists $T_m \leq T$ such that $I(t) \geq 0$ for all $t \in [0, T_m]$.This implies that,for all $t \in [0, T_m]$,

$$J(t) = \frac{1}{p+1} I(t) + \frac{p-1}{2(p+1)} ||u||_{\mathbb{W}_0^{1,2}}^2, \tag{2.11}$$

by $I(t) > 0$, we obtain

$$J(t) \geq \frac{(p-1)}{2(p+1)} ||u||_{\mathbb{W}_0^{1,2}}^2, \tag{2.12}$$

then, from $E(t)$ and $E^{'}(t)$, we get

$$\begin{aligned} ||u||_{\mathbb{W}_0^{1,2}}^2 &\leq \frac{2(p+1)}{(p-1)} J(t) \\ &\leq \frac{2(p+1)}{(p-1)} E(t) \\ &\leq \frac{2(p+1)}{(p-1)} E(0). \end{aligned} \tag{2.13}$$

Now, we use the embedding $\mathbb{W}_0^{1,2}(\Omega,\alpha) \hookrightarrow L^{p+1}(\Omega)$, we obtain

$$\begin{aligned} ||u||_{p+1}^{p+1} &\leq C_*||u||_{\mathbb{W}_0^{1,2}}^{p+1} \\ &= C_*||u||_{\mathbb{W}_0^{1,2}}^{p-1}||u||_{\mathbb{W}_0^{1,2}}^{2} \\ &\leq C_*\big(\frac{2(p+1)}{(p-1)}E(0)\big)^{\frac{p-1}{2}}||u||_{\mathbb{W}_0^{1,2}}^{2} \\ &= \beta||u||_{\mathbb{W}_0^{1,2}}^{2} \\ &< ||u||_{\mathbb{W}_0^{1,2}}^{2} \quad \forall t \in [0,T_m]. \end{aligned} \tag{2.14}$$

As a result, by (2.4) and (2.10), we get: $I(t) > 0$, $\forall\ t \in [0, T_m]$.

Therefore, $T_m$ can be extended to $T$. The proof is completed. ■

**Lemma 2.3** *If the assumptions of the Lemma 2.2 holds, then, there exists $\eta_1 = 1-\beta$ such that*

$$||u||_{p+1}^{p+1} \leq (1-\eta_1)||u||_{\mathbb{W}_0^{1,2}}^{2}.$$

**Proof.** We have

$$||u||_{p+1}^{p+1} \leq \beta||u||_{\mathbb{W}_0^{1,2}}^{2},$$

when, we put $\eta_1 = 1-\beta$, we obtain the result. ■

**Remark 2.1** We have

$$||u||_{p+1}^{p+1} \leq (1-\eta_1)||u||_{\mathbb{W}_0^{1,2}}^{2},$$

then, we can deduce that

$$||u||_{\mathbb{W}_0^{1,2}}^{2} \leq \frac{1}{\eta_1}I(t). \tag{2.15}$$

**Theorem 2.2** *Let $u_0 \in W$ satisfying the assumptions of the Lemma 2.2 and we suppose that $2 < p+1 < \frac{2n}{n-2}$, $n > 2$ holds, then the solution of problem (2.1) is global in time.*

**Proof.** It suffices to show that $||u_t||_2^2 + ||u||_{\mathbb{W}_0^{1,2}}^{2}$ is bounded independently of $t$, we have

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}||u_t||_2^2 + \frac{1}{2}||u||_{\mathbb{W}_0^{1,2}}^{2} - \frac{1}{p+1}||u||_{p+1}^{p+1} \\ &= \frac{1}{2}||u_t||_2^2 + \frac{p-1}{2(p+1)}||u||_{\mathbb{W}_0^{1,2}}^{2} + \frac{1}{p+1}I(t) \\ &\geq \frac{1}{2}||u_t||_2^2 + \frac{(p-1)}{2(p+1)}||u||_{\mathbb{W}_0^{1,2}}^{2}, \end{aligned}$$

since $I(t) \geq 0$. Thus

$$||u_t||_2^2 + ||u||^2_{\mathbb{W}_0^{1,2}} \leq CE(0),$$

where $C = \max\{2; \frac{2(p+1)}{(p-1)}\}$. By using Theorem 2.1, we have the global existence result.

■

### 2.1.2 Decay of solutions

In this section, we prove the decay of energy by using Nakao's inequality:

**Theorem 2.3** *Let $u_0 \in W$, we suppose that $2 < p+1 < \frac{2n}{n-2}$, $n > 2$ and (2.10) holds. Then, we have the following decay estimates:*

$$E(t) \leq \begin{cases} E(0)e^{-w_0[t-1]^+}, & \text{if } q = 1, \\ \left(E(0)^{-\lambda} + C_7^{-1}\lambda[t-1]^+\right)^{-\frac{1}{\lambda}}, & \text{if } q > 1, \end{cases}$$

*where $w_0, \lambda$ and $C_7$ are positive constants which will be defined later.*

**Proof.** We integrate $E'(t)$ over $[t, t+1]$, $t > 0$, we obtain

$$E(t) - E(t+1) = D^{q+1}(t), \tag{2.16}$$

where

$$D^{q+1}(t) = \int_t^{t+1} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|_2^2\right) d\tau. \tag{2.17}$$

Using (2.17) and Hölder's inequality, we observe that

$$\int_t^{t+1} \int_\Omega |u_t|^2 \, dxdt \leq C_0 D^2(t), \tag{2.18}$$

where $C_0 > 0$.

Then, there exists $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$||u_t(t_i)||_2 \leq CD(t), \quad i = 1, 2. \tag{2.19}$$

By multiplying the first equation of (2.1) by $u$ and integrate over $\Omega \times [t_1, t_2]$, using integration by parts, we obtain

$$\int_{t_1}^{t_2} I(t)dt = -\left[\int_{t_1}^{t_2}\int_\Omega u.u_{tt}dxdt + \int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u dxdt \right. \\ \left. + \int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dxdt\right]. \tag{2.20}$$

Now, we use (2.1) and integrating by parts, then we apply the Cauchy-Schwarz inequality in the first term and Hölder inequality in the second term of the right hand side of (2.20), we get

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \leq & ||u_t(t_1)||_2||u(t_1)||_2 + ||u_t(t_2)||_2||u(t_2)||_2 \\ & + \int_{t_1}^{t_2} ||u_t(t)||_2^2 dt + \int_{t_1}^{t_2} ||\nabla u_t||_2||\nabla u||_2 dt \\ & - \int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dx dt. \end{aligned} \tag{2.21}$$

Our goal now is to estimate the last term in the right hand side of the inequality. Using Hölder inequality, we get

$$\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dx dt \leq \int_{t_1}^{t_2} ||u_t||_{q+1}^q ||u(t)||_{q+1} dt. \tag{2.22}$$

After that, using (2.13), we find

$$\begin{aligned}\int_{t_1}^{t_2} ||u_t(t)||_{q+1}^q ||u(t)||_{q+1} dt &\leq C_* \int_{t_1}^{t_2} ||u_t(t)||_{q+1}^q ||u||_{\mathbb{W}_0^{1,2}} dt \\ &\leq C_* \int_{t_1}^{t_2} ||u_t(t)||_{q+1}^q ||u||_{\mathbb{W}_0^{1,2}}^2 dt \\ &\leq C_* \Big(\frac{2(p+1)}{(p-1)}E(0)\Big)^{\frac{1}{2}} \int_{t_1}^{t_2} ||u_t(t)||_{q+1}^q E^{\frac{1}{2}}(s) dt \\ &\leq C_* \Big(\frac{2(p+1)}{(p-1)}E(0)\Big)^{\frac{1}{2}} \sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s) D^q(t). \end{aligned} \tag{2.23}$$

Next, we estimate the fourth term of the right hand side of (2.21), we get

$$\begin{aligned}\int_{t_1}^{t_2} ||\nabla u_t||_2||\nabla u||_2 dt &\leq C_* \int_{t_1}^{t_2} ||\nabla u_t||_2 ||u||_{\mathbb{W}_0^{1,2}}^2 dt \\ &\leq C_* \Big(\frac{2(p+1)}{(p-1)}E(0)\Big)^{\frac{1}{2}} \int_{t_1}^{t_2} ||\nabla u_t||_2 E^{\frac{1}{2}}(s) dt \\ &\leq C_* \Big(\frac{2(p+1)}{(p-1)}E(0)\Big)^{\frac{1}{2}} \sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s) \int_{t_1}^{t_2} ||\nabla u_t||_2 dt. \end{aligned}$$

Which implies

$$\begin{aligned}\int_{t_1}^{t2} \|\nabla u_t\|_2 \, dt &\leq \left(\int_{t_1}^{t_2} 1 dt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\nabla u_t\|_2^2 \, dt\right)^{\frac{1}{2}} \\ &\leq CD(t), \end{aligned}$$

then

$$\int_{t_1}^{t_2} ||\nabla u_t||_2||\nabla u||_2 dt \leq CC_* \left(\frac{2(p+1)}{(p-1)} E(0)\right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) D(t), \tag{2.24}$$

after that, from (2.13) and (2.19), we get

$$||u_t(t_i)||_2||u(t_i)||_2 \leq C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s), \tag{2.25}$$

with $C_1 = 2C_* \left(\frac{2(p+1)}{(p-1)} E(0)\right)^{\frac{1}{2}}$.
Then, by using (2.20)-(2.25), we obtain

$$\begin{aligned} \int_{t_1}^{t_2} I(t)dt \leq & C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) + D^2(t) \\ & + CC_* \left(\frac{2(p+1)}{(p-1)} E(0)\right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) D(t) \\ & + C_* \left(\frac{2(p+1)}{(p-1)} E(0)\right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) D^q(t). \end{aligned} \tag{2.26}$$

On the other hand, from Remark 2.1, we have

$$E(t) \leq \frac{1}{2} \|u_t\|_2^2 + C_3 I(t), \tag{2.27}$$

with $C_3 = \frac{1}{\eta_1} \frac{(p-1)}{2(p+1)} + \frac{1}{p+1}$.

By integration (2.27) over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} E(t)dt \leq \frac{1}{2} \int_{t_1}^{t_2} \|u_t\|_2^2 dt + C_3 \int_{t_1}^{t_2} I(t)dt. \tag{2.28}$$

Hence, from (2.19), (2.26) and (2.28), we get

$$\begin{aligned} \int_{t_1}^{t_2} E(t)dt \leq & \frac{1}{2} CD^2(t) + C_3 \Bigg[ C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) + D^2(t) \\ & + CC_* \left(\frac{2(p+1)}{(p-1)} E(0)\right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) D(t) \\ & + C_* \left(\frac{2(p+1)}{(p-1)} E(0)\right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) D^q(t) \Bigg]. \end{aligned} \tag{2.29}$$

Next, we integrate over $[t, t_2]$, we obtain

$$E(t) = E(t_2) + \int_t^{t_2} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|_2^2\right) d\tau, \tag{2.30}$$

since $t_2 - t_1 \geq \frac{1}{2}$, we conclude that

$$\int_{t_1}^{t_2} E(t)dt \geq (t_2 - t_1) E(t_2) \geq \frac{1}{2} E(t_2),$$

then

$$E(t_2) \leq 2 \int_{t_1}^{t_2} E(t)dt. \tag{2.31}$$

Consequently, since $t_1, t_2 \in [t, t+1]$, we obtain

$$\begin{aligned} E(t) &\leq 2 \int_{t_1}^{t_2} E(t)dt + \int_{t_1}^{t_2} \left(||u_\tau||_{q+1}^{q+1} + ||\nabla u_\tau||^2\right) d\tau \\ &= 2 \int_{t_1}^{t_2} E(t)dt + D^{q+1}(t). \end{aligned} \tag{2.32}$$

Hence, by (2.29), we have

$$E(t) \leq \left(\frac{1}{2}C + C_3\right) D^2(t) + D^{q+1}(t) + C_4 \left[D(t) + D^q(t)\right] E^{\frac{1}{2}}(t). \tag{2.33}$$

Then, we obtain

$$E(t) \leq C_5 \left[2D^2(t) + D^{q+1}(t) + D^{2q}(t)\right], \tag{2.34}$$

since $E(t)$ is non-increasing function and $E(t) \geq 0$ on $[0, \infty)$,

$$D^{q+1}(t) = E(t) - E(t+1) \leq E(0). \tag{2.35}$$

Then, we have

$$D(t) \leq E^{\frac{1}{q+1}}(0). \tag{2.36}$$

Consequently, by (2.34) and (2.36), we get

$$\begin{aligned} E(t) &\leq C_5 \left[2D^2(t) + D^{q+1}(t) + D^{2q}(t)\right] \\ &\leq C_5 D^2(t) \left[2 + D^{q-1}(t) + D^{2(q-1)}(t)\right] \\ &\leq C_5 D^2(t) \left[2 + E^{(q-1)\times\frac{1}{q+1}}(0) + E^{2(q-1)\left(\frac{1}{q+1}\right)}(0)\right] \\ &= C_6 D^2(t). \end{aligned}$$

Then, we obtain

$$E^{1+\frac{(q-1)}{2}}(t) \leq C_7 D^{q+1}(t). \tag{2.37}$$

**case 1 :** When $q = 1$ from (2.37), we get

$$E(t) \leq C_7 D^2(t) = C_7[E(t) - E(t+1)],$$

by Lemma 1.4, we obtain

$$E(t) \leq E(0)e^{-w_0[t-1]^+},$$

where $w_0 = \ln(\frac{C_7}{C_7-1})$.

**case 2** : When $q > 1$, we apply Lemma 1.4 to (2.37), we obtain

$$E(t) \leq \big(E(0)^{-\lambda} + C_7^{-1}\lambda[t-1]^+\big)^{-\frac{1}{\lambda}},$$

where $\lambda = \frac{q-1}{2}$. The proof of Theorem 2.3 is completed.

■

### 2.1.3 Blow up results

**Definition 2.1** A solution $u$ of the problem (2.1) is called blow-up with ($q = 1$), if there exists a finite time $T^*$ such that

$$\lim_{t \to T^{*-}} A(t) = \infty. \tag{2.38}$$

Where

$$A(t) = \int_\Omega u^2 dx + \int_0^t (||u||_2^2 + ||\nabla u||_2^2) d\tau \quad \text{for} \quad t \geq 0. \tag{2.39}$$

**Lemma 2.4** *We assume that:* $2 < p+1 < \dfrac{2n}{n-2}$, $n > 2$ *and* $0 \leq 4\delta \leq p-1$, *then, we have*

$$A^{''}(t) \geq 4(\delta+1)\int_\Omega u_t^2 dx - 4(2\delta+1)E(0) + 4(2\delta+1)\int_0^t (||\nabla u_t||_2^2 + ||u_t||_2^2) d\tau. \tag{2.40}$$

**Proof.** By taking the first and second derivative of $A(t)$, we obtain

$$A^{'}(t) = 2\int_\Omega u u_t dx + ||\nabla u||_2^2 + ||u||_2^2, \tag{2.41}$$

$$\begin{aligned} A^{''}(t) =& 2\int_\Omega u_t^2 dx + 2\int_\Omega u u_{tt} dx + 2\int_\Omega (\nabla u \nabla u_t) dx + 2\int_\Omega (|u| u_t) dx. \\ &= 2||u_t||_2^2 - 2||u||^2_{\mathbb{W}_0^{1,2}} + 2||u||_{p+1}^{p+1}. \end{aligned} \tag{2.42}$$

From (2.42) and (2.6), we get

$$\begin{aligned} A^{''}(t) =& 4(\delta+1)\int_\Omega u_t^2 dx - 4(2\delta+1)E(0) + 4(2\delta+1)\int_0^t (||\nabla u_t||_2^2 + ||u_t||_2^2) d\tau \\ &+ 4\delta||u||^2_{\mathbb{W}_0^{1,2}} + \big(2 - \frac{8\delta+4}{p+1}\big)||u||_{p+1}^{p+1}, \end{aligned}$$

since $0 < 4\delta \leq p-1$, we obtain (2.40). ■

**Lemma 2.5** *We assume that* $2 < p+1 < \frac{2n}{n-2}$ *with* $n > 2$ *and one of the following statements are satisfied*

1. *If* $E(0) < 0$*, then* $A^{'}(t) > ||u_0||_2^2$ *for* $t > t^*$*, where* $t_0 = t^*$ *is given by*
$$t^* = \max\{0, \frac{A^{'}(t) - ||u_0||_2^2}{(8\delta+4)E(0)}\}.$$

2. *If* $E(0) > 0$ *and*
$$A^{'}(0) > r_2\left[A(0) + \frac{G}{4\delta+4}\right] + ||u_0||_2^2, \tag{2.43}$$
*holds, where* $G$ *and* $t^*$ *will be defined later, then,* $A^{'}(t) > ||u_0||_2^2$ *where* $t_0 = 0$.

**Proof.**

1. When $E(0) < 0$, then, $A^{''}(t) \geq -4(1+2\delta)E(0)$ for $t \geq 0$, by integration over $[0,t]$, we get
$$A^{'}(t) \geq A'(0) - 4(2\delta+1)E(0)t, \quad t \geq 0.$$
Then, for $t > t^*$ we get $A^{'}(t) > ||u_0||_2^2$ with
$$t^* = \max\{0, \frac{A^{'}(t) - ||u_0||_2^2}{4(2\delta+1)E(0)}\}. \tag{2.44}$$

2. When $E(0) > 0$, we find
$$2\int_0^t \int_\Omega u u_t dx d\tau = ||u||_2^2 - ||u_0||_2^2. \tag{2.45}$$
Using the Hölder inequality and the Young inequality, we get
$$||u||_2^2 \leq ||u_0||_2^2 + \int_0^t (||u||_2^2 + ||u_t||_2^2) d\tau. \tag{2.46}$$
Then, using (2.46), Hölder's inequality and Young's inequality, we obtain
$$A^{'}(t) \leq A(t) + \int_\Omega u_t^2 dx + \int_0^t ||u_t||_2^2 d\tau + ||u_0||_2^2. \tag{2.47}$$
By (2.46) and (2.40), we have
$$A''(t) - 4(\delta+1)A'(t) + ||u_0||_2^2 A(t) + (8\delta+4)E(0) + (4\delta+4)||u_0||_2^2 \geq 0, \tag{2.48}$$
where
$$G = (8\delta+4)E(0) + (4\delta+4)||u_0||_2^2. \tag{2.49}$$

Let

$$b(t) = A(t) + \frac{G}{4(\delta+1)}, \text{for} \quad t > 0.$$

Where $b(t)$ satisfies Lemma 1.5.

Finally, from (2.43) we get $A^{'}(t) > ||u_0||_2^2$ for $t > 0$ and $r_2$ is given in Lemma 1.5.

■

**Theorem 2.4** *We assume that* $2 < p+1 < \frac{2n}{n-2}$ *and* $2 < n$, *then, we have two cases*

1. *Case 1: If* $E(0) < 0$, *then the solution* $u$ *blows up in finite time* $T^*$ *in the sense of* $\lim_{t\longrightarrow T^{*-}} A(t) = \infty$ *and*

$$T^* \leq t_0 - \frac{L(t_0)}{L'(t_0)}. \tag{2.50}$$

*Moreover, if* $L(t_0) < \min\left|1, (-\frac{a}{b})^{\frac{1}{2}}\right|$, *we have:*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\kappa}{\kappa - L(t_0)}, \tag{2.51}$$

*where* $\kappa = (-\frac{a}{b})^{\frac{1}{2}}$.

2. *Case 2: If* $0 < E(0) < \dfrac{\left(F^{'}(t_0)\right)^2}{8F(t_0)}$ *and* (2.43) *holds, then, the solution* $u$ *blows up in finite time* $T^*$ *in the sense of* $\lim_{t\longrightarrow T^{*-}} A(t) = \infty$, *and*

$$T^* \leq \frac{L(t_0)}{\sqrt{a}} \text{ or } T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\mu\delta}{\sqrt{a}}\{1 - [1 + \mu L(t_0)]^{-\frac{1}{2\delta}}\}. \tag{2.52}$$

*With:*

$$a = \delta^2 L^{2+\frac{2}{\delta}}(t_0)\left[\left(F'(t_0)\right)^2 - 8E(0)L^{-\frac{1}{\delta}}(t_0)\right] > 0, \tag{2.53}$$

$$b = 8\delta^2 E(0), \tag{2.54}$$

$$\mu = (\frac{a}{b})^{2+\frac{1}{\delta}}. \tag{2.55}$$

**Proof.** Let

$$L(t) = [F(t)]^{-\delta}, t \in [0, T_1], \tag{2.56}$$

where

$$F(t) = A(t) + (T_1 - t)||u_0||_2^2,$$

with $T_1$ is a strictly positive constant which will be defined later.

Then, by taking the second derivative of $L(t)$, we obtain

$$\begin{aligned} L'(t) &= -\delta \left[A(t) + (T_1 - t)\|u_0\|_2^2\right]^{-\delta-1} \left[A'(t) - \|u_0\|_2^2\right] \\ &= -\delta L^{1+\frac{1}{\delta}}(t) \left[A'(t) - \|u_0\|_2^2\right]. \qquad (2.57) \\ L''(t) &= -\delta L^{1+\frac{2}{\delta}}(t) A''(t) \left[A(t) + (T_1 - t)\|u_0\|_2^2\right] \\ &\quad + \delta L^{1+\frac{2}{\delta}}(t)(1+\delta)[A'(t) - ||u_0||_2^2]^2 \\ L''(t) &= -\delta L^{1+\frac{2}{\delta}}(t) W(t), \qquad (2.58) \end{aligned}$$

where

$$W(t) = A''(t)F(t) - (1+\delta)(F'(t))^2. \qquad (2.59)$$

Now, we define

$$M_u = \int_\Omega u^2 dx, \quad N_u = \int_\Omega u_t^2 dx,$$
$$P_u = \int_0^t \|u\|^2 dt, \quad Q_u = \int_0^t \|u_t\|^2 dt.$$

From (2.41), (2.44) and by using Hölder's inequality, we obtain

$$\begin{aligned} A'(t) &= 2\int_\Omega u u_t dx + ||u_0|\,|_2^2 + 2\int_0^t \int_\Omega u u_t dx dt \\ &\le 2\left((N_u M_u)^{\frac{1}{2}} + (P_u Q_u)^{\frac{1}{2}}\right) + ||u_0||_2^2. \qquad (2.60) \end{aligned}$$

When case 1 holds and by using Lemma 2.4, we get

$$A''(t) \ge -4(1+2\delta)E(0) + 4(1+\delta)\left(N_u + Q_u\right). \qquad (2.61)$$

Then, from (2.56), (2.59) and (2.61), we obtain

$$\begin{aligned} W(t) \ge& \left[-4(1+2\delta)E(0) + 4(1+\delta)\left(N_u + Q_u\right)\right] L^{-\frac{1}{\delta}}(t) \\ &-4(1+\delta)\left((N_u M_u)^{\frac{1}{2}} + (P_u Q_u)^{\frac{1}{2}}\right)^2. \end{aligned}$$

We have

$$A(t) = \int_\Omega u^2 dx + \int_0^t \int_\Omega u^2 dx ds = M_u,$$

and by using $L(t)$, we get

$$W(t) \geq -4(1+2\delta)E(0)L^{-\frac{1}{\delta}}(t) + 4(1+\delta)[(N_u + Q_u)(T_1 - t)||u_0||_2^2 + \Lambda(t)],$$

where

$$\Lambda(t) = (N_u + Q_u)(M_u + P_u) - \left((N_u M_u)^{\frac{1}{2}} + (P_u Q_u)^{\frac{1}{2}}\right)^2.$$

Where $\Lambda(t)$ being non-negative function and by the Schwartz inequality, we obtain

$$W(t) \geq -4(1+2\delta)E(0)L^{-\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0.$$

Thus, by $L^{''}(t)$, we get

$$L^{''}(t) \leq 4\delta(1+2\delta)E(0)L^{1+\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0. \tag{2.62}$$

We have

$$L'(t) < 0, \quad t \geq t_0.$$

By multiplying (2.62) by $L'(t)$ and integrate over $[t_0; t]$, we get

$$L^{'2}(t) \geq a + bL^{2+\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0,$$

with $a, b$ are defined.

Using the steps of case 1, if case 2 holds, then if and only if

$$E(0) < \frac{(F'(t_0))^2}{8F(t_0)},$$

we obtain $a > 0$.

After that, we use Lemma 1.6, then, there exists $T^*$ such that $\lim_{t \longrightarrow T^*-} L(t) = 0$ and according to the sign of $E(0)$, the upper bound of $T^*$ is estimated. This means that (2.38) holds. ■

## 2.2 Conclusion

In this chapter, we obtained the local and global existence, decay of solutions and blow up time for a quasilinear hyperbolic equation with source terms in a bounded domain. This improves and extends many results in the literature.

# CHAPTER 3

# ON THE DECAY AND BLOW UP OF SOLUTIONS FOR A QUASILINEAR HYPERBOLIC EQUATIONS INVOLVING THE WEIGHTED $P-$LAPLACIAN OPERATOR

The main prupose of this chapter is to prove the global existence, the exponential stability results and blow up of solutions with negative-initial-energy for a quasilinear hyperbolic equation involving the weighted $p-$Laplacian. The results appearing in this context have been published throughout [8].

In this chapter, we study the following value problem related to the quasilinear hyperbolic equation involving the weighted $p-$Laplacian with source terms:

$$\begin{cases} u_{tt} - div\big(\alpha(x)|\nabla u|^{p-2}\nabla u\big) - \Delta u_t + |u_t|^{q-1}u_t = \xi |u|^{m-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Where $\Omega$ is bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n (n \geq 1)$, $\xi > 0$, $m, q \geq 1$, $p \geq 2$ and $p'$ the conjugate exponent of $p$ satisfies

$$p' = \frac{p}{p-1},$$

$$p^*_\gamma = \frac{pn}{n-p+\gamma}, \quad \text{for} \gamma \in \mathbb{R}^n,$$

and $\alpha(x)$ is a weight function, which is a strictly positive and continuously differentiable function in $\Omega$. Problems of weighted $p-$Laplacian type arise in many applications in physics, mechanics, biology (for example, we represents the gas flows in porous media and spread of biological populations...etc).

Many authors have studied the existence and uniqueness of local, global and blow up of solution for quasilinear hyperbolic equations involving the $p-$Laplacian operator. For example, in [23] the authors have studied the existence of solutions of the following equations

$$u_{tt} - \Delta u_t - div(|\nabla u|^{p-2}\nabla u) = f. \tag{3.2}$$

Then, in [20] the researchers studied the lower bound for the blow up time of solutions in the following $p-$Laplacian equation with strong and weak damping terms

$$u_{tt} - div(|\nabla u|^{p-2}\nabla u) - a\Delta u_t + bu_t = |u|^{q-2}u. \tag{3.3}$$

In [39], the authors proved the global existence, the growth and the decay estimates of solutions for the equation

$$u_{tt} - div(|\nabla u|^{p-2}\nabla u) - u_t + |u|^{p-2}u = |u|^{p-2}u \ln |u|. \tag{3.4}$$

In this chapter, we start by showing the local existence theorem, then we get the global existence and the decay of solutions. Finally, we obtain the finite time blow up of solutions with non-positive-initial energy by using the same techniques in [35] and [45].

## 3.1 The main of results

### 3.1.1 Global Existence results

In this section, we study the global existence (3.1) with $\xi = 1$. We start by showing the local existence theorem

**Theorem 3.1 (Local existence)** *We suppose that:* $2 \leq p < m+1 < \dfrac{np}{n-p+\gamma} < n$ *and* $(u_0, u_1) \in (\mathbb{W}_0^{1,p}(\Omega), L^2(\Omega))$ *then, the problem* (3.1) *has a unique local solution such that*

$$u \in C\big([0,T); \mathbb{W}_0^{1,p}(\Omega)\big),$$

*and*

$$u_t \in C\big([0,T); L^2(\Omega)\big) \cap L^{q+1}\big([0,T) \times \Omega\big).$$

Now, we define the potential energy and Nihari's functionals as follow

$$J(t) = \frac{1}{p}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} - \frac{1}{m+1}||u||^{m+1}_{m+1}, \tag{3.5}$$

$$I(t) = ||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} - ||u||^{m+1}_{m+1}, \tag{3.6}$$

and

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{p}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} - \frac{1}{m+1}||u||^{m+1}_{m+1}, \tag{3.7}$$

we denote the Nihari space

$$W = \{u : u \in \mathbb{W}_0^{1,p}(\Omega), I(u) > 0\} \cup \{0\}. \tag{3.8}$$

The next lemma shows that our energy functional (3.7) is a non-increasing function along the solution of (3.1)

**Lemma 3.1** *For $t \geq 0$, $E(t)$ is a non-increasing function and*

$$E^{'}(t) = -(||u_t||^{q+1}_{q+1} + ||\nabla u_t||^2) \leq 0. \tag{3.9}$$

**Proof.** We multiply the first equation of (3.1) by $u_t$ and we integrate over $\Omega$ and by using integration by parts, we obtain

$$E(t) - E(0) = -\int_0^t ||u_\tau||^{q+1}_{q+1} d\tau - \int_0^t ||\nabla u_\tau||^2_2 d\tau \quad \text{for} \quad t \geq 0. \tag{3.10}$$

■

**Lemma 3.2** *Let $u_0 \in W$ and $u_1 \in L^2(\Omega)$, we suppose that $(m+1 > p)$ and*

$$\beta = C_*\Big(\frac{p(m+1)}{(m+1-p)}E(0)\Big)^{\frac{m+1-p}{p}} < 1, \tag{3.11}$$

*then, $u \in W$ for each $t \geq 0$.*

**Proof.** Since $I(0) > 0$ and it follows by the continuity of $u(t)$ that: $I(t) > 0$, for some interval near $t = 0$. Let $T_m > 0$ the maximal time, when (3.6) holds on $[0, T_m]$. From (3.5) and (3.6), we get

$$J(t) = \frac{1}{m+1}I(t) + \frac{m+1-p}{p(m+1)}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}, \tag{3.12}$$

then, we have

$$J(t) \geq \frac{(m+1-p)}{p(m+1)}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}, \tag{3.13}$$

from $E(t)$ and $E^{'}(t)$, we obtain

$$\begin{aligned} ||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} &\leq \frac{p(m+1)}{(m+1-p)}J(t) \\ &\leq \frac{p(m+1)}{(m+1-p)}E(t) \\ &\leq \frac{p(m+1)}{(m+1-p)}E(0). \end{aligned} \tag{3.14}$$

Using Lemma 3.2 and (3.14), we get

$$\begin{aligned} ||u||^{m+1}_{m+1} &\leq C_*^{m+1}||u||^{m+1}_{\mathbb{W}_0^{1,p}(\Omega)} \\ &= C_*^p||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}||u||^{m+1-p}_{\mathbb{W}_0^{1,p}(\Omega)} \\ &\leq C_*^p\Big(\frac{p(m+1)}{(m+1-p)}E(0)\Big)^{\frac{m+1-p}{p}}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} \\ &= \beta||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} \\ &< ||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} \quad \forall t \in [0, T_m], \end{aligned} \tag{3.15}$$

by using (3.7), we conclude that $I(t) > 0$, $\forall\, t \in [0, T_m]$.

When we repeat the procedure, $T_m$ is extended to $T$. ■

**Lemma 3.3** *When the assumptions of the Lemma 3.2 hold, then, there exists* $\eta_1 = 1-\beta$ *such that*

$$||u||^{m+1}_{m+1} \leq (1-\eta_1)||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}.$$

**Proof.** We have

$$||u||^{m+1}_{m+1} \leq \beta||u||^p_{\mathbb{W}_0^{1,p}(\Omega)},$$

we put $\eta_1 = 1-\beta$, we obtain the following results. ■

**Remark 3.1** We have

$$||u||^{m+1}_{m+1} \leq (1-\eta_1)||u||^p_{\mathbb{W}_0^{1,p}(\Omega)},$$

then, we deduce that

$$||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} \leq \frac{1}{\eta_1}I(t). \tag{3.16}$$

**Theorem 3.2** *Let $u_0 \in W$ satisfying Lemma 3.2 and we suppose that $2 \leq p < m+1 < \dfrac{np}{n-p+\gamma} = p^*_\gamma$, $n > p$ holds, then the solution of problem (3.1) is global.*

**Proof.** It is sufficient to show that $||u_t||_2^2 + ||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}$ is bounded independently of $t$, to achieve this we use (3.6) and (3.7) to get

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}||u_t||_2^2 + \frac{1}{p}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} - \frac{1}{m+1}||u||_{m+1}^{m+1} \\ &= \frac{1}{2}||u_t||_2^2 + \frac{m+1-p}{p(m+1)}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} + \frac{1}{m+1}I(t) \\ &\geq \frac{1}{2}||u_t||_2^2 + \frac{(m+1-p)}{p(m+1)}||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}, \end{aligned}$$

since $I(t) \geq 0$, then

$$||u_t||_2^2 + ||u||^p_{\mathbb{W}_0^{1,p}(\Omega)} \leq CE(0),$$

where $C = \max\{2; \frac{p(m+1)}{(m+1-p)}\}$. Therefore by Theorem 3.1, we get the global existence result. ■

### 3.1.2 Exponential and Polynomial Decay of solutions

**Theorem 3.3** *Let $(u_0, u_1) \in (W, L^2(\Omega))$, we suppose that $2 \leq p < m+1 < \frac{np}{n-p+\gamma}$, $n > p$ and (3.11) hold.*

*Thus, we have the following decay estimates:*

$$E(t) \leq \begin{cases} E(0)e^{-w_0[t-1]^+}, & \text{if } q = 1, \\ \left(E(0)^{-\lambda} + C_7^{-1}\lambda[t-1]^+\right)^{-\frac{1}{\lambda}}, & \text{if } q > 1, \end{cases}$$

*where $w_0, \lambda$, and $C_7$ are positive constants which will be defined later.*

**Proof.** We integrate $E'(t)$ over $[t, t+1]$ with $t > 0$, we obtain

$$E(t) - E(t+1) = D^{q+1}(t), \tag{3.17}$$

where

$$D^{q+1}(t) = \int_t^{t+1} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|_2^2\right) d\tau. \tag{3.18}$$

Now, by using $D^{q+1}(t)$ and Hölder's inequality, we observe that

$$\int_t^{t+1} \int_\Omega |u_t|^2 \, dx dt \leq C_\Omega D^2(t), \tag{3.19}$$

where $C_\Omega > 0$. Then, there exists $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$||u_t(t_i)||_2 \leq CD(t), \quad i = 1, 2. \tag{3.20}$$

By multiplying the first equation of (3.1) by $u$ and integrate over $\Omega \times [t_1, t_2]$ and by using integration by parts, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt = &-\left[\int_{t_1}^{t_2}\int_\Omega u.u_{tt}dxdt + \int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u dxdt\right.\\ &\left.+\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dxdt\right].\end{aligned} \tag{3.21}$$

Using (3.1) and integrate by parts then, we apply the Cauchy-Schwartz inequality in the first term and we use Hölder inequality in the second term, we get

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \leq\ & ||u_t(t_1)||_2||u(t_1)||_2 + ||u_t(t_2)||_2||u(t_2)||_2\\ &+\int_{t_1}^{t_2}||u_t(t)||_2^2dt + \int_{t_1}^{t_2}||\nabla u_t||_2||\nabla u||_2 dt\\ &-\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dxdt.\end{aligned} \tag{3.22}$$

Now, Our goal is to estimate the last term in the inequality, by using Hölder inequality, we obtain

$$\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dxdt \leq \int_{t_1}^{t_2}||u_t||_{q+1}^q||u(t)||_{q+1}dt. \tag{3.23}$$

After that, we use (3.14), we have

$$\begin{aligned}\int_{t_1}^{t_2}||u_t(t)||_{q+1}^q||u(t)||_{q+1}dt &\leq C_*\int_{t_1}^{t_2}||u_t(t)||_{q+1}^q||u||_{\mathbb{W}_0^{1,p}(\Omega)}dt\\ &\leq C_*\int_{t_1}^{t_2}||u_t(t)||_{q+1}^q||u||^p_{\mathbb{W}_0^{1,p}(\Omega)}dt\\ &\leq C_*^p\Big(\frac{p(m+1)}{(m+1-p)}E(0)\Big)^{\frac{1}{p}}\int_{t_1}^{t_2}||u_t(t)||_{q+1}^qE^{\frac{1}{p}}(s)dt\\ &\leq C_*^p\Big(\frac{p(m+1)}{(m+1-p)}E(0)\Big)^{\frac{1}{p}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}}(s)D^q(t).\end{aligned} \tag{3.24}$$

Next, we estimate the fourth term of the right hand side of (3.22), we find

$$\begin{aligned}\int_{t_1}^{t_2}||\nabla u_t||_2||\nabla u||_2dt &\leq C_*^p\int_{t_1}^{t_2}||\nabla u_t||_2||u||^p_{\mathbb{W}_0^{1,p}}dt\\ &\leq C_*^p\Big(\frac{p(m+1)}{(m+1-p)}E(0)\Big)^{\frac{1}{p}}\int_{t_1}^{t_2}||\nabla u_t||_2E^{\frac{1}{p}}(s)dt\\ &\leq C_*^p\Big(\frac{p(m+1)}{(m+1-p)}E(0)\Big)^{\frac{1}{p}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}(s)}\int_{t_1}^{t_2}||\nabla u_t||_2dt,\end{aligned}$$

then, we get

$$\begin{aligned}\int_{t_1}^{t2}\|\nabla u_t\|_2\,dt &\leq\left(\int_{t_1}^{t_2}1dt\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\|\nabla u_t\|_2^2\,dt\right)^{\frac{1}{2}}\\ &\leq kD(t).\end{aligned}$$

Consequently, we obtain

$$\int_{t_1}^{t_2}||\nabla u_t||_2||\nabla u||_2dt\leq kC_*^p\left(\frac{p(m+1)}{(m+1-p)}E(0)\right)^{\frac{1}{p}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}}(s)D(t), \tag{3.25}$$

on the other hand, we have

$$||u_t\left(t_i\right)||_2||u\left(t_i\right)||_2\leq C_1D(t)\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}}(s), \tag{3.26}$$

where $C_1=kC_*^p\left(\frac{p(m+1)}{(m+1-p)}E(0)\right)^{\frac{1}{p}}$.

Then, we find

$$\begin{aligned}\int_{t_1}^{t_2}I(t)dt\leq &C_1D(t)\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}}(s)+D^2(t)\\ &+kC_*^p\left(\frac{p(m+1)}{(m+1-p)}E(0)\right)^{\frac{1}{p}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}}(s)D(t)\\ &+C_*^p\left(\frac{p(m+1)}{(m+1-p)}E(0)\right)^{\frac{1}{p}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{p}}(s)D^q(t).\end{aligned} \tag{3.27}$$

On the other hand, we have

$$E(t)\leq\frac{1}{2}\left\|u_t\right\|_2^2+C_3I(t), \tag{3.28}$$

where $C_3=\frac{1}{\eta_1}\frac{(m+1-p)}{p(m+1)}+\frac{1}{m+1}$.

Integrating over $[t_1,t_2]$, we get

$$\int_{t_1}^{t_2}E(t)dt\leq\frac{1}{2}\int_{t_1}^{t_2}\left\|u_t\right\|_2^2dt+C_3\int_{t_1}^{t_2}I(t)dt. \tag{3.29}$$

Then, we get

$$\begin{aligned}\int_{t_1}^{t_2} E(t)dt \leq & \frac{1}{2}C_\Omega D^2(t) + C_3 \left[C_1 D(t) \sup_{t_1\leq s\leq t_2} E^{\frac{1}{p}}(s) + D^2(t)\right. \\ & + kC_*^p \left(\frac{p(m+1)}{(m+1-p)}E(0)\right)^{\frac{1}{p}} \sup_{t_1\leq s\leq t_2} E^{\frac{1}{p}}(s)D(t) \\ & \left. + C_*^p \left(\frac{p(m+1)}{(m+1-p)}E(0)\right)^{\frac{1}{p}} \sup_{t_1\leq s\leq t_2} E^{\frac{1}{p}}(s)D^q(t)\right]. \end{aligned} \tag{3.30}$$

By integrating over $[t, t_2]$, we obtain

$$E(t) = E(t_2) + \int_t^{t_2} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|_2^2\right) d\tau, \tag{3.31}$$

since $t_2 - t_1 \geq \frac{1}{2}$, we conclude that

$$\int_{t_1}^{t_2} E(t)dt \geq (t_2 - t_1) E(t_2) \geq \frac{1}{2}E(t_2),$$

then, we find

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)dt. \tag{3.32}$$

Consequently, we get

$$\begin{aligned} E(t) &\leq 2\int_{t_1}^{t_2} E(t)dt + \int_{t_1}^{t_2} \left(||u_\tau||_{q+1}^{q+1} + ||\nabla u_\tau||^2\right) d\tau \\ &= 2\int_{t_1}^{t_2} E(t)dt + D^{q+1}(t). \end{aligned} \tag{3.33}$$

Hence, we have

$$E(t) \leq \left(\frac{1}{2}C_\Omega + C_3\right) D^2(t) + D^{q+1}(t) + C_4 \left[D(t) + D^q(t)\right] E^{\frac{1}{p}}(t). \tag{3.34}$$

Then, we obtain

$$E(t) \leq C_5 \left[D^2(t) + D^{q+1}(t) + D^{\frac{p}{p-1}}(t) + D^{\frac{p}{p-1}q}(t)\right], \tag{3.35}$$

since $E(t)$ is non-increasing function and $E(t) \geq 0$ on $[0, \infty)$, then

$$D^{q+1}(t) = E(t) - E(t+1) \leq E(0). \tag{3.36}$$

After that, we have

$$D(t) \leq E^{\frac{1}{q+1}}(0). \tag{3.37}$$

Consequently, we get

$$\begin{aligned} E(t) &\leq C_5\left[D^2(t)+D^{q+1}(t)+D^{\frac{p}{p-1}}(t)+D^{\frac{p}{p-1}q}(t)\right] \\ &\leq C_5 D^{\frac{p}{p-1}}(t)\left[D^{\frac{p-2}{p-1}}(t)+D^{q-\frac{1}{p-1}}(t)+1+D^{\frac{p(q-1)}{p-1}}(t)\right] \\ &= C_6 D^{\frac{p}{p-1}}(t). \end{aligned}$$

Then, we obtain

$$E^{1+\frac{(p-1)q-1}{p}}(t) \leq C_7 D^{q+1}(t). \tag{3.38}$$

**case 1** : When $q=1$ and $p=2$, we obtain

$$E(t) \leq C_7 D^2(t) = C_7[E(t)-E(t+1)],$$

then, by lemma 1.4 we get

$$E(t) \leq E(0)e^{-w_0[t-1]^+},$$

where $w_0=\ln(\frac{C_7}{C_7-1})$.

**case 2** : When $(p-1)q>1$, we apply Lemma 1.4 to (3.38), we get

$$E(t) \leq \left(E(0)^{-\lambda}+C_7^{-1}\lambda[t-1]^+\right)^{-\frac{1}{\lambda}},$$

where $\lambda=\frac{(p-1)q-1}{p}$.

The proof of Theorem 3.3 is completed.

■

### 3.1.3 Blow up results

In this section, we obtain the blow up results with non-positive initial energy of problem (3.1) with $(q=1)$.

**Definition 3.1** A solution $u$ of the problem (3.1) is called blow up if there exists $T^*$ a finite time such that

$$\lim_{t\to T^{*-}} A(t)=\infty. \tag{3.39}$$

Where

$$A(t)=\int_\Omega u^2dx+\int_0^t\int_\Omega(|u|^2+|\nabla u|^2)dxd\tau \quad \text{for} \quad t\geq 0. \tag{3.40}$$

**Lemma 3.4** *We assume that:* $p < m+1 < \frac{np}{n-p}$, $n > p$ *and* $p-2 \leq 4\delta \leq m-1$ *then, we have*

$$A^{''}(t) \geq (4\delta+4)\int_\Omega u_t^2 dx - (8\delta+4)E(0) + (8\delta+4)\int_0^t (||\nabla u_t||_2^2 + ||u_t||_2^2)d\tau. \quad (3.41)$$

**Proof.** We have

$$A^{'}(t) = 2\int_\Omega uu_t dx + ||\nabla u||_2^2 + ||u||_2^2, \quad (3.42)$$

$$\begin{aligned} A^{''}(t) =& 2\int_\Omega u_t^2 dx + 2\int_\Omega uu_{tt}dx + 2\int_\Omega (\nabla u \nabla u_t)dx + 2\int_\Omega (|u|u_t)dx. \\ &= 2||u_t||_2^2 - 2||u||^2_{\mathbb{W}_0^{1,p}} + 2||u||_{m+1}^{m+1}. \end{aligned} \quad (3.43)$$

From (3.43) and (3.7), we obtain

$$\begin{aligned} A^{''}(t) =& (4\delta+4)\int_\Omega u_t^2 dx - (8\delta+4)E(0) + (8\delta+4)\int_0^t (||\nabla u_t||_2^2 + ||u_t||_2^2)d\tau \\ &+ (\frac{8\delta+4-2p}{p})||u||^2_{\mathbb{W}_0^{1,p}} + (\frac{2(m+1)-8\delta-4}{m+1})||u||_{m+1}^{m+1}. \end{aligned}$$

Where $p-2 < 4\delta \leq m-1$, we obtain (3.41). ■

**Lemma 3.5** *We assume that* $p < m+1 < \frac{pn}{n-p}$ *and* $n > p$*, then we have*

1. *If* $E(0) < 0$*, then* $A^{'}(t) > ||u_0||_2^2$ *for* $t > t^*$*, where* $t_0 = t^*$ *is given by*
$$t^* = \max\{0, \frac{A^{'}(t) - ||u_0||_2^2}{(8\delta+4)E(0)}\}.$$
2. *If* $\int_\Omega u_0 u_1 dx > 0$ *and* $E(0) = 0$*, then,* $A^{'}(t) > ||u_0||_2^2$ *where* $t_0 = 0$.

**Proof.**

1. When $E(0) < 0$ and for $t \geq 0$, then $A^{''}(t) \geq -(4+8\delta)E(0)$, by integration over $[0,t]$, we get
$$A^{'}(t) \geq A'(0) - (8\delta+4)E(0)t, \quad t \geq 0.$$
Then, we obtain $A^{'}(t) > ||u_0||_2^2$ for $t > t^*$, where:
$$t^* = \max\{0, \frac{A^{'}(t) - ||u_0||_2^2}{(8\delta+4)E(0)}\}. \quad (3.44)$$
2. When $E(0) = 0$, and $\int_\Omega u_0 u_1 dx > 0$. Then, $A^{''} \geq 0$ for $t \geq 0$. We have $A^{'} \geq ||u_0||_2^2$, $t \geq 0$.

■

**Theorem 3.4** *We assume that* $p < m+1 < \frac{pn}{n-p}$ *and* $p < n$ *then, we have two cases*

1. *Case 1: If* $E(0) < 0$*, then the solution* $u$ *blows up in finite time* $T^*$ *in the sense of* $\lim_{t\longrightarrow T^*-} A(t) = \infty$ *and*

$$T^* \leq t_0 - \frac{L(t_0)}{L'(t_0)}. \tag{3.45}$$

*Moreover, if* $L(t_0) < \min\left|1, (-\frac{a}{b})^{\frac{1}{2}}\right|$*, we have:*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\kappa}{\kappa - L(t_0)}, \tag{3.46}$$

*where* $\kappa = (-\frac{a}{b})^{\frac{1}{2}}$.

2. *Case 2: If* $E(0) = 0$ *and* $\int_\Omega u_0 u_1 dx > 0$*. then the solution* $u$ *blows up in finite time* $T^*$ *in the sense of* $\lim_{t\longrightarrow T^*-} A(t) = \infty$*, and*

$$T^* \leq t_0 - \frac{L(t_0)}{L'(t_0)}. \tag{3.47}$$

*Where:*

$$a = \delta^2 L^{2+\frac{2}{\delta}}(t_0)\left[(F'(t_0))^2 - 8E(0)L^{-\frac{1}{\delta}}(t_0)\right] > 0, \tag{3.48}$$

$$b = 8\delta^2 E(0). \tag{3.49}$$

**Proof.** Let

$$L(t) = \left[A(t) + (T_1 - t)\,||u_0||_2^2\right]^{-\delta}, t \in [0, T_1], \tag{3.50}$$

where

$$F(t) = A(t) + (T_1 - t)||u_0||_2^2,$$

with $T_1$ is a strictly positive constant which will be defined later.

By taking the second derivative of $L(t)$, we obtain

$$\begin{aligned} L'(t) &= -\delta\left[A(t)+(T_1-t)\|u_0\|_2^2\right]^{-\delta-1}\left[A'(t)-\|u_0\|_2^2\right] \\ &= -\delta L^{1+\frac{1}{\delta}}(t)\left[A'(t)-\|u_0\|_2^2\right]. \end{aligned} \tag{3.51}$$

$$\begin{aligned} L''(t) &= -\delta L^{1+\frac{2}{\delta}}(t)A''(t)\left[A(t)+(T_1-t)\|u_0\|_2^2\right] \\ &\quad +\delta L^{1+\frac{2}{\delta}}(t)(1+\delta)[A'(t)-||u_0||_2^2]^2 \\ L''(t) &= -\delta L^{1+\frac{2}{\delta}}(t)W(t), \end{aligned} \tag{3.52}$$

where

$$W(t)=A''(t)F(t)-(1+\delta)(F'(t))^2. \tag{3.53}$$

We define

$$M_u=\int_\Omega u^2dx,\quad N_u=\int_\Omega u_t^2dx,$$
$$P_u=\int_0^t\|u\|^2dt,\quad Q_u=\int_0^t\|u_t\|^2dt.$$

From (3.42) and by Hölder's inequality, we get

$$\begin{aligned} A'(t) &= 2\int_\Omega uu_tdx+||u_0|\,|_2^2+2\int_0^t\int_\Omega uu_tdxdt \\ &\leq 2\left((N_uM_u)^{\frac{1}{2}}+(P_uQ_u)^{\frac{1}{2}}\right)+||u_0||_2^2. \end{aligned} \tag{3.54}$$

When case 1 holds and using Lemma 3.4, we get

$$A''(t)\geq -(4+8\delta)E(0)+(4+4\delta)\left(N_u+Q_u\right). \tag{3.55}$$

Then, from (3.50), (3.53) and (3.55), we obtain

$$\begin{aligned} W(t) &\geq \left[-(4+8\delta)E(0)+(4+4\delta)\left(N_u+Q_u\right)\right]L^{-\frac{1}{\delta}}(t) \\ &\quad -(4+4\delta)\left((N_uM_u)^{\frac{1}{2}}+(P_uQ_u)^{\frac{1}{2}}\right)^2. \end{aligned}$$

From $A(t)$, we have

$$A(t)=\int_\Omega u^2dx+\int_0^t\int_\Omega u^2dxds=M_u,$$

and by $L(t)$, we get

$$W(t)\geq -(4+8\delta)E(0)L^{-\frac{1}{\delta}}(t)+(4+4\delta)[(N_u+Q_u)(T_1-t)||u_0||_2^2+\Lambda(t)],$$

where

$$\Lambda(t) = (N_u + Q_u)(M_u + P_u) - \left((N_u M_u)^{\frac{1}{2}} + (P_u Q_u)^{\frac{1}{2}}\right)^2.$$

Where $\Lambda(t)$ being non-negative function and by the Schwartz inequality, we obtain

$$W(t) \geq -(4+8\delta)E(0)L^{-\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0.$$

Thus, by $L^{''}(t)$, we get

$$L^{''}(t) \leq (4\delta + 8\delta^2)E(0)L^{1+\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0. \tag{3.56}$$

We have

$$L'(t) < 0, \quad t \geq t_0.$$

We multiply (3.56) by $L'(t)$ and integrating over $[t_0; t]$, we obtain

$$L^{'2}(t) \geq a + bL^{2+\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0,$$

with $a, b$ are defined.

Finally, by using Lemma 1.6: there exists $T^*$ such that $\lim_{t \longrightarrow T^*-} L(t) = 0$ and according to the sign of $E(0)$, the upper bound of $T^*$ is estimated. So (3.39) holds. ■

## 3.2 Numerical example

In this section, we give some numerical to illustrate the theoretical results to the our problem. We solve the problem under specific initial data and Dirichlet boundary conditions. We use a numerical schema based on the finite element method and we present one numerical example which represents by problem (3.1). The error of problem (3.1) between the exact solution and the approximate solution in the example is draw by using FreeFEM++.

We consider problem (3.1)in two space-dimension and take $\varphi(x) = 1$ and we put $\xi|u|^{m-1}u = f(u)$.

**Test** : We consider the domain $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ with a triangulation discretization which consists 1682 triangle and 888 vertices, and we use the initial conditions

$$u_0(x, y) = \cosh(x) * \cosh(y), \quad u_1 = -2 * \cosh(x) * \cosh(y).$$

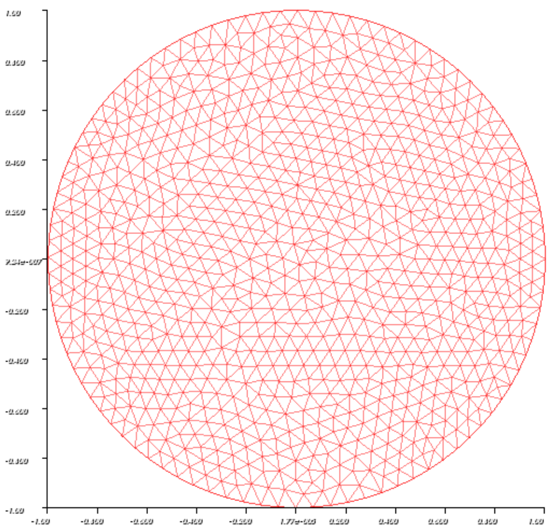
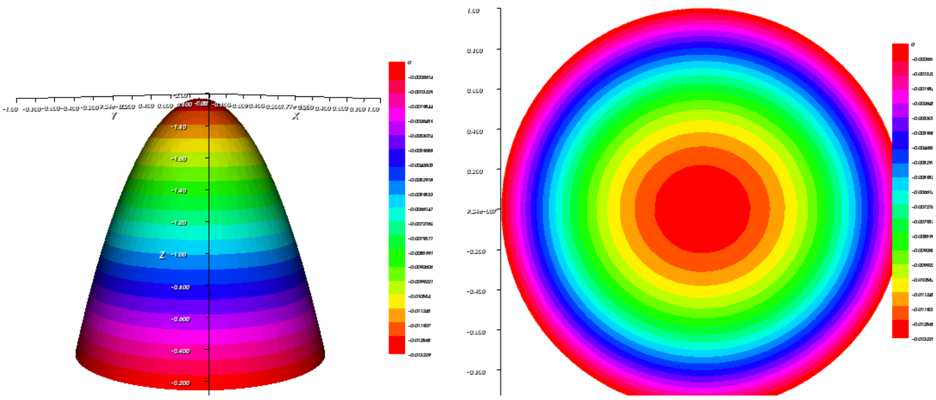
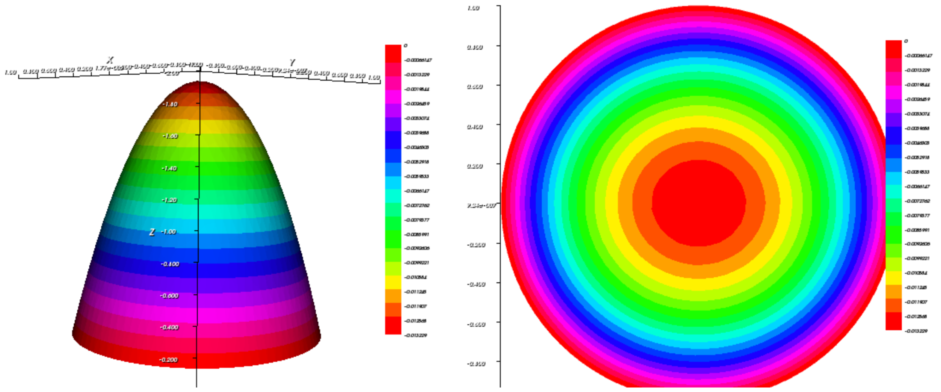


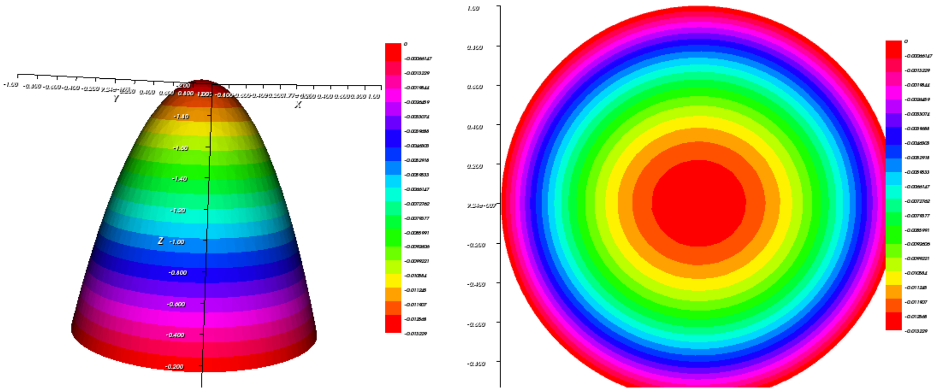
Figure 3.1: Uniforme mesh grid



(A): t=0

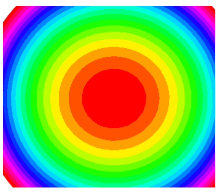
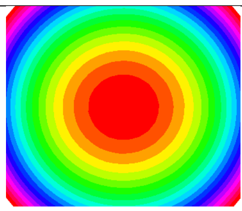
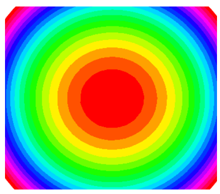


(B): t=0.5



(C): t=0.9

In (A), (B) and (C) we show the numerical approximation of solutions $u$ at different time $t = 0$, $t = 0.5$ and $t = 0.9$.

| $t$ | $t = 0$ | $t = 0.5$ | $t = 0.9$ |
|---|---|---|---|
| Error | 2.2836 | 0.829166 | 0.3749696 |
| Zoom |  |  |  |

It is noticed that the value of the error decreases if $t$ increases, which guarantees us the stability and the convergence of the method.

CHAPTER 4

# ON THE BLOW UP OF SOLUTIONS FOR HYPERBOLIC EQUATION INVOLVING THE FRACTIONAL LAPLACIAN WITH SOURCE TERMS

## 4.1 Introduction

The results obtained in this chapter have been published in [9].

In this chapter, we study the following boundary value problem related to the hyperbolic equation involving the fractional Laplacian with source terms:

$$\begin{cases} u_{tt} + (-\Delta)^s u - (-\Delta)^s u_t + |u_t|^{q-1} u_t = \alpha |u|^{p-1} u, & x \in \Omega, t > 0 \\ u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{4.1}$$

Where $\Omega \subset \mathbb{R}^n$ be an open domain with smooth boundary $\partial\Omega$, $(-\Delta)^s$ is the fractional Laplacian such that $s \in ]0,1[$, $\alpha = 1$ and $1 \leq q < p \leq p^*$ such that the exponent $p^*$ satisfies

$$p^* \leq \frac{2n}{n-2s} = 2_s^*, \quad n > 2s.$$

In recent years, many mathematical models involving fractional and non-local operators have been actively studied because these types of operators arise in various applications, such as physics, image processing, population dynamics, etc(see [13, 30]).

The following equation

$$\partial_t^2 u + [u]_s^{2(\theta-1)}(-\Delta)^s u = |u|^{p-1}u, \tag{4.2}$$

where $\theta \in [1, 2_s^*]$ and $[u]_s$ defined by

$$\Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\Big)^{\frac{1}{2}}, \tag{4.3}$$

in [44], the researchers obtained the global existence and blow up of solutions for (4.2) by using Galerkin method combined the potential wells. Also, they showed the global existence of solutions under initial conditions.

Then, in [43] the authors studied the global existence, behavior and blow up of solutions of the following equation

$$\partial_t^2 u + [u]_s^{2(\theta-1)}(-\Delta)^s u + |\partial_t u|^{\alpha-1}\partial_t u + u = |u|^{p-2}u, \tag{4.4}$$

where $2 < \alpha < 2\theta < p < 2_s^* < s$.

In [37], the authors studied the blow up by a modified concavity method in finite time (see [11, 54]).

In [5] the researches obtained the blow-up of solutions for the following equation

$$\partial_t^2 u + (-\Delta)^s u + (-\Delta)^s \partial_t u = u|u|^{p-2}, \quad x \in \Omega, t > 0. \tag{4.5}$$

In the absence of the nonlinear damping $|u_t|^{q-1}u_t$, the authors studied the blow up of solutions for the wave equation involving the fractional Laplacian by using the concavity method in [5].

In this chapter we show the global existence of solutions. Then, we prove the blow up of solutions by using the concavity method.

## 4.2 The Potential wells

In this section, we consider problem (4.1) in stationary case. In fact, if we replace $u$ in this section by $u(t)$ for any $t \in [0, T)$, all the facts are still valid.

We define the Nihari's functionals as follow

$$J(t) = \frac{1}{2}||u||^2_{W_0^{s,2}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1}, \tag{4.6}$$

$$I(t) = ||u||^2_{W_0^{s,2}(\Omega)} - ||u||^{p+1}_{p+1}. \tag{4.7}$$

We also define the energy function

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||u||^2_{W_0^{s,2}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1}, \tag{4.8}$$

and we introduce now the stable set as follows

$$W = \{u : u \in W_0^{s,2}(\Omega), I(u) > 0; J(u) < d\} \cup \{0\}, \tag{4.9}$$

where the montain pass level

$$d = \inf_{u \in W_0^{s,2}(\Omega)/0} \{\sup_{\mu \geq 0} J(\mu u)\}. \tag{4.10}$$

And the Nihari manifold

$$N = \{u \in W_0^{s,2}(\Omega)/0 : I(u) = 0\}, \tag{4.11}$$

with the potential depth $d$

$$d = \inf_{u \in N} J(u), \tag{4.12}$$

which implies that

$$dist(0, N) = \min_{u \in N} ||u||_{W_0^{s,2}(\Omega)}. \tag{4.13}$$

**Definition 4.1** Let us assume $s \in ]0, 1[, 1 \leq q < p \leq p^*, u_0 \in W_0^{s,2}(\Omega)$ and $u_1 \in L^2(\Omega)$ then, there exists a function $u = u(t, x)$ be a weak global solution of problem (4.1), if

$$u \in L^\infty\big([0, T); W_0^{s,2}(\Omega)\big) \quad \text{and} \quad u_t \in L^\infty\big([0, T); L^2(\Omega)\big) \cap L^{q+1}\big([0, T) \times \Omega\big),$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{a.e. in} \quad \Omega.$$

If a weak global solution $u$ belongs to $C\left((0,\infty);W_0^{s,2}(\Omega)\right)$, we say that $u$ is a strong global solution of problem (4.1).

In the next lemma, we show the energy function which is a nonincreasing function along the solution of (4.1)

**Lemma 4.1** *Let $u$ be a weak solution of* (4.1)*, if $u_0 \in W$ and $u_1 \in L^2(\Omega)$, then $E(t)$ is a non-increasing function and*

$$E(t) \leq E(0), \quad \forall t \in [0,T). \tag{4.14}$$

**Proof.** We have

$$E^{'}(t) = -\left(||u_t||_{q+1}^{q+1} + ||u_t||^2_{W_0^{s,2}(\Omega)}\right) \leq 0. \tag{4.15}$$

By integration over $[0,t]$, we get $E(t) - E(0) \leq 0$. ∎

**Lemma 4.2** *If $u_0 \in W$ and $u_1 \in L^2(\Omega)$. We suppose that $p^* \leq \dfrac{2n}{n-2s} = 2_s^*, n > 2s$ and $s \in ]0,1[$, then, the solution $u \in W$, $\forall t \geq 0$.*

**Proof.** Since $u_0 \in W$, we have

$$I(u_0) = ||u_0||^2_{W_0^{s,2}(\Omega)} - ||u_0||_{p+1}^{p+1} > 0,$$

and by the continuity of $u(t)$, so: $I(t) > 0$, for some interval near $t = 0$. Let $T_m > 0$.

$$J(t) = \frac{1}{p+1} I(t) + \frac{p-1}{2(p+1)} ||u||^2_{W_0^{s,2}(\Omega)}, \tag{4.16}$$

since $I(t) > 0$, we have

$$J(t) \geq \frac{p-1}{2(p+1)} ||u||^2_{W_0^{s,2}(\Omega)}, \tag{4.17}$$

then from $E(t)$ and $E^{'}(t)$, we obtain

$$\begin{aligned} ||u||^2_{W_0^{s,2}(\Omega)} &\leq \frac{2(p+1)}{p-1} J(t) \\ &\leq \frac{2(p+1)}{p-1} E(t) \\ &\leq \frac{2(p+1)}{p-1} E(0). \end{aligned} \tag{4.18}$$

Also, we have

$$\begin{aligned}
||u||_{p+1}^{p+1} &\leq C_{*s}||u||^{p+1}_{W_0^{s,2}(\Omega)} \\
&= C_{*s}||u||^{p-1}_{W_0^{s,2}(\Omega)}||u||^2_{W_0^{s,2}(\Omega)} \\
&\leq C_{*s}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{p-1}{2}}||u||^2_{W_0^{s,2}(\Omega)} \\
&= \beta||u||^2_{W_0^{s,2}(\Omega)} \\
&< ||u||^2_{W_0^{s,2}(\Omega)} \quad \forall t \in [0, T_m]. \qquad (4.19)
\end{aligned}$$

With $C_{*s} = C_{2_s^*}^{p+1}$.

When we repeat the procedure, $T_m$ is extended to $T$. So the proof is completed. ■

## 4.3 Blow up results

In this section, we prove the blow up result to the problem (4.1), when ($q = 1$).

**Definition 4.2** A solution $u$ of the problem (4.1) is called blow up if there exists $T^*$ a finite time such that

$$\lim_{t\to T^{*-}} ||u(t,x)||_2 = \infty. \qquad (4.20)$$

We put

$$A(t) = ||u(t,x)||_2 \quad \text{for} \quad t \geq 0. \qquad (4.21)$$

**Lemma 4.3** *Let $u_0 \in W$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) > 0$ and $\int_\Omega u_0u_1dx > 0$, then any solution $u$ blows up in finite time.*

**Proof.** We have

$$A^{'}(t) = 2 < u_t, u >_{L^2(\Omega)}, \quad t \in [0, T]. \qquad (4.22)$$

and

$$A^{''}(t) = 2 < u_{tt}, u >_{L^2(\Omega)} +2||u_t||_2, \quad t \in [0, T]. \qquad (4.23)$$

Now, we multiply the problem (4.1) by $u$ then we integrate over $\Omega$, we get

$$< u_{tt}, u >_{L^2(\Omega)} = -||u||_{W_0^{s,2}(\Omega)} - < u_t, u >_{W_0^{s,2}(\Omega)} - \int_\Omega u_tudx + \int_\Omega |u|^{p-1}udx, \qquad (4.24)$$

by Cauchy-Schwartz inequality, we obtain

$$A^{'2}(t) = 4 < u, u_t >^2_{L^2} \leq 4||u||_2^2||u_t||_2^2, \quad t \in [0, T]. \tag{4.25}$$

Which implies that

$$\begin{aligned}
&A^{''}(t)A(t) - (1+\delta)(A^{'}(t))^2 \\
&\geq \big(2||u_t||_2^2 - 2||u||^2_{W_0^{s,2}(\Omega)} - 2 < u_t, u >_{L^2(\Omega)} + 2\int_\Omega u|u|^{p-1}u\big)A(t) \\
&- 4(1+u) < u_t, u >^2_{L^2} \\
&\geq \big(2||u_t||_2^2 - 2||u||^2_{W_0^{s,2}(\Omega)} - 2 < u_t, u >_{L^2(\Omega)} + 2\int_\Omega u|u|^{p-1}u\big)A(t) \\
&- 4(1+u)||u_t||_2^2||u||_2^2 \\
&= \big( - 2||u||^2_{W_0^{s,2}(\Omega)} - 2 < u_t, u >_{L^2(\Omega)} + 2\int_\Omega u|u|^{p-1}u \\
&- 4(1+2\delta)||u_t||_2^2\big)A(t),
\end{aligned}$$

where $\delta > 0$.

Then, we put

$$M(t) = -2||u||^2_{W_0^{s,2}(\Omega)} + 2\int_\Omega u|u|^{p-1}u - 4(1+2\delta)||u_t||_2^2. \tag{4.26}$$

After that

$$\begin{aligned}
M(t) &= -2||u||^2_{W_0^{s,2}(\Omega)} + 2||u||^{p+1}_{p+1} - 4(1+2\delta)||u_t||_2^2 \\
&\geq -4(1+2\delta)||u_t||_2^2 - 2||u||^2_{W_0^{s,2}(\Omega)} + (p+1)||u_t||_2^2 + (p+1)||u||^2_{W_0^{s,2}(\Omega)} - 2(p+1)E(0) \\
&= -(8\delta - p + 1)||u_t||_2^2 + (p-1)||u||^2_{W_0^{s,2}(\Omega)} - 2(p+1)E(0) \\
&\geq -(8\delta - p + 1)||u_t||_2^2 + (p-1)||u||^2 + (p-1)||u||_2^2 - 2(p+1)E(0),
\end{aligned}$$

for $t \in [0, T)$. Set $\delta = \dfrac{p-1}{8} > 0$, then we get

$$M(t) \geq (p-1)||u||_2^2 - 2(p+1)E(0) \geq 0. \tag{4.27}$$

Therefore, we obtain

$$A^{''}(t)A(t) - (1+\delta)(A^{'}(t))^2 > 0, \quad t \in [0, T). \tag{4.28}$$

This implies that

$$(A^{-\delta})' = -\delta A^{-\delta-1}A'(t) < 0, \tag{4.29}$$

$$(A^{-\delta})'' = -\delta A^{-\delta-2}\big(A''(t)A(t) - (1+\delta)(A'(t))^2\big) < 0, \tag{4.30}$$

for all $t \in [0, T)$, which means that the function $A^{-\delta}$ is concave. Obviously, $A(0) > 0$, then there must exist $T^*$ such that

$$\lim_{t \longrightarrow T^*} A^{-\delta}(t) = 0.$$

So, that

$$\lim_{t \longrightarrow T^{*-}} A(t) = \infty.$$

Thus, the proof is completed. ■

## 4.4 Conclusion

In this work, we obtained the global existence results and the blow up of solutions for a hyperbolic equation involving the fractional Laplacian with source terms in a bounded domain. This improves and extends many results in the literature.

# CHAPTER 5

# ON THE HYPERBOLIC WAVE EQUATION WITH NONREGULAR BOUNDARY CONDITIONS

## 5.1 Introduction

In this chapter, we study the following value problem related to the wave hyperbolic equation with nonregular boundary conditions:

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \Gamma_0, t > 0, \\ u(t,x) = -|u_t|^{q-2}u_t, & x \in \Gamma_1, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{5.1}$$

Where $\Omega \subset \mathbb{R}^n$, $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \phi$, with $\Gamma_0$ and $\Gamma_1$ are measurable over $\partial\Omega$ endowed with the $(n-1)-$dimensional Lebesgue measures $\lambda_{n-1}(\Gamma_0)$ and $\lambda_{n-1}(\Gamma_1)$; $q$, $p \geq 2$. These properties of $\Omega$, $\Gamma_0$, $\Gamma_1$, $\lambda_{n-1}(\Gamma_0) > 0$ and $\lambda_{n-1}(\Gamma_1) > 0$ will be assumed throughout this chapter.

A lot of mathematical models, including those for the diffusion phenomena, hydrodynamics, thermoelasticity and hydrologic filtration process use nonregular boundary problems.

Since the paper of Lions in 1969 [38], the evolution problems with a nonregular boundary conditions (first order equations in time), have been thoroughly investigated. Later, it was the subject of lengthy study and successful innovative endeavors by physicists and mathematicians, in [6, 22, 36].

In [36], the authors proved the local and global existence under suitable condition. Then, in [55] they studied the blow-up of the solution under the condition $E(0) < d$ in the case of the initial data are in the unstable set and in [18], the authors obtained the blow-up of the solution for a finite time at a critical energy level or high-energy level for the same problem. In this chapter, firstly, we show the local existence of solutions of the problem (5.1). Secondly, we get the finite time blow up of solutions.

## 5.2 Preliminaries

In this section, we give some lemmas and assumptions which we will be used in this chapter.

Let $||.||_2$, $||.||_q$ and $||.||_{q,\Gamma_1}$ denote the usual $L^2(\Omega)$, $L^q(\Omega)$ and $L^q(\Gamma_1)$ norm, respectively. Let

$$H^1_{\Gamma_0} = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}.$$

Now, we define some useful functional

$$J(t) = \frac{1}{2}||\nabla u||_2^2 - \frac{1}{p}||u||_p^p, \tag{5.2}$$

and

$$I(t) = ||\nabla u||_2^2 - ||u||_p^p. \tag{5.3}$$

Also, we define the energy function

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{1}{2}||\nabla u||_2^2 - \frac{1}{p}||u||_p^p, \tag{5.4}$$

according to the definition of energy function $E(t)$, we have

$$E(0) = \frac{1}{2}||u_1||_2^2 + \frac{1}{2}||\nabla u_0||_2^2 - \frac{1}{p}||u_0||_p^p.$$

We will use the embedding (see [2]), for $2 \le p < r$ introduced, where

$$r = \begin{cases} \dfrac{2(n-1)}{n-2}, & \text{if} \quad n \ge 3, \\ +\infty, & \text{if} \quad n = 1, \quad 2. \end{cases} \tag{5.5}$$

We also introduce the unstable set $W$ defined by

$$W = \{(u_0, u_1) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega), I(0) < 0, 0 < E(0) = d\}, \tag{5.6}$$

where $d$ is the mountain pass level, characterized by the following equation

$$d = \inf_{u \in H^1_{\Gamma_0}(\Omega), u|\Gamma_1 \neq 0} \{\sup_{\lambda \geq 0} J(\lambda u)\}. \tag{5.7}$$

And, we define

$$\lambda_1 := C_*^{\frac{-p}{p-2}}. \tag{5.8}$$

In [48], it has been proved that $\left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2$ is the potential well depth with

$$d = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2. \tag{5.9}$$

In [48] and [15], the authors have been proved the local and global existence of solutions for problem (5.1). Next, we introduce some results in [48] and [15] as follows:

**Theorem 5.1 (Local existence)** *There exists $T > 0$, such that the problem* (5.1) *has a unique local weak solution $u$:*

$$u \in C\big([0, T); H^1_{\Gamma_0}(\Omega)\big) \cap C^1\big([0, T); L^2(\Omega)\big) \quad \text{and} \quad u_t \in L^q\big([0, T) \times \Gamma_1\big).$$

*And*

$$E(t) + \int_s^t ||u_\tau(\tau)||^q_{q,\Gamma_1} d\tau = E(s),$$

*holds, for $0 \leq s \leq t \leq T$.*

## 5.3 Finite time blow up

In order to study the finite time blow up of solution, first we prove some basic lemmas.

**Lemma 5.1** *(Invariant manifolds) Suppose that $2 \leq p < r$, $\frac{r}{r+1-p} < q$ and*

$$W' = \{(u_0, u_1) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega), ||\nabla u_0|| > \lambda_1, 0 < E(0) = d\},$$

*then, we have $W = W'$.*

**Proof.** Firstly, we show that $W \subset W'$, let $(u_0, u_1) \in W$ then, we have $I(0) < 0$, by using the embedding inequality, we obtain

$$||\nabla u_0||_2^2 < ||u_0||_p^p \leq C_*^p ||\nabla u_0||_2^p.$$

Then, we have $||\nabla u_0||_2 > C_*^{\frac{-p}{p-2}} = \lambda_1$, so we obtain that $W \subset W'$.
Secondly, we show that $W' \subset W$, let $(u_0, u_1) \in W'$ then, we have $||\nabla u_0||_2 > \lambda_1$ and $E(0) = d$, we suppose by contradiction that $I(0) \geq 0$, we have

$$||\nabla u_0||_2^2 \geq ||u_0||_p^p,$$

combining

$$E(0) = \frac{1}{2}||u_1||_2^2 + \frac{1}{2}||\nabla u_0||_2^2 - \frac{1}{p}||u_0||_p^p,$$

we obtain

$$\frac{1}{2}||\nabla u_0||_2^2 - \frac{1}{p}||u_0||_p^p \leq d,$$

next, we get

$$d \geq (\frac{1}{2} - \frac{1}{p})||\nabla u_0||_2^2,$$

since $||\nabla u_0||_2 > \lambda_1$, it means that

$$d > (\frac{1}{2} - \frac{1}{p})\lambda_1^2 = d,$$

which is a contradiction. This completes the proof. ■

**Lemma 5.2** *(Invariant manifolds and boundness) We suppose that $2 \leq p < r$, $\frac{r}{r+1-p} < q$ and let $(u_0, u_1) \in W$ and $u$ be a weak solution of the problem (5.1) on $[0, T_{\max})$, then $(u, u_t)$ remains inside $W$ for any $[0, T_{\max})$. Furthermore, we have*

$$||\nabla u(t)||_2^2 < ||u(t)||_p^p, \quad t \in [0, t_{\max}),$$

$$||u(t)||_p > C_* \lambda_1, \quad t \in [0, t_{\max}),$$

$$||\nabla u(t)||_2 > \lambda_1, \quad t \in [0, t_{\max}).$$

**Proof.** From $I(0) < 0$ and by the continuity of $I(u)$respecting to $t$, so there exists a $t_1 > 0$ such that $I(u) < 0$ for $0 < t < t_1$, we set $d_1 = E(t_1)$, we have

$$0 < d_1 = d - \int_0^{t_1} ||u_\tau(\tau)||_{q,\Gamma_1}^q d\tau < d.$$

We put $t = t_1$ as the initial time, we get $(u, u_t)$ remains inside $W$ for any $t \in [0, T_{\max})$, according to the definition of $I(u)$, we have

$$||\nabla u(t)||_2^2 < ||u(t)||_p^p, \quad t \in [0, T_{\max}),$$

moreover, by the Lemma 5.1, we obtain

$$||\nabla u(t)||_2 > \lambda_1, \quad t \in [0, T_{\max}), \tag{5.10}$$

then, we get

$$||u(t)||_p > C_* \lambda_1, \quad t \in [0, T_{\max}). \tag{5.11}$$

This completes the proof. ■

**Lemma 5.3** *We suppose that* $2 \leq p < r$, $\frac{r}{r+1-p} < q$ *and let* $(u_0, u_1) \in W$ *and* $u$ *be a weak solution of the problem* (5.1) *on* $[0, T_{\max})$, *so, there is a* $\lambda_2 > \lambda_1$ *such that* $||\nabla u(t)||_2 \geq \lambda_2 > \lambda_1$.

**Proof.** We have $I(u) < 0$ for all $t \in [0, T_{\max})$, by Sobolev-Poincaré inequality, we get

$$E(t) \geq \frac{1}{2}||\nabla u||_2^2 - \frac{1}{p}||u||_p^p \geq \frac{1}{2}||\nabla u||_2^2 - \frac{1}{p}C_*^p||\nabla u||_2^p := g(||\nabla u||_2), \tag{5.12}$$

with $g(\lambda) = \frac{1}{2}\lambda^2 - \frac{1}{p}C_*^p\lambda^p$ for $\lambda \geq 0$, it is clear that $g$ takes the maximum at $\lambda = \lambda_1$, with $g(\lambda_1) = d$, being strictly decreasing for $\lambda \geq \lambda_1$ and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. In the fact, $E(t)$ is decreasing when $t \in [0, T_{\max})$ and $E(0) = d$, we continue to argue as follows, by the continuity of $||\nabla u(.)||_2$. There are only two possibilities:

1. There is a $t_0 \geq 0$ such that $E(t_0) < d$ and $||\nabla u(.)||_2 > \lambda_1$.

2. There is a $\epsilon_0 \geq 0$ such that $E(t) = d$ on $[0, \epsilon_0)$.

In case 1, we choose the initial time $t_0$, due to the fact that $E(t)$ and $g(\lambda)$ are both decreasing and continuous, so there exists a $\lambda_2 > \lambda_1$ such that $E(t_0) = g(\lambda_2)$, now we claim that

$$||\nabla u(t_0)||_2 \geq \lambda_2.$$

We suppose for contradiction that for some $t \in [0, T_{\max})$: $||\nabla u(t_0)||_2 < \lambda_2$ and by using (5.12) and the fact that the function $g(\lambda)$ is a decreasing function. We also have

$$E(t_0) \geq g(||\nabla u(t_0)||_2) > g(\lambda_2) = E(t_0)$$

, which leads to a contradiction, we can also get that

$$\frac{1}{2}||\nabla u||_2^2 \geq \frac{1}{2}\lambda_2^2 > \frac{1}{2}\lambda_1^2.$$

Then, we choose $\lambda_0 = \frac{1}{2}\lambda_1^2$ and $E_1 = (1 - \frac{p}{2})\lambda_0$. So, this is the proof for the first case. In the second case, we have $E(t) = d$ for $t \in [0, \epsilon_0)$ and

$$\int_0^{t_0} ||u_\tau(\tau)||_{q,\Gamma_1}^q d\tau = 0, \quad t \in [0, \epsilon_0). \tag{5.13}$$

Due $||u_t(t)||_{q,\Gamma_1}^q \geq 0$, we have $u_t = 0$ and $u(t) = u_0$ on $[0, \epsilon_0)$. We suppose for contradiction that $||\nabla u_0||_2 < \lambda_2$ for $t \in [0, \epsilon_0)$. We can get that

$$d = E(0) = \frac{1}{2}||\nabla u_0||_2^2 - \frac{1}{p}||u_0||_p^p \geq g(||\nabla u_0||_2) > g(\lambda_2) = E(0) = d,$$

which leads to a contradiction. This completes the proof. ■

**Theorem 5.2** *We assume that* $1 < q < p$, $q > \frac{r}{r+1-p}$, $2 \leq p < r$. *If* $(u_0, u_1) \in W$, *then the solution of problem* (5.1) *blows up in finite time.*

**Proof.** Arguing by contradiction, we assume that there exists a global weak solution of problem (5.1). We set $H(t) = d - E(t)$. We have, $E(t)$ is decreasing about $t$, so $H(t)$ is an increasing function, then we have

$$H(t) \geq H(0) = d - E(0) = 0, \quad t \geq 0. \tag{5.14}$$

By using the definition of $E(t)$, we obtain

$$H(t) \leq d - \frac{1}{2}||\nabla u(t)||_2^2 + \frac{1}{p}||u(t)||_p^p, \quad t \geq 0. \tag{5.15}$$

Using (5.10) and (5.9), we can get that

$$d - \frac{1}{2}||\nabla u(t)||_2^2 \leq d - \frac{1}{2}\lambda_1^2 = -\frac{1}{p}\lambda_1^2 < 0. \tag{5.16}$$

Then, we get

$$H(t) - \frac{1}{p}||u(t)||_p^p \leq d - \frac{1}{2}||\nabla u(t)||_2^2 < 0,$$

from (5.14), we obtain

$$H(0) \leq H(t) < \frac{1}{p}||u(t)||_p^p, \quad t \geq 0. \tag{5.17}$$

After that, by using the definition of $E(t)$ and (5.15), we have

$$\frac{d}{dt}(u,u_t) = ||u_t||_2^2 - ||\nabla u(t)||_2^2 + ||u(t)||_p^p - \int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma$$
$$= 2||u_t||_2^2 + \big(1 - \frac{2}{p}||u(t)||_p^p - 2E(t) - \int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma$$
$$= 2||u_t||_2^2 + \big(1 - \frac{2}{p}||u(t)||_p^p - 2d + 2H(t) - \int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma.$$

We choose a $\lambda_2$ such that $||\nabla u(t)||_2 \geq \lambda_2 > \lambda_1$, so we obtain

$$\frac{d}{dt}(u,u_t) \geq 2||u_t||_2^2 + \big(1 - \frac{2}{p} - 2d(C_*\lambda_2)^{-p}\big)||u(t)||_p^p + 2H(t) - \int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma \quad (5.18)$$
$$= 2||u_t||_2^2 + C_1||u(t)||_p^p + 2H(t) - \int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma. \quad (5.19)$$

Where $C_1 = \big(1 - \frac{2}{p} - 2d(C_*\lambda_2)^{-p}\big) > 0$, now we estimate the last term, by using Hölder's inequality, Young inequality and the fact that $H^{'}(t) = ||u_t||_{q,\Gamma_1}^q$, we get

$$|\int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma| \leq ||u_t||_{q,\Gamma 1}^q ||u||_p \quad (5.20)$$
$$\leq ||u||_p^{1-\frac{p}{q}} ||u||_p^{\frac{p}{q}} ||u_t||_{q,\Gamma 1}^{q-1} \quad (5.21)$$
$$\leq C_2 H^{\frac{1}{p}-\frac{1}{q}}(t)||u||_p^{\frac{p}{q}} ||u_t||_{q,\Gamma 1}^{q-1} \quad (5.22)$$
$$\leq C_3\big(\epsilon^q ||u||_p^p + \epsilon^{-q'} H^{'}(t)\big)H^{-\alpha^*}(t), \quad (5.23)$$

where $\alpha = \frac{1}{q} - \frac{1}{p}$ and $\frac{1}{q} - \frac{1}{q'} = 1$, for any $\epsilon > 0$ and we denote $C_1, C_2,...,$ as suitable positive constants.

Let $0 < \alpha < \alpha^*$, we have

$$|\int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma| \leq C_3\big(\epsilon^q H^{-\alpha^*}(0)||u||_p^p + \epsilon^{-q'} H^{'}(t)H^{-\alpha^*}(t)H^{\alpha-\alpha^*}(0)\big). \quad (5.24)$$

Now, we define the auxiliary function

$$Z(t) = H^{1-\alpha}(t) + \delta \int_{\Omega} u u_t dx,$$

where $\delta$ is a small positive constant which will be defined later. We have

$$Z^{'}(t) \geq (1-\alpha)H^{-\alpha}(t) + \delta\big(2||u_t||_2^2 + C_1||u(t)||_p^p + 2H(t) - \int_{\Gamma_1} |u_t|^{q-2}u_t u d\sigma\big)$$
$$\geq (1 - \alpha - \delta C_3 \epsilon^{-q'} H^{\alpha-\alpha^*}(0))H^{-\alpha}(t)H^{'}(t) + \delta(C_1 - C_3\epsilon^q H^{-\alpha^*}(0))||u||_p^p$$
$$+ 2\delta||u_t||_2^2 + 2\delta H(t).$$

Let $\delta < (1-\alpha)C_3^{-1}\epsilon^{q'}H^{\alpha^*-\alpha}(0)$, consequently, the first term on the right hand side is positive. Further,, if we choose $\epsilon$ sufficiently small, we can get that

$$C_1 - C_3\epsilon^q H^{-\alpha^*}(0) \geq \frac{1}{2}C_1.$$

Moreover, we obtain

$$Z'(t) \geq \frac{1}{2}C_0\delta||u||_p^p + 2\delta||u_t||_2^2 + 2\delta H(t) \geq C_4\delta(||u||_p^p + ||u_t||_2^2 + H(t)). \tag{5.25}$$

Let $\delta$ sufficiently small, we have $Z'(0) > 0$ and noting that $H(t)$is an increasing function, we also have $Z(t) \geq Z(0)$ for $t \geq 0$. Next we set $r = \frac{1}{1-\alpha}$ and since $\alpha < \alpha^* < 1$, now we use Young inequality and Cauchy-Schwarz inequality, we get

$$Z^r(t) \leq 2^{r-1}(H(t) + \delta^r||u_t||_2^r||u||_2^r) \leq C_4(H(t) + ||u_t||_2^2 + ||u||_2^{\frac{1}{\frac{1}{2}-\alpha}}), \tag{5.26}$$

by choosing $\alpha$ sufficiently small, we obtain

$$||u||_2^{\frac{1}{\frac{1}{2}-\alpha}} \leq 1 + ||u||_2^2, \tag{5.27}$$

using Poincaré inequality and combining (5.26) and (5.27), we get

$$Z^r(t) \leq C_5(H(t) + ||u_t||_2^2 + ||\nabla u||_2^2) \leq C_6(||u||_p^p + ||u_t||_2^2 + H(t)), \tag{5.28}$$

then, we obtain

$$Z'(t) \geq C_7 Z^r(t), \tag{5.29}$$

so, there exist positive constants $C_8$ and $C_9$, such that

$$Z^{r-1}(t) \geq \frac{1}{-C_8(t) + C_9},$$

then, we get

$$\lim_{t \to \frac{C_9}{C_8}} Z^{r-1}(t) = \infty.$$

So $Z(t)$ is not global. This completes the proof. ■

**Remark:**The results presented in this chapter have been submitted for publication in International Journal.

# CONCLUSION AND FUTURE WORK

In conclusion, in our thesis, we studied the well-posedness and the asymptotic behavior of solutions for some quasi-linear and nonlinear hyperbolic equations with regular and nonregular boundary conditions. We proved the global existence, exponential and polynomial decay and blow up finite time for the quasi-linear hyperbolic equation involving the weighted Laplacian, the weighted $p-$Laplacian, fractional Laplacian operator and hyperbolic wave equation with nonregular boundary conditions. We also gave some numerical tests to illustrate our theoretical results.

**Future work**

The following open questions can be addressed in our future work. Firstly, we will study the existence, decay and blow up of solutions for a quasi-linear system with source terms. Also, we will use these studies for extend more general boundary value systems involving fractional $p-$Laplacian and we will find the appropriate numerical methods. We can also try to find an application of these models in image processing.

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