Algerian Democratic and Popular Republic
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

20 August 1955 university of Skikda
Faculty of Sciences
Department of Mathematics
Ref:………………

جامعة 20 أوت 1955 -سكيكدة
كلية العـلـوم
قسم الرياضيات
المرجع:..........

Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: Applied Mathematics

Qualitative Study of some Classes of Evolution Problems

Presented by:

KAREK Widad

Publicly discussed:

In front of the Jury:

1.	**Hebhoub Fahima**	**M.C.A**	**20 August 1955 University of Skikda**	**President**
2.	**Bouzettouta Lamine**	**M.C.A**	**20 August 1955 University of Skikda**	**Supervisor**
3.	**Lallouche Abdallah**	**M.C.B**	**20 August 1955 University of Skikda**	**Co-supervisor**
4.	**Chaoui Abderrezak**	**Pro.**	**University of May 8, 1945 – Guelma**	**Examiner**
5.	**Aggoune Fateh**	**Pro.**	**University of May 8, 1945 – Guelma**	**Examiner**
6.	**Bouakkaz Ahlème**	**M.C.A**	**20 August 1955 University of Skikda**	**Examiner**

University year : 2024/2025

Abstract

In this thesis, we present our results about the existence and exponential decay of certain classes of evolution problems. The first problem focuses on a one-dimensional Lord-Schulman thermoelastic system coupled with porous damping and time delay. The heat conduction in this type of systems is described by the Lord–Shulman theory. The second problem focuses on the swelling porous system with the Gurtin-Pipkin thermal effect as the only source of damping with delay. In general, the study shows that the dissipation obtained from the Guertin-Pipkin heat law is sufficient to stabilize the system exponentially, regardless of the system parameters. The third problem focuses on a one-dimensional swelling porous-heat system with time-varying delay in a bounded domain under Dirichlet boundary conditions, with thermodiffusion effects and frictional damping. Overall, using the semi-group approach, the variable norm technique of T. Kato, and the appropriate assumptions on the weight of delay, we establish the well-posedness of the considered systems. Then, we show that our systems are exponentially stable by employing an appropriate Lyapunov functional. We point out that our results are achieved without taking into account that we have the equal of speeds.

Keywords: Swelling porous systems, Lord-Shulman, Gurtin-Pipkin thermal law ,Porous damping, Thermodiffusion effects, Delay term, Time-varying delay, Well-posedness, Semigroup theory, Exponential stability.

ملخص

في هذه الأطروحة، نقدم نتائج حول الوجود والانحلال الأسي لبعض أصناف المسائل التطورية. تركز المسألة الأولى على نظام حراري مرن للورد-شولمان أحادي البعد مقترنا بالتخميد المسامي وحد التأخر. يتم وصف التوصيل الحراري في هذا النوع من الأنظمة بقانون لورد-شولمان. أما المسألة الثانية، فتتناول نظاما مساميا منتفخا مع تأثير حراري لجورتين-بيبكن كمصدر وحيد للتخميد مع حد التأخر. تظهر الدراسة أن التخميد الناتج عن قانون جورتين-بيبكن للتوصيل الحراري كاف لاستقرار النظام أسيا بغض النظر على معلمات النظام. تركز المسألة الثالثة على نظاما مساميا حراريا منتفخا أحادي البعد مع تأخير متغير زمنيا وبشروط حدية لديريشليه، مع تأثيرات الانتشار الحراري وتخميد احتكاكي.

بشكل عام باستخدام نظرية أنصاف الزمر، وتقنية التظيم المتغير لـ T. Kato، والافتراضات المناسبة على وزن التأخر، نثبت وجود ووحدانية الحل لمسائلنا ثم باستخدام دالة ليابونوف المناسبة نثبت أن مسائلنا مستقرة أسيا. نشير إلى أن نتائجنا تم تحقيقها دون أخد بعين الاعتبار تساوي السرعات.

الكلمات المفتاحية

نظام مسامي منتفخ، لورد-شولمان، قانون التوصيل الحراري لجورتين-بيبكن، تخميد مسامي، تأثيرات الانتشار الحراري، حد التأخر، تأخر متغير زمنيا، الوضع الجيد، نظرية أنصاف الزمر، الاستقرار الأسي.

Résumé

Dans cette thèse, nous présentons nos résultats sur l'existence et la décroissance exponentielle de certains problèmes d'évolution. Le premier problème se concentre sur un système thermoélastique unidimensionnel de Lord Shulman avec un amortissement poreux et un retard temporel. La conduction thermique dans ce système est décrite par la théorie de Lord–Shulman. Le deuxième problème se concentre sur un système poreux gonflant avec l'effet thermique de Gurtin-Pipkin comme une seule source d'amortissement avec retard. En général, l'étude montre que la dissipation obtenue à partir de la loi thermique de Guertin-Pipkin est suffisante pour stabiliser le système de manière exponentielle, quels que soient les paramètres du système. Le troisième problème se concentre sur un système poreux gonflant unidimensionnel avec un retard variable dans le temps dans un domaine borné sous des conditions aux limites de type Dirichlet, avec des effets de thermodiffusion et un amortissement par frottement. Dans l'ensemble, en utilisant l'approche de semi-groupe, La technique de la norme variable de T. Kato, et les hypothèses appropriées sur le poids du retard, nous établissons la bien-poséité des systèmes considérés. Ensuite, nous montrons que nos systèmes sont exponentiellement stables d'après une fonctionnelle de Lyapunov bien choisée. Notons que nos résultats sont obtenus sans prendre en considération l'égalité des vitesses.
mots-clés: Systèmes poreux gonflés, Lord-Shulman, Loi thermique de Gurtin-Pipkin, Amortissement poreux, Effets de thermodiffusion, Terme de retard, Retard temporel variable, Bien posé, Théorie de semi-groupe, Stabilité exponentielle.

Acknowledgment

First and foremost, I would like to express my deepest gratitude to God for granting me the strength, wisdom, and perseverance to complete this PhD journey. I would like to extend my heartfelt thanks to my supervisor, Dr. BOUZETTOUTA Lamine, and my co-supervisor, Dr. LALLOUCHE Abdallah, for their valuable advice, continuous encouragement, and expertise, which have greatly contributed to the quality and progress of this work. Their support and perspective were instrumental in refining this thesis.

I would also like to extend my heartfelt thanks to the members of my thesis committee, for their constructive feedback, thoughtful suggestions, and the time they dedicated to reviewing my research. Your input has been invaluable in enhancing this work.

To the faculty and staff of University of 20 August 1955 Skikda, I am deeply thankful for their support and for fostering a stimulating academic environment. A special thanks to Dr. KAREK chafia and Dr. FAR zina for their encouragement and guidance during this journey.

To my colleagues and fellow researchers, thank you for your collaboration, thought-provoking discussions, and shared moments of learning. Your camaraderie has been a constant source of motivation.

Lastly, I extend my sincere thanks to all my friends and everyone who has contributed in any way to the completion of this thesis. Your kindness and encouragement have meant the world to me.

This achievement is not mine alone but shared with all those who have supported me along the way. Thank you.

Dedicate

To God, for granting me strength, wisdom, and perseverance.

To my parents, for their sacrifices and unwavering faith in my potential.

To my husband, who has walked every step of this journey with me.

To my boys, whose smiles and presence inspired me to persevere and achieve this milestone.

To my siblings, for their constant support and encouragement.

To my professors, for imparting knowledge and guidance throughout this journey.

To my friends, for their companionship and motivation during challenging times.

To my colleagues, for their collaboration and shared inspiration.

To my alma mater, for fostering an environment of learning and growth.

To the future generations, for whom I hope this work serves as a source of inspiration.

To everyone who contributed, directly or indirectly, to the completion of this work—thank you.

Contents

Introduction

Swelling porous soils

Swelling porous soils are types of soils that expand in volume when subjected to moisture. These soils typically contain clay minerals that attract and absorb water, leading to an increase in soil pore pressure. The behavior of swelling porous soils is studied under the framework of porous media theory, which includes the swelling of soils, plants, drying of fibers, wood, and paper. The swelling of soils is a complex phenomenon that involves the interaction between the soil matrix and the absorbed water. The swelling pressure is a critical factor in understanding the behavior of these soils, particularly in engineering applications such as foundation design and soil stabilization. Recently, many researches have focused on developing mathematical models that describe very accurately the behavior of swollen porous soils, such as the multiscale theory which has been developed in order to model the behavior of swelling clay soils, taking into account the incorporating physicochemical effects, see [2]. The studies have been based on the linear case of the swelling porous elastic soils problem, considering the formulation within the theory of mixtures for porous elastic materials. Overall, the study of swelling porous soils is crucial for understanding and mitigating the effects of soil swelling which enters into many engineering and environmental applications. For further details, we refer to [16, 30, 38, 41, 61]. The basic model of the linear swelling porous elastic soils problem is given by the following system of equations

$$\begin{cases} \rho_{\mathcal{U}} \mathcal{U}_{tt} &= \mathcal{P}_{1x} + \mathcal{G}_1 + \mathcal{F}_1 \\ \rho_{\mathcal{V}} \mathcal{V}_{tt} &= \mathcal{P}_{2x} + \mathcal{G}_2 + \mathcal{F}_2 \end{cases} \tag{0.1}$$

where $\mathcal{U}, \mathcal{V}$ are the displacement of the fluid and the elastic solid material and the density of each component is respectively $\rho_{\mathcal{U}}, \rho_{\mathcal{V}} > 0$. The functions $\mathcal{P}_1$, $\mathcal{G}_1$, $\mathcal{F}_1$ represent respectively the partial tension, internal body forces, and eternal forces acting on the displacement. The same thing about $\mathcal{P}_2$, $\mathcal{G}_2$, $\mathcal{F}_2$ but acting on the elastic solid. Furthermore, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}}_{\mathcal{A}} \begin{pmatrix} \mathcal{U}_x \\ \mathcal{V}_x \end{pmatrix}, \tag{0.2}$$

where a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number. $\mathcal{A}$ is a positive-definite matrix such as $a_1 a_3 > a_2^2$.

The paper [61] investigated (0.1) taking into account

$$\mathcal{G}_1 = \mathcal{G}_2 = \xi\left(\mathcal{U}_t - \mathcal{V}_t\right), \quad \mathcal{F}_1 = a_3 \mathcal{U}_{xxt}, \quad \mathcal{F}_2 = 0.$$

The authors found an exponential stability where $\xi > 0$. Similarly, Wang and Guo [75] examined (0.1) utilizing initial and partially mixed boundary conditions and taking

$$\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}_2 = 0, \quad \mathcal{F}_1 = -\rho_{\mathcal{U}} \gamma(x) \mathcal{U}_t$$

where $\gamma(x)$ is an internal viscous damping function with positive mean. They used the spectral theory approach to establish some exponential stability results.

In [66], Ramos et al considered (0.1) with

$$\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}_1 = 0, \quad \mathcal{F}_2 = -\gamma(x) g(\mathcal{V}_t)$$

and they succeeded to show an exponential decay rate when the wave speeds of the system are equal. In [65], the authors considered (0.1) under the following new conditions

$$\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}_1 = 0, \quad \mathcal{F}_2 = -\mu_1 \mathcal{V}_t - \mu_2 \mathcal{V}_t\left(x, t - \tau\right)$$

They obtained the stability of the system as both energy decay and exponential stability were showed. Recently, Apalara [12] looked into (0.1), with viscoelastic damping acting on the domain. So, he took

$$\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}_1 = 0, \quad \mathcal{F}_2 = -\int_0^t g\left(t-s\right) \mathcal{V}_{xx}(s) ds,$$

and established a general decay rate irrespective of the wave speed of the system. Very recently, Al-Mahdi et al. [1] also considered (0.1) with

$$\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}_1 = 0, \quad \mathcal{F}_2 = -\int_0^{\infty} g\,(s)\,\mathcal{V}_{xx}(t-s)ds,$$

and established a general decay results considering a wider class of relaxation functions, and they also performed several numerical tests to illustrate their theoretical results. For more details, see [11,43,51,62,63] and the references therein.

Thermoelasticity

Thermoelastic systems deal with the elastic and thermal behavior of elastic heat-conducting media, focusing on the reciprocal actions of elastic stresses and temperature variations. Duhamel [26] developed the theory of thermoelasticity in 1838, establishing equations for strain in an elastic body with temperature gradients. Neumann [52] found similar results in 1841. However, the hypothesis assumed that thermal and mechanical impacts were independent. The total strain was determined by superimposing the elastic strain and the thermal expansion caused by the temperature distribution only. The hypothesis did not account for thermal state motion or strain-temperature distribution interactions. In 1857, Thomson [74] pioneered the application of thermodynamic theories to predict the stresses and strains in an elastic body as temperatures changed. In 1954, Landau and Lifshitz [45] used classical thermodynamics to develop the coupled equations of thermoelasticity.

The thermoelastic equations describe the behavior of an elastic, heat-conducting material. The classical model incorporates the hyperbolic elastic system and the classical equivalent model of thermal conductivity which creates a parabolic coupling system. So, thermoelasticity refers to the interactions between elastic stresses and temperature differences. When the proposed materials are present in the linear case of a homogeneous and isotropic medium with a zero of external body forces and a zero of external heat, and subject to Fourier's law for the conduction of internal heat, we can introduce the coupled system as follows:

$$\begin{cases} \mathcal{U}_{tt} - \mathcal{U}_{xx} + \gamma_1 \theta_x = 0, & \text{in } (0,L)\times(0,T), \\ \theta_t - \theta_{xx} + \gamma_2 \mathcal{U}_{tx} = 0, & \text{in } (0,L)\times(0,T), \end{cases} \tag{0.3}$$

with the initial and boundary conditions

$$\begin{cases} \mathcal{U}(x,0) = \mathcal{U}_0(x),\ \mathcal{U}_t(x,0) = \mathcal{U}_1(x),\ \theta(x,0) = \theta_0(x), & \text{in } (0,L), \\ \mathcal{U}(0,t) = \mathcal{U}(L,t) = \theta(0,t) = \theta(L,t) = 0, & \text{in } (0,T), \end{cases} \tag{0.4}$$

where u and θ are the displacement and the temperature difference, α and β are coupling constants. In the classical theory of thermoelasticity the constitutive equation for the heat flux is expressed through the Fourier's law of heat conduction

$$q = -k\theta_x \tag{0.5}$$

where θ is the temperature, q is the heat flux vector and $k > 0$ is the coefficient of thermal conductivity.

Regarding the diffusion equations defined by Foueier's law, it is widely known that the thermal perturbation at any point in a material body would be felt quickly but irregularly at all points of the body. This physical inconsistency results in the paradox of limitless heat propagation speed. Several theories have been proposed to modify the relation between heat flux and temperature, including Cattaneo's law (developed by Lord and Shulman [49]), Gurtin and Pipkin's theory, Jeffrey's law, Green and Naghdi's theory, and more. All of these theories use hyperbolic differential equations and model heat flow as thermal waves with finite speed. For more information, see [22,39].

Using the following Cattaneo's law of heat conduction

$$\tau_0 q_t + q + k\theta_x = 0. \tag{0.6}$$

instead of (0.5) leads to thermoelasticity systems with second sound, as shown

$$\begin{cases} u_{tt} - u_{xx} + \gamma_1\theta_x = 0, & \text{in } (0,L)\times(0,T), \\ \theta_t - q_x + \gamma_2 u_{tx} = 0, & \text{in } (0,L)\times(0,T), \\ \tau_0 q_t + q + k\theta_x = 0, & \text{in } (0,L)\times(0,T). \end{cases} \tag{0.7}$$

Using the following Gurtin-Pipkin's law of heat conduction

$$\beta q + \int_0^1 g(s)\theta_x(t-s)ds = 0, \tag{0.8}$$

where g, called the memory kernel, is a bounded convex summable function on $[0,\infty)$of total mass

$$\int_0^1 g(s)ds = 1,$$

instead of (0.5) leads to thermoelasticity systems with Gurtin-Pipkin thermal effects, as shown,

$$\begin{cases} \mathcal{U}_{tt} - \mathcal{U}_{xx} + \gamma_1 \theta_x = 0, & \text{in } (0,L)\times(0,T), \\ \theta_t - \dfrac{1}{\beta}\displaystyle\int_0^1 g(s)\theta_{xx}(t-s)ds + \gamma_2 \mathcal{U}_{tx} = 0, & \text{in } (0,L)\times(0,T). \end{cases} \tag{0.9}$$

In [68], Mu noz Rivera considered system (0.3). He used the energy approach and proved that there exist a dissipation of energy caused by the heat equation and that is strong enough to exponentially stabilize the system. Also, Mu noz Rivera and Racke [67] considered the classical Timoshenko system subject to thermo-elastic effect of Fourier's law

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_3 \theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{0.10}$$

with different boundary conditions and proved that the solution of the system is exponentially stable if and only if

$$\frac{k}{\rho_1} - \frac{b}{\rho_2} = 0. \tag{0.11}$$

We note that for Timoshenko systems, φ is the displacement and ψ is the rotation angle of filament of the beam and $\rho_1, \rho_2, \rho_3, k, b, \gamma, \kappa$ are constitutive constants.

In [20], PS Casas, R Quintanilla considered the one dimensional problem of the porous-thermo-elasticity with Fourier's law

$$\begin{cases} \rho \mathcal{U}_{tt} = \mu \mathcal{U}_{xx} + b\varphi_x - \beta\theta_x, \\ J\varphi_{tt} = \alpha\varphi_{xx} - b\mathcal{U}_x - \xi\varphi + m\theta - \tau\varphi_t, \\ c\theta_t = \kappa\theta_{xx} - \beta\mathcal{U}_{xt} - m\varphi_t, \end{cases}$$

The authors established the exponential decay of the above system. In [21], the authors demonstrated that neglecting porous damping ($\tau = 0$) results in delayed energy decay. According to [70], the heat effect alone cannot cause exponential decay without condition $\frac{\mu}{\rho} = \frac{\alpha}{J}$. Han and Xu [36] analyzed a nonuniform Lord-Shulman-type porous thermoelastic model. The authors used spectral theory to calculate the exponential decay of energy, assuming C^1 coefficients. Messaoudi and Fareh [50] used the energy approach instead of spectral theory to achieve the same stability results as in [36], but dealing with constant coefficients. Fernández Sare and

Racke [32] considered the Maxwell-Cattaneo-Timoshenko system given by

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, & \text{in } (0,L)\times(0,\infty) \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,L)\times(0,\infty). \end{cases} \tag{0.12}$$

They proved that the solution of Timoshenko system coupled with the heat equation modeled by Cattaneo's law losses the exponential stability under the condition (0.11). On the other hand, Santos et al. [71] considered the same system (0.12) and they found a new number that characterizes the stabilization. Precisely, they proved that the exponential stability holds if and only if

$$\chi_0 = \left(\tau - \frac{\rho_1}{\rho_3 k}\right)\left(\rho_2 - \frac{b\rho_1}{k}\right) - \frac{\tau\rho_1\delta^2}{\rho_3 k} = 0.$$

Dell'Oro and Pata [23] considered the following system by applying the Gurtin-Pipkin thermal law

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)ds + \delta\psi_{xt}, & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{0.13}$$

where $g(.)$ is the memory kernal.They demonstrated that the corresponding semigroup associated with the above system is exponentially stable if and only

$$\chi_g = \left(\frac{\rho_1}{k\rho_3} - \frac{\beta}{g(0)}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\beta}{g(0)}\frac{\rho_1\delta^2}{bk\rho_3} = 0,$$

Aouadi el al. [5] presented the classical Timoshenko beam model within the framework of thermodiffusion theory, the resulting system is given by

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k\left(\varphi_x + \psi\right) - \gamma_1\theta_x - \gamma_2\mathcal{P}_x = 0, & \text{in } (0,L)\times(0,\infty), \\ c\theta_t + d\mathcal{P}_t - k\theta_{xx} - \gamma_1\psi_{xt} = 0, & \text{in } (0,L)\times(0,\infty), \\ d\theta_t + r\mathcal{P}_t - h\mathcal{P}_{xx} - \gamma_2\psi_{xt} = 0, & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{0.14}$$

where θ is the temperature and $\mathcal{P}$ is the chemical potential and the constant $\rho_1, \rho_2, k, \alpha$, $\gamma_1, \gamma_2, c, r, d, h$ and k are positive physical parameters. They demonstrated the lack of exponential stability for the Neumann problem without assuming the well-known equal wave speeds condition, meanwhile, one linear frictional damping is sufficient to ensure exponential stability for the Dirichlet problem.

Thermodiffusion effect

Diffusion is defined as the spontaneous movement of particles from a high-concentration zone to a low-concentration region in response to a concentration gradient, which is the change in concentration owing to location change. Thermal diffusion is the process of transferring heat across a thin liquid or gas to achieve isotopic separation. Thermal diffusion is still a useful method for separating noble gases and light isotopes for research. Fick's law is commonly used to compute concentrations in various applications. This basic law ignores the interaction between the substance and the medium, as well as the impact of temperature. Temperature and gradients have a certain degree of interaction, as temperature accelerates diffusion. Elastic solids undergo thermodiffusion through a combination of temperature, mass diffusion, strain, and heat exchange with the surrounding environment. Nowacki [58] developed the theory of thermoelastic diffusion by using a coupled thermoelastic model. Dudziak and Kowalski [25] and Olesiak and Pyryev [59] discussed respectively the theory of thermodiffusion and coupled quasistationary problems of thermal diffusion for an elastic layer.The study examined the impact of cross-effects caused by the interaction of temperature, mass diffusion, and strain. Thermal excitation causes increased mass concentration, which in turn generates additional temperature fields. Sherief et al. demonstrated the uniqueness and reciprocity theorems for the extended thermoelastic diffusion problem in isotropic medium.

Thermodiffusion is a critical area of study with broad implications. Continued research and technological progress will further unravel its complexities, enhancing our ability to harness this effect in practical applications.

Aouadi and Castejón [6] considered a new Timoshenko beam model with thermal and mass diffusion effects according to the Gurtin-Pinkin model with linear frictional damping and nonlinear source terms

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + f_1(\varphi,\psi) + \mu_1\varphi_t = h_1, & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k\left(\varphi_x + \psi\right) - \gamma_1\theta_x - \gamma_2 P_x + f_2(\varphi,\psi) = h_2, & \text{in } (0,1)\times(0,\infty), \\ c\theta_t + dP_t - \int_0^\infty \kappa(s)\theta_{xx}(t-s)ds - \gamma_1\psi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \\ d\theta_t + rP_t - \int_0^\infty \hbar(s)P_{xx}(t-s)ds - \gamma_2\psi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \end{cases}$$

where $\mu_1\varphi_t$ is a linear damping term, $f_1(\varphi,\psi), f_2(\varphi,\psi)$ are two nonlinear source terms and h_1, h_2 are two external forces. They proved without assuming the well-known equal wave speeds condition, that the thermal and chemical potential coupling is strong enough to guarantee the quasistability. Also Aouadi et all. [10] considered a new Timoshenko beam model with thermal and mass diffusion effects where the heat and the mass diffusion flux are governed by Cattaneo's law. The system is given by

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, & \text{in } (0,L)\times(0,\infty),\\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x+\psi) - \gamma_1\theta_x - \gamma_2 P_x = 0, & \text{in } (0,L)\times(0,\infty),\\ c\theta_t + dP_t + q_x - \gamma_1\psi_{tx} = 0, & \text{in } (0,L)\times(0,\infty),\\ \tau_0 q_t + q + \kappa\theta_x = 0, & \text{in } (0,L)\times(0,\infty)\\ rP_t + d\theta_t + \eta_x - \gamma_2\psi_{tx} = 0, & \text{in } (0,L)\times(0,\infty),\\ \tau_1\eta_t + \eta + \hbar P_x = 0, & \text{in } (0,L)\times(0,\infty). \end{cases}$$

They determined the required and sufficient conditions for exponential stability based on the model's physical parameters. More specifically, the relationship is given by

$$\chi_0 = (d\gamma_2 - r\gamma_1)\frac{\kappa}{\tau_0\gamma_1} - (d\gamma_1 - c\gamma_2)\frac{\hbar}{\tau_1\gamma_2},$$
$$\chi_1 = \xi\left(\rho_2 - \frac{\alpha\rho_1}{k}\right) - \left(\frac{\tau_0\gamma_1^2}{\kappa} + \frac{\tau_1\gamma_2^2}{\hbar}\right)(1-\xi),$$

where $\xi = 1 - \frac{\rho_1\Gamma}{\delta k}, \delta = cr - d^2 > 0$ and $\Gamma = (d\gamma_2 - c\gamma_1)\frac{\kappa}{\tau_0\gamma_1} = (d\gamma_1 - c\gamma_2)\frac{\hbar}{\tau_1\gamma_2}$
If $\chi_0 = 0$, play the main role for getting the stabilization properties.
Elhindi et al. [28] introduced the Bresse-Timoshenko beam model with thermal, mass diffusion and theormoelastic effects

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, & \text{in } (0,L)\times(0,\infty),\\ -\rho_2\psi_{ttx} - \alpha\psi_{xx} + k(\varphi_x+\psi) - \xi_1\theta_x - \xi_2 P_x = 0, & \text{in } (0,L)\times(0,\infty),\\ c\theta_t - dP_t - \kappa\theta_{xx} - \xi_1\psi_{tx} = 0, & \text{in } (0,L)\times(0,\infty),\\ d\theta_t + rP_t - hP_{xx} - \gamma_2\psi_{tx} = 0, & \text{in } (0,L)\times(0,\infty), \end{cases}$$

they showed an exponential stability of the problem
Douib and Zitouni. [24] considered a one-dimensional swelling porous-heat system with ther-

modiffusion effects and single time-delay

$$\begin{cases} \rho_1 \mathcal{U}_{tt} - a_1 \mathcal{U}_{xx} - a_2 \varphi_{xx} = 0, & \text{in } (0,1) \times (0,\infty) \\ \rho_2 \varphi_{tt} - a_3 \varphi_{xx} - a_2 \mathcal{U}_{xx} - \gamma_1 \theta_x - \gamma_2 \mathcal{P}_x + \mu_1 \varphi_t + \mu_2 \varphi_t(x, t-\tau) = 0, & \text{in } (0,1) \times (0,\infty) \\ c\theta_t + d\mathcal{P}_t - k\theta_{xx} - \gamma_1 \varphi_{xt} = 0, & \text{in } (0,1) \times (0,\infty) \\ d\theta_t + r\mathcal{P}_t - h\mathcal{P}_{xx} - \gamma_2 \varphi_{xt} = 0, & \text{in } (0,1) \times (0,\infty). \end{cases}$$

Using the energy approach and under suitable assumptions on the weight of the delay, they obtained the exponential decay result without imposing the usual condition of equal wave speeds of the system.

Recently, Rahmoune et al [64] considered a new Timoshenko beam model with thermal and mass diffusion effects combined with a time-varying delay

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x - \mu_1 \varphi_t + \mu_2 \varphi(x, t - \tau(t)) = 0, & \text{in } (0,L) \times (0,\infty), \\ \rho_2 \psi_{tt} - \alpha \psi_{xx} + k(\varphi_x + \psi) - \gamma_1 \theta_x - \gamma_2 \mathcal{P}_x = 0, & \text{in } (0,L) \times (0,\infty), \\ c\theta_t + d\mathcal{P}_t - \kappa \theta_{xx} - \gamma_1 \psi_{tx} = 0, & \text{in } (0,L) \times (0,\infty), \\ d\theta_t + r\mathcal{P}_t - \hbar \mathcal{P}_{xx} - \gamma_2 \psi_{tx} = 0, & \text{in } (0,L) \times (0,\infty), \end{cases}$$

The authors demonstrated exponential stability in a system with unequal wave propagation speeds using a multiplier technique combined with appropriate Lyapunov functions under an appropriate assumption on the weights of the delay and the damping. For more information, see [4,8,9,27].

On the other hand, time delays are common in many applications as they are influenced by both current conditions and past events. Recent researches focuse on controlling PDEs with time delay effects which can cause instability without extra conditions or control terms. The stability of systems with delay is a significant theoretical and practical issue. There are three versions of this term:

(i) discrete/constant delay $\frac{d}{dt}x(t) = f\left(t, x(t-\tau)\right)$,

(ii) distributed delay $\frac{d}{dt}x(t) = f\left(t, \int_0^{\tau} \mu(s)x(t-s)ds\right)$, and

(iii) time-varying delay $\frac{d}{dt}x(t) = f\left(t, x(t-\tau(t))\right)$.

Where $\frac{d}{dt}x(t) = f(t, x_t)$, $x(t) \in \mathbb{R}^n$, $x_t = \{x(\tau) : \tau \leq t\}$ represents the trajectory of the solution in the past. The functional operator f takes a time input and a continuous function x_t and generates a real number $\frac{d}{dt}x(t)$ as its output and μ is a L^∞ function. See for further information on this type [17,18,33,35,53,54,56,57,73].

The main results of this thesis:

This thesis contains four chapters.

In chapter 1, We recall some basic notations and some mathematical concepts that will be used throughout this thesis.

In chapter 2, we consider a one-dimensional Lord Shulman thermoelastic system with porous damping and time delay

$$\begin{cases} \rho_1 \mathcal{U}_{tt} = \mu^* \mathcal{U}_{xx} + \mu_0 \mathcal{V}_x - \beta_0(\tau_1 \theta_{xt} + \theta_x), & \text{in } (0,1) \times (0,\infty), \\ J \mathcal{V}_{tt} = a_0 \mathcal{V}_{xx} - \mu_0 \mathcal{U}_x - \xi \mathcal{V} + \beta_1(\tau \theta_t + \theta) - \zeta_1 \mathcal{V}_t - \zeta_2 \mathcal{V}_t(x, t-\tau), & \text{in } (0,1) \times (0,\infty), \\ a(\tau_1 \theta_t + \theta)_t = -\beta_0 \mathcal{U}_{xt} - \beta_1 \mathcal{V}_t + \kappa \theta_{xx}, & \text{in } (0,1) \times (0,\infty). \end{cases} \tag{0.15}$$

We include the initial conditions provided by the above system

$$\begin{cases} \mathcal{U}(x,0) = \mathcal{U}_0(x), \quad \mathcal{U}_t(x,0) = \mathcal{U}_1(x), \quad \mathcal{V}(x,0) = \mathcal{V}_0(x), & x \in (0,1), \\ \mathcal{V}_t(x,0) = \mathcal{V}_1(x), \quad \theta(x,0) = \theta_0(x), & x \in (0,1), \\ \mathcal{V}_t = f_0(x,t), & x \in (0,1),\ t \in (0,\tau), \end{cases} \tag{0.16}$$

and the boundary conditions

$$\begin{cases} \mathcal{U}_x(0,t) = \mathcal{U}_x(1,t) = \mathcal{V}(0,t) = \mathcal{V}(1,t) = 0, & t > 0, \\ \theta(0,t) = \theta(1,t) = 0, & t > 0. \end{cases} \tag{0.17}$$

We use the semigroup theory to establish the well-posedness, then we employ the multiplier method to prove an exponential decay result without imposing the usual condition of equal wave speeds of the system.

In chapter 3, We investigate a swelling porous system with the Gurtin-Pupkin thermal law and time delay,

$$\begin{cases} \rho_1 \mathcal{U}_{tt} - a_1 \mathcal{U}_{xx} - a_2 \mathcal{V}_{xx} = 0, & \text{in } (0,1) \times (0,\infty), \\ \rho_2 \mathcal{V}_{tt} - a_3 \mathcal{V}_{xx} - a_2 \mathcal{U}_{xx} - \delta \mathcal{T}_x + \gamma_1 \mathcal{V}_t + \gamma_2 \mathcal{V}_t(x, t-\tau) = 0, & \text{in } (0,1) \times (0,\infty), \\ \rho_3 \mathcal{T}_t - \dfrac{1}{\tau_1} \displaystyle\int_0^\infty \mu(s) \mathcal{T}_{xx}(x, t-s) ds - \delta \mathcal{V}_{xt} = 0, & \text{in } (0,1) \times (0,\infty), \end{cases}$$

$$\begin{cases} \mathcal{U}(x,0)=\mathcal{U}_0(x),\ \mathcal{U}_t(x,0)=\mathcal{U}_1(x), & \text{in } (0,1),\\ \mathcal{V}(x,0)=\mathcal{V}_0(x),\ \mathcal{V}_t(x,0)=\mathcal{V}_1(x), & \text{in } (0,1),\\ \mathcal{T}(x,-t)=\mathcal{T}_0(x,t), & \text{in } (0,1)\times(0,\infty),\\ \mathcal{U}_x(0,t)=\mathcal{U}_x(1,t)=\mathcal{V}_x(0,t)=\mathcal{V}_x(1,t)=\mathcal{T}(0,t)=\mathcal{T}(1,t)=0, & \text{in}(0,\infty). \end{cases} \tag{0.18}$$

We use Lumer-Phillips Theorem to demonstrate the well-posedness result under appropriate delay term weight assumptions. Furthermore using an appropriate Lyapunov functional leads to exponential stability without relying on wave speeds assumptions.

In chapter 4, We examine a one-dimensional swelling porous-heat system with thermodiffusion effects and time-varying delay

$$\begin{cases} \rho_1\mathcal{U}_{tt}-a_1\mathcal{U}_{xx}-a_2\varphi_{xx}=0,\\ \rho_2\varphi_{tt}-a_3\varphi_{xx}-a_2\mathcal{U}_{xx}-\zeta_1\theta_x-\zeta_2\mathcal{P}_x+\mu_1\varphi_t+\mu_2\varphi_t(x,t-\tau(t))=0,\\ c\theta_t+d\mathcal{P}_t-k\theta_{xx}-\zeta_1\varphi_{xt}=0,\\ d\theta_t+r\mathcal{P}_t-h\mathcal{P}_{xx}-\zeta_2\varphi_{xt}=0. \end{cases} \tag{0.19}$$

Where $(x,t)\in(0,1)\times(0,\infty)$ with the initial data and boundary conditions

$$\begin{cases} \mathcal{U}(x,0)=\mathcal{U}_0(x),\ \mathcal{U}_t(x,0)=\mathcal{U}_1(x), & x\in(0,1),\\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x), & x\in(0,1),\\ \theta(x,0)=\theta_0(x),\ \mathcal{P}(x,0)=\mathcal{P}_0(x), & x\in(0,1),\\ \varphi_t(x,t-\tau(0))=f_0(x,t-\tau(0)), & (x,t)\in(0,1)\times(0,\tau(0)),\\ \mathcal{U}(0,t)=\varphi(0,t)=\theta(0,t)=\mathcal{P}(0,t)=0, & \forall t\geq 0,\\ \mathcal{U}_x(1,t)=\varphi_x(1,t)=\theta(1,t)=\mathcal{P}(1,t)=0, & \forall t\geq 0. \end{cases} \tag{0.20}$$

Using the variable norm technique of T. Kato, we prove that the system is well posed under some hypothesis adopted. Then, using the energy method, which consists of establishing an appropriate Lyapunov functional based on the multiplier technique, we demonstrate that the considered dissipation in which we relied is strong enough to guarantee an exponential decay result, this result is obtained without the need for equal-speed.

CHAPTER 1

Preliminary and basic concepts

1.1 Definitions and Preliminary notions

1.1.1 Lebesgue and Sobolev spaces

In this chapter, we recall some fundamental mathematical concepts that form the groundwork for this thesis, including "Sobolev spaces, Green's formula, Lax-Milgram Theorem, m-dissipative operators, semigroups of bounded linear operators, and Hille-Yosida Theorem." We suppose that X is a Banach space, H is a Hilbert space and Ω is an open subset of $\mathbb{R}^n$ unless otherwise noted.

Definition 1.1. [3] For $1 \leq p < +\infty$, $L^p(\Omega)$ is the space of measurable fonctions whose p-th powers are integrable over Ω. Equipped with the norm:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

$L^p(\Omega)$ is a Banach space.
For $p = +\infty$, $L^\infty(\Omega)$ is the space of measurable functions f which are essentially bounded over Ω, that is, there exists a constant $C > 0$ such that $|f(x)| \leq C$ almost everywhere in Ω. Equipped with the norm:

$$\|f\|_{L^\infty(\Omega)} = inf \left\{ C \in \mathbb{R}^+ \ \text{ such that } \ |f(x)| \leq C \ \text{ a.e in } \ \Omega \right\}.$$

$L^\infty(\Omega)$ is a Banach space.

Definition 1.2. [3] We define by $L^2(\Omega)$ the space of measurable functions which are of square integrable in Ω. Equipped with the scalar product:

$$\langle f,g\rangle = \int_{\Omega} f(x)g(x)dx,$$

and the norm,

$$\|f\|_{L^2(\Omega)} = \left(\int_{\Omega} |f(x)|^2 \, dx\right)^{1/2}.$$

$L^2(\Omega)$ is a Hilbert space.

Notation: [3] Let $\alpha = (\alpha_i)_{i=1}^{n} \in \mathbb{N}^n$ be a multi-index, that is, a vector of n components which are non-negative integres $\alpha_i \geq 0$, we denote by $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and, for a function v,

$$D^{\alpha} v\,(x) = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}\,(x)\,.$$

Definition 1.3. [3] For every enteger $m \geq 0$, the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) \;\text{ such that }\; ,\forall\alpha \;\text{ with }\; |\alpha| \leq m, D^{\alpha} v \in L^p(\Omega)\}\,,$$

where the partial derivative $D^{\alpha}v$ is taken in the weak sense. Equipped with the norm:

$$\|v\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^{\alpha} v\|_p^p\right)^{1/p}.$$

$W^{m,p}(\Omega)$ is a Banach space.

Definition 1.4. [3] For an enteger $m \geq 0$, the Sobolev space $H^m(\Omega)$ is defined by:

$$H^m(\Omega) = \left\{v \in L^2(\Omega) \;\text{ such that, }\; \forall\alpha \;\text{ with }\; |\alpha| \leq m, D^{\alpha} v \in L^2(\Omega)\right\},$$

where the partial derivative $D^{\alpha}v$ is to be taken in the weak sens, Equipped with the scalar product:

$$\langle u,v\rangle = \int_{\Omega} \sum_{|\alpha|\leq m} D^{\alpha} u(x) D^{\alpha} v(x) dx$$

and the norm $\|v\|_{H^m(\Omega)} = \sqrt{\langle v,v\rangle}$. $H^m(\Omega)$ is a Hilbert space.

Definition 1.5. [3] The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \left\{v \in L^2(\Omega) \;\text{ such that }\; \forall i \in \{1,...,n\}, \frac{\partial v}{\partial x_i} \in L^2(\Omega)\right\},$$

where $\frac{\partial v}{\partial x_i}$ is the weak partial derivative of v. Equipped with the scalar product:

$$\langle u, v\rangle = \int_{\Omega} (u(x)v(x) + \nabla u(x)\nabla v(x))\, dx$$

and with the norm,

$$\|v\|_{H^1(\Omega)} = \left(\int_{\Omega} \left(|v(x)|^2 + |\nabla v(x)|^2\right) dx\right)^{1/2}.$$

The Sobolev space $H^1(\Omega)$ is a Hilbert space.

Let us now define another Sobolev space which is a subspace of $H^1(\Omega)$ and which will be very useful for problems with Dirichlet boundary conditions.

Definition 1.6. [3] Let $C_c^{\infty}(\Omega)$ be the space of functions of class C^{∞} with compact support in Ω. The Sobolev spase $H_0^1(\Omega)$ is defined as the closure of $C_c^{\infty}(\Omega)$ in $H^1(\Omega)$. Equipped with the scalar product of $H^1(\Omega)$. The Sobolev space $H_0^1\,(\Omega)$ is a Hilbert space.

Corollary 1.1. *[3] Let Ω be an open bounded regular subset $\mathbb{R}^n$, of class C^1. The space $H_0^1(\Omega)$ coincides with the subspace of $H^1(\Omega)$ composed of functions which are zero on the boundary $\partial\Omega$.*

1.1.2 Bochner Space

Definition 1.7. [19,72] Let (M, Σ, μ) be a measure space. A function $f : M \to X$ is called **Bochner-measurable** if there exists a sequence of simple functions $f_n : M \to X$ such that

$$f_n(x) \to f(x) \quad \text{in the norm of } X \text{ for almost every } x \in M.$$

Equivalently, f is Bochner-measurable if it is the pointwise limit (almost everywhere) of measurable simple functions.

Definition 1.8. [19,72] Let $1 \le p \le \infty$. The Bochner space $L^p(M; X)$ is defined as the set of all Bochner-measurable functions $f : M \to X$ such that:

- For $1 \le p < \infty$:

$$\|f\|_{L^p(M;X)} := \left(\int_M \|f(x)\|_X^p \, d\mu(x)\right)^{1/p} < \infty.$$

- For $p = \infty$:

$$\|f\|_{L^\infty(M;X)} := \text{ess sup}_{x\in M}\|f(x)\|_X < \infty.$$

Lemma 1.1. *[19,72] Let $1 \leq p \leq \infty$. The Bochner space $L^p(M; X)$ has the following properties:*

- ***Linearity:*** *$L^p(M; X)$ is a vector space.*
- ***Banach space:*** *for $1 \leq p \leq \infty$, the space $L^p(M; X)$ is a Banach space under the norm $\| \cdot \|_{L^p}$.*
- ***Hilbert Structure for $p = 2$:*** *If X is a Hilbert space, then $L^2(M; X)$ is also a Hilbert space with inner product*

$$(f, g)_{L^2(M;X)} := \int_M (f(x), g(x))_X \, d\mu(x).$$

where $(\cdot, \cdot)_X$ denotes the inner product in the Hilbert space X.

1.1.3 Some important inequalities

Theorem 1.1. *(Young's inequality) Let $a, b \geq 0$. For any $\epsilon > 0$ we have*

$$ab \leq \frac{a^2}{4\epsilon} + \epsilon b^2.$$

Theorem 1.2. *[19](Holder's inequality) Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ with $1 \leq p, q \leq +\infty$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then*

$$uv \in L^1(\Omega) \quad \text{and} \quad \int_\Omega |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} .$$

Theorem 1.3. *[19](Cauchy-Schwarz's inequality) We put $p = q = 2$ in Holder's inequality, we obtain the Cauchy-Schwarz inequality:*

$$\int_\Omega |uv| \, dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} .$$

Proposition 1.1. *[19](Poincaré's inequality) There exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} ,$$

for any $u \in H_0^1(\Omega)$.

Proposition 1.2. *[19](Poincaré–Wirtinger Inequality) There exists a constant $C > 0$ such that*

$$\|u - u_\Omega\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)},$$

where

$$u_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x) \, dx$$

for any $u \in H^1(\Omega)$.

Corollary 1.2. *[3] Let* Ω *be an open subset of* $\mathbb{R}^n$*, bounded in at least one space direction. Then the seminorm*

$$|v|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla v(x)|^2 \, dx \right)^{1/2}$$

is a norm over $H_0^1(\Omega)$ *which is equivalent to the usual norm of* $H^1(\Omega)$*.*

1.1.4 Green's formula

Theorem 1.4. *[3] (Green's formula) Let* Ω *be an open bounded regular subset* $\mathbb{R}^n$*, of class* C^1*. If* u *and* v *are functions of* $H^1(\Omega)$*, then,*

$$\int_\Omega u(x) \frac{\partial v}{\partial x_i}(x) dx = - \int_\Omega v(x) \frac{\partial u}{\partial x_i}(x) dx + \int_{\partial\Omega} u(x) v(x) \eta_i(x) ds,$$

where $\eta = (\eta_i)_{1 \leq i \leq n}$ *is the outward unit normal to* $\partial\Omega$*.*

1.1.5 Lax-Milgram Theorem

Definition 1.9. [19] A bilinear form $a : H \times H \to \mathbb{R}$ is said to be:

1. Continuous if there is a constant $C > 0$ such that:

$$|a(u, v)| \leq C \|u\|_H \|v\|_H, \quad \forall u, v \in H.$$

2. Coercive if there is a constant $\alpha > 0$ such that:

$$a(v, v) \geq \alpha \|v\|_H^2, \quad \forall v \in H.$$

Theorem 1.5. *[19] (Lax-Milgram) Assume that* $a(u, v)$ *is a continuous coercive bilinear form on* H*. Then, given any* $\phi \in H^*$ *(* H^* *is the topological duality of* H*), there exists a unique element* $u \in H$ *such that:*

$$a(u, v) = \langle \phi, v \rangle .$$

1.1.6 Operator basics

Definition 1.10. [29] Let $A : D(A) \subset X \to X$ be a linear operator. We say that A is closed if and only if $G(A) = \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$.

Lemma 1.2. *[29] Let $A : D(A) \subset X \to X$ be a linear operator. Then, A is closed if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \to x \in X$ and $Ax_n \to y \in X$ then, $x \in D(A)$ and $Ax = y$.*

1.1.7 m-dissipative operator

Definition 1.11. [29] An unbounded linear operator $A : D(A) \subset X \to X$ is said to be dissipative if it satisfies $\forall \lambda > 0, \forall x \in D(A)$:

$$\|(\lambda I - A)x\| \geq \lambda \|x\|.$$

Lemma 1.3. *[19] An unbounded linear operator $A : D(A) \subset H \to H$ is said to be dissipative if it satisfies*

$$(Av, v) \leq 0, \quad \forall v \in D(A).$$

Proposition 1.3. *[29] Let $A : D(A) \subset X \to X$, be a dissipative operator. Then, the following property holds:*

$$\lambda I - A \qquad is\ \ injective, \qquad \forall \lambda > 0.$$

Definition 1.12. [19] A unbounded linear operator $A : D(A) \subset X \to X$ is said to be m-dissipative if it satisfies,

1. A is dissipative,
2. $Im(\lambda I - A) = X$ for all $\lambda > 0$. i.e;

$$\forall \lambda > 0, \forall f \in X, \exists u \in D(A), \qquad \text{such that } \lambda u - Au = f.$$

Proposition 1.4. *[19] Let $A : D(A) \subset H \to H$ be a m-dissipative operator. Then,*

1. *$D(A)$ is dense in H,*
2. *A is a closed operator.*

1.1.8 Spectrum of operator

Definition 1.13. [29] Let $A : D(A) \subset X \to X$ be a closed linear operator. We call the following subset $\rho(A) = \left\{\lambda \in \mathbb{C} : (\lambda I - A) : D(A) \to X \text{ is bijective and} (\lambda I - A)^{-1} \text{ is bounded}\right\}$ by the resolvent set and its complement $\sigma(A) = \mathbb{C}/\rho(A)$ by the spectrum of A. For $\lambda \in \rho(A)$, the inverse

$$R(\lambda, A) = (\lambda I - A)^{-1},$$

is called the resolvent of A.

Definition 1.14. [60,76] Let $A : D(A) \subset X \to X$ be a linear operator and let λ be a complex number. The spectrum of A consists of three mutually exclusive parts:

1. The point spectrum $\sigma_p(A)$ defined as: $\lambda \in \sigma_p(A)$ such that $(\lambda I - A)$ does not admit of an inverse.

2. The residual spectrum $\sigma_r(A)$ defined as: $\lambda \in \sigma_r(A)$ such that $D\left((\lambda I - A)^{-1}\right)$ is not dense in X.

3. The continuous spectrum $\sigma_c(A)$ defined as: $\lambda \in \sigma_c(A)$ such that $D\left((\lambda I - A)^{-1}\right)$ is dense in X but $(\lambda I - A)^{-1}$ is not continuous.

1.2 Semigroups of bounded linear operators

Definition 1.15. [60] A one parameter family $\{T(t)\}_{t\geq 0}$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if and only if the following property hold:

$$T(0) = I,$$

$$T(t+s) = T(t)T(s), \qquad \forall t, s \geq 0.$$

Remark 1.1. When the second property is true for all $t, s \in \mathbb{R}$, then we say that $\{T(t)\}_{t\in\mathbb{R}}$ is a group.

1.2.1 Uniformly continuous semigroups

Definition 1.16. [60] A semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ is uniformly continuous if it satisfy,

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

1.2.2 Strongly continuous semigroups

Definition 1.17. [60] A semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X is a strongly continuous semigroup if it satisfy,

$$\lim_{t\to 0} T(t)\, x = x, \qquad \forall x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

Definition 1.18. [60] The linear operator A defined by:

$$D(A) = \left\{ x \in X, \quad \lim_{t \to 0} \frac{T(t)x - x}{t} \quad \text{exists} \right\},$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}, \quad \forall x \in D(A).$$

is called the infinitesimal generator of $\{T(t)\}_{t \geq 0}$.

Theorem 1.6. *[60] Let $\{T(t)\}_{t \geq 0}$ be a C_0 semigroup. Then, there exist two constants $\omega \geq 0$ and $M \geq 1$ such that,*

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall 0 \leq t < \infty.$$

Notation: We denote by $\mathcal{SG}(M, \omega)$ the set of C_0 semigroup $\{T(t)\}_{t \geq 0}$ for which there exists $\omega \geq 0$ and $M \geq 1$ such that,

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Definition 1.19. [60] Let $\{T(t)\}_{t \geq 0}$ be a C_0 semigroup. From Theorem 1.6 it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$.

1. If $\omega = 0$, then, $\{T(t)\}_{t \geq 0}$ is called uniformly bounded.
2. If $\omega = 0$ and $M = 1$, then, $\{T(t)\}_{t \geq 0}$ is called a C_0 semigroup of contractions.

Theorem 1.7. *[60] A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded operator.*

1.3 Hille-Yosida Theorem

Theorem 1.8. *[60](Hille-Yosida) A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of C_0 semigroup $\{T(t)\}_{t \geq 0} \in \mathcal{SG}(M, \omega)$ if and only if:*

1. *A is closed,*
2. *$\overline{D(A)} = X$,*
3. *The set $\Lambda_\omega = \{\lambda \in \mathbb{C}$ such that $Re(\lambda) > \omega\} \subset \rho(A)$ and for any $\lambda \in \Lambda_\omega$, we have,*

$$\|R(\lambda; A)^n\| \leq \frac{M}{(Re(\lambda) - \omega)^n}, \quad \forall n \in \mathbb{N}^*.$$

Theorem 1.9. *[60](Lumer phillips) Let $A : D(A) \subset X \to X$ be an unbounded linear operator of dense domain in X:*

1. *If A is dissipative and there exists a $\lambda_0 > 0$ such that $Im\ (\lambda_0 I - A) = X$ then, A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

2. *If A is the generator of a C_0 semigroupe of contractions on X then, A is dissipative and $Im\ (\lambda I - A) = X$ for all $\lambda > 0$.*

CHAPTER 2

Exponential stability of Lord Shulman thermoelastic system with porous damping and delay term

2.1 Introduction

In this chapter, we consider a one-dimensional Lord Shulman thermoelastic system with porous damping and time delay without microtemperature effect . We will established the well-posedness of the system and we prove the exponential stability result. Our problem that we will study in this chapter is written as follows

$$\begin{cases} \rho_1 \mathcal{U}_{tt} = \mu^* \mathcal{U}_{xx} + \mu_0 \mathcal{V}_x - \beta_0(\tau_1\theta_{xt} + \theta_x), & \text{in } (0,1)\times(0,\infty), \\ J\mathcal{V}_{tt} = a_0\mathcal{V}_{xx} - \mu_0\mathcal{U}_x - \xi\mathcal{V} + \beta_1(\tau\theta_t + \theta) & \\ \qquad\qquad -\zeta_1\mathcal{V}_t - \zeta_2\mathcal{V}_t(x,t-\tau), & \text{in } (0,1)\times(0,\infty), \\ a(\tau_1\theta_t + \theta)_t = -\beta_0\mathcal{U}_{xt} - \beta_1\mathcal{V}_t + \kappa\theta_{xx}, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{2.1}$$

Where $\mathcal{U}$ and $\mathcal{V}$ constitute the displacement of the solid elastic material and the volume fraction, respectively, θ is the temperature and τ_1 is the relaxation parameter, which is assumed to be small but strictly positive, ρ_1 is the mass density, J is the product of the mass density by the equilibrated inertia. The coefficients $\beta_0, \beta_1, \mu_0, \kappa, a_0$ denote respectively, the coupling between the displacement and the temperature, the coupling between the displacement and the volume fraction, the coupling between the displacement and the porosity, the thermal conductivity and the thermal capacity. The remaining parameters, $k_1, k_2, k_3, \xi, \mu^*$ and μ_2 define the characteristics of the material where

$$\mu_0^2 < \mu^* \xi,$$

including respectively the following initial and boundary conditions

$$\begin{cases} \mathcal{U}(x,0) = \mathcal{U}_0(x), & \mathcal{U}_t(x,0) = \mathcal{U}_1(x), \quad \mathcal{V}(x,0) = \mathcal{V}_0(x), \quad x \in (0,1), \\ \mathcal{V}_t(x,0) = \mathcal{V}_1(x), & \theta(x,0) = \theta_0(x), \qquad x \in (0,1), \\ \mathcal{V}_t(x,-t) = f_0(x,t), & \qquad x \in (0,1),\ t \in (0,\tau). \end{cases} \tag{2.2}$$

$$\begin{cases} \mathcal{U}_x(0,t) = \mathcal{U}_x(1,t) = \mathcal{V}(0,t) = \mathcal{V}(1,t) = 0, & t > 0, \\ \theta(0,t) = \theta(1,t) = 0, & t > 0. \end{cases} \tag{2.3}$$

Here $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{V}_0$, $\mathcal{V}_1$ end θ_0 are given functions, and f_0 is a history function. In the system, $\zeta_1 \mathcal{V}_t$ represents a porous term. The time delay is given by $\zeta_2 \mathcal{V}_t(x, t-\tau)$, where ζ_1, ζ_2, τ are positive constants such that

$$|\zeta_2| \leq \zeta_1. \tag{2.4}$$

This investigation aims to study the exponential stability of the system, independent of the wave speeds, in the presence of dissipations caused by thermal effects and porous damping. From the boundary conditions (2.1) and the first equation of (2.3), we obtain

$$\frac{d^2}{dt^2} \int_0^1 \mathcal{U}(x,t)dx = 0, \quad \forall t \geq 0, \tag{2.5}$$

end therefore

$$\int_0^1 \mathcal{U}(x,t)dx = t \int_0^1 \mathcal{U}_1(x)dx + \int_0^1 \mathcal{U}_0(x)dx, \quad \forall t \geq 0.$$

Consequently, if we set

$$\bar{\mathcal{U}}(x,t) = \mathcal{U}(x,t) - t \int_0^1 \mathcal{U}_1(x)dx - \int_0^1 \mathcal{U}_0(x)dx, \quad \forall t \geq 0,\ x \in [0,1],$$

we end up with

$$\int_0^1 \bar{\mathcal{U}}(x,t)dx = 0, \quad t \geq 0.$$

Poincaré's inequality can therefore be applied to $\bar{\mathcal{U}}$. Furthermore, the solutions to problems (2.1)-(2.3) can be found by performing a simple substitution, where the initial data for $\bar{\mathcal{U}}$ is given as

$$\bar{\mathcal{U}}_0(x) = \mathcal{U}_0(x) - \int_0^1 \mathcal{U}_0(x)dx \text{ and } \bar{\mathcal{U}}_1(x) = \mathcal{U}_1(x) - \int_0^1 \mathcal{U}_1(x)dx.$$

We shall use $\bar{\mathcal{U}}$ in the following, however for convenience, we write $\mathcal{U}$ instead of $\bar{\mathcal{U}}$. Throughout this work, the symbol c_0 denotes a general positive constant.

2.2 Well-posedness

In this section, we give an existence and uniqueness results for the system(2.1)-(2.3) using the semi-group theory see [60]. Motivated by [54, 69], we define the following new dependent variable:

$$\mathcal{Z}(x,\rho,t) = \mathcal{V}_t(x, t-\tau\rho), \text{ in } (0,1)\times(0,1)\times(0,\infty).$$

Then, the function $\mathcal{Z}$ satisfies

$$\tau\mathcal{Z}_t(x,\rho,t) + \mathcal{Z}_\rho(x,\rho,t) = 0, \text{ in } (0,1)\times(0,1)\times(0,\infty).$$

Thus, equations (2.1) are transformed to

$$\begin{cases} \rho_1 \mathcal{U}_{tt} = \mu^* \mathcal{U}_{xx} + \mu_0 \mathcal{V}_x - \beta_0(\tau_1\theta_{xt} + \theta_x), & \text{in } (0,1)\times(0,\infty), \\ J\mathcal{V}_{tt} = a_0\mathcal{V}_{xx} - \mu_0\mathcal{U}_x - \xi\mathcal{V} + \beta_1(\tau\theta_t + \theta) & \\ \qquad\qquad -\zeta_1\mathcal{V}_t - \zeta_2\mathcal{Z}(x,1,t), & \text{in } (0,1)\times(0,\infty), \\ a(\tau_1\theta_t + \theta)_t = -\beta_0\mathcal{U}_{tx} - \beta_1\mathcal{V}_t + \kappa\theta_{xx}, & \text{in } (0,1)\times(0,\infty), \\ \tau\mathcal{Z}_t(x,\rho,t) = -\mathcal{Z}_\rho(x,\rho,t), & \text{in } (0,1)\times(0,1)\times(0,\infty), \end{cases} \tag{2.6}$$

and the initial and boundary conditions are

$$\begin{cases} \mathcal{Z}(x,0,t) = \mathcal{V}_t(x,t), & x\in(0,1), t\in(0,1), \\ \mathcal{U}(x,0) = \mathcal{U}_0(x), \mathcal{U}_t(x,0) = \mathcal{U}_1(x), \mathcal{V}_t(x,0) = \mathcal{V}_1(x), & x\in(0,1), \\ \mathcal{V}(x,0) = \mathcal{V}_0(x), \theta(x,0) = \theta_0(x), & x\in(0,1), \\ \mathcal{Z}(x,\rho,0) = f_0(x,\tau\rho), & x\in(0,1), t\in(0,\tau), \end{cases}$$

$$\begin{cases} \mathcal{U}_x(0,t) = \mathcal{U}_x(1,t) = 0, \mathcal{V}(0,t) = \mathcal{V}(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = 0, \qquad\qquad t > 0. \end{cases} \tag{2.7}$$

Introducing the vector function $\mathscr{U} = (\mathcal{U}, \mathcal{W}, \mathcal{V}, \mathcal{Q}, \theta, \mathcal{S}, \mathcal{Z})^t$ where $\mathcal{W} = \mathcal{U}_t$, $\mathcal{Q} = \mathcal{V}_t$ and $\mathcal{S} = \theta_t$ then, system (2.6)-(2.7) can be rewritten as follow:

$$\begin{cases} \mathscr{U}_t = \mathscr{A}\mathscr{U} \quad t > 0, \\ \mathscr{U}(0) = \mathscr{U}_0 = (\mathcal{U}_0, \mathcal{U}_1, \mathcal{V}_0, \mathcal{V}_1, \theta_0, \theta_1, f_0), \end{cases} \tag{2.8}$$

where the operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} \longrightarrow \mathscr{H}$ is defined by

$$\mathscr{A}\mathscr{U} = \begin{pmatrix} \mathcal{W} \\ \frac{\mu^*}{\rho_1}\mathcal{U}_{xx} + \frac{\mu_0}{\rho_1}\mathcal{V}_x - \frac{\beta_0}{\rho_1}(\tau_1\theta_{tx} + \theta_x) \\ \mathcal{Q} \\ \frac{a_0}{J}\mathcal{V}_{xx} - \frac{\mu_0}{J}\mathcal{U}_x - \frac{\xi}{J}\mathcal{V} + \frac{\beta_1}{J}(\tau_1\theta_t + \theta) - \frac{\zeta_1}{J}\mathcal{V}_t - \frac{\zeta_2}{J}\mathcal{Z}(x,1,t) \\ \mathcal{S} \\ \frac{-1}{\tau_1}\theta_t - \frac{\beta_0}{a\tau_1}\mathcal{U}_{tx} - \frac{\beta_1}{a\tau_1}\mathcal{V}_t + \frac{\kappa}{a\tau_1}\theta_{xx} \\ \frac{-1}{\tau}\mathcal{Z}_\rho(x,\rho,t) \end{pmatrix}. \tag{2.9}$$

We consider the following spaces:

$$\begin{aligned} L^2_*(0,1) &= \left\{\varpi \in L^2(0,1),\ \int_0^1 \varpi(x)dx = 0\right\}, \\ H^2_*(0,1) &= \left\{\varpi \in H^2(0,1),\ \varpi_x(0) = \varpi_x(1) = 0\right\}, \\ H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \end{aligned}$$

and the Hilbert space
$\mathscr{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2((0,1);L^2(0,1))$ equipped with the inner product

$$\begin{aligned} \left\langle \mathscr{U}, \widetilde{\mathscr{U}} \right\rangle_{\mathscr{H}} : &= \rho_1 \int_0^1 \mathcal{W}\widetilde{\mathcal{W}}dx + J\int_0^1 \mathcal{Q}\widetilde{\mathcal{Q}}dx + \mu_0 \int_0^1 \left(\mathcal{U}_x\widetilde{\mathcal{V}} + \mathcal{V}\widetilde{\mathcal{U}}_x\right)dx \\ &+\xi\int_0^1 \mathcal{V}\widetilde{\mathcal{V}}dx + a_0\int_0^1 \mathcal{V}_x\widetilde{\mathcal{V}}_xdx + a\int_0^1 (\theta + \tau_1\mathcal{S})\left(\widetilde{\theta} + \tau_1\widetilde{\mathcal{S}}\right)dx \\ &+\mu^*\int_0^1 \mathcal{U}_x\widetilde{\mathcal{U}}_xdx + \tau_1\kappa\int_0^1 \theta_x\widetilde{\theta}_xdx + \tau|\zeta_2|\int_0^1\int_0^x \mathcal{Z}\widetilde{\mathcal{Z}}dx, \end{aligned} \tag{2.10}$$

for $\mathscr{U} = (\mathcal{U}, \mathcal{W}, \mathcal{V}, \mathcal{Q}, \theta, \mathcal{S}, \mathcal{Z})^t \in \mathscr{H}$ and $\widetilde{\mathscr{U}} = \left(\widetilde{\mathcal{U}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{Q}}, \widetilde{\theta}, \widetilde{\mathcal{S}}, \widetilde{\mathcal{Z}}\right)^t \in \mathscr{H}$. The domain of $\mathscr{A}$ is given by

$$\mathscr{D}(\mathscr{A}) = \left(\begin{array}{ll} \mathscr{U} \in \mathscr{H} \backslash & \mathcal{U} \in H^1_*(0,1) \cap H^2_*(0,1)\ ,\ \mathcal{W} \in H^1_0(0,1) \cap H^2(0,1), \\ & \mathcal{V} \in H^1_0(0,1) \cap H^2(0,1)\ ,\ \mathcal{Q} \in H^1_0(0,1), \\ & \theta \in H^1_0(0,1) \cap H^2(0,1)\ ,\ \mathcal{S} \in H^1_0(0,1), \\ & \mathcal{Z}, \mathcal{Z}_\rho \in L^2((0,1), L^2(0,1)),\ \ \mathcal{Z}(x,0) = \mathcal{Q}(x). \end{array} \right). \tag{2.11}$$

Clearly, the domain $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$ We have the following existence and uniqueness the result:

Theorem 2.1. *[60] Let $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, Problem (2.8) has a unique solution $\mathscr{U} \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H})$ Moreover if $\mathscr{U}_0 \in \mathscr{H}$, then the solution $\mathscr{U} \in C(\mathbb{R}_+, \mathscr{H})$.*

Proof. If $\mathscr{A}$ is a dissipative operator, the Lumer-Phillips theorem yields the desired conclusion. In the following, we establish the dissipative nature of A. Using the inner product and for any $\mathscr{U} \in \mathscr{D}(\mathscr{A})$, we get

$$\begin{aligned} \langle \mathscr{A}\mathscr{U}, \mathscr{U} \rangle_{\mathscr{H}} &= -\kappa \int_0^1 \theta_x^2 dx - \zeta_1 \int_0^1 \mathcal{V}_t^2 dx - \frac{|\zeta_2|}{2} \int_0^1 \mathcal{Z}^2(x,1,t) dx \\ &+ \frac{|\zeta_2|}{2} \int_0^1 \mathcal{V}_t^2 dx - \zeta_2 \int_0^1 \mathcal{Z}(x,1,t) \mathcal{V}_t dx. \end{aligned} \tag{2.12}$$

Using Young's inequality, the last term in (2.12), we have

$$-\zeta_2 \int_0^1 \mathcal{Z}(x,1,t)\mathcal{V}_t dx \leq \frac{|\zeta_2|}{2} \int_0^1 \mathcal{Z}^2(x,1,t) dx + \frac{|\zeta_2|}{2} \int_0^1 \mathcal{V}_t^2 dx. \tag{2.13}$$

Substituting (2.13) in (2.12) yields

$$\langle \mathscr{A}\mathscr{U}, \mathscr{U} \rangle_{\mathscr{H}} \leq -\kappa \int_0^1 \theta_x^2 dx - (\zeta_1 - |\zeta_2|) \int_0^1 \mathcal{V}_t^2 dx.$$

It follows that $\langle \mathscr{A}\mathscr{U}, \mathscr{U} \rangle_{\mathscr{H}} \leq 0$, under assumption (2.4), which implies that $\mathscr{A}$ is dissipative. Next we prove that the operator $I - \mathscr{A}$ is surjective. Given $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^t \in \mathscr{H}$, we prove that there exists $\mathscr{U} \in \mathscr{D}(\mathscr{A})$ satisfying

$$\mathscr{U} + \mathscr{A}(\mathscr{U}) = G; \tag{2.14}$$

that is,

$$
\begin{cases}
\mathcal{U}-\mathcal{W}=g_1\in H^1_*(0,1),\\
\rho_1\mathcal{W}-\mu_*\mathcal{U}_{xx}-\mu_0\mathcal{V}_x+\beta_0\left(\tau_1\mathcal{S}_x+\theta_x\right)=\rho_1 g_2\in L^2_*(0,1),\\
\mathcal{V}-\mathcal{Q}=g_3\in H^1_0(0,1),\\
J\mathcal{Q}-a_0\mathcal{V}_{xx}+\mu_0\mathcal{U}_x+\xi\mathcal{V}-\beta_1\left(\tau_1\mathcal{S}+\theta\right)+\zeta_1\mathcal{Q}+\zeta_2\mathcal{Z}\left(x,1,t\right)=Jg_4\in L^2(0,1),\\
\theta-\mathcal{S}=g_5\in H^1_0(0,1),\\
a\tau_1\mathcal{S}+a\mathcal{S}+\beta_0\mathcal{W}_x+\beta_1\mathcal{Q}-\kappa\theta_{xx}=a\tau_1 g_6\in L^2(0,1),\\
\mathcal{Z}_\rho+\tau\mathcal{Z}=\tau g_7\in L^2\left((0,1);L^2\left(0,1\right)\right).
\end{cases}
\tag{2.15}
$$

We note that the last equation in (2.15) with $\mathcal{Z}\left(x,0,t\right)=\mathcal{Q}$ has a unique solution.

$$
\mathcal{Z}\left(x,\rho,t\right)=\mathcal{Q}(x)e^{-\tau\rho}-e^{-\tau\rho}g_3(x)+\tau e^{-\tau\rho}\int_0^\rho e^{\tau s}g_7(x,s)ds.
$$

Inserting $\mathcal{W}=\mathcal{U}-g_1, \mathcal{Q}=\mathcal{V}-g_3,\ \mathcal{S}=\theta-g_5$ in $(2.15)_2$, $(2.15)_4$, $(2.15)_6$, we obtain

$$
\begin{cases}
\rho_1\mathcal{U}-\mu^*\mathcal{U}_{xx}-\mu_0\mathcal{V}_x+\beta_0\left(\tau_1+1\right)\theta_x=\hbar_1\in L^2_*(0,1),\\
\eta\mathcal{V}-a_0\mathcal{V}_{xx}+\mu_0 u_x-\beta_1\left(\tau_1+1\right)\theta=\hbar_2\in L^2(0,1),\\
a\left(\tau_1+1\right)\theta+\beta_0\mathcal{U}_x+\beta_1\mathcal{V}-\kappa\theta_{xx}=\hbar_3\in L^2(0,1),
\end{cases}
\tag{2.16}
$$

where

$$
\eta=\left(J+\xi+\zeta_1+\zeta_2 e^{-\tau}\right),
$$

$$
\hbar_1=\rho_1\left(g_1+g_2\right)+\tau_1\beta_0 g_{5x},
$$

$$
\hbar_2=\left(J-\zeta_1+\zeta_2 e^{-\tau}\right)g_3-\beta_1\tau_1 g_5+Jg_4-\zeta_2\tau e^{-\tau}\int_0^1 e^s g_7\left(x,s\right)ds,
$$

$$
\hbar_3=a\tau_1\left(g_5+g_6\right)+\beta_0 g_{1x}+\beta_1 g_3+ag_5.
$$

To solve (2.16) we consider

$$
\mathcal{B}\left(\left(\mathcal{U},\mathcal{V},\theta\right),\left(\tilde{\mathrm{u}},\widetilde{\mathcal{V}},\widetilde{\theta}\right)\right)=\mathcal{L}\left(\widetilde{\mathcal{U}},\widetilde{\mathcal{V}},\widetilde{\theta}\right),
\tag{2.17}
$$

where $\mathcal{B} : \left[H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form

$$\begin{aligned}
\mathcal{B}\left((\mathcal{U},\mathcal{V},\theta),\left(\widetilde{\mathrm{u}},\widetilde{\mathcal{V}},\widetilde{\theta}\right)\right) &= \rho_1\int_0^1 \mathcal{U}\widetilde{\mathcal{U}}dx + \eta\int_0^1 \mathcal{V}\widetilde{\mathcal{V}}dx + a(\tau_1+1)^2\int_0^1 \theta\widetilde{\theta}dx \\
&+\mu*\int_0^1 \mathcal{U}_x\widetilde{\mathcal{U}_x}dx + a_0\int_0^1 \mathcal{V}_x\widetilde{\mathcal{V}_x}dx + \kappa(\tau_1+1)\int_0^1 \theta_x\widetilde{\theta}_xdx \\
&+\mu_0\int_0^1\left(\mathcal{V}\widetilde{\mathcal{U}_x} + \mathcal{U}_x\widetilde{\mathcal{V}}\right)dx + \beta_0(\tau_1+1)\int_0^1\left(\mathcal{U}_x\widetilde{\theta} - \theta\widetilde{\mathcal{U}_x}\right)dx \\
&+\beta_1(\tau_1+1)\int_0^1\left(\mathcal{V}\widetilde{\theta} - \theta\widetilde{\mathcal{V}}\right)dx,
\end{aligned}$$

and $\mathcal{L} : \left[H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)\right] \longrightarrow \mathbb{R}$ is the linear form

$$\mathcal{L}\left(\widetilde{\mathcal{U}},\widetilde{\mathcal{V}},\widetilde{\theta}\right) = \int_0^1 \hbar_1\widetilde{\mathcal{U}}dx + \int_0^1 \hbar_2\widetilde{\mathcal{V}}dx + (\tau+1)\int_0^1 \hbar_3\widetilde{\theta}dx.$$

Now, for $\mathscr{H}_* = H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$ equipped with the norm

$$\|(\mathcal{U},\mathcal{V},\theta)\|_{\mathscr{H}_*}^2 = \|\mathcal{U}\|_2^2 + \|\mathcal{U}_x\|_2^2 + \|\mathcal{V}\|_2^2 + \|\theta_x\|_2^2 + \|\theta\|_2^2.$$

One can easily see that $\mathcal{B}$ and $\mathcal{L}$ are bonded. Furthermore, using integration by parts, we obtain

$$\begin{aligned}
\mathcal{B}\left((\mathcal{U},\mathcal{V},\theta),(\mathcal{U},\mathcal{V},\theta)\right) &= \rho_1\int_0^1 \mathcal{U}^2dx + a(\tau_1+1)^2\int_0^1 \theta^2dx + a_0\int_0^1 \mathcal{V}_x^2dx \\
&+k(\tau_1+1)\int_0^1 \theta_x^2dx + \frac{1}{2}\int_0^1\left[\mu^*\left(\mathcal{U}_x + \frac{\mu_0}{\mu_1}\mathcal{V}\right)^2\right. \\
&\left.+\eta\left(\mathcal{V} + \frac{\mu_0}{\eta}\mathcal{U}_x\right)^2 + \left(\mu^* - \frac{\mu_0^2}{\eta}\right)\mathrm{u}_x^2 + \left(\eta - \frac{\mu_0^2}{\mu^*}\right)\mathcal{V}^2\right]dx \\
&\geq \alpha\|(\mathcal{U},\mathcal{V},\theta)\|_{\mathscr{H}_*}^2.
\end{aligned}$$

Where

$$\begin{aligned}
\mu^*\mathcal{U}_x^2 + \eta\mathcal{V}^2 + 2\mu_0\mathcal{U}_x\mathcal{V} &= \frac{1}{2}\int_0^1\left[\mu^*\left(\mathcal{U}_x + \frac{\mu_0}{\mu_1}\mathcal{V}\right)^2 + \eta\left(\mathcal{V} + \frac{\mu_0}{\eta}\mathcal{U}_x\right)^2\right. \\
&\left.+\left(\mu^* - \frac{\mu_0^2}{\eta}\right)\mathcal{U}_x^2 + \left(\eta - \frac{\mu_0^2}{\mu^*}\right)\mathcal{V}^2\right]dx,
\end{aligned}$$

and using the fact $\mu_0^2 < \mu^*\xi$, we get

$$\mu^*\mathcal{U}_x^2 + \eta\mathcal{V}^2 + 2\mu_0\mathcal{U}_x\mathcal{V} > 0.$$

Next, for certain $\alpha > 0$. $\mathcal{B}$ is hence coercive. Thus, system (2.16) under Lax-Milgram Lemma, has a unique solution

$$\mathcal{U} \in H_*^1(0,1), \quad \mathcal{V} \in H_0^1(0,1), \quad \theta \in H_0^1(0,1).$$

Substituting $\mathcal{U}, \mathcal{V}$, and θ in $(2.15)_1$,$(2.15)_3$,$(2.15)_5$and $\mathcal{Q}$ into $(2.15)_7$ respectively, we obtain

$$\mathcal{W} \in H_*^1(0,1), \mathcal{Q} \in H_0^1(0,1), \mathcal{S} \in H_0^1(0,1) \text{ and } \mathcal{Z}, \mathcal{Z}_\rho \in L^2\left((0,1), L^2(0,1)\right).$$

Now, if $\left(\widetilde{\mathcal{V}}, \widetilde{\theta}\right) = (0,0) \in H_0^1(0,1) \times H_0^1(0,1)$ then (2.17) reduces to

$$\rho_1 \int_0^1 \mathcal{U}\widetilde{\mathcal{U}}dx + \mu^* \int_0^1 \mathcal{U}_x\widetilde{\mathcal{U}}_x dx + \mu_0 \int_0^1 \mathcal{V}\widetilde{\mathcal{U}}_x dx - \beta_0(\tau+1)\int_0^1 \theta\widetilde{\mathcal{U}}_x dx = \int_0^1 \hbar_1\widetilde{\mathcal{U}}dx, \quad \forall \widetilde{\mathcal{U}} \in H_*^1(0,1), \tag{2.18}$$

which implies

$$\mu^* \mathcal{U}_{xx} = -\mu_0\mathcal{V}_x - \beta_0(\tau+1)\theta + \rho_1\mathcal{U} - \hbar_1 \in L^2(0,1). \tag{2.19}$$

Consequently, by the elliptic regularity theory, it follow that

$$\mathcal{U} \in H^2(0,1) \cap H_*^1(0,1).$$

Moreover (2.18) is also true for any $\mathcal{Q} \in C^1([0,1]) \subset H_*^1(0,1)$. Hence we have

$$\mu^* \int_0^1 \mathcal{U}_x \mathcal{Q} dx + \int_0^1 (\mu_0\mathcal{V}_x + \beta_0(\tau+1)\theta - \rho_1\mathcal{U} + \hbar_1)\mathcal{Q}dx = 0,$$

for all $\mathcal{Q} \in C^1([0,1])$. Thus, using integration par part and bearing mind (2.19), we obtain

$$\mathcal{U}_x(1)\mathcal{Q}(1) - \mathcal{U}_x(0)\mathcal{Q}(0) = 0, \quad \forall \mathcal{Q} \in C^1([0,1]).$$

Therefor, $\mathcal{U}_x(1) = \mathcal{U}_x(0) = 0$. Consequently, we obtain

$$\mathcal{U} \in H_*^2(0,1) \cap H_*^1(0,1).$$

Similarly, we obtain

$$\mathcal{V}_{xx} = \mu_0\mathcal{U}_x + \eta\mathcal{V} + \beta_1(\tau_1+1)\theta - \hbar_2 \in L^2(0,1),$$

$$\kappa\theta_{xx} = \beta_0\mathcal{U}_x + a(\tau_1+1)\theta - \beta_1\mathcal{V} - \hbar_3 \in L^2(0,1),$$

thus, we have

$$\mathcal{V},\theta\in H^2(0,1)\cap H_0^1(0,1).$$

Lastly, the existence of a unique $\mathscr{U}\in\mathscr{D}(\mathscr{A})$ such that (2.14) is satisfied is ensured by using regularity theory to the linear elliptic equation. As a result, A is a dissipative operator. Therefore, the outcome of The Lumer-Phillips theorem (see [48, 60]) yields Theorem 2.1. □

2.3 Exponential stability

In this section, we demonstrate the exponential stability of system (2.6)-(2.7) using the energy approach.

Theorem 2.2. *Let $(\mathcal{U},\mathcal{V},\theta,\mathcal{Z})$ be a solution of (2.6)-(2.7). Then there exist positive constants a_1,a_2 such that*

$$\mathcal{E}(t)\le a_1e^{-a_2t},\quad \forall t\ge 0. \tag{2.20}$$

Initially, we establish and validate a few technical lemmas required for our result's proof.

Lemma 2.1. *Let $(\mathcal{U},\mathcal{V},\theta,\mathcal{Z})$ be a solution of (2.6)-(2.7). Then, the energy functional $\mathcal{E}(t)$, defined by*

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int_0^1\Big(\rho_1\mathcal{U}_t^2+J\mathcal{V}_t^2+a_0\mathcal{V}_x^2+a(\tau_1\theta_t+\theta)^2+\tau_1\kappa\theta_x^2\\ &\quad+\left(\xi-\frac{\mu_0}{\mu^*}\right)\mathcal{V}^2+\mu^*\left(\frac{\mu_0}{\mu^*}\mathcal{V}+\mathcal{U}_x\right)^2+\frac{\tau|\zeta_2|}{2}\int_0^1 \mathcal{Z}^2(x,\rho,t)\,d\rho\Big)dx,\end{aligned}$$

satisfies

$$\mathcal{E}'(t)\le-\kappa\int_0^1\theta_x^2dx-(\zeta_1-|\zeta_2|)\int_0^1\mathcal{V}_t^2dx. \tag{2.21}$$

Proof. By multiplying $(2.6)_1$, $(2.6)_2$, $(2.6)_3$ by $\mathcal{U}_t$, $\mathcal{V}_t$ and $(\tau_1\theta_t+\theta)$ respectively, and integrating over $(0,1)$ with the boundary condition and integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Big(\rho_1\mathcal{U}_t^2+J\mathcal{V}_t^2+a_0\mathcal{V}_x^2+a(\tau_1\theta_t+\theta)^2\\ &\qquad+\tau_1\kappa\theta_x^2+\left(\xi-\tfrac{\mu_0^2}{\mu^*}\right)\mathcal{V}^2+\mu^*\left(\tfrac{\mu_0}{\mu^*}\mathcal{V}+\mathcal{U}_x\right)^2\Big)dx\\ &\quad=-\zeta_1\int_0^1\mathcal{V}_t^2dx-\zeta_2\int_0^1\mathcal{Z}(x,1,t)\,\mathcal{V}_tdx-\kappa\int_0^1\theta_x^2dx.\end{aligned} \tag{2.22}$$

Now, multiplying the $(2.6)_4$ by $|\zeta_2|\mathcal{Z}$ and integrating over $(0,1)\times(0,1)$, we obtain

$$\frac{\tau|\zeta_2|}{2}\frac{d}{dt}\int_0^1\int_0^1\mathcal{Z}^2(x,\rho,t)\,d\rho dx=\frac{|\zeta_2|}{2}\int_0^1\mathcal{V}_t^2dx-\frac{|\zeta_2|}{2}\int_0^1\mathcal{Z}^2(x,1,t)\,dx. \tag{2.23}$$

A combination of (2.22) and (2.23) give as

$$\begin{aligned} \mathcal{E}'(t) &= -\zeta_1 \int_0^1 \mathcal{V}_t^2 dx - \kappa \int_0^1 \theta_x^2 dx + \frac{|\zeta_2|}{2} \int_0^1 \mathcal{V}_t^2 dx \\ &\quad -\frac{|\zeta_2|}{2} \int_0^1 \mathcal{Z}^2(x,1,t)\, dx - \zeta_2 \int_0^1 \mathcal{Z}(x,1,t)\, \mathcal{V}_t dx. \end{aligned} \tag{2.24}$$

Meanwhile, we have by applying Young's inequality

$$-\zeta_2 \int_0^1 \mathcal{Z}(x,1,t)\, \mathcal{V}_t dx \le \frac{\zeta_2}{2} \int_0^1 \mathcal{Z}^2(x,1,t)\, dx + \frac{\zeta_2}{2} \int_0^1 \mathcal{V}_t^2 dx. \tag{2.25}$$

Sample substitution of (2.25) into (2.24) and using $|\zeta_2| < \zeta_1$ give (2.21), which concludes the proof. □

Lemma 2.2. *Let* $(\mathcal{U}, \mathcal{V}, \theta, \mathcal{Z})$ *be a solution of (2.6)-(2.7). Then the functional*

$$\mathcal{F}_1(t) := \rho_1 \int_0^1 \mathcal{U}_t \mathcal{U} dx - \beta_0 \tau_1 \int_0^1 \theta\, \mathcal{U}_x dx, \quad t \ge 0,$$

satisfies, the estimate

$$\mathcal{F}_1'(t) \le -\frac{\mu^*}{2} \int_0^1 \mathcal{U}_x^2 dx + c_0 \int_0^1 \left(\theta_x^2 + \mathcal{U}_t^2 + \mathcal{V}^2\right) dx, \quad t \ge 0. \tag{2.26}$$

Proof. Taking the derivation of $\mathcal{F}_1$, using $(2.2)_1$ and integration by parts, we get

$$\mathcal{F}_1'(t) = -\mu^* \int_0^1 \mathcal{U}_x^2 dx + \rho_1 \int_0^1 \mathcal{U}_t^2 dx - \mu_0 \int_0^1 \mathcal{V}\mathcal{U}_x dx - \beta_0 \int_0^1 \theta_x \mathcal{U} dx + \beta_0 \tau_1 \int_0^1 \mathcal{U}_t \theta_x dx.$$

Estimate (2.26) is established by applying Young's and Poincaré inequalities. □

Lemma 2.3. *Let* $(\mathcal{U}, \mathcal{V}, \theta, \mathcal{Z})$ *be a solution of (2.6)-(2.7). Then the functional*

$$\mathcal{F}_2(t) := J \int_0^1 \mathcal{V}_t \mathcal{V} dx - \frac{\beta}{2} \int_0^1 \mathcal{V}^2 dx - \frac{\mu_0 \rho_1}{\mu^*} \int_0^1 \mathcal{U}_t \left(\int_0^x \mathcal{V}(y) dy\right) dx + \frac{\zeta_1}{2} \int_0^1 \mathcal{V}^2 dx,$$

satisfies, for any $\varepsilon_1 \ge 0$, *the estimate*

$$\begin{aligned} \mathcal{F}_2'(t) &\le -a_0 \int_0^1 \mathcal{V}_x^2 dx - \frac{\mu_1}{2} \int_0^1 \mathcal{V}^2 dx + \varepsilon_1 \int_0^1 \mathcal{U}_t^2 dx + c_0 \int_0^1 (\tau_1 \theta_t + \theta)^2 dx \\ &\quad c_0 \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \mathcal{V}_t^2 dx + \frac{\zeta_2^2}{2\mu_1} \int_0^1 \mathcal{Z}^2(x,1,t)\, dx, \end{aligned} \tag{2.27}$$

where $\mu_1 = \left(\xi - \frac{\mu_0^2}{\mu^*}\right)$.

Proof. A simple differentiation of $\mathcal{F}_2$, using (2.2)$_2$ and integration by parts, we obtain

$$\begin{aligned}\mathcal{F}_2'(t) &= -a_0\int_0^1 \mathcal{V}_x^2dx-\left(\xi-\frac{\mu_0^2}{\mu^*}\right)\int_0^1 \mathcal{V}^2dx+J\int_0^1 \mathcal{V}_t^2dx+\left(\beta_1-\frac{\beta_0\mu_0}{\mu^*}\right)\int_0^1(\tau_1\theta_t+\theta)\,\mathcal{V}dx\\ &\quad -\frac{\mu_0\rho_1}{\mu^*}\int_0^1 \mathcal{U}_t\left(\int_0^x \mathcal{V}_t(y)dy\right)dx-\zeta_2\int_0^1 z\,(x,1,t)\,\mathcal{V}dx+\beta\int_0^1 \mathcal{V}_t\mathcal{V}dx.\end{aligned}$$

Estimate (2.27) is obtained by using Young's and Poincaré inequalities with $\varepsilon_1 \geq 0$. □

Lemma 2.4. *Let $(\mathcal{U},\mathcal{V},\theta,z)$ be a solution of (2.6)-(2.7). Then the functional*

$$\mathcal{F}_3(t) := -a\int_0^1 \tau_1^2\theta_t\theta dx-\frac{a\tau_1}{2}\int_0^1\theta^2dx,$$

satisfies, for any $\varepsilon_2 \geq 0$, the estimate

$$\mathcal{F}_3'(t)\leq \frac{-a}{2}\int_0^1(\tau_1\theta_t+\theta)^2dx+\varepsilon_2\int_0^1 \mathcal{U}_t^2dx+c_0\int_0^1 \mathcal{V}_t^2dx+c_0\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1\theta_x^2dx. \tag{2.28}$$

Proof. Using the third equation in (2.6), differentiation of $\mathcal{F}_3$ and integration by parts, we obtain

$$\mathcal{F}_3'(t) = -a\tau_1^2\int_0^1\theta_t^2dx-\beta_0\tau_1\int_0^1 \mathcal{U}_t\theta_xdx+\beta_1\tau_1\int_0^1 \mathcal{V}_t\theta dx+\kappa\tau_1\int_0^1\theta_x^2dx. \tag{2.29}$$

Utilizing the Poincaré and Young's inequalities with $\varepsilon_2 \geq 0$, and the fact that

$$-\int_0^1(\tau_1\theta_t)^2dx\leq -\frac{1}{2}\int_0^1(\tau_1\theta_t+\theta)^2dx+\int_0^1\theta_x^2dx,$$

we obtain estimate (2.28). □

Lemma 2.5. *Let $(\mathcal{U},\mathcal{V},\theta,z)$ be a solution of (2.6)-(2.7). Then the functional*

$$\mathcal{F}_4(t) := \rho_1 a\int_0^1\left(\int_0^x \mathcal{U}_tdy\right)(\tau_1\theta_t+\theta)\,dx, \tag{2.30}$$

satisfies the estimate for each $\varepsilon_3 \geq 0$

$$\begin{aligned}\mathcal{F}_4'(t) &\leq \frac{-\beta_0\rho_1}{2}\int_0^1 \mathcal{U}_t^2dx+\varepsilon_3\int_0^1 \mathcal{U}_x^2dx+c_0\int_0^1 \mathcal{V}^2dx+c_0\int_0^1 \mathcal{V}_t^2dx\\ &\quad +c_0\int_0^1\theta_x^2dx+c_0\left(1+\frac{1}{\varepsilon_3}\right)\int_0^1(\tau_1\theta_t+\theta)^2dx.\end{aligned} \tag{2.31}$$

Proof. Direct computations using integration by parts and the fact that $\int_0^1 \mathcal{U}dy = 0$, we get

$$\begin{aligned}\mathcal{F}_4'(t) &= -\beta_0\rho_1 \int_0^1 \mathcal{U}_t^2 dx + \beta_0 a \int_0^1 (\tau_1\theta_t + \theta)^2 dx + \kappa\rho_1 \int_0^1 \mathcal{U}_t\theta_x dx \\ &\quad -a\mu^* \int_0^1 \mathcal{U}_x (\tau_1\theta_t + \theta) dx - \mu_0 a \int_0^1 \mathcal{V}(\tau_1\theta_t + \theta) dx + \beta_1\rho_1 \int_0^1 \left(\int_0^x \mathcal{U}_t dy\right) \mathcal{V}_t dx.\end{aligned}$$

Estimate (2.31) is obtained by using Young's and Poincaré inequalities with $\varepsilon_3 \geq 0$. □

Lemma 2.6. *Let $(\mathcal{U}, \mathcal{V}, \theta, \mathcal{Z})$ be a solution of (2.6)-(2.7). Then the functional*

$$\mathcal{F}_5'(t) := \tau \int_0^1 \int_0^1 e^{-\tau\rho} \mathcal{Z}^2(x,\rho,t)\, d\rho dx.$$

The following estimate is satisfied for some positive constant m

$$\mathcal{F}_5'(t) \leq -m\left(\int_0^1 \mathcal{Z}^2(x,1,t)\, dx + \tau \int_0^1 \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho dx\right) + \int_0^1 |\mathcal{V}_t|^2 dx. \tag{2.32}$$

Proof. By differentiating $\mathcal{F}_5$ and using the fourth equation in (2.6), we obtain

$$\begin{aligned}\mathcal{F}_5'(t) &= -2\int_0^1 \int_0^1 e^{-\tau\rho} \mathcal{Z}(x,\rho,t)\, \mathcal{Z}_\rho(x,\rho,t)\, d\rho dx \\ &= -\frac{d}{d\rho}\int_0^1 \int_0^1 e^{-\tau\rho} \mathcal{Z}^2(x,\rho,t)\, d\rho dx - \tau \int_0^1 \int_0^1 e^{-\tau\rho} \mathcal{Z}^2(x,\rho,t)\, d\rho dx \\ &= -\int_0^1 \left(e^{-\tau}\mathcal{Z}^2(x,1,t) - \mathcal{Z}^2(x,0,t)\right) dx - \tau \int_0^1 \int_0^1 e^{-\tau\rho}\mathcal{Z}^2(x,\rho,t)\, d\rho dx.\end{aligned}$$

By using the fact that $\mathcal{Z}(x,0,t) = \mathcal{V}_t(x,t)$ and $e^{-\tau} \leq e^{-\tau\rho} \leq 1, \ \forall\rho \in [0,1]$, we get

$$\mathcal{F}_5'(t) = e^{-\tau}\left(\int_0^1 \mathcal{Z}^2(x,1,t)\, dx + \tau \int_0^1 \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho dx\right) + \int_0^1 |\mathcal{V}_t|^2 dx,$$

setting $m = e^{-\tau}$ yields (2.32). □

We can now state and demonstrate the following result on exponential stability.

Lemma 2.7. *for N sufficiently large, the functional define by*

$$\ell(t) := N\mathcal{E}(t) + N_1\mathcal{F}_1(t) + N_2\mathcal{F}_2(t) + N_3\mathcal{F}_3(t) + N_4\mathcal{F}_4(t) + N_5\mathcal{F}_5(t). \tag{2.33}$$

Where $N_1, N_2, ..., N_5$ are positive real numbers to be chosen appropriately later, satisfies

$$b_1\mathcal{E}(t) \leq \ell(t) \leq b_2\mathcal{E}(t), \ \ \forall t \geq 0, \tag{2.34}$$

for two positive constants b_1 and b_2.

Proof. We have

$$|\ell(t) - N\mathcal{E}(t)| \leq N_1|\mathcal{F}_1(t)| + N_2|\mathcal{F}_2(t)| + N_3|\mathcal{F}_3(t)| + N_4|\mathcal{F}_4(t)| + N_5|\mathcal{F}_5(t)|.$$

On account of the Cauchy-Schwartz, Young and Poincaré inequalities, we get

$$\begin{aligned} |\ell(t) - N\mathcal{E}(t)| &\leq c\int_0^1 \left(\mathcal{U}_t^2 + \mathcal{V}_t^2 + \mathcal{V}_x^2 + (\tau_1\theta_t + \theta)^2 + \theta_x^2 + \mathcal{V}^2 + (\mathcal{V} + \mathcal{U}_x)^2\right) dx \\ &\quad + \int_0^1 \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho dx \\ &\leq c\mathcal{E}(t). \end{aligned} \tag{2.35}$$

This implies

$$(N - c)\,\mathcal{E}(t) \leq \ell(t) \leq (N + c)\,\mathcal{E}(t).$$

If we choose N to be sufficiently large, we get estimate (2.34).

Now we are prepared to state and demonstrate the main result of this section. By differentiating (2.33) and recalling (2.26), (2.27), (2.28), (2.31) and (2.32) yields

$$\begin{aligned} \ell'(t) \leq &- \left[N\kappa - N_1c_0 - N_3c_0\left(1 + \frac{1}{\varepsilon_2}\right) - N_4c_0\right] \int_0^1 \theta_x^2 dx \\ &- \left[N(\zeta_1 - |\zeta_2|) - N_2c_0\left(1 + \frac{1}{\varepsilon_1}\right) - N_3c_0 - N_4c_0\right] \int_0^1 \mathcal{V}_t^2 dx \\ &- \left[N_1\frac{\mu^*}{2} - N_4\varepsilon_3\right] \int_0^1 \mathcal{U}_x^2 dx - \left[N_2\frac{\mu_1}{2} - N_1c_0 - N_4c_0\right] \int_0^1 \mathcal{V}^2 dx \\ &- N_2a_0 \int_0^1 \mathcal{V}_x^2 dx - \left[N_4\frac{\beta_0\rho_1}{2} - N_1c_0 - N_2\varepsilon_1 - N_3\varepsilon_2 - N_5\right] \int_0^1 \mathcal{U}_t^2 dx \\ &- \left[N_3\frac{a}{2} - N_2c_0 - N_4c_0\left(1 + \frac{1}{\varepsilon_3}\right)\right] \int_0^1 (\tau_1\theta_t + \theta)^2 dx \\ &- \left[N_2\frac{\zeta_2^2}{2\mu_1} - N_5m\right] \int_0^1 \mathcal{Z}^2(x,1,t)\, dx - N_5m\tau \int_0^1 \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho dx, \end{aligned}$$

now, letting $\varepsilon_1 = \frac{\beta_0\rho_1}{8}\frac{N_4}{N_2}$, $\varepsilon_2 = \frac{\beta_0\rho_1}{4}\frac{N_4}{N_3}, \varepsilon_3 = \frac{\mu_*}{4}\frac{N_1}{N_4}$ and $m = e^{-\tau}$, we end up with

$$\begin{aligned} \ell'(t) \leq &- \left[N\kappa - N_1c_0 - N_3c_0\left(1 + \frac{4N_3}{\beta_0\rho_1N_4}\right) - N_4c_0\right] \int_0^1 \theta_x^2 dx \\ &- \left[N(\zeta_1 - |\zeta_2|) - N_2c_0\left(1 + \frac{8N_2}{\beta_0\rho_1N_4}\right) - N_3c_0 - N_4c_0\right] \int_0^1 \mathcal{V}_t^2 dx \\ &- N_1\frac{\mu^*}{4} \int_0^1 u_x^2 dx - \left[N_2\frac{\mu_1}{2} - N_1c_0 - N_4c_0\right] \int_0^1 \mathcal{V}^2 dx \end{aligned}$$

$$-N_2 a_0 \int_0^1 \mathcal{V}_x^2 dx - \left[N_4 \frac{\beta_0 \rho_1}{8} - N_1 c_0\right] \int_0^1 \mathcal{U}_t^2 dx$$
$$- \left[N_3 \frac{a}{2} - N_2 c_0 - N_4 c_0 \left(1 + \frac{4N_4}{\mu^* N_1}\right)\right] \int_0^1 (\tau_1 \theta_t + \theta)^2 dx$$
$$- \left[N_2 \frac{\zeta_2^2}{2} - N_5 m\right] \int_0^1 z^2(x,1,t)\, dx - N_5 m \tau \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho dx.$$

At this point, we choose N_4 large enough to

$$N_4 \frac{\beta_0 \rho_1}{8} - N_1 c_0 > 0,$$

then, we chooseN_2 large enough such that

$$N_2 \tfrac{\mu_1}{2} - N_1 c_0 - N_4 c_0 > 0,$$
$$N_2 \tfrac{\zeta_2^2}{2\mu_1} - N_5 m > 0.$$

Similarly, we choose N_3 large enough to ensure that

$$N_3 \frac{a}{2} - N_2 c_0 - N_4 c_0 \left(1 + \frac{4N_4}{\mu^* N_1}\right) > 0.$$

Finally, we choose N large enough so that (2.34) stays valid, and

$$N\kappa - N_1 c_0 - N_3 c_0 \left(1 + \tfrac{4N_3}{\beta_0 \rho_1 N_4}\right) - N_4 c_0 > 0,$$
$$N(\zeta_1 - |\zeta_2|) - N_2 c_0 \left(1 + \tfrac{8N_2}{\beta_0 \rho_1 N_4}\right) - N_3 c_0 - N_4 c_0 > 0.$$

Thus, we get to

$$\begin{aligned} \ell'(t) &\leq -\lambda_1 \int_0^1 \Big(\mathcal{U}_t^2 + \mathcal{U}_x^2 + \mathcal{V}_t^2 + \mathcal{V}_x^2 + \mathcal{V}^2 + \theta_x^2 + (\tau_1 \theta_t + \theta)^2 + z^2(x,1,t) \\ &\quad + \int_0^1 z^2(x,\rho,t)\, d\rho \Big) dx \\ &\leq -\lambda_2 \mathcal{E}(t). \end{aligned} \tag{2.36}$$

where λ_1, λ_2 are some positive constant. Considering the statement regarding the equivalency of $E(t)$ and $\ell'(t)$, we deduce that

$$\ell'(t) \leq -\alpha_1 \ell(t), \quad t > 0, \tag{2.37}$$

where $\alpha_1 = \frac{\lambda_2}{b_2} > 0$. A simple integration of (2.37) gives

$$\ell(t) \leq \ell(0)e^{-\alpha_1 t}, \quad t > 0.$$

Using the other side of the equivalence relation produces the desired result (2.20). □

CHAPTER 3

Well-Posedness and Exponential Stability of Swelling Porous with Gurtin-Pipkin Thermoelasticity and delay term

3.1 Introduction

In this chapter, we consider the following problem,

$$\begin{cases} \rho_1 \mathcal{U}_{tt} - a_1 \mathcal{U}_{xx} - a_2 \mathcal{V}_{xx} = 0, & \text{in } (0,1) \times (0,\infty), \\ \rho_2 \mathcal{V}_{tt} - a_3 \mathcal{V}_{xx} - a_2 \mathcal{U}_{xx} - \delta \mathcal{T}_x + \gamma_1 \mathcal{V}_t + \gamma_2 \mathcal{V}_t(x,t-\tau) = 0, & \text{in } (0,1) \times (0,\infty), \\ \rho_3 \mathcal{T}_t - \dfrac{1}{\tau_1} \displaystyle\int_0^\infty \mu(s) \mathcal{T}_{xx}(x,t-s)ds - \delta \mathcal{V}_{xt} = 0, & \text{in } (0,1) \times (0,\infty), \\ \mathcal{U}(x,0) = \mathcal{U}_0(x),\ \mathcal{U}_t(x,0) = \mathcal{U}_1(x), & \\ \mathcal{V}(x,0) = \mathcal{V}_0(x),\ \mathcal{V}_t(x,0) = \mathcal{V}_1(x), & \text{in } (0,1), \\ \mathcal{T}(x,-t) = \mathcal{T}_0(x,t), & \text{in } (0,1) \times (0,\infty), \\ \mathcal{V}(x,t-\tau) = f_0(x,t-\tau), & \text{in } (0,1) \times (0,\tau), \\ \mathcal{U}(0,t) = \mathcal{U}(1,t) = \mathcal{V}(0,t) = \mathcal{V}(1,t) = \mathcal{T}(0,t) = \mathcal{T}(1,t) = 0, & \text{in} (0,\infty), \end{cases} \tag{3.1}$$

Where $\mathcal{U}(x,t)$ is the displacement of fluid, $\mathcal{V}(x,t)$ is the elastic solid material and $\mathcal{T}(x,t)$ is the temperature respectively, $\rho_1, \rho_2, \rho_3, a_1, a_3, \delta > 0,\ a_2 \neq 0$ and $a_1 a_3 > a_2^2$. The initial conditions $\mathcal{U}_0, \mathcal{U}_1, \mathcal{V}_0, \mathcal{V}_1,$ and $\mathcal{T}_0$ are fixed data. The Gurtin-Pipkin heat conduction law defines the heat flow q as

$$q(t) = -\frac{1}{\tau_1}\int_0^\infty \mu(s)\mathcal{T}_x(x,t-s)ds, \tag{3.2}$$

where τ_1 is a positive constant that represents the relaxation time and $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ denotes the memory kernel will have its specific characteristics listed in the section that follows. As a swelling porous system with the Gurtin-Pupkin thermal law and time delay. We will establish the existence and uniqueness of solutions and also will show the exponential stability under a suitable assumptions on the delay term without commitment to any stability number. Our work extend the stability results in [14], by adding a frictional damping term $\gamma_1\mathcal{V}_t$ and a time delay term $\gamma_2\mathcal{V}_t(x,t-\tau)$ in the second equation where $\gamma_1 \in \mathbb{R}_+, \gamma_2 \in \mathbb{R}$ with $|\gamma_2| < \gamma_1$.

3.2 Assumption and Transformation

In this section, we give some material required in the proof of our result.
The memory kernel $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, is a $C^2(\mathbb{R}_+)$ convex summable non-increasing function satisfying

$(\mathcal{A}_1)$ $\mu(0) > 0, \lim_{s\to\infty}\mu(s) = 0,$ and $\int_0^\infty \mu(s)ds = 1.$ Furthermore, there exists a positive real number i such that

$$-\mu''(s) \leq i\mu'(s), \tag{3.3}$$

where $\mu''(s) = \frac{d^2}{ds^2}\mu(s)$. Taking $g = -\mu'$, Hence, the following condition ensues

$(\mathcal{A}_2)$ $g : \mathbb{R}_+ \to \mathbb{R}_+,\ g \in C^1(\mathbb{R}_+),\ g(0) > 0,\ g_0 = \int_0^\infty g(s)ds = \mu(0) > 0,$ and $\int_0^\infty sg(s)ds = 1.$
Furthermore, (3.3) implies

$$g'(s) \leq -ig(s). \tag{3.4}$$

We introduce a variable $\eta : (0,1) \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined as

$$\eta(x,t,s) = \int_{t-s}^t \mathcal{T}(x,\sigma)d\sigma, \tag{3.5}$$

with initial datum $\eta(x,0,s) = \int_0^s \mathcal{T}_0(x,\sigma)d\sigma$. Simple inspection show that η satisfies

$$\begin{aligned} &\eta_t(x,t,s) + \eta_s(x,t,s) = \mathcal{T}(x,t), && (0,1)\times\mathbb{R}_+\times\mathbb{R}_+, \\ &\eta(0,t,s) = \eta(1,t,s) = \eta(x,t,0) = 0, && (0,1)\times\mathbb{R}_+\times\mathbb{R}_+. \end{aligned}$$

Direct calculations we get

$$\begin{aligned}\int_0^\infty \mu(s)\mathcal{T}_{xx}(x,t-s)ds &= \lim_{b\to\infty}\mu(s)\int_{t-s}^t \mathcal{T}_{xx}(x,\sigma)d\sigma\Big|_{s=0}^{s=b} - \int_0^\infty \mu'(s)\int_{t-s}^t \mathcal{T}_{xx}(x,\sigma)d\sigma ds\\ &= \int_0^\infty g(s)\eta_{xx}(x,t,s)ds,\end{aligned}$$

where $\lim_{b\to\infty}\mu(s)\int_{t-s}^t \mathcal{T}_{xx}(x,\sigma)d\sigma\Big|_{s=0}^{s=b} = 0.$
Moreover, as in [7,15], let us introduce the following new variable:

$$\mathcal{Z}(x,\rho,t) = \mathcal{V}_t(x,t-\tau\rho), \text{ in } (0,1)\times(0,1)\times(0,\infty). \tag{3.6}$$

Then, the function $\mathcal{Z}$ satisfies

$$\tau\mathcal{Z}_t(x,\rho,t) + \mathcal{Z}_\rho(x,\rho,t) = 0, \text{ in } (0,1)\times(0,1)\times(0,\infty). \tag{3.7}$$

Hence, By gathering our transformations, the system (3.1) becomes

$$\begin{cases}\rho_1\mathcal{U}_{tt} - a_1\mathcal{U}_{xx} - a_2\mathcal{V}_{xx} = 0, & \text{in } (0,1)\times(0,\infty),\\ \rho_2\mathcal{V}_{tt} - a_3\mathcal{V}_{xx} - a_2\mathcal{U}_{xx} - \delta\mathcal{T}_x + \gamma_1\mathcal{V}_t + \gamma_2\mathcal{Z}(x,1,t) = 0, & \text{in } (0,1)\times(0,\infty),\\ \rho_3\mathcal{T}_t - \dfrac{1}{\tau_1}\displaystyle\int_0^\infty g(s)\eta_{xx}(x,t,s)ds - \delta\mathcal{V}_{xt} = 0, & \text{in } (0,1)\times(0,\infty)\times(0,\infty),\\ \eta_t(x,t,s) + \eta_s(x,t,s) = \mathcal{T}(x,t), & \text{in } (0,1)\times(0,\infty)\times(0,\infty),\\ \tau\mathcal{Z}_t(x,\rho,t) + \mathcal{Z}_\rho(x,\rho,t) = 0, & \text{in } (0,1)\times(0,1)\times(0,\infty),\\ \mathcal{U}(x,0) = \mathcal{U}_0(x),\ \mathcal{U}_t(x,0) = \mathcal{U}_1(x), & \\ \mathcal{V}(x,0) = \mathcal{V}_0(x),\ \mathcal{V}_t(x,0) = \mathcal{V}_1(x), & \text{in } (0,1),\\ \mathcal{Z}(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } (0,1)\times(0,1),\\ \mathcal{T}(x,-t) = \mathcal{T}_0(x,t),\ \eta_0(x,s) = \displaystyle\int_0^s \mathcal{T}_0(x,\sigma)d\sigma, & \text{in } (0,1)\times(0,\infty),\\ \mathcal{U}(0,t) = \mathcal{U}(1,t) = \mathcal{V}(0,t) = \mathcal{V}(1,t) = 0 & \\ \mathcal{T}(0,t) = \mathcal{T}(1,t) = 0, & \text{in}(0,\infty),\\ \eta(0,t,s) = \eta(1,t,s) = \eta(x,t,0) = 0, & \text{in } (0,1)\times(0,\infty)\times(0,\infty),\\ \mathcal{Z}(x,0,t) = \mathcal{V}_t(x,t), & \text{in } (0,1)\times(0,\infty).\end{cases} \tag{3.8}$$

Moving on, we take into account system (3.8) and determine the exponential stability result, disregarding any stability number or other connections between the system's parameters. To prevent any ambiguity coming forward, we will only mention the variables x, t and s when necessary.

3.3 Well-posedness

In this section we use the Lumer-Phillips theorem to verify the existence and uniqueness of solution for (3.8). Taking $\mathcal{W} = \mathcal{U}_t,\ \mathcal{N} = \mathcal{V}_t$ and let $\mathscr{U} = (\mathcal{U}, \mathcal{W}, \mathcal{V}, \mathcal{N}, \mathcal{T}, \eta, \mathcal{Z})^T$ then, the problem (3.8) can be written as

$$\begin{cases} \mathscr{U}_t\,(t) = \mathscr{A}\mathscr{U}\,(t)\,, \quad t > 0, \\ \mathscr{U}\,(0) = \mathscr{U}_0 = \left(\mathcal{U}_0, \mathcal{U}_1, \mathcal{V}_0, \mathcal{V}_1, \mathcal{T}_0, \eta_0, f_0\right)^T, \end{cases} \tag{3.9}$$

where the operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} \longrightarrow \mathscr{H}$ is defined by

$$\mathscr{A}\mathscr{U} = \begin{pmatrix} \mathcal{W} \\ \frac{a_1}{\rho_1}\mathcal{U}_{xx} + \frac{a_2}{\rho_1}\mathcal{V}_{xx} \\ \mathcal{N} \\ \frac{a_3}{\rho_2}\mathcal{V}_{xx} + \frac{a_2}{\rho_2}\mathcal{U}_{xx} + \frac{\delta}{\rho_2}\mathcal{T}_x - \frac{\gamma_1}{\rho_2}\mathcal{N} - \frac{\gamma_2}{\rho_2}\mathcal{Z}(x,1,t) \\ \dfrac{1}{\tau_1\rho_3}\displaystyle\int_0^\infty g(s)\eta_{xx}(x,t,s)ds + \dfrac{\delta}{\rho_3}\mathcal{N}_x \\ \mathcal{T}(x,t) - \eta_s(x,t,s) \\ -\frac{1}{\tau}\mathcal{Z}_\rho(x,\rho,t) \end{pmatrix}. \tag{3.10}$$

We consider the following space

$$L_g = \left\{ \eta : \mathbb{R}_+ \longrightarrow H_0^1(0,1),\ \|\eta\|_{L_g}^2 := \int_0^\infty g(s)\|\eta_x(s)\|^2 ds < \infty \right\},$$

where $L_g = L_g^2\left(\mathbb{R}_+, H_0^1(0,1)\right)$ be a Hilbert space equipped with the inner product

$$(\theta, \vartheta)_{L_g^2\left(\mathbb{R}_+, H_0^1(0,1)\right)} = \int_0^1 \int_0^{+\infty} g(s)\theta_x(s)\vartheta_x(s)dsdx,$$

and let

$$\mathscr{H} = H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L_g(\mathbb{R}_+, H_0^1(0,1)) \times L^2((0,1), L^2(0,1)),$$

be a Hilbert space equipped with the inner product,

$$\begin{aligned}\left\langle \mathscr{U}, \widetilde{\mathscr{U}} \right\rangle_{\mathscr{H}} &= a_1 \int_0^1 \mathcal{U}_x \widetilde{\mathcal{U}_x} dx + \rho_1 \int_0^1 \mathcal{W} \widetilde{\mathcal{W}} dx + a_3 \int_0^1 \mathcal{V}_x \widetilde{\mathcal{V}_x} dx \\ &+ \rho_2 \int_0^1 \mathcal{N} \widetilde{\mathcal{N}} dx + \rho_3 \int_0^1 \mathcal{T} \widetilde{\mathcal{T}} dx + a_2 \int_0^1 \left(\mathcal{U}_x \widetilde{\mathcal{V}_x} + \mathcal{V}_x \widetilde{\mathcal{U}_x} \right) dx \\ &+ \frac{1}{\tau_1} \int_0^1 \int_0^\infty g(s) \eta_x \widetilde{\eta}_x ds dx + \xi \int_0^1 \int_0^1 \mathcal{Z}(x,\rho) \widetilde{\mathcal{Z}}(x,\rho) d\rho dx,\end{aligned}$$

for $\mathscr{U} = (\mathcal{U}, \mathcal{W}, \mathcal{V}, \mathcal{N}, \mathcal{T}, \eta, \mathcal{Z})^T \in \mathscr{H}$, $\widetilde{\mathscr{U}} = (\widetilde{\mathcal{U}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{T}}, \widetilde{\eta}, \widetilde{\mathcal{Z}})^T \in \mathscr{H}$ and ξ is a positive constant such that

$$\tau |\gamma_2| < \xi < \tau(2\gamma_1 - |\gamma_2|). \tag{3.11}$$

The domain of $\mathscr{A}$ is

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{l} \mathscr{U} \in \mathscr{H} /\ \mathcal{U}, \mathcal{V} \in H^2(0,1) \cap H_0^1(0,1), \quad \mathcal{W}, \mathcal{N}, \mathcal{T} \in H_0^1(0,1), \\ \\ \eta, \eta_s \in L_g, \quad \int_0^\infty g(s)\eta(s) ds \in H^2(0,1), \\ \\ \mathcal{Z}, \mathcal{Z}_\rho \in L^2((0,1), L^2(0,1)), \ \ \mathcal{Z}(x,0) = \mathcal{N}(x) \end{array} \right\}. \tag{3.12}$$

The following theorem gives the well-posedness of our problem.

Theorem 3.1. *Assume that g satisfies $(\mathcal{A}_2)$ and (3.11) holds, then for any $\mathscr{U}_0 \in \mathscr{H}$, there exists a unique solution $\mathscr{U} \in C(\mathbb{R}_+, \mathscr{H})$ of problem (3.9). Furthermore, if $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, then $\mathscr{U} \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H})$.*

Proof. We show that $\mathscr{A}$ is dissipative. Let $\mathscr{U} \in \mathscr{D}(\mathscr{A})$, then,

$$\begin{aligned}\langle \mathscr{A}\mathscr{U}, \mathscr{U} \rangle_{\mathscr{H}} &= \frac{-1}{\tau_1} \int_0^1 \int_0^\infty g(s) \eta_{xs}(s) \eta_x(s) ds dx - \gamma_1 \int_0^1 \mathcal{N}^2 dx \\ &- \gamma_2 \int_0^1 \mathcal{Z}(x,1,t) \mathcal{N} dx - \frac{\xi}{\tau} \int_0^1 \int_0^1 \mathcal{Z}_\rho \mathcal{Z} d\rho dx,\end{aligned} \tag{3.13}$$

by applying Young's inequality to the third term of (3.13), we get

$$-\gamma_2 \int_0^1 \mathcal{Z}(x,1,t) \mathcal{N} dx \leq \frac{|\gamma_2|}{2} \int_0^1 \mathcal{Z}^2(x,1,t) dx + \frac{|\gamma_2|}{2} \int_0^1 \mathcal{N}^2 dx. \tag{3.14}$$

Substituting (3.14) in (3.13) and integration the last term of (3.13) with the fact $\mathcal{Z}(x,0) = \mathcal{N}(x)$

we get

$$\begin{aligned}
\langle \mathscr{A}\mathscr{U},\mathscr{U}\rangle_{\mathscr{H}} &= \frac{-1}{2\tau_1}\int_0^\infty g(s)\frac{\partial}{\partial s}\|\eta_x(s)\|^2 ds - \left(\gamma_1 - \frac{\xi}{2\tau} - \frac{|\gamma_2|}{2}\right)\int_0^1 \mathcal{N}^2 dx \\
&\quad - \left(\frac{\xi}{2\tau} - \frac{|\gamma_2|}{2}\right)\int_0^1 \mathcal{Z}^2(x,1,t)dx \\
&= -\frac{1}{2\tau_1}\frac{\partial}{\partial s}\int_0^\infty g(s)\|\eta_x(s)\|^2 ds + \frac{1}{2\tau_1}\int_0^\infty g'(s)\|\eta_x(s)\|^2 ds \\
&\quad - r_0\int_0^1 \left(\mathcal{N}^2 + \mathcal{Z}^2(x,1,t)\right) dx \\
&= \frac{-1}{2\tau_1}\lim_{b\to\infty} g(s)\|\eta_x(s)\|^2\Big|_{s=0}^{s=b} + \frac{1}{2\tau_1}\int_0^\infty g'(s)\|\eta_x(s)\|^2 ds \\
&\quad - r_0\int_0^1 \left(\mathcal{N}^2 + \mathcal{Z}^2(x,1,t)\right) dx,
\end{aligned}$$

for some constant r_0 such that $r_0 > 0$ since (3.11) is holds. It follows from (3.5) that $\eta_x(x,t,0) = 0$. Furthermore, by solving (3.4), we obtain

$$g(s) \leq g(0)e^{-is} \rightsquigarrow g(s)\|\eta_x(s)\|^2 \leq g(0)e^{-is}\|\eta_x(s)\|^2,$$

where $\lim\limits_{s\to\infty} g(s)\|\eta_x(s)\|^2 = 0$. Hence,

$$\langle \mathscr{A}\mathscr{U},\mathscr{U}\rangle_{\mathscr{H}} \leq \frac{1}{2\tau_1}\int_0^\infty g'(s)\|\eta_x(s)\|^2 ds - r_0\int_0^1 \left(\mathcal{N}^2 + \mathcal{Z}^2(x,1,t)\right) dx \leq 0.$$

So, $\mathscr{A}$ is a dissipative operator. It remains to demonstrate that for any

$$\mathscr{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathscr{H},$$

there exists a unique $\mathscr{U} \in \mathscr{D}(\mathscr{A})$ such that

$$\mathscr{A}\mathscr{U} = \mathscr{F}, \tag{3.15}$$

which is equivalent to

$$\begin{cases}
\mathcal{W} = f_1, & \text{in } H^1, \\
a_1\mathcal{U}_{xx} + a_2\mathcal{V}_{xx} = \rho_1 f_2, & \text{in } L^2, \\
\mathcal{N} = f_3, & \text{in } H^1, \\
a_2\mathcal{U}_{xx} + a_3\mathcal{V}_{xx} + \delta\mathcal{T}_x - \gamma_1\mathcal{N} - \gamma_2\mathcal{Z}(.,1) = \rho_2 f_4, & \text{in } L^2,
\end{cases}$$

$$\begin{cases} \delta \mathcal{N}_x + \dfrac{1}{\tau_1}\displaystyle\int_0^\infty g(s)\eta_{xx} = \rho_3 f_5, & \text{in } L^2, \\ \mathcal{T} - \eta_s = f_6, & \text{in } L_g, \\ -\mathcal{Z}_\rho = \tau f_7, & \text{in } L^2\left((0,1), L^2(0,1)\right). \end{cases} \tag{3.16}$$

Using $(3.16)_1$,$(3.16)_3$ and $(3.16)_7$, respectively we get

$$\mathcal{W}, \mathcal{N} \in H^1, \quad \mathcal{Z}_\rho \in L^2\left((0,1), L^2(0,1)\right).$$

By substituting $(3.16)_3$ into $(3.16)_5$, we obtain

$$\int_0^\infty g(s)\eta_{xx} ds = \delta\tau_1 f_{3x} - \tau_1\rho_3 f_5 \in L^2(0,1).$$

Hence, we have

$$\int_0^\infty g(s)\eta(s) ds \in H^2(0,1).$$

It is clear that

$$\eta(s) = s\mathcal{T} - \int_0^s f_6(y)dy, \tag{3.17}$$

satisfies $(3.16)_6$ and the boundary condition $\eta(0) = 0$.
Note that $\int_0^\infty sg(s)ds = 1$, by using (3.17) we obtain,

$$\mathcal{T} = \int_0^\infty g(s)\eta(s)ds + \int_0^\infty g(s)\int_0^s f_6(y)dyds. \tag{3.18}$$

Meanwhile, by using the Fubini Theorem and the Cauchy-Schwarz inequality, for $0 < i_1 < i$, we have

$$\begin{aligned} \int_0^\infty g(s)\left\| \int_0^s f_{6x}(y)dy \right\|^2 ds &= \int_0^\infty g(s)\left\| \int_0^s e^{\frac{i_1}{2}y} e^{-\frac{i_1}{2}y} f_{6x}(y)dy \right\|^2 ds \\ &\le \int_0^\infty g(s)\left(\int_0^s e^{i_1 y}dy\right)\int_0^s e^{-i_1 y}\left\|f_{6x}(y)\right\|^2 dyds \\ &\le \frac{1}{i_1}\int_0^\infty \left(\int_y^\infty g(s)\left(e^{i_1 s}-1\right)ds\right)e^{-i_1 s}\left\|f_{6x}(y)\right\|^2 dy. \end{aligned} \tag{3.19}$$

We set

$$h(y) = \int_y^\infty g(s)\left(e^{i_1 s}-1\right)ds,$$

and define

$$H(y) =: h(y) - \frac{e^{i_1 y}}{i - i_1} g(y).$$

Obviously that $\lim_{y\to\infty} h(y) = 0$. From the non-negativity of g and $g(y) \le g(0)e^{-iy}$, we obtain $\lim_{y\to\infty} \frac{e^{i_1 y}}{i-i_1} g(y) = 0$, it implies that $\lim_{y\to\infty} H(y) = 0$. Furthermore, from (3.4), we get

$$\begin{aligned} H'(y) &= -g(y)e^{i_1 y} + g(y) - \frac{i_1 e^{i_1 y}}{i - i_1} g(y) - \frac{e^{i_1 y}}{i - i_1} g'(y) \\ &\ge -g(y)e^{i_1 y} + g(y) - \frac{i_1 e^{i_1 y}}{i - i_1} g(y) + \frac{i e^{i_1 y}}{i - i_1} g(y) \\ &= g(y) \ge 0, \end{aligned}$$

hence, H is non-decreasing so, $H(y) \le 0$, on the other hand we have

$$e^{-i_1 y} h(y) = e^{-i_1 y} \int_y^\infty g(s) \left(e^{i_1 s} - 1 \right) ds \le \frac{1}{i - i_1} g(y). \tag{3.20}$$

The substitution of (3.20) into (3.19) yields

$$\int_0^\infty g(s) \Big\| \int_0^s f_{6x}(y) dy \Big\|^2 ds \le \frac{1}{i\,(i_1 - i)} \int_0^\infty g(y) \Big\| f_{6x}(y) \Big\|^2 dy = \frac{1}{i\,(i_1 - i)} \|f_6\|^2_{L_g} < \infty. \tag{3.21}$$

The result (3.21) implies that $f_6 \in L_g$. Then, by definition we have $\int_0^s f_6(y)dy \in L_g$. Moreover by using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} \Big\| \int_0^\infty g(s) \int_0^s f_{6x}(y) dy ds \Big\|^2 &\le \int_0^\infty g(s) ds \int_0^\infty g(s) \Big\| \int_0^s f_{6x}(y) dy \Big\| \\ &\le g_0 \int_0^\infty g(s) \Big\| \int_0^s f_{6x}(y) dy \Big\| \\ &< \infty. \end{aligned} \tag{3.22}$$

It follows from the last above inequality that $\Big\| \int_0^\infty g(s) \int_0^s f_{6x}(y) dy ds \Big\|^2$ is bounded. Than, $\int_0^\infty g(s) \int_0^s f_{6x}(y) dy ds \in H_0^1(0,1)$. Consequently, from (3.18) we have

$$\mathcal{T} \in H_0^1(0,1).$$

Meanwhile, we observe that

$$\int_0^\infty g(s)\|\mathcal{T}_x\|^2 ds = g_0\|\mathcal{T}_x\|^2 < \infty \Longrightarrow \mathcal{T} \in L_g.$$

Additionally, by applying (3.4), integration by parts, the fact that $\int_0^\infty sg(s) = 1$, and L'Hôpital's rule, we obtain

$$\int_0^\infty s^2 g(s)ds \leq \frac{2}{i} < \infty.$$

Therefore,

$$\int_0^\infty g(s)\|s\mathcal{T}_x\|^2 ds = \left(\int_0^\infty s^2 g(s)ds\right)\|\mathcal{T}_x\|^2 < \infty.$$

It implies that $s\mathcal{T} \in L_g$. Consequently, from $(3.16)_6$ and (3.18), we get

$$\eta, \eta_s \in L_g.$$

According to Lax-Milgram theorem and classical elliptic regularity, the system $(3.16)_2 - (3.16)_4$ has a solution such that $(\mathcal{U}, \mathcal{V}) \in H^2(0,1) \times H^2(0,1)$. Moreover,

$$\begin{aligned}
\mathcal{U}_{xx} &= \frac{1}{a_1a_3 - a_2^2}\left(a_3\rho_1 f_2 - a_2\rho_2 f_4 + a_2\delta\mathcal{T}_x - a_2\gamma_1\mathcal{N} - a_2\gamma_2\mathcal{Z}(x,1,t)\right) \in L^2,\\
\mathcal{V}_{xx} &= \frac{1}{a_1a_3 - a_2^2}\left(a_1\rho_2 f_4 - a_2\rho_1 f_2 - a_1\delta\mathcal{T}_x + a_1\gamma_1\mathcal{N} + a_1\gamma_2\mathcal{Z}(x,1,t)\right) \in L^2.
\end{aligned}$$

Additionally, it is clear that

$$\begin{aligned}
\mathcal{U}_x &= \frac{1}{a_1a_3 - a_2^2}\left(a_3\rho_1\int_0^x f_2(y)dy - a_2\rho_2\int_0^x f_4(y)dy\right.\\
&\quad \left. - a_2\gamma_1\int_0^x \mathcal{N}(y)dy - a_2\gamma_2\int_0^x \mathcal{Z}(y,1,t)dy + a_2\delta\mathcal{T}\right),\\
\mathcal{V}_x &= \frac{1}{a_1a_3 - a_2^2}\left(a_1\rho_2\int_0^x f_4(y)dy - a_2\rho_1\int_0^x f_2(y)dy\right.\\
&\quad \left. + a_1\gamma_1\int_0^x \mathcal{N}(y)dy + a_1\gamma_2\int_0^x \mathcal{Z}(y,1,t)dy - a_1\delta\mathcal{T}\right).
\end{aligned}$$

Hence, $\mathcal{U}$ and $\mathcal{V}$ satisfy the boundary conditions since $f_2, f_4 \in L^2$ and $\mathcal{T} \in H_0^1(0,1)$, that is

$$\mathcal{U}_x(0) = \mathcal{U}_x(1) = \mathcal{V}_x(0) = \mathcal{V}_x(1) = 0.$$

Then, (3.15) has a unique solution in $\mathscr{D}(\mathscr{A})$ moreover $\|\mathscr{U}\|_{\mathscr{H}} \leq k\|\mathscr{F}\|_{\mathscr{H}}$, thus, $0 \in \rho(A)$. finally, since $0 \in \rho(\mathscr{A})$ and $\mathscr{A}$ is dissipative we conclude that the operator $\mathscr{A}$ is the infinitesimal generator of a c_0-semigroup of contraction in $\mathscr{H}$ by using Theorem 1.2.4 of [54] (Lumer-Phillips Theorem). □

3.4 Exponential stability

In this section we assume that $(\mathcal{U}, \mathcal{V}, \mathcal{T}, \eta, z)$ is a solution of (3.8) and we demonstrate the exponential stability of our problem by using the energy technique.

Lemma 3.1. *Assume that g satisfies $(\mathcal{A}_2)$. Then, the energy functional E defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left(\rho_1 \mathcal{U}_t^2 + a_1 \mathcal{U}_x^2 + \rho_2 \mathcal{V}_t^2 + a_3 \mathcal{V}_x^2 + 2a_2 \mathcal{U}_x \mathcal{V}_x + \rho_3 \mathcal{T}^2\right) dx \\ &\quad + \frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx + \frac{1}{2\tau_1}\int_0^1\int_0^\infty g(s)|\eta_x|^2 ds dx, \end{aligned} \tag{3.23}$$

satisfies,

$$E'(t) \leq -m_0\left(\int \mathcal{V}_t^2(x,t)\,dx + \int_0^1 z^2(x,1,t)\,dx\right) + \frac{1}{2\tau_1}\int_0^1\int_0^\infty g'(s)|\eta_x|^2 ds dx, \tag{3.24}$$

for some $m_0 > 0$.

Proof. We multiplying the first and three equations of (3.8) by $\mathcal{U}_t$, $\mathcal{V}_t$ and $\mathcal{T}$, respectively, and integration by part over $(0,1)$, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho_1 \mathcal{U}_t^2 + a_1 \mathcal{U}_x^2 + \rho_2 \mathcal{V}_t^2 + a_3 \mathcal{V}_x^2 + \rho_3 \mathcal{T}^2 + 2a_2 \mathcal{U}_x \mathcal{V}_x\right) dx \\ &\qquad + \gamma_1 \int_0^1 \mathcal{V}_t^2 dx + \gamma_2 \int_0^1 \mathcal{V}_t z(x,1,t)\,dx + \frac{1}{\tau_1}\int_0^1 \mathcal{T}_x \int_0^\infty g(s)\eta_x ds dx = 0. \end{aligned} \tag{3.25}$$

Using $(3.8)_4$ where is $\eta_t(x,t,s) + \eta_s(x,t,s) = \mathcal{T}(x,t)$, by simple calculation we find

$$\begin{aligned} \int_0^1 \mathcal{T}_x \int_0^\infty g(s)\eta_x ds dx &= \int_0^1\int_0^\infty g(s)\left(\eta_{ts} + \eta_{sx}\right)\eta_x ds dx \\ &= \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^\infty g(s)|\eta_x|^2 ds dx + \frac{1}{2}\int_0^1\int_0^\infty g(s)\frac{\partial}{\partial s}|\eta_x|^2 ds dx \\ &= \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^\infty g(s)|\eta_x|^2 ds dx - \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\eta_x|^2 ds dx. \end{aligned} \tag{3.26}$$

Now, multiplying $(3.8)_5$ by $\frac{\xi}{\tau}\mathcal{Z}$ and integrating the result over $(0,1)\times(0,1)$ we get

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 \mathcal{Z}^2(x,\rho,t)\,d\rho dx &= -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\mathcal{Z}^2(x,\rho,t)\,d\rho dx\\ &= \frac{\xi}{2\tau}\int_0^1\left(\mathcal{Z}^2(x,0,t)-\mathcal{Z}^2(x,1,t)\right)\\ &= \frac{\xi}{2\tau}\int_0^1\left(\mathcal{V}_t^2-\mathcal{Z}^2(x,1,t)\right).\end{aligned} \tag{3.27}$$

Substituting (3.26), (3.27) in (3.25) we find

$$\begin{aligned}\frac{dE(t)}{dt} &= -\left(\gamma_1-\frac{\xi}{2\tau}\right)\int_0^1 \mathcal{V}_t^2(x,t)dx-\frac{\xi}{2\tau}\int_0^1 \mathcal{Z}^2(x,1,t)dx\\ &\quad -\gamma_2\int_0^1 \mathcal{V}_t(x,t)\mathcal{Z}(x,1,t)dx+\frac{1}{2\tau_1}\int_0^1\int_0^\infty g'(s)\,|\eta(x)|^2dsdx.\end{aligned} \tag{3.28}$$

By applying Young's inequality in (3.28), we obtain

$$\begin{aligned}\frac{dE(t)}{dt} &\le -\left(\gamma_1-\frac{\xi}{2\tau}-\frac{|\gamma_2|}{2}\right)\int_0^1 \mathcal{V}_t^2(x,t)dx-\left(\frac{\xi}{2\tau}-\frac{|\gamma_2|}{2}\right)\int_0^1 \mathcal{Z}^2(x,1,t)dx\\ &\quad +\frac{1}{2\tau_1}\int_0^1\int_0^\infty g'(s)\,|\eta(x)|^2dsdx.\end{aligned} \tag{3.29}$$

By using (3.11) and (3.23) we obtain (3.24). □

Lemma 3.2. *The functional*

$$F_1(t)=\frac{\rho_2}{a_2}\int_0^1\left(\mathcal{V}_t\mathcal{U}-\mathcal{V}\mathcal{U}_t\right),$$

satisfies, the estimate

$$\begin{aligned}F_1'(t) &\le -\frac{1}{2}\int_0^1 \mathcal{U}_x^2dx+\left(\frac{1}{a_2^2}\left(\frac{a_1\rho_2}{\rho_1}-a_3\right)^2+\frac{\rho_2}{\rho_1}\right)\int_0^1 \mathcal{V}_x^2dx+\frac{\delta^2}{a_2^2}\int_0^1 \mathcal{T}^2dx\\ &\quad +\frac{1}{2}\int_0^1 \mathcal{U}^2dx+\frac{\gamma_1^2}{a_2^2}\int_0^1 \mathcal{V}_t^2dx+\frac{\gamma_2^2}{a_2^2}\int_0^1 \mathcal{Z}^2(x,1,t)\,dx.\end{aligned} \tag{3.30}$$

Proof. Taking the derivative of F_1, by using $(3.8)_1$ and $(3.8)_2$, we obtain

$$\begin{aligned}F_1'(t) &= -\int_0^1 \mathcal{U}_x^2dx+\frac{\rho_2}{\rho_1}\int_0^1 \mathcal{V}_x^2dx-\frac{\delta}{a_2}\int_0^1 \mathcal{U}_x\mathcal{T}dx+\frac{1}{a_2}\left(\frac{a_1\rho_2}{\rho_1}-a_3\right)\int_0^1 \mathcal{U}_x\mathcal{V}_xdx\\ &\quad -\frac{\gamma_1}{a_2}\int_0^1 \mathcal{V}_t\mathcal{U}dx-\frac{\gamma_2}{a_2}\int_0^1 \mathcal{Z}(x,1,t)\,\mathcal{U}dx.\end{aligned} \tag{3.31}$$

By using Young's inequality we obtain

$$-\frac{\delta}{a_2}\int_0^1 \mathcal{U}_x\mathcal{T}dx \leq \frac{1}{4}\int_0^1 \mathcal{U}_x^2dx+\frac{\delta^2}{a_2^2}\int_0^1 \mathcal{T}^2dx, \tag{3.32}$$

$$\frac{1}{a_2}\left(\frac{a_1\rho_2}{\rho_1}-a_3\right)\int_0^1 \mathcal{U}_x\mathcal{V}_xdx \leq \frac{1}{4}\int_0^1 \mathcal{U}_x^2dx+\frac{1}{a_2^2}\left(\frac{a_1\rho_2}{\rho_1}-a_3\right)^2\int_0^1 \mathcal{V}_x^2dx, \tag{3.33}$$

$$-\frac{\gamma_1}{a_2}\int_0^1 \mathcal{V}_t\mathcal{U}dx \leq \frac{1}{4}\int_0^1 \mathcal{U}^2dx+\frac{\gamma_1^2}{a_2^2}\int_0^1 \mathcal{V}_t^2dx, \tag{3.34}$$

$$-\frac{\gamma_2}{a_2}\int_0^1 z(x,1,t)\,\mathcal{U}dx \leq \frac{1}{4}\int_0^1 \mathcal{U}^2dx+\frac{\gamma_2^2}{a_2^2}\int_0^1 z^2(x,1,t)\,dx. \tag{3.35}$$

By substituting (3.32)-(3.35) into (3.31), we obtain (3.30). □

Lemma 3.3. *The functional*

$$F_2(t)=\frac{1}{\lambda}\left(a_1\rho_2\int_0^1 \mathcal{V}_t\mathcal{V}dx-a_2\rho_1\int_0^1 \mathcal{U}_t\mathcal{V}dx\right)+\frac{\gamma_1 a_1}{2\lambda}\int_0^1 \mathcal{V}^2dx,$$

satisfies, for any $\varepsilon_1>0$ small, the estimate

$$\begin{aligned}F_2'(t) \ &\leq\ \frac{-1}{2}\int_0^1 \mathcal{V}_x^2dx+\varepsilon_1\int_0^1 \mathcal{U}_t^2dx+\left(\frac{a_1\rho_2}{\lambda}+\frac{a_2^2\rho_1^2}{4\varepsilon_1\lambda^2}\right)\int_0^1 \mathcal{V}_t^2dx\\ &+\frac{a_1^2\delta^2}{\lambda^2}\int_0^1 \mathcal{T}^2dx+\frac{1}{2}\int_0^1 \mathcal{V}^2dx+\frac{a_1^2\gamma_2^2}{2\lambda^2}\int_0^1 z^2(x,1,t)\,dx,\end{aligned} \tag{3.36}$$

where $\lambda=a_1a_3-a_2^2>0$.

Proof. Taking the derivative of F_2, by using $(3.8)_1$ and $(3.8)_2$, we obtain

$$\begin{aligned}F_2'(t) \ &=\ -\int_0^1 \mathcal{V}_x^2dx+\frac{a_1\rho_2}{\lambda}\int_0^1 \mathcal{V}_t^2dx-\frac{a_2\rho_1}{\lambda}\int_0^1 \mathcal{U}_t\mathcal{V}_tdx-\frac{a_1\delta}{\lambda}\int_0^1 \mathcal{V}_x\mathcal{T}dx\\ &-\frac{a_1\gamma_2}{\lambda}\int_0^1 z(x,1,t)\,\mathcal{V}dx.\end{aligned} \tag{3.37}$$

Using Young's inequalities with $\varepsilon_1>0$,

$$-\frac{a_2\rho_1}{\lambda}\int_0^1 \mathcal{U}_t\mathcal{V}_tdx \leq \varepsilon_1\int_0^1 \mathcal{U}_t^2dx+\frac{a_2^2\rho_1^2}{4\varepsilon_1\lambda^2}\int_0^1 \mathcal{V}_t^2dx, \tag{3.38}$$

$$-\frac{a_1\delta}{\lambda}\int_0^1 \mathcal{V}_x\mathcal{T}dx \leq \frac{1}{2}\int_0^1 \mathcal{V}_x^2dx+\frac{a_1^2\delta^2}{\lambda^2}\int_0^1 \mathcal{T}^2dx, \tag{3.39}$$

$$-\frac{a_1\gamma_2}{\lambda}\int_0^1 z(x,1,t)\,\mathcal{V}dx \leq \frac{1}{2}\int_0^1 \mathcal{V}^2dx+\frac{a_1^2\gamma_2^2}{2\lambda^2}\int_0^1 z^2(x,1,t)\,. \tag{3.40}$$

By substituting (3.38)-(3.40) into (3.37), we obtain (3.36). □

Lemma 3.4. *Assume that g satisfies $(\mathcal{A}_2)$ then, the functional*

$$F_3(t) = -\frac{1}{g_0}\int_0^1 \mathcal{T}\int_0^\infty g(s)\eta(s)dsdx,$$

satisfies, for any $\varepsilon_3 > 0$ small, the estimate

$$\begin{aligned} F_3'(t) \leq & -\frac{1}{2}\int_0^1 \mathcal{T}^2dx + \varepsilon_3\int_0^1 \mathcal{V}_t^2dx + \left(\frac{1}{\rho_3\tau_1} + \frac{\delta^2}{4\rho_3^2 g_0\varepsilon_3}\right)\int_0^1\int_0^\infty g(s)|\eta_x|^2dsdx \\ & -\frac{g(0)}{2g_0^2}\int_0^1\int_0^\infty g'(s)|\eta_x|^2dsdx. \end{aligned} \tag{3.41}$$

Proof. Taking the derivative of F_3 and using $(3.8)_3$ and $(3.8)_4$, we obtain

$$\begin{aligned} F_3'(t) = & -\int_0^1 \mathcal{T}^2dx + \frac{1}{\rho_3 g_0\tau_1}\int_0^1\left(\int_0^{i\infty} g(s)\eta_x ds\right)^2dx + \frac{\delta}{\rho_3 g_0}\int_0^1 \mathcal{V}_t\int_0^\infty g(s)\eta_x dsdx \\ & +\frac{1}{g_0}\int_0^1 \mathcal{T}\int_0^\infty g(s)\eta_s dsdx. \end{aligned} \tag{3.42}$$

Applying Young's and Cauchy-Schwarz inequalities, with $\varepsilon_3 > 0$ we get

$$\frac{1}{\rho_3 g_0\tau_1}\int_0^1\left(\int_0^\infty g(s)\eta_x ds\right)^2dx \leq \frac{1}{\rho_3\tau_1}\int_0^1\int_0^\infty g(s)|\eta_x|^2dsdx, \tag{3.43}$$

$$\frac{\delta}{\rho_3 g_0}\int_0^1 \mathcal{V}_t\int_0^\infty g(s)\eta_x dsdx \leq \varepsilon_3\int_0^1 \mathcal{V}_t^2dx + \frac{\delta^2}{4\rho_3^2 g_0\varepsilon_3}\int_0^1\int_0^\infty g(s)|\eta_x|^2dsdx, \tag{3.44}$$

$$\frac{1}{g_0}\int_0^1 \mathcal{T}\int_0^\infty g(s)\eta_s dsdx \leq \frac{1}{2}\int_0^1 \mathcal{T}^2dx - \frac{g(0)}{2g_0^2}\int_0^1\int_0^\infty g'(s)|\eta_x|^2dsdx. \tag{3.45}$$

By substituting (3.43)-(3.45) into (3.42), we obtain (3.41). □

Lemma 3.5. *The functional*

$$F_4(t) = -\int_0^1 \mathcal{U}_t\mathcal{U}dx,$$

satisfies, the estimate

$$F_4'(t) \leq -\int_0^1 \mathcal{U}_t^2dx + \frac{1}{2\rho_1}(2a_1 + a_2)\int_0^1 \mathcal{U}_x^2dx + \frac{a_2}{2\rho_1}\int_0^1 \mathcal{V}_x^2dx. \tag{3.46}$$

Proof. Taking the derivative of F_4, using $(3.8)_1$ and Young's inequality, we obtain

$$\begin{aligned} F_4'(t) &= -\int_0^1 \mathcal{U}_t^2 dx + \frac{a_1}{\rho_1}\int_0^1 \mathcal{U}_x^2 dx + \frac{a_2}{\rho_1}\int_0^1 \mathcal{U}_x \mathcal{V}_x dx, \\ &\leq \int_0^1 \mathcal{U}_t^2 dx + \frac{1}{2\rho_1}(2a_1 + a_2)\int_0^1 \mathcal{U}_x^2 dx + \frac{a_2}{2\rho_1}\int_0^1 \mathcal{V}_x^2 dx. \end{aligned}$$

□

Lemma 3.6. *The functional*

$$F_5(t) = \tau \int_0^1 \int_0^1 e^{-\tau\rho} \mathcal{Z}^2(x,\rho,t)\, d\rho dx,$$

satisfies, for some positive constant m_1, the estimate

$$F_5'(t) \leq -m_1\left(\int_0^1 \mathcal{Z}^2(x,1,t)\, dx + \tau\int_0^1\int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho dx\right) + \int_0^1 \mathcal{V}_t^2 dx. \tag{3.47}$$

Proof. Taking the derivative of F_5 and using $(3.8)_5$, we obtain

$$\begin{aligned} F_5'(t) &= -2\tau\int_0^1\int_0^1 e^{-\tau\rho}\mathcal{Z}(x,\rho,t)\,\mathcal{Z}_\rho(x,\rho,t)\, d\rho dx \\ &= -\int_0^1\int_0^1 e^{-\tau\rho}\frac{\partial}{\partial\rho}\mathcal{Z}^2(x,\rho,t)\, d\rho dx \\ &= -\int_0^1\left(e^{-\tau}\mathcal{Z}^2(x,1,t) - \mathcal{Z}^2(x,0,t)\right)dx - \tau\int_0^1\int_0^1 e^{-\tau\rho}\mathcal{Z}^2(x,\rho,t)\, d\rho dx \\ &= -e^{-\tau}\int_0^1 \mathcal{Z}^2(x,1,t)\, dx - \tau\int_0^1\int_0^1 e^{-\tau\rho}\mathcal{Z}^2(x,\rho,t)\, d\rho dx + \int_0^1 \mathcal{V}_t^2 dx. \end{aligned} \tag{3.48}$$

By using the fact that $\mathcal{Z}(x,0,t) = \mathcal{V}_t(x,t)$ and $e^{-\tau} \leq e^{-\tau\rho} \leq 1$ for all $\rho \in [0,1]$, we get

$$F_5'(t) \leq -e^{-\tau}\left(\int_0^1 \mathcal{Z}^2(x,1,t)\, dx + \tau\int_0^1\int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho dx\right) + \int_0^1 \mathcal{V}_t^2 dx,$$

setting $m_1 = e^{-\tau}$, we obtain (3.47). □

Now, we define the Lyapunov functional by

$$\mathfrak{L}(t) = NE(t) + \sum_{j=1}^{4} N_j F_j(t) + F_5(t). \tag{3.49}$$

for N and $N_j, j = 1, ..., 4$ are positive constants and we show that $\mathfrak{L}(t)$ is equivalent with the energy functional E.

Lemma 3.7. *Assume that g satisfies $(\mathcal{A}_2)$. Then, there exist two positive constants $a, b > 0$, such that*

$$aE(t) \leq \mathfrak{L}(t) \leq bE(t), \qquad t \geq 0. \tag{3.50}$$

Proof. by doing a simple calculations based on the definition of $F_j, j = 1, ..., 5$ we get

$$\begin{aligned} |\mathfrak{L}(t) - NE(t)| \leq\ & N_1 \frac{\rho_2}{|a_2|} \int_0^1 |\mathcal{V}_t \mathcal{U} - \mathcal{V}\mathcal{U}_t| dx + \frac{N_2}{\lambda} \left(a_1 \rho_2 \int_0^1 |\mathcal{V}_t \mathcal{V}| dx + |a_2| \rho_1 \int_0^1 |\mathcal{U}_t \mathcal{V}| dx \right) \\ & + \frac{a_1 \gamma_1}{2\lambda} \int_0^1 |\mathcal{V}|^2 dx + \frac{N_3}{g_0} \int_0^1 \left| \mathcal{T} \int_0^\infty g(s) \eta(s) \, ds \right| dx \\ & + N_4 \int_0^1 |\mathcal{U}_t \mathcal{U}| dx + \tau \int_0^1 \left| e^{-\tau\rho} z^2(x, \rho, t) \right| d\rho dx. \end{aligned}$$

By using Cauchy-Schwarz and Young's inequalities, we arrive to

$$\begin{aligned} |\mathfrak{L}(t) - NE(t)| \leq\ & \left(\frac{\rho_2 N_1}{2a_2^2} + \frac{a_1^2 \rho_2^2 N_2}{2\lambda \rho_1} \right) \int_0^1 \mathcal{V}_t^2 dx + \left(\frac{\rho_2 N_1}{2} + \frac{\rho_1 N_2}{\lambda} + \frac{\gamma_1 a_1 N_2}{2\lambda} \right) \int_0^1 \mathcal{V}^2 dx \\ & + \left(\frac{\rho_2 N_1}{2a_2^2} + \frac{\rho_1 a_2^2 N_2}{2\lambda} + \frac{N_4}{2} \right) \int_0^1 \mathcal{U}_t^2 dx + \left(\frac{\rho_2 N_1}{2} + \frac{N_4}{2} \right) \int_0^1 \mathcal{U}^2 dx \\ & + \frac{N_3}{2} \int_0^1 \mathcal{T}^2 dx + \frac{N_3}{2g_0} \int_0^1 \int_0^\infty g(s) |\eta_x|^2 ds dx + \tau \int_0^1 \int_0^1 z^2(x, \rho, t) \, d\rho dx. \end{aligned} \tag{3.51}$$

Using Poincaré's inequality, it is evident that

$$\begin{aligned} |\mathfrak{L}(t) - NE(t)| \leq\ & \tau_2 \int_0^1 \Big(\mathcal{V}_t^2 + \mathcal{U}_t^2 + \mathcal{V}_x^2 + \mathcal{U}_x^2 + \mathcal{T}^2 \\ & + \int_0^1 g(s) |\eta_x|^2 ds + \int_0^1 z^2(x, \rho, t) \, d\rho \Big) dx. \end{aligned} \tag{3.52}$$

Where

$$\begin{aligned} \tau_2 = \max \Big\{ & \frac{\rho_2 N_1}{2a_2^2} + \frac{\rho_2^2 a_1^2 N_2}{2\lambda \rho_1}, \frac{\rho_2 N_1}{2a_2^2} + \frac{\rho_1 a_2^2 N_2}{2\lambda} + \frac{N_4}{2}, c_p \left(\frac{\rho_2 N_1}{2} + \frac{\rho_1 N_2}{\lambda} + \frac{\gamma_1 a_1 N_2}{2\lambda} \right), \\ & c_p \left(\frac{\rho_2 N_1}{2} + \frac{N_4}{2} \right), \frac{N_3}{2}, \frac{N_3}{2g_0}, \tau \Big\} > 0, \end{aligned}$$

where c_p is a Poincaré's constant. However, the energy functional provided by (3.23) can be defined as

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1 \mathcal{U}_t^2 + \rho_2 \mathcal{V}_t^2 + \frac{a_1}{2}\left(\mathcal{U}_x + \frac{a_2}{a_1}\mathcal{V}_x\right)^2 + \frac{a_3}{2}\left(\mathcal{V}_x + \frac{a_2}{a_3}\mathcal{U}_x\right)^2 + \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right)\mathcal{U}_x^2 \right. \\ &\quad \left. + \frac{1}{2}\left(a_3 - \frac{a_2^2}{a_1}\right)\mathcal{V}_x^2 + \rho_3 \mathcal{T}^2 + \frac{1}{\tau_1}\int_0^\infty g(s)|\eta_x|^2 ds + \xi \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho \right] dx \\ &\geq \frac{1}{2}\int_0^1 \left[\rho_1 \mathcal{U}_t^2 + \rho_2 \mathcal{V}_t^2 + \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right)\mathcal{U}_x^2 + \frac{1}{2}\left(a_3 - \frac{a_2^2}{a_1}\mathcal{V}_x^2\right) \right. \\ &\quad \left. + \rho_3 \mathcal{T}^2 + \frac{1}{\tau_1}\int_0^\infty g(s)|\eta_x|^2 ds + \xi \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho \right] dx. \end{aligned} \tag{3.53}$$

It follows from $a_1 a_3 > a_2^2$ and (3.53) that

$$E(t) \geq \tau_3 \int_0^1 \left[\mathcal{V}_t^2 + \mathcal{U}_t^2 + \mathcal{V}_x^2 + \mathcal{U}_x^2 + \mathcal{T}^2 + \int_0^\infty g(s)|\eta_x|^2 ds + \int_0^1 \mathcal{Z}^2(x,\rho,t)\, d\rho\right] dx, \tag{3.54}$$

where

$$\tau_3 = \min \frac{1}{2}\left\{\rho_1,\ \rho_2,\ \rho_3,\ \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right),\ \frac{1}{2}\left(a_3 - \frac{a_2^2}{a_1}\right),\ \frac{1}{\tau_1},\ \xi\right\} > 0.$$

Combining (3.52) and (3.54) we find

$$|\mathfrak{L}(t) - NE(t)| \leq \frac{\tau_2}{\tau_3} E(t),$$

which implies that

$$\left[N - \frac{\tau_2}{\tau_3}\right] E(t) \leq \mathfrak{L}(t) \leq \left[N + \frac{\tau_2}{\tau_3}\right] E(t). \tag{3.55}$$

For $N = a + \frac{\tau_2}{\tau_3}$ results in (3.50), where $a > 0$ and $b = a + \frac{2\tau_2}{\tau_3}$. □

Now, we state and demonstrate our stability result.

Theorem 3.2. *Assume that g satisfies$(\mathcal{A}_2)$. Then, there exist two positive constants α_1 and α_2 such that the energy designated by (3.23) satisfies*

$$E(t) \leq \alpha_1 e^{-\alpha_2 t}, \qquad for all t \geq 0. \tag{3.56}$$

Proof. By differentiating equation (3.49), recalling the equations (3.24), (3.30), (3.36), (3.41), (3.46), (3.47) and taking

$$N_4 = 2, \qquad \varepsilon_1 = \frac{1}{N_2}, \qquad \varepsilon_3 = \frac{1}{N_3},$$

we arrive to

$$\begin{aligned}
\mathfrak{L}'(t) \leq & \left[\frac{N}{2\tau_1} - \frac{g(0)N_3}{2g_0^2}\right] \int_0^1 \int_0^\infty g'(s)|\eta_x|^2 ds dx - \left[\omega N_1 - \frac{2a_1+a_2}{\rho_1}\right] \int_0^1 \mathcal{U}_x^2 dx \\
& - \left[\omega N_2 - \left(\frac{1}{a_2^2}\left(\frac{a_1\rho_2}{\rho_1} - a_3\right)^2 + \frac{\rho_2}{\rho_1}\right) N_1 - \frac{a_2}{\rho_1}\right] \int_0^1 \mathcal{V}_x^2 dx \\
& - \left[m_0 N - \frac{\gamma_1^2 N_1}{a_2^2} - \left(\frac{a_1\rho_2}{\lambda} + \frac{a_2^2\rho_1^2 N_2}{4\lambda^2}\right) N_2 - 2\right] \int_0^1 \mathcal{V}_t^2 dx \\
& - \left[\frac{N_3}{2} - \frac{\delta^2 N_1}{a_2^2} - \frac{a_1^2\delta^2 N_2}{\lambda^2}\right] \int_0^1 \mathcal{T}^2 dx \\
& - \left[m_0 N + m_1 - \frac{\gamma_2^2 N_1}{a_2^2} - \frac{a_1^2\gamma_2^2 N_2}{2\lambda^2}\right] \int_0^1 z^2(x,1,t)\, dx - \int_0^1 \mathcal{U}_t^2 dx \\
& - \tau m_1 \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho dx + \left[\left(\frac{1}{\rho_3\tau_1} + \frac{\delta^2 N_3}{4\rho_3^2 g_0}\right) N_3\right] \int_0^1 \int_0^\infty g(s)|\eta_x|^2 ds dx,
\end{aligned} \tag{3.57}$$

where $\omega = \dfrac{1}{2} - \dfrac{c_p}{2} > 0$, since $c_p < 1$. Now, we choose N_1, N_2 and N_3 three constants are large enough such that

$$\begin{cases}
\omega N_1 - \frac{2a_1+a_2}{\rho_1} > 0, \\
\omega N_2 - \left(\frac{1}{a_2^2}\left(\frac{a_1\rho_2}{\rho_1} - a_3\right)^2 + \frac{\rho_2}{\rho_1}\right) N_1 - \frac{a_2}{\rho_1} > 0, \\
\frac{N_3}{2} - \frac{\delta^2 N_1}{a_2^2} - \frac{a_1^2\delta^2 N_2}{\lambda^2} > 0.
\end{cases}$$

Finally, by using $(\mathcal{A}_2)$, combining first and last term in (3.57) and choose N large enough we obtain,

$$\begin{aligned}
\mathfrak{L}'(t) & \leq -\beta_1 \int_0^1 \left(\mathcal{U}_t^2 + \mathcal{U}_x^2 + \mathcal{V}_t^2 + \mathcal{V}_x^2 + \mathcal{T}^2 + \int_0^1 z^2(x,\rho,t)\, d\rho + \int_0^\infty g(s)|\eta_x^2| ds\right) dx \\
& \leq -\beta_2 E(t), \qquad \forall t \geq 0,
\end{aligned}$$

for some positive constant β_1 and β_2. From Lemma 3.7, we conclude that

$$\mathfrak{L}'(t) \leq -r_1 \mathfrak{L}'(t), \forall t \geq 0, \tag{3.58}$$

where $r_1 = \frac{\beta_2}{b} > 0$. A simple integration of (3.58) yields

$$\mathfrak{L}(t) \leq \mathfrak{L}(0) e^{-r_1 t}, \qquad \forall t \geq 0.$$

By applying Lemma 3.7 again we obtain the desired result (3.56). □

CHAPTER 4

Well-posedness and exponential stability for a swelling porous-heat system with thermodiffusion effects and time-varying delay

4.1 Introduction

In this chapter, we investigate the swelling porous-heat system with thermodiffusion effects and time-varying delay

$$\begin{cases} \rho_1 \mathcal{U}_{tt} - a_1 \mathcal{U}_{xx} - a_2 \varphi_{xx} = 0, \\ \rho_2 \varphi_{tt} - a_3 \varphi_{xx} - a_2 \mathcal{U}_{xx} - \zeta_1 \theta_x - \zeta_2 \mathcal{P}_x + \mu_1 \varphi_t + \mu_2 \varphi_t(x, t - \tau(t)) = 0, \\ c\theta_t + d\mathcal{P}_t - k\theta_{xx} - \zeta_1 \varphi_{xt} = 0, \\ d\theta_t + r\mathcal{P}_t - h\mathcal{P}_{xx} - \zeta_2 \varphi_{xt} = 0, \end{cases} \tag{4.1}$$

where $(x, t) \in (0, 1) \times (0, \infty)$ with the following initial and boundary conditions

$$\begin{cases} \mathcal{U}(x, 0) = \mathcal{U}_0(x), \ \mathcal{U}_t(x, 0) = \mathcal{U}_1(x), & x \in (0, 1), \\ \varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x), & x \in (0, 1), \\ \theta(x, 0) = \theta_0(x), \ \mathcal{P}(x, 0) = \mathcal{P}_0(x), & x \in (0, 1), \\ \varphi_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), & (x, t) \in (0, 1) \times (0, \tau(0)), \\ \mathcal{U}(0, t) = \varphi(0, t) = \theta(0, t) = \mathcal{P}(0, t) = 0, & \forall t \geq 0, \\ \mathcal{U}(1, t) = \varphi(1, t) = \theta(1, t) = \mathcal{P}(1, t) = 0, & \forall t \geq 0. \end{cases} \tag{4.2}$$

$\mathcal{U}(x,t)$ is the displacement of the fluid, $\varphi(x,t)$ is the elastic solid material, $\theta(x,t)$ is the temperature difference, $\mathcal{P}(x,t)$ is the chemical potential. ρ_1, ρ_2 are the densities of $\mathcal{U}$ and φ, respectively, where $\mathcal{k}$ and $\mathcal{h}$ represent respectively the heat and the mass diffusion conductivity coefficients. $a_1, a_3, \zeta_1, \zeta_2, \mu_1, r, c$ and d are positive constants, and $\mu_2, a_2 \neq 0$ are real numbers, where

$$a_1 a_3 > a_2^2, \tag{4.3}$$

and

$$\lambda = rc - d^2 > 0. \tag{4.4}$$

Equations (4.1) result from the fundamental field equations for the theory of swelling of one-dimensional porous elastic soils, which are given by ([37])

$$\begin{cases} \rho_1 \mathcal{U}_{tt} = t_{1x} - \mathcal{H}_1 + \mathcal{G}_1, \\ \rho_2 \varphi_{tt} = t_{2x} - \mathcal{H}_2 + \mathcal{G}_2, \end{cases} \tag{4.5}$$

the terms t_i, $\mathcal{G}_i$ and $\mathcal{H}_i$ represent the partial tensions, external forces, and internal body forces associated with the dependent variables $\mathcal{U}$ and φ. Furthermore, the constitutive equations for partial tensions are defined by

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}}_{\mathcal{A}} \begin{pmatrix} \mathcal{U}_x \\ \varphi_x \end{pmatrix}, \tag{4.6}$$

where $\mathcal{A}$ is a positive definite symmetric array, meaning that $a_1 a_3 > a_2^2$, and $\mathcal{H}_1 = \mathcal{H}_2 = 0$ represents the null hypothesis of the body's internal forces. In final form, we selected

$$\mathcal{G}_1 = 0 \text{ and } \mathcal{G}_2 = \zeta_1 \theta_x - \zeta_2 \mathcal{P}_x + \mu_1 \varphi_t + \mu_2 \varphi_t(x, t - \tau(t)),$$

we assume that as in [55], the time-varying delay $\tau(t)$ is considered to satisfy the following conditions:

$$\begin{cases} 0 < \tau_1 \leq \tau(t) \leq \tau_2, & \forall t > 0, \\ \tau'(t) \leq r_1 < 1, & \forall t > 0, \\ \tau \in W^{2,\infty}([0,T]), & \forall T > 0, \end{cases} \tag{4.7}$$

where r_1 is a positive constant, and that μ_1, μ_2 satisfy

$$|\mu_2| < \sqrt{1 - r_1}\mu_1. \tag{4.8}$$

Numerous results about the polynomial or exponential decline of thermoelastic solution were obtained [13,20,31,34,40,46].

The main contribution of this study is to examine the system (4.1)-(4.2), where the introduction of the time-varying delay term and thermodiffusion effects can differentiate and be important in the problem compared to other literature addressed.

4.2 Well-posedness

In order to prove the existence and uniqueness of a unique of problem (4.1)-(4.2), using semi group theory. as in [44], we introduce the new variable

$$\mathcal{Z}(x,\rho,t) = \varphi_t(x, t - \tau(t)\rho), \quad (x,\rho,t) \in (0,1)^2 \times (0,\infty). \tag{4.9}$$

which satisfies

$$\tau(t)\mathcal{Z}_t(x,\rho,t) + (1 - \tau'(t)\rho)\mathcal{Z}_\rho(x,\rho,t) = 0, \tag{4.10}$$

for $(x,\rho,t) \in (0,1)^2 \times (0,\infty)$.

Hence, problem (4.1) is equivalent to

$$\begin{cases} \rho_1 \mathcal{U}_{tt} - a_1 \mathcal{U}_{xx} - a_2 \varphi_{xx} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \rho_2 \varphi_{tt} - a_3 \varphi_{xx} - a_2 \mathcal{U}_{xx} - \zeta_1 \theta_x - \zeta_2 \mathcal{P}_x & \\ \quad + \mu_1 \varphi_t + \mu_2 \mathcal{Z}(x,1,t) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \tau(t)\mathcal{Z}_t(x,\rho,t) + (1 - \tau'(t)\rho)\mathcal{Z}_\rho(x,\rho,t) = 0, & (x,\rho,t) \in (0,1)^2 \times (0,\infty), \\ c\theta_t + d\mathcal{P}_t - k\theta_{xx} - \zeta_1 \varphi_{xt} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ d\theta_t + r\mathcal{P}_t - h\mathcal{P}_{xx} - \zeta_2 \varphi_{xt} = 0, & (x,t) \in (0,1) \times (0,\infty). \end{cases} \tag{4.11}$$

with the following initial and boundary conditions

$$\begin{cases} \mathcal{U}(x,0) = \mathcal{U}_0(x), \ \mathcal{U}_t(x,0) = \mathcal{U}_1(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \end{cases}$$

$$\begin{cases} \theta(x,0)=\theta_0(x),\ \mathcal{P}(x,0)=\mathcal{P}_0(x), & x\in(0,1),\\ \mathcal{Z}(x,0,t)=\varphi_t(x,t), & x\in(0,1),\\ \mathcal{Z}(x,\rho,0)=f_0(x,-\rho\tau(0)), & x,\rho\in(0,1),\\ \mathcal{U}(0,t)=\varphi(0,t)=\theta(0,t)=\mathcal{P}(0,t)=0, & \forall t\geq 0,\\ \mathcal{U}(1,t)=\varphi(1,t)=\theta(1,t)=\mathcal{P}(1,t)=0, & \forall t\geq 0. \end{cases} \tag{4.12}$$

Now, we set $\mathcal{V}=\mathcal{U}_t, \psi=\varphi_t$ and let $\mathscr{U}=(\mathcal{U},\mathcal{V},\varphi,\psi,\mathcal{Z},\theta,\mathcal{P})^T$, then system (4.11)-(4.12) can be written as

$$\begin{cases} \mathscr{U}_t=\mathscr{A}(t)\mathscr{U}, & t>0,\\ \mathscr{U}(0)=\mathscr{U}_0=(\mathcal{U}_0,\mathcal{U}_1,\varphi_0,\varphi_1,f_0(x,-\rho\tau(0)),\theta_0,\mathcal{P}_0)^T. \end{cases} \tag{4.13}$$

where the operator $\mathscr{A}(t):\mathscr{D}(\mathscr{A}(t))\subset\mathscr{H}\longrightarrow\mathscr{H}$ is defined by

$$\mathscr{A}(t)\mathscr{U}=\begin{pmatrix} \mathcal{V}\\ \frac{1}{\rho_1}\left(a_1\mathcal{U}_{xx}+a_2\varphi_{xx}\right)\\ \psi\\ \frac{1}{\rho_2}\left(a_3\varphi_{xx}+a_2\mathcal{U}_{xx}+\zeta_1\theta_x+\zeta_2\mathcal{P}_x-\mu_1\psi-\mu_2\mathcal{Z}(x,1,t)\right)\\ -\frac{1-\tau'(t)\rho}{\tau(t)}\mathcal{Z}_\rho(x,\rho,t)\\ \left(\frac{rk}{\lambda}\right)\theta_{xx}-\left(\frac{hd}{\lambda}\right)\mathcal{P}_{xx}+\left(\frac{r\zeta_1-d\zeta_2}{\lambda}\right)\psi_x\\ \left(\frac{ch}{\lambda}\right)\mathcal{P}_{xx}-\left(\frac{kd}{\lambda}\right)\theta_{xx}+\left(\frac{c\zeta_2-d\zeta_1}{\lambda}\right)\psi_x \end{pmatrix}.$$

And where the energy space $\mathscr{H}$ is :

$$\mathscr{H}=\left(H_0^1(0,1)\times L^2(0,1)\right)^2\times L^2\left((0,1)\times(0,1)\right)\times\left(L^2(0,1)\right)^2,$$

For $\mathscr{U}=(\mathcal{U},\mathcal{V},\varphi,\psi,\mathcal{Z},\theta,\mathcal{P})^T$ and $\widetilde{\mathscr{U}}=\left(\widetilde{\mathcal{U}},\widetilde{\mathcal{V}},\widetilde{\varphi},\widetilde{\psi},\widetilde{\mathcal{Z}},\widetilde{\theta},\widetilde{\mathcal{P}}\right)^T$, we define the inner product in $\mathscr{H}$ by

$$\begin{aligned}\left\langle\mathscr{U},\widetilde{\mathscr{U}}\right\rangle_{\mathscr{H}} &= \rho_1\int_0^1\mathcal{V}\widetilde{\mathcal{V}}dx+a_1\int_0^1\mathcal{U}_x\widetilde{\mathcal{U}}_xdx+\rho_2\int_0^1\psi\widetilde{\psi}dx\\ &\quad+a_3\int_0^1\varphi_x\widetilde{\varphi}_xdx+a_2\int_0^1\left(\mathcal{U}_x\widetilde{\varphi}_x+\varphi_x\widetilde{\mathcal{U}}_x\right)dx+\int_0^1 c\theta\widetilde{\theta}dx\\ &\quad+d\int_0^1\left(\mathcal{P}\widetilde{\theta}+\theta\widetilde{\mathcal{P}}\right)dx+r\int_0^1\mathcal{P}\widetilde{\mathcal{P}}dx+\int_0^1\int_0^1\mathcal{Z}(x,\rho)\widetilde{\mathcal{Z}}(x,\rho)dxd\rho.\end{aligned}$$

Next, we define $\mathscr{D}(\mathscr{A})$ by

$$\mathscr{D}(\mathscr{A}(t)) = \left\{ \mathscr{U} \in \mathcal{H} / \quad \psi = z(.,0) \right\}.$$

for $t > 0$, where

$$\begin{aligned} \mathcal{H} = \quad & \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1) \times \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1) \\ & \times L^2((0,1), H^1(0,1)) \times \left(H^2(0,1) \cap H_0^1(0,1)\right) \times \left(H^2(0,1) \cap H_0^1(0,1)\right) \end{aligned}$$

Note that the domain of $\mathscr{A}(t)$ is independent of the time $t > 0$.

The existence and uniqueness results are provided as follows:

Theorem 4.1. *Let (4.7) be satisfied and assume that (4.8) holds. Then, for any $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}(0))$, there exists a unique solution $\mathscr{U}$ of problem (4.11)-(4.12) satisfying*

$$\mathscr{U} \in C\left([0,+\infty), \mathscr{D}(\mathscr{A}(0))\right) \cap C^1\left([0,+\infty), \mathscr{H}\right).$$

In order to prove Theorem 4.1, we will use the variable norm technique developed by Kato in [42], where the following theorem is proved:

Theorem 4.2. *Assume that:*

(i) $\mathscr{D}(\mathscr{A}(0))$ *is a dense subset of* $\mathscr{H}$;

(ii) $\mathscr{D}(\mathscr{A}(t)) = \mathscr{D}(\mathscr{A}(0)), \quad \forall t > 0;$

(iii) for all $t \in [0,T]$, $\mathscr{A}(t)$ generates a strongly continuous semigroup on $\mathscr{H}$ and the family $\mathscr{A} = \{\mathscr{A}(t) : t \in [0,T]\}$ is stable with stability constants C and m independent of t. i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathscr{A}(t)$ satisfies

$$\|S_t(s)(\mathscr{U})\|_{\mathscr{H}} \leq Ce^{ms}\|\mathscr{U}\|_{\mathscr{H}}, \quad \forall \mathscr{U} \in \mathscr{H}, s \geq 0;$$

(iv) $\partial_t\mathscr{A}(t) \in L_*^\infty\left([0,T], B\left(\mathscr{D}(\mathscr{A}(0)), \mathscr{H}\right)\right)$, where $L_*^\infty\left([0,T], B\left(\mathscr{D}(\mathscr{A}(0)), \mathscr{H}\right)\right)$ is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B\left(\mathscr{D}(\mathscr{A}(0)), \mathscr{H}\right)$ of bounded operators from $\mathscr{A}(0))$ into $\mathscr{H}$

Then problem (4.13) has a unique solution

$$\mathscr{U} \in C\left([0,T), \mathscr{D}(\mathscr{A}(0))\right) \cap C^1\left([0,T), \mathscr{H}\right),$$

for any initial datum in $\mathcal{D}(\mathcal{A}(0))$.

Proof of Theorem 4.2. We will apply the technique described in [47, 55], making suitable modifications based on the specifics of our problem.

(i) First, we show that $\mathcal{D}(\mathcal{A}(0))$ is dense in $\mathcal{H}$. For, let $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$ be orthogonal to all elements of $\mathcal{D}(\mathcal{A}(0))$ with respect to the inner product $\langle .,. \rangle_{\mathcal{H}}$:

$$0 = \langle \mathcal{U}, \mathcal{F} \rangle_{\mathcal{H}} = \int_0^1 \left(\rho_1 \mathcal{V} f_2 + a_1 \mathcal{U}_x f_{1x} + \rho_2 \psi f_4 + a_3 \varphi_x f_{3x} + a_2 \left(\mathcal{U}_x f_{3x} + \varphi_x f_{1x}\right)\right) dx \\ + \int_0^1 \left(c\theta f_6 + d\left(\mathcal{P} f_6 + \theta f_7\right) + r\mathcal{P} f_7\right) dx + \int_0^1 \int_0^1 z(x,\rho) f_5(x,\rho) d\rho dx, \tag{4.14}$$

for all $\mathcal{U} = (\mathcal{U}, \mathcal{V}, \varphi, \psi, z, \theta, \mathcal{P})^T \in \mathcal{D}(\mathcal{A}(0))$, Our objective is to demonstrate that $f_i, i = 1, .., 7$.

Let us first take $z \in \mathcal{D}((0,1) \times (0,1))$ and $\mathcal{U} = \mathcal{V} = \varphi = \psi = \theta = \mathcal{P} = 0$, so the vector $\mathcal{U} = (0,0,0,0,z,0,0)^T \in \mathcal{D}(\mathcal{A}(0))$ and therefore, from (4.14), we deduce that

$$\int_0^1 \int_0^1 z(x,\rho) f_5(x,\rho) d\rho dx = 0.$$

Since $\mathcal{D}((0,1) \times (0,1))$ is dense in $L^2((0,1) \times (0,1))$, then, $f_5 = 0$.

Let $\mathcal{V} \in \mathcal{D}(0,1)$, then, $\mathcal{U} = (0, \mathcal{V}, 0, 0, 0, 0, 0)^T \in \mathcal{D}(\mathcal{A}(0))$, which implies from (4.14) that

$$\int_0^1 \mathcal{V} f_2 dx = 0,$$

so, as above, $f_2 = 0$.

similarly, we have $f_4, f_6, f_7 = 0$. Next, let $\mathcal{U} = (\mathcal{U}, 0, 0, 0, 0, 0, 0)^T$ then we obtain from (4.14) that

$$\int_0^1 \mathcal{U}_x f_{1x} dx = 0.$$

It is obvious that $(\mathcal{U}, 0, 0, 0, 0, 0, 0)^T \in \mathcal{D}(\mathcal{A}(0))$, if and only if $\mathcal{U} \in H^2(0,1) \cap H_0^1(0,1)$ and since $H^2(0,1) \cap H_0^1(0,1)$ is dense in $H_0^1(0,1)$ with respect to the inner product

$$\langle g, h \rangle_{H_0^1(0,1)} = \int_0^1 g_x h_x dx,$$

we get $f_1 = 0$. We can also demonstrate that $f_3 = 0$ using the same concepts as above.

(ii) With our choice, $\mathscr{D}(\mathscr{A}(t))$ is independent of t, and therefore

$$\mathscr{D}(\mathscr{A}(t)) = \mathscr{D}(\mathscr{A}(0)), \quad \forall t > 0.$$

(iii) Now, we show that the operator $\mathscr{A}(t)$ generates a C_0-semigroup in $\mathscr{H}$ for a fixed t. We define the time-dependent inner-product on $\mathscr{H}$, (which is equivalent to the classical inner product)

$$\begin{aligned}
\left\langle \mathscr{U}, \widetilde{\mathscr{U}} \right\rangle_t &= \int_0^1 \left\{ \rho_1 \mathcal{V}\widetilde{\mathcal{V}} + a_1 \mathcal{U}_x \widetilde{\mathcal{U}}_x + \rho_2 \psi \widetilde{\psi} + a_3 \varphi_x \widetilde{\varphi}_x + a_2 \left(\mathcal{U}_x \widetilde{\varphi}_x + \varphi_x \widetilde{\mathcal{U}}_x \right) \right\} dx \\
&+ \int_0^1 \left\{ c\theta\widetilde{\theta} + d \left(\mathcal{P}\widetilde{\theta} + \theta \widetilde{\mathcal{P}} \right) + r\mathcal{P}\widetilde{\mathcal{P}} \right\} dx \\
&+ \eta \tau(t) \int_0^1 \int_0^1 z(x,\rho) \widetilde{z}(x,\rho) d\rho dx.
\end{aligned}$$

where η satisfies

$$\frac{|\zeta_2|}{\sqrt{1-r_1}} \leq \eta \leq 2\zeta_1 - \frac{|\zeta_2|}{\sqrt{1-r_1}}, \tag{4.15}$$

thanks to hypothesis (4.8). Let us set

$$\hbar(t) = \frac{\left(\tau'(t)^2 + 1\right)^{\frac{1}{2}}}{2\tau(t)}.$$

The following step involves proving the dissipativity of the operator $\widetilde{\mathscr{A}}(t) = \mathscr{A}(t) - \hbar(t) I$. For a fixed t and $\mathscr{U} = (\mathcal{U}, \mathcal{V}, \varphi, \psi, z, \theta, p)^T \in \mathscr{D}(\mathscr{A}(t))$, we get

$$\begin{aligned}
\langle \mathscr{A}(t)\mathscr{U}, \mathscr{U} \rangle_t = &- \mu_1 \int_0^1 \psi^2 dx - \mu_2 \int_0^1 z(x,1,t)\psi dx - k \int_0^1 \theta_x^2 dx - h \int_0^1 \mathcal{P}_x^2 dx \\
&- \eta \int_0^1 \int_0^1 \left(1 - \tau'(t)\rho\right) z(x,\rho) z_\rho(x,\rho) dx d\rho.
\end{aligned} \tag{4.16}$$

Observe that

$$\begin{aligned}
&\int_0^1 \int_0^1 \left(1 - \tau'(t)\rho\right) z(x,\rho) z_\rho(x,\rho) dx d\rho \\
&= \int_0^1 \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} z^2 \left(1 - \tau'(t)\rho\right) dx d\rho \\
&= \frac{\tau'(t)}{2} \int_0^1 \int_0^1 z^2(x,\rho) dx d\rho + \frac{1}{2} \int_0^1 \left\{ z^2(x,1)(1 - \tau'(t) - z^2(x,0) \right\} dx.
\end{aligned} \tag{4.17}$$

Whereupon

$$\begin{aligned}\langle \mathscr{A}(t)\mathscr{U},\mathscr{U}\rangle_t = &-\mu_1\int_0^1 \psi^2 dx - \mu_2\int_0^1 \mathcal{Z}(x,1)\psi dx - \kappa\int_0^1 \theta_x^2 dx \\ &- \hbar\int_0^1 \mathcal{P}_x^2 dx - \frac{\eta\tau'(t)}{2}\int_0^1\int_0^1 \mathcal{Z}^2(x,\rho)dxd\rho \\ &- \frac{\eta}{2}\int_0^1 \mathcal{Z}^2(x,1)(1-\tau'(t))dx + \frac{\eta}{2}\int_0^1 \psi^2 dxd\rho. \end{aligned} \tag{4.18}$$

By using Cauchy-Schwarz inequality and $(4.7)_2$, we get

$$\begin{aligned}\langle \mathscr{A}(t)\mathscr{U},\mathscr{U}\rangle_t \leq &\left(-\mu_1 + \frac{|\mu_2|}{2\sqrt{1-r_1}} + \frac{\eta}{2}\right)\int_0^1 \psi^2 dx - \kappa\int_0^1 \theta_x^2 dx - -\hbar\int_0^1 \mathcal{P}_x^2 dx \\ &+ \left(\frac{|\mu_2|\sqrt{1-r_1}}{2} - \frac{\eta}{2}(1-r_1)\right)\int_0^1 \mathcal{Z}^2(x,1)dx + \hbar(t)\langle \mathscr{U},\mathscr{U}\rangle_t .\end{aligned}$$

Condition (4.15) allows writing.

$$-\mu_1 + \frac{|\mu_2|}{2\sqrt{1-r_1}} + \frac{\eta}{2} \leq 0, \quad \frac{|\mu_2|\sqrt{1-r_1}}{2} - \eta\frac{(1-r_1)}{2} \leq 0.$$

As a result, the operator $\widetilde{\mathscr{A}}(t)$ is dissipative.

Now, we prove the surjectivity of the operator $(I - \mathscr{A}(t))$ for fixed $t > 0$. For, let $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathscr{H}$, we seek $\mathscr{U} = (\mathcal{U}, \mathcal{V}, \varphi, \psi, \mathcal{Z}, \theta, \mathcal{P})^T \in \mathscr{D}(\mathscr{A}(t))$ resolution of the following equations system

$$\begin{cases} \mathcal{U} - \mathcal{V} = f_1, \\ \mathcal{V} - \frac{a_1}{\rho_1}\mathcal{U}_{xx} - \frac{a_2}{\rho_1}\varphi_{xx} = f_2, \\ \varphi - \psi = f_3, \\ \psi - \frac{a_3}{\rho_2}\varphi_{xx} - \frac{a_2}{\rho_2}\mathcal{U}_{xx} - \frac{\zeta_1}{\rho_2}\theta_x - \frac{\zeta_2}{\rho_2}\mathcal{P}_x + \frac{\mu_1}{\rho_2}\psi + \frac{\mu_2}{\rho_2}\mathcal{Z}(x,1) = f_4, \\ \mathcal{Z} + \frac{(1-\tau'(t)\rho)}{\tau(t)}\mathcal{Z}_\rho = f_5, \\ \theta - \left(\frac{r\kappa}{\lambda}\right)\theta_{xx} + \left(\frac{\hbar d}{\lambda}\right)\mathcal{P}_{xx} - \left(\frac{r\zeta_1 - d\zeta_2}{\lambda}\right)\psi_x = f_6, \\ \mathcal{P} - \left(\frac{c\hbar}{\lambda}\right)\mathcal{P}_{xx} - \left(\frac{\hbar d}{\lambda}\right)\theta_{xx} + \left(\frac{c\zeta_2 - d\zeta_1}{\lambda}\right)\psi_x = f_7. \end{cases} \tag{4.19}$$

Assume we find $\mathcal{U}$ and φ. Then, the first and third equations in (4.19) yield

$$\begin{cases} \mathcal{V} = \mathcal{U} - f_1, \\ \psi = \varphi - f_3. \end{cases} \tag{4.20}$$

Furthermore, using (4.19), we find $\mathcal{Z}$ as:

$$\mathcal{Z}(x,0) = \psi(x), \quad \text{for } x \in (0,1). \tag{4.21}$$

Following the same procedure as in [55] , we get, by using the last equation in $(4.19)_5$,

$$\mathcal{Z}(x,\rho) = \psi(x)e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \int_0^1 f_5(x,\sigma)\, e^{\sigma\tau(t)} d\sigma, \quad \text{if } \tau'(t) = 0,$$

and

$$\mathcal{Z}(x,\rho) = \psi(x)e^{\vartheta_\rho(t)} + e^{\vartheta_\rho(t)} \int_0^1 \frac{f_5(x,\sigma)\,\tau(t)}{1-\tau'(t)\sigma} e^{-\vartheta_\sigma(t)} d\sigma, \quad \text{if } \tau'(t) \neq 0,$$

where $\vartheta_\rho(t) = \frac{\tau(t)}{\tau'(t)} ln\,(1-\tau'(t)\rho)$. Then, from (4.20), we get

$$\mathcal{Z}(x,\rho) = \varphi(x)e^{-\rho\tau(t)} - f_3 e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \int_0^1 f_5(x,\sigma)\, e^{\sigma\tau(t)} d\sigma, \quad \text{if } \tau'(t) = 0, \tag{4.22}$$

and

$$\mathcal{Z}(x,\rho) = \varphi(x)e^{\vartheta_\rho(t)} - f_3 e^{\vartheta_\rho(t)} + e^{\vartheta_\rho(t)} \int_0^1 \frac{f_5(x,\sigma)\,\tau(t)}{1-\tau'(t)\sigma} e^{-\vartheta_\sigma(t)} d\sigma, \quad \text{if } \tau'(t) \neq 0. \tag{4.23}$$

However, we also have

$$\mathcal{Z}(x,1) = \begin{cases} \varphi(x)e^{-\tau(t)} + \mathcal{Z}_0(x), & \text{if } \tau'(t) = 0, \\ \varphi(x)e^{\vartheta_1(t)} + \mathcal{Z}_0(x), & \text{if } \tau'(t) \neq 0, \end{cases} \tag{4.24}$$

where $x \in (0,1)$ and

$$\mathcal{Z}_0(x) = \begin{cases} -f_3 e^{-\tau(t)} + \tau(t)e^{-\tau(t)} \int_0^1 f_5(x,\sigma)e^{\sigma\tau(t)} d\sigma, & \text{if } \tau'(t) = 0, \\ -f_3 e^{\vartheta_1(t)} + e^{\vartheta_1(t)} \int_0^1 \frac{f_5(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\vartheta_\sigma(t)} d\sigma, & \text{if } \tau'(t) \neq 0. \end{cases} \tag{4.25}$$

If we substitute (4.20) and (4.24), into (4.19) we obtain

$$\begin{cases} \rho_1 \mathcal{U} - a_1 \mathcal{U}_{xx} - a_2 \varphi_{xx} = g_1, \\ \mathcal{M}\varphi - a_3 \varphi_{xx} - a_2 \mathcal{U}_{xx} - \zeta_1 \theta_x - \zeta_2 \mathcal{P}_x = g_2, \\ \lambda\theta - rk\theta_{xx} + hd\mathcal{P}_{xx} - (r\zeta_1 - d\zeta_2)\varphi_x = g_3, \\ \lambda\mathcal{P} - ch\mathcal{P}_{xx} + kd\theta_{xx} - (c\zeta_2 - d\zeta_1)\varphi_x = g_4, \end{cases} \tag{4.26}$$

where

$$\begin{cases} \mathcal{M} = \rho_2 + \mu_1 + \mu_2 e^{-\tau(t)}, \\ g_1 = \rho_1 (f_2 + f_1), \\ g_2 = \rho_2 (f_4 + f_3) + \mu_1 f_3 - \mu_2 z_0(x), \qquad \text{if } \tau'(t) = 0, \\ g_3 = \lambda f_6 - (r\zeta_1 - d\zeta_2) f_{3x}, \\ g_4 = \lambda f_7 - (c\zeta_2 - d\zeta_1) f_{3x}, \end{cases} \tag{4.27}$$

and,

$$\begin{cases} \mathcal{M} = \rho_2 + \mu_1 + \mu_2 e^{\vartheta_1(t)}, \\ g_1 = \rho_1 (f_2 + f_1), \\ g_2 = \rho_2 (f_4 + f_3) + \mu_1 f_3 - \mu_2 z_0(x), \qquad \text{if } \tau'(t) \neq 0, \\ g_3 = \lambda f_6 - (r\zeta_1 - d\zeta_2) f_{3x}, \\ g_4 = \lambda f_7 - (c\zeta_2 - d\zeta_1) f_{3x}, \end{cases} \tag{4.28}$$

We multiply $(4.26)_1$ by $\widetilde{\mathcal{U}}$, $(4.26)_2$ by $\widetilde{\varphi}$, $(4.26)_3$ by $\frac{c}{\lambda}\widetilde{\theta}$, $(4.26)_4$ by $\frac{r}{\lambda}\widetilde{\mathcal{P}}$, $(4.26)_3$ by $\frac{d}{\lambda}\widetilde{\mathcal{P}}$ and $(4.26)_4$ by $\frac{d}{\lambda}\widetilde{\theta}$ and integrate their sum over $(0,1)$ to obtain the following variational formulation

$$\mathscr{B}\left((\mathcal{U},\varphi,\theta,\mathcal{P}),\left(\widetilde{\mathcal{U}},\widetilde{\varphi},\widetilde{\theta},\widetilde{\mathcal{P}}\right)\right) = \mathscr{L}\left(\widetilde{\mathcal{U}},\widetilde{\varphi},\widetilde{\theta},\widetilde{\mathcal{P}}\right), \tag{4.29}$$

where the bilinear form $\mathscr{B} : \left(H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)\right)^2 \longrightarrow \mathbb{R}$, and the linear form $\mathscr{L} : \left(H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)\right) \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} \mathscr{B}\left((\mathcal{U},\varphi,\theta,\mathcal{P}),\left(\widetilde{\mathcal{U}},\widetilde{\varphi},\widetilde{\theta},\widetilde{\mathcal{P}}\right)\right) &= \int_0^1 \left(\rho_1 \mathcal{U}\widetilde{\mathcal{U}} + a_1 \mathcal{U}_x \widetilde{\mathcal{U}}_x + a_2\left(\varphi_x \widetilde{\mathcal{U}}_x + \mathcal{U}_x \widetilde{\varphi}_x\right)\right. \\ &\quad \left. + \left(\rho_2 + \mu_1 + \mu_2 e^{-\tau(t)}\right)\varphi\widetilde{\varphi}_x + a_3 \varphi_x \widetilde{\varphi}_x + c\theta\widetilde{\theta}\right) dx \end{aligned}$$

$$+\int_0^1 \left(k\theta_x\widetilde{\theta}_x + r\mathcal{P}\widetilde{\mathcal{P}} + h\mathcal{P}_x\widetilde{\mathcal{P}}_x + d\left(\theta\widetilde{\mathcal{P}} + \mathcal{P}\widetilde{\theta}\right)\right.$$
$$\left. +\zeta_1\left(\theta\widetilde{\varphi}_x - \varphi_x\widetilde{\theta}\right) + \zeta_2\left(\mathcal{P}\widetilde{\varphi}_x - \varphi_x\widetilde{\mathcal{P}}\right)\right)dx,$$

and

$$\mathcal{L}\left(\widetilde{\mathcal{U}},\widetilde{\varphi},\widetilde{\theta},\widetilde{\mathcal{P}}\right) = \int_0^1 \left(g_1\widetilde{\mathcal{U}} + g_2\widetilde{\varphi} + \left(\frac{c}{\lambda}\right)g_3\widetilde{\theta} + \left(\frac{r}{\lambda}\right)g_4\widetilde{\mathcal{P}} + \left(\frac{d}{\lambda}\right)g_3\widetilde{\mathcal{P}} + \left(\frac{d}{\lambda}\right)g_4\widetilde{\theta}\right)dx, \tag{4.30}$$

if $\tau'(t) = 0$.

If $\tau'(t) \neq 0$, we define

$$\mathcal{B}\left((\mathcal{U},\varphi,\theta,\mathcal{P}),\left(\widetilde{\mathcal{U}},\widetilde{\varphi},\widetilde{\theta},\widetilde{\mathcal{P}}\right)\right)$$
$$= \int_0^1 \left(\rho_1\mathcal{U}\widetilde{\mathcal{U}} + a_1\mathcal{U}_x\widetilde{\mathcal{U}}_x + a_2\left(\varphi_x\widetilde{\mathcal{U}}_x + \mathcal{U}_x\widetilde{\varphi}_x\right) + \left(\rho_2 + \mu_1 + \mu_2 e^{\vartheta_1(t)}\right)\varphi\widetilde{\varphi}_x + a_3\varphi_x\widetilde{\varphi}_x + c\theta\widetilde{\theta}\right)dx$$
$$+ \int_0^1 \left(k\theta_x\widetilde{\theta}_x + r\mathcal{P}\widetilde{\mathcal{P}} + h\mathcal{P}_x\widetilde{\mathcal{P}}_x + d\left(\theta\widetilde{\mathcal{P}} + \mathcal{P}\widetilde{\theta}\right) + \zeta_1\left(\theta\widetilde{\varphi}_x - \varphi_x\widetilde{\theta}\right) + \zeta_2\left(\mathcal{P}\widetilde{\varphi}_x - \varphi_x\widetilde{\mathcal{P}}\right)\right)dx,$$

the operator $\mathcal{L}$ has the same formula (4.30), with $\mathcal{Z}_0$ satisfying the second equation in (4.25). It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $\left(\widetilde{\mathcal{U}},\widetilde{\varphi},\widetilde{\theta},\widetilde{\mathcal{P}}\right) \in H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$, problem (4.29) admits a unique solution $(\mathcal{U},\varphi,\theta,\mathcal{P}) \in H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$. The application of the classical regularity theory, it follows from (4.26) that $(\mathcal{U},\varphi,\theta,\mathcal{P}) \in H^2(0,1) \times H^2(0,1) \times H^2(0,1) \times H^2(0,1)$. Hence, the operator $I - \mathcal{A}(t)$ is surjective for any fixed $t > 0$. Since $\hbar(t) > 0$ and

$$I - \widetilde{\mathcal{A}}(t) = (1 + \hbar(t))\, I - \mathcal{A}(t),$$

Thus, for $t > 0$, we also conclude that the operator $I - \widetilde{\mathcal{A}}(t)$ is surjective. To complete the proof of (iii), it's suffices to show that

$$\frac{\|\Psi\|_t}{\|\Psi\|_s} \leq e^{\frac{b}{2\tau_1}|t-s|}, \qquad \forall t,s \in [0,T], \tag{4.31}$$

where b is a positive constant, $\Psi = (\mathcal{U},\mathcal{V},\varphi,\psi,\mathcal{Z},\theta,\mathcal{P})$ and $\|.\|_t$ is the norm associated to the inner product (4.15). For $t, s \in [0,1]$, we have from (4.15)

$$\begin{aligned} &\|\Psi\|_s^2 - \|\Psi\|_s^2 e^{\frac{\mathscr{b}}{\tau_1}|t-s|} \\ &= \left(1 - e^{\frac{\mathscr{b}}{\tau_1}|t-s|}\right)\int_0^1 \left(\rho_1 \mathcal{V}^2 + a_1 \mathcal{U}_x^2 + \rho_2 \psi^2 + a_3 \varphi_x^2 + 2a_2 \mathcal{U}_x \varphi_x \right. \\ &\left. + c\theta^2 + 2d\mathcal{P}\theta + r\mathcal{P}^2\right) dx + \eta\left(\tau(t) - \tau(s) e^{\frac{\mathscr{b}}{\tau_1}|t-s|}\right)\int_0^1\int_0^1 \mathcal{Z}^2(x,\rho)\, dx d\rho. \end{aligned} \tag{4.32}$$

We notice that $1 - e^{\frac{\mathscr{b}}{\tau_1}|t-s|} \leq 0$. Now, it remains to show that $\tau(t) - \tau(s)e^{\frac{\mathscr{b}}{\tau_1}|t-s|} \leq 0$ for some $\mathscr{b} > 0$. To do this, we have

$$\tau(t) = \tau(s) + \tau'(a)(t-s),$$

where $a \in (s,t)$ which implies

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau'(a)|}{\tau(s)}|t-s|.$$

Using $(4.7)_3$, we conclude that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{\mathscr{b}}{\tau_1}|t-s| \leq e^{\frac{\mathscr{b}}{\tau_1}|t-s|}.$$

which his demonstrates (4.31) and therefore (iii).

(iv) Clearly, that

$$\frac{d}{dt}\mathscr{A}(t)\mathscr{U} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{(\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1))}{\tau^2(t)}\mathcal{Z}_\rho \\ 0 \\ 0 \end{pmatrix}.$$

Then, by using $(4.7)_3$ and $(4.7)_1$, (iv) holds exactly as in [55]. Consequently, from the above analysis, we deduce that the problem

$$\begin{cases} \widetilde{\mathscr{U}_t} = \widetilde{\mathscr{A}}(t)\widetilde{\mathscr{U}}, \\ \widetilde{\mathscr{U}}(0) = \mathscr{U}_0, \end{cases} \tag{4.33}$$

has a unique solution $\widetilde{\mathscr{U}} \in \mathcal{C}\left([0,\infty), \mathscr{H}\right)$ and if $\mathscr{U}_0 \in \mathscr{D}\left(\mathscr{A}(0)\right)$, then

$$\widetilde{\mathscr{U}} \in \mathcal{C}\left([0,\infty), \mathscr{D}\left(\mathscr{A}(0)\right)\right) \cap \mathcal{C}^1\left([0,\infty), \mathscr{H}\right).$$

Now, we let

$$\mathscr{U}(t) = e^{\mathcal{N}(t)}\widetilde{\mathscr{U}}(t),$$

with $\mathcal{N}(t) = \int_0^1 \hbar(s)ds$, then we have by using (4.33)

$$\begin{aligned}
\mathscr{U}_t(t) &= \hbar(t)e^{\mathcal{N}(t)}\widetilde{\mathscr{U}}(t) + e^{\mathcal{N}(t)}\widetilde{\mathscr{U}_t}(t) \\
&= \hbar(t)e^{\mathcal{N}(t)}\widetilde{\mathscr{U}}(t) + e^{\mathcal{N}(t)}\widetilde{\mathscr{A}}(t)\widetilde{\mathscr{U}}(t) \\
&= e^{\mathcal{N}(t)}\left(\hbar(t)\widetilde{\mathscr{U}}(t) + \widetilde{\mathscr{A}}(t)\widetilde{\mathscr{U}}(t)\right) \\
&= e^{\mathcal{N}(t)}\mathscr{A}(t)\widetilde{\mathscr{U}}(t) \\
&= \mathscr{A}(t)e^{\mathcal{N}(t)}\widetilde{\mathscr{U}}(t) \\
&= \mathscr{A}(t)\mathscr{U}(t).
\end{aligned}$$

Consequently, $\mathscr{U}(t)$ is the unique solution of (4.13). This ends the proof of Theorem 4.1. □

4.3 Exponential stability

In this section, we suppose that $(\mathcal{U}, \varphi, \mathcal{Z}, \theta, \mathcal{P})$ is a solution of (4.11)-(4.12) and we demonstrate the exponential decay for our problem. The energy approach will be used to establish an appropriate Lyapunov function.

We define the energy functional $\mathcal{E}(t)$ as

$$\begin{aligned}
\mathcal{E}(t) = \; & \frac{1}{2}\int_0^1 \left(\rho_1 \mathcal{U}_t^2 + a_1 \mathcal{U}_x^2 + \rho_2 \varphi_t^2 + a_3 \varphi_x^2 + 2a_2 \mathcal{U}_x \varphi_x + c\theta^2 + 2d\theta\mathcal{P} + r\mathcal{P}^2\right) dx \\
& + \frac{\eta}{2}\tau(t)\int_0^1\int_0^1 \varphi_t^2\left(x, t-\tau(t)\rho\right) dxd\rho.
\end{aligned} \tag{4.34}$$

Noting (4.4), we have for $\theta, \mathcal{P} \neq 0$,

$$c\theta^2 + 2d\theta\mathcal{P} + r\mathcal{P}^2 = \frac{\lambda}{r}\theta^2 + \left(\frac{d}{\sqrt{r}}\theta + \sqrt{r}\mathcal{P}\right)^2 > 0,$$

then we get that the energy E (t) is positive. We have the following result.

Lemma 4.1. *Assume that (4.8) holds and the hypotheses (4.7) are satisfied. Then the energy $\mathcal{E}(t)$ is non-increasing, and there exists a positive constant $\mathcal{C}_*$ such that for any solution of (4.11)-(4.12), and for any $t \geq 0$, we have*

$$\frac{d\mathcal{E}(t)}{dt} \leq -\mathcal{C}_* \left(\int_0^1 \varphi_t^2(x,t)dx + \int_0^1 \varphi_t^2(x,t-\tau(t))dx \right) - k \int_0^1 \theta_x^2 dx - \hbar \int_0^1 \mathcal{P}_x^2 dx. \tag{4.35}$$

Proof. By multiplying $(4.11)_1$, $(4.11)_2$, $(4.11)_4$ and $(4.11)_5$ by $\mathcal{U}_t, \varphi_t, \theta$ and $\mathcal{P}$, respectively, then integrating over $(0,1)$ with respect to x, using integration by parts and the boundary conditions, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left(\rho_1 \mathcal{U}_t^2 + a_1 \mathcal{U}_x^2 + \rho_2 \varphi_t^2 + a_3 \varphi_x^2 + 2a_2 \mathcal{U}_x \varphi_x + c\theta^2 + 2d\theta\mathcal{P} + r\mathcal{P}^2 \right)(x,t)dx \right) \\ &= -k \int_0^1 \theta_x^2 dx - \hbar \int_0^1 \mathcal{P}_x^2 dx - \mu_1 \int_0^1 \varphi_t^2 dx - \mu_2 \int_0^1 \varphi_t \mathcal{Z}(x,1,t)dx. \end{aligned} \tag{4.36}$$

Now, we multiplying $(4.11)_3$ by $\eta\mathcal{Z}$ and integrate the resulting equation over $(0,1)\times(0,1)$ with respect to ρ and x, respectively, to get

$$\begin{aligned} &\frac{\eta}{2}\frac{d}{dt}\int_0^1\int_0^1 \tau(t)\mathcal{Z}^2(x,\rho,t)d\rho dx \\ &\quad = -\eta \int_0^1\int_0^1 \left(1-\tau'(t)\rho\right)\mathcal{Z}(x,\rho,t)\mathcal{Z}_t(x,\rho,t)dxd\rho + \frac{\eta}{2}\tau'(t)\int_0^1\int_0^1 \mathcal{Z}^2(x,\rho,t)dxd\rho \\ &\quad = -\frac{\eta}{2}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(1-\tau'(t)\rho\right)\mathcal{Z}^2(x,\rho,t)dxd\rho \\ &\quad = \frac{\eta}{2}\int_0^1 \left(\mathcal{Z}^2(x,0,t)-\mathcal{Z}^2(x,1,t)\right)dx + \frac{\eta\tau'(t)}{2}\int_0^1 \mathcal{Z}^2(x,1,t)dx. \end{aligned} \tag{4.37}$$

From (4.34),(4.36) and (4.37), we obtain

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} = &-k\int_0^1 \theta_x^2(x,t)dx - \hbar\int_0^1 \mathcal{P}_x^2(x,t)dx - \left(\mu_1 - \frac{\eta}{2}\right)\int_0^1 \varphi_t^2(x,t)dx \\ &+\frac{\eta}{2}\left(\tau'(t)-1\right)\int_0^1 \mathcal{Z}^2(x,1,t)dx - \mu_2\int_0^1 \varphi_t(x,t)\mathcal{Z}(x,1,t)dx. \end{aligned} \tag{4.38}$$

Using Young's inequality, the last term in (4.38) can be estimated as follows:

$$\mu_2\int_0^1 \varphi_t(x,t)\mathcal{Z}(x,1,t)dx \leq \frac{\mu_2}{2\sqrt{1-r_1}}\int_0^1 \varphi_t^2(x,t)dx + \frac{\mu_2\sqrt{1-r_1}}{2}\int_0^1 \mathcal{Z}^2(x,1,t)dx. \tag{4.39}$$

By inserting (4.39) into (4.13), we get

$$\begin{aligned}\frac{d\mathcal{E}(t)}{dt} \leq\ & -\kappa\int_0^1 \theta_x^2(x,t)dx - \hbar\int_0^1 \mathcal{P}_x^2(x,t)dx - \left(\mu_1 - \frac{\eta}{2} - \frac{\mu_2}{2\sqrt{1-r_1}}\right)\int_0^1 \varphi_t^2(x,t)dx \\ & + \left(\frac{\eta}{2}\left(\tau'(t)-1\right) + \frac{\mu_2\sqrt{1-r_1}}{2}\right)\int_0^1 z^2(x,1,t)dx.\end{aligned} \tag{4.40}$$

Then, using (4.15) and $(4.7)_2$, our conclusion holds. □

The main objective of this part is proof of the following stability result.

Theorem 4.3. *Let $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A}(0))$. Assume that (4.8) holds. Then under the hypotheses (4.7), any solution of problem (4.11)-(4.12) satisfies*

$$\mathcal{E}(t) \leq \mathcal{C}e^{-\gamma t}, \quad \forall t \geq 0, \tag{4.41}$$

for some positive constants $\mathcal{C}$ and γ independent of t.

To obtain the exponential decay of the solution in Theorem 4.3, we create a functional $\mathcal{L}(t)$ that is equivalent to the energy $\mathcal{E}(t)$ satisfying

$$\frac{d\mathcal{L}(t)}{dt} \leq -v\mathcal{L}(t), \quad \forall t \geq 0,$$

where v represents a positive constant. This requires several lemmas.

Lemma 4.2. *The functional*

$$\mathcal{F}_1(t) = -\rho_1\int_0^1 \mathcal{U}\mathcal{U}_t dx,$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\mathcal{F}_1'(t) \leq -\rho_1\int_0^1 \mathcal{U}_t^2 dx + \left(a_1 + \frac{a_2^2}{4\varepsilon_1}\right)\int_0^1 \mathcal{U}_x^2 dx + \varepsilon_1\int_0^1 \varphi_x^2 dx. \tag{4.42}$$

Proof. By differentiating $\mathcal{F}_1(t)$ with respect to t, using $(4.11)_1$ and integrating by parts, we get

$$\mathcal{F}_1'(t) = -\rho_1\int_0^1 \mathcal{U}_t^2 dx + a_1\int_0^1 \mathcal{U}_x^2 dx + a_2\int_0^1 \mathcal{U}_x\varphi_x dx,$$

Then, using Young's inequality, we get the result. □

Lemma 4.3. *The functional*

$$\mathcal{F}_2(t) = a_1\rho_2\int_0^1 \varphi\varphi_t dx - a_2\rho_1\int_0^1 \varphi\mathcal{U}_t dx,$$

satisfies, for any $\varepsilon_2 > 0$*, the estimate*

$$\begin{aligned}\mathcal{F}_2'(t) \leq & -\frac{a}{2}\int_0^1 \varphi_x dx + \mathcal{C}_3(\varepsilon_2)\int_0^1 \varphi_t^2 dx + \varepsilon_2 \int_0^1 \mathcal{U}_t^2 dx + \frac{2a_1^2\zeta_1^2}{a}\int_0^1 \theta_x^2 dx \\ & + \frac{2a_1^2\zeta_2^2}{a}\int_0^1 \mathcal{P}_x^2 dx + \frac{2a_1^2\mu_2^2}{a}\int_0^1 \mathcal{Z}^2(x,1,t)dx,\end{aligned} \tag{4.43}$$

where

$$a = a_1 a_3 - a_2^2 > 0, \qquad \mathcal{C}_3(\varepsilon_2) = a_1\rho_2 + \frac{2a_1^2\mu_1^2}{a} + \frac{a_2^2\rho_1^2}{4\varepsilon_2}.$$

Proof. By differentiating $\mathcal{F}_2(t)$ with respect to t, using $(4.11)_1$ and $(4.11)_2$ and integrating by parts, we get

$$\begin{aligned}\mathcal{F}_2'(t) = & -a\int_0^1 \varphi_x^2 dx + a_1\rho_2\int_0^1 \varphi_t^2 dx - a_2\rho_1\int_0^1 \varphi_t \mathcal{U}_t dx + \zeta_1 a_1 \int_0^1 \varphi\theta_x dx \\ & + \zeta_2 a_1 \int_0^1 \varphi \mathcal{P}_x dx - \mu_1 a_1 \int_0^1 \varphi\varphi_t dx - \mu_2 a_1 \int_0^1 \varphi \mathcal{Z}(x,1,t)dx.\end{aligned}$$

Estimate (4.43) is established through the application of Young's and Poincaré inequalities. □

Lemma 4.4. *The functional*

$$\mathcal{F}_3(t) = \frac{a_1\rho_2}{a_2}\int_0^1 \varphi_t \mathcal{U} dx - \frac{a_3\rho_1}{a_2}\int_0^1 \mathcal{U}_t \varphi dx,$$

satisfies, for any $\varepsilon_3 > 0$*, the estimate*

$$\begin{aligned}\mathcal{F}_3'(t) \leq & -\frac{a_1}{2}\int_0^1 \mathcal{U}_x^2 dx + a_3\int_0^1 \varphi_x^2 dx + \frac{2a_1^2\zeta_1^2}{a_2^2}\int_0^1 \theta_x^2 dx + \frac{2a_1^2\zeta_2^2}{a_2^2}\int_0^1 \mathcal{P}_x^2 dx \\ & + \mathcal{C}_4(\varepsilon_3)\int_0^1 \varphi_t^2 dx + \frac{2a_1^2\mu_2^2}{a_2^2}\int_0^1 \mathcal{Z}^2(x,1,t)dx + \varepsilon_3\int_0^1 \mathcal{U}_t^2 dx,\end{aligned} \tag{4.44}$$

where

$$\mathcal{C}_4(\varepsilon_3) = \frac{2a_1^2\mu_1^2}{a_2^2} + \frac{1}{4\varepsilon_3}\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)^2.$$

Proof.

$$\begin{aligned}\mathcal{F}_3'(t) = & -a_1\int_0^1 \mathcal{U}_x^2 dx + a_3\int_0^1 \varphi_x^2 dx + \frac{\zeta_1 a_1}{a_2}\int_0^1 \theta_x \mathcal{U} dx + \frac{\zeta_2 a_1}{a_2}\int_0^1 \mathcal{P}_x \mathcal{U} dx \\ & - \frac{\mu_1 a_1}{a_2}\int_0^1 \varphi_t \mathcal{U} dx - \frac{\mu_2 a_1}{a_2}\int_0^1 \mathcal{Z}(x,1,t)\mathcal{U} dx + \left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)\varphi_t \mathcal{U}_t dx.\end{aligned}$$

Estimate (4.44) is established through the application of Young's and Poincaré inequalities. □

Lemma 4.5. *The functional*

$$\mathcal{F}_4(t) = \eta\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho} z(x,\rho,t)dxd\rho,$$

satisfies, the estimate

$$\mathcal{F}_4'(t) \leq -2\mathcal{F}_4(t) + \eta\int_0^1 \varphi_t^2(x,t)dx. \tag{4.45}$$

Proof. Differentiating $\mathcal{F}_4(t)$ with respect to t, we get

$$\begin{aligned}\mathcal{F}_4'(t) = \; & \eta\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)dxd\rho \\ & -2\eta\tau(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\rho z^2(x,\rho,t)dxd\rho \\ & +2\eta\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho} z_t(x,\rho,t)z(x,\rho,t)dxd\rho.\end{aligned} \tag{4.46}$$

By using $(4.11)_3$, the last term in (4.46) can be rewritten as follows

$$\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(z_t z)(x,\rho,t)dxd\rho = \int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(\tau'(t)\rho - 1\right)(z_\rho z)(x,\rho,t)dxd\rho. \tag{4.47}$$

In addition, it is evident that

$$\begin{aligned}&\int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(\tau'(t)\rho - 1\right) z_\rho(x,\rho,t)z(x,\rho,t)dxd\rho \\ &= \frac{1}{2}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau(t)\rho}\left(\tau'(t)\rho - 1\right) z^2(x,\rho,t)\right)dxd\rho \\ &+\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(\tau'(t)\rho - 1\right) z^2(x,\rho,t)dxd\rho \\ &-\frac{\tau'(t)}{2}\int_0^1\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)dxd\rho.\end{aligned} \tag{4.48}$$

Using (4.48) and (4.47), equation (4.46) gives the form

$$\begin{aligned}\mathcal{F}_4'(t) = \; & -2\eta\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)dxd\rho + \eta\int_0^1 \varphi_t^2(x,t)dx \\ & -\eta\left(1 - \tau'(t)\right)e^{-2\tau(t)}\int_0^1 z^2(x,\rho,t)dx,\end{aligned} \tag{4.49}$$

This is followed immediately by (4.45). □

Proof of Theorem 4.3. Let us present a Lyapunov functional.

$$\mathcal{L}(t) := \mathcal{N}\mathcal{E}(t) + \sum_{i=1}^{3} \mathcal{N}_i \mathcal{L}_i(t) + \mathcal{L}_4(t), \tag{4.50}$$

where $\mathcal{N}$ and $\mathcal{N}_i, i = 1,2,3,$ are positive constants that will be selected later.

By differentiating $\mathcal{L}(t)$, exploiting (4.35) and (4.42)-(4.45), and sitting $\varepsilon_i = \frac{1}{\mathcal{N}_i}, i = 1,2,3$ we have

$$\begin{aligned}
\mathcal{L}'(t) \leq \; & -[\rho_1 \mathcal{N}_1 - 2] \int_0^1 \mathcal{U}_t^2 dx \\
& - \left[\frac{a_1}{2}\mathcal{N}_3 - \left(a_1 + \frac{a_2^2 \mathcal{N}_1}{4}\right)\mathcal{N}_1\right] \int_0^1 \mathcal{U}_x^2 dx \\
& - [\mathcal{C}_* \mathcal{N} - \mathcal{C}_3(\varepsilon_2)\mathcal{N}_2 - \mathcal{C}_4(\varepsilon_3)\mathcal{N}_3 - \eta] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{a}{2}\mathcal{N}_2 - a_3 \mathcal{N}_3 - 1\right] \int_0^1 \varphi_x^2 dx \\
& - \left[\kappa \mathcal{N} - \frac{2a_1^2 \zeta_1^2}{a}\mathcal{N}_2 - \frac{2a_1^2 \zeta_1^2}{a_2^2}\mathcal{N}_3\right] \int_0^1 \theta_x^2 dx \\
& - \left[h \mathcal{N} - \frac{2a_1^2 \zeta_2^2}{a}\mathcal{N}_2 - \frac{2a_1^2 \zeta_2^2}{a_2^2}\mathcal{N}_3\right] \int_0^1 \mathcal{P}_x^2 dx \\
& - \left[\mathcal{C}_* \mathcal{N} - \frac{2a_1^2 \mu_2^2}{a}\mathcal{N}_2 - \frac{2a_1^2 \mu_2^2}{a_2^2}\mathcal{N}_3\right] \int_0^1 z^2(x,1,t) dx \\
& -2\mathcal{F}_4(t).
\end{aligned}$$

□

First, we choose $\mathcal{N}_3$ large enough so that

$$\rho_1 \mathcal{N}_1 - 2 > 0.$$

Next, we choose $\mathcal{N}_3$ large enough such that

$$\frac{a_1}{2}\mathcal{N}_3 - \left(a_1 + \frac{a_2^2 \mathcal{N}_1}{4}\right)\mathcal{N}_1 > 0.$$

Then, we choose $\mathcal{N}_2$ large enough so that

$$\frac{a}{2}\mathcal{N}_2 - a_3 \mathcal{N}_3 - 1 > 0.$$

Finally, we choose $\mathcal{N}$ large enough such that

$$\mathcal{C}_*\mathcal{N} - \mathcal{C}_3(\varepsilon_2)\mathcal{N}_2 - \mathcal{C}_4(\varepsilon_3)\mathcal{N}_3 - \eta > 0,$$

$$\mathcal{k}\mathcal{N} - \frac{2a_1^2\zeta_1^2}{a}\mathcal{N}_2 - \frac{2a_1^2\zeta_1^2}{a_2^2}\mathcal{N}_3 > 0,$$

$$\mathcal{h}\mathcal{N} - \frac{2a_1^2\zeta_2^2}{a}\mathcal{N}_2 - \frac{2a_1^2\zeta_2^2}{a_2^2}\mathcal{N}_3 > 0,$$

$$\mathcal{C}_*\mathcal{N} - \frac{2a_1^2\mu_2^2}{a}\mathcal{N}_2 - \frac{2a_1^2\mu_2^2}{a_2^2}\mathcal{N}_3 > 0.$$

Based on the previous information, we can therefore conclude that there is a positive constant β_0 such that

$$\mathcal{L}'(t) \leq -\beta_0\mathcal{E}(t). \tag{4.51}$$

However, the fact that $\mathcal{L}(t) \sim \mathcal{E}(t)$ is easily observed, meaning that there are two positive constants, β_1 and β_2, such that

$$\beta_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq \beta_2\mathcal{E}(t), \;\; \forall t \geq 0. \tag{4.52}$$

Integrating (4.51) and (4.52), we arrive at the following

$$\mathcal{L}'(t) \leq -\gamma\mathcal{E}(t), \;\; \forall t \geq 0. \tag{4.53}$$

Where $\gamma = \frac{\beta_0}{\beta_2}$. A simple integration of (4.53) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\gamma t}, \;\; \forall t \geq 0. \tag{4.54}$$

Combining the equivalence of $\mathcal{L}(t)$ and $\mathcal{E}(t)$ yields the desired result of Theorem 4.2.

Conclusion

In this thesis, we have investigated the existence and exponential decay of solutions for specific classes of evolution problems. The study addressed three distinct systems: a one-dimensional Lord-Schulman thermoelastic system with porous damping and time delay, a swelling porous system influenced by the Gurtin-Pipkin thermal effect, and a swelling porous-heat system with time-varying delay under Dirichlet boundary conditions. By employing the semi-group approach and appropriate assumptions on the delay, we established the well-posedness of these systems. Furthermore, through the construction of suitable Lyapunov functionals, we demonstrated that the systems exhibit exponential stability, regardless of system parameters or equal speeds. These findings contribute valuable insights into the stability mechanisms of complex thermoelastic and porous systems. Future research could explore extending these results to higher-dimensional domains, non-linear systems. Additionally, exploring numerical methods to verify and simulate the theoretical findings for practical implementation. The research advances the understanding of coupled thermoelastic systems and provides a foundation for both theoretical and applied developments in this field.

Bibliography

[1] A. M. Al-Mahdi, M. M. Al-Gharabli, and M. Alahyane. Theoretical and numerical stability results for a viscoelastic swelling porous-elastic system with past history. *AIMS Math*, 6(11):11921–11949, 2021.

[2] A. M. Al-Mahdi, S. A. Messaoudi, and M. M. Al-Gharabli. A stability result for a swelling porous system with nonlinear boundary dampings. *Journal of Function Spaces*, 2022(1):8079707, 2022.

[3] G. Allaire. *Numerical analysis and optimization: an introduction to mathematical modelling and numerical simulation*. OUP Oxford, 2007.

[4] M. Aouadi. Theory of generalized micropolar thermoelastic diffusion under lord–shulman model. *Journal of Thermal Stresses*, 32(9):923–942, 2009.

[5] M. Aouadi, M. Campo, M. I. Copetti, and J. R. Fernández. Existence, stability and numerical results for a timoshenko beam with thermodiffusion effects. *Zeitschrift für angewandte Mathematik und Physik*, 70(4):117, 2019.

[6] M. Aouadi and A. Castejón. Properties of global and exponential attractors for nonlinear thermo-diffusion timoshenko system. *Journal of Mathematical Physics*, 60(8), 2019.

[7] M. Aouadi, M. Ciarletta, and F. Passarella. Thermoelastic theory with microtemperatures and dissipative thermodynamics. *Journal of Thermal Stresses*, 41(4):522–542, 2018.

[8] M. Aouadi and A. Miranville. Quasi-stability and global attractor in nonlinear thermoelastic diffusion plate with memory. *Evol. Equ. Control Theory*, 4(3):241–263, 2015.

[9] M. Aouadi and A. Miranville. Smooth attractor for a nonlinear thermoelastic diffusion thin plate based on gurtin–pipkin's model. *Asymptotic Analysis*, 95(1-2):129–160, 2015.

[10] M. Aouadi, A. Ramos, and A. Castejón. Stability conditions for thermodiffusion timoshenko system with second sound. *Zeitschrift für angewandte Mathematik und Physik*, 72(4):151, 2021.

[11] T. Apalara, A. Soufyane, M. Afilal, and M. Alahyane. A general stability result for swelling porous elastic media with nonlinear damping. *Applicable Analysis*, 102(4):1183–1198, 2023.

[12] T. A. Apalara. General stability result of swelling porous elastic soils with a viscoelastic damping. *Zeitschrift für angewandte Mathematik und Physik*, 71(6):200, 2020.

[13] T. A. Apalara and A. Soufyane. Energy decay for a weakly nonlinear damped porous system with a nonlinear delay. *Applicable Analysis*, 101(17):6113–6135, 2022.

[14] T. A.-A. Apalara and O. B. Almutairi. Well-posedness and exponential stability of swelling porous with gurtin–pipkin thermoelasticity. *Mathematics*, 10(23):4498, 2022.

[15] N. Bazarra, J. R. Fernández, and R. Quintanilla. Lord–shulman thermoelasticity with microtemperatures. *Applied Mathematics & Optimization*, 84(2):1667–1685, 2021.

[16] A. Bedford and D. S. Drumheller. Theories of immiscible and structured mixtures. *International Journal of Engineering Science*, 21(8):863–960, 1983.

[17] L. Bouzettouta. Stabilization of a type iii thermoelastic bresse system with distributed delay-time. *Iranian Journal of Mathematical Sciences and Informatics*, 18(1):1–18, 2023.

[18] L. Bouzettouta, S. Zitouni, K. Zennir, and A. Guesmia. Stability of bresse system with internal distributed delay. *J. Math. Comput. Sci.*, 7(1):92–118, 2016.

[19] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011.

[20] P. S. Casas and R. Quintanilla. Exponential decay in one-dimensional porous-thermo-elasticity. *Mechanics Research Communications*, 32(6):652–658, 2005.

[21] P. S. Casas and R. Quintanilla. Exponential stability in thermoelasticity with microtemperatures. *International Journal of Engineering Science*, 43(1-2):33–47, 2005.

[22] D. Chandrasekharaiah. Hyperbolic thermoelasticity: a review of recent literature. 1998.

[23] F. Dell'Oro and V. Pata. On the stability of timoshenko systems with gurtin–pipkin thermal law. *Journal of Differential Equations*, 257(2):523–548, 2014.

[24] M. Douib and S. Zitouni. Exponential stability for a swelling porous-heat system with thermodiffusion effects and delay. *Vladikavkaz Mathematical Journal*, 25(2):65–77, 2023.

[25] W. Dudziak and S. J. Kowalski. Theory of thermodiffusion for solids. *International Journal of Heat and Mass Transfer*, 32(11):2005–2013, 1989.

[26] J. M. C. Duhamel. *Mémoire sur le calcul des actions moléculaires développées par les changements de température dans les corps solides*. 1836.

[27] T. El Arwadi, B. Feng, and W. Youssef. Theoretical analysis and numerical simulation for a laminated beam system with thermodiffusion effects. 2021.

[28] M. Elhindi, K. Zennir, D. Ouchenane, A. Choucha, and T. E. Arwadi. Bresse-timoshenko type systems with thermodiffusion effects: well-possedness, stability and numerical results. *Rendiconti del Circolo Matematico di Palermo Series 2*, pages 1–26, 2021.

[29] K. Engel. One-parameter semigroups for linear evolution equations, 2000.

[30] A. C. Eringen. A continuum theory of swelling porous elastic soils. *International journal of engineering science*, 32(8):1337–1349, 1994.

[31] B. Feng, M. Freitas, D. Almeida Júnior, and A. Ramos. Quasi-stability and attractors for a porous-elastic system with history memory. *Applicable Analysis*, 101(17):6237–6254, 2022.

[32] H. D. Fernández Sare and R. Racke. On the stability of damped timoshenko systems: Cattaneo versus fourier law. *Archive for Rational Mechanics and Analysis*, 194:221–251, 2009.

[33] F. Foughali, S. Zitouni, L. Bouzettouta, and H. E. Khochemane. Well-posedness and general decay for a porous-elastic system with microtemperatures effects and time-varying delay term. *Zeitschrift für angewandte Mathematik und Physik*, 73(5):183, 2022.

[34] A. Green and P. Naghdi. Thermoelasticity without energy dissipation. *Journal of elasticity*, 31(3):189–208, 1993.

[35] J. Hale. History of delay equations. *Delay Differential Equations and Applications*, pages 1–28, 2006.

[36] Z.-J. Han and G.-Q. Xu. Exponential decay innon-uniform porous-thermo-elasticity model of lord-shulman type. *Discrete and Continuous Dynamical Systems-B*, 17(1):57–77, 2011.

[37] D. Ieşan and R. Quintanilla. Existence and continuous dependence results in the theory of interacting continua. *Journal of elasticity*, 36(1):85–98, 1994.

[38] L. D. Jones and I. Jefferson. Expansive soils. 2012.

[39] D. D. Joseph and L. Preziosi. Heat waves. *Reviews of modern physics*, 61(1):41, 1989.

[40] D. A. Júnior, A. Ramos, M. Freitas, M. Dos Santos, and T. E. Arwadi. Polynomial stability for the equations of porous elasticity in one-dimensional bounded domains. *Mathematics and Mechanics of Solids*, 27(2):308–318, 2022.

[41] B. Kalantari et al. Engineering significant of swelling soils. *Research Journal of Applied Sciences, Engineering and Technology*, 4(17):2874–2878, 2012.

[42] T. Kato. Linear and quasi-linear equations of evolution of hyperbolic type. In *Hyperbolicity: Lectures given at the Centro Internazionale Matematico Estivo (CIME), held in Cortona (Arezzo), Italy, June 24–July 2, 1976*, pages 125–191. Springer, 2011.

[43] A. Keddi, S. A. Messaoudi, and M. Alahyane. Well-posedness and stability results for a swelling porous-heat system of second sound. *Journal of Thermal Stresses*, 44(12):1427–1440, 2021.

[44] M. Kirane, B. Said-Houari, and M. N. Anwar. Stability result for the timoshenko system with a time-varying delay term in the internal feedbacks. *Commun. Pure Appl. Anal*, 10(2):667–686, 2011.

[45] L. Landau and E. M. Lifshits. Mechanics of continuous media. *Gostekhizdat, Moscow*, 81, 1954.

[46] G. Li, Y. Luan, and W. Lıu. Well-posedness and exponential stability of a thermoelastic-bresse system with second sound and delay. *Hacettepe Journal of Mathematics and Statistics*, 49(2):523–538, 2019.

[47] W. Liu and M. Chen. Well-posedness and exponential decay for a porous thermoelastic system with second sound and a time-varying delay term in the internal feedback. *Continuum Mechanics and Thermodynamics*, 29:731–746, 2017.

[48] Z. Liu and S. Zheng. *Semigroups associated with dissipative systems*, volume 398. CRC Press, 1999.

[49] H. W. Lord and Y. Shulman. A generalized dynamical theory of thermoelasticity. *Journal of the Mechanics and Physics of Solids*, 15(5):299–309, 1967.

[50] S. A. Messaoudi and A. Fareh. Exponential decay for linear damped porous thermoelastic systems with second sound. *Discrete Contin. Dyn. Syst. Ser. B*, 20(2):599–612, 2015.

[51] M. A. Murad and J. H. Cushman. Thermomechanical theories for swelling porous media with microstructure. *International Journal of Engineering Science*, 38(5):517–564, 2000.

[52] K. Neumann. Die gesetze der doppelbrechung des lichts in comprimirten oder ungleichförmig erwärmten unkrystallinischen körpern. *Annalen der Physik*, 130(12):449–476, 1841.

[53] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM Journal on Control and Optimization*, 45(5):1561–1585, 2006.

[54] S. Nicaise and C. Pignotti. Stabilization of the wave equation with boundary or internal distributed delay. 2008.

[55] S. Nicaise, C. Pignotti, and J. Valein. Exponential stability of the wave equation with boundary time-varying delay. *Discrete and Continuous Dynamical Systems-Series S*, 4(3):693–722, 2011.

[56] S. Nicaise, J. Valein, and E. Fridman. Stability of the heat and of the wave equations with boundary time-varying delays. *Discrete and Continuous Dynamical Systems-Series S*, 2(3):559–581, 2009.

[57] S.-I. Niculescu and K. Gu. *Advances in time-delay systems*, volume 38. Springer Science & Business Media, 2004.

[58] W. Nowacki. Dynamic problems of diffusion in solids. *Engineering fracture mechanics*, 8(1):261–266, 1976.

[59] Z. Olesiak and Y. A. Pyryev. A coupled quasi-stationary problem of thermodiffusion for an elastic cylinder. *International Journal of Engineering Science*, 33(6):773–780, 1995.

[60] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[61] R. Quintanilla. Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation. *Journal of computational and applied mathematics*, 145(2):525–533, 2002.

[62] R. Quintanilla. On the linear problem of swelling porous elastic soils with incompressible fluid. *International journal of engineering science*, 40(13):1485–1494, 2002.

[63] R. Quintanilla. Exponential stability of solutions of swelling porous elastic soils. *Meccanica*, 39(2):139–145, 2004.

[64] A. Rahmoune, O. Khaldi, D. Ouchenane, and F. Yazid. Well-posedness and stability result for timoshenko system with thermodiffusion effects and time-varying delay term. 16(2):90–104, 2024.

[65] A. Ramos, D. Almeida Júnior, M. Freitas, A. Noé, and M. Dos Santos. Stabilization of swelling porous elastic soils with fluid saturation and delay time terms. *Journal of Mathematical Physics*, 62(2), 2021.

[66] A. Ramos, M. Freitas, D. Almeida, A. Noé, and M. Dos Santos. Stability results for elastic porous media swelling with nonlinear damping. *Journal of Mathematical Physics*, 61(10), 2020.

[67] J. E. M. Rivera and R. Racke. Mildly dissipative nonlinear timoshenko systems—global existence and exponential stability. *Journal of Mathematical Analysis and Applications*, 276(1):248–278, 2002.

[68] J. M. Rivera. Energy decay rates in linear thermoelasticity. *Funkcial. Ekvac*, 35(1):19–30, 1992.

[69] B. Said-Houari and Y. Laskri. A stability result of a timoshenko system with a delay term in the internal feedback. *Applied Mathematics and Computation*, 217(6):2857–2869, 2010.

[70] M. Santos, A. Campelo, and M. Oliveira. On porous-elastic systems with fourier law. *Applicable Analysis*, 98(6):1181–1197, 2019.

[71] M. Santos, D. A. Júnior, and J. M. Rivera. The stability number of the timoshenko system with second sound. *Journal of Differential Equations*, 253(9):2715–2733, 2012.

[72] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49. American Mathematical Soc., 2013.

[73] H. L. Smith. *An introduction to delay differential equations with applications to the life sciences*, volume 57. springer New York, 2011.

[74] W. Thomson. *On the Thermo-Elastic and Thermo-Magnetic Properties of Matter*. Quart. J. Math., 1, 57-77, 1857.

[75] J.-M. Wang and B.-Z. Guo. On the stability of swelling porous elastic soils with fluid saturation by one internal damping. *IMA journal of applied mathematics*, 71(4):565–582, 2006.

[76] K. Yoshida and M. S. Narasimhan. *Lectures on semi-group theory and its application to Cauchy's problem in partial differential equations*, volume 8. Tata Institute of Fundamental Research Bombay, India, 1957.