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On The Maximum Number Of Limit Cycles Of Generalized Polynomial Liénard Differential Systems Via Averaging Theory*

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Abstract

In this paper, we apply the averaging theory of first and second order for studying the limit cycles of generalized polynomial Liénard systems of the form

$$\dot{x} = y - l(x)y, \quad \dot{y} = -x - f(x) - g(x)y - h(x)y^2 - p(x)y^3,$$

where $l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x)$, $f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x)$, $g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x)$, $h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x)$ and $p(x) = \epsilon p_1(x) + \epsilon^2 p_2(x)$ where $l_k(x)$ has degree m, $f_k(x)$, $g_k(x)$, $h_k(x)$ and $p_k(x)$ have degree n for each $k = 1,\, 2$, and ϵ is a small parameter.

1 Introduction

Polynomial Liénard systems occur as models or at least as simplifications of models in many domains in science. According to Smale [16], they are also a good starting point to try to solve the second part of Hilbert's 16th problem, which asks for the maximal number of limit cycles that a polynomial planar vector fields can have, depending on the degree of the system [8]. Recall that a limit cycle of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system.

The notion of a center of real planar polynomial differential system is an isolated equilibrium point having a neighborhood such that all the orbits of this neighborhood are periodic orbits with the unique exception of the singular point, which is defined by Poincaré in [14]. A classical way of producing limit cycles is by perturbing a system which has a center. The techniques used for studying the limit cycles that can bifurcate from the periodic orbits of a center are: Poincaré return map [2], Abelian integrals or Melnikov integrals (note that for systems in the plane the two notions are equivalent) [7], inverse integrating factor [5] and averaging theory ([3, 15]). In the plane at same order all these techniques produce the same results, but the computations can change with the different technique.

In this paper, using averaging theory we want to study the number of limit cycles which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ of the more generalized Liénard polynomial differential system

$$\begin{cases} \dot{x} = y - l(x)y, \\ \dot{y} = -x - f(x) - g(x)y - h(x)y^2 - p(x)y^3, \end{cases} \tag{1}$$

where $l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x)$, $f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x)$, $g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x)$, $h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x)$ and $p(x) = \epsilon p_1(x) + \epsilon^2 p_2(x)$ where $l_k(x)$ has degree m, $f_k(x)$, $g_k(x)$, $h_k(x)$ and $p_k(x)$ have degree n for each $k = 1, 2$, and ϵ is a small parameter. Note that system (1) was been studied in [4] when $l(x) = 0$ and $p(x) = 0$.

There are some results concerning the maximum number of limit cycles bifurcating from the linear center $\dot{x} = y, \dot{y} = -x$ of generalized polynomial Liénard differential systems using averaging theory.

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- In [10], Llibre et al. studied how many limit cycles of system $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$ can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of order k for $k = 1, 2, 3$, where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively.
- In [11], the authors studied using the averaging theory of first and second order the following system

$$\begin{cases} \dot{x} = y - \epsilon(g_{11}(x) + f_{11}(x)y) - \epsilon^2(g_{12}(x) + f_{12}(x)y), \\ \dot{y} = -x - \epsilon(g_{21}(x) + f_{21}(x)y) - \epsilon^2(g_{22}(x) + f_{22}(x)y), \end{cases} \tag{2}$$

  where g_{1i}, f_{1i}, g_{2i}, f_{2i} have degree l, k, m and n respectively for each $i = 1, 2$, and ϵ is a small parameter.
- In [1], Alavez-Ramirez et al. studied the polynomial differential system

$$\begin{cases} \dot{x} = y - \epsilon g_{11}(x) - \epsilon^2 g_{12}(x), \\ \dot{y} = -x - \epsilon(g_{21}(x) + f_{21}(x)y) - \epsilon^2(g_{22}(x) + f_{22}(x)y), \end{cases} \tag{3}$$

  where g_{1i}, g_{2i}, f_{2i} have degree l, m and n respectively for each $i = 1, 2$, and ϵ is a small parameter.
- In [12, 13], the authors studied using the averaging theory of first second and third order the number of limit cycles of the polynomial differential system

$$\begin{cases} \dot{x} = y - f_1(x)y, \\ \dot{y} = -x - g_2(x) - f_2(x)y, \end{cases} \tag{4}$$

  where $f_1(x) = \epsilon f_{11}(x) + \epsilon^2 f_{12}(x) + \epsilon^3 f_{13}(x)$, $g_2(x) = \epsilon g_{21}(x) + \epsilon^2 g_{22}(x) + \epsilon^3 g_{23}(x)$ and $f_2(x) = \epsilon f_{21}(x) + \epsilon^2 f_{22}(x) + \epsilon^3 f_{23}(x)$ where f_{1i}, f_{2i} and g_{2i} have degree l, n and m respectively for each $i = 1, 2, 3$, and ϵ is a small parameter.

In what follows we present our main results.

Theorem 1 *For $|\epsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized polynomial Liénard differential systems (1) bifurcating from the periodic orbits of linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of first order is*

$$\lambda_1 = \max \left[\frac{n+2}{2} \right].$$

The proof of Theorem 1 is given in Section 3.

Theorem 2 *For $|\epsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized polynomial Liénard differential systems (1) bifurcating from the periodic orbits of linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of second order is*

$$\lambda_2 = \max \left\{ n+1, \left[\frac{n-1}{2} \right] + \left[\frac{m-1}{2} \right] + 2 \right\}.$$

The proof of Theorem 2 is given in Section 4.

In [4] it has been shown that there exist generalized Liénard systems (1) with $l(x) = p(x) = 0$, having at least n limit cycles. The result in Theorem 2 improves this lower estimate $(\lambda_2 > n)$. Note that we do not know if the upper bound λ_1 (respectively λ_2) for the number of limit cycles of the polynomial differential system (1), which bifurcate from the periodic solutions of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first order (respectively second order) is reached. In Section 5 we prove that the upper bound is reached for $1 \le n \le 9$ in the first order averaging theory and for $1 \le n \le 4$, $1 \le m \le 4$ in the second order averaging theory (see Examples 1 and 2).

In Section 2, we introduce the averaging theory of first and second order.

2 Averaging Theory of First and Second Order

We consider the differential system

$$\dot{x} = \epsilon F_1(t,x) + \epsilon^2 F_2(t,x) + \epsilon^3 R(t,x,\epsilon), \tag{5}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\epsilon_f, \epsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i), (ii) hold.

(i) $F_1(t,.) \in C^2(D)$, $F_2(t,.) \in C^1(D)$: for all :$t \in \mathbb{R}$,: F_1, F_2, R are locally Lipschitz with respect to x, and R is twice differentiable with respect to ϵ.

We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_z F_1(s,z) y_1(s,z) + F_2(s,z)\right] ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\, dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f)\backslash\{0\}$, there exists $a \in V$ such that $F_{10}(a) + \epsilon F_{20}(a) = 0$ and $d_B(F_{10} + \epsilon F_{20}, V, a) \neq 0$. The expression $d_B(F_{10} + \epsilon F_{20}, V, a) \neq 0$ means that the Brouwer degree (see [3]) of the function $F_{10} + \epsilon F_{20} : V \to \mathbb{R}$ at the fixed point a is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \epsilon F_{20}$ at a is not zero.

Then, for $|\epsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(., \epsilon)$ of the equation (5) such that $\varphi(0, \epsilon) \to a$ when $\epsilon \to 0$.
- If F_{10} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the averaging theory of first order.
- If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{20} for ϵ sufficiently small. In this case the previous result provides the averaging theory of second order.

For a general introduction to averaging theory see ([9, 17]).

3 Proof of Theorem 1

We shall need the first-order averaging theory to prove Theorem 1. We write system (1) in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$. If we take

$$l_1(x) = \sum_{i=0}^{m} e_{i,1}x^i, \;\; f_1(x) = \sum_{i=0}^{n} a_{i,1}x^i, \;\; g_1(x) = \sum_{i=0}^{n} b_{i,1}x^i, \;\; h_1(x) = \sum_{i=0}^{n} c_{i,1}x^i, \;\; p_1(x) = \sum_{i=0}^{n} d_{i,1}x^i,$$

the system (1) becomes

$$\begin{cases} \dot{r} = -\epsilon\left(\sum\limits_{i=0}^{n} A_{i,1}(\theta,r)r^i + \sum\limits_{i=0}^{m} e_{i,1}R_{i+1}(\theta)r^{i+1}\right) + O(\epsilon^2), \\ \dot{\theta} = -1 - \frac{\epsilon}{r}\left(\sum\limits_{i=0}^{n} D_{i,1}(\theta,r)r^i - \sum\limits_{i=0}^{m} e_{i,1}T_i(\theta)r^{i+1}\right) + O(\epsilon^2), \end{cases} \tag{6}$$

where

$$\begin{aligned} A_{i,1}(r,\theta) &= a_{i,1}R_i(\theta)+b_{i,1}T_i(\theta)r+c_{i,1}S_i(\theta)r^2+d_{i,1}U_i(\theta)r^3,\\ R_j(\theta) &= \cos^j\theta\sin\theta,\\ T_j(\theta) &= \cos^j\theta\sin^2\theta = cos^j\theta - cos^{j+2}\theta,\\ S_j(\theta) &= \cos^j\theta\sin^3\theta = cos^j\theta\sin\theta - cos^{j+2}\theta\sin\theta,\\ U_j(\theta) &= \cos^j\theta\sin^4\theta = cos^j\theta - 2cos^{j+2}\theta + cos^{j+4}\theta,\\ D_{i,1}(r,\theta) &= a_{i,1}\cos^{i+1}\theta + b_{i,1}R_{i+1}(\theta)r + c_{i,1}T_{i+1}(\theta)r^2 + d_{i,1}S_{i+1}(\theta)r^3. \end{aligned}$$

Now taking θ as the new independent variable, system (6) becomes

$$\frac{dr}{d\theta} = \frac{\dot r}{\dot\theta} = \epsilon F_1(r,\theta) + O(\epsilon^2), \tag{7}$$

where

$$F_1(r,\theta) = \sum_{i=0}^{n} A_{i,1}(\theta,r)r^i + \sum_{i=0}^{m} e_{i,1}R_{i+1}(\theta)r^{i+1}.$$

Therefore, from Section 2 we must study the simple positive zeros of the function

$$\begin{aligned} F_{10}(r) &= \frac{1}{2\pi}\left(\sum_{i=0}^{n} r^i(a_{i,1}J_i(2\pi) + b_{i,1}r\tilde I_i(2\pi) + c_{i,1}r^2\tilde J_i(2\pi) + d_{i,1}r^3\tilde{\tilde I}_i(2\pi))\right.\\ &+ \left.\sum_{i=0}^{m} e_{i,1}J_{i+1}(2\pi)r^{i+1}\right), \end{aligned}$$

where

$$J_k(2\pi) = \int_0^{2\pi} R_k(\theta)d\theta, \quad \tilde I_k(2\pi) = \int_0^{2\pi} T_k(\theta)d\theta, \quad \tilde J_k(2\pi) = \int_0^{2\pi} S_k(\theta)d\theta, \quad \tilde{\tilde I}_k(2\pi) = \int_0^{2\pi} U_k(\theta)d\theta.$$

To calculate the exact expression of $F_{10}(r)$, we use the following integrals (see for more details [6]):

$$J_i(\theta) = \int_0^{\theta} \cos^i t\sin t dt = \frac{1}{i+1}\left(1-\cos^{i+1}\theta\right), \tag{8}$$

$$\tilde J_i(\theta) = \int_0^{\theta} \cos^i t\sin^3 t dt = \frac{2}{(i+1)(i+3)} - \frac{1}{i+1}\cos^{i+1}\theta + \frac{1}{i+3}\cos^{i+3}\theta, \tag{9}$$

$$J_i(2\pi) = \tilde J_i(2\pi) = 0, \tag{10}$$

$$I_i(\theta) = \int_0^{\theta} \cos^i t dt = \begin{cases} \sum\limits_{l=0}^{k} \gamma_{k,l}\sin(2l+1)\theta & \text{if } i = 2k+1,\\ \delta_k + \sum\limits_{l=1}^{k} \beta_{k,l}\sin(2l\theta) & \text{if } i = 2k, \end{cases} \tag{11}$$

where

$$\delta_i = \frac{1}{2^{2i}}\begin{pmatrix} 2i\\ i\end{pmatrix}\theta, \quad \gamma_{k,l} = \frac{1}{2^{2i}}\begin{pmatrix} 2i+1\\ i-l\end{pmatrix}\frac{1}{2l+1}, \quad \beta_{k,l} = \begin{pmatrix} 2i\\ i+1\end{pmatrix}\frac{1}{l}.$$

$$I_i(2\pi) = \begin{cases} 0 & \text{if } i = 2k+1,\\ \frac{\pi\alpha_k}{2^{k-1}k!} & \text{if } i = 2k, \end{cases}$$

where $\alpha_k = 3 \cdot 5 \cdot \cdots \cdot (2k-1)$, $\alpha_{k+1} = (2k+1)\alpha_k$. And

$$I_{2k+2}(2\pi) = \frac{2k+1}{2k+2} I_{2k}(2\pi), \tag{12}$$

$$\tilde{I}_i(2\pi) = I_i(2\pi) - I_{i+2}(2\pi), \tag{13}$$

$$\tilde{\tilde{I}}_i(2\pi) = I_i(2\pi) - 2I_{i+2}(2\pi) + I_{i+2}(2\pi). \tag{14}$$

And thus we have

$$F_{10}(r) = r \sum_{i=0}^{[\frac{n}{2}]} r^{2i} \frac{\alpha_i}{2^{i+1}(i+1)!} \left(b_{2i,1} + d_{2i,1} r^2 \frac{3}{2(i+2)} \right). \tag{15}$$

From (15) the function $F_{10}(r)$ has at most $[\frac{n+2}{2}]$ simple positive zeros.

4 Proof of Theorem 2

For proving Theorem 2 we shall use the second order averaging theory. If we write

$$l_k(x) = \sum_{i=0}^{m} e_{i,k} x^i, \quad f_k(x) = \sum_{i=0}^{n} a_{i,k} x^i, \quad g_k(x) = \sum_{i=0}^{n} b_{i,k} x^i, \quad h_k(x) = \sum_{i=0}^{n} c_{i,k} x^i, \quad p_k(x) = \sum_{i=0}^{n} d_{i,k} x^i,$$

for each $k = 1, 2$. Then system (1) in polar coordinates (r, θ), $r > 0$ becomes

$$\begin{cases} \dot{r} & = -\epsilon(\sum_{i=0}^{n} A_{i,1}(\theta,r) r^i + \sum_{i=0}^{m} e_{i,1} R_{i+1}(\theta) r^{i+1}) - \epsilon^2(\sum_{i=0}^{n} A_{i,2}(\theta,r) r^i \\ & \quad + \sum_{i=0}^{m} e_{i,2} R_{i+1}(\theta) r^{i+1}) + O(\epsilon^3), \\ \dot{\theta} & = -1 - \frac{\epsilon}{r}(\sum_{i=0}^{n} D_{i,1}(\theta,r) r^i - \sum_{i=0}^{m} e_{i,1} T_i(\theta) r^{i+1}) - \frac{\epsilon^2}{r}(\sum_{i=0}^{n} D_{i,2}(\theta,r) r^i \\ & \quad - \sum_{i=0}^{m} e_{i,2} T_i(\theta) r^{i+1}) + O(\epsilon^3), \end{cases} \tag{16}$$

where

$$A_{i,j}(\theta,r) = a_{i,j} R_i(\theta) + b_{i,j} T_i(\theta) r + c_{i,j} S_i(\theta) r^2 + d_{i,j} U_i(\theta) r^3, \quad j = 1, 2,$$

$$D_{i,j}(\theta,r) = a_{i,j} \cos^{i+1}\theta + b_{i,j} R_{i+1}(\theta) r + c_{i,j} T_{i+1}(\theta) r^2 + d_{i,j} S_{i+1}(\theta) r^3, \quad j = 1, 2.$$

Taking θ as the new independent variable system (16) writes

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\theta} = \epsilon F_1(r,\theta) + \epsilon^2 F_2(r,\theta) + O(\epsilon^3), \tag{17}$$

where

$$F_1(r,\theta) = \sum_{i=0}^{n} A_{i,1}(\theta,r) r^i + \sum_{i=0}^{m} e_{i,1} R_{i+1}(\theta) r^{i+1}, \tag{18}$$

$$\begin{aligned} F_2(r,\theta) \quad = \quad & \sum_{i=0}^{n} A_{i,2}(\theta,r) r^i + \sum_{i=0}^{m} e_{i,2} R_{i+1}(\theta) r^{i+1} \\ & -\frac{1}{r} \left(\sum_{i=0}^{n} A_{i,1}(\theta,r) r^i + \sum_{i=0}^{m} e_{i,1} R_{i+1}(\theta) r^{i+1} \right) \left(\sum_{i=0}^{n} D_{i,1}(\theta,r) r^i - \sum_{i=0}^{m} e_{i,1} T_i(\theta) r^{i+1} \right). \end{aligned} \tag{19}$$

In order to apply the averaging theory of second order, $F_{10}(r)$ must be identically zero. From (15), F_{10} is identically zero if and only if

$$\begin{cases} b_0 = d_{2[\frac{n}{2}]} = 0, \\ b_{2i} = \frac{-3}{2i-1} d_{2i-2}, \quad 1 \le i \le [\frac{n}{2}], \\ b_{2i} = \frac{-3}{2i-1} d_{2i-2}, \quad 1 \le i \le [\frac{n}{2}]. \end{cases} \tag{20}$$

In the next Proposition we obtain the value of $F_1(r,\theta)$ when $F_{10}(r) \equiv 0$.

Proposition 1 *If $F_{10}(r) \equiv 0$, then*

$$\begin{aligned} F_1(r,\theta) &= \sum_{i=0}^{n} r^i \left(a_{i,1}R_i(\theta) + c_{i,1}r^2 S_i(\theta)\right) + \sum_{i=0}^{[\frac{n-1}{2}]} r^{2i+2}\left(b_{2i+1,1}T_{2i+1}(\theta) + d_{2i+1,1}r^2 U_{2i+1}(\theta)\right) \\ &\quad + \sum_{i=0}^{m} e_{i,1} r^{i+1} R_{i+1}(\theta) + \sum_{i=0}^{[\frac{n}{2}]} d_{2i-2} r^{2i+1}\left(T_{2i-2} - \frac{2i+2}{2i-1} T_{2i}(\theta)\right). \end{aligned} \tag{21}$$

Proof. Since $F_{10}(r) \equiv 0$, we have the relation (20). If the relations $U_k(\theta) = T_k(\theta) - T_{k+2}(\theta)$ and (20) are substituted in the expression (18) we obtain (21). ■

Now we continue our proof for Theorem 2. If we derive the expression of (21) with respect to r, we obtain

$$\begin{aligned} \frac{dF_1(r,\theta)}{dr} &= \sum_{i=0}^{n} r^{i-1}(ia_{i,1}R_i(\theta) + (i+2)c_{i,1}r^2 S_i(\theta)) \\ &\quad + \sum_{i=0}^{[\frac{n-1}{2}]} r^{2i+1}\left[(2i+2)b_{2i+1,1}T_{2i+1}(\theta) + (2i+4)d_{2i+1,1}r^2 U_{2i+1}(\theta)\right] \\ &\quad + \sum_{i=0}^{m} e_{i,1}(i+1)r^i R_{i+1}(\theta) \\ &\quad + \sum_{i=0}^{[\frac{n}{2}]} (2i+1) d_{2i-2} r^{2i}\left(T_{2i-2} - \frac{2i+2}{2i-1}T_{2i}(\theta)\right). \end{aligned} \tag{22}$$

In the next Proposition we obtain $y(r\theta) = \int_0^\theta F(r,s)ds$.

Proposition 2 *We have*

$$\begin{aligned} y(r,\theta) &= \sum_{i=0}^{n} r^i \left(a_{i,1}J_i(\theta) + c_{i,1}r^2 \tilde{J}_i(\theta)\right) + \sum_{i=0}^{[\frac{n-1}{2}]} r^{2i+2}\left(b_{2i+1,1}\sum_{l=0}^{i+1}\tilde{\gamma}_{i,l}\sin(2l+1)\theta \right. \\ &\quad \left. + d_{2i+1,1}\sum_{l=0}^{i+2}\tilde{\tilde{\gamma}}_{i,l}\sin(2l+1)\theta r^2\right) + \sum_{i=0}^{m} e_{i,1} r^{i+1} J_{i+1}(\theta) \\ &\quad + \sum_{i=0}^{[\frac{n}{2}]} d_{2i-2} r^{2i+1} \sum_{l=0}^{i+1} \tilde{\beta}_{i,l}\sin(2l\theta), \end{aligned} \tag{23}$$

where

$$\tilde{\gamma}_{i,l} = \begin{cases} \gamma_{i,l} - \gamma_{i+1,l}, : & 0 \le l \le i, \\ -\gamma_{i+1,i+1}, : & l = i+1. \end{cases}, \quad \tilde{\tilde{\gamma}}_{i,l} = \begin{cases} \gamma_{i,l} - 2\gamma_{i+1,l} + \gamma_{i+2,l}, : & 0 \le l \le i, \\ -2\gamma_{i+1,i+1} + \gamma_{i+2,i+1}, : & l = i+1, \\ \gamma_{i+2,i+2}, : & l = i+2, \end{cases}$$

$$\tilde{\beta}_{i,l} = \begin{cases} \beta_{i-1,l} - \frac{4i+1}{2i-1}\beta_{i,l} + \frac{2i+2}{2i-1}\beta_{i+1,l}, & 0 \le l \le i-1, \\ -\frac{4i+1}{2i-1}\beta_{i,i} + \frac{2i+2}{2i-1}\beta_{i+1,i}, & l = i, \\ \frac{2i+2}{2i-1}\beta_{i+1,i+1}, & l = i+1. \end{cases}$$

Proof. Simplifying by using the relations $\tilde{I}_k(\theta) = I_k(\theta) - I_{k+2}(\theta)$ and $\tilde{\tilde{I}}_k(\theta) = I_k(\theta) - 2I_{k+2}(\theta) + I_{k+4}(\theta)$. Taking into account that

$$\delta_{i-1}\theta - \frac{4i+1}{2i-1}\delta_i\theta + \frac{2i+2}{2i-1}\delta_{i+1}\theta = 0,$$

and using the integrals (8), (9) and (11), we obtain (23). Thus, the Proposition flows. ■

Now we determine the corresponding function

$$F_{20}(r) = \frac{1}{2\pi}(II(r) + III(r)), \tag{24}$$

where

$$II(r) = \int_0^{2\pi} \frac{\partial F_1(r,\theta)}{\partial r}.y(r,\theta)d\theta \quad \text{and} \quad III(r) = \int_0^{2\pi} F_2(r,\theta)d\theta.$$

First we shall compute the integrals $II(r)$. Using the expressions (22) and (23), We have

$$\begin{aligned} II(r) \;=\; & \sum_{i=0}^{n}\sum_{i=0}^{n} r^{i+j-1}M^1_{i,j} + \sum_{i=0}^{n}\sum_{k=0}^{[\frac{n-1}{2}]} r^{i+2k+1}M^2_{i,j} + \sum_{i=0}^{n}\sum_{j=0}^{m} r^{i+j}M^3_{i,j} \\ & + \sum_{i=0}^{n}\sum_{k=0}^{[\frac{n}{2}]} r^{i+2k}M^4_{i,j} + \sum_{s=0}^{[\frac{n-1}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k+3}M^5_{i,j} + \sum_{k=0}^{[\frac{n-1}{2}]}\sum_{i=0}^{m} r^{i+2k+2}M^6_{i,j} \\ & + \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k+2}M^7_{i,j} + \sum_{i=0}^{m}\sum_{j=0}^{m} r^{i+j+1}M^8_{i,j} + \sum_{k=0}^{[\frac{n}{2}]}\sum_{i=0}^{m} r^{2k+i+1}M^9_{i,j} \\ & + \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n}{2}]} r^{2s+2k+1}M^{10}_{i,j}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} M^1_{i,j} \;=\; & ia_{i,1}a_{j,1}\int_0^{2\pi} R_i(\theta)J_j(\theta)d\theta + ia_{i,1}c_{j,1}r^2\int_0^{2\pi} R_i(\theta)\tilde{J}_j(\theta)d\theta \\ & +(i+2)c_{i,1}a_{j,1}\int_0^{2\pi} S_i(\theta)J_j(\theta)d\theta + (i+2)c_{i,1}c_{j,1}r^4\int_0^{2\pi} S_i(\theta)\tilde{J}_j(\theta)d\theta, \end{aligned}$$

$$\begin{aligned} M^2_{i,j} \;=\; & a_{i,1}b_{2k+1,1}\left(i\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}\int_0^{2\pi} R_i(\theta)\sin(2l+1)\theta d\theta + (2k+2)\int_0^{2\pi} T_{2k+1}(\theta)\right. \\ & \left.\times J_i(\theta)d\theta\right) + a_{i,1}d_{2k+1,1}r^2\left(i\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}\int_0^{2\pi} R_i(\theta)\sin(2l+1)\theta d\theta + (2k+4)\right. \\ & \left.\times\int_0^{2\pi} U_{2k+1}(\theta)J_i(\theta)d\theta\right) + c_{i,1}b_{2k+1,1}r^2\left((i+2)\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}\int_0^{2\pi} S_i(\theta)\sin(2l+1)\right. \\ & \left.\times\theta d\theta + (2k+2)\int_0^{2\pi} T_{2k+1}(\theta)\tilde{J}_i(\theta)d\theta\right) + c_{i,1}d_{2k+1,1}r^4\left((i+2)\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}\right. \\ & \left.\times\int_0^{2\pi} S_i(\theta)\sin(2l+1)\theta d\theta + (2k+4)\int_0^{2\pi} U_{2k+1}(\theta)\tilde{J}_i(\theta)d\theta\right), \end{aligned}$$

$$M^3_{i,j} = a_{i,1}e_{j,1}\left(i\int_0^{2\pi} R_i(\theta)J_{j+1}(\theta)d\theta + (j+1)\int_0^{2\pi} R_{j+1}(\theta)J_i(\theta)d\theta\right) + c_{i,1}e_{j,1}r^2 \times\left((i+2)\int_0^{2\pi} S_i(\theta)J_{j+1}(\theta)d\theta + (j+1)\int_0^{2\pi} R_{j+1}(\theta)\tilde{J}_i(\theta)d\theta\right),$$

$$M^4_{i,j} = a_{i,1}d_{2k-2,1}\left(i\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\int_0^{2\pi} R_i(\theta)\sin(2l\theta)d\theta + (2k+1)\left(\int_0^{2\pi} T_{2k-2} \times (\theta)J_i(\theta)d\theta - \frac{2k+2}{2k-1}\int_0^{2\pi} T_{2k}(\theta)J_i(\theta)d\theta)\right)\right) + c_{i,1}d_{2k-2,1}r^2\left((i+2) \times\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\int_0^{2\pi} S_i(\theta)\sin(2l\theta)d\theta + (2k+1)\left(\int_0^{2\pi} T_{2k-2}(\theta)\tilde{J}_i(\theta)d\theta - \frac{2k+2}{2k-1}\int_0^{2\pi} T_{2k}(\theta)\tilde{J}_i(\theta)d\theta\right)\right),$$

$$M^5_{i,j} = (2k+2)\left(b_{2s+1,1}b_{2k+1}\sum_{l=0}^{s+1}\tilde{\gamma}_{s,l}\int_0^{2\pi} T_{2k+1}(\theta)\sin(2l+1)\theta d\theta + d_{2s+1,1}b_{2k+1}r^2 \times\sum_{l=0}^{s+2}\tilde{\tilde{\gamma}}_{s,l}\int_0^{2\pi} T_{2k+1}(\theta)\sin(2l+1)\theta d\theta\right) + (2k+4)\left(b_{2s+1,1}d_{2k+1}r^2\sum_{l=0}^{s+1}\tilde{\gamma}_{s,l} \times\int_0^{2\pi} U_{2k+1}(\theta)\sin(2l+1)\theta d\theta + d_{2s+1,1}d_{2k+1}r^4\sum_{l=0}^{s+2}\tilde{\tilde{\gamma}}_{s,l}\int_0^{2\pi} U_{2k+1}(\theta) \times\sin(2l+1)\theta d\theta\right),$$

$$M^6_{i,j} = e_{i,1}b_{2k+1}\left((2k+2)\int_0^{2\pi} T_{2k+1}(\theta)J_{i+1}(\theta)d\theta + (j+1)\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}\int_0^{2\pi} R_{i+1}(\theta) \times\sin(2l+1)\theta)\, d\theta + e_{i,1}d_{2k+1}r^2\left((2k+4)\int_0^{2\pi} U_{2k+1}(\theta)J_{i+1}(\theta)d\theta + (i+1) \times\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}\int_0^{2\pi} R_{i+1}(\theta)\sin(2l+1)\theta\right) d\theta,$$

$$M^7_{i,j} = d_{2s-2,1}b_{2k+1}\left((2k+2)\sum_{l=0}^{s+1}\tilde{\beta}_{s,l}\int_0^{2\pi} T_{2k+1}(\theta)\sin(2l\theta)d\theta + (2s+1)\left(\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l} \times\int_0^{2\pi} T_{2s-2}(\theta)\sin(2l+1)\theta d\theta - \frac{2s+2}{2s-1}\int_0^{2\pi} T_{2s}(\theta)\sin((2l+1)\theta)d\theta\right)\right) + d_{2s-2,1}d_{2k+1}r^2\left((2k+4)\sum_{l=0}^{s+1}\tilde{\beta}_{s,l}\int_0^{2\pi} U_{2k+1}(\theta)\sin(2l\theta)d\theta + (2s+1)\left(\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l} \times\int_0^{2\pi} T_{2s-2}(\theta)\sin(2l+1)\theta d\theta - \frac{2s+2}{2s-1}\int_0^{2\pi} T_{2s}(\theta)\sin(2l+1)\theta)d\theta)\right)\right),$$

$$M^8_{i,j} = (j+1)e_{i,1}e_{j,1}\int_0^{2\pi} R_{j+1}(\theta)J_{i+1}(\theta)d\theta,$$

$$\begin{aligned} M^9_{i,j} &= e_{i,1}d_{2k-2}\Bigg[(i+1)\sum_{l=0}^{k+1}\tilde{\beta}_{k,l}\int_0^{2\pi}R_{i+1}(\theta)\sin(2l\theta)d\theta+(2k+1)\Bigg(\Bigg(\int_0^{2\pi}T_{2k-2}(\theta)\\ &\quad\times-\frac{2k+2}{2k-1}\int_0^{2\pi}T_{2k}(\theta)J_{i+1}(\theta)d\theta)\Bigg)\Bigg], \end{aligned}$$

and

$$\begin{aligned} M^{10}_{i,j} &= (2s+1)d_{2s-2,1}d_{2k-2,1}\sum_{l=0}^{k+1}\tilde{\beta}_{k,l}\Big[\int_0^{2\pi}T_{2s-2}(\theta)\sin(2l\theta)d\theta-\frac{2s+2}{2s-1}\int_0^{2\pi}T_{2s}(\theta)\\ &\quad\times\sin(2l\theta)d\theta)\Big]. \end{aligned}$$

In the next propositions we obtain some results of the integrals on the right hand part of (25).

Proposition 3 *For $i,j\in\mathbb{N}$ and $\theta\in\mathbb{R}$ the following statements hold.*

(a) $\cos^i R_j(\theta)=R_{i+j}(\theta)$,

(b) $\cos^i T_j(\theta)=R_i(\theta)R_j(\theta)=T_{i+j}(\theta)$,

(c) $\cos^i S_j(\theta)=R_i(\theta)T_j(\theta)=S_{i+j}(\theta)$,

(d) $\cos^i U_j(\theta)=R_i(\theta)S_j(\theta)=T_i(\theta)T_j(\theta)=U_{i+j}(\theta)$,

(e) $R_i(\theta)U_j(\theta)=T_i(\theta)S_j(\theta)=S_{i+j}(\theta)-S_{i+j+2}(\theta)$,

(f) $T_i(\theta)U_j(\theta)=S_i(\theta)S_j(\theta)=U_{i+j}(\theta)-U_{i+j+2}(\theta)$,

(g) $S_i(\theta)U_j(\theta)=S_{i+j}(\theta)-2S_{j+j+2}(\theta)+S_{i+j+4}(\theta)$.

Proof. By using the elemental trigonometric relations we obtain easily the previous equalities. ■

Corollary 1 *For $i,j\in\mathbb{N}$, the following statements hold.*

(a)

$$\begin{aligned} \int_0^{2\pi}\cos^i\theta R_j(\theta)d\theta &= \int_0^{2\pi}\cos^i\theta S_j(\theta)d\theta=\int_0^{2\pi}R_i(\theta)T_j(\theta)d\theta\\ &= \int_0^{2\pi}R_i(\theta)U_j(\theta)d\theta\int_0^{2\pi}T_i(\theta)S_j(\theta)d\theta=\int_0^{2\pi}S_i(\theta)U_j(\theta)d\theta=0. \end{aligned}$$

(b)

$$\int_0^{2\pi}\cos^i T_j(\theta)d\theta=\int_0^{2\pi}R_i(\theta)R_j(\theta)d\theta=\tilde{I}_{i+j}(2\pi)=\begin{cases}0, & \textit{if } i+j \textit{ is odd},\\ \tilde{I}_{2k}(2\pi), & \textit{if } i+j \textit{ is even}.\end{cases}$$

(c)

$$\begin{aligned} \int_0^{2\pi}\cos^i U_j(\theta)d\theta &= \int_0^{2\pi}R_i(\theta)S_j(\theta)d\theta=\int_0^{2\pi}T_i(\theta)T_j(\theta)d\theta\\ &= \tilde{\tilde{I}}_{i+j}(2\pi)=\begin{cases}0, & \textit{if } i+j \textit{ is odd},\\ \tilde{\tilde{I}}_{2k}(2\pi), & \textit{if } i+j \textit{ is even}.\end{cases} \end{aligned}$$

(d)

$$\int_0^{2\pi} T_i(\theta)U_j(\theta)d\theta = \int_0^{2\pi} S_i(\theta)S_j(\theta)d\theta = \tilde{\tilde{I}}_{i+j}(2\pi) - \tilde{\tilde{I}}_{i+j+2}(2\pi)$$
$$= \begin{cases} 0, & \text{if } i+j \text{ is odd,} \\ \tilde{\tilde{I}}_{2k}(2\pi) - \tilde{\tilde{I}}_{2k+2}(2\pi), & \text{if } i+j \text{ is even.} \end{cases}$$

Proof. Using Proposition 3 and using (10) for all $i, j \in \mathbb{N}$, we obtain (a). Again from Proposition 3 and using (13) and (14) we obtain (b), (c) and (d). Hence the Corollary follows. ■

Proposition 4 *For $i, j \in \mathbb{N}$ and $\theta \in \mathbb{R}$ the following statements hold.*

(a) $R_i(\theta)J_j(\theta) = \frac{1}{j+1}(R_i(\theta) - R_{i+j+1}(\theta))$.

(c) $R_i(\theta)\tilde{J}_j(\theta) = \frac{2R_i(\theta)}{(j+1)(j+3)} - \frac{R_{i+j+1}(\theta)}{j+1} + \frac{R_{i+j+3}(\theta)}{j+3}$.

(e) $T_i(\theta)J_j(\theta) = \frac{1}{j+1}(T_i(\theta) - T_{i+j+1}(\theta))$.

(g) $T_i(\theta)\tilde{J}_j(\theta) = \frac{2T_i(\theta)}{(j+1)(j+3)} - \frac{T_{i+j+1}(\theta)}{j+1} + \frac{T_{i+j+3}(\theta)}{j+3}$.

(i) $S_i(\theta)J_j(\theta) = \frac{1}{j+1}(S_i(\theta) - S_{i+j+1}(\theta))$.

(k) $S_i(\theta)\tilde{J}_j(\theta) = \frac{2S_i(\theta)}{(j+1)(j+3)} - \frac{S_{i+j+1}(\theta)}{j+1} + \frac{S_{i+j+3}(\theta)}{j+3}$.

(m) $U_i(\theta)\tilde{j}_j(\theta) = \frac{1}{j+1}(U_i(\theta) - U_{i+j+1}(\theta))$.

(o) $U_i(\theta)\tilde{I}_j(\theta) = \frac{2U_i(\theta)}{(j+1)(j+3)} - \frac{U_{i+j+1}(\theta)}{j+1} + \frac{U_{i+j+3}(\theta)}{j+3}$.

Proof. The Proposition follows if we use the integrals (8) and (9) and Proposition 3. In the next Corollary we use

$$\int_0^{2\pi} \cos^i(\theta)\sin^j(\theta)\sin((2l+1)\theta)d\theta \neq 0, \text{ if } i \text{ even and } j \text{ odd,}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin^j(\theta)\sin((2l+1)\theta)d\theta = \begin{cases} 0, & \text{if } i \text{ odd or } j \text{ even,} \\ \pi C_{k,l}, & \text{if } i = 2k,\ j = 1 \text{ and } l \geq 0, \\ \pi K_{k,l}, & \text{if } i = 2k,\ j = 3 \text{ and } l \geq 0, \end{cases} \tag{26}$$

where $C_{k,l}$, $K_{k,l}$ are non-zero constants.

$$\int_0^{2\pi} \cos^i(\theta)\sin^j(\theta)\sin(2l\theta)d\theta \neq 0, \text{ if } i \text{ and } j \text{ odd,}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin^j(\theta)\sin(2l\theta)d\theta = \begin{cases} 0, & \text{if } i \text{ odd or } j \text{ even,} \\ \pi\tilde{C}_{k,l}, & \text{if } i = 2k+1,\ j = 1 \text{ and } l \geq 0, \\ \pi\tilde{K}_{k,l}, & \text{if } i = 2k+1,\ j = 3 \text{ and } l \geq 0, \end{cases} \tag{27}$$

where $\tilde{C}_{k,l}$, $\tilde{K}_{k,l}$ are non-zero constants. ■

Corollary 2 *For $i, j \in \mathbb{N}$, the following statements hold.*

(a)

$$\begin{aligned}\int_0^{2\pi} T_i(\theta)\sin(2l+1)\theta d\theta &= \int_0^{2\pi} T_i(\theta)\sin(2l\theta)d\theta = \int_0^{2\pi} U_i(\theta)\sin(2l+1)\theta d\theta \\ &= \int_0^{2\pi} U_i(\theta)\sin(2l\theta)d\theta = 0.\end{aligned}$$

(b) $\int_0^{2\pi} R_i(\theta)\sin(2l+1)\theta d\theta = \pi C_{i,l}$.

(c) $\int_0^{2\pi} R_i(\theta)\sin(2l\theta)d\theta = \pi \tilde{C}_{i,l}$.

(d) $\int_0^{2\pi} S_i(\theta)\sin(2l+1)\theta d\theta = \pi K_{i,l}$.

(e) $\int_0^{2\pi} S_i(\theta)\sin(2l\theta)d\theta = \pi \tilde{K}_{i,l}$.

Proof. The Corollary follows if we use the integrals (26) and (27). ■

Corollary 3 *For $i, j \in \mathbb{N}$ the following statements hold.*

(a)

$$\int_0^{2\pi} R_i(\theta)J_j(\theta)d\theta = \int_0^{2\pi} R_i(\theta)\tilde{J}_j(\theta)d\theta = \int_0^{2\pi} S_i(\theta)J_j(\theta)d\theta = \int_0^{2\pi} S_i(\theta)\tilde{J}_j(\theta)d\theta = 0.$$

(b)

$$\int_0^{2\pi} T_i(\theta)J_j(\theta)d\theta = \begin{cases} \frac{1}{2s+1}\tilde{I}_{2k}(2\pi), & \text{if } i = 2k,\ j = 2s, \\ \frac{1}{2s+2}(\tilde{I}_{2k}(2\pi) - \tilde{I}_{2k+2s+2}(2\pi)), & \text{if } i = 2k,\ j = 2s+1, \\ -\frac{1}{2s+2}\tilde{I}_{2k+2s+2}(2\pi), & \text{if } i = 2k+1,\ j = 2s, \\ 0, & \text{if } i = 2k+1,\ j = 2s+1. \end{cases}$$

(c)

$$\begin{aligned}&\int_0^{2\pi} T_i(\theta)\tilde{J}_j(\theta)d\theta \\ = &\begin{cases} \frac{2}{(2s+1)(2s+3)}\tilde{I}_{2k}(2\pi), & \text{if } i = 2k,\ j = 2s, \\ \frac{2}{(2s+2)(2s+4)}\tilde{I}_{2k}(2\pi) - \frac{2}{(2s+2)}\tilde{I}_{2k+2s+2}(2\pi) + \frac{2}{(2s+4)}\tilde{I}_{2k+2s+4}(2\pi), & \text{if } i = 2k,\ j = 2s+1, \\ -\frac{2}{(2s+1)}\tilde{I}_{2k+2s+2}(2\pi) + \frac{2}{(2s+3)}\tilde{I}_{2k+2s+4}(2\pi), & \text{if } i = 2k+1,\ j = 2s, \\ 0, & \text{if } i = 2k+1,\ j = 2s+1. \end{cases}\end{aligned}$$

(d)

$$\int_0^{2\pi} U_i(\theta)J_j(\theta)d\theta = \begin{cases} \frac{1}{2s+1}\tilde{\tilde{I}}_{2k}(2\pi), & \text{if } i = 2k,\ j = 2s, \\ \frac{1}{2s+2}(\tilde{\tilde{I}}_{2k}(2\pi) - \tilde{\tilde{I}}_{2k+2s+2}(2\pi)), & \text{if } i = 2k,\ j = 2s+1, \\ -\frac{1}{2s+2}\tilde{\tilde{I}}_{2k+2s+2}(2\pi), & \text{if } i = 2k+1,\ j = 2s, \\ 0, & \text{if } i = 2k+1,\ j = 2s+1. \end{cases}$$

(e)

$$\int_0^{2\pi} U_i(\theta)\tilde{J}_j(\theta)d\theta$$
$$= \begin{cases} \frac{2}{(2s+1)(2s+3)}\tilde{\tilde{I}}_{2k}(2\pi), & \text{if } i=2k,\ j=2s, \\ \frac{2}{(2s+2)(2s+4)}\tilde{\tilde{I}}_{2k}(2\pi) - \frac{2}{(2s+2)}\tilde{\tilde{I}}_{2k+2s+2}(2\pi) + \frac{2}{(2s+4)}\tilde{\tilde{I}}_{2k+2s+4}(2\pi), & \text{if } i=2k,\ j=2s+1, \\ -\frac{2}{(2s+1)}\tilde{\tilde{I}}_{2k+2s+2}(2\pi) + \frac{2}{(2s+3)}\tilde{\tilde{I}}_{2k+2s+4}(2\pi), & \text{if } i=2k+1,\ j=2s, \\ 0, & \text{if } i=2k+1,\ j=2s+1. \end{cases}$$

Proof. Using Proposition 4 and using (10) for all $i, j \in \mathbb{N}$, we obtain (a). Again from Proposition 4 and using (13) and (14) one has (b), (c), (d) and (e) and the Corollary follows. ■

Now we can compute the integral $II(r)$.

Lemma 1 *The integral $II(r)$ is given by*

$$\begin{aligned} II(r) &= \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k+1}M^2_{2s,2k+1} + \sum_{s=0}^{[\frac{n-1}{2}]}\sum_{k=0}^{[\frac{n}{2}]} r^{2s+2k+1}M^4_{2s+1,2k} \\ &\quad + \sum_{k=0}^{[\frac{n-1}{2}]}\sum_{s=0}^{[\frac{m-1}{2}]} r^{2s+2k+3}M^6_{2s+1,2k+1} + \sum_{k=0}^{[\frac{n}{2}]}\sum_{s=0}^{[\frac{m}{2}]} r^{2k+2s+1}M^9_{2s,2k}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} M^2_{2s,2k+1} &= a_{2s,1}b_{2k+1,1}(2s\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}\pi C_{s,l} - \frac{(k+1)\pi\alpha_{k+s+1}}{2^{k+s}(2s+1)(k+s+2)!}) + a_{2s,1}d_{2k+1,1}r^2 \\ &\quad \times(2s\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}\pi C_{i,l} - \frac{3(k+2)\pi\alpha_{k+s+1}}{2^{k+s+1}(2s+1)(k+s+3)!}) + c_{2s,1}b_{2k+1,1}r^2((2s+2) \\ &\quad \times\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}\pi K_{s,l} - \frac{(k+1)(4k+10s+15)\pi\alpha_{k+s+1}}{2^{k+s+1}(2s+1)(2s+3)(k+s+3)!}) + c_{2s,1}d_{2k+1,1}r^4 \\ &\quad \times((2s+2)\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}\pi K_{2s,l} - \frac{3(k+2)(4k+14s+21)\pi\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(2s+3)(k+s+4)!}), \end{aligned} \tag{29}$$

$$\begin{aligned} M^4_{2s+1,2k} &= a_{2s+1,1}d_{2k-2,1}((2s+1)\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\pi\tilde{C}_{s,l} + \frac{3(2k+1)\pi\alpha_{k+s}}{2^{k+s+1}(2k-1)(k+s+2)!}) \\ &\quad +c_{2s+1,1}d_{2k-2,1}r^2((2s+3)\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\pi\tilde{K}_{s,l} + \frac{15(2k+1)\pi\alpha_{k+s}}{2^{k+s+2}(2k-1)(k+s+3)!}), \end{aligned} \tag{30}$$

$$\begin{aligned} M^6_{2s+1,2k+1} &= e_{2s+1,1}b_{2k+1}(-\frac{(k+1)\pi\alpha_{k+s+2}}{2^{k+s+1}(2s+3)(k+s+3)!} + (2s+2)\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l} \\ &\quad \times\pi C_{s+1,l}) + e_{2s+1,1}d_{2k+1}r^2(-\frac{3(k+2)\pi\alpha_{k+s+2}}{2^{k+s+2}(2s+3)(k+s+4)!} \\ &\quad +(2s+2)\sum_{l=0}^{k+1}\tilde{\tilde{\gamma}}_{k,l}\pi C_{s+1,l}), \end{aligned} \tag{31}$$

$$M^9_{2s,2k} = e_{2s,1}d_{2k-2}\left((2s+1)\sum_{l=0}^{k+1}\tilde{\beta}_{k,l}\pi\tilde{C}_{s,l} + \frac{3(2k+1)}{2^{k+s+1}(2k-1)}\right)\frac{\pi\alpha_{k+s}}{(k+s+2)!} \tag{32}$$

Proof. Using the expression (25) and the results of Corollaries 2 and 3, we have

- $M^5_{2s+1,2k+1} = M^7_{2s,2k+1} = M^{10}_{2s,2k} = 0$ for all $s,\ k \in N$.
- $M^1_{i,j} = M^3_{i,j} = M^8_{i,j} = 0$ for all $i,\ j \in N$.
- If $i = 2s$ and by substituting in (25), we obtain $M^2_{2s,2k+1} = M^9_{2s,2k} = 0$ and also the expressions (30) and (31).
- Finally if $i = 2s+1$ and by substituting in (25) we obtain $M^4_{2s+1,2k} = M^6_{2s+1,2k+1} = 0$, (29) and (32).

The proof of the Lemma is completed. ■

In order to complete the computation of $F_{20}(r)$ we must determine the function $III(r)$ of the expression of $F_{20}(r)$ given in (24). In the next Proposition we obtain the value of $F_2(r,\theta)$ when $F_{10}(r) \equiv 0$.

Proposition 5 *If $F_{10}(r) \equiv 0$, then*

$$\begin{aligned} F_2(r,\theta) \quad = \quad & \sum_{i=0}^{n} r^i\left(a_{i,2}R_i(\theta) + b_{i,2}rT_i(\theta) + c_{i,2}r^2S_i(\theta) + d_{i,2}r^3U_{2i+1}(\theta)\right) \\ & + \sum_{i=0}^{m} e_{i,2}r^{i+1}R_{i+1}(\theta) - \frac{1}{r}\left[\sum_{i=0}^{n} r^i(a_{i,1}R_i(\theta) + c_{i,1}r^2S_i(\theta)) + \sum_{k=0}^{[\frac{n-1}{2}]} r^{2k+2}\right. \\ & \times(b_{2k+1,1}T_{2k+1}(\theta) + d_{2k+1,1}r^2U_{2k+1}(\theta)) + \sum_{i=0}^{m} e_{i,1}r^{i+1}R_{i+1}(\theta) \\ & \left.+ \sum_{k=0}^{[\frac{n}{2}]} d_{2k-2}r^{2i+1}\left(T_{2k-2}(\theta) - \frac{2k+2}{2k-1}T_{2k}(\theta)\right)\right] \times \left[\sum_{i=0}^{n} r^i(a_{i,1}\cos^{i+1}(\theta)\right. \\ & + c_{i,1}r^2T_{i+1}(\theta)) + \sum_{k=0}^{[\frac{n-1}{2}]} r^{2k+2}(b_{2k+1,1}R_{2k+2}(\theta)d_{2k+1,1}r^2S_{2k+2}(\theta)) \\ & \left.+ \sum_{i=0}^{m} e_{i,1}r^{i+1}T_i(\theta) + \sum_{k=0}^{[\frac{n}{2}]} d_{2k-2}r^{2k+1}\left(R_{2k-1} - \frac{2k+2}{2k-1}R_{2k+1}(\theta)\right)\right]. \end{aligned} \tag{33}$$

Proof. If $F_{10}(r) \equiv 0$, from (15) we have that $b_0 = d_{2[\frac{n}{2}]} = 0$, and $b_{2i} = \frac{-3}{2i-1}d_{2i-2}$. By substituting the relation $U_k(\theta) = T_k(\theta) - T_{k+2}(\theta)$ in (19) we obtain (33).

Now we shall compute the integrals $III(r)$. Using the expression of (33) we have

$$\begin{aligned}
III(r) &= \sum_{i=0}^{n} r^i(a_{i,2}\int_0^{2\pi} R_i(\theta)d\theta + b_{i,2}r\int_0^{2\pi} T_i(\theta)d\theta + c_{i,2}r^2\int_0^{2\pi} S_i(\theta)d\theta \\
&+d_{i,2}r^3\int_0^{2\pi} U_i(\theta)d\theta) + \sum_{i=0}^{n}\sum_{i=0}^{n} r^{i+j-1}\tilde{M}^1_{i,j} + \sum_{i=0}^{n}\sum_{k=0}^{[\frac{n-1}{2}]} r^{i+2k+1}\tilde{M}^2_{i,j} \\
&+\sum_{i=0}^{n}\sum_{j=0}^{m} r^{i+j}\tilde{M}^3_{i,j} + \sum_{i=0}^{n}\sum_{s=0}^{[\frac{n}{2}]} r^{i+2s}\tilde{M}^4_{i,j} + \sum_{s=0}^{[\frac{n-1}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k+3}\tilde{M}^5_{i,j} \\
&+\sum_{k=0}^{[\frac{n-1}{2}]}\sum_{i=0}^{m} r^{i+2k+2}\tilde{M}^6_{i,j} + \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k+2}\tilde{M}^7_{i,j} + \sum_{i=0}^{m}\sum_{j=0}^{m} r^{i+j+1}\tilde{M}^8_{i,j} \\
&+\sum_{s=0}^{[\frac{n}{2}]}\sum_{i=0}^{m} r^{2s+i+1}\tilde{M}^9_{i,j} + \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n}{2}]} r^{2s+2k+1}\tilde{M}^{10}_{i,j},
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\tilde{M}^1_{i,j} &= a_{i,1}a_{j,1}\int_0^{2\pi} R_i(\theta)\cos^{j+1}\theta d\theta + a_{i,1}c_{j,1}r^2\int_0^{2\pi} R_i(\theta)T_{j+1}(\theta)d\theta + c_{i,1}a_{j,1}r^2 \\
&\times\int_0^{2\pi} S_i(\theta)\cos^{j+1}\theta d\theta + c_{i,1}c_{j,1}r^4\int_0^{2\pi} S_i(\theta)T_{j+1}(\theta)d\theta,
\end{aligned}$$

$$\begin{aligned}
\tilde{M}^2_{i,j} &= a_{i,1}b_{2k+1,1}(\int_0^{2\pi} T_{2k+1}(\theta)\cos^{i+1}\theta d\theta + \int_0^{2\pi} R_{2k+2}(\theta)R_i(\theta)d\theta) + a_{i,1}d_{2k+1,1}r^2 \\
&\times(\int_0^{2\pi} U_{2k+1}(\theta)\cos^{i+1}\theta d\theta + \int_0^{2\pi} S_{2k+2}(\theta)R_i(\theta)d\theta) + c_{i,1}b_{2k+1,1}r^2 \\
&\times(\int_0^{2\pi} T_{2k+1}(\theta)T_i(\theta)d\theta + \int_0^{2\pi} R_{2k+2}(\theta)S_i(\theta)d\theta) + c_{i,1}d_{2k+1,1}r^4\theta \\
&\times(\int_0^{2\pi} U_{2k+1}(\theta)T_i(\theta)d + \int_0^{2\pi} S_{2k+2}(\theta)S_i(\theta)d\theta),
\end{aligned}$$

$$\begin{aligned}
\tilde{M}^3_{i,j} &= a_{i,1}e_{j,1}(\int_0^{2\pi} R_i(\theta)T_j(\theta)d\theta + \int_0^{2\pi} R_{j+1}(\theta)\cos^{i+1}\theta d\theta) + c_{i,1}e_{j,1}r^2 \\
&\times(\int_0^{2\pi} S_i(\theta)T_j(\theta)d\theta + \int_0^{2\pi} R_{j+1}(\theta)T_i(\theta)d\theta),
\end{aligned}$$

$$\begin{aligned}
\tilde{M}^4_{i,j} &= a_{i,1}d_{2k-2,1}(\int_0^{2\pi} R_i(\theta)R_{2k-1}(\theta)d\theta + \int_0^{2\pi} T_{2k-2}(\theta)\cos^{i+1}\theta d\theta - \frac{2k+2}{2k-1} \\
&\times(\int_0^{2\pi} R_i(\theta)R_{2k+1}(\theta)d\theta + \int_0^{2\pi} T_{2k}(\theta)\cos^{i+1}\theta d\theta)) + c_{i,1}d_{2k-2,1}r^2 \\
&\times(\int_0^{2\pi} S_i(\theta)R_{2k-1}(\theta)d\theta + \int_0^{2\pi} T_{2k-2}(\theta)T_i(\theta)d\theta - \frac{2k+2}{2k-1}\int_0^{2\pi} S_i(\theta)R_{2k+1}(\theta)d\theta \\
&-\frac{2k+2}{2k-1}\int_0^{2\pi} T_{2k}(\theta)T_i(\theta)d\theta),
\end{aligned}$$

$$\begin{aligned}\tilde{M}^5_{i,j} &= b_{2s+1,1}b_{2k+1}\int_0^{2\pi} T_{2s+1}(\theta)R_{2k+2}(\theta)d\theta + d_{2s+1,1}b_{2k+1}r^2\int_0^{2\pi} U_{2s+1}(\theta)\\ &\quad\times R_{2k+2}(\theta)d\theta + b_{2s+1,1}d_{2k+1}r^2\int_0^{2\pi} T_{2s+1}(\theta)S_{2k+2}(\theta)d\theta + d_{2s+1,1}d_{2k+1}\\ &\quad\times r^4\int_0^{2\pi} U_{2s+1}(\theta)S_{2k+2}(\theta)d\theta,\end{aligned}$$

$$\begin{aligned}\tilde{M}^6_{i,j} &= e_{i,1}b_{2k+1}\left(\int_0^{2\pi} R_{2k+2}(\theta)R_{i+1}(\theta)d\theta + \int_0^{2\pi} T_i(\theta)T_{2k+1}(\theta)d\theta\right)\\ &\quad+e_{i,1}d_{2k+1}r^2\left(\int_0^{2\pi} S_{2k+2}(\theta)R_{i+1}(\theta)d\theta + \int_0^{2\pi} T_i(\theta)U_{2k+1}(\theta)d\theta\right),\end{aligned}$$

$$\begin{aligned}\tilde{M}^7_{i,j} &= d_{2s-2,1}b_{2k+1}\Bigg[\int_0^{2\pi} T_{2s-2}(\theta)R_{2k+2}(\theta)d\theta + \int_0^{2\pi} R_{2s-1}(\theta)T_{2k+1}(\theta)d\theta\\ &\quad-\frac{2s+2}{2s-1}\left(\int_0^{2\pi} T_{2s}(\theta)R_{2k+2}(\theta)d\theta + \int_0^{2\pi} R_{2s+1}(\theta)T_{2k+1}(\theta)d\theta\right)\Bigg]\\ &\quad+d_{2s-2,1}d_{2k+1}r^2\Bigg[\int_0^{2\pi} T_{2s-2}(\theta)S_{2k+2}(\theta)d\theta + \int_0^{2\pi} R_{2s-1}(\theta)U_{2k+1}(\theta)d\theta\\ &\quad-\frac{2s+2}{2s-1}\left(\int_0^{2\pi} T_{2s}(\theta)S_{2k+2}(\theta)d\theta + \int_0^{2\pi} R_{2s+1}(\theta)U_{2k+1}(\theta)d\theta\right)\Bigg],\end{aligned}$$

$$\tilde{M}^8_{i,j} = e_{i,1}e_{j,1}\int_0^{2\pi} R_{j+1}(\theta)T_i(\theta)d\theta,$$

$$\begin{aligned}\tilde{M}^9_{i,j} &= e_{i,1}d_{2k-2}\Bigg[\int_0^{2\pi} R_{i+1}(\theta)R_{2k-1}(\theta)d\theta + \int_0^{2\pi} T_{2k-2}(\theta)T_i(\theta)d\theta\\ &\quad-\frac{2k+2}{2k-1}\left(\int_0^{2\pi} R_{2k+1}(\theta)R_{i+1}(\theta)d\theta + \int_0^{2\pi} T_{2k}(\theta)T_i(\theta)d\theta\right)\Bigg],\end{aligned}$$

$$\begin{aligned}\tilde{M}^{10}_{i,j} &= d_{2s-2,1}d_{2k-2,1}\Bigg[\int_0^{2\pi} T_{2s-2}(\theta)R_{2k-1}(\theta)d\theta - \frac{2s+2}{2s-1}\\ &\quad\times\left(\int_0^{2\pi} T_{2s-2}(\theta)R_{2k+1}(\theta)d\theta + \int_0^{2\pi} T_{2s}(\theta)R_{2k-1}(\theta)d\theta\right)\\ &\quad+\frac{(2s+2)^2}{(2s-1)^2}\int_0^{2\pi} T_{2s}(\theta)R_{2k+1}(\theta)d\theta\Bigg].\end{aligned}$$

■

In the next propositions we obtain some results on the integrals of the right hand part of (34).

Proposition 6 *If $F_{10}(r) \equiv 0$, then*

$$\begin{aligned}III(r) &= \sum_{s=0}^{[\frac{n}{2}]} r^{2s+1}\frac{\pi\alpha_s}{2^s(s+1)!}\left(b_{2s,2} + \frac{3}{2(s+2)}d_{2s,2}r^2\right) + \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k+1}\times\tilde{M}^2_{2s,2k+1}\\ &\quad+\sum_{s=0}^{[\frac{n-1}{2}]}\sum_{k=0}^{[\frac{n}{2}]} r^{2s+2k+1}\tilde{M}^4_{2s+1,2k} + \sum_{k=0}^{[\frac{n-1}{2}]}\sum_{s=0}^{[\frac{m-1}{2}]} r^{2s+2k+3}\times\tilde{M}^6_{2s+1,2k+1}\\ &\quad+\sum_{k=0}^{[\frac{n}{2}]}\sum_{s=0}^{[\frac{m}{2}]} r^{2k+2s+1}\tilde{M}^9_{2s,2k},\end{aligned}\qquad(35)$$

where

$$\begin{aligned}\tilde{M}^2_{2s,2k+1} &= -a_{2s,1}b_{2k+1,1}\frac{\pi\alpha_{s+k+1}}{2^{s+k}(s+k+2)!} - a_{2s,1}d_{2k+1,1}r^2\frac{3\pi\alpha_{s+k+1}}{2^{s+k+1}(s+k+3)!} \\ &\quad -c_{2s,1}b_{2k+1,1}r^2\frac{3\pi\alpha_{s+k+1}}{2^{s+k+1}(s+k+3)!} - c_{2s,1}d_{2k+1,1}r^4 \\ &\quad \times\frac{15\pi\alpha_{s+k+1}}{2^{s+k+2}(s+k+4)!}, \qquad (36)\\ \tilde{M}^4_{2s+1,2k} &= a_{2s+1,1}d_{2k-2,1}\frac{3\pi\alpha_{s+k}}{2^{s+k-1}(2k-1)(s+k+2)!} + c_{2s+1,1}d_{2k-2,1}r^2 \\ &\quad \times\frac{3\pi(3s-2k+4)\alpha_{s+k}}{2^{s+k}(2k-1)(s+k+3)!}, \qquad (37)\\ \tilde{M}^6_{2s+1,2k+1} &= -e_{2s+1,1}b_{2k+1}\frac{\pi(s+k)\alpha_{s+k+1}}{2^{s+k+1}(s+k+3)!} - e_{2s+1,1}d_{2k+1}r^2 \\ &\quad \times\frac{3\pi(s+k-1)\alpha_{s+k+1}}{2^{s+k+2}(s+k+4)!}, \qquad (38)\\ \tilde{M}^9_{2s,2k} &= e_{i,1}d_{2k-2}\frac{3\pi(s^2-s+2k+sk-1)\alpha_{s+k-1}}{2^{s+k-1}(2k-1)(s+k+2)!}. \qquad (39)\end{aligned}$$

Proof. Using the expression (34) and the Corollaries 1 and 3, we have

- $\tilde{M}^5_{2s+1,2k+1} = \tilde{M}^7_{2s,2k+1} = \tilde{M}^{10}_{2s,2k} = 0$ for all $s,\ k \in N$.
- $\tilde{M}^1_{i,j} = \tilde{M}^3_{i,j} = \tilde{M}^8_{i,j} = 0.$ for all $i,\ j \in N$.
- If $i = 2s$ and by substituting in (34) we obtain $\tilde{M}^2_{2s,2k+1} = \tilde{M}^9_{2s,2k} = 0$ and (37) and (38).
- Finally if $i = 2s+1$ and by substituting in (34) we obtain $\tilde{M}^4_{2s+1,2k} = \tilde{M}^6_{2s+1,2k+1} = 0$ and (36) and (39). The proof of the Proposition is completed. ∎

The following Proposition provide us an expression of $F_{20}(r)$.

Proposition 7 *The function $F_{20}(r)$ defined in (24) can be written as*

$$\begin{aligned}F_{20}(r) &= \frac{r}{2\pi}\left[\sum_{s=0}^{[\frac{n}{2}]} r^{2s}\frac{\pi\alpha_s}{2^s(s+1)!}\left(b_{2s,2}+\frac{3}{2(s+2)}d_{2s,2}r^2\right)+\right.\\ &\quad \sum_{s=0}^{[\frac{n}{2}]}\sum_{k=0}^{[\frac{n-1}{2}]} r^{2s+2k}(M^2_{2s,2k+1}+\tilde{M}^2_{2s,2k+1}) + \sum_{s=0}^{[\frac{n-1}{2}]}\sum_{k=0}^{[\frac{n}{2}]} r^{2s+2k}(M^4_{2s+1,2k}+\tilde{M}^4_{2s+1,2k})\\ &\quad + \sum_{k=0}^{[\frac{n-1}{2}]}\sum_{s=0}^{[\frac{m-1}{2}]} r^{2s+2k}(M^6_{2s+1,2k+1}+\tilde{M}^6_{2s+1,2k+1})\\ &\quad \left.+\sum_{k=0}^{[\frac{n}{2}]}\sum_{s=0}^{[\frac{m}{2}]} r^{2k+2s}(M^9_{2s,2k}+\tilde{M}^9_{2s,2k})\right], \qquad (40)\end{aligned}$$

where $M^2_{2s,2k+1}+\tilde{M}^2_{2s,2k+1}$, $M^4_{2s+1,2k}+\tilde{M}^4_{2s+1,2k}$, $M^6_{2s+1,2k+1}+\tilde{M}^6_{2s+1,2k+1}$ and $M^9_{2s,2k}+\tilde{M}^9_{2s,2k}$ have degree $4, 2, 2$ and 0 respectively.

Proof. The proof follows by Lemmas 1 and 2 and considering that $2\pi F_{20} = II(r) + III(r)$. ∎
Proof of Theorem 2. Note that in order to find the positive roots of $F_{20}(r)$ we must find the zeros of a polynomial in r^2 of degree equal to the

$$\max\left\{[\frac{n-1}{2}]+[\frac{n}{2}]+2,[\frac{n-1}{2}]+[\frac{m-1}{2}]+2\right\}.$$

Since $\left[\frac{n-1}{2}\right]+\left[\frac{n}{2}\right]+2$ takes the values $\left[\frac{n-1}{2}\right]+\left[\frac{n-1}{2}\right]+2$ if n is odd, and $\left[\frac{n-2}{2}\right]+\left[\frac{n}{2}\right]+2$ if n is even, that is $n+1$ in both cases. we conclude that $F_{20}(r)$ has at most

$$\max\left\{n+1,[\frac{n-1}{2}]+[\frac{m-1}{2}]+2\right\}$$

simple positive zeros. ∎

5 Examples

In this section we prove that the upper bound is reached in the following examples. The computations have been obtained by using Maple.

Example 1 *We denote by $F_{10}^{m,n}(r)$ the function $F_{10}(r)/r$ corresponding to degree n, an adequate computation allows to obtain that*

$$F_{10}^{m,1}(r) = \frac{1}{2}b_{0,1} + \frac{3}{8}d_{0,1}r^2,$$

now if we fixed $b_{0,1} = 2$ and $d_{0,1} = -\frac{8}{3}$ we have that $-F_{10}^{m,1}(r) = 1 - r^2$, that has exactly one positive zero.

$$F_{10}^{m,2}(r) = F_{10}^{m,3}(r) = \frac{1}{2}b_{0,1} + (\frac{3}{8}d_{0,1} + \frac{1}{8}b_{2,1})r^2 + \frac{1}{16}d_{2,1}r^4,$$

now if we fixed $b_{0,1} = 2$, $d_{0,1} = -\frac{8}{3}$, $d_{2,1} = 8$ and $b_{2,1} = -4$ we have that $F_{10}^{m,1}(r) = F_{10}^{m,3}(r) = \frac{1}{2}(1-r^2)(2-r^2)$, that has exactly two positive zeros.

$$F_{10}^{m,4}(r) = F_{10}^{m,5}(r) = \frac{1}{2}b_{0,1} + (\frac{3}{8}d_{0,1} + \frac{1}{8}b_{2,1})r^2 + (\frac{1}{16}d_{2,1} + \frac{1}{16}b_{4,1})r^4 + \frac{3}{128}d_{4,1}r^6,$$

now if we fixed $b_{0,1} = 2$, $d_{0,1} = -\frac{8}{3}$, $d_{2,1} = 8$, $d_{4,1} = \frac{-64}{9}$, $b_{2,1} = \frac{-20}{3}$ and $b_{4,1} = 8$ we have that

$$F_{10}^{m,4}(r) = F_{10}^{m,5}(r) = \frac{1}{6}(1-r^2)(2-r^2)(3-r^2),$$

that has exactly three positive zeros.

$$\begin{aligned} F_{10}^{m,6}(r) &= F_{10}^{m,7}(r) \\ &= \frac{1}{2}b_{0,1} + (\frac{3}{8}d_{0,1} + \frac{1}{8}b_{2,1})r^2 + (\frac{1}{16}d_{2,1} + \frac{1}{16}b_{4,1})r^4 \\ &\quad +(\frac{3}{128}d_{4,1} + \frac{5}{128}b_6)r^6 + \frac{3}{256}d_{6,1}r^8, \end{aligned}$$

now if we fixed $b_{0,1} = 2$, $d_{0,1} = -\frac{8}{3}$, $d_{2,1} = 8$, $d_{4,1} = \frac{-64}{9}$, $d_{6,1} = \frac{32}{9}$, $b_{2,1} = \frac{-26}{3}$, $b_{4,1} = \frac{46}{3}$ and $b_6 = -\frac{32}{5}$ we have that

$$F_{10}^{m,6}(r) = F_{10}^{m,7}(r) = \frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2),$$

that has exactly four positive zeros.

$$\begin{aligned} F_{10}^{m,8}(r) &= F_{10}^{m,9}(r) \\ &= \frac{1}{2}b_{0,1} + (\frac{3}{8}d_{0,1} + \frac{1}{8}b_{2,1})r^2 + (\frac{1}{16}d_{2,1} + \frac{1}{16}b_{4,1})r^4 + (\frac{3}{128}d_{4,1} + \frac{5}{128}b_6)r^6 \\ &\quad +(\frac{3}{256}d_{6,1} + \frac{7}{256}b_8)r^8 + \frac{7}{1024}d_8r^{10}, \end{aligned}$$

now if we fixed $b_{0,1} = 2$, $d_{0,1} = -\frac{8}{3}$, $d_{2,1} = 8$, $d_{4,1} = \frac{-64}{9}$, $d_{6,1} = \frac{32}{9}$, $d_{8,1} = -\frac{128}{105}$, $b_{2,1} = -\frac{154}{15}$, $b_{4,1} = 22$, $b_{6,1} = -\frac{208}{15}$ *and* $b_{8,1} = \frac{64}{21}$ *we have that*

$$F_{10}^{m,8}(r) = F_{10}^{m,9}(r) = \frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2),$$

that has exactly five positive zeros.

Example 2 *We denote by* $F_{20}^{m,n}(r)$ *the function* $F_{20}(r)/r$ *corresponding, an adequate computation allows to obtain that*

$$F_{20}^{m,1}(r) = \frac{1}{2}b_{0,2} - \frac{1}{2}a_{0,1}b_{1,1} + (\frac{3}{8}d_{0,2} - \frac{1}{8}c_{0,1}b_{1,1} - \frac{3}{8}d_{1,1}a_{0,1})r^2 - \frac{3}{16}d_{1,1}, c_{0,1}r^4,$$

now if we fixed $b_{0,2} = 2$, $b_{1,1} = 0$, $d_{1,1} = -\frac{8}{3}$, $d_{0,2} = -\frac{20}{3}$ *and* $a_{0,1} = 1$ *we have that* $F_{20}^{m,1}(r) = \frac{1}{2}(1-r^2)(2-r^2)$, *that has exactly two positive zeros.*

$$\begin{aligned} F_{20}^{1,2}(r) &= \frac{1}{2}b_{0,2} - \frac{1}{2}a_{0,1}b_{1,1} \\ &\quad +(\frac{3}{8}d_{0,2} - \frac{1}{8}c_{0,1}b_{1,1} - \frac{3}{8}d_{1,1}a_{0,1} - \frac{1}{8}b_{1,1}a_{2,1} + \frac{3}{8}d_{0,1}a_{1,1} + \frac{3}{8}e_{0,1}d_{0,1} + \frac{1}{8}b_{2,2})r^2 \\ &\quad -(\frac{3}{16}d_{1,1}c_{0,1} - \frac{1}{48}b_{1,1}c_{2,1} + \frac{1}{16}d_{0,1}c_{1,1} + \frac{1}{16}d_{2,2} - \frac{1}{16}d_{1,1}a_{2,1})r^4 \\ &\quad -\frac{3}{128}d_{1,1}c_{2,1}r^6, \end{aligned}$$

now if we fixed $b_{0,2} = 2$, $b_{1,1} = 0$, $c_{0,1} = 1$, $d_{1,1} = -\frac{8}{3}$, $c_{2,1} = -\frac{8}{3}$, $d_{0,2} = -4$, $d_{0,1} = -8$, $a_{1,1} = 0$, $c_{1,1} = 0$, $e_{0,1} = \frac{1}{3}$, $a_{2,1} = -8$, $e_{1,1} = 1$, $b_{2,2} = -\frac{8}{3}$, $d_{2,2} = \frac{88}{3}$ *and* $a_{0,1} = 1$ *we have that* $F_{20}^{1,2}(r) = \frac{1}{6}(1-r^2)(2-r^2)(3-r^2)$, *that has exactly three positive zeros.*

$$F_{20}^{2,2}(r) = F_{20}^{3,2}(r) = F_{20}^{1,2}(r) + \frac{1}{16}d_{0,1}e_{2,1}r^4,$$

now if we fixed $b_{0,2} = 2$, $b_{1,1} = 0$, $c_{0,1} = 1$, $d_{1,1} = -\frac{8}{3}$, $c_{2,1} = -\frac{8}{3}$, $d_{0,2} = -4$, $d_{0,1} = -8$, $a_{1,1} = 0$, $c_{1,1} = 0$, $e_{0,1} = \frac{1}{3}$, $a_{2,1} = -8$, $e_{1,1} = 1$, $b_{2,2} = -\frac{8}{3}$, $d_{2,2} = \frac{40}{3}$, $a_{0,1} = 1$ *and* $e_{2,1} = -2$ *we have that* $F_{20}^{2,2}(r) = \frac{1}{6}(1-r^2)(2-r^2)(3-r^2)$, *that has exactly three positive zeros.*

$$F_{20}^{4,2}(r) = F_{20}^{2,2}(r) + \frac{3}{128}d_{0,1}e_{4,1}r^6,$$

now if we fixed $b_{0,2} = 2$, $b_{1,1} = 0$, $c_{0,1} = 1$, $d_{1,1} = -\frac{8}{3}$, $c_{2,1} = -1$, $d_{0,2} = -4$, $d_{0,1} = -8$, $a_{1,1} = 0$, $c_{1,1} = 0$, $e_{0,1} = \frac{1}{3}$, $a_{2,1} = -8$, $e_{1,1} = 1$, $b_{2,2} = -\frac{8}{3}$, $d_{2,2} = \frac{40}{3}$, $a_{0,1} = 1$, $e_{2,1} = -2$, *and* $e_{4,1} = \frac{5}{9}$ *we have that*

$F_{20}^{2,2}(r)=\frac{1}{6}(1-r^2)(2-r^2)(3-r^2)$, *that has exactly three positive zeros.*

$$\begin{aligned}F_{20}^{1,3}(r) &= \frac{1}{2}b_{0,2}-\frac{1}{2}a_{0,1}b_{1,1}\\ &+\left(\frac{3}{8}d_{0,2}-\frac{1}{8}c_{0,1}b_{1,1}-\frac{3}{8}d_{1,1}a_{0,1}-\frac{1}{8}b_{1,1}a_{2,1}+\frac{3}{8}d_{0,1}a_{1,1}+\frac{3}{8}e_{0,1}d_{0,1}+\frac{1}{8}b_{2,2}-\frac{3}{8}a_{0,1}b_{3,1}\right)r^2\\ &+\Bigg(-\frac{3}{16}d_{1,1}c_{0,1}-\frac{1}{48}b_{1,1}c_{2,1}+\frac{1}{16}d_{0,1}c_{1,1}+\frac{1}{16}d_{2,2}-\frac{1}{16}d_{1,1}a_{2,1}-\frac{1}{24}b_{3,1}e_{1,1}\\ &-\frac{7}{48}c_{0,1}b_{3,1}+\frac{3}{16}d_{0,1}a_{3,1}-\frac{3}{16}a_{0,1}d_{3,1}-\frac{5}{48}b_{3,1}a_{2,1}\Bigg)r^4\\ &+\left(-\frac{3}{128}d_{1,1}c_{2,1}-\frac{1}{64}d_{3,1}e_{1,1}-\frac{3}{128}b_{3,1}c_{2,1}-\frac{5}{128}d_{3,1}a_{2,1}-\frac{13}{128}d_{3,1}c_{0,1}+\frac{3}{128}d_{0,1}c_{3,1}\right)r^6\\ &-\frac{19}{1280}d_{3,1}c_{2,1}r^8,\end{aligned}$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-\frac{8}{3}$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=0$, $b_{2,2}=-\frac{8}{3}$, $d_{2,2}=8$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$ *and* $c_{3,1}=\frac{307}{342}$ *we have that* $F_{20}^{1,1}(r)=\frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2)$, *that has exactly four positive zeros.*

$$F_{20}^{2,3}(r)=F_{20}^{1,3}(r)+\frac{1}{16}d_{0,1}e_{2,1}r^4,$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-\frac{8}{3}$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=0$, $b_{2,2}=-\frac{8}{3}$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $c_{3,1}=\frac{307}{342}$, $d_{2,2}=-8$ *and* $e_{2,1}=-2$ *we have that* $F_{20}^{2,3}(r)=\frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2)$, *that has exactly four positive zeros.*

$$F_{20}^{3,3}(r)=F_{20}^{2,3}(r)-\frac{1}{64}b_{3,1}e_{3,1}r^6-\frac{3}{640}d_{3,1}e_{3,1}r^8,$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-2$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=0$, $b_{2,2}=-\frac{8}{3}$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $c_{3,1}=\frac{3637}{3078}$, $d_{2,2}=-8$, $e_{2,1}=-2$ *and* $e_{3,1}=-\frac{19}{9}$ *we have that* $F_{20}^{2,3}(r)=\frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2)$, *that has exactly four positive zeros.*

$$F_{20}^{4,3}(r)=F_{20}^{3,3}(r)+\frac{3}{128}d_{0,1}e_{4,1}r^6,$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-2$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=0$, $b_{2,2}=-\frac{8}{3}$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $c_{3,1}=-\frac{5597}{3078}$, $d_{2,2}=-8$, $e_{2,1}=-2$, $e_{3,1}=-\frac{19}{9}$ *and* $e_{4,1}=3$ *we have that* $F_{20}^{2,3}(r)=\frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2)$, *that has exactly* *four positive zeros.*

$$
\begin{aligned}
F_{20}^{1,4}(r) &= \frac{1}{2}b_{0,2} - \frac{1}{2}a_{0,1}b_{1,1} + \\
&+ \left(\frac{3}{8}d_{0,2} - \frac{1}{8}c_{0,1}b_{1,1} - \frac{3}{8}d_{1,1}a_{0,1} - \frac{1}{8}b_{1,1}a_{2,1} + \frac{3}{8}d_{0,1}a_{1,1} + \frac{3}{8}e_{0,1}d_{0,1} + \frac{1}{8}b_{2,2} - \frac{3}{8}a_{0,1}b_{3,1}\right)r^2 \\
&+ \Big(\frac{1}{16}d_{4,2} + \frac{1}{16}d_{2,1}e_{0,1} + \frac{1}{16}d_{2,1}a_{1,1} - \frac{1}{16}a_{4,1}b_{1,1} - \frac{3}{16}d_{1,1}c_{0,1} - \frac{1}{48}b_{1,1}c_{2,1} + \frac{1}{16}d_{0,1}c_{1,1} \\
&+ \frac{1}{16}d_{2,2} - \frac{1}{16}d_{1,1}a_{2,1} - \frac{1}{24}b_{3,1}e_{1,1} - \frac{7}{48}c_{0,1}b_{3,1} + \frac{3}{16}d_{0,1}a_{3,1} - \frac{1}{16}a_{0,1}d_{3,1} - \frac{5}{48}b_{3,1}a_{2,1}\Big)r^4 \\
&+ \Big(\frac{3}{128}d_{4,2} + \frac{5}{128}a_{3,1}d_{2,1} - \frac{1}{128}c_{4,1}b_{1,1} - \frac{3}{128}a_{4,1}d_{1,1} - \frac{7}{128}a_{4,1}b_{3,1} + \frac{1}{128}d_{2,1}c_{1,1} \\
&- \frac{3}{128}d_{1,1}c_{2,1} - \frac{1}{64}d_{3,1}e_{1,1} - \frac{3}{128}b_{3,1}c_{2,1} - \frac{5}{128}d_{3,1}a_{2,1} - \frac{13}{128}d_{3,1}c_{0,1} + \frac{3}{128}d_{0,1}c_{3,1}\Big)r^6 \\
&+ \left(-\frac{19}{1280}d_{3,1}c_{2,1} - \frac{9}{1280}c_{4,1}d_{1,1} - \frac{11}{1280}c_{4,1}b_{3,1} + \frac{1}{256}d_{2,1}c_{3,1} - \frac{21}{1280}a_{4,1}d_{3,1}\right)r^8 \\
&- \frac{5}{1024}c_{4,1}d_{3,1}r^{10},
\end{aligned}
$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-\frac{8}{3}$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=-8$, $d_{2,2}=8$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $d_{4,1}=\frac{72}{5}$, $b_{2,2}=-\frac{64}{15}$, $b_{4,2}=8$, $d_{2,1}=\frac{100}{3}$, $c_{4,1}=\frac{608}{375}$, $a_{4,1}=\frac{89731}{28350}$, $d_{4,2}=-\frac{524269}{484785}$, *and* $c_{3,1}=\frac{307}{342}$ *we have that* $F_{20}^{1,4}(r)=\frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, *that has exactly five positive zeros.*

$$F_{20}^{2,4}(r) = F_{20}^{1,4}(r) + \frac{1}{16}d_{0,1}e_{2,1}r^4 + \frac{3}{128}d_{2,1}e_{2,1}r^6,$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-\frac{8}{3}$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=0$, $d_{2,2}=-8$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $d_{4,1}=\frac{72}{5}$, $b_{2,2}=-\frac{64}{15}$, $b_{4,2}=8$, $d_{2,1}=\frac{100}{3}$, $c_{4,1}=\frac{608}{375}$, $a_{4,1}=\frac{89731}{28350}$, $d_{4,2}=\frac{31794731}{484785}$, $c_{3,1}=\frac{307}{342}$ *and* $e_{2,1}=-2$ *we have that* $F_{20}^{2,4}(r)=\frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, *that has exactly five positive zeros.*

$$F_{20}^{3,4}(r) = F_{20}^{2,4}(r) - \frac{1}{64}b_{3,1}e_{3,1}r^6 - \frac{3}{640}d_{3,1}e_{3,1}r^8,$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-\frac{8}{3}$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=-8$, $d_{2,2}=-8$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $d_{4,1}=\frac{72}{5}$, $b_{2,2}=-\frac{64}{15}$, $b_{4,2}=8$, $d_{2,1}=\frac{100}{3}$, $c_{4,1}=\frac{608}{375}$, $a_{4,1}=\frac{89731}{28350}$, $d_{4,2}=\frac{157161523}{2423925}$, $c_{3,1}=\frac{7351}{8550}$, $e_{2,1}=-2$, *and* $e_{3,1}=-1$ *we have that* $F_{20}^{2,4}(r)=\frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, *that has exactly five positive zeros.*

$$F_{20}^{4,4}(r) = F_{20}^{3,4}(r) + \frac{3}{128}d_{0,1}e_{4,1}r^6 + \frac{3}{256}d_{2,1}e_{4,1}r^8,$$

now if we fixed $b_{0,2}=2$, $b_{1,1}=0$, $d_{1,1}=-\frac{8}{3}$, $c_{2,1}=-\frac{8}{3}$, $d_{0,2}=-4$, $d_{0,1}=-8$, $e_{0,1}=\frac{1}{3}$, $c_{0,1}=1$, $c_{1,1}=0$, $a_{2,1}=-8$, $d_{2,2}=-8$, $a_{0,1}=1$, $b_{3,1}=\frac{2}{3}$, $d_{3,1}=\frac{20}{19}$, $a_{3,1}=-\frac{89}{171}$, $a_{1,1}=0$, $e_{1,1}=1$, $d_{4,1}=\frac{72}{5}$, $b_{2,2}=-\frac{64}{15}$, $b_{4,2}=8$, $d_{2,1}=\frac{100}{3}$, $c_{4,1}=\frac{608}{375}$, $a_{4,1}=\frac{89731}{28350}$, $d_{4,2}=\frac{40813123}{2423925}$, $c_{3,1}=-\frac{69599}{8550}$, $e_{2,1}=-2$, *and* $e_{3,1}=-1$, *and* $e_{4,1}=3$ *we have that* $F_{20}^{2,4}(r)=\frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, *that has exactly five positive zeros.*

References

[1] J. Alavez-Ramirez, G. Blé, J. Llibre and J. Lopez-Lopez, On the maximum number of limit cycles of a class of generalized Liénard differential systems, Int J Bifurcation Chaos, 22(2012), 14pp.

[2] T. R. Blows and L. M. Perko, Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems, SIAM Review, 36(1994), 341–376.

[3] A. Buicã and J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bull. Sci. Math., 128(2004), 7–22.

[4] B. Garcia, J. Llibre and J. S Pérez del Rio, Limit cycles of generalized Liénard polynomial differential systems via averaging theory, Chaos Solitons & Fractals, 62-63(2014), 1–9.

[5] M. Viano, J. Llibre and H. Giacomini, Arbitrary order bifurcation for perturbed Hamiltonian planar systems via the reciprocal of an integrating factor, Nonlinear Anal., 48(2002), 117–136.

[6] I. S. Gradshteyn and I. M. Ryshik, Table of Integrals, Series and Products, Edited by A. Jeffrey and D. Zwillinger, Academic Press, New York, 7th edition, 2007.

[7] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Appl. Math. Sci. 42, Springer-Verlag, New York, 1986.

[8] D. Hilbert, Mathematische problems, Lecture in: SecondInternat. Cong. Math. Paris, 1900, Nachr. Ges. Wiss. Gottingen. Math. Phys. Ki 5 (1900), 253–297; English Transl: Bull. Amer. Math. Soc., 8(1902), 437–479.

[9] J. Llibre, Averaging theory and limit cycles for quadratic systems, Rad. Mat., 11(2002), 215–228.

[10] J. Llibre, A. C. Mereu and M. A. Teixeira, Limit cycles of the generalized polynomial Liénard differential equations, Math. Proc. Camb. Phil. Soc., 148(2010), 363–383.

[11] J. Llibre and C. Valls, On the number of limit cycles of a class of polynomial differential systems, Proc A: R Soc., 468(2012), 2347–2360.

[12] J. Llibre and C. Valls, Limit cycles for a generalization of Liénard polynomial differential systems, Chaos Solitons Fractals, 46(2013), 65–74.

[13] J. Llibre and C. Valls, On the number of limit cycles for a generalization of Liénard polynomial differential systems, Int J Bifurcation Chaos, 23(2013), 16pp.

[14] H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, in: Oeuvreus de Henri Poincaré, Gauthiers-Villars, Paris, 1(1951), 95–114.

[15] J. A. Sanders and F. Verhulst, Averaging Methods in Nonlinear Dynamical Systems, Appl. Math. Sci, 59 Springer-Verlag, New York- Berlin- Heidelberg- Tokyo, 1985.

[16] S. Smale, Mathematical Problems for the next century, Math. Intelligencer, 20(1998), 7–15.

[17] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Springer-Verlag, Berlin Heidelberg, New York, Second Edition, 1991.