Algerian Democratic and Popular Republic

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

20 August 1955 University of Skikda

Faculty of Sciences

Department of Mathematics

Ref:………………

جامعة 20 أوت 1955 –سكيكدة

كلية العـلوم

قسم الرياضيات

المرجع:..........

Thesis

A view to obtaining the diploma of

Master of 2° cycle (LMD) in Mathematics

Option: ***Numerical Analysis of PDEs***

The Convergence of Caratheodory's Approximate Solutions of Stochastic Differential Equation

Presented by:

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CONTENTS

Abstract in English

Abstract

In this memory, we give the proof of the convergence of the caratheodory's approximate solution to the unique solution of stochastic differential equations by using the carathéodory's approximation scheme under the non-uniform lipschitz and non-linear growth conditions.

Key words: Stochastic process, Brownian motion, Itô's stochastic integral, Itô's formula, stochastic differential equation.

Abstract in French

Résumé

Dans ce mémoire, nous avons donné la démonstration de la convergence de la solution approximative de carathéodory vers l'unique solution d'une modèle des équations différentielles stochastiques, eu utilisant le schéma d'approximation de carathéodory sous les conditions qui ne sont pas lipschitziennes et ne sont pas croissance linéaires

Mots clés: Processus stochastique, mouvement Brownien, intégrale stochastique d'Itô , Formule d'Itô', Equation différentielle stochastique.

Abstract in Arabic

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ملخص

في هذه المذكرة قمنا بإعطاء برهان لتقارب الحل التقريبي لكاراثيودوري نحو الحل الوحيد لنموذج من المعادلات التفاضلية العشوائية وذلك باستعمال طريقة كاراثيودوري التقريبية و كذلك باستعمال شروط ليست ليبشيتزية و كذلك ليست تزايديه خطية.

الكلمات المفتاحية: العمليات العشوائية، الحركة براونية، تكامل إيتو العشوائي، معادلة إيتو العشوائية، المعادلات التفاضلية العشوائية.

Dedication

To:

Our parents

Our siblings

All our families

Our teachers

Our friends

All the shining stars we lost

Acknowledgements

We dedicate this modest work

First and foremost, we praise and thank Allah, the Almighty, for giving us strength and patience to finish this humble work.

I would like to express my sincere thanks and gratitude to my honourable parents who have been my main support in my academic journey

I also express my gratitude to Dr El-Hacène Chalabi, the supervisor of this thesis, for his attention, advice and guidance, and for the use of his library, so God rewarded him with the best reward for his students.

I thank all my honourable professors who spared no effort in guiding me and providing me with what I needed from their libraries.

I would also like to thank the Board of Co-supervisor AbdAllah Lallouche who helped me and gave me a lot of advice. I also thank Dr Karek who agrees to chair the jury and Dr Amar Ouaoua for the valuable time they devoted to evaluating this research and for contributing their suggestions and comments to improve the quality of this modest study.

We would also like to extend our gratitude to all those who contributed to the realisation of this research, even with a prayer

CHAPTER 1

Introduction

The theorie of stochastic differential equations is a branch very imprtant in mathematic It has a many applications in finance, physics, medicine and biology etc ...

Mao [11] studied the estimate on difference between the caratheodory's approximate solution and the unique solution of the differential equation under the lipschitz and linear growth conditions.

a several mathematicians established a many results to the stochadtic differential equations, see for exemple [6, 7, 9],[10, 11, 12, 14, 15],[18],[17]

Ren and Xia [15] proved the existence and uniqueness of solution to the folowing stochastic differential equation

$$X_t = X_0 + \int_0^T a\,(x,t)\,dt + \int_0^T b\,(x,t)\,dB_t$$

under non-lipschitz and non-linear growth conditions.

The method of Caratheodory's approximation sheme was used by many mathematician for proving the existence and uniqueness of solution pf differential equations and of stochastic differential equations under diffrent conditions see for example [3], [4], [5], [8]

In this memory, we consider the follwing stochastic differential equation.

$$dX_t = X_0 + \int_{t_0}^{T} f\left(t, X\left(t - \alpha\left(t\right)\right)\right) dt + \int_{t_0}^{T} g\left(t, X\left(t - \alpha\left(t\right)\right)\right) dB_t \qquad (1,1)$$

such that X_0 is given initial condition and (B_t) is a Brownian motion and the coeffecients f and g are non-lipschitz and non-linear growth conditions.We proved the convergence of the carathéodory's approximate solutin to the unique solution of $(1,1)$, we used the caratheodory's approximation scheme.

This memory is divited as follow: In the second chapter we shall recall some basic notions and results from probability where we use in the previous chapters.

In the third chapter, we gived five sections, in the first two, we come into the first contact with the concept of stochastic processes. the brownian motion, and Martingales. The last two sections are devoted to both stochastic integration .

In the last chapter, a result on the convergence of the Caratheodory's approximate solution to the unique solution for stochastic differential equation is given .

CHAPTER 2

Preliminaries

In this chapter we shall recall some basic notions and results from probability where we use in the previous chapters.

2.1 Mesured space

Definition 2.1 ($\sigma - algebra$) : Let Ω be a non-empty set, We call a $\sigma-$algebra on Ω a family $\mathcal{F}$ of parts of Ω possessing the following properties if:

1. $\Omega \in \mathcal{F}$.
2. $\mathcal{F}$ is stable by complementarity: if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ (where $A^c = \Omega - A$).
3. $\mathcal{F}$ is stable by countable reunion: if $A_n \in \mathcal{F}$ then for any $n \in \mathbb{N}^\star$, $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{F}$.

   Definition 2.2 (Filtration): a filtration $\{\mathcal{F}_t;\ 0 \leq t \leq \infty\}$ is an increasing family of subtribes of $\mathcal{F}$: for any $0 \leq s \leq t \leq \infty$, $\mathcal{F}_s \leq \mathcal{F}_t$.

The elements of $\mathcal{F}$ are called the measurable parts of Ω. We say that $(\Omega, \mathcal{F})$is a measurable space.

Definition 2.3 Let $\mathcal{F}$ be a family of parts of X, $\mathcal{M}$ $\sigma-$algebra over X We denote

$$\sigma(\mathcal{F}) = \bigcap_{\mathcal{F} \subset \mathcal{M}} \mathcal{M}$$

Then $\sigma(\mathcal{F})$ is a $\sigma-$algebra on X called the $\sigma-$algebra generated by $\mathcal{F}$. It is the smallest $\sigma-$algebra on X that contains $\mathcal{F}$.

Definition 2.4 A **topology** on X is a family $\mathcal{T}$ of parts of X such that :

1. $\phi \in \mathcal{T}$, $X \in \mathcal{T}$
2. If $O_1, ..., O_n \in \mathcal{T}$, then $\bigcap\limits_{i=1}^{n} O_i \in \mathcal{T}$.
3. If $\{O_i\}_{i \in I}$ is any family of elements of $\mathcal{T}$ then $\bigcup\limits_{i \in I}^{n} O_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called **the opens** of X. We say that $(X, \mathcal{T})$ is **a topological space.**

Definition 2.5 (Borel $\sigma-$algebra): Let $(X, \mathcal{T})$ be a topological space. The $\sigma-$algebra generated by the openings of X is called **a Borel $\sigma-$algebra** on X: $\mathcal{M} = \sigma(\mathcal{T})$.

Proposition 2.1 *The $\sigma-$algebra $\mathcal{B}(\mathbb{R})$ is generated by the intervals $]a, +\infty[$ for $a \in \mathbb{R}$.*

Definition 2.6 Let $(X, \mathcal{M})$ be a measurable space.We call **a positive measure** on X an application $\mu : \mathcal{M} \to [0, +\infty]$ verifying :

1. $\mu(\phi) = 0$

2. (Countable additivity): if $\{A_n\}_{n\in\mathbb{N}}$ is a countable family of pairwise disjoint measurable sets then:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

We say that $(X, \mathcal{M}, \mu)$ is **a measured space.**

2.1.1 Probability space

Definition 2.7 We call probability on (Ω, C) (where Ω is the set of events and C a class of parts of Ω), any application P: $\mathcal{C} \longrightarrow [0;1]$ such that :

1. $P(\Omega) = 1$

2. for any countable set of incompatible events $A_1, A_2, ..., A_n$, ... we have

$$P(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} P(A_i) \qquad (\sigma - \text{additivity of } P)$$

Definition 2.8 We call the triplet $(\Omega, \mathcal{C}, P)$ a probability space where Ω is the fundamental set, C is a collection of subsets of Ω (the collection of events), which has the previous structure of a Boorlean $\sigma-$algebra and $P : \mathcal{C} \longrightarrow [0; 1]$ is a probability measure on C.

Independence

Two random variables X and Y are independent if the occurrence of one of them does not change the probability density of the other. More precisely,if for any two open intervals A, $B \subset \mathbb{R}$, the events

$$E = \{\omega,\ X(\omega) \in A\}, \quad F = \{\omega, Y(\omega) \in B\}$$

are independent, $i.e$: $P(E \cap F) = P(E)P(F)$, then X and Y are called independent random variables.

Mathematical Expectation

The expected value of a random variable is denoted by $\mathbb{E}[X]$. The expected value can be thought of as the "average" value attained by the random variable. In fact, the expected value of a random variable is also called its mean.

The Discrete Case If X is a discrete random variable having a probability mass function $P(x)$, then the expected value of X is defined by

$$\mathbb{E}[X] = xP(x).$$

In other words, the expected value of X is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes that value.

The Continuous Case Wemayalso define the expected value of a continuous random variable. This is done as follows. If X is a continuous random variable having a probability density function $f(x)$, then the expected value of X is defined by

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx.$$

Variance and Covariance

Definition 2.9 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable. Assume that the mathematical expectation $\mathbb{E}[X]$ exists and is finite. The variance is defined as the mean of the square of the difference $X - \mathbb{E}[X]$ i.e.

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

The standard deviation is defined as

$$\sigma = \sigma(X) = \sqrt{var(X)}$$

Definition 2.10 Let $X, Y : \Omega \to \mathbb{R}$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$, and having finite means $\mathbb{E}[X]$, $\mathbb{E}[Y]$. The number

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

is called the covariance of X and Y.

Definition 2.11 (Gaussian Random Variable) A continous random variable X with probability density function of the from

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \mathbb{R}$$

is called the Gaussian random variable or (Normal random variable), where μ is the mean and σ^2 is the variance of the random variable X.

Notation

- $\mathcal{N}(\mu, \sigma)$ denotes a Gaussian distribution with mean μ and variance σ^2.
- $X \sim \mathcal{N}(\mu, \sigma) \Rightarrow X$ is a Gaussian r.v with mean μ and variance σ^2.
- if $X \sim \mathcal{N}(0, 1)$, then X is a standard Gaussian r.v.

A process X is Gaussian if any finite linear combination of $(X_t, t \geq 0)$ is a Gaussian random variable, *i.e.* if

$$\forall n \ \forall t_i,\ 1 \le i \le n,\ \forall a_i,\ \sum_{i=1}^{n} a_i X_{t_i} \text{is a Gaussian r.v.}$$

A Gaussian process is characterised by its expectation and covariance.

A Gaussian space is a (closed vector) subspace of $L^2(\Omega)$ formed by centred Gaussian r.v.s.

The Gaussian space generated by a Gaussian process is the subspace of $L^2(\Omega)$ generated by the centred r.v.s $(X_t - \mathbb{E}(X_t), t \ge 0)$, *i.e.* the subspace formed by linear combinations of these centred variables and their mean-squared limits..

Convergence of sequences of random variables

Let X, $(X_n)_{n\in N}$ be a variable and a sequence of random variables with values in $\mathbb{R}^d$ on $(\Omega, \mathcal{F}, P)$. There are several types of convergence:

- If there exists a set $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 0$ such that for each $\omega \notin \Omega_0$, the follows $(X_n(\omega))_{n\in N}$ converges to $X(\omega)$ in the usual sense in $\mathbb{R}^d$ then (X_n) is said to converge to X almost surely or with probability 1, and we write $\lim_{n\to\infty} X_n = X$ a.s.

- If for each $\epsilon > 0$

$$\lim_{n\to\infty} P\{\omega : |Xn(\omega) - X(\omega)| > \epsilon\} = 0\ .$$

  Then $(X_n)_{n\in N}$ converges to X in probability.

- If X and $(X_n)_{n\in N}$ belong to L^p with

$$\mathbb{E}(|X_n - X|^p) = 0\ .$$

  Then $(X_n)_{n\in \mathbb{N}}$ converges to X in L^p

- If for each continuous bounded function f with value in $\mathbb{R}^d$

$$\lim_{n\to\infty}\mathbb{E}\left[f(X_n)\right]=\mathbb{E}\left[f(X)\right].$$

Then $(X_n)_{n\in N}$ converges in law to X.

2.1.2 Conditional expectation

Definition 2.12 Y is random variable on a probability space $(\Omega,\mathcal{F},P)$ and let ς be a sub $\sigma-algebra$ of $\mathcal{F}$ .let X be random variable satisfying

1. X is $\varsigma-$measurable.
2. $\int_{\mathcal{B}} YdP=\int_{\mathcal{B}} XdP$ for all sets $B\in\varsigma$

The random variable X is called the conditional expectation of Y given ς and denoted by $\mathbb{E}\left(Y|\varsigma\right)$.

Properties of conditional Expectation

1. If $\mathcal{F}_t=\{\phi,\Omega\}$, then $\mathbb{E}\left(X|\mathcal{F}_t\right)=\mathbb{E}\left(X\right)$.
2. $\mathbb{E}\left\{E\left(X|\mathcal{F}_t\right)\right\}=\mathbb{E}\left(X\right)$.
3. If $\mathcal{F}_s\subset\mathcal{F}_t$, then $\mathbb{E}\left\{\mathbb{E}\left(X|\mathcal{F}_s\right)|\mathcal{F}_t\right\}=\mathbb{E}\left(X|\mathcal{F}_s\right)$
4. If Y is $\mathcal{F}_t$ measurable, then $\mathbb{E}\left(XY|\mathcal{F}_t\right)=Y\mathbb{E}\left(X|\mathcal{F}_t\right)$
5. If X and Y are independent, then $\mathbb{E}\left(Y|\sigma\left(X\right)\right)=\mathbb{E}\left(Y\right)$
6. $\mathbb{E}\left(aX+bY|\mathcal{F}_t\right)=a\mathbb{E}\left(X|\mathcal{F}_t\right)+b\mathbb{E}\left(Y|\mathcal{F}_t\right)$
7. If $g\left(.\right)$ is a convex function, then $\mathbb{E}\left\{g\left(X\right)|\mathcal{F}_t\right\}\geq g\left\{\mathbb{E}\left(X|\mathcal{F}_t\right)\right\}$
8. $\mathbb{E}\left(Y|\sigma\left(X_t,t\in\Gamma\right)\right)=\mathbb{E}\left(Y|X_t,t\in\Gamma\right)$

2.2 Hölder, Gronwall and Doob inequalities

Theorem 2.1 *(Hölder inequality). Let p and q be real numbers strictly greater than 1 such that $\frac{1}{p}+\frac{1}{q}=1$. If the variables $|X|^p$ and $|Y|^q$ are integrable, we have réels*

$$\mathbb{E}\,|XY| \leq (\mathbb{E}\,|X|^p)^{\frac{1}{p}}\,(\mathbb{E}\,|Y|^q)^{\frac{1}{q}}$$

Theorem 2.2 *(Gronwall inequality). Let $T>0$, $C\geq 0$, $u\,(.)$ be a positive bounded function with Borel measure on $[0,T]$ and $v\,(.)$ an integrable function*

$$u\,(t) \leq C + \int_0^t v\,(s)\,u(s)ds, \quad \forall t \in [0,T]\,.$$

then

$$u\,(t) \leq C\exp\left(\int_0^t v\,(s)\,ds\right), \quad \forall t \in [0,T]\,.$$

Theorem 2.3 *(Doob inequality): If X is a continuous martingale. Then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] \leq 4 \sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2\right].$$

Theorem 2.4 *(moment inequality) Let $p\geq 2$. Let $f \in \mathbf{M}^2([0,T];R^{d\times m})$ such that*

$$\mathbb{E}\int_0^T |f(s)|^p ds < \infty.$$

Then

$$\mathbb{E}|\int_0^T f(s)dB(s)|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\int_0^T |f\,(s)\,|^p ds$$

Theorem 2.5 *(Cauchy-Schwarz Inequality) If $f(t)$ and $g(t)$ are any real-valued functions, then*

$$\left\{\int_L^U f(t)g(t)dt\right\}^2 \leq \left(\int_L^U f^2(t)dt\right)\left(\int_L^U g^2(t)dt\right)$$

CHAPTER 3

Stochastic Calculus

3.1 Stochastic process

A stochastic process is a mathematical model that can be used to describe the behaviour of a random phenomenon at any time after the initial instant.

At any time after the initial instant (for example $t_0 = 0$), the behaviour of a random phenomenon can be described. We define this concept as follows

Definition 3.1 A stochastic process $X = (X_t)_{t \in I}$ is a family of random variables, indexed by I and denoted on the probability space $(\Omega; \mathcal{F}; P)$ with values in a measurable space $(E; B)$, called state space.

1. For a fixed t,$\omega \in \Omega \to X_t(\omega)$ is a random variable.
2. For a fixed $\omega, t \in T \to X_t(\omega)$ is a real function, called the trajectory of the process.

   - $T \subseteq N$ the process is discrete time

   - $T = [0, a]$ such that $a > 0$ the process is continuous time.

Definition 3.2 A stochastic process $(X(t))_{t\geq 0}$ is stationary increasing if the distribution of $X(t_1+s)-X(t_1)$ depends only on s, i.e.

$$\forall t_1,\ t_2,\ s\geq 0, \qquad X(t_1+s)-X(t_1)\overset{\mathcal{L}}{=}X(t_2+s)-X(t_2)$$

Definition 3.3 $(X(t))_{t\geq 0}$is independently increasing if for any $0\leq t_0< t_1<...<t_n$, the family of random variables $(X(t_1)-X(t_0),\ ...,\ X(t_n)-X(t_{n-1}))$ independent.$(X(t))_{t\geq 0}$ is a Gaussian (or normal) process if for any $0\leq t_0<t_1<...<t_n$, the vector $(X(t_0),...,X(t_n))$ is Gaussian.

3.1.1 Brownian motion

Brownian motion is a stochastic process (a random function of time). Originally introduced by the botanist Robert Brown in the $19th$ century (1827) to model the movement of pollen grains in suspension, it is now an essential Gaussian process, particularly in stochastic calculus.

Brownian motion is both a natural phenomenon and a mathematical object. The natural phenomenon is the disordered movement of particles suspended in a liquid. The result is a highly irregular movement of the large particle.

Let $(\Omega,\mathcal{F},\mathcal{F},P)$ be a restricted probability space

Definition 3.4 A process $B:\Omega\rightarrow \boldsymbol{T}=[0;T]$ is a standard Brownian motion (BM) if:

1. $B_0=0$,P -p.s.
2. $\forall s\leq t$,$B_{s,t}:=B_t-B_s\sim\mathcal{N}(0,t-s)$.
3. For all $0=t_0<t_1\ldots<t_n\leq T$; the variables $B_{t_0};B_{t_1-t_0},\ldots,B_{t_n-t_{n-1}}$are independent.

-We call $B = (B_1, \ldots, B_n)^\intercal$ $n-$dimensional Brownian motion if $B_1, \ldots, B_n$ are independent Brownian motions.

3.1.2 Stochastic integral

We want to give meaning to the random variable:

$$\int_0^T \theta_s dB_s$$

when integrating a function g with respect to a derivative function f, if g is regular we denote its integral as:

$$\int_0^T g(s)\, df(s) = \int_0^T g(s)\, f' ds,$$

If ever f is not derivable but simply a bounded variation, we can get out of it again by denoting the integral by :

$$\int_0^T g(s)\, df(s) = \lim_{n\to\infty} \sum_{i=1}^{n} g(t_i)\,(f(t_{i+1}) - f(t_i))$$

.

The resulting integral is called the Stieljes integral.

We will therefore construct the stochastic integral on the set

$$L^2_{\mathcal{F}}(\Omega, [0,T]) = \left\{ (\theta_t)_{0\leq t\leq T}\,,\ \textit{cadlag} \text{ process } \mathcal{F}\text{-adapted such that } \mathbb{E}\left(\int_0^T \theta_s^2 ds\right) < \infty \right\}$$

Definition 3.5 Y is random variable on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}$ be a sub $\sigma - algebra$ of $\mathcal{F}$ .let X be random variable satisfying

1. X is $\mathcal{F}$- measurable.

2. $\int_{\mathcal{B}} Y dP = \int_{\mathcal{B}} X dP$ for all sets $B \in \mathcal{F}$

The random variable X is the called the conditional expectation of Y given $\mathcal{F}$ and denoted by $\mathbb{E}\left(Y|\mathcal{F}\right)$.

3.1.3 Martingale process

Definition 3.6 a process $(X_t)_{t\geq 0}$adapted with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$and such that for all $t \geq 0$, is called:

★ a martingale if for all:

$$s \leq t : \mathbb{E}(X_t/\mathcal{F}) = X_s.$$

Definition 3.7 a process$(X_t)_{t\geq 0}$ adapted with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$and such that for all t $>$ 0, is called:

★ a super martingale if for all:

$$s \leq t : \mathbb{E}(X_t/\mathcal{F}) \leq X_s$$

★ a sub martingale if for all:

$$s \leq t : \mathbb{E}(X_t/\mathcal{F}) \geq X_s$$

3.1.4 Integral and ito's formula

The following integral is defined by

$$\mathbf{I}(\theta) = \int_0^T \theta_s dB_s \tag{3.1}$$

or $\{\theta_t, t \geq 0\}$is a certain process and $(B_t,)_{t\geq 0}$a Brownian motion. The problem, of course, is to give meaning to the differential element dB_s since the function

$s \longrightarrow B_s(\omega)$ is non-drivable.

Wiener integral

The Wiener integral is simply an integral of the type (3.1) with a deterministic function, i.e. does not depend on ω. We set a horizon $T > 0$

and note

$L^2\left([0,T],\mathbb{R}\right) = \left\{\theta : [0,T] \rightarrow \mathbb{R}, \text{such that} \int_0^T |\theta_s|^2 ds < \infty\right\}$

★ **Staircase**: If θ_n is a staircase function given by

$$\theta_n\left(t\right) = \sum_{i=1}^{T_n} \alpha_i\left(t\right) \mathbf{1}_{[t_i,t_{i+1}]} \quad \alpha_i \in \mathbb{R}$$

and $\{t_i^n\}$ an increasing sequence in $[0,T]$. It is very easy to derive the Wiener integral:

$$I\left(\theta^n\right) = \int_0^T \theta_s^n dB_s = \sum_{i=1}^{T_n} \alpha_i\left(t\right)\left(B_{t_{i+1}} - B_{t_i}\right)$$

Remark 3.1 the random variable $I\left(\theta^n\right)$ is a Gaussian variable of zero expectation and variance

$$\begin{aligned} Var\left(\mathbf{I}\left(\theta^n\right)\right) &= \sum_{i=1}^{T_n} \alpha_i^2\left(t\right) var\left(B_{t_{i+1}} - B_{t_i}\right) \\ &= \sum_{i=1}^{T_n} \alpha_i^2\left(t\right)\left(t_{_{i+1}} - t_{_i}\right) = \int_0^T \left(\theta_s^n\right)^2 dB_s \end{aligned}$$

★ **In general** :either $\theta \in L\left([0,T],\mathbb{R}\right)$,there exists a sequence θ^n of escalating functions converges to

$$\int_0^T \theta dx = \lim_{n\to+\infty} \int_0^T |\theta_n - \theta|(x)\, dx$$

the limit taken in $L^2(\Omega)$.We say that $\mathbf{I}(\theta^n)$ is the stochastic integral (or Wiener integral) of θ with respect to B.

Stochastic integral

We are now trying to define the integral (1.1), the randomness of θ will require additional conditions compared to the case of the Wiene integral.

★**Staged processes**: These are the processes: $\mathbf{I}(\theta^n) = \sum_{i=1}^{T_n} \theta_i(t)\, 1_{]t_i,t_{i+1}]}$ or $0 \le t_1 \le \ldots \le t_{\tau_n}$and $\theta \in L^2(\Omega, \mathcal{F}, P)$ for all $i = 0, \ldots \tau_n$. So we say

$$\mathbf{I}(\theta^n) = \int_0^T \theta_s^n dB_s = \sum_{i=1}^{T_n} \alpha_i(t)\left(B_{t_{i+1}} - B_{t_i}\right),$$
$$\mathbb{E}(\mathbf{I}(\theta^n)) = 0 \text{ and } var(\mathbf{I}(\theta^n)) = \mathbb{E}\left(\int_0^T (\theta_s^n)^2 dB_s\right)$$

Properties of the stochastic integral

1. $X \to \int_0^T X_s dB_s$ is linear.
2. The process$\left(\int_0^T X_s dB_s\right)_{t\in[0,T]}$has a continuous path
3. The processus $\left(\int_0^T X_s dB_s\right)_{t\in[0,T]}$is adapted to respect $\left(\mathcal{F}_t^B\right)_{t\in[0,T]}$.
4. $\mathbb{E}\left[\int_0^T X_s dB_s\right] = 0$ and $var\left[\int_0^T X_s dB_s\right] = \mathbb{E}\left[\int_0^T X_s^2 dB_s\right]$
5. For $0 \le s \le t \le T$, $\mathbb{E}\left[\int_0^T X_u dB_u|\mathcal{F}_s^W\right] = 0$ and $\mathbb{E}\left[\left(\int_0^T X_s dB_s\right)^2 |\mathcal{F}_s^B\right] = \mathbb{E}\left[\int_0^T X_u^2 du|\mathcal{F}_s^B\right]$.
6. The process$\left(\int_0^T X_s dB_s\right)_{t\in[0,T]}$is an $\left(\mathcal{F}_t^B\right)_{t\in[0,T]}$continuous martingale of integrable square.

$$\mathbb{E}\left[\sup_{0\le t\le T} |\int_0^T X_s dB_s|^2\right] \le 4\mathbb{E}\left[\int_0^T X_s^2 dB_s\right]$$

Itô process

Definition 3.8 It is a process in the form of :

$$X_t = X_0 + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW_s$$

or b is an adapted $\mathcal{F}_t^W$ process such that:$\int_0^t b(s)\,ds < +\infty \quad P-p.s$,for all $t \geq 0$ and σ a good local process. We use this notation:

$$\begin{cases} dX_t = b(t)\,dt + \sigma(t)\,dWt \\ X_0 = x \end{cases}$$

Theorem 3.1 $1st$ ***ito formula***: *Assume f of class C^2then:*

$$f(X_t) = f(x) + \int_0^t f'(X_s)\,dX_s + \tfrac{1}{2}\int_0^t f''(X_s)\,\sigma_{(s)}^2 ds$$

This formula can be written as :

$$\begin{aligned} df(X_t) &= f'(Xt)\,dX_t + \frac{1}{2}f''(X_t)\,\sigma_{(t)}^2 dt \\ &= f'(X_t)\,b(t)\,dt + \frac{1}{2}f''(X_t)\,\sigma_{(t)}^2 dt + f'(Xt)\,\sigma(t)\,dW_t \\ &= f'(X_t)\,b(t)\,dt + \frac{1}{2}f''(X_t)\,dX_t + f'(X_t)\,\sigma(t)\,dW_t \end{aligned}$$

$2nd$ **Ito formula** :Let f be defined on $R_+ \times R$ of classC^1 with respect to t, of class C^2 with respect to x. We have:

$$f(t, X_t) =$$
$$f(0, X_0) + \int_0^t f_t'(s, X_s)\,ds + \int_0^t f_x'(s, X_s)\,dX_s + \tfrac{1}{2}\int_0^t f_{xx}''(s, X_s)\,\sigma_{(s)}^2 ds$$

As before, this formula can be written in differential form:

$$\begin{aligned} df(t, X_t) &= \left[f_t'(t, X_t) + \tfrac{1}{2}f_{xx}''(t, X_t)\,\sigma_{(t)}^2\right] dt + f_x'(t, X_t)\,dX_t \\ &= f_t'(t, X_t)\,dt + f_x'(t, X_t)\,dX_t + \tfrac{1}{2}f_{xx}''(t, X_t)\,d\langle X\rangle_t \end{aligned}$$

3.2 Stochastic Differential Equations

Stochastic differential equation (SDE) is an important mean to construct stochastic processes. It plays an important role in the study of economy, biology and physics. For example, in economy, option pricing can be calculated by stochastic differential equation. In the market sales, we can determine the stochastics variables by analyzing large amounts of sample data and building the mathematical model from the SDE.The general form of such equation is

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{4.1}$$

where B_t is a Brownian motion and b and σ are called the coefficients of the equation. In fact, this differential relation has the following integral meaning:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{4.2}$$

where the last integral is taken in the Itô sense. Relation (4.2) is taken as the definition for the SDE (4.1) (note that if $\sigma = 0$, then (4.1) reduces to an ODE).

3.2.1 Existence and Uniqueness Result

We now to the existence and uniqueness question (A) above.

Theorem 3.2 (***Existence and Uniqueness theorem for stochastic differential equation***

Let $T > 0$ and b(.,.) :$[0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma(.,.) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|); \qquad x \in \mathbb{R}^n, t \in [0,T] \tag{4.3}$$

for some constant C,(where $|\sigma|^2 = \sum |\sigma_{ij}|^2$)and such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|; \quad x,y \in \mathbb{R}^n, t \in [0,T] \tag{4.4}$$

for some constant D.Let Z be a random variable which is independent of the $\sigma-$algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(.), s \geq 0$ and such that

$$\mathbb{E}[|Z|^2] < \infty.$$

Then the stochastic differential equation.

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t, \qquad 0 \leq t \leq T, X_0 = Z \tag{4.5}$$

has a unique t-continuous solution $X_t(\omega)$ with the property that
$X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(.); s \leq t$
and

$$\mathbb{E}\left[\int_0^T |X_t|^2 dt\right] < \infty \tag{4.6}$$

Remark 3.2 Conditions (4.3) and (4.4) are natural in view of the following two simple examples from deterministic differential equations $(i.e.\sigma = 0)$:

a) The equation

$$\frac{dX_t}{dt} = X_t^2, \qquad X_0 = 1 \tag{4.7}$$

corresponding to $b(x) = x^2$ (which does not satisfy (5.2.1))has the (unique)solution

$$X_t = \frac{1}{1-t}; \qquad 0 \leq t < 1.$$

Thus it is impossible to find a global solution (defined for all t)in this case.

More generally (4.3)ensures that the solution $X_t(\omega)$of (4.5)does not explode,$i.e.$that $|X_t(\omega)|$does not tend to ∞ in a finite time.

b) The equation

$$\frac{dX_t}{dt} = 3X_t^{2/3}; \qquad X_0 = 0 \tag{4.8}$$

has more than one solution.In fact,for any $a > 0$ the function

$$X_t = \begin{cases} 0 & \text{for} \quad t \leq a \\ (t-a)^3 & \text{for} \quad t > a \end{cases}$$

solves(4.8).In this case $b(x) = 3x^{2/3}$ does not satisfy the Lipschitz condition (4.4) at $x = 0$.

Thus condition (4.4)guarantees that equation (4.5)has a unique solution.Here uniqueness means that if $X_1(t,\omega)$and $X_2(t,\omega)$are two t-continuous processes satisfying(4.5)and (4.6)then

$X_1(t,\omega) = X_2(t,\omega) \quad$ for all $t \leq T, a.s.$

CHAPTER 4

Approximate Solutions

4.1 Assumptions

A_1) Suppose that there exist a constant positive K, such that for all $t \in [t_0,\ T]$

$$|f(t,0,0)|^2 + g(t,0,0)|^2 \leq K$$

A_2) for all $x_1,\ x_2,\ y_1,\ y_2$

$$|f(t,x_1,y_1) - f(t,x_2,y_2)|^2 + |g(t,x_1,y_1) - g(t,x_2,y_2)|^2 \leq \varphi\left(|x_2 - x_1|^2 + |y_2 - y_1|^2\right)$$

4.2 Carathéodory's approximation scheme

The Caratheodory approximation sheme is defined as follows:

For each integer $n \geq 1$;

$$X_n(t_0 + \theta) = X(\theta), \quad \text{if } \theta \in (-\infty,\ 0]$$

and for $t \in [t_0,\ T]$

$$\begin{aligned} x_n(t) &= \varepsilon(0) + \int_{t_0}^{t} I_{\overline{D}_n}(s) f\left(s, x(s-\frac{1}{n}), x(s-\alpha(s))\right) ds \\ &+ \int_{t_0}^{t} I_{D_n}(s) f\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) ds \\ &+ \int_{t_0}^{t} I_{\overline{D}_n}(s) g(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s)) dB(s) \\ &+ \int_{t_0}^{t} I_{D_n}(s) g\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) dB(s), \end{aligned}$$

where $D_n = \left\{t:\ t_0 \le t \le T;\ \alpha(s) < \frac{1}{n}\right\},\ \ \overline{D}_n = [t_0,\ T] - D_n$.

4.3 Lemmas and theorems

Lemma 4.1 *under the assumptions A_1 and A_2, we have for all integer $n \geq 1$, $t \in [t_0,\ T]$, we have*

$$\mathbb{E}\left(\sup_{-\infty \le t \le T} |x_n(t)|^2\right) \le B_1,$$

where

$$B_1 = \frac{1}{2} + 6\mathbb{E}|\varepsilon(0)|^2 + KC_1(T-t_0)\exp(2a(T-t_0)C_1),$$

and

$$C_1 = 10(T - t_0 + 4)$$

Proof. we have by definition of Caratheodory's apprixmation scheme:

$$\begin{aligned} x_n(t) &= \varepsilon(0) + \int_{t_0}^{t} I_{\overline{D}_N}(s) f\left(s, x(s-\frac{1}{n}), x(s-\alpha(s))\right) ds \\ &+ \int_{t_0}^{t} I_{D_n}(s) f\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) ds \\ &+ \int_{t_0}^{t} I_{\overline{D}_n}(s) g(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s)) dB(s) \\ &+ \int_{t_0}^{t} I_{D_n}(s) g\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) dB(s), \end{aligned}$$

by using the fact $(\sum_{i=1}^{6} a_i)^2 \leq 6\sum_{i=1}^{6} a_i^2$, we get

$$\begin{aligned}|x_n(t)|^2 \;\leq\;& 5|\varepsilon(0)|^2 + 5|\int_{t_0}^{t} I_{\overline{D}_n}(s) f\left(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s))\right) ds|^2 \\ &+5|\int_{t_0}^{t} I_{D_n}(s) f\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) ds|^2 \\ &+5|\int_{t_0}^{t} I_{\overline{D}_n}(s)\, g(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s)) dB(s)|^2 \\ &+5|\int_{t_0}^{t} I_{D_n}(s)\, g\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) dB(s)|^2\end{aligned}$$

By Hölder's and Doob's martingale inequalities, we derive that for $t_0 \in [t_0,\ t]$

$$\begin{aligned}&\mathbb{E}\left(\sup_{t_0\leq s\leq t} |x_n(s)|^2\right) \\ \leq\;& 5|\varepsilon(0)|^2 + 5(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left|f\left(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s))\right)\right|^2 ds \\ &+5(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left|f\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right)\right|^2 ds \\ &+5\times 4\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left|g(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s))\right|^2 ds \\ &+5\times 4\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left|g\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right)\right|^2 ds.\end{aligned}$$

Adding and deleting $f(s,0,0)$, $g(s,0,0)$

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le t}|x_n(s)|^2\right)\\
\le\;& 5|\varepsilon(0)|^2+5(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left|f\left(s,x\left(s-\frac{1}{n}\right),x(s-\alpha(s))\right)\right.\\
&\left.-f(s,0,0)+f(s,0,0)\right|^2 ds\\
&+5(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left|f\left(s,x\left(s-\frac{1}{n}\right),x\left(s-\alpha(s)-\frac{1}{n}\right)\right)\right.\\
&\left.-f(s,0,0)+f(s,0,0)\right|^2 ds\\
&+5\times 4\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left|g(s,x\left(s-\frac{1}{n}\right),x(s-\alpha(s))\right.\\
&\left.-g(s,0,0)+g(s,0,0)\right|^2 ds\\
&+5\times 4\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left|g\left(s,x\left(s-\frac{1}{n}\right),x\left(s-\alpha(s)-\frac{1}{n}\right)\right)\right.\\
&\left.-g(s,0,0)+g(s,0,0)\right|^2 ds,
\end{aligned}
$$

and by $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\
\leq\ & 5|\varepsilon(0)|^2 + 10(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left|f\left(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s))\right) - f(s,0,0)\right|^2 ds \\
&+10(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|f(s,0,0)|^2\, ds \\
&+10(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left(\left|f\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) - f(s,0,0)\right|^2\right) ds \\
&+10(T-t_0)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,|f(s,0,0)|^2\, ds \\
&+40\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left|g(s, x\left(s-\frac{1}{n}\right), x(s-\alpha(s)) - g(s,0,0)\right|^2 ds \\
&+40\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|g(s,0,0)|^2\, ds \\
&+40\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left|g\left(s, x\left(s-\frac{1}{n}\right), x\left(s-\alpha(s)-\frac{1}{n}\right)\right) - g(s,0,0)\right|^2 ds \\
&+40\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,|g(s,0,0)|^2\, ds
\end{aligned}
$$

By assumptions A_1 and A_2, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\
\leq\ & 5|\varepsilon(0)|^2 + 10(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left[K + \varphi\left(\left|x\left(s-\frac{1}{n}\right)\right|^2 + |x(s-\alpha(s))|^2\right)\right] ds \\
&+10(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left[K + \varphi\left(\left|x\left(s-\frac{1}{n}\right)\right|^2 + \left|x\left(s-\alpha(s)-\frac{1}{n}\right)\right|^2\right)\right] ds
\end{aligned}
$$

Or $\varphi(.)$ is a concave function and $\varphi(0) = 0$, then, there exist a positive constants a such that $\varphi(v) = a(1+v)$ for all $v \geq 0$: Therefore

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\
\leq\ & 5|\varepsilon(0)|^2 + 10(T - t_0 + 4)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left[K + a\left(1 + \left|x\left(s - \frac{1}{n}\right)\right|^2 + |x(s - \alpha(s))|^2\right)\right] ds \\
&+10(T - t_0 + 4)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left[K + a\left(1 + \left|x\left(s - \frac{1}{n}\right)\right|^2 + \left|x\left(s - \alpha(s) - \frac{1}{n}\right)\right|^2\right)\right] ds,
\end{aligned}$$

then,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\
\leq\ & 5|\varepsilon(0)|^2 + 10(T - t_0 + 4)\,K(T - t_0) \\
&+10(T - t_0 + 4)\,a\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left[a\left(1 + \left|x\left(s - \frac{1}{n}\right)\right|^2 + |x(s - \alpha(s))|^2\right)\right] ds \\
&+10(T - t_0 + 4)\,a\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left[a\left(1 + \left|x\left(s - \frac{1}{n}\right)\right|^2 + \left|x\left(s - \alpha(s) - \frac{1}{n}\right)\right|^2\right)\right] ds,
\end{aligned}$$

we drive

$$\begin{aligned}
\mathbb{E}\left(\sup_{-\infty \leq s \leq t} |x_n(s)|^2\right) \leq\ & 6|\varepsilon(0)|^2 + 10(T - t_0 + 4)\,K(T - t_0) \\
&+10(T - t_0 + 4)\,a\int_{t_0}^{t}\left[\left(1 + 2E\left(\sup_{-\infty \leq u \leq S} |x_n(u)|^2\right)\right)\right] ds,
\end{aligned}$$

Mutipling in 2 and added 1, we derive

$$\begin{aligned}
1 + 2\mathbb{E}\left(\sup_{-\infty \leq s \leq t} |x_n(s)|^2\right) \leq\ & 1 + 12|\varepsilon(0)|^2 + 20(T - t_0 + 4)\,K(T - t_0) \\
&+20(T - t_0 + 4)\,a\int_{t_0}^{t}\left[\left(1 + 2\mathbb{E}\left(\sup_{-\infty \leq u \leq s} |x_n(u)|^2\right)\right)\right] ds
\end{aligned}$$

Using Gronawl's lamma, we obtain

$$\begin{aligned}
1 + 2\mathbb{E}\left(\sup_{-\infty \leq t \leq T} |x_n(t)|^2\right) \leq\ & \left(1 + 12|\varepsilon(0)|^2 + 20(T - t_0 + 4)\,K(T - t_0)\right) \times \\
& \exp\left(20a(T - t_0 + 4)(T - t_0)\right).
\end{aligned}$$

Consequently,

$$\mathbb{E}\left(\sup_{-\infty\leq s\leq t}|x_n\left(t\right)|^2\right) \leq \left(\frac{1}{2}+6|\varepsilon\left(0\right)|^2+10\left(T-t_0+4\right)K\left(T-t_0\right)\right)\times \exp\left(20a\left(T-t_0+4\right)\left(T-t_0\right)\right).$$

■

Lemma 4.2 *Inder assumptions A_1 and A_2, we have*

$$\mathbb{E}\left(\sup_{-\infty\leq t\leq T}|x\left(t\right)|^2\right)\leq B_2$$

where

$$B_2=\left(\frac{1}{2}+4|\varepsilon\left(0\right)|^2+6\left(T-t_0+4\right)K\left(T-t_0\right)\right)\exp\left(12a\left(T-t_0+4\right)\left(T-t_0\right)\right).$$

Proof. we have by definition of Caratheodory appriximation scheme:

$$\begin{aligned} x\left(t\right) &= \varepsilon\left(0\right)+\int_{t_0}^{t}f\left(s,x(s),x\left(s-\alpha\left(s\right)\right)\right)ds \\ &+\int_{t_0}^{t}g\left(s,x\left(s\right),x\left(s-\alpha(s)\right)\right)dB\left(s\right) \end{aligned}$$

by using the fact $(a+b+C)^2\leq 3\left(a^2+b^2+c^2\right)$, we get

$$\begin{aligned} |x\left(t\right)|^2 &\leq 3|\varepsilon\left(0\right)|^2+3|\int_{t_0}^{t}f\left(s,x\left(s\right),x\left(s-\alpha\left(s\right)\right)\right)ds|^2 \\ &+3|\int_{t_0}^{t}f\left(s,x\left(s\right),x\left(s-\alpha(s)\right)\right)dB\left(s\right)|^2 \end{aligned}$$

By Hölder's inequality, Doob's martingale inequality, we derive that for $t_0\in[t_0,\ t]$

$$\begin{aligned} \mathbb{E}\left(\sup_{t_0\leq s\leq t}|x\left(s\right)|^2\right) &\leq 3|\varepsilon\left(0\right)|^2+3\left(T-t_0\right)\mathbb{E}\int_{t_0}^{t}\left|f\left(s,x\left(s\right),x\left(s-\alpha\left(s\right)\right)\right)\right|^2ds \\ &+12\mathbb{E}\int_{t_0}^{t}\left|f\left(s,x\left(s\right),x\left(s-\alpha(s)\right)\right)\right|^2ds \end{aligned}$$

Adding and deleting $f(s,0,0)$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|x(s)|^2\right)\\
\leq\ & 3|\varepsilon(0)|^2+3(T-t_0)\,\mathbb{E}\int_{t_0}^{t}|f(s,x(s),x(s-\alpha(s)))-f(s,0,0)+f(s,0,0)|^2\,ds\\
& 12\mathbb{E}\int_{t_0}^{t}|f(s,x(s),x(s-\alpha(s)))-f(s,0,0)+f(s,0,0)|^2\,ds
\end{aligned}$$

and by $(a+b)^2\leq 2(a^2+b^2)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|x(s)|^2\right)\\
\leq\ & 3|\varepsilon(0)|^2+6(T-t_0)\,\mathbb{E}\int_{t_0}^{t}|f(s,x(s),x(s-\alpha(s)))-f(s,0,0)|^2\,ds\\
&+6(T-t_0)\,\mathbb{E}\int_{t_0}^{t}|f(s,0,0)|^2\,ds\\
&+12\mathbb{E}\int_{t_0}^{t}\left(|f(s,x(s),x(s-\alpha(s)))-f(s,0,0)|^2\right)ds\\
&+12\mathbb{E}\int_{t_0}^{t}|f(s,0,0)|^2\,ds,
\end{aligned}$$

by assumptions A_1 and A_2, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|x(s)|^2\right)\\
\leq\ & 3|\varepsilon(0)|^2+6(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t}\left[K+\varphi\left(|x(s)|^2+|x(s-\alpha(s))|^2\right)\right]ds
\end{aligned}$$

Or $\varphi(.)$ is a concave function and $\varphi(0)=0$, then, there exist a positive constants a such that $\varphi(v)=a(1+v)$ for all $v\geq 0$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|x(s)|^2\right)\\
\leq\ & 3|\varepsilon(0)|^2+6(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t}\left[K+a\left(1+|x(s)|^2+|x(s-\alpha(s))|^2\right)\right]ds
\end{aligned}$$

then,

$$\begin{aligned} & \mathbb{E}\left(\sup_{t_0\leq s\leq t}|x(s)|^2\right) \\ \leq\ & 3|\varepsilon(0)|^2+6(T-t_0+4)K(T-t_0) \\ & +6(T-t_0+4)a\mathbb{E}\int_{t_0}^{t}\left[\left(1+|x(s)|^2+|x(s-\alpha(s))|^2\right)\right]ds \end{aligned}$$

we drive

$$\begin{aligned} \mathbb{E}\left(\sup_{-\infty\leq s\leq t}|x(s)|^2\right) \leq\ & 4|\varepsilon(0)|^2+6(T-t_0+4)K(T-t_0) \\ & +6(T-t_0+4)a\int_{t_0}^{t}\left[\left(1+2\mathbb{E}\left(\sup_{-\infty\leq u\leq S}|x(u)|^2\right)\right)\right]ds, \end{aligned}$$

Mutipling in 2 and added 1, we derive

$$\begin{aligned} 1+2\mathbb{E}\left(\sup_{-\infty\leq s\leq t}|x(s)|^2\right) \leq\ & 1+8|\varepsilon(0)|^2+12(T-t_0+4)K(T-t_0) \\ & +12(T-t_0+4)a\int_{t_0}^{t}\left[\left(1+2\mathbb{E}\left(\sup_{-\infty\leq u\leq S}|x(u)|^2\right)\right)\right]ds \end{aligned}$$

Using Gronawl's lamma, we obtain

$$\begin{aligned} 1+2\mathbb{E}\left(\sup_{-\infty\leq t\leq T}|x(t)|^2\right) \leq\ & \left(1+8|\varepsilon(0)|^2+12(T-t_0+4)K(T-t_0)\right)\times \\ & \exp\left(12a(T-t_0+4)(T-t_0)\right). \end{aligned}$$

consequencly

$$\begin{aligned} \mathbb{E}\left(\sup_{-\infty\leq t\leq T}|x(t)|^2\right) \leq\ & \left(\frac{1}{2}+4|\varepsilon(0)|^2+6(T-t_0+4)K(T-t_0)\right)\times \\ & \exp\left(12a(T-t_0+4)(T-t_0)\right). \end{aligned}$$

■

Lemma 4.3 *under assumptions A_1 and A_2, we have for all $t\geq t_{\grave{a}}$, for any $t_0\leq s<t\leq T$, wich $t-s<1$*

$$\mathbb{E}|X(t)-X(s)|^2\leq B_3(t-s),$$

where

$$B_3 = 8\left(K + a\left(1 + 2B_2\right)\right).$$

Proof. we have

$$\begin{aligned} X(t) - X(s) &= \int_s^t f\left(w, X(w), X\left(w - \alpha\left(w\right)\right)\right) dw \\ &\quad + \int_s^t g\left(w, X(w), X\left(w - \alpha\left(w\right)\right)\right) dB\left(w\right) \end{aligned}$$

$$\begin{aligned} \left|X(t) - X(s)\right|^2 &\leq 2\left|\int_s^t f\left(w, X(w), X\left(w - \alpha\left(w\right)\right)\right) dw\right|^2 \\ &\quad +2\left|\int_s^t g\left(w, X(w), X\left(w - \alpha\left(w\right)\right)\right) dB\left(w\right)\right|^2. \end{aligned}$$

By using the Hölder's, and moment inequalities, we drive

$$\begin{aligned} \mathbb{E}\left|X(t) - X(s)\right|^2 &\leq 4\left(t-s\right)\mathbb{E}\int_s^t \left|f\left(w, X(w), X\left(w - \alpha\left(w\right)\right)\right) - f\left(s, 0, 0\right)\right|^2 dw \\ &\quad +4\mathbb{E}\int_s^t \left|g\left(w, X(w), X\left(w - \alpha\left(w\right)\right)\right) - g\left(s, 0, 0\right)\right|^2 dw \\ &\quad +4\left(t-s\right)\mathbb{E}\int_s^t \left|f\left(s, 0, 0\right)\right|^2 ds + 4\int_s^t \left|g\left(s, 0, 0\right)\right|^2 dw \end{aligned}$$

and by assumption A_1 and A_2, we deduce

$$E\left|X(t) - X(s)\right|^2 \leq 4\left(t - s + 1\right)\mathbb{E}\int_s^t \left|K + \varphi\left(\left|X(w)\right|^2 + \left|X\left(w - \alpha\left(w\right)\right)\right|^2\right)\right| dw$$

Since φ is a concave function and $\varphi(0) = 0$, then, we have $\varphi(v) = a(1 + v)$ for all $v \geq 0$,

$$\mathbb{E}\left|X(t) - X(s)\right|^2 \leq 4\left(t - s + 1\right)\mathbb{E}\int_s^t K + a\left(1 + \left|X(w)\right|^2 + \left|X\left(w - \alpha\left(w\right)\right)\right|^2\right) dw,$$

thus

$$\mathbb{E}\left|X(t) - X(s)\right|^2 \leq 4\left(t - s + 1\right)\mathbb{E}\int_s^t K + a\left(1 + 2\mathbb{E}\sup_{-\infty < u \leq w}\left(X\left(u\right)\right)\right) dw,$$

by using thelemma 3.2, we get

$$\mathbb{E}\left|X(t)-X(s)\right|^2 \leq 4(t-s+1)(K+(1+2B_2))(t-s),$$

or $t-s\leq 1$, we obtain

$$\mathbb{E}\left|X(t)-X(s)\right|^2 \leq 8(K+1+2B_2)(t-s)$$

■

Theorem 4.1 *under assumptions A_1, A_1, we have for all integer $n\geq 1$*

$$\mathbb{E}\left(\sup_{t_0\leq t\leq T}\left|X(t)-X_n(t)\right|\right)\leq B_n$$

where

$$B_n = 4(T-t_0+4)\left(1+\frac{4}{n}B_3\right)(T-t_0)\exp\left(16a(T-t_0+4)(T-t_0)\right)$$

Proof.

$$\begin{aligned} X(t)-X_n(t) &= \int_{t_0}^{t} I_{\overline{D}_n}(s)\,(f(s,X(s),X(s-\alpha(s))) \\ &\quad -f\left(s,X_n(s-\frac{1}{n}),X_n(s-\alpha(s))\right)ds \\ &\quad +\int_{t_0}^{t} I_{D_n}(s)\,(f(s,X(s),X(s-\alpha(s))) \\ &\quad -f\left(s,X_n(s-\frac{1}{n}),X_n\left(s-\alpha(s)-\frac{1}{n}\right)\right)\Bigg)ds \\ &\quad +\int_{t_0}^{t} I_{\overline{D}_n}(s)\,(g(s,X(s),X(s-\alpha(s))) \\ &\quad -g\left(s,X_n(s-\frac{1}{n}),X_n(s-\alpha(s))\right)dB(s) \\ &\quad +\int_{t_0}^{t} I_{D_n}(s)\,(gs,X(s),X(s-\alpha(s)) \\ &\quad -g\left(s,X_n(s-\frac{1}{n}),X_n\left(s-\alpha(s)-\frac{1}{n}\right)\right)dB(s) \end{aligned}$$

$$
\begin{aligned}
|X(t) - X_n(t)|^2 \leq 4 \bigg| \int_{t_0}^{t} & I_{\overline{D}_n}(s) f(s, X(s), X(s - \alpha(s))) \\
& - f\left(s, X_n(s - \frac{1}{n}), X_n(s - \alpha(s))\right) ds \bigg|^2 \\
& + 4 \bigg| \int_{t_0}^{t} I_{D_n}(s) f(s, X(s), X(s - \alpha(s))) \\
& - f\left(s, X_n(s - \frac{1}{n}), X_n\left(s - \alpha(s) - \frac{1}{n}\right)\right) ds \bigg|^2 \\
& + 4 \bigg| \int_{t_0}^{t} I_{\overline{D}_n}(s) g(s, X(s), X(s - \alpha(s))) \\
& - g\left(s, X_n(s - \frac{1}{n}), X_n(s - \alpha(s))\right) dB(s) \bigg|^2 \\
& + 4 \bigg| \int_{t_0}^{t} I_{D_n}(s) g(s, X(s), X(s - \alpha(s))) \\
& - g\left(s, X_n(s - \frac{1}{n}), X_n\left(s - \alpha(s) - \frac{1}{n}\right)\right) dB(s) \bigg|^2
\end{aligned}
$$

By using the Hölder's, and moment inequalities, we drive

$$
\begin{aligned}
\mathbb{E}|X(t) - X_n(t)|^2 \leq 4(T - t_0)\, \mathbb{E} \int_{t_0}^{t} & I_{\overline{D}_n}(s) \, |f(s, X(s), X(s - \alpha(s))) \\
& - f\left(s, X_n(s - \frac{1}{n}), X_n(s - \alpha(s))\right)\bigg|^2 ds \\
& + 4(T - t_0)\, \mathbb{E} \int_{t_0}^{t} I_{D_n}(s) \, |f(s, X(s), X(s - \alpha(s))) \\
& - f\left(s, X_n(s - \frac{1}{n}), X_n\left(s - \alpha(s) - \frac{1}{n}\right)\right)\bigg|^2 ds \\
& + 4 \times 4\mathbb{E} \int_{t_0}^{t} I_{\overline{D}_n}(s) \, |g(s, X(s), X(s - \alpha(s))) \\
& - g\left(s, X_n(s - \frac{1}{n}), X_n(s - \alpha(s))\right)\bigg|^2 ds \\
& + 4 \times 4\mathbb{E} \int_{t_0}^{t} I_{D_n}(s) \, |g(s, X(s), X(s - \alpha(s))) \\
& - g\left(s, X_n(s - \frac{1}{n}), X_n\left(s - \alpha(s) - \frac{1}{n}\right)\right)\bigg|^2 ds,
\end{aligned}
$$

by using A_2

$$\begin{aligned}\mathbb{E}\left|X(t)-X_n(t)\right|^2 \leq 4(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,\varphi\Big(|X(s)-X_n\left(s-\frac{1}{n}\right)|^2 \\ +|X(s-\alpha(s))-X_n(s-\alpha(s))|^2\Big)\,ds \\ +4(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,\varphi\Big(|X(s)-X_n\left(s-\frac{1}{n}\right)|^2 \\ +|X(s-\alpha(s))-X_n\left(s-\alpha(s)-\frac{1}{n}\right)|^2\Big)\,ds\end{aligned}$$

Since $\varphi(.)$ is a concave fonction, then, we have

$$\begin{aligned}&\mathbb{E}\left|X(t)-X_n(t)\right|^2 \\ \leq\; & 4(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s) \\ &\left(1+|X(s)-X_n\left(s-\frac{1}{n}\right)|^2+|X(s-\alpha(s))-X_n(s-\alpha(s))|^2\right)ds \\ &+4(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{D_n}(s) \\ &\left(1+|X(s)-X_n\left(s-\frac{1}{n}\right)|^2+|X(s-\alpha(s))-X_n\left(s-\alpha(s)-\frac{1}{n}\right)|^2\right)ds\end{aligned}$$

and

$$
\begin{aligned}
& \mathbb{E}\left|X(t)-X_n(t)\right|^2 \\
\leq\ & 4\left(T-t_0+4\right) a \mathbb{E} \int_{t_0}^t I_{\overline{D}_n}(s) \\
& \times\left(1+|X(s)-X_n\left(s-\frac{1}{n}\right)+X\left(s-\frac{1}{n}\right)-X\left(s-\frac{1}{n}\right)|^2\right) ds \\
& +4\left(T-t_0+4\right) a \mathbb{E} \int_{t_0}^t I_{\overline{D}_n}(s)\left(1+|X(s-\alpha(s))-X_n(s-\alpha(s))|^2\right) ds \\
& +4\left(T-t_0+4\right) a \mathbb{E} \int_{t_0}^t I_{D_n}(s) \\
& \times\left(1+|X(s)-X_n\left(s-\frac{1}{n}\right)+X\left(s-\frac{1}{n}\right)-X\left(s-\frac{1}{n}\right)|^2\right) ds \\
& +4\left(T-t_0+4\right) a \mathbb{E} \int_{t_0}^t I_{D_n}(s) \\
& \times\left(1+|X(s-\alpha(s))-X_n\left(s-\alpha(s)-\frac{1}{n}\right)+\right. \\
& \left. X\left(s-\alpha(s)-\frac{1}{n}\right)-X\left(s-\alpha(s)-\frac{1}{n}\right)|^2\right) ds
\end{aligned}
$$

thus

$$\begin{aligned}
&\mathbb{E}\left|X(t)-X_n(t)\right|^2 \\
\leq\ & 4(T-t_0+4)\,a\,(T-t_0) \\
&+8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left(|X\left(s-\frac{1}{n}\right)-X_n\left(s-\frac{1}{n}\right)|^2\right)ds \\
&+8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left(|X(s)-X\left(s-\frac{1}{n}\right)|^2\right)ds \\
&+4(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left(|X(s-\alpha(s))-X_n(s-\alpha(s))|^2\right)ds \\
&8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{T} I_{D_n}(s)\left(|X\left(s-\frac{1}{n}\right)-X_n\left(s-\frac{1}{n}\right)|^2\right)ds \\
&+8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left(|X(s)-X\left(s-\frac{1}{n}\right)|^2\right)ds \\
&+8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left(|X\left(s-\alpha(s)-\frac{1}{n}\right)-X_n\left(s-\alpha(s)-\frac{1}{n}\right)|^2\right)ds \\
&+8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left(|X(s-\alpha(s))-X\left(s-\alpha(s)-\frac{1}{n}\right)|^2\right)ds
\end{aligned}$$

wich implies

$$\begin{aligned}
&\mathbb{E}\left|X(t)-X_n(t)\right|^2 \\
\leq\ & 4(T-t_0+4)\,a\,(T-t_0) \\
&+12(T-t_0+4)\,a\int_{t_0}^{t} I_{\overline{D}_n}(s)\left(\mathbb{E}\sup_{t_0\leq u\leq s}|X(u)-X_n(u)|^2\right)ds \\
&+8(T-t_0+4)\,a\int_{t_0}^{t} I_{\overline{D}_n}(s)\left(\mathbb{E}|X(s)-X\left(s-\frac{1}{n}\right)|^2\right)ds \\
&16(T-t_0+4)\,a\int_{t_0}^{t} I_{D_n}(s)\left(\mathbb{E}\sup_{t_0\leq u\leq s}|X(u)-X_n(u)|^2\right)ds \\
&+8(T-t_0+4)\,a\int_{t_0}^{t} I_{D_n}(s)\left(\mathbb{E}|X(s)-X\left(s-\frac{1}{n}\right)|^2\right)ds \\
&+8(T-t_0+4)\,a\int_{t_0}^{t} I_{D_n}(s)\left(\mathbb{E}|X(s-\alpha(s))-X\left(s-\alpha(s)-\frac{1}{n}\right)|^2\right)ds
\end{aligned}$$

by using the lemma 3, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le t}|X(s)-X_n(s)|^2\right)\le 4(T-t_0+4)\,a\,(T-t_0)\\
&+12(T-t_0+4)\,a\int_{t_0}^{t}I_{\overline{D}_n}(s)\left(\mathbb{E}\sup_{t_0\le u\le s}|X(u)-X_n(u)|^2\right)ds\\
&+8(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t}\frac{B_3}{n}I_{\overline{D}_n}(s)\,ds\\
&16(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t}I_{D_n}(s)\left(\mathbb{E}\sup_{t_0\le u\le s}|X(u)-X_n(u)|^2\right)ds\\
&+16(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t}\frac{B_3}{n}I_{D_n}(s)\,ds,
\end{aligned}$$

finally we conclued

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le t}|X(s)-X_n(s)|^2\right)\\
\le\;& 4(T-t_0+4)\,a\,(T-t_0)\left(1+4\frac{B_3}{n}\right)\\
& 16(T-t_0+4)\,a\int_{t_0}^{t}\mathbb{E}\left(\sup_{t_0\le u\le s}|X(u)-X_n(u)|^2\right)ds,
\end{aligned}$$

by the Gronwall's lemma, we arrive

$$\begin{aligned}
\mathbb{E}\left(\sup_{t_0\le tT}|X(t)-X_n(t)|^2\right)\;\le\;& 4(T-t_0+4)\,a\,(T-t_0)\\
&\left(1+4\frac{B_3}{n}\right)\exp\left[16(T-t_0+4)\,a\,(T-t_0)\right]
\end{aligned}$$

■

Theorem 4.2 *Under assumptions A_1, A_1, we have for all integers $m,\ n\ge 1$ such that $n\le m$*

$$\mathbb{E}\left(\sup_{t_0\le t\le T}|X_n(t)-X_m(t)|\right)\le B_{n,m}$$

where

$$B_{n,m}=4(T-t_0+4)\left(1+3B_n+B_m+3B_3\left(\frac{1}{n}-\frac{1}{m}\right)\right)(T-t_0)$$

Proof. We have, for all $t \in [t_0, T]$

$$
\begin{aligned}
& X_n(t) - X_m(t) \\
= & \int_{t_0}^{t} I_{\overline{D}_n}(s) \times f\left(s, X_n(s-\frac{1}{n}), X_n(s-\alpha(s))\right) - f\left(s, X_m(s-\frac{1}{m}), X_m(s-\alpha(s))\right) ds \\
& + \int_{t_0}^{t} I_{D_n}(s) \times \\
& f\left(s, X_n(s-\frac{1}{n}), X\left(s-\alpha(s)-\frac{1}{n}\right)\right) - f\left(s, X_m(s-\frac{1}{m}), X_m\left(s-\alpha(s)-\frac{1}{m}\right)\right) ds \\
& + \int_{t_0}^{t} I_{\overline{D}_n}(s) \times \\
& g\left(s, X_n(s-\frac{1}{n}), X_n(s-\alpha(s))\right) - g\left(s, X_m(s-\frac{1}{m}), X_m(s-\alpha(s))\right) dB(s) \\
& + \int_{t_0}^{t} I_{D_n}(s) \times \\
& g\left(s, X_n(s-\frac{1}{n}), X_n\left(s-\alpha(s)-\frac{1}{n}\right)\right) - g\left(s, X_m(s-\frac{1}{m}), X_m\left(s-\alpha(s)-\frac{1}{m}\right)\right) dB(s)
\end{aligned}
$$

$$
\begin{aligned}
& |X_n(t) - X_m(t)|^2 \\
\leq & 4\left|\int_{t_0}^{t} I_{\overline{D}_n}(s) f\left(s, X_n(s-\frac{1}{n}), X_n(s-\alpha(s))\right)\right. \\
& \left. - f\left(s, X_m(s-\frac{1}{m}), X_m(s-\alpha(s))\right) ds\right|^2 \\
& + 4\left|\int_{t_0}^{t} I_{D_n}(s) f\left(s, X_n(s-\frac{1}{n}), X\left(s-\alpha(s)-\frac{1}{n}\right)\right)\right. \\
& \left. - f\left(s, X_m(s-\frac{1}{m}), X_m\left(s-\alpha(s)-\frac{1}{m}\right)\right) ds\right|^2 \\
& + 4\left|\int_{t_0}^{t} I_{\overline{D}_n}(s) g\left(s, X_n(s-\frac{1}{n}), X_n(s-\alpha(s))\right)\right. \\
& \left. - g\left(s, X_m(s-\frac{1}{m}), X_m(s-\alpha(s))\right) dB(s)\right|^2 \\
& + 4\left|\int_{t_0}^{t} I_{D_n}(s) g\left(s, X_n(s-\frac{1}{n}), X_n\left(s-\alpha(s)-\frac{1}{n}\right)\right)\right. \\
& \left. - g\left(s, X_m(s-\frac{1}{m}), X_m\left(s-\alpha(s)-\frac{1}{m}\right)\right) dB(s)\right|^2
\end{aligned}
$$

By using the Hölder's, and moment inequalities, we drive

$$
\begin{aligned}
&\mathbb{E}\left|X\left(t\right)-X_n\left(t\right)\right|^2\\
\leq\;& 4\left(T-t_0\right)\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}\left(s\right)\bigg|f\left(s,X_n(s-\frac{1}{n}),X_n\left(s-\alpha\left(s\right)\right)\right)\\
&-f\left(s,X_m(s-\frac{1}{m}),X_m\left(s-\alpha\left(s\right)\right)\right)ds\bigg|^2 ds\\
&+4\left(T-t_0\right)\mathbb{E}\int_{t_0}^{t} I_{D_n}\left(s\right)\bigg|f\left(s,X_n(s-\frac{1}{n}),X\left(s-\alpha\left(s\right)-\frac{1}{n}\right)\right)\\
&-f\left(s,X_m(s-\frac{1}{m}),X_m\left(s-\alpha\left(s\right)-\frac{1}{m}\right)\right)ds\bigg|^2 ds\\
&+4\times 4\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}\left(s\right)\bigg|g\left(s,X_n(s-\frac{1}{n}),X_n\left(s-\alpha\left(s\right)\right)\right)\\
&-g\left(s,X_m(s-\frac{1}{m}),X_m\left(s-\alpha\left(s\right)\right)\right)dB\left(s\right)\bigg|^2 ds\\
&+4\times 4\mathbb{E}\int_{t_0}^{t} I_{D_n}\left(s\right)|\;\bigg|-g\left(s,X_m(s-\frac{1}{m}),X_m\left(s-\alpha\left(s\right)-\frac{1}{m}\right)\right)dB\left(s\right)\bigg|^2 ds,
\end{aligned}
$$

by using A_2

$$
\begin{aligned}
&\mathbb{E}\left|X_n\left(t\right)-X_m\left(t\right)\right|^2\\
\leq\;& 4\left(T-t_0+4\right)\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}\left(s\right)\varphi\left(|X_n\left(s-\frac{1}{n}\right)-X_m\left(s-\frac{1}{m}\right)|^2\right.\\
&\left.+|X_n\left(s-\alpha\left(s\right)\right)-X_m\left(s-\alpha\left(s\right)\right)|^2\right)ds\\
&+4\left(T-t_0+4\right)\mathbb{E}\int_{t_0}^{t} I_{D_n}\left(s\right)\varphi\left(|X_n\left(s-\frac{1}{n}\right)-X_m\left(s-\frac{1}{m}\right)|^2\right.\\
&\left.+|X_n\left(s-\alpha\left(s\right)-\frac{1}{n}\right)-X_m\left(s-\alpha\left(s\right)-\frac{1}{m}\right)|^2\right)ds
\end{aligned}
$$

Since $\varphi(.)$ is a concave fonction, then, we have

$$\begin{aligned}
&\mathbb{E}\left|X_n(t)-X_m(t)\right|^2 \\
\leq\ & 4(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\left(1+|X_n\left(s-\frac{1}{n}\right)-X_m\left(s-\frac{1}{m}\right)|^2\right. \\
&\left.+|X_n(s-\alpha(s))-X_m(s-\alpha(s))|^2\right)ds \\
&+4(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left(1+|X_n\left(s-\frac{1}{n}\right)-X_m\left(s-\frac{1}{m}\right)|^2\right. \\
&\left.+|X_n\left(s-\alpha(s)-\frac{1}{n}\right)-X_m\left(s-\alpha(s)-\frac{1}{m}\right)|^2\right)ds
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left|X_n(t)-X_m(t)\right|^2 \\
\leq\ & 4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|X\left(s-\frac{1}{n}\right)-X_n\left(s-\frac{1}{n}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|X\left(s-\frac{1}{n}\right)-X_m\left(s-\frac{1}{m}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|X\left(s-\frac{1}{n}\right)-X\left(s-\frac{1}{m}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 2\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|X(s-\alpha(s))-X_n(s-\alpha(s))|^2 ds \\
&+4(T-t_0+4)\,a\times 2\mathbb{E}\int_{t_0}^{t} I_{\overline{D}_n}(s)\,|X(s-\alpha(s))-X_m(s-\alpha(s))|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,|X\left(s-\frac{1}{n}\right)-X_n\left(s-\frac{1}{n}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\left(X\left(s-\frac{1}{m}\right)-|X_m\left(s-\frac{1}{m}\right)|^2\right)ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,X\left(s-\frac{1}{n}\right)-|X\left(s-\frac{1}{m}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,|X\left(s-\alpha(s)-\frac{1}{n}\right)-X_n\left(s-\alpha(s)-\frac{1}{n}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,|X\left(s-\alpha(s)-\frac{1}{m}\right)-X_m\left(s-\alpha(s)-\frac{1}{m}\right)|^2 ds \\
&+4(T-t_0+4)\,a\times 3\mathbb{E}\int_{t_0}^{t} I_{D_n}(s)\,|X\left(s-\alpha(s)-\frac{1}{m}\right)-X\left(s-\alpha(s)-\frac{1}{n}\right)|^2 ds
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{E}\sup_{t_0\le s\le t}|X_n(s)-X_m(s)|^2\\
\le\ & 4(T-t_0+4)\,a\,(T-t_0)\\
&+12(T-t_0+4)\,a\int_{t_0}^{t}\mathbb{E}\sup_{t_0\le u\le s}\left(|X(u)-X_n(u)|^2\right)ds\\
&+12(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t}\left(|X(u)-X_m(u)|^2\right)ds\\
&+12(T-t_0+4)\,a\mathbb{E}\int_{t_0}^{t}\left(|X\left(s-\alpha(s)-\frac{1}{m}\right)-X\left(s-\alpha(s)-\frac{1}{n}\right)|^2\right)ds,
\end{aligned}$$

anb by using the lemmas 3 and theorem 4, we derive

$$\begin{aligned}
&\mathbb{E}\sup_{t_0\le s\le t}|X_n(s)-X_m(s)|^2\\
\le\ & 4(T-t_0+4)\,a\left(1+3B_n+3B_m+3B_3\left(\frac{m-n}{nm}\right)\right)(T-t_0)
\end{aligned}$$

Theorem 4.3 *assume that $X(t)$, $Y(t)$ two solutions of the equation* (1.1)*. Then, we have*

$$\mathbb{E}\left(\sup_{t_0\le t\le T}|X(t)-Y(t)|\right)\le B_4,$$

where

$$B_4=2a(T-t_0+4)\exp\left(4a(T-t_0+4)(T-t_0)\right)$$

■

Proof. We have

$$\begin{aligned}
&|X(t)-Y(t)|^2\le\\
&\quad 2|\int_{t_0}^{t}f(s,X(s),X(s-\alpha(s)))-f(s,Y(s),Y(s-\alpha(s)))ds|^2\\
&\quad +2|\int_{t_0}^{t}g(s,X(s),X(s-\alpha(s)))-g(s,Y(s),Y(s-\alpha(s)))dB(s)|^2
\end{aligned}$$

by using the Hölder's and the moment inequalites, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{t_0\leq s\leq t}|X(s)-Y(s)|^2 \leq \\ &2\,(T-t_0)\int_{t_0}^{t}|f(s,X(s),X(s-\alpha\,(s)))-f(s,Y(s),Y(s-\alpha\,(s)))|^2 ds \\ &+2\int_{t_0}^{t}|g(s,X(s),X(s-\alpha\,(s)))-g(s,Y(s),Y(s-\alpha\,(s)))|^2 ds,\end{aligned}$$

and the assumption (A_2), we have

$$\begin{aligned}&\mathbb{E}\sup_{t_0\leq s\leq t}|X(s)-Y(s)|^2 \leq \\ &2\,(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t}\varphi\left(|X(s)-Y(s)|^2+|X(s-\alpha\,(s)))-f(s,,Y(s-\alpha\,(s))|^2\right)ds\end{aligned}$$

Since φ is a concave fonction, then, we have

$$\begin{aligned}&\mathbb{E}\sup_{t_0\leq s\leq t}|X(s)-Y(s)|^2 \leq \\ &2\,(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t}a\left(1+|X(s)-Y(s)|^2+|X(s-\alpha\,(s)))-f(s,,Y(s-\alpha\,(s))|^2\right)ds\end{aligned}$$

thus

$$\mathbb{E}\sup_{t_0\leq s\leq t}|X(s)-Y(s)|^2 \leq 2\,(T-t_0+4)\,\mathbb{E}\int_{t_0}^{t}a\left(1+2\sup_{t_0\leq u\leq s}|X(u)-Y(u)|^2\right)ds$$

wich implies

$$\begin{aligned}\mathbb{E}\sup_{t_0\leq s\leq t}|X(s)-Y(s)|^2 &\leq 2\,(T-t_0+4)\,a\,(T-t_0) \\ &\quad +4\,(T-t_0+4)\,a\int_{t_0}^{t}\mathbb{E}\left(\sup_{t_0\leq u\leq s}|X(u)-Y(u)|^2\right)ds\end{aligned}$$

Finally, by the Gronwall lemas we have

$$\mathbb{E}\sup_{t_0\leq s\leq t}|X(s)-Y(s)|^2 \leq 2a\,(T-t_0+4)\exp\left(4a\,(T-t_0+4)\,(T-t_0)\right)$$

■

4.4 Conclusion

InThis memory, we have proved the convergence of carathèodory's approximate solutions to the unique solution of stochastic differential equations model with under conditions that are not lipschitziennes and not linear growth. We used the approximation sheme of Carathèodory'. We proved the existence and uniquenesses of theire solutions. These equations can be useful in many applications where we assume that there are deterministic changes combined with random ones. The study of such equations needs a full knowledge of probability theory and stochastic calculus including stochastic processes, stochastic integration (Itô integral) and stochastic differentiation (Itô formula).

in my prespectives generalize this work on other equation models with other broader conditions and we want to search for concrete applications for the equations, for example in finance, physics, modeling and biology, ect....

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