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Thesis

A view to obtaining the diploma of

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Option: ***Mathématics***

Mathematical Study of a Class of PDE and Fractional Derivatives Equations with Application in Images Processing

Presented by:

Hana Matallah

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Thesis *for the fulfilment of the requirements of Doctorate of 3rd cycle degree in Mathematics*
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Topic:

Mathematical study of a class of PDE and fractional derivatives equations with application in images processing

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بسم الله الرحمن الرحيم

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First of all, I thank God the Almighty for the will, the health and the patience he gave me during all the long years of study so that I could get there.

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Matallah Hana

Dedication

With the expression of my gratitude, I dedicate this modest work to those who, whatever the terms embraced, I would never manage to express my sincere love to them.

To the man who is offering to me from god, who owes my life, my success and all my respect; my dear father.

To the woman who suffered without letting me suffer, who never said no to my demands and who spared no effort to make me happy; my lovely mother.

To my dear husband who never stopped advising, encouraging and supporting me throughout my studies. May god protect them and given them luck and happiness.

To my grandmother, my brother, uncles and aunts. May god them gives a long and happy life.

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To all my teachers, their generosity and their support obliges me to show them all my deep respect and my loyalty consideration.

To all my friends and colleagues, they will find here testimony of infinite fidelity and friendship.

To all those I love and those who love me.

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Abstract

The works presented in this thesis focuses on the study of some partial differential equations ($PDEs$) of the parabolic type, where we give the proof of the existence results fot three principal problems using Feado-Galerkin approximation. On the first one, we prove the existence of the generalized solution for a class of quasilinear parabolic system with nonlocal boundary conditions. The second one is the study of a reaction diffusion system, where we give the existence of global weak solution in the case of the positivity and the total mass conditions on the nonlinearities funtions. For the last problem, we propose a new approaches of a reaction diffusion model of fractional order, which is based on image processing.

Keywords:Fractional order derivitive, Quasilinear parabolic equations; Reaction-diffusion systems; Image prosessing;
AMS subject classification (2010): 35K59; 37C25; 35J70; 35D05; 68U10.

Titre: **Étude Mathématiques d'une classe des EDP et Équations aux dérivées fractionnaire avec application en traitement d'images**

Résumé

Les travaux présentés dans cette thèse portent sur l'étude de quelques équations aux dérivées partielles (EDP) de type parabolique, où nous donnons la preuve des résultats d'existence pour trois problèmes principaux en utilisant l'approximation de Feado-Galerkin. Sur la première, nous prouvons l'existence de la solution généralisée pour une classe de système parabolique quasi linéaire avec des conditions aux limites non locales. La seconde est l'étude d'un système de diffusion de réaction, où l'on donne l'existence d'une solution faible globale dans le cas de la positivité et les conditions de masse totale sur les fonctions de non-linéarités. Pour le dernier problème, nous proposons une nouvelle approche d'un modèle de diffusion de réaction d'ordre fractionnaire, qui est basé sur le traitement d'images.

Mots clés : Dérivée d'ordre fractionnaire; Équations paraboliques quasi linéaires; Systémes de réaction-diffusion; Traitement d'image.

ملخص

تركز الأعمال المقدمة في هذه الرسالة على دراسة بعض المعادلات و الجمل التفاضلية الجزئية من النوع المكافئ، حيث أثبتنا وجود حلول لثلاث مسائل رئيسية باستخدام تقريب $Feado - Galerkin$. في المسألةالأول، أثبتنا وجود الحل المعمم لنوع من الجمل شبه الخطية ذات الشروط الحديةالغير محلية. الثاني هو دراسة مسألة لجمل تفاعل الإنتشار، حيث نعطي وجود حل ضعيف في حالة الإيجابية وظروف الكتلة الكلية على الدوال اللاخطية. بالنسبة للمشكلة الأخيرة، نقترح طرقًا جديدة لنموذج تفاعل الإنتشار برتبة إشتقاق كسرية، والذي يعتمد على معالجة الصور.

كلمات مفتاحية: المشتقات الكسرية؛ معادلات مكافئة شبه خطية؛ مسائل تفاعل الإنتشار؛ معالجة الصورة.

Notations

• $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \dots\}$.

• $\mathbb{R}$: the set of real numbers.

• $\mathbb{R}^N$: the real space of dimension N.

• Ω : open in $\mathbb{R}^N$, $N \in \mathbb{N}^*$.

• $\partial\Omega$: topological boundary of Ω.

•$x = (x_1, x_2, ..., x_N)$: generic point of $\mathbb{R}^N$.

• $dx = dx_1 dx_2 ... dx_N$: measure of Lebesgue on Ω.

• $(.,.)$: usual inner product in $L^2(\Omega)$.

•$Q_T =]0, T[\times \Omega, \quad T > 0$.

•$\Sigma_T =]0, T[\times \partial\Omega$.

• G_σ : the gaussian filter.

• σ : the variance.

• $f * g$: the convolution product.

• Δ : is the Laplace operator.

• ∇u : gradient of u.

• div : the divergence operator.

• $\frac{\partial}{\partial x}$: partial derivative.

• $\frac{\partial}{\partial \eta}$: outward normal derivative.

• $\rightarrow$: designates the strong convergence.

• $\rightharpoonup$: indicates the weak convergence.

• $W^{k,p}(\Omega)$: sobolev space on Ω, $H^r(\Omega) = W^{k,2}(\Omega)$.

• $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^r(\Omega)}$: the usual norms on $L^2(\Omega)$ and $H^r(\Omega)$ respectively.

• $H^{-r}(\Omega)$: dual space of $H^r(\Omega)$, $\|\cdot\|_{H^{-r}(\Omega)}$: norm in $H^{-r}(\Omega)$.

•$L^p(\Omega)$: space of the functions of power p-th integrable on for measurement dx; $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$.

• $K(x)$: norm of $k(x,y)$ in $L^q(\Omega)$ with respect to y, $K(x) = \left(\int_\Omega |k(x,y)|^q dy\right)^{1/q}$.

• $K_i(x)$: norm of $D_i k(x,y)$ in $L^q(\Omega)$ with respect to y, $K_i(x) = \left(\int_\Omega |\frac{\partial k(x,y)}{\partial x_i}|^q dy\right)^{1/q}$.

• $\mathrm{W} = L^2(\Omega) \times L^2(\Omega)$ which is a Banach space endowed with the norm

$$\|(u,v)\|^2_{\mathrm{W}} = \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)},$$

.

• $\mathrm{V} = L^p(\Omega) \times L^p(\Omega)$ and $\mathrm{Y} = L^q(\Omega) \times L^q(\Omega)$.

• $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(\partial\Omega)}$: the usual norms of $L^q(\Omega)$, $L^2(\Omega)$, $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively.

If X is a Banach space

•$L^p\left(]0,T[\,;X\right) = \left\{ u :]0,T[\, \rightarrow X \text{ measurable; } \int_0^T \|u\|^p_X \, dt < \infty \right\}$.

•$L^\infty\left(]0,T[\,;X\right) = \left\{ u :]0,T[\, \rightarrow X \text{ measurable; } ess - sup_{t\in(0,T)} \|u\|_X \, dt < \infty \right\}$.

Contents

General introduction

Mathematics always has the benefit of participating in the development of several scientific fields, like physics, biology, biomedical and engineering. For many scientist and researchers, thess areas offers a new and exciting branches of research, while for the specialist and the mathematical modeling offers another research tool commensurate with new laboratory techniques. These laws are typically expressed as balance sheets, which formally correspond to ordinary differential equations or partial differential equations ($PDEs$). The latter, make it possible to approach observed phenomena from a mathematical point of view, for exemple; In the field of physics and chimistry used to model reaction, where the time depent situations translate more particularly into evolution equations taking into account possible interaction between objects and events. Then also are used in many other fields: in economics to study market behavior, in finance to study derivatives. More recently, this differential equations ($PDEs$) have found unsuspected application in the field of image processing, that field in recent years become the focus of many mathematician attention [[3], [22], [1], [27], [29]] and been very active, which is defined as the use of computerized algorithms for the analyses of digital images with respect to an application to get an enhanced image and to recovery lost information. The one of the most active topic of this field is the restoration of a digital image see [[5], [25], [22], [26]]. Not long a go, this Topic was included by a reaction difusion system [[3], [4]], which this system is a mathematical model that describes the evolution of the concentrations of one or more substances spatially distributed and subjected to two processes: a process of local chemical reactions, in which the different substances are transformed, and a process of diffusion which causes a distribution of these substances in space, where the general form of the reaction diffusion system is the follow:

$$\frac{\partial u}{\partial t} - div(D(t,x,u,\nabla u)\cdot \nabla u) + f(t,x,u,\nabla u), \quad x \in \Omega,\ t \geq 0,$$

where $u(t,x) = (u_1, \dots, u_m) : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}^m$ is a vector of variable, f is a linear or nonlinear vector function, which is called the reaction terms and $D : \mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \longrightarrow \mathbb{R}^m$ is a regular function, when D is a square matrix it is called the diffusion matrix, in this case

$$div(D(t,x,u,\nabla u)\cdot \nabla u) = D\Delta u;$$

are the diffusion terms.
In another meanong, the reaction terms are the result of any interaction between the components of u; for example u can be a vector of chemical concentrations, and f is the effect of the

chemical reactions of these concentrations, or the components of u can be densities of plant or animal populations, and f represents the effect of relationships (competitive or symbiotic) between predators and prey, for the diffusion terms can represent molecular diffusions or some random movements of individuals in a population [36]. Since many theoretical and practical contributions has shown the most important of fractional order calculus and there interest in the restoration of images, which this latter occuped the studies of many researchers thanks to its wide importance in several field.

This thesis is mainly devoted to the study of a class of problems nonlinear and quasilinear partial differential equation with respectivly a Carathéodory function and nonlocal boundary conditions, furthermore the study of fractionail equation by using Feado-Galrkin method, we prove the existence result of weak solution of this problems and for fractionail problem, we give a numerical result in image processing.

⋆ The first class of problem is the investigation of the work studing the quasilinear parabolic equations with nonlocal boundary condition, this study is a generalized of the Chen result in 2011 [13], where we construct an approximate solution and we derive a priori estimates and show the convergence of the approximate solution.

⋆ The second class is the system of reaction diffusion using in image processing, in fact is interpolation of the result given by [[3], [15], [1]], where we give the proof of existence result to the global weak solution for the new generic model, we can cite the leading works of some authors, In 2020 [32], A. Ouaoua, A. Khaldi and M. Maouni give a new study of the stabilization to the solution for a Kirchhoff type reaction-diffusion equation. In 2013, Maouni and Nouri [25] used a new model based on p-gradient using to restore a digital image. Lecheheb, Maouni and Lakhal in 2021 [21] used a novel model combining the Perona-Malik equation and heat equation in image restoration.

⋆ the last class is the fractional reaction-diffusion process for restoration of image. Our main is this part, which is the approches of to the investigation of a reaction- diffusion model of fractional order in which we apply the fractional derivative in the sense of the Caputo by contribution to time on the model proposed by Nourddine Alaa in 2014 [3], this study is based on the restoration of digital image such that a digital result is given on a noisy image in which this model is found to be effective in eliminating noise.

This thesis can be rougly divided into four chapters.

In the first chapter, we recall some definitions and important results in the Banach spaces and Sobolev spaces that has an essential role in the supsequent chapters.

In the second chapter, we studie the existence of the generalized solution for the class of quasilinear parabolic equation with nonlocal boundary condition using Feado-Galerkin approximation.

In the third chapter, we prove the existence of global weak solution for new generic reaction-diffusion system, where we trancate the system and by using Schauder fixed point theorem in

Banach spaces and show the existence of a solution for this approched problem, finally by making some estimations we prove that the solution of the truncated equation converge to the solution of our problem.

In the last chater, we show how fractional order differential equations are used to restore a digital image, where first we prove the existence of the weak solution, then we give a numerical result given in image processing.

Chapter 1

Preliminaries

This chapter collects several basic tools that will be required throughout this work. The common link between all of the results in this chapter is that they are preparatory for the main results, which are contained in the following chapters. The following will be an open bounded $\mathbb{R}^N$ and we will use the measure of Lesbegue.

1.1 Functional Spaces

In this section, we will review the fundamental concepts of functional spaces, specifically L^p spaces and Sobolev spaces. Then, We also provide some important definitions and results for the chapters that follow.
For any nonnegative integer m, let $C^m(\Omega)$ [16] be the spaces of m continuously differentiable functions on Ω, where

$$C^\infty(\Omega) = \bigcap_{m\in\mathbb{N}} C^m(\Omega).$$

We abbreviate $C^0(\Omega) \equiv C(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f \quad \text{continuous}\}$.

1.1.1 The $L^p(X; E)$ spaces

In all that follows, $(X, \mathcal{A}, \mu)$ is a measured space and E is a Banach space (on $\mathbb{R}$) where the norm note by $\|\cdot\|$.

Definition 1.1. *[16]*
i) For $1 \le p < +\infty$, we define:

$$L^p(X; E) = \left\{ f : X \longrightarrow E \ \mu\text{-mesurable and } \int_X \|f\|^p d\mu < +\infty \right\},$$

with

$$\|f\|_{L^p(X;E)} = \left(\int_X \|f\|^p d\mu \right)^{1/p}.$$

ii) If $p = \infty$, let

$$L^\infty(X;E) = \left\{ f : X \longrightarrow E \ \mu\text{-mesurable and } \exists C > 0, \quad \|f(x)\| \leq C \text{ for } \mu\text{-a.e. } x \right\},$$

with

$$\|f\|_{L^\infty(X;E)} = \inf\{C > 0; \quad \|f(x)\| \leq C \text{ for } \mu\text{-a.e. } x\}.$$

iii) In particulary, if $p = 2$, $L^2(X;E)$ equipped with the inner product

$$(f,g) = \int_X f(x)g(x)d\mu,$$

is a Hilbert space.
Recall that

$$L^1_{Loc}(X;E) = \left\{ f \ : \ f \in L^1(K) \text{ for every compact } K \text{ of } X \right\}$$

Remark 1.1. *[16]*

1)- When the space E is not specified, this means that we take $E = \mathbb{R}$, where $L^p = L^p(X;\mathbb{R})$.

2)- If E is continuously injected to F, $L^p(X;E)$ continuously injects into $L^p(X;F)$.

3)- By Hôlder's inequality, we see that if $r \in [p,q]$, then $L^p(X;E) \cap L^q(X;E) \subset L^r(X;E)$, If X is the finite measure, we have $L^p(X;E) \hookrightarrow L^q(X;E)$, where $p \geq q$.

1.1.2 The Sobolev spaces

The powers of Sobolev spaces functions and the powers of their derivatives (in the sense of transposition or in a weaker meaning that we will explain later) are Lebesgue-integrable, making them functional spaces, that is, spaces containing functions. These spaces are Banach spaces, much as the Lebesgue spaces. It is critical to demonstrate the existence of solutions to some partial differential equations that the Sobolev spaces are complete.

For all $k \in \mathbb{N}$, $H^k(\Omega)$ is the set of functions u in Ω such that u and it's derivatives of the order $D^s u$, where $|s| = \sum_{j=1}^n (s_j) \leq k$, are in $L^2(\Omega)$, $H^k(\Omega)$ is the Hilbert space for the norm:

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|s|\leq k} \int_\Omega |D^s u|^2 dx \right)^{1/2}.$$

Definition 1.2 (The weak derivative). *[16] Let $f \in L^1_{loc}(\Omega)$. We say that f is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{loc}(\Omega)$, such that*

$$\int_\Omega f \partial_i \varphi dx = -\int_\Omega g_i \varphi dx, \text{ for all } \varphi \in C_c^\infty(\Omega).$$

The function g_i is called the weak ith partial derivative of f, and is denoted by $\partial_i f$.

Proposition 1.1 (**The integration by parts formula**). *[16] Let u, v in $H^1(\Omega)$ and $\partial\Omega \in C^1(\Omega)$, then for $1 \leq i \leq N$, we have*

$$\int_\Omega \frac{\partial u(x)}{\partial x_i} v(x)dx = -\int_\Omega u(x)\frac{\partial v(x)}{\partial x_i}dx + \int_{\partial\Omega} u(s)v(s)\eta_i ds.$$

If $v \in H^1(\Omega)$ and $u_i \in H^1(\Omega)$, where u_i the components of the vector $\vec{u}$ then we have

$$\int_\Omega div(\vec{u}(x)).v(x)dx = -\int_\Omega \big(\vec{u}(x), \nabla v(x)\big)dx + \int_{\partial\Omega} \big(\vec{u},\vec{\eta}\big)v ds.$$

By the notation

$$\Delta u = div(\nabla \vec{u}),$$

we get Green's formula.

Proposition 1.2. *[16] For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u(x)\nabla v(x)dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v ds.$$

1.1.3 The $L^p(0,T;H^r(\Omega))$, $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;C^\infty(\Omega))$ spaces

$L^p(0,T;H^k(\Omega))$ [3] is the set of functions u, such that for all $t \in (0,T)$ and u in $H^k(\Omega)$ for the norm:

$$\|u\|_{L^p(0,T;H^k(\Omega))} = \left(\int_0^T \|u(t)\|^p_{H^k(\Omega)} dt\right)^{1/p}, \;\; 1 < p < \infty, \;\; k \in \mathbb{N}.$$

$L^\infty(0,T;L^2(\Omega))$ [3] is the set of functions u such that for all $t \in (0,T)$ and $u(t)$ in $L^2(\Omega)$ for the norm:

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} = \sup_{0<t<T} \|u(t)\|^2_{L^2(\Omega)}.$$

$L^\infty(0,T;C^\infty(\Omega))$ [3] is the set of functions u such that, for all $t \in (0,T)$, $u(t)$ in $C^\infty(\Omega)$ for the norm:

$$\|u\|_{L^\infty(0,T;C^\infty(\Omega))} = \inf\{c, \|u(t)\|_{C^\infty(\Omega)} \leq c, a.e.t \text{ in } (0,T)\}.$$

We have that

$$\|(u,v)\|_{L^p(0,T,H^k(\Omega))^2} = \|u\|_{L^p(0,T,H^k(\Omega))} + \|v\|_{L^p(0,T,H^k(\Omega))}, \quad 1 < p < \infty, \quad k \in \mathbb{N}. \tag{1.1}$$

$$\|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} = \|u\|_{L^\infty(0,T,L^2(\Omega))} + \|v\|_{L^\infty(0,T,L^2(\Omega))}. \tag{1.2}$$

$$\|(u,v)\|_{L^\infty(0,T,C^\infty(\Omega))} = \|u\|_{L^\infty(0,T,C^\infty(\Omega))} + \|v\|_{L^\infty(0,T,C^\infty(\Omega))}. \tag{1.3}$$

1.1.4 Some outcomes concerning integration and duality

Theorem 1.1 (**Dominated convergence theorem, Lebesgue**). *[10] Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy*

1. $f_n(x) \longrightarrow f(x)$ *a.e. on* Ω,

2. *There is a function $g \in L^1(\Omega)$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω,*

$$\text{then } f \in L^1(\Omega) \text{ and } \|f_n - f\|_{L^1(\Omega)} \longrightarrow 0.$$

Theorem 1.2. *[10] Let (f_n) be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$, such that $\|f_n - f\|_{L^p(\Omega)} \longrightarrow 0$.*
Then, there exist a subsequence f_{n_k} and a function $h \in L^p(\Omega)$, such that

1. $f_{n_k} \longrightarrow f$ *a.e. on* Ω,

2. $|f_{n_k}(x)| \leq h(x)$, $\forall k$, *a.e. on* Ω.

The Vitali's theorem

The Vitali's theorem is another fundamental result of integration theory, and it is based on the following definition:

Definition 1.3. *[16] Let $1 \leq p < \infty$. We say that a sequence of functions $(f_n)_n$ of $L^p(X : E)$ is p-equi-integrable if it satisfies the following conditions:*

(i)- $\forall \varepsilon > 0$, $\exists K \subset X$ *measurement finished as* $\forall n \geq 1$ *we have:*

$$\int_{X/K} \|f_n\|^p d\mu < \varepsilon.$$

(ii)- $\forall \varepsilon > 0$, $\exists \delta > 0$ *with* $\forall n \geq 1$, $\forall A \subset X$, *where* $\mu(A) < \delta$ *we have:*

$$\int_A \|f_n\|^p d\mu < \varepsilon.$$

Theorem 1.3 (**The Vitali's theorem**). *[16] Let $1 \leq p < \infty$. If $(f_n)_n$ is a sequence of $L^p(X; E)$ converging almost everywhere to f, then $f_n \to f$ in $L^p(X; E)$ if, and only if $(f_n)_n$ is p-equi-integrable.*

Theorem 1.4 (**The Schauder fixed point theorem**). *[16] Let $(X, \|.\|)$ be a Banach space and let Ω be a closed convex, bounded and impty subset of X. If $T : \Omega \longrightarrow \Omega$ is a (completly continuous) compact and continuous mapping, then T has a fixed point in Ω, i.e,*

$$\exists x \in \Omega : \ Tx = X.$$

1.1.5 Some useful inequalities and lemmas

In this part, we will review several inequalities and lemmas that will be used in later chapters.

The Cauchy-Schwarz inequality

Theorem 1.5. *[11]By standing in space $E = L^2(I;\mathbb{R}) \cap \mathcal{C}(I;\mathbb{R})$ continuous functions of square integrable on I (with I any real interval) endowed with the scalar product*

$$(f.g) \longmapsto < f|g >= \int_I fg,$$

then

$$\forall (f.g) \in (L^2(I;\mathbb{R}))^2, \quad \left|\int_I fg\right| \leq \left(\int_I f^2\right)^{1/2} . \quad \left(\int_I g^2\right)^{1/2} .$$

The Young inequality

Theorem 1.6. *[11] Let a, $b \in \mathbb{R}_+$ and p, $q \in]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The generalized Young inequality

Theorem 1.7. *[11]Let a, $b \in \mathbb{R}_+$ and for all $\varepsilon > 0$, we have :*

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

The Holder inequality

Theorem 1.8. *[11]Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$fg \in L^1(\Omega) \text{ and } \|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

The Minkowski inequality

Theorem 1.9. *[11]Let $f, g \in L^p(\Omega)$ and $p \geq 1$ then*

$$f + g \in L^p(\Omega) \text{ and } \|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Lemma 1.1. *[17] Let E a Banach space and F a Hilbert space, which $E \subset F$ with continuous injection, let E dense in F, We identify F with F' ($E \subset F = F' \subset E'$). Let $u \in L^2_E(]0, T[)$. We assume thet $u_t \in L^2_{E'}(]0, T[)$. Then $u \in C([0, T]; F)$, and for all t_1, $t_2 \in [0, T]$ we have:*

$$\|u(t_1)\|^2_F - \|u(t_2)\|^2_F = 2 \int_{t_2}^{t_1} < u_t, u >_{E',E} dt.$$

The Gronwall's lemma

Gronwall's lemma plays a major role in estimating integro-differential terms and is frequently used to obtain a priori estimates in the norms of the aforementioned spaces and others.

Lemma 1.2. *[**The Gronwall's lemma**][33] If a, b are non-negative and integrable functions on $(0,T)$, where the function b is not-decreasing on $(0,T)$, and $\lambda \in L^1(0,T)$, $\lambda > 0$, it follows from:*

$$a(t) \leq b(t) + \int_0^t \lambda(s)a(s)ds,$$

then

$$a(t) \leq b(t)\exp(\Lambda(t)),$$

where

$$\Lambda(t) = \int_0^t \lambda(s)ds.$$

1.2 Fractional calculus mathematical basis

1.2.1 The specific function for fractional derivation

The Gamma function

The Gamma function is in mathematics, a complex function, also considered as a special function. It extends the factorial function to all complex numbers (except at certain points).

Definition 1.4. *[19] For $\alpha \in \mathbb{C}$, such that $Re(\alpha) > 0$, the Gamma function given by:*

$$\Gamma \ : \ \alpha \longrightarrow \int_0^{+\infty} \exp^{(-t)} t^{\alpha-1}dt.$$

With $\Gamma(1) = 1$, $\Gamma(0_+) = +\infty$, where this integral converges absolutely on the complex half-plane where the real part is strictly positive.
Integrating by parts, we can see that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \ \ Re(\alpha) > 0.$$

Especially

$$\Gamma(n+1) = n!, \ \ \forall n \in \mathbb{N}.$$

The Bêta function

The Bêta function (which is a type of Euler integral, just like the Gamma function) is a function defined by:

Definition 1.5. *[19] For p,$q \in \mathbb{C}$, such that $Re(p) > 0$, $Re(q) > 0$, the Bêta function given by:*

$$B(p,q) = \int_0^1 \tau^{p-1}(\tau-1)^{q-1}d\tau.$$

The links between the Gamma function and the Beta function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\ Re(p) > 0,\ Re(q) > 0.$$

1.2.2 The fractional derivatives

The Grnwald-Letnikov derivative

Definition 1.6. *[19] Let $\alpha > 0$. The Grnwald-Letnikov fractional derivative of the order α is given by*

$$\forall t \in \mathbb{R}, \quad {}_a^G\mathcal{D}_t^\alpha f(t) = \lim_{h \longrightarrow 0^+} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh).$$

The Riemann-Liouville derivative

Definition 1.7. *[19] Let $\alpha > 0$ and $n = [\alpha] + 1$. The Riemann-Liouville fractional derivative of the order α is given by*

$$\forall t \in [a,b], \quad {}_a\mathcal{D}_t^\alpha f(t) = \frac{1}{\mathit{\Gamma}(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-1-\alpha} f(\tau) d\tau.$$

The Caputo derivative

Definition 1.8. *[19] Let $\alpha > 0$ and $n = [\alpha] + 1$. The Caputo derivative of the order α is given by*

$$\forall t \in [a,b], \quad {}_a^c\mathcal{D}_t^\alpha f(t) = \frac{1}{\mathit{\Gamma}(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau) d\tau.$$

Chapter 2

Existence results for quasilinear parabolic systems with nonlocal boundary conditions

2.1 Introduction

This chapter present the results of the generalized solution to the class of quasilinear parabolic system with nonlocal boundary conditions by constructing approximate solution using Feado-Galerkin method. The proposed model is an inerpolation of the Chen result in 2011 [13], where the Chen model given by:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(|u|^{p-2}\frac{\partial u}{\partial x_i}\big) + |u|^{p-2}u = f(x,t) \quad \text{in } Q_T, \tag{2.1}$$

$$u(x,t) = \int_\Omega k(x,y)u(y,t)dy \quad \text{in } \Sigma_T, \tag{2.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{2.3}$$

This study is the extension of the problem in Lion's book [[24], p.140] in which the boundary conditions are homegeneous, but the main difficulty of there problem is related to the presence of both quasilinear term and nonlocal boundary condition, where the quasilinear term makes it difficult to apply classical methods like semi group method or method of upper and lower solutions, However, they used the Faedo-Galerkin approximation to prove the existence of a generalized solution for the problem, in which this kind of problem are very limited, where we only found a paper given in 1999 [15], which the authors study quasilinear parabolic equation with nonlocal boundary conditions different from Chen condition (2.2).
This chapter, is organized as follows. In second section, we define the proposed model with some assumptions using in this study. And the last section we prove the existence of the solution of the problem, where we state the art method for $PDEs$ with the definition of the generalized solution, then we demonstrate the construction of an approximate solution and we

derive a priori estimates for the approximate solution. Finally, we show the convergence of the approximate solution.

2.2 The proposed model

In this section, we define the proposed model considered this chapter as follow:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(|u|^{\rho}\frac{\partial u}{\partial x_i}\big) + |v|^{\rho} v = f_1(x,t) \quad \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(|v|^{\rho}\frac{\partial v}{\partial x_i}\big) + |u|^{\rho} u = f_2(x,t) \quad \text{in } Q_T \\ u(x,t) = \displaystyle\int_{\Omega} k(x,y)u(y,t)dy \quad \text{in } \Sigma_T \\ v(x,t) = \displaystyle\int_{\Omega} k(x,y)v(y,t)dy \quad \text{in } \Sigma_T \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } \Omega. \end{cases} \tag{2.4}$$

Where Ω is a regular and bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, T is a fixed real number ($T > 0$) and $\rho = p-2$. Where $(u,v)(x,t)$ are the solution of this system.

In this study, we need the following assumptions:

(H_1) $n \geq 2,\ p > n,\ r > \dfrac{n}{2} + 2.$

(H_2) $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

(H_3) $f = (f_1, f_2) \in L^q(0,T,L^q(\Omega))^2$ and u_0, v_0 are in $L^{\infty}(\Omega)$.

(H_4) For any $x \in \partial\Omega,\ K(x) < \infty$ and $K_i(x) < \infty$.

(H_5) $\displaystyle\sum_{i=1}^{n} \int_{\partial\Omega} K(x)^{p-1} K_i(x) d\partial\Omega < 1 - \frac{1}{p}.$

2.3 The Main of Results

In this part, we explore the notation of generalized solutions as a definition, and present the approximate solutions and a priori estimations, then show that the approximate solution converges to the issue solution. In the beginning, let us express clearly our concept to a generalized

solution of the problem (2.4), we define the following space:
$\mathrm{U} = \{\forall(\psi,\varphi) \in (H^r(\Omega))^2,\ \psi(x) = \int_\Omega k(x,y)\psi(y)dy \text{ and } \varphi(x) = \int_\Omega k(x,y)\varphi(y)dy, \text{ for } x \in \partial\Omega\}.$

Definition 2.1. *Let (u,v) be the generalized solution of the problem (2.4) if:*
(i) $(u,v) \in (L^\infty(0,T,L^2(\Omega)))^2 \cap (C(0,T,H^{-r}(\Omega)))^2$,
(ii) $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in (L^q(0,T,H^{-r}(\Omega)))^2$,
(iii) $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$,
(iv) for all $(\psi,\varphi) \in U$ *and a.e.* $t \in [0,T]$,

$$\begin{aligned}
&\int_\Omega \frac{\partial u}{\partial t}\psi dx + \int_\Omega \frac{\partial v}{\partial t}\varphi dx - \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|u|^{p-2}\frac{\partial u}{\partial x_i}\big)\psi dx \\
&- \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|v|^{p-2}\frac{\partial v}{\partial x_i}\big)\varphi dx + \int_\Omega |v|^{p-2}v\psi dx \\
&+ \int_\Omega |u|^{p-2}u\varphi dx = \int_\Omega f_1(x,t)\psi dx + \int_\Omega f_2(x,t)\varphi dx,
\end{aligned} \tag{2.5}$$

2.3.1 The approximate solution and a priori estimates

It is easy to see that U is a subspace of $(H^r(\Omega))^2$, which is separable, then we can choose a countable set of distinct basis elements $(w_j, \widetilde{w}_j)$, where $j = 1, 2, \ldots$, which generate U and are orthonormal in W. Let U_m be the subspace of U generated by the first m elements, we constuct the approximate solution of the problem in the following form:

$$\begin{cases} u_m(x,t) = \sum_{j=1}^m g_{jm}(t)w_j(x), \quad (x,t) \in \Omega \times [0,T], \\ \\ v_m(x,t) = \sum_{j=1}^m h_{jm}(t)\widetilde{w}_j(x), \quad (x,t) \in \Omega \times [0,T], \end{cases} \tag{2.6}$$

where $\big((g_{jm}(t))_{j=1}^m, (h_{jm}(t))_{j=1}^m\big)$ remains to be determined.
Let the orthogonal projection of (u_0, v_0) on U_m given by $(u_m^0, v_m^0) = (P\mathrm{U}_m u_0, P\mathrm{U}_m v_0)$, then $(u_m^0, v_m^0) \longrightarrow (u_0, v_0)$, as $m \longrightarrow \infty$ in U. Let $\big((g_{jm}^0)_{j=1}^m, (h_{jm}^0)_{j=1}^m\big)$ be the coordinate of (u_m^0, v_m^0) in the basis $\big((w_j)_{j=1}^m, (\widetilde{w}_j(t))_{j=1}^m\big)$ of U_m, such as

$$\begin{cases} u_m^0 = \sum_{j=1}^m g_{jm}^0 w_j(x), \text{ where } g_{jm}^0 = g_{jm}(0), \\ \\ v_m^0 = \sum_{j=1}^m h_{jm}^0 \widetilde{w}_j(x), \text{ where } h_{jm}^0 = h_{jm}(0). \end{cases}$$

Now, we have to determinate $\big((g_jm(t))_{j=1}^m, (h_jm(t))_{j=1}^m\big)$ to satisfy

$$\begin{aligned}
&\int_\Omega \frac{\partial u_m}{\partial t} w_j dx - \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big) w_j dx \\
+ \ &\int_\Omega |v_m|^{p-2} v_m w_j dx = \int_\Omega f_1(x,t) w_j dx, \quad 1 \le j \le m, \\
&\int_\Omega \frac{\partial v_m}{\partial t} \widetilde{w}_j dx - \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big) \widetilde{w}_j dx \\
+ \ &\int_\Omega |u_m|^{p-2} u_m \widetilde{w}_j dx = \int_\Omega f_2(x,t) \widetilde{w}_j dx, \quad 1 \le j \le m.
\end{aligned} \tag{2.7}$$

The system (2.7) is a system of ordinary differential equations in $g_{jm}(t)$ respectively $h_{jm}(t)$, by Caratheodory theorem [14]. There exists solution $(g_{jm}(t), h_{jm}(t))_{j=1}^m$, $t \in [0, t_m)$.
by multiply both sides of (2.7) by $g_{jm}(t)$ and $h_{jm}(t)$ respectively, then sum over j from 0 to m, getting:

$$\begin{aligned}
&\int_\Omega \frac{\partial u_m}{\partial t} u_m dx + \int_\Omega \frac{\partial v_m}{\partial t} v_m dx - \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big) u_m dx \\
&- \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big) v_m dx + \int_\Omega |v_m|^{p-2} v_m u_m dx \\
&+ \int_\Omega |u_m|^{p-2} u_m v_m dx = \int_\Omega f_1(x,t) u_m dx + \int_\Omega f_2(x,t) v_m dx, \quad 1 \le j \le m.
\end{aligned} \tag{2.8}$$

Integrating by parts (2.8), we have:

$$\begin{aligned}
&\int_\Omega \frac{\partial u_m}{\partial t} u_m dx + \int_\Omega \frac{\partial v_m}{\partial t} v_m dx + \sum_{i=1}^n \int_\Omega \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u_m}{\partial x_i} dx \\
&+ \sum_{i=1}^n \int_\Omega \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\frac{\partial v_m}{\partial x_i} dx + \int_\Omega |v_m|^{p-2} v_m u_m dx \\
&+ \int_\Omega |u_m|^{p-2} u_m v_m dx = \int_\Omega f_1(x,t) u_m dx + \int_\Omega f_2(x,t) v_m dx \\
&+ \sum_{i=1}^n \int_{\partial\Omega} \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big) u_m d\partial\Omega + \sum_{i=1}^n \int_{\partial\Omega} \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big) v_m d\partial\Omega.
\end{aligned} \tag{2.9}$$

Integrating with respect to t from 0 to T on the both sides of (2.9), to get:

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u_m}{\partial t}u_m dxdt + \int_0^T\int_\Omega \frac{\partial v_m}{\partial t}v_m dxdt + \int_0^T\sum_{i=1}^n\int_\Omega \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u_m}{\partial x_i}dxdt \\
&+\int_0^T\sum_{i=1}^n\int_\Omega \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\frac{\partial v_m}{\partial x_i}dxdt + \int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt \\
&+\int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt = \int_0^T\int_\Omega f_1(x,t)u_m dxdt + \int_0^T\int_\Omega f_2(x,t)v_m dxdt \\
&+\int_0^T\sum_{i=1}^n\int_{\partial\Omega} \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\partial\Omega dt + \int_0^T\sum_{i=1}^n\int_{\partial\Omega} \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\partial\Omega dt,
\end{aligned} \tag{2.10}$$

then

$$\begin{aligned}
&\frac{1}{2}\Big[\|u_m(T)\|^2_{L^2(\Omega)} + \|v_m(T)\|^2_{L^2(\Omega)}\Big] + \frac{4}{p^2}\Big[\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\big)^2 dxdt \\
+\ &\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\big)^2 dxdt\Big] + \int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt \\
+\ &\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt = \int_0^T\int_\Omega f_1(x,t)u_m dxdt + \int_0^T\int_\Omega f_2(x,t)v_m dxdt \\
+\ &\int_0^T\int_{\partial\Omega}\sum_{i=1}^n\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\partial\Omega dt + \int_0^T\int_{\partial\Omega}\sum_{i=1}^n\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\partial\Omega dt \\
+\ &\frac{1}{2}\Big[\|u_m(0)\|^2_{L^2(\Omega)} + \|v_m(0)\|^2_{L^2(\Omega)}\Big],
\end{aligned}$$

this gives

$$\begin{aligned}
&\|(u_m(T),v_m(T)\|^2_{\mathrm{W}} + \int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt + \int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt \\
+\ &\frac{4}{p^2}\Big[\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\big)^2 dxdt + \int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\big)^2 dxdt\Big] \\
\leq\ &\int_0^T\int_\Omega f(x,t)(u_m,v_m)dxdt + \int_0^T\int_{\partial\Omega}\sum_{i=1}^n\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\partial\Omega dt \\
+\ &\int_0^T\int_{\partial\Omega}\sum_{i=1}^n\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\partial\Omega dt + \|(u_m(0),v_m(0))\|^2_{\mathrm{W}}.
\end{aligned} \tag{2.11}$$

We derive some a priori estimates for the approximate solution as follows:

The first term in the right-hand side of (2.11) can be estimated as follows:

$$\begin{aligned}
|\int_0^T\int_\Omega(f_1(x,t),f_2(x,t))(u_m,v_m)dxdt| &\le \int_0^T\int_\Omega |f||(u_m,v_m)|dxdt\\
&\le \int_0^T\Big[\big(\int_\Omega |f|^q dx\big)^{1/q}\big(\int_\Omega |(u_m,v_m)|^p dx\big)^{1/p}\Big]dt\\
&\le \int_0^T \|f\|_{\mathrm{Y}}\|(u_m,v_m)\|_{\mathrm{V}}dt\\
&\le \frac{1}{p}\int_0^T \|(u_m,v_m)\|_{\mathrm{V}}^p dt+\frac{p-1}{p}\int_0^T \|f\|_{\mathrm{Y}}^{\frac{p}{p-1}}dt,
\end{aligned}$$

hence

$$|\int_0^T\int_\Omega(f_1(x,t),f_2(x,t))(u_m,v_m)dxdt| \le \frac{1}{p}\int_0^T \|(u_m,v_m)\|_{\mathrm{V}}^p dt+\frac{p-1}{p}\int_0^T \|f\|_{\mathrm{Y}}^{\frac{p}{p-1}}dt. \quad (2.12)$$

For the second and the third terms in the right-hand side of (2.11), we have for $x\in\partial\Omega$:

$$\begin{cases}|u_m(x,t)|\le K(x)\|u_m\|_{L^p(\Omega)},\\ \\ |v_m(x,t)|\le K(x)\|v_m\|_{L^p(\Omega)},\end{cases}$$

and

$$\begin{cases}|\frac{\partial}{\partial x_i}u_m(x,t)|\le K_i(x)\|u_m\|_{L^p(\Omega)},\\ \\ |\frac{\partial}{\partial x_i}v_m(x,t)|\le K_i(x)\|v_m\|_{L^p(\Omega)}.\end{cases}$$

Using Hôlder's inequality and the assumptions (H_4) and (H_5), we have:

$$\begin{aligned}
|\int_0^T\sum_{i=1}^n\int_{\partial\Omega}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\partial\Omega dt| &\le \int_0^T\sum_{i=1}^n\int_{\partial\Omega}|u_m|^{p-1}|\frac{\partial u_m}{\partial x_i}|d\partial\Omega dt\\
&\le \int_0^T\Big[\sum_{i=1}^n\int_{\partial\Omega}K(x)^{p-1}K_i(x)d\partial\Omega\Big]\|u_m\|_{L^p(\Omega)}^p dt,
\end{aligned}$$

hence

$$|\int_0^T\sum_{i=1}^n\int_{\partial\Omega}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\partial\Omega dt+\int_0^T\sum_{i=1}^n\int_{\partial\Omega}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\partial\Omega dt|\le c\int_0^T\|(u_m,v_m)\|_V^p dt, \quad (2.13)$$

where $c=2\sum_{i=1}^n\int_{\partial\Omega}K(x)^{p-1}K_i(x)d\partial\Omega<2(1-\frac{1}{p})$

For the second and the third terms in the left-hand side of (2.11), we have:

$$\begin{aligned}
|\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt| &\leq \int_0^T\int_\Omega |v_m|^{p-1}|u_m|dxdt \\
&\leq \int_0^T\Big[\big(\int_\Omega |v_m^{p-1}|^q dx\big)^{\frac{1}{q}}\int_\Omega |u_m|^p dx\big)^{\frac{1}{p}}\Big]dt \\
&\leq \int_0^T \|v_m\|_{L^p(\Omega)}^{\frac{p}{q}}\|u_m\|_{L^p(\Omega)}dt, \\
&\leq \frac{p-1}{p}\int_0^T \|v_m\|_{L^p(\Omega)}^p dt + \frac{1}{p}\int_0^T \|u_m\|_{L^p(\Omega)}^p dt,
\end{aligned}$$

hence

$$|\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt + \int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt| \leq 2\int_0^T \|(u_m,v_m)\|_{\mathrm{V}}^p dt. \tag{2.14}$$

From the form (2.12), (2.13) and (2.14), we have:

$$\begin{aligned}
&\|(u_m(T),v_m(T)\|_{\mathrm{W}}^2 + \frac{4}{p^2}\Big[\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\big)^2 dxdt \\
+ &\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\big)^2 dxdt\Big] + C\int_0^T \|(u_m,v_m)\|_{\mathrm{V}}^p dt \\
\leq &\frac{p-1}{p}\int_0^T \|f\|_{\mathrm{Y}}^{\frac{p}{p-1}}dt + \|(u_m(0),v_m(0))\|_{\mathrm{W}}^2,
\end{aligned} \tag{2.15}$$

where $C = 2 + c + \frac{1}{p}$.
From the above estimates (2.12), (2.13) and (2.14) and for any finite $T > 0$, we have the following a priori estimates:
(i) (u_m, v_m) are bounded in $(L^\infty(0,T,L^2(\Omega)))^2$.
(ii) $|u_m|^{\frac{p-2}{2}}u_m$ and $|v_m|^{\frac{p-2}{2}}v_m$ are bounded in $L^2(0,T,H^1(\Omega))$.
(iii) (u_m, v_m) are bounded in $(L^p(0,T,L^p(\Omega)))^2$.
Since T is an artbitrary positive number, we have

$$\|(u_m,v_m)\|_{\mathrm{V}}^p < \infty, \quad a.e.t \tag{2.16}$$

2.3.2 The existence of a generalized solution

The main focus of this part is to prove existence of a generalized solution. First, we have to prove the following lemma.

Lemma 2.1. *Let* (u_m, v_m) *be the approximate solution of the problem* (2.4) *in the sense of the definition* (2.1), *then* $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ *are bounded in* $(L^q(0,T,H^{-r}(\Omega)))^2$.

Proof. Let U is a subset of $H^r(\Omega)$ and for $(\psi,\varphi)\in$ U satisfy:

$$\begin{aligned}&\int_\Omega \frac{\partial u_m}{\partial t}\psi dx+\int_\Omega \frac{\partial v_m}{\partial t}\varphi dx+\sum_{i=1}^n\int_\Omega \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial \psi}{\partial x_i}dx\\&+\sum_{i=1}^n\int_\Omega \big(|v_m|^{p-2}\frac{\partial \varphi}{\partial x_i}\big)\frac{\partial v_m}{\partial x_i}dx+\int_\Omega |v_m|^{p-2}v_m\psi dx\\&+\int_\Omega |u_m|^{p-2}u_m\varphi dx=\int_\Omega f_1(x,t)\psi dx+\int_\Omega f_2(x,t)\varphi dx\\&+\sum_{i=1}^n\int_{\partial\Omega}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\psi d\partial\Omega+\sum_{i=1}^n\int_{\partial\Omega}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\varphi d\partial\Omega.\end{aligned}\tag{2.17}$$

Using Hôlder inequality to estimate the two terms in the left-hand side in the last of (2.17), this give:

$$\begin{aligned}|\int_\Omega |v_m|^{p-2}v_m\psi dx+\int_\Omega |u_m|^{p-2}u_m\varphi dx| &\le \|v_m\|^{\frac{p}{q}}_{L^p(\Omega)}|\psi|_{L^p(\Omega)}+\|u_m\|^{\frac{p}{q}}_{L^p(\Omega)}\|\varphi\|_{L^p(\Omega)}\\ &\le 2\|(u_m,v_m)\|^{\frac{p}{q}}_{\mathrm{V}}\|(\psi,\varphi)\|_{\mathrm{V}}.\end{aligned}$$

Since $H^r(\Omega)\hookrightarrow L^p(\Omega)$, then

$$|\int_\Omega |v_m|^{p-2}v_m\varphi dx+\int_\Omega |u_m|^{p-2}u_m\varphi dx|\le c_1\|(u_m,v_m)\|^{\frac{p}{q}}_{\mathrm{V}}\|(\psi,\varphi)\|_{(H^r(\Omega))^2},\quad c_1>0.$$

Then

$$\||v_m|^{p-2}v_m\|_{H^{-r}(\Omega)}+\||u_m|^{p-2}u_m\|_{H^{-r}(\Omega)}\le c_1\|(u_m,v_m)\|^{\frac{p}{q}}_{\mathrm{V}},\quad c_1>0.$$

The norm of $|v_m|^{p-2}v_m$ and $|u_m|^{p-2}u_m$ in $L^q(0,T,H^{-r}(\Omega))$ are bounded by:

$$\Big(\int_0^T (c_1(\|(u_m,v_m)\|^p_{\mathrm{V}})^{1/q})^q dt\Big)^{\frac{1}{q}}=\Big(\int_0^T c_1^q\|(u_m,v_m)\|^p_{\mathrm{V}}dt\Big)^{\frac{1}{q}}<\infty,\tag{2.18}$$

Therefore, $|u_m|^{p-2}u_m$ and $|v_m|^{p-2}v_m$ are bounded in $L^q(0,T,H^{-r}(\Omega))$.
Next, we estimate the last two terms in the right-hand side of (2.17) by:

$$\big(a(u_m,v_m),(\psi,\varphi)\big)=\int_{\partial\Omega}\sum_{i=1}^n\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\psi d\partial\Omega+\int_{\partial\Omega}\sum_{i=1}^n\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\varphi d\partial\Omega,$$

$$\begin{aligned}
|\big(a(u_m,v_m),(\psi,\varphi)\big)| &= |\int_{\partial\Omega}\sum_{i=1}^{n}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\psi d\partial\Omega+\int_{\partial\Omega}\sum_{i=1}^{n}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\varphi d\partial\Omega| \\
&\leq \sum_{i=1}^{n}\|(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i})\|_{L^q(\partial\Omega)}\|\psi\|_{L^p(\partial\Omega)}+\sum_{i=1}^{n}\|(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i})\|_{L^q(\partial\Omega)}\|\varphi\|_{L^p(\partial\Omega)} \\
&\leq \sum_{i=1}^{n}\|K(x)^{p-2}K_i(x)\|_{L^q(\partial\Omega)}\|K(x)\|_{L^p(\partial\Omega)}\|u_m\|^{p-1}_{L^p(\Omega)}\|\psi\|_{L^p(\Omega)} \\
&+ \sum_{i=1}^{n}\|K(x)^{p-2}K_i(x)\|_{L^q(\partial\Omega)}\|K(x)\|_{L^p(\partial\Omega)}\|v_m\|^{p-1}_{L^p(\Omega)}\|\varphi\|_{L^p(\Omega)} \\
&\leq 2\sum_{i=1}^{n}\|K(x)^{p-2}K_i(x)\|_{L^q(\partial\Omega)}\|K(x)\|_{L^p(\partial\Omega)}\|(u_m,v_m)\|^{p-1}_{\mathrm{V}}\|(\psi,\varphi)\|_{\mathrm{V}} \\
&\leq 2\sum_{i=1}^{n}\|K(x)^{p-2}K_i(x)\|_{L^q(\partial\Omega)}\|K(x)\|_{L^p(\partial\Omega)}\|(u_m,v_m)\|^{p-1}_{\mathrm{V}}c_1\|(\psi,\varphi)\|_{(H^r(\Omega))^2}.
\end{aligned}$$

Hence

$$\|a(u_m,v_m)\|_{(H^{-r}(\Omega))^2}\leq 2\sum_{i=1}^{n}\|K(x)^{p-2}K_i(x)\|_{L^q(\partial\Omega)}\|K(x)\|_{L^p(\partial\Omega)}\|(u_m,v_m)\|^{p-1}_{\mathrm{V}}c_1<\infty.$$

The norm of $a(u_m,v_m)$ in $(L^q(0,T,H^{-r}(\Omega)))^2$ is bounded by

$$\Big(2\int_0^T\sum_{i=1}^{n}\big(\|K(x)^{p-2}K_i(x)\|_{L^q(\partial\Omega)}\|K(x)\|_{L^p(\partial\Omega)}c_1\big)^q\|(u_m,v_m)\|^p_{\mathrm{V}}dt\Big)^{\frac{1}{q}}<\infty, \tag{2.19}$$

Therefore, $a((u_m,v_m))$is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

Next, we consider the third and the fourth terms in the left-hand side of (2.17) and by integrating by parts, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Omega}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial\psi}{\partial x_i}dx+\sum_{i=1}^{n}\int_{\Omega}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\frac{\partial\varphi}{\partial x_i}dx \\
&=\frac{1}{p-1}\Big(\sum_{i=1}^{n}\int_{\partial\Omega}|u_m|^{p-2}u_m\frac{\partial\psi}{\partial x_i}d\partial\Omega-\int_{\Omega}|u_m|^{p-2}u_m\Delta\psi dx \qquad (2.20)\\
&+\sum_{i=1}^{n}\int_{\partial\Omega}|v_m|^{p-2}v_m\frac{\partial\varphi}{\partial x_i}d\partial\Omega-\int_{\Omega}|v_m|^{p-2}v_m\Delta\varphi dx\Big).
\end{aligned}$$

Let

$$\big(b(u_m,v_m),(\psi,\varphi)\big)=\sum_{i=1}^{n}\int_{\partial\Omega}|u_m|^{p-2}u_m\frac{\partial\psi}{\partial x_i}d\partial\Omega+\sum_{i=1}^{n}\int_{\partial\Omega}|v_m|^{p-2}v_m\frac{\partial\varphi}{\partial x_i}d\partial\Omega,$$

$$\begin{aligned}
|\big(b(u_m,v_m),(\psi,\varphi)\big)| &= |\sum_{i=1}^{n}\int_{\partial\Omega}|u_m|^{p-2}u_m\frac{\partial\psi}{\partial x_i}d\partial\Omega+\sum_{i=1}^{n}\int_{\partial\Omega}|v_m|^{p-2}v_m\frac{\partial\varphi}{\partial x_i}d\partial\Omega| \\
&\leq 2\sum_{i=1}^{n}\|K(x)^{p-1}K_i(x)\|_{L^q(\partial\Omega)}\|(u_m,v_m)\|_{\mathrm{V}}^{p-1}\|(\psi,\varphi)\|_{\mathrm{V}} \\
&\leq \sum_{i=1}^{n}\|K(x)^{p-1}K_i(x)\|_{L^q(\partial\Omega)}\|(u_m,v_m)\|_{\mathrm{V}}^{p-1}c_1\|(\psi,\varphi)\|_{(H^r(\Omega))^2}.
\end{aligned}$$

Then

$$\|b(u_m,v_m)\|_{(H^{-r}(\Omega))^2}\leq 2\sum_{i=1}^{n}\|K(x)^{p-1}\|_{L^q(\partial\Omega)}\|K_i(x)\|_{L^p(\partial\Omega)}\|(u_m,v_m)\|_{\mathrm{V}}^{p-1}c_1<\infty.$$

The norm of $b(u_m,v_m)$ in $(L^q(0,T,H^{-r}(\Omega)))^2$ is bounded by

$$\left(2\int_0^T\sum_{i=1}^{n}\big(\|K(x)^{p-1}\|_{L^q(\partial\Omega)}\|K_i(x)\|_{L^p(\partial\Omega)}c_1\big)^q\|(u_m,v_m)\|_{\mathrm{V}}^{p}dt\right)^{\frac{1}{q}}<\infty. \tag{2.21}$$

Hence $b(u_m,v_m)$is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

Next, we consider

$$\big(\tilde{b}(u_m,v_m),(\psi,\varphi)\big)=\int_\Omega|u_m|^{p-2}u_m\Delta\psi dx+\int_\Omega|v_m|^{p-2}v_m\Delta\varphi dx,$$

$$\begin{aligned}
|\big(\tilde{b}(u_m,v_m),(\psi,\varphi)\big) &= |\int_\Omega|u_m|^{p-2}u_m\Delta\psi dx+\int_\Omega|v_m|^{p-2}v_m\Delta\varphi dx| \\
&\leq \int_\Omega|u_m|^{p-1}|\Delta\psi|dx+\int_\Omega|v_m|^{p-1}|\Delta\varphi|dx \\
&\leq \|u_m\|_{L^p(\Omega)}^{p-1}\|\Delta\psi\|_{L^p(\Omega)}+\|v_m\|_{L^p(\Omega)}^{p-1}\|\Delta\varphi\|_{L^p(\Omega)} \\
&\leq \|(u_m,v_m)\|_{\mathrm{V}}^{\frac{p}{q}}\|(\Delta\psi,\Delta\varphi)\|_{\mathrm{V}};
\end{aligned}$$

From the proof of [[24], Theorem 12.2 p 140] we have

$$|\big(\tilde{b}(u_m,v_m),(\psi,\varphi)\big)|\leq c_1\|(u_m,v_m)\|_{\mathrm{V}}^{\frac{p}{q}}\|(\psi,\varphi)\|_{(H^r(\Omega))^2}<\infty. \tag{2.22}$$

Therefore

$$\left(\int_0^T\|\tilde{b}(u_m,v_m)\|_{(H^{-r}(\Omega))^2}^{q}dt\right)^{\frac{1}{q}}\leq\left(\int_0^T c_1^q\|(u_m,v_m)\|_{\mathrm{V}}^{p}dt\right)^{\frac{1}{q}}<\infty. \tag{2.23}$$

So, we have $\tilde{b}(u_m,v_m)$ is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.
Finally, we consider

$$\begin{aligned}
|\int_\Omega(f_1(x,t),f_2(x,t))(\psi,\varphi)dx| &= \int_\Omega|f(x,t)||(\psi,\varphi)|dx \\
&\leq \|f\|_{\mathrm{Y}}\|(\psi,\varphi)\|_{\mathrm{V}} \\
&\leq \|f\|_{\mathrm{Y}}c_1\|(\psi,\varphi)\|_{H^r(\Omega)}.
\end{aligned}$$

Then

$$\|f\|_{(H^{-r}(\Omega))^2} \leq c_1\|f\|_{\mathrm{Y}} < \infty. \tag{2.24}$$

The norm of f in $(L^q(0,T,H^{-r}(\Omega)))^2$ is bounded by

$$\Big(\int_0^T \|f\|^q_{(H^{-r}(\Omega))^2} dt\Big)^{\frac{1}{q}} \leq \Big(\int_0^T c_1^q\|f\|^q_{\mathrm{Y}} dt\Big)^{\frac{1}{q}} < \infty. \tag{2.25}$$

Then, we have that f is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

From the result (2.18), (2.19), (2.21), (2.23) and (2.25), we have $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ are bounded in $L^q(0,T,H^{-r}(\Omega))^2$ □

2.3.3 The convergence

From the lemmas 4.3 to lemma 4.7 of [13], we quote the following lemmas.

Lemma 2.2. *Let (u_m, v_m) be the approximate solution of (2.4), constructed as in (2.6), then*

$$(u_m, v_m) \to (u, v),$$

in $(L^p(0,T,L^p(\Omega)))^2$ strongly and almost everywhere.

Proof. We quote the Theorem here from the Lemma (2.1) and the use of [[24], Theorem 12.1] to prove the previous Lemma.

Theorem 1. *Let B, B_1 be the Banach spaces, and S be a set, we define:*

$$M(\varphi_1,\varphi_2) = \Big(\sum_{i=1}^n \int_\Omega |\varphi_1|^{p-2}(\frac{\partial\varphi_1}{\partial x_i})^2 dx\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^n \int_\Omega |\varphi_2|^{p-2}(\frac{\partial\varphi_2}{\partial x_i})^2 dx\Big)^{\frac{1}{p}},$$

On S with:
(a)− $S \subset B \subset B_1$, and $M(\varphi_k) \geq 0$ on S, $M(\lambda_k\varphi_k) = |\lambda_k|M(\varphi_k)$, where $k = 1,2$.
(b)− The set $\{\varphi_k|\varphi_k \in S, M(\varphi_k) \leq 1\}$ is relatively compact in B.
Define the set $F = \{\varphi_k : \varphi_k$ are locally summable on $[0,T]$ with value in B_1, $\int_0^T \big(M(\varphi_k(t))\big)^{p_0} dt \leq C, \varphi_k'$ bounded in $(L^{p_1}(0,T,B_1))^2\}$, where $1 < p_j < \infty$, $j = 0,1$ and $k = 1,2$. Then $F \subset L^{p_0}(0,T,B)$ and F is relatively compact in $L^{p_0}(0,T,B)$.

We define the set S as follows:
$S = \{\varphi_k : |\varphi_k|^{\frac{p-2}{2}}\varphi_k \in H^1(\Omega)$, where $k = 1,2\}$.
$H^1(\Omega)$ is also compactly embedded in $L^2(\Omega)$, with the proof of [Proposition 12.1 p 143, [24]], we have (b) in the Theorem (1).
From the Lemma(2.1), $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ are bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$, then we have to prove now that

$$\int_0^T \big(M(u_m(t), v_m(t)\big)^{p_0} dt \leq c, \quad c > 0.$$

Let $B = L^p(\Omega)$, $B_1 = H^{-r}(\Omega)$, $p_0 = p$ and $p_1 = q$ then

$$\int_0^T \big(M(u_m(t), v_m(t))\big)^{p_0} dt = \frac{4}{p^2}\Big[\int_0^T \sum_{i=1}^n \int_\Omega \Big(\frac{\partial}{\partial x_i}\big(|u_m|^{\frac{p-2}{2}} u_m\big)\Big)^2 dxdt + \sum_{i=1}^n \int_\Omega \Big(\frac{\partial}{\partial x_i}\big(|v_m|^{\frac{p-2}{2}} v_m\big)\Big)^2 dxdt\Big].$$

Let $|u_m|^{\frac{p-2}{2}} u_m$ and $|v_m|^{\frac{p-2}{2}} v_m$ are bounded in $L^2(0, T, H^1(\Omega))$, so

$$\int_0^T \big(M(u_m(t), v_m(t))\big)^{p_0} dt < \infty.$$

Therefore, conclusion follows easily from application of Theorem(1), $F \subset L^p(0, T, L^p(\Omega))$ and F is relatively compact in $L^p(0, T, L^p(\Omega))$. □

Next, we prove that we can pass to the limit on (2.17).

Lemma 2.3. *Let (u_m, v_m) be the approximate solution of* (2.4)*, constructed as in* (2.6)*, then*

$$(|u_m|^{p-2}u_m + |v_m|^{p-2}v_m, \varphi) \to (|u|^{p-2}u + |v|^{p-2}v, \varphi), \quad m \to \infty.$$

Proof. To prove this Lemma, we need to show that $|u_m|^{p-2}u_m \rightharpoonup |u|^{p-2}u$ and $|v_m|^{p-2}v_m \rightharpoonup |v|^{p-2}v$ weakly in $L^q(\Omega)$, this is a consequence of [[24], Lemma 1.3].
First, let prove that.

$$\begin{aligned} \||u_m|^{p-2}u_m\|_{L^q(\Omega)} &\le \Big(\int_\Omega (|u_m|^{p-1})^q dx\Big)^{\frac{1}{q}} \\ &\le \Big(\int_\Omega |u_m|^p dx\Big)^{\frac{1}{q}} \\ &\le \big(\|u_m\|^p_{L^p(\Omega)}\big)^{\frac{1}{q}}. \end{aligned}$$

Then, from the Lemma (2.2) we have $|u_m|^{p-2}u_m \to |u_m|^{p-2}u_m$ almost everywhere, for $x \in \Omega$. Similarly for $|v_m|^{p-2}v_m$.

□

Lemma 2.4. *Let (u_m, v_m) be the approximate solution of* (2.4)*, constructed as in* (2.6)*, then*

$$\int_{\partial\Omega} \Big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i} + |v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\Big)\varphi d\partial\Omega \to \int_{\partial\Omega} \Big(|u|^{p-2}\frac{\partial u}{\partial x_i} + |v|^{p-2}\frac{\partial v}{\partial x_i}\Big)\varphi d\partial\Omega, \quad m \to \infty.$$

Proof. From the result, (u_m, v_m) are bounded $(L^p(\Omega))^2$, for almost every where t, then there exists a subsequence of (u_m, v_m) still denote (u_m, v_m) converge in $(L^p(\Omega))^2$, for almost everywhere t.
From the assumptions, $K(x) = \big(\int_\Omega |k(x,t)|^q dx\big)^{\frac{1}{q}} < \infty$, for fixed $x \in \partial\Omega$, where

$$\int_\Omega k(x, y)u_m(y, t)dy \to \int_\Omega k(x, y)u(y, t)dt, \quad m \to \infty,$$

$$\int_{\Omega}\frac{\partial k(x,y)}{\partial x_i}u_m(y,t)dy \to \int_{\Omega}\frac{\partial k(x,y)}{\partial x_i}u(y,t)dt, \quad m\to\infty.$$

similarly for $v_m(x,t)$ for fixed $x\in\partial\Omega$.

First we prove that $\int_{\partial\Omega}|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}d\partial\Omega \rightharpoonup \int_{\partial\Omega}|u|^{p-2}\frac{\partial u}{\partial x_i}d\partial\Omega$ weakly in $L^q(\partial\Omega)$.

Since, $K(x)\in L^p(\partial\Omega)$,we have $\|u_m\|_{L^p(\partial\Omega)}<\infty$. Similarly, we have $\|\frac{\partial u_m}{\partial x_i}\|_{L^q(\partial\Omega)}<\infty$ then

$$\begin{aligned}\||u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\|_{L^q(\partial\Omega)} &= \||u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\|_{L^{\frac{p}{p-2}+p}(\partial\Omega)}\\ &\le \||u_m|^{p-2}\|_{L^{\frac{p}{p-2}}(\partial\Omega)}|\frac{\partial u_m}{\partial x_i}|_{L^p(\partial\Omega)}\\ &\le \|u_m\|^{p-2}_{L^p(\partial\Omega)}\|\frac{\partial u_m}{\partial x_i}\|_{L^p(\partial\Omega)}<\infty.\end{aligned}$$

According to the lemma 1.3 [24], $\int_{\partial\Omega}|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}d\partial\Omega \rightharpoonup \int_{\partial\Omega}|u|^{p-2}\frac{\partial u}{\partial x_i}d\partial\Omega$ weakly in $L^q(\partial\Omega)$, for $a.e, t\in[0,T]$, since $\|\varphi\|_{L^p(\partial\Omega)}$. Similarly, we have $\int_{\partial\Omega}|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}d\partial\Omega \rightharpoonup \int_{\partial\Omega}|v|^{p-2}\frac{\partial v}{\partial x_i}d\partial\Omega$ weakly in $L^q(\partial\Omega)$, for $a.e, t\in[0,T]$, since $\|\varphi\|_{L^p(\partial\Omega)}$.

□

Lemma 2.5. *Let* (u_m,v_m) *be the approximate solution of* (2.4), *constructed as in* (2.6), *then*

$$\int_{\Omega}\left(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}+|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\right)(\frac{\partial\varphi}{\partial x_i})dx \to \int_{\Omega}\left(|u|^{p-2}\frac{\partial u}{\partial x_i}+|v|^{p-2}\frac{\partial v}{\partial x_i}\right)(\frac{\partial\varphi}{\partial x_i})dx, \quad m\to\infty.$$

Proof. From (2.20), we have to prove

(i) $$\int_{\partial\Omega}\left(|u_m|^{p-2}u_m+|v_m|^{p-2}v_m\right)\frac{\partial\varphi}{\partial x_i}d\partial\Omega \to \int_{\partial\Omega}\left(|u|^{p-2}u+|v|^{p-2}v\right)\frac{\partial\varphi}{\partial x_i}d\partial\Omega, \quad m\to\infty.$$

(ii) $$\int_{\Omega}\left(|u_m|^{p-2}u_m+|v_m|^{p-2}v_m\right)\Delta\varphi dx \to \int_{\Omega}\left(|u|^{p-2}u+|v|^{p-2}v\right)\Delta\varphi dx, \quad m\to\infty.$$

(i) From the proof of the lemma (2.2), lemma (2.4) and for fixed $x\in d\partial\Omega$,

$$|u_m(x,t)|^{p-2}u_m(x,t)\to|u(x,t)|^{p-2}u(x,t),$$

almost everywhere, and

$$\||u_m(x,t)|^{p-2}u_m(x,t)\|_{L^q(\partial\Omega)} \le \|u_m\|^{p-1}_{L^p(\partial\Omega)}<\infty.$$

According to the consequence of lemma1.3 [24], we have $|u_m(x,t)|^{p-2}u_m(x,t)\rightharpoonup|u(x,t)|^{p-2}u(x,t)$, weakly in $L^q(\partial\Omega)$, since $\frac{\partial\varphi}{\partial x_i}\in L^q(\partial\Omega)$, and the same for $|v_m(x,t)|^{p-2}v_m(x,t)\rightharpoonup|v(x,t)|^{p-2}v(x,t)$, weakly in $L^q(\partial\Omega)$, since $\frac{\partial\varphi}{\partial x_i}\in L^q(\partial\Omega)$.

(ii) From the proof of lemma (2.2), we have the convergence almost everywhere, for $x \in \Omega$, then

$$\| |u_m(x,t)|^{p-2} u_m(x,t)\|_{L^q(\Omega)} \leq \|u_m\|^{p-1}_{L^p(\Omega)} < \infty.$$

From the lemma1.3 [24], $|u_m(x,t)|^{p-2}u_m(x,t) \rightharpoonup |u(x,t)|^{p-2}u(x,t)$, weakly in $L^q(\Omega)$, since $\Delta\varphi \in L^q(\Omega)$.
Similarly, $|v_m(x,t)|^{p-2}v_m(x,t) \rightharpoonup |v(x,t)|^{p-2}v(x,t)$, weakly in $L^q(\Omega)$, since $\Delta\varphi \in L^q(\Omega)$.

□

Lemma 2.6. *Let (u_m, v_m) be the approximate solution of (2.4), constructed as in (2.6), then*

$$\Big((\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t}), (\psi, \varphi) \Big) \to \Big((\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}), (\psi, \varphi) \Big), \quad m \to \infty.$$

And $(u(t), v(t))$ are continuous on $[0, T]$.

Proof. From the previous result, we have that $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ are bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$, by Alaoglu's Theorem, there exists a subsequences, still denoted by $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$, converging to $(\chi, \tilde{\chi})$ weak star in $L^q(0,T,H^{-r}(\Omega))$. Then by modifying the proof of [[9], Theorem 1] (with the space $L^q(0,T,H^{-r}(\Omega))$, instead of $L^2(0,T,B^{\frac{1}{2}}(0,1))$). We have $(\chi, \tilde{\chi}) = (\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t})$ and $(u(t), v(t))$ are continuous on $[0, T]$. □

Chapter 3

Global weak solution to a generic reaction-diffusion nonlinear parabolic system

3.1 Introduction

In the present chapter, we are interested in the study of the existence of the global weak solution for a new generic reaction-diffusion parabolic system, it is in fact a generalization of the work presented by[[3], [15], [1]]. Where the study of Alaa and all in 2014 [3], based in the image processing, which they suggested to modify the model proposed by Morfu in 2006 [31] by appling a gaussian filter on the gradient of the noisy during the calculation of coefficient of anisotropic diffusion. The authors demonstrated the existence and consistency of their proposed model, in which They used a new technique recently introduced by Pierre [36] for study of semi-linear isotropic systems. The model is given by:

$$\begin{array}{c} \frac{\partial u}{\partial t} - div(g(|\nabla u_\sigma|)\nabla u) = f(t,x,u) \quad \text{in } Q_T \\ u(0,x) = u_0(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{in } \Sigma_T, \end{array} \tag{3.1}$$

with $\Omega =]0,1[\times]0,1[$ the picture domain with boundary $\partial\Omega$, with Neumann boundary conditions, $u(t,x)$ is the restored image and u_0 is the original image to be processed. $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, $(T > 0)$, $\sigma > 0$ and G_σ is the gaussian filter, where is given as follow :

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-\frac{|x|^2}{4\sigma}), \;\; x \in \mathbb{R}^2.$$

They consider the gradient norm of ω as:

$$|\nabla\omega| = \left(\sum_{i=1}^{i=2} (\frac{\partial\omega}{\partial x_i})^2 \right)^{1/2},$$

$\nabla\omega_\sigma$ is the smoothed version of gradient norme where :

$$\nabla\omega_\sigma = \nabla(\omega * G_\sigma) = \omega * \nabla G_\sigma.$$

The diffusivity g is smooth decreasing function defined by

$$g(0) = 1, \lim_{s \longrightarrow \infty} g(s) = 0, \tag{3.2}$$

one of the diffusivities Perona and Malik [34] proposed is:

$$g(s) = \frac{d}{\sqrt{1 + v\left(\frac{s}{\lambda}\right)^2}},$$

where $v \geq 0$, $d > 0$ and λ is a threshold (contrast) parameter that separates forward and backward diffusion [42]. In 2016, Bassam Al-Hamzah and Naji Yabari [15] proposed a new reaction-diffusion model in image processing, which they proved the existence of global solution for the nonlinear reaction-diffusion model. this study deals with the equation:

$$\begin{array}{c} \frac{\partial u}{\partial t} - div(g(|\nabla u_\sigma|)\nabla u) = f(t, x, u, \nabla u) \quad \text{in } Q_T \\ u(0, x) = u_0(x) \geq 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \Sigma_T, \end{array} \tag{3.3}$$

with $\Omega =]0, 1[\times]0.1[$, $Q_T =]0, T[\times\Omega$ and $\Sigma_T =]0, T[\times\partial\Omega$.
In fact the results $f(t, x, u, \nabla u)$ are a generalization of the work $f = 0$ presented by Catté [12], and the work $f = f(t, x, u)$ presented by Alaa [3]. In 2018 Aaraba, Alaa and Khalfi [1] provided the existence of global solution to a generic reaction-diffusion system with application to image restoration and enhancement. This study is a generalization of the work presented by[[3], [12], [35]] in the case of reaction-diffusion equations. They gave an example of the application demonstrated on a novel bio-inspired image restoration model [1], there proposed model is the follow:

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\big(g\big(|\nabla(u_\sigma)|\big)\nabla u\big) = f(t, x, u, v) \quad \text{in} \quad Q_T \\ \\ \dfrac{\partial v}{\partial t} - d_v \Delta v = p(t, x, u, v) \quad \text{in} \quad Q_T \\ \\ \dfrac{\partial u}{\partial \eta} = 0, \dfrac{\partial v}{\partial \eta} = 0 \quad \text{in} \quad \Sigma_T \\ \\ u(0, .) = u_0, v(0, .) = v_0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.4}$$

In this Chapter we give a proof to the global weak solution of the proposed model given in the next section. In our main results given in the last section, we truncate the system and show that there is a solution by using Schauder fixed point theorem in Banach spaces. Finally by making some estimations we prove that the solution of the truncated system converge to the solution of the proposed problem.

3.2 The proposed model

We consider a new generic reaction-diffusion system given as the follow:

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\big(g\big(|\nabla(u_\sigma)|\big)\nabla u\big) = f(t,x,u,v,\nabla v) \quad \text{in} \quad Q_T \\ \\ \dfrac{\partial v}{\partial t} - d_v \Delta v = p(t,x,u,v,\nabla u) \quad \text{in} \quad Q_T \\ \\ u(0,.) = u_0, v(0,.) = v_0 \quad \text{in} \quad \Omega \\ \\ \dfrac{\partial u}{\partial \eta} = 0, \dfrac{\partial v}{\partial \eta} = 0 \quad \text{in} \quad \Sigma_T, \end{cases} \tag{3.5}$$

where the initial conditions u_0, v_0 are only assumed to be square integrable, η is an outward normal to domain Ω. Let $\sigma > 0$, ∇u_σ be a regularization by convolution of ∇u. It is defined as $\nabla u_\sigma = \nabla(G_\sigma * u)$ and the diffusivity g check the same properties provided by Alaa [3], which is given in the equation (3.2).
The nonlinear functions $f, p : Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ mesurable for (t,x) and locally lipshitz continuous for u and v, $\exists r > 0$ for almost $(t,x) \in Q_T$ such that
$|f(t,x,u,v,q) - f(t,x,\bar{u},v,q_1)| \leq k_1(r)[|u-\bar{u}| + \|q-q_1\|]$
$|p(t,x,u,v,q) - p(t,x,u,\bar{v},q_2)| \leq k_2(r)[|v-\bar{v}| + \|q-q_2\|]$
for all $0 \leq |u|, |\bar{u}|, \|q_1\|, \|q_2\| \leq r$ and $0 \leq |v|, |\bar{v}|, \|q_1\|, \|q_2\| \leq r$.
In addition the nonlinearities satisfy the following main properties:
$(H_{\tilde{1}})$- the positivity property:
For almost $(t,x) \in Q_T$

$$f(t,x,0,s_1,\nabla s_1) \geq 0, \quad \text{and} \quad p(t,x,s_2,0,\nabla s_2) \geq 0, \quad \forall s_i \geq 0, i = 1,2.$$

$(H_{\tilde{2}})$ $-$ $\forall (u,v,q_1) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, $uf(t,x,u,v,q_1) \leq 0$ and $vf(t,x,u,v,q_1) \leq 0$,

$-$ $\forall (u,v,q_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, $up(t,x,u,v,q_2) \leq 0$ and $vp(t,x,u,v,q_2) \leq 0$,

$$(u+v)(f+p)(t,x,u,v,q_1,q_2) \leq 0.$$

Furthermore,

$$\sup_{|r|+|s| \leq R} (|f(t,x,l_1,l_2,\nabla l_2)| + |p(t,x,l_1,l_2,\nabla l_1)|) \in L^1(Q_T), \quad \text{for} \quad R = 2r > 0.$$

3.2.1 The Main of Results

In this section, we discuss the notions of weak solutions and the existence result. First, we define the folowing spaces:

$$X = \{u, v \in L^2(0, T, H^1(\Omega)) \cap C(0, T, L^2(\Omega)),\ u(0, .) = u_0 \text{ and } v(0, .) = v_0\ \}.$$

$$Z = \{\phi, \psi \in C^1(Q_T) \text{ such that } \phi(T, .) = 0 \text{ and } \psi(T, .) = 0\}.$$

$$D = \{u, v \in L^2(0, T, H^1(\Omega)) \cap C(0, T, L^2(\Omega))\}.$$

Definition 3.1. *We call (u, v) a weak solution of system* (3.5) *if*
• $\forall u, v \in X$ *and* $\forall \phi, \psi \in Z$ *we have*

$$\begin{cases} -\displaystyle\int_{Q_T} u\frac{\partial \phi}{\partial t}dxdt + \int_{Q_T} g(|\nabla u_\sigma|)\nabla u \nabla \phi dxdt = \int_{Q_T} f(t, x, u, v, \nabla v)\phi dxdt + \int_\Omega u_0\phi(0, x)dx \\ -\displaystyle\int_{Q_T} v\frac{\partial \psi}{\partial t}dxdt + \int_{Q_T} d_v \nabla v \nabla \psi dxdt = \int_{Q_T} p(t, x, u, v, \nabla u)\psi dxdt + \int_\Omega v_0\psi(0, x)dx, \end{cases} \tag{3.6}$$

where $f(t, x, u, v, \nabla v)$, $p(t, x, u, v, \nabla u) \in L^2(Q_T)$.

The Principal result is the following existence theorem:

Theorem 3.1. *Under the assumptions* $(H_{\tilde{1}})$, $(H_{\tilde{2}})$ *and for the diffusivity g given in* (3.2). *The reaction-diffusion system* (3.5) *admits a weak positive solution* (u, v) *in the sense defined in* (3.6) *for all* u_0, $v_0 \in L^2(\Omega)$ *such that* $u_0, v_0 \geq 0$.

Proof. The proof of the Theorem (4.1) is done in four step:

3.2.2 Step1: Existence result for bounded nonlinearities

In this part we will show the existence result for bounded source terms f, p, we define the following lemma.

Lemma 3.1. *Under the assumptions* $(H_{\tilde{1}})$ *and* $(H_{\tilde{2}})$ *of the nonlinearities, if there exists* M_f, M_p *such that for almost every* $(t, x) \in Q_T$,

$$|f(t, x, s_1, s_2, \nabla s_2)| \leq M_f, \quad |p(t, x, s_1, s_2, \nabla s_1)| \leq M_p \quad \forall (s_1, s_2) \in \mathbb{R}^2, \tag{3.7}$$

then for every u_0,v_0 *in* $L^2(\Omega)$, *there exists a weak solution* (u, v) *to the considered system* (3.5). *Moreover there exists* $C(M_f, M_p, T, \sigma, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)})$ *such that*

$$\|(u, v)\|_{L^\infty(0,T,L^2(\Omega))^2} + \|(u, v)\|_{L^2(0,T,H^1(\Omega))^2} \leq C. \tag{3.8}$$

Furthermore, if u_0, v_0 *are positive and* f, p *are quasi-positive, then* $u(t, x) \geq 0$ *and* $v(t, x) \geq 0$ *for almost every* $(t, x) \in Q_T$.

Proof. We show the existence of a weak solution by the classical Schauder fixed point theorem. We introduce the space

$$W(0, T) : \{u, v \in L^2(0, T, H^1(\Omega)) \cap L^\infty(0, T, L^2(\Omega)), \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in L^2(0, T, (H^1(\Omega))')\}$$

Let $w = (w_1, w_2) \in W(0,T)^2$ and (u, v) be the solution of a linearization of problem (3.5) given by:

• $\forall u, v \in D$ and $\forall \phi, \psi \in Z$

$$\begin{cases} -\int_{Q_T} u\frac{\partial \phi}{\partial t}dxdt + \int_{Q_T} g(|\nabla (w_1)_\sigma|)\nabla u \nabla \phi dxdt = \int_{Q_T} f(t,x,w_1,w_2,\nabla w_2)\phi dxdt + \int_\Omega u_0\phi(0,x)dx \\ -\int_{Q_T} v\frac{\partial \psi}{\partial t}dxdt + \int_{Q_T} d_v \nabla v \nabla \psi dxdt = \int_{Q_T} p(t,x,w_1,w_2,\nabla w_1)\psi dxdt + \int_{Q_T} v_0\psi(0,x)dx. \end{cases} \tag{3.9}$$

The application $w \in W(0,T) \longrightarrow (u,v) \in W(0,T)$ is clearly well defined. In fact $w_1 \in L^\infty(0,T,L^2(\Omega))$ and g, G_σ are in $C^\infty(\Omega)$ so $g(|\nabla(w_1)_\sigma|) \in L^\infty(0,T,C^\infty(Q_T))$ and since g is non-increasing it satisfied

$$a \leq g(|\nabla (w_1)_\sigma|) \leq d, \tag{3.10}$$

where $d > 0$ and a is a positive constant that depends only on σ and g. Let (u,v) the solution of the linearized problem (3.9).

Now we establish some important estimates to construct the functional setting where Schauder fixed point theory is applicable.

For all $0 < t < T$ and for $\phi = u$ and $\psi = v$

$$\begin{cases} -\int_{Q_t} u\frac{\partial u}{\partial t}dxdt + \int_{Q_t} g(|\nabla (w_1)_\sigma|)\nabla u \nabla u dxdt = \int_{Q_t} f(t,x,w_1,w_2,\nabla w_2)u dxdt + \int_\Omega u_0^2 dx \\ -\int_{Q_t} v\frac{\partial v}{\partial t}dxdt + d_v \nabla v \nabla v dxdt = \int_{Q_t} p(t,x,w_1,w_2,\nabla w_1)v dxdt + \int_{Q_T} v_0^2 dx. \end{cases}$$

Using lemma(1.1), we have:

$$\begin{cases} \frac{1}{2}\int_\Omega u^2(t)dx + \int_{Q_t} g(|\nabla (w_1)_\sigma|)|\nabla u|^2 dxdt = \int_{Q_t} f(t,x,w_1,w_2,\nabla w_2)u dxdt + \frac{1}{2}\int_\Omega u_0^2 dx \\ \frac{1}{2}\int_\Omega v^2(t)dx + d_v\int_{Q_t} |\nabla v|^2 dxdt = \int_{Q_t} p(t,x,w_1,w_2,\nabla w_1)v dxdt + \frac{1}{2}\int_\Omega v_0^2 dx, \end{cases} \tag{3.11}$$

$$\begin{cases} \frac{1}{2}\int_\Omega u^2(t)dx + a\int_{Q_t} |\nabla u|^2 dxdt \leq \int_{Q_t} |M_f||u| dxdt + \frac{1}{2}\int_\Omega u_0^2 dx \\ \frac{1}{2}\int_\Omega v^2(t)dx + d_v\int_{Q_t} |\nabla v|^2 dxdt \leq \int_{Q_t} |M_p||v| dxdt + \frac{1}{2}\int_\Omega v_0^2 dx, \end{cases} \tag{3.12}$$

with $a\int_{Q_t} |\nabla u|^2 \geq 0$ and $d_v \int_{Q_t} |\nabla v|^2 \geq 0$.

$$\begin{cases} \frac{1}{2}\int_\Omega u^2(t)dx \leq \int_{Q_T} |M_f||u| dxdt + \frac{1}{2}\int_\Omega u_0^2 dx \\ \frac{1}{2}\int_\Omega v^2(t)dx \leq \int_{Q_T} |M_p||v| dxdt + \frac{1}{2}\int_\Omega v_0^2 dx. \end{cases}$$

Hence

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega u^2(t)dx \leq \Big(\int_{Q_T} |M_f|^2 dxdt\Big)^{\frac{1}{2}} \Big(\int_{Q_T} |u|^2 dxdt\Big)^{\frac{1}{2}} + \dfrac{1}{2}\int_\Omega u_0^2 dx \\ \dfrac{1}{2}\displaystyle\int_\Omega v^2(t)dx \leq \Big(\int_{Q_T} |M_p|^2 dxdt\Big)^{\frac{1}{2}} \Big(\int_{Q_T} |v|^2 dxdt\Big)^{\frac{1}{2}} + \dfrac{1}{2}\int_\Omega v_0^2 dx, \end{cases}$$

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega u^2(t)dx \leq \dfrac{\varepsilon}{2}\int_{Q_T} |M_f|^2 dxdt + \dfrac{1}{2\varepsilon}\int_{Q_T} u^2 dxdt + \dfrac{1}{2}\int_\Omega u_0^2 dx \\ \dfrac{1}{2}\displaystyle\int_\Omega v^2(t)dx \leq \dfrac{\varepsilon}{2}\int_{Q_T} |M_p|^2 dxdt + \dfrac{1}{2\varepsilon}\int_{Q_T} v^2 dxdt + \dfrac{1}{2}\int_\Omega v_0^2 dx. \end{cases}$$

With $\int_{Q_T} |M_f|^2 dxdt = M_f$, $\int_{Q_T} |M_p|^2 dxdt = M_p$

$$\begin{cases} \displaystyle\int_\Omega u^2(t)dx \leq |M_f| + \int_{Q_T} u^2 dxdt + \int_\Omega u_0^2 dx \\ \displaystyle\int_\Omega v^2(t)dx \leq |M_p| + \int_{Q_T} v^2 dxdt + \int_\Omega v_0^2 dx. \end{cases} \tag{3.13}$$

Using Gronwall's lemma (1.2) to obtain

$$\begin{cases} \displaystyle\int_{Q_T} u^2 dxdt \leq \Big(\exp(T) - 1\Big)\Big(M_f + \int_\Omega u_0^2 dx\Big) \\ \displaystyle\int_{Q_T} v^2 dxdt \leq \Big(\exp(T) - 1\Big)\Big(M_p + \int_\Omega v_0^2 dx\Big), \end{cases} \tag{3.14}$$

$$\begin{cases} \displaystyle\sup_{0\leq t\leq T}\int_\Omega u^2(t)dx \leq M_f + \Big(\exp(T) - 1\Big)\Big(M_f + \int_\Omega u_0^2 dx\Big) + \int_\Omega u_0^2 dx := c_1 \\ \displaystyle\sup_{0\leq t\leq T}\int_\Omega v^2(t)dx \leq M_p + \Big(\exp(T) - 1\Big)\Big(M_p + \int_\Omega v_0^2 dx\Big) + \int_\Omega v_0^2 dx := c_2. \end{cases}$$

Therefore by setting $C_1 = c_1 + c_2$ we get

$$\|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} \leq C_1. \tag{3.15}$$

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega u^2(t)dx + a\int_{Q_T} |\nabla u|^2 dxdt \leq \int_{Q_T} |M_f||u| dxdt + \dfrac{1}{2}\int_\Omega u_0^2 dx \\ \dfrac{1}{2}\displaystyle\int_\Omega v^2(t)dx + d_v\int_{Q_T} |\nabla v|^2 dxdt \leq \int_{Q_T} |M_p||v| dxdt + \dfrac{1}{2}\int_\Omega v_0^2 dx, \end{cases}$$

$$\begin{cases} \min(\dfrac{1}{2}, a)\displaystyle\int_{Q_T} |u|^2 dxdt + \int_{Q_T} |\nabla u|^2 dxdt \leq M_f + \int_\Omega u^2(t)dx + \int_\Omega u_0^2 dx \\ \min(\dfrac{1}{2}, d_v)\displaystyle\int_{Q_T} |v|^2 dxdt + \int_{Q_T} |\nabla v|^2 dxdt \leq M_p + \int_\Omega v^2(t)dx + \int_\Omega v_0^2 dx, \end{cases}$$

$$\begin{cases} \displaystyle\int_{Q_T} |u|^2 dxdt + \int_{Q_T} |\nabla u|^2 dxdt \leq \frac{M_f + \int_\Omega u^2(t)dx + \int_\Omega u_0^2 dx}{\min(\frac{1}{2}, a)} := c_4 \\ \\ \displaystyle\int_{Q_T} |v|^2 dxdt + \int_{Q_T} |\nabla v|^2 dxdt \leq \frac{M_p + \int_\Omega v^2(t)dx + \int_\Omega v_0^2 dx}{\min(\frac{1}{2}, d_v)} := c_5, \end{cases}$$

setting $C_2 = c_4 + c_5$

$$\|(u,v)\|_{L^2(0,T,H^1(\Omega))^2} \leq C_2. \tag{3.16}$$

Next we estimate the $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ in $L^2(0,T,(H^1(\Omega))')$ we have

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(g(|\nabla u_\sigma|)\nabla u) + f(t,x,u,v,\nabla v) \\ \\ \dfrac{\partial v}{\partial t} = d_v \Delta v + p(t,x,u,v,\nabla u), \end{cases}$$

$$\|\partial u_t\|_{L^2(0,T,(H^1(\Omega))')} \leq c\|\nabla u\|_{L^2(Q_T)} + M_f T := C_3,$$

$$\|\partial v_t\|_{L^2(0,T,(H^1(\Omega))')} \leq d_v\|\nabla v\|_{L^2(Q_T)} + M_p T := C_4,$$

setting $C_5 = C_3 + C_4$

$$\|(\partial u_t, \partial v_t)\|_{L^2(0,T,(H^1(\Omega))')^2} \leq C_5. \tag{3.17}$$

Now we are in a position to apply Schauder fixed point in the functional space

$$W_0(0,T) = \{(u,v) \in L^2(0,T,H^1(\Omega))^2 \cap L^\infty(0,T,(H^1(\Omega))')^2, \|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} \leq C_1,$$

$$\|(u,v)\|_{L^2(0,T,H^1(\Omega))^2} \leq C_2, \|(\partial u_t, \partial v_t)\|_{L^2(0,T,(H^1(\Omega))')^2} \leq C_5, u(.,0) = u_0, v(.,0) = v_0\}.$$

We can verify that $W_0(0,T)$ is a nonemply closed convex in $W(0,T)$ [1] to use Schauder fixed point, we will show that the aplication

$$\begin{array}{rcl} F : W_0(0,T) & \longrightarrow & W_0(0,T) \\ w & \longmapsto & F(w), \end{array} \tag{3.18}$$

is a weakly continuous.
We consider a sequence $w_n \in W_0(0,T)$ such that $w_n \rightharpoonup w$ in $W_0(0,T)$ and let $F(w_n) = (u_n, v_n)$ thus

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = div(g(|\nabla w_{1_{n\sigma}}|)\nabla u_n) + f(t,x,w_{1n},w_{2n},\nabla w_{2n}) \\ \\ \dfrac{\partial v_n}{\partial t} = d_v \Delta v_n + p(t,x,w_{1n},w_{2n},\nabla w_{1n}), \end{cases}$$

we have that (u_n, v_n) is bounded in $L^2(0,T,H^1(\Omega))^2$ and $(\partial_t u_n, \partial_t v_n)$ is bounded in $L^2(0,T,(H^1(\Omega))')^2$ then by Simon [38] is relatively compact in $L^2(Q_T)$ which means we can extract a subsequence denoted w_n such that:
•$u_n \rightharpoonup u$ in $L^2(0,T,H^1(\Omega))$.
•$v_n \rightharpoonup v$ in $L^2(0,T,H^1(\Omega))$.
•$f(t,x,w_n,\nabla v_n) \to f(t,x,w,\nabla v)$ in $L^2(Q_T)$.
•$p(t,x,w_n,\nabla u_n) \to p(t,x,w,\nabla u)$ in $L^2(Q_T)$.
•$u_n \to u$ in $L^2(0,T,L^2(\Omega))$ and almost everywhere in Q_T.
•$u_n \to u$ in $L^2(0,T,L^2(\Omega))$ and almost everywhere in Q_T.
•$\nabla u_n \rightharpoonup \nabla u$ in $L^2(0,T,L^2(\Omega))$.
•$\nabla v_n \rightharpoonup \nabla v$ in $L^2(0,T,L^2(\Omega))$.
•$w_n \to w$ in $L^2(0,T,L^2(\Omega))$ and almost everywhere in Q_T.
•$g(|\nabla (w_1)_{n\sigma}|) \to g(|\nabla (w_1)_\sigma|)$ in $L^2(0,T,H^1(\Omega))$.
•$\partial_t u_n \rightharpoonup \partial_t u$ in $L^2(0,T,(H^1(\Omega))')$.
•$\partial_t v_n \rightharpoonup \partial_t v$ in $L^2(0,T,(H^1(\Omega))')$.
Using these convergence, we can pass to the limit and show that the limit (u,v) are solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(g(|\nabla w_{1_\sigma}|)\nabla u) + f(t,x,w_1,w_2,\nabla w_2) \\ \\ \dfrac{\partial v}{\partial t} = d_v \Delta v + p(t,x,w_1,w_2,\nabla w_1). \end{cases}$$

Thus $F(w) = (u,v)$ then F is weakly continuous, then we deduce the existence of $w = (u,v) \in W_0(0,T)$ such as $w = F(w)$ and thus the existence of $(u,v) \in W(0,T)$. □

3.2.3 Step2: Existence result for truncated nonlinearities

In this case, we truncate f and p using truncation function $\Psi_n \in C_c^\infty(\mathbb{R})$, such that $0 \le \Psi_n \le 1$ and

$$\Psi_n(r) = \begin{cases} 1 & \text{if} \quad |r| \le n \\ 0 & \text{if} \quad |r| \ge n+1. \end{cases} \tag{3.19}$$

Thus, we can state that the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = div(g(|\nabla u_{n_\sigma}|)\nabla u_n) + f_n(t, x, u_n, v_n, \nabla v_n) \\ \\ \dfrac{\partial v_n}{\partial t} = d_v \Delta v_n + p_n(t, x, u_n, v_n, \nabla u_n), \end{cases} \tag{3.20}$$

where

$$f_n(t, x, u_n, v_n, \nabla v_n) = \Psi_n(|u|)f(t, x, u, v, \nabla v),$$

and

$$p_n(t, x, u_n, v_n, \nabla u_n) = \Psi_n(|u|)p(t, x, u, v, \nabla u).$$

By the means of theorem (3.1) the problem (4.13) admits a weak solution. Now we show that a subsequence (u_n, v_n) converges to the weak solution (u, v) of problem (3.5), for this we need to prove the following results.

Lemma 3.2. *Let* (u_n, v_n) *be the solution of the approximate system* (4.13) *then*

(1) There exists a constant $M\left(\int_\Omega u_{n_0}dx, \int_\Omega v_{n_0}dx, T\right)$ *such that*

$$\int_{Q_T} (u_n + v_n)dxdt \le M \quad \forall t \in [0, T].$$

(2) There exists $M_1 > 0$ *such that*

$$\int_{Q_T} |\nabla u_n|^2 + |\nabla v_n|^2 dxdt \le M_1.$$

(3) There exists $M_2 > 0$ *such that*

$$\int_{Q_T} |f_n| + |p_n| dxdt \le M_2.$$

Proof. (1) We have the approximatif problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n_\sigma}|)\nabla u_n) = f_n(t, x, u_n, v_n, \nabla v_n) \\ \\ \dfrac{\partial v_n}{\partial t} - d_v \Delta v_n = p_n(t, x, u_n, v_n, \nabla u_n), \end{cases}$$

$$\begin{cases} (u_n + v_n)\dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n_\sigma}|)\nabla u_n)(u_n + v_n) = (u_n + v_n)f_n(t, x, u_n, v_n, \nabla v_n) \\ \\ (u_n + v_n)\dfrac{\partial v_n}{\partial t} - d_v \Delta v_n(u_n + v_n) = (u_n + v_n)p_n(t, x, u_n, v_n, \nabla u_n), \end{cases}$$

$$\int_{Q_T}(u_n+v_n)\frac{\partial(u_n+v_n)}{\partial t}dxdt \leq \int_{Q_T}(u_n+v_n)\big[(|f_n|+|p_n|)(t,x,u_n,v_n,\nabla v_n,\nabla u_n)\big]dxdt,$$

$$\begin{aligned}\frac{1}{2}\int_\Omega(u_n(t)+v_n(t))^2dx-\frac{1}{2}\int_\Omega(u_{n_0}+v_{n_0})^2dx &\leq \int_{Q_T}(u_n+v_n)\big[(|f_n|+|p_n|)\big]dxdt\\ &\leq \int_{Q_T}(u_n+v_n)(|M_f|+|M_p|)dxdt,\end{aligned}$$

setting $|M_f|+|M_p|=c$

$$\frac{1}{2}\int_\Omega(u_n(t)+v_n(t))^2dx-\frac{1}{2}\int_\Omega(u_{n_0}+v_{n_0})^2dx \leq c\int_{Q_T}(u_n+v_n)dxdt,$$

$$\int_\Omega(u_n(t)+v_n(t))^2dx \leq 2c\int_{Q_T}(u_n+v_n)dxdt+\int_\Omega(u_{n_0}+v_{n_0})^2dx.$$

Using a standard Gronwall's lemma (1.2) we get:

$$\begin{aligned}\int_{Q_T}(u_n+v_n)dxdt &\leq \Big(\int_\Omega(u_{n_0}+v_{n_0})^2dx\Big)\Big(\exp(\int_0^T 1ds)\Big)\\ &\leq \Big(\int_\Omega(u_{n_0}+v_{n_0})^2dx\Big)\Big(\exp(T)\Big),\end{aligned}$$

$$\int_{Q_T}(u_n+v_n)dxdt \leq M.$$

(2) We have:

$$\frac{\partial u_n}{\partial t}-div(g(|\nabla u_{n_\sigma}|)\nabla u_n)=f_n(t,x,u_n,v_n,\nabla v_n),$$

$$\begin{aligned}\int_{Q_t}u_n\frac{\partial u_n}{\partial t}dxdt+\int_{Q_t}g(|\nabla u_{n_\sigma}|)|\nabla u_n|^2dxdt &\leq \int_{Q_t}|u_n||f_n(t,x,u_n,v_n,\nabla v_n)|dxdt\\ &\leq \int_{Q_t}|u_n||M_f|dxdt,\end{aligned}$$

$$\frac{1}{2}\int_\Omega u_n(t)^2dx-\frac{1}{2}\int_\Omega u_{n_0}^2dx+a\int_{Q_T}|\nabla u_n|^2dxdt \leq \int_{Q_T}|u_n||M_f|dxdt. \tag{3.21}$$

Therefore

$$\begin{aligned}\frac{1}{2}\int_\Omega u_n(t)^2dx-\frac{1}{2}\int_\Omega u_{n_0}^2dx &\leq \big(\int_{Q_T}|u_n|^2dxdt\big)^{\frac{1}{2}}\big(\int_{Q_T}|M_f|^2dxdt\big)^{\frac{1}{2}}\\ &\leq \frac{\varepsilon}{2}\int_{Q_T}|u_n|^2dxdt+\frac{M_f}{2\varepsilon}mes(Q_T)\\ &\leq \frac{1}{2}\int_{Q_T}u_n^2dxdt+\frac{M_f}{2}mes(Q_T).\end{aligned}$$

Hence

$$\int_\Omega u_n^2(t)dx \leq \int_{Q_T} u_n^2 dxdt + \int_\Omega u_{n_0}^2 dx + M_f mes(Q_T). \tag{3.22}$$

Using Gronwall's lemma (1.2) to get

$$\int_{Q_T} u_n^2 dxdt \leq \left(\int_\Omega u_{n_0}^2 dx + M_f mes(Q_T) \right) \exp(T), \tag{3.23}$$

$$\int_{Q_T} u_n^2 dxdt \leq c_1, \tag{3.24}$$

we have that $\int_{Q_T} u_n^2 dxdt$ is bounded.

Let

$$\frac{\partial v_n}{\partial t} - d_v \Delta v_n = p_n(t, x, u_n, v_n, \nabla u_n),$$

$$\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt + d_v \int_{Q_t} \nabla v_n \nabla v_n dxdt = \int_{Q_t} v_n p_n(t, x, u_n, v_n, \nabla u_n) dxdt,$$

$$\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt + d_v \int_{Q_t} |\nabla v_n|^2 dxdt \leq \int_{Q_t} |v_n||p_n(t, x, u_n, v_n, \nabla u_n)| dxdt, \tag{3.25}$$

hence

$$\begin{aligned} \int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt &\leq \int_{Q_t} |v_n||M_p| dxdt \\ &\leq M_p \left(\int_{Q_T} |1|^2 dxdt \right)^{\frac{1}{2}} \left(\int_{Q_t} |v_n|^2 dxdt \right)^{\frac{1}{2}} \\ &\leq \frac{M_p \varepsilon}{2} mes(Q_T) + \frac{1}{2\varepsilon} \int_{Q_t} v_n^2 dxdt. \end{aligned}$$

$$\int_\Omega v_n^2(t)dx - \int_\Omega v_{n_0}^2 dx \leq M_p mes(Q_T) + \int_{Q_T} v_n^2 dxdt. \tag{3.26}$$

Using Gronwall's lemma (1.2)

$$\int_{Q_T} v_n^2 dxdt \leq \left(M_p mes(Q_T) + \int_\Omega v_{n_0}^2 dx \right) exp(T), \tag{3.27}$$

$$\int_{Q_T} v_n^2 dxdt \leq c_2 \tag{3.28}$$

we have that $\int_{Q_T} v_n^2 dxdt$ is bounded.
From (3.25) we have:

$$\begin{aligned}\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt + d_v \int_{Q_t} |\nabla v_n|^2 dxdt &\leq \int_{Q_t} |v_n||p_n(t,x,u_n,v_n,\nabla u_n)|dxdt \\ &\leq \int_{Q_t} |v_n||M_p|dxdt \\ &\leq \frac{M_p\varepsilon}{2} mes(Q_t) + \frac{1}{2\varepsilon}\int_{Q_t} |v_n|^2 dxdt,\end{aligned}$$

$$\frac{1}{2}\int_\Omega v_n^2(t)dx - \frac{1}{2}\int_\Omega v_{n_0}^2 dx + d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq \frac{M_p\varepsilon}{2} mes(Q_T) + \frac{1}{2\varepsilon}\int_{Q_T} |v_n|^2 dxdt,$$

$$\int_\Omega v_n^2(t)dx + 2d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq M_p mes(Q_T) + \int_{Q_T} |v_n|^2 dxdt + \int_\Omega v_{n_0}^2 dx,$$

from the result that $\int_{Q_T} |v_n|^2 dxdt$ is bounded with $c_2 > 0$

$$\int_\Omega v_n^2(t)dx + 2d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq M_p mes(Q_T) + c_2 + \int_\Omega v_{n_0}^2 dx. \tag{3.29}$$

Using Gronwall's lemma (1.2):

$$\begin{aligned}\int_{Q_T} |\nabla v_n|^2 dxdt &\leq \left(M_p mes(Q_T) + c_2 + \int_\Omega v_{n_0}^2 dx \right) \exp(T) \\ &\leq C_1,\end{aligned}$$

we have that $\int_{Q_T} |\nabla v_n|^2 dxdt$ is bounded, now let us show that $\int_{Q_T} |\nabla u_n| dxdt$ is bounded.
From (3.21), we have:

$$\begin{aligned}\frac{1}{2}\int_\Omega u_n(t)^2 dx - \frac{1}{2}\int_\Omega u_{n_0}^2 dx + a\int_{Q_T} |\nabla u_n|^2 dxdt &\leq \int_{Q_T} |u_n||M_f|dxdt \\ &\leq \frac{M_f\varepsilon}{2} mes(Q_T) + \frac{1}{2\varepsilon}\int_{Q_T} |u_n|^2 dxdt,\end{aligned}$$

$$\int_\Omega u_n^2(t)dx + 2a\int_{Q_T} |\nabla u_n|^2 dxdt \leq M_f mes(Q_T) + \int_{Q_T} |u_n|^2 dxdt + \int_\Omega u_{n_0}^2 dx,$$

we have that $\int_{Q_T} |u_n|^2 dxdt$ is bounded with $c_1 > 0$:

$$\int_\Omega u_n^2(t)dx + 2a\int_{Q_T} |\nabla u_n|^2 dxdt \leq M_f mes(Q_T) + c_1 + \int_\Omega u_{n_0}^2 dx. \tag{3.30}$$

Using Gronwall's lemma (1.2):

$$\begin{aligned}\int_{Q_T} |\nabla u_n|^2 dxdt &\leq \left(M_f mes(Q_T) + c_1 + \int_\Omega u_{n_0}^2 dx \right) exp(T) \\ &\leq C_2,\end{aligned}$$

with $\int_{Q_T} |\nabla u_n|^2 dxdt$ bounded.

$$\int_{Q_T} |\nabla u_n|^2 + |\nabla v_n|^2 dxdt \leq M_1, \quad \text{where} \quad M_1 = C_1 + C_2 > 0. \tag{3.31}$$

(3) Let:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n_\sigma}|)\nabla u_n) = f_n(t, x, u_n, v_n, \nabla v_n) \\ \\ \dfrac{\partial v_n}{\partial t} - d_v \Delta v_n = p_n(t, x, u_n, v_n, \nabla u_n), \end{cases}$$

where

$$\begin{cases} a \displaystyle\int_{Q_T} |\nabla u_n|^2 dxdt \leq \int_{Q_T} |u_n||f_n(t, x, u_n, v_n, \nabla v_n)| dxdt + \frac{1}{2} \int_\Omega u_{n_0}^2 dx \\ \\ d_v \displaystyle\int_{Q_T} |\nabla v_n|^2 dxdt \leq \int_{Q_T} |v_n||p_n(t, x, u_n, v_n, \nabla u_n)| dxdt + \frac{1}{2} \int_\Omega v_{n_0}^2 dx, \end{cases}$$

hence

$$\begin{cases} a \displaystyle\int_{Q_T} |\nabla u_n|^2 dxdt \leq \left(\int_{Q_T} |u_n|^2 dxdx \right)^{\frac{1}{2}} \left(\int_{Q_T} |f_n(t, x, u_n, v_n, \nabla v_n)|^2 dxdt \right)^{\frac{1}{2}} + \frac{1}{2} \int_\Omega u_{n_0}^2 dx \\ \\ d_v \displaystyle\int_{Q_T} |\nabla v_n|^2 dxdt \leq \left(\int_{Q_T} |v_n|^2 dxdt \right)^{\frac{1}{2}} \left(\int_{Q_T} |p_n(t, x, u_n, v_n, \nabla u_n)|^2 dxdt \right)^{\frac{1}{2}} + \frac{1}{2} \int_\Omega v_{n_0}^2 dx, \end{cases}$$

$$\begin{cases} 2a \displaystyle\int_{Q_T} |\nabla u_n|^2 dxdt \leq \int_{Q_T} |u_n|^2 dxdx + \int_{Q_T} |f_n(t, x, u_n, v_n, \nabla v_n)|^2 dxdt + \int_\Omega u_{n_0}^2 dx \\ \\ 2d_v \displaystyle\int_{Q_T} |\nabla v_n|^2 dxdt \leq \int_{Q_T} |v_n|^2 dxdt + \int_{Q_T} |p_n(t, x, u_n, v_n, \nabla u_n)|^2 dxdt + \int_\Omega v_{n_0}^2 dx, \end{cases}$$

from the previous result that $\int_{Q_T} |\nabla u_n|^2 dxdt$ and $\int_{Q_T} |\nabla v_n|^2 dxdt$ are bounded, moreover $\int_\Omega |u_n|^2 dxdt$ and $\int_\Omega |v_n|^2 dxdt$ are bounded. Hence

$$\int_{Q_T} |f_n| + |p_n| dxdt \leq M_2, \quad M_2 > 0. \tag{3.32}$$

□

3.2.4 Step3: Convergence

According to the lemma (3.2) we have (u_n, v_n) is bounded in $L^2(0,T,H^1(\Omega))^2$ and $(\frac{\partial u_n}{\partial t}, \frac{\partial v_n}{\partial t})$ is bounded in $(L^2(0,T,(H^1(\Omega))')^2 + L^1(Q_T))^2$, then with Simon [38] (u_n, v_n) is relatively compact in $(L^2(Q_T))^2$, then we can extract a subsequence (u_n, v_n) in $(L^2(Q_T))^2$ such that:
•$u_n \rightharpoonup u$ in $L^2(Q_T)$ and for almost everywhere in Q_T.
•$v_n \rightharpoonup v$ in $L^2(Q_T)$ and for almost everywhere in Q_T.
•$\nabla G_\sigma * u_n \rightharpoonup \nabla G_\sigma * u$ in $L^2(Q_T)$ and for almost everywhere in Q_T.
•$g(|\nabla G_\sigma * u_n|) \longrightarrow g(|\nabla G_\sigma * u|)$ in $L^2(Q_T)$.
•$f_n(t,x,u_n,v_n,\nabla v_n) \longrightarrow f(t,x,u,v,\nabla v)$ for almost everywhere in Q_T.
•$p_n(t,x,u_n,v_n,\nabla v_n) \longrightarrow p(t,x,u,v,\nabla v)$ for almost everywhere in Q_T.
To prove that (u,v) is a weak solution of system (3.5), we actually need to prove that $f_n(t,x,u_n,v_n,\nabla v_n)$ converges strongly toward $f(t,x,u,v,\nabla v)$ in $L^1(Q_T)$ and this convergence is given by the following lemma.

Lemma 3.3. *Under the additional assumptions that, for $R>0$:*

$$\sup_{|r|+|s|\le R}\Big(|f(t,x,r,s,\nabla s)| + |p(t,x,r,s,\nabla r)|\Big) \in L^1(Q_T).$$

(1) There exists $C>0$ such that:

$$\int_{Q_T}(u_n+2v_n)(|f_n|+|p_n|)dxdt \le M_3.$$

(2) f_n and p_n converges strongly toward f and p in $L^1(Q_T)$.

Proof. (1) Let:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) = f_n(t,x,u_n,v_n,\nabla v_n) \\ \\ \dfrac{\partial v_n}{\partial t} - d_v\Delta v_n = p_n(t,x,u_n,v_n,\nabla u_n), \end{cases}$$

Therefore

$$(u_n+2v_n)\frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)(u_n+2v_n) = (u_n+2v_n)f_n(t,x,u_n,v_n,\nabla v_n), \quad (3.33)$$

$$(u_n+2v_n)\frac{\partial(2v_n)}{\partial t} - 2d_v\Delta v_n(u_n+2v_n) = 2(u_n+2v_n)p_n(t,x,u_n,v_n,\nabla u_n), \quad (3.34)$$

from (3.33) and (3.34) we have:

$$\begin{aligned}
\int_{Q_t}(u_n+2v_n)\frac{\partial(u_n+2v_n)}{\partial t}dxdt &\leq \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt+\int_{Q_t}(u_n+2v_n)|p_n|dxdt\\
&\leq \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt+\int_{Q_t}|u_n||p_n|dxdt\\
&+ 2\int_{Q_t}|v_n||p_n|dxdt\\
&\leq \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt+M_p\int_{Q_t}|u_n|dxdt\\
&+ 2M_p\int_{Q_t}|v_n|dxdt,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\int_{\Omega}\big(u_n(t)+2v_n(t)\big)^2dx-\frac{1}{2}\int_{\Omega}\big(u_{n_0}+2v_{n_0}\big)^2dx &\leq \int_{Q_T}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt\\
&+ M_p\int_{Q_T}|u_n|dxdt+2M_p\int_{Q_T}|v_n|dxdt.
\end{aligned}$$

From the previous result of lemma (3.2) that $\int_{Q_T}|u_n|dxdt$ and $\int_{Q_T}|v_n|dxdt$ are bounded.

$$\int_{Q_T}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt\leq\frac{1}{2}\int_{\Omega}\big(u_{n_0}+2v_{n_0}\big)^2dx+M_p\int_{Q_T}|u_n|dxdt+2M_p\int_{Q_T}|v_n|dxdt, \tag{3.35}$$

$$\int_{Q_T}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt\leq M_3,\quad M_3>0. \tag{3.36}$$

(2) We have f_n, p_n converge almost everywhere toward f, p. We will show that f_n and p_n are equi-integrable in $L^1(Q_T)$. The proof will be given for f_n however the same result for p_n. Let $\varepsilon>0$ and prove that there exists $\delta>0$ such that $|E|<\delta$ with $E\subset Q_T$ implies that

$$\int_E|f_n(t,x,u_n,v_n,\nabla v_n)|dxdt<\varepsilon.$$

For all $K\geq 0$:

$$\begin{aligned}
\int_E|f_n(t,x,u_n,v_n,\nabla v_n)|dxdt &\leq \int_{[E\cap|u_n+2v_n|\leq K]}|f_n(t,x,u_n,v_n,\nabla v_n)|dxdt\\
&+ \int_{[E\cap|u_n+2v_n|>K]}|f_n(t,x,u_n,v_n,\nabla v_n)|dxdt,
\end{aligned}$$

with

$$\int_{[E\cap|u_n+2v_n|\leq K]}|f_n(t,x,u_n,v_n,\nabla v_n)|dxdt\leq\int_E\sup_{|u_n+2v_n|\leq K}|f_n(t,x,u_n,v_n,\nabla v_n)|dxdt,$$

and $\sup\limits_{|u_n+2v_n|\leq K} |f_n(t,x,u_n,v_n,\nabla v_n)| \leq \sup\limits_{|u_n+v_n|\leq K} |f_n(t,x,u_n,v_n,\nabla v_n)|$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|E| < \delta$ we obtain

$$\int_E \sup_{|u_n+v_n|\leq K} |f_n(t,x,u_n,v_n,\nabla v_n)| dxdt \leq \frac{\varepsilon}{2}, \tag{3.37}$$

we have $|u_n + 2v_n| > K \Longrightarrow |u_n + v_n| > K \Longrightarrow \dfrac{1}{K} < \dfrac{1}{|u_n + v_n|}$ then we have:

$$\int_{[E\cap|u_n+2v_n|>K]} |f_n(t,x,u_n,v_n,\nabla v_n)| dxdt \leq \frac{1}{K}\int_{Q_T} (u_n+2v_n)|f_n(t,x,u_n,v_n,\nabla v_n)| dxdt,$$

with $E \subset Q_T$

$$\int_{[E\cap|u_n+2v_n|>K]} |f_n(t,x,u_n,v_n,\nabla v_n)| dxdt \leq \frac{1}{K}\int_{E} (u_n+2v_n)|f_n(t,x,u_n,v_n,\nabla v_n)| dxdt, \tag{3.38}$$

and since (3.36) ensures that $\displaystyle\int_E (u_n+2v_n)|f_n| dxdt$ is bounded, we obtain:

$$\int_E |f_n(t,x,u_n,v_n,\nabla v_n)| dxdt \leq \varepsilon. \tag{3.39}$$

The same thing holds for p_n as well. □

3.2.5 Step 4: The positivity of the solution

Consider the function:

$$sign^-(s) = \begin{cases} -1 & \text{if} \quad s < 0 \\ 0 & \text{if} \quad s \geq 0. \end{cases} \tag{3.40}$$

Let $\varepsilon > 0$ we build a sequence of regular convex functions $j_\varepsilon(s)$ such as $j'_\varepsilon(s)$ is bounded and for all $s \in \mathbb{R}$, where $j'_\varepsilon(r) \to sing^-(s)$, when $\varepsilon \to 0$.
To prove the positivity of the solution (u, v), we proved that the bilinear form is definite and positive, the definite is relies on the quasi-positivity of nonlinearities given in the hypothese $(H_{\tilde{1}})$, so it remains to show the positivity, for this we take $u = v$.

$$2\frac{\partial u}{\partial t} - [div(g(|\nabla u_\sigma|)\nabla u) + d_v \Delta u] = (f+p)(t,x,u,u,\nabla u). \tag{3.41}$$

Let u be a solution of (3.41), we multiply both of the equation by $j'_\varepsilon(u)$, and integrating on $Q_t =]0,t[\times\Omega$ for $t \in [0,T[$

$$\begin{aligned} 2\int_{Q_t} \frac{\partial u}{\partial t} j'_\varepsilon(u) dxdt \quad &- \quad \int_{Q_t} div(g(|\nabla u_\sigma|)\nabla u) j'_\varepsilon(u) dxdt - \int_{Q_t} d_v \Delta u j'_\varepsilon(v) dxdt \\ &= \quad \int_{Q_t} (f+p)(s,x,u,u,\nabla u) j'_\varepsilon(u) dxds, \end{aligned}$$

$$\begin{aligned} 2\int_{Q_t}\frac{\partial u}{\partial t}j'_\varepsilon(u)dxdt &+ \int_{Q_t} g(|\nabla u_\sigma|)\nabla u\nabla j'_\varepsilon(u)dxdt + d_v\int_{Q_t}\nabla u\nabla j'_\varepsilon(v)dxdt \\ &= \int_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds. \end{aligned}$$

From (3.10)

$$\begin{aligned} 2\int_{Q_t}\frac{\partial j_\varepsilon(u(t,x))}{\partial t}dxdt &+ a\int_{Q_t}|\nabla u|^2 j''_\varepsilon(u)dxdt + d_v\int_{Q_t}|\nabla u|^2 j''_\varepsilon(v)dxdt \\ &\leq \int_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds. \end{aligned}$$

$$2\int_\Omega [j_\varepsilon(u)(t) - j_\varepsilon(u)(0)]dx + (a+d_v)\int_{Q_t}|\nabla u|^2 j''_\varepsilon(u)dxdt \leq \int_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

Since $u(0,x) = u_0$, then $j_\varepsilon(u_0(x)) = j_\varepsilon(u)(0) = 0$, $\int_\Omega j_\varepsilon(u)(0)dx = 0$ and $\int_{Q_t}|\nabla u|^2 j''_\varepsilon(u)dxdt \geq 0$, then we have

$$2\int_\Omega j_\varepsilon(u)(t)dx \leq \int_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

$$\begin{aligned} \int_\Omega j_\varepsilon(u)(t)dx &\leq \frac{1}{2}\int_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds \\ &\leq \frac{1}{2}\int_{(0,t)\times[u<0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds \\ &+ \frac{1}{2}\int_{(0,t)\times[u\leq 0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds. \end{aligned}$$

Where $u \leq 0$ we have $j'_\varepsilon(u) = 0$, so $\int_{(0,t)\times[u\leq 0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds = 0$, therefore

$$\int_\Omega j_\varepsilon(u)(t)dx \leq \frac{1}{2}\int_{(0,t)\times[u<0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds,$$

pass to the limit when $\varepsilon \longrightarrow 0$, we obtain

$$\int_\Omega (u)^-(t)dx \leq -\frac{1}{2}\int_{(0,t)\times[u<0]}(f+p)(s,x,u,u,\nabla u)dxds \leq 0, \tag{3.42}$$

by the hypothese $(H_{\bar{5}})$ of nonlinearities, we obtain $(u)^-(t)dx \geq 0$, then $(u)^-(t) = 0$ on Ω, therefore $u \geq 0$ and $v \geq 0$ in Q_T.

□

Chapter 4

Image restoration by fractional reaction-diffusion process

4.1 Introduction

The purpose of this chapter is to show the use of the fractional differential equations to restore a digital image. It is actually a generalization of the work presented by Alaa and all in 2014 [3], which is based on the model proposed by Perona-Malik in 1987 [34] which answered to the most popular question in the restoration of the image; how to preserve the contours of an image while the elimination of the noise, but the basic model of Perona-Malik is ill-posed in the sense of Hadamard. To overcome this problem, Catté and the others in 1992 [12] who suggested to introduce the regularization in space and time directly into the continuous equation in order to obtain a related well-posed model. In this way Morfu proposed a new model in 2006 [31] which is given by :

$$\begin{array}{c} \frac{\partial u}{\partial t} - div(g(|\nabla u|)\nabla u) = f(u) \quad \text{in } Q_T \\ u(0,x) = u_0(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \Sigma_T, \end{array} \tag{4.1}$$

where u_0 is the original image and $f(s) = s(s-a)(1-s)$ with $0 < a < 1$.
The model of Morfu [31] has two disadvantages, the first one is the sensitivity to noise and the second is that no results of existence and consistency in proved. To overcome this problem, in 2014 [3] Alaa and all combining the regulaisation procedure in Catté [12] and Morfu [31] model, There model is given by (3.1). In 2018, Alaa and Zirhem [4], proposed a new model of nonlinear and anisotropic reaction diffudion system applied to image restoration and to contrast enhancement. This Model is based on a system of partial differential equations of type Fitzhugh-Nagumo. They compared the performance of their alghorithm with the classical Fizhugh- Nagumo model.
This chapter is based on the existence of the solution for our model given in the next section. Then by the principal result, we truncate the equation and show that it can be solved in the sense of the Schauder fixed point theorem. Finally by making some estimations, we prove that the solution of the approximate problem converge to the solution of the problem proposed,

finally we give a straightforward application of our result in the fractional reaction diffusion model for image restoration.

4.2 The Proposed Model

We consider the fractional nonlinear reaction-diffusion equation proposed as follows:

$$\begin{array}{c} {}_0^C\mathcal{D}_t^{\alpha}u(t,x) - div(g(|\nabla u_\sigma|)\nabla u) = f(t,x,u) \quad \text{in } Q_T \\ u(0,x) = u_0(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \Sigma_T, \end{array} \tag{4.2}$$

with $0 < \alpha < 1$, $u(t,x)$ is the solution to the problem, u_0 is the original image, the diffusivity g check the same properties provided by Alaa [3], which is given in the equation (3.2) and the function $f(t,x,u)$ is used to represent sources. ${}_0^C\mathcal{D}_t^{\alpha}u(t,x)$ is the fractional derivative in the Caputo sense of order α of $u(t,x)$ defined as the definition (1.8).
Now, we need the following assumptions and properties:
$(H_{\dot{1}})$- $f : Q_T \times \mathbb{R} \longrightarrow \mathbb{R}$ mesurable for (t,x) and continous for u.
$(H_{\dot{2}})$- $\forall (t,x) \in Q_T$, $f(t,x,0)$ is a positive function.
$(H_{\dot{3}})$- $\forall u \in \mathbb{R}$ and for all $(t,x) \in Q_T$, $uf(t,x,u)$ is negative.
$(H_{\dot{4}})$- Assumed that $u(t,x)$ is differentiable in the sense of the gâteau, so

$${}_0^C\mathcal{D}_t^{\alpha}u(t,x) = t^{1-\alpha}\frac{\partial u(t,x)}{\partial t}.$$

$(H_{\dot{5}})$- Let $B(t) = t^{\alpha-1}$ and $\sup\limits_{0<t<T} | B(t) | \leq C_B$, where $C_B > 0$ and $0 < \alpha < 1$.
Therefore, the equation (4.2) is given as follows:

$$\begin{array}{c} \frac{\partial u}{\partial t} - B(t)div(g(|\nabla u_\sigma|)\nabla u) = B(t)f(t,x,u) \quad \text{in } Q_T \\ u(0,x) = u_0(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{in } \Sigma_T. \end{array} \tag{4.3}$$

4.3 The Main of result

This section goes through the concepts of weak solutions and the existence result. First, we provide the following spaces:

$$\tilde{X} = \{u \text{ in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad u(0,.) = u_0\}.$$

$$\tilde{Z} = \{\varphi \text{ in } C^1(Q_T), \text{ where } \varphi(T,.) = 0\}.$$

$$\tilde{D} = \{u \text{ in } C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\}.$$

Definition 4.1. *Let u the weak solution of the problem* (4.3) *if*
For all $u \in \tilde{X}$ and $\varphi \in \tilde{Z}$ with $f(t,x,u) \in L^1(Q_T)$

$$\int_{Q_T} -u\frac{\partial \varphi}{\partial t}dxdt + \int_{Q_T} B(t)g(|\nabla u_\sigma|)\nabla u \nabla \varphi dxdt = \int_{Q_T} B(t)f(t,x,u)\varphi dxdt + \int_\Omega u_0\varphi(0,x)dx. \tag{4.4}$$

Theorem 4.1. *Under the assumptions $(H_1) - (H_5)$ and $\forall\ R \geq 0$,*

$$\sup_{|u|\leq R}\big(|f(t,x,u)|\big) \in L^1(Q_T). \tag{4.5}$$

Then with fixed $T > 0$, $\sigma > 0$ and $0 < \alpha < 1$ and for any $0 \leq u_0 \in L^2(\Omega)$, the equation (4.4) *admits a weak positive solution. If moreover, $\forall\ r \geq 1$, $f(t,x,r) \leq 0$ and $u_0(x) \leq 1$, we have $0 \leq u(t,x) \leq 1$ in Q_T.*

The proof of Theorem(4.1) is done in four steps.

Proof. **4.3.1 Step1: The positivity of the solution**

Let the function $sign^-$ defined as:

$$sign^-(r) = \begin{cases} -1 & \text{if} \quad r < 0, \\ 0 & \text{if} \quad r \geq 0. \end{cases} \tag{4.6}$$

We build a sequence of convex function $j_\varepsilon(r)$, where $j'_\varepsilon(r)$ is bounded and $\forall\ r \in \mathbb{R}$, $j'_\varepsilon(r)$ converge to $sign^-(r)$ when $\varepsilon \longrightarrow 0$.
We consider u the solution of (4.4), we multiply the equation by $j'_\varepsilon(u)$ and by integration on $Q_t =]0,t[\times\Omega$ and $t \in [0,T[$ we get:

$$\int_{Q_t} \frac{\partial u}{\partial t} j'_\varepsilon(u)dxds + \int_{Q_t} B(t)g(|\nabla u_\sigma|)\nabla u \nabla j'_\varepsilon(u)dxds = \int_{Q_t} B(t)f(s,x,u)j'_\varepsilon(u)dxds,$$

we set $A(t,x) = g(|\nabla u_\sigma|)$ and with

$$\|\nabla u_\sigma\|_{L^\infty(Q_t)} \leq C_0,$$

for the properties of g, we have $a = g(C_0)$, where C_0 depend to σ and $\|u_0\|_{L^\infty(\Omega)}$, such that

$$A(t,x) \geq a, \quad \forall (t,x) \in Q_T. \tag{4.7}$$

$$\int_\Omega [j_\varepsilon(u(t)) - j_\varepsilon(u(0))]dx + a\int_{Q_T} B(t)|\nabla u|^2 j''_\varepsilon(u)dxdt \leq \int_{Q_T} B(t)f(t,x,u)j'_\varepsilon(u)dxdt,$$

then

$$\begin{aligned}\int_\Omega j_\varepsilon(u(t))dx &\leq C_B \int_{Q_T} f(t,x,u) j'_\varepsilon(u)dxdt \\ &\leq C_B \int_{[u<0]} f(t,x,u) j'_\varepsilon(u)dxdt + C_B \int_{[u\geq 0]} f(t,x,u) j'_\varepsilon(u)dxdt,\end{aligned}$$

then

$$\int_\Omega j_\varepsilon(u(t)) \leq C_B \int_{[u<0]} f(t,x,u) j'_\varepsilon(u)dxdt. \tag{4.8}$$

By crossing in the limit, when $\varepsilon \to 0$

$$\int_\Omega (u)^-(t)dx \leq -C_B \int_{[u\leq 0]} f(t,x,u)dxdt, \tag{4.9}$$

then $(u)^-(t) \geq 0$, so $(u)^-(t) = 0$, hence $u \geq 0$.
Then, we have to prove the following lemma:

Lemma 4.1. *We consider* u *the weak solution of* (4.4)*, and assume that* $0 \leq u_0 \leq 1$ *in* Ω *then* $0 \leq u \leq 1$ *in* Q_T.

Proof. In the previous results, we have obtained the positivity of the weak solution if the initial data is positive, so, we assume that $u_0 \leq 1$ and prove that $u \leq 1$.
We take $\bar{u} = 1 - u$, where $\nabla \bar{u} = \nabla u$.
For all $\bar{u} \in \tilde{X}$ and $\varphi \in \tilde{Z}$ with $f(t,x,1-\bar{u}) \in L^1(Q_T)$

$$-\int_{Q_T} \bar{u} \frac{\partial \varphi}{\partial t} dxdt + \int_{Q_T} B(t) g(|\nabla \bar{u}_\sigma|) \nabla \bar{u} \nabla \varphi dxdt = \int_{Q_T} B(t) f(t,x,1-\bar{u}) \varphi dxdt.$$

Let $j_\varepsilon(r)$ a sequence of convex function, where $j'_\varepsilon(r)$ is bounded and $\forall r \in \mathbb{R}$, $j'_\varepsilon(r) \to Sing^-(r)$ when $\varepsilon \to 0$, Let $j'_\varepsilon(\bar{u}) = \varphi$

$$\int_{Q_t} \frac{\partial j_\varepsilon(\bar{u})}{\partial t} dxds + \int_{Q_t} B(t) g(|\nabla \bar{u}_\sigma|) \nabla \bar{u} \nabla j'_\varepsilon(\bar{u}) dxds = \int_{Q_t} B(t) f(t,x,1-\bar{u}) j'_\varepsilon(\bar{u}) dxds,$$

$$\begin{aligned}\int_\Omega j_\varepsilon(\bar{u}(t))dx &\leq C_B \int_{Q_T} f(t,x,1-\bar{u}) j'_\varepsilon(\bar{u})dxdt \\ &\leq C_B \int_{[\bar{u}<0]} f(t,x,1-\bar{u}) j'_\varepsilon(\bar{u})dxdt + C_B \int_{[\bar{u}\geq 1]} f(t,x,1-\bar{u}) j'_\varepsilon(\bar{u})dxdt,\end{aligned}$$

pass to the limit when $\varepsilon \to 0$

$$-\int_\Omega (\bar{u})^-(t,x) \leq C_B \int_{[\bar{u}\geq 1]} f(t,x,1-\bar{u}) j'_\varepsilon(\bar{u})dxdt. \tag{4.10}$$

Hence

$$\int_\Omega (\bar{u})(t,x)dx \geq 0, \tag{4.11}$$

by result, $(\bar{u})(t) \geq 0$, so $u = 1 - \bar{u} \leq 1$. □

4.3.2 Step2: Existence result for bounded nonlinearity

First, we will show the existence result for bounded source term f.

Lemma 4.2. *Under the above assumptions of the nonlinearity f and $(H_{\dot{5}})$, if there exists $\tilde{M}_f \geq 0$, for almost $(t,x) \in Q_T$ and every $r \in \mathbb{R}$, we have*

$$|f(t,x,r)| \leq \tilde{M}_f, \tag{4.12}$$

then for all $u_0 \in L^2(\Omega)$, the problem (4.4) admits a weak solution. Moreover, there exists $C = C\big(\tilde{M}_f, a, C_B, T, \|u\|_{L^2(\Omega)}\big)$ where:

$$\|u(t)\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{4.13}$$

Proof. We introduce the space $W(0,T)$ to show the existence of a weak solution with the classical Schauder fixed point theorem:

$$W(0,T) = \{v \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) : \frac{\partial v}{\partial t} \in L^2(0,T;(H^1(\Omega))')\},$$

Let $v \in W(0,T)$ and u be the solution of a linearization of problem (4.3) given by
• $\forall u \in \tilde{D}$ and $\forall \varphi \in \tilde{Z}$

$$\int_{Q_T} \Big(- u\frac{\partial \varphi}{\partial t} + B(t)g(|\nabla v_\sigma|)\nabla u \nabla \varphi \Big) dxdt = \int_{Q_T} B(t)f(t,x,v)\varphi dxdt + \int_\Omega u_0\varphi(0,x)dx. \tag{4.14}$$

We take u as a test function φ in (4.14), with $0 < t < T$

$$\frac{1}{2}\int_\Omega u^2(t)dx + \int_{Q_T} B(t)g(|\nabla v_\sigma|)|\nabla u|^2 dxdt = \int_{Q_T} B(t)f(t,x,v)udxdt + \frac{1}{2}\int_\Omega u_0^2 dx.$$

with (4.12) and (4.7)

$$\int_\Omega u^2(t)dx + 2a\int_{Q_T} |B(t)||\nabla u|^2 dxdt \leq C_B\tilde{M}_f \int_{Q_T} u^2 dxdt + \int_\Omega u_0^2 dx, \tag{4.15}$$

using Gronwall's lemma(1.2)

$$\int_\Omega u^2(t)dx \leq \|u_0\|^2_{L^2(\Omega)}\big(\exp(C_B\tilde{M}_f T) - 1\big),$$

$$\sup_{0<t<T}\int_\Omega u^2(t)dx \leq C_1,$$

then

$$\|u\|_{L^\infty(0,T,L^2(\Omega))} \leq C_1. \tag{4.16}$$

From (4.15), we have

$$2a\int_{Q_T} |B(t)||\nabla u|^2 dxdt + C_B\tilde{M}_f\int_{Q_T} |u|^2 dxdt \leq \int_\Omega |u(t)|^2 dx + \int_\Omega u_0^2 dx$$

using $(H_{\dot{5}})$ to obtain that

$$\int_{Q_T} |\nabla u|^2 dxdt + \int_{Q_T} |u|^2 dxdt \leq \frac{\|u_0\|^2_{L^2(\Omega)}}{\min(2a, C_B \tilde{M}_f)},$$

$$\|u\|_{L^2(0,T,H^1(\Omega))} \leq C_2, \quad C_2 > 0. \tag{4.17}$$

From the previous estimates we introduce the space

$W_0(0,T) = \{v \in L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T,L^2(\Omega)),\ \|u\|_{L^2(0,T,H^1(\Omega))} + \|u\|_{L^\infty(0,T,L^2(\Omega))} \leq C,\ v(0) = u_0\}$,
where $C(\tilde{M}_f, C_B, T, a, \|u_0\|_{L^2(\Omega)})$.
The space $W_0(0,T)$ is nonemply closed convex in $W(0,T)$ [3], moreover it injects with a compact way in $L^2(0,T,L^2(\Omega))$, we define the application

$$\begin{array}{rcl} F : W_0(0,T) & \longrightarrow & W_0(0,T) \\ w & \longmapsto & F(w). \end{array} \tag{4.18}$$

F is well defined, to apply the Schauder fixed point theorem (1.4), we have to show that the application F is weakly continuous from $W_0(0,T)$ in $W_0(0,T)$ we consider a sequence $v_n \in W_0(0,T)$ where $v_n \rightharpoonup v$ in $W_0(0,T)$ and let $u_n = f(v_n)$. According to the classical result of compactness, we can extract from the sequence (u_n) a subsequence yet denoted (u_n) such taht

- $u_n \rightharpoonup u$ weakly in $L^2(0,T;L^2(\Omega))$.
- $u_n \longrightarrow u$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T.
- $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(0,T;L^2(\Omega))$.
- $v_n \longrightarrow v$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T.
- $\nabla G_\sigma * v_n \longrightarrow \nabla G_\sigma * v$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T.
- $B(t)g(|\nabla G_\sigma * v_n|) \longrightarrow B(t)g(|\nabla G_\sigma * v|)$ strongly in $L^2(0,T;L^2(\Omega))$.
- $f(t,x,v_n) \to f(t,x,v)$ strongly in $L^1(Q_T)$.
- $B(t)f(t,x,v) \to B(t)f(t,x,v)$ strongly in $L^1(Q_T)$.

The latter is obtained by applying the dominated convergence theorem (1.1). We can then pass to the limit, then the sequene $u_n = F(v_n)$ converges weakly to $u = F(v)$ in $W_0(0,T)$, then we deduce the existence of $u \in W_0(0,T)$ such that $u = F(u)$ and thus the existence of $u \in W(0,T)$. □

4.3.3 Step3: The truncated problem and a priori estimates

We consider the truncated function Ψ_n in $C_c^\infty(\mathbb{R})$, such that $0 \leq \Psi_n \leq 1$ and defined by:

$$\Psi_n(r) = \begin{cases} 1 & \text{if} \quad |r| \leq n, \\ 0 & \text{if} \quad |r| \geq n+1. \end{cases} \tag{4.19}$$

We truncate the nonlinearity f by Ψ_n

$$f_n(t,x,u_n) = \Psi_n(|u|)f(t,x,u),$$

thus, we can earily check that f_n satisfies $(H_{\dot{1}})-(H_{\dot{3}})$ with $\tilde{M}_f = \tilde{M}_{f_n}$ and for every $(t,x) \in Q_T$, $\forall r \in \mathbb{R}$

$$f_n(t,x,u_n) \to f(t,x,u),$$

since $u_0 \in L^2(\Omega)$ and $|f_n(t,x,u_n)| \leq \tilde{M}_{fn}$, so lemma (4.2) is applied, then we get the existence of a weak solution of the problem

$$\begin{array}{c} \dfrac{\partial u_n}{\partial t} - B(t)div(g(|\nabla(u_{n_\sigma})|)\nabla u_n) = B(t)f_n(t,x,u_n) \quad \text{in } Q_T, \\ u_n(0,x) = u_{n_0} \quad \text{in } \Omega, \\ \frac{\partial u_n}{\partial \eta} = 0 \quad \text{in } \Sigma_T. \end{array} \tag{4.20}$$

Now, we are in the case to prove that a subsequence u_n converge to the weak solution u of the problem (4.3), for this we have to prove that lemma:

Lemma 4.3. *We consider u_n as sequence of weak solutions given in* (4.4)*:*
(i) $\displaystyle\int_{Q_T} |f_n(t,x,u_n)|dxdt \leq \int_\Omega |u_{n0}|dx.$
(ii) u_n is bounded in $L^2(0,T,H^1(\Omega))$ and

$$\int_{Q_T} |u_n f_n(t,x,u_n)|dxdt \leq \int_\Omega u_{n0}^2 dx.$$

(iii) u_n is relatively compact in $L^2(Q_T)$.

Proof. (i) We have:

$$\frac{\partial u_n}{\partial t} - B(t)div(g(|\nabla(u_{n_\sigma})|)\nabla u_n) = B(t)f_n(t,x,u_n),$$

$$\int_{Q_T} \frac{\partial u_n}{\partial t}dxdt - \int_{Q_T} B(t)div(g(|\nabla(u_{n_\sigma})|)\nabla u_n)dxdt = \int_{Q_T} B(t)f_n(t,x,u_n)dxdt,$$

$$\int_\Omega |u_n(t)|dx + C_B \int_{Q_T} |f_n(t,x,u_n)|dxdt \leq \int_\Omega |u_{n_0}|dx, \tag{4.21}$$

$$\int_{Q_T} |f_n(t,x,u_n)|dxdt \leq \int_\Omega |u_{n_0}|dx. \tag{4.22}$$

(ii) First, we prove that u_n is bounded in $L^2(Q_T)$, for this, we consider u_n as a test function φ in the approximate problem

$$\frac{1}{2}\int_\Omega |u_n|^2(t)dx + C_B a \int_{Q_T} |\nabla u_n|^2 dxdt \leq C_B \int_{Q_T} |u_n||f(t,x,u_n)|dxdt + \frac{1}{2}\int_\Omega |u_{n_0}|^2 dx,$$

we have that $C_B a \displaystyle\int_{Q_T} |\nabla u_n|^2 dxdt \geq 0$ then

$$\int_{Q_T} |u_n f_n(t,x,u_n)|dxdt \leq \frac{1}{2}\int_\Omega |u_{n_0}|dx, \tag{4.23}$$

where

$$\sup_{0<t<T} \|u_n(\Omega)\|_{L^2(\Omega)} \leq \|u_{n_0}\|^2_{L^2(\Omega)}. \tag{4.24}$$

Let from the previous result

$$\frac{1}{2}\int_{\Omega} u_n^2(t)dx + aC_B \int_{Q_T} |\nabla u_n|^2 dxdt \leq \frac{C_B}{2}\int_{Q_T} u_n^2 dxdt + \big(\frac{C_B}{2}+\frac{1}{2}\big)\int_{\Omega} u_{n_0}^2,$$

$$\min(aC_B, \frac{C_B}{2})\int_{Q_T} |\nabla u_n|^2 dxdt + \int_{Q_T} |u_n|^2 dxdt \leq \big(C_B+1\big)\int_{\Omega} u_{n_0}^2 dx.$$

Setting $C_1 = \min(aC_B, \frac{C_B}{2})$

$$\int_{Q_T} |\nabla u_n|^2 dxdt + \int_{Q_T} |u_n|^2 dxdt \leq C_2 \int_{\Omega} u_{n_0}^2 dx, \tag{4.25}$$

$$\|u_n\|_{L^2(0,T,H^1(\Omega))} \leq C_2 \|u_{n_0}\|^2_{L^2(\Omega)}. \tag{4.26}$$

(iii) Let $f_n(t,x,u_n)$ bounded in $L^1(Q_T)$, u_n bounded in $L^2(0,T,H^1(\Omega))$ and $\dfrac{\partial u_n}{\partial t}$ bounded in $L^2(0,T,(H^1(\Omega))')$, with Simon [38], u_n is relatively compact in $L^2(Q_T)$. □

4.3.4 Step4: Convergence

According to the previous result of Lemma (4.3), u_n is relatively compact in $L^2(Q_T)$, we can extract a subsequence still denoted u_n where:
- $u_n \rightharpoonup u$ weakly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T.
- $\nabla G_\sigma * u_n \longrightarrow \nabla G_\sigma * u$ strongly in $L^2(Q_T)$ and almost everywhere in Q_T.
- $B(t)g(|\nabla G_\sigma * u_n|) \longrightarrow B(t)g(|\nabla G_\sigma * u|)$ strongly in $L^2(Q_T)$.
- $f(t,x,u_n) \rightharpoonup f(t,x,u)$ for almost everywhere Q_T .
- $B(t)f(t,x,u) \rightharpoonup B(t)f(t,x,u)$ for almost everywhere in Q_T.

To prove that u is a weak solution of (4.3), it suffices to prove that $f_n(t,x,u_n) \to f(t,x,u)$ in $L^1(Q_T)$, since $f_n(t,x,u_n) \rightharpoonup f(t,x,u)$ almost every where in Q_T. We will prove that $f_n(t,x,u_n)$ is uniformly integrable in $L^1(Q_T)$. For this we use the Vitali theorem [16] where:
$\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall E \subset Q_T$ measurable with $|E| < \delta$ we have:

$$\int_E |f_n(t,x,u_n)| \leq \varepsilon.$$

$\forall K \geq 0$:

$$\int_E |f_n(t,x,u_n)|dxdt \leq \int_{[E\cap|u_n|\leq K]} |f_n(t,x,u_n)|dxdt + \int_{[E\cap|u_n|>k]} |f_n(t,x,u_n)|dxdt.$$

Where:

$$\int_{[E\cap|u_n|\leq K]} |f_n(t,x,u_n)|dxdt \leq \int_E \sup_{|u_n|\leq K} |f_n(t,x,u_n)|dxdt. \tag{4.27}$$

$$\int_E |f_n(t,x,u_n)|dxdt \leq \int_E \sup_{|u_n|\leq K} |f_n(t,x,u_n)|dxdt + \int_{[E\cap|u_n|>k]} |f_n(t,x,u_n)|dxdt,$$

we have that $\sup_{|u_n|\leq K} |f_n(t,x,u_n)| \in L^1(Q_T)$, $\forall\varepsilon > 0$, $\exists\delta > 0$ such as $|E| < \delta$ then:

$$\int_E \sup_{|u_n|\leq K} |f_n(t,x,u)|dxdt \leq \frac{\varepsilon}{2}. \tag{4.28}$$

We have $|u_n| > K$

$$\begin{aligned}
\int_{[E\cap|u_n|>K]} |f_n(t,x,u_n)|dxdt &\leq \frac{1}{K}\int_{Q_T} u_n f_n(t,x,u_n)dxdt,\\
&\leq \int_E \sup_{|u_n|\leq K} |f_n(t,x,u_n)|dxdt + \frac{1}{K}\int_{E\cap|u_n|>K} |u_n f_n(t,x,u_n)|dxdt,\\
&\leq \frac{\varepsilon}{2} + \frac{1}{K}\int_E |u_n f_n(t,x,u_n)|dxdt,
\end{aligned}$$

if $K \geq \dfrac{\|u\|^2_{L^2(\Omega)}}{\varepsilon}$ then

$$\int_{[E\cap|u_n|>K]} |f_n(t,x,u_n)| \leq \frac{\varepsilon}{2}, \tag{4.29}$$

hence

$$\int_E |f_n(t,x,u_n)| \leq \varepsilon. \tag{4.30}$$

□

4.4 Numerical application and result

In this section, we show how to use our result of the fractional reaction diffusion model in image restoration.

We apply finite differences method to the problem (4.3) with respectively h and dt the spatial and time steps sizes and $u^n_{i,j}$ is the approximation of $u(i,j)$ in the pixel (i,j) at time ndt. In the follow, we take $h = 1$ and we introduce a discrete gradient operator ∇. We can write

$$\nabla u^n(i,j) = \begin{pmatrix}\nabla_x u^n(i,j)\\ \nabla_y u^n(i,j)\end{pmatrix} = \begin{pmatrix}u^n(i+1,j) - u^n(i,j)\\ u^n(i,j+1) - u^n(i,j)\end{pmatrix}.$$

For every field $P = (P_1, P_2) \in \mathbb{R}^2$, a discretisation possible to the divergence is given by

$$div\big(P(i,j)\big) = P_1(i,j) - P_1(i-1,j) + P_2(i,j) - P_2(i,j-1),$$

where

$$P(i,j) = \begin{pmatrix} P_1(i,j) \\ P_2(i,j) \end{pmatrix} = \begin{pmatrix} g\big(|h(i,j)|\big)\nabla_x u^n(i,j) \\ g\big(|h(i,j)|\big)\nabla_y u^n(i,j) \end{pmatrix},$$

with

$$h(i,j) = \begin{pmatrix} h_1(i,j) \\ h_2(i,j) \end{pmatrix} = \begin{pmatrix} \big(G_\sigma * \nabla_x u^n\big)(i,j) \\ \big(G_\sigma * \nabla_y u^n\big)(i,j) \end{pmatrix}.$$

Furthermore, the convolution with the Gaussian kernel can be discretised as:

$$\big(G_\sigma * X\big)(i,j) = \sum_{k_1=-1}^{k_1=1} \sum_{k_2=-1}^{k_2=1} G_\sigma(k_1,k_2) X(i-k_1, j-k_2),$$

with $X(i,j) = \nabla_x u^n(i,j)$ or $\nabla_y u^n(i,j)$. The diffusivity g given in which checking the some properties defined in (3.2) .
The nonlinearity f defined by

$$f(t,x,u) = \frac{2}{\Gamma(2.3)} \exp(x)(2-x)t^{1.3} - u(x,t) - 2\exp(x)t^2.$$

For the exact solution given by:

$$u(x,t) = \exp(x)(2-x)t^2.$$

The discretisation of the proposed model (4.4) is given as follow:

$$\begin{aligned} t_n^{1-\alpha} \frac{u^{n+1}(i,j) - u^n(i,j)}{dt} &- div\big(g\big(|\nabla(G_\sigma * u^n(i,j))|\big)\nabla u^n(i,j)\big) \\ &= (\frac{2}{\Gamma(2.3)} t_n^{-0.7} - 1)u^n(i,j) - 2\exp(i,j)t_n^2, \end{aligned}$$

where $0 < \alpha < 1$, $0 < n < N$, $u^n(i,j) = u(t_n, x_i, y_j)$, $x_i = ih$, $y_j = ih$, $t_n = ndt$ and $dt = \dfrac{T}{N}$.
First of all, we consider an original image without noise then we apply the fractional model on a noisy image. The choice for our numerical test are: for noise we use the gaussian noise (the probability density of noise k with zero mean and variance $\sigma = 0.09$), we set the processing time T at 9×10^{-3}, $dt = 10^{-3}$ and α by 0.7.

Figure 4.1. *Cameraman image without noise.*

Figure 4.2. *Noisy image with $\sigma = 0.09$.*

Figure 4.3. *Restored image with proposed model.*

Figure 4.4. *Peppers image without noise.*

Figure 4.5. *Noisy image with* $\sigma = 0.09$.

Figure 4.6. *Restored image with proposed model.*

Conclusion

The focus of this thesis is the study of the existence results for some partial differential equation problems of parabolic type using Faedo-Galerkin approximation and some functional analysis tools. The first study is the proof of generalised solution for the class of quasilinear parabolic system with nonlocal boundary conditions. The second is a class of nonlinear parabolic system which is a system of reaction-diffusion, where we give a proof of the weak solution, then the result of this system can be using in the application of image restoration and enhencement. For the last is a new model for image denoising which can be using to show how fractional order differential equations are used to restore a digital image.
In the near future, I want to focus much more on the numerical part and the image processing application and I hope that the works given in this thesis will be of considerable benefit for future promotions, which will enrich the university documentation.

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