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***Ministry of Higher Education and Scientific Research***

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Faculty of Sciences
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# Thesis

A view to obtaining the diploma of

## Doctorate of 3° cycle (LMD) in Mathematics

**Option:** ***Applied Functional Analysis***

# On the study of the existence of nontrivial solutions for a parabolic fractional problem

Presented by:

## Chaima SAADI

**Publicly discussed:**

**In front of the Jury**:

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FACULTY OF SCIENCES

**DEPARTEMENT OF MATHEMATICS**
OPTION: APPLIED FONCTIONAL ANNALYSIS

**THESIS**

A view to obtaining the diploma of

**Doctorate of $3^o$ cycle (LMD) in Mathematics**

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# On the study of the existence of nontrivial solutions for a parabolic fractional problem

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# Dedication

## ✎... *I dedicate this work*

To
my life's man, my eternal example, my moral support and source of
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my father: **MOHAMED**.

The light of my days, the source of my efforts, the flame of my heart, has
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# Contents

# PUBLICATIONS RESULTING FROM THIS THESIS

❒ Saadi Chaima, Lakhal Hakim, Slimani Kamel, Dob Sara.

✎ [43] **Existence and uniqueness of distributional solution for semilinear fractional elliptic equation involving new operator and some numerical results.**

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# *ABSTRACT IN ENGLISH*

The fundamental focus of this thesis is to investigate certain type of nonlinear fractional partial differential equations (FPDEs), both elliptic and parabolic in nature. Where we interest in this works under certain assumptions on the nonlinear terms to study the existence of weak solutions to five classes of fractional partial differential equations. We use the technique of the Leray-Schauder degree theory together with the application of Schauder fixed point theorem for demonstrate this. Then, for the uniqueness of weak solutions, we suggest the Banach contraction principle theorem. we also use the Galerkin approach to prove the existence and uniqueness results. The first class is the semilinear fractional elliptic problem involving the distributional Riesz fractional gradient in Bessel potential spaces. The second class is semilinear fractional system involving a nonlocal operator. Therefor, the third class in this thesis focuses on adding the transport term in the nonlinear fractional problem involving the distributional Riesz fractional derivative. Then the primary objective in the forth class is time fractional semilinear equations involving Riemann-Liouville time fractional derivative with fractional Laplacian. Finally, the the last class is the time fractional semilinear equation containing the Riemann-Liouville derivative.

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☞ ***Key words :*** *Distributional Riesz fractional gradient, fixed point theorems, topological degree theory, Galerkin method, fractional Laplacian, Riemann-Liouville time fractional derivative, Sobolev fractional space.*

# ABSTRACT IN FRENCH

L' objective fondamental de cette thèse est l'investigation certains types d'équations aux dérivées partielles fractionnaires non-linéaires de type paraboliques et elliptiques. Nous intéressons dans ce travail avec certaines hypothèses sur les termes non-linéaires pour étudier l'existence de solutions faible à cinq classes d'équations aux dérivées partielles fractionnaires et en utilisant la technique du degré de Leray-Schauder avec le théorème du point fixe. Ensuite, pour l'unicité de solution faible, nous suggérons le thèoréme du principe de contraction de Banach et l'approche de Galerkin. La première classe est le problème elliptique fractionnaire semi-linéaire qui contient le gradient fractionnaire de Riesz dans les espace de Bessel. La deixième classe est un système fractionnaire semi-linéaire qui contient un opérateur non local. Par conséquent, la troisième classe de cette thèse avons pour ajouté le terme de transporte dans le problème fractionnaire non-linéaire impliquant la dérivée fractionnaire de Riesz. La quatrième classe est l'équation dévolution semi-linéaires fractionnaire impliquant la dérivée fractionnaire qui contien la dérivée de Riemann-Liouville avec Laplacien fractionnaire. Enfin, la dernièr classe est l'équation semi-linéaire temporelle fractionnaire contenant la dérivée de Riemann-Liouville.

☞ ***Mots-clés :*** *Dérivée fractionnaire de Riesz distributionnelle, théorème du point fixe, degré de Leray-Schauder, methode de Galerkin, Laplacien fractionnaire, dérivée de Riemann-Liouville, espace de Sobolev fractionnaire.*

# *ABSTRACT IN ARABIC*

الهدف الرئيسي لهذه الاطروحة هو دراسة نوع معين من المعادلات التفاضلية غير الخطية الكسرية ذات الطبيعة الناقصية (*elliptic*) والقطع المكافئ (*parabolic*) . حيث نركز في هذا العمل تحت بعض الشروط المناسبة على الجزء غير خطي على دراسة وجود الحلول الضعيفة لخمسة أنواع من المعادلات التفاضلية الجزئية الكسرية ونطبق درجة ليراي شودر (*Leray-Schauder degree*) ونظرية النقطة الثابتة (*fixed point theorem*) ، ثم من اجل وحدانية هذه الحلول نقترح تطبيق مبدأ انكماش بناخ (*Banach contraction theorem*) كما نستعمل أيضا نظرية غاليركن (*Galerkin method*) لإثبات وجود ووحدانية الحلول. فالمسألة الأولى هي المعادلة الناقصية الجزئية شبه الخطية ذات الطبيعة الكسرية التي تحتوي على تدرج ريز كسري التوزيعي (*distributional Riesz gradient*)، ثم المسألة الثانية هي عبارة عن جملة معادلات كسرية شبه خطي يشتمل على مؤثر غير محلي، في المسألة الثالثة، نركز على إضافة حد النقل (*transport term*) في المعادلة الكسرية غير خطية التي تتضمن على تدرج ريز كسري التوزيعي. بعد ذلك نتطرق في المسألة الرابعة الى المعادلة الشبه الخطية التطورية حيث المشتقة بالنسبة للزمن من نوع ريمان-ليوفيل (*Riemann-Liouville*) و بالنسبة للموضع متعلق ب لابلاسيان الكسري (*fractional Laplacian*). وفي الأخير المسألة الخامسة هي المعادلة الشبه الخطية التطورية المتعلقة بمشتقة ريمان-ليوفيل.

**الكلمات المفتاحية:** تدرج ريز الكسري التوزيعي، نظرية النقطة الثابتة، نظرية ليراي شودر، طريقة غاليركن، لابلاسيان الكسري، مشتقة ريمان-ليوفيل، فضاء سوبولاف الكسري.

# NOTATION

- $\mathbb{R}$ the set of real numbers (1-dimensional real Euclidean space).
- $\mathbb{R}^d$: the real space of dimension $d$ or $d$-dimensional real Euclidean space.
- $L^p(\Omega)$: Lebesgue space.
- $L^p_{loc}(\Omega)$: set of locally integrable functions.
- $W^{k,p}(\Omega)$: the Sobolev space.
- $C^k(\Omega)$: the space of $k$ times continously differentiable functions.
- $C^\infty(\Omega) = \cap_{k\in\mathbb{N}} C^k(\Omega)$.
- $C_c^\infty(\Omega)$: the space of $C^\infty(\Omega)$ functions with compact support.
- $\Delta^s$: the fractional Laplace operator of order $s$.
- $D^s$: the distributional Riesz fractional gradient of order $s$.
- ${}^{RL}\mathscr{D}^s_{0,t}$: the Riemann-Liouville time fractional derivative.
- $div^s = D^s \cdot D^s$: the fractional divergence.
- $\Gamma$: the Gamma function.
- $< .,. >$: the scalar product.
- $\|.\|$: the norm.
- $\rightarrow$ the strong convergence.
- $\rightharpoonup$: the weak convergence.
- $\hookrightarrow$: the continuous embedding.
- $B_R$: the open ball with center 0 and the radius $R$.
- $p.v.$: the abbreviation for (in the principal value sense).

- *a.e.*: the abbreviation for (almost everywhere).
- FPDEs: the the abbreviation for " fractional partial differential equations".

# GENERAL INTRODUCTION

## FRACTIONAL CALCULUS

Over the past year, scientists have focused more on studying the branch of fractional partial differential equations (FPDEs) which are a category type of partial differential equations (PDEs) that involve derivatives of non-integer order. Therefore, it can be asserted that fractional partial differential equation is generalizations and extensions of partial differential equations. In recent year, it has apparent that fractional calculus is an effective and powerful tool for modeling a lots of actual problems in numerous fields, including engineering and science [53], physics [28], biological [34] and other phenomena. Frequently, the modeling mentioned previously is characterized as fractional differential equation.
In the upcoming ling, we will discuss the primary history and the basic development of fractional calculus

### ■ History of fractional calculus

On September 30, ***1695*** the notion of fractional calculus emerged through a letter between the french scientists Marquis de L'Hôpital (1661-1704) and Gottfried Wilhelm Leibniz (1646-1716). On that day, L'Hôpital posed a question to Leibniz about the results when taking the derivative of a function with an order 1/2. Leibniz replied, "*This is an apparent paradox from which one day useful consequences will be drawn*". This mathematical riddler has draw the attention of numerous scientists, who have significant accomplishments in this field including: Leonhard Euler (1707-1783), Joseph-Louis Lagrange (1736-1813), Pierre-Simon Laplace (1749-1827), Sylvestre-François Lacroix (1765-1843), Joseph Fourier (1768-1830), George Green (1793-1841), Niels Henrik Abel (1802-1829), Joseph Liouville (1809-1882), Bernhard Riemann (1826-1866), Nikolay Sonin (1849-1915), Erik-Albert Holmgren (1872-1943), Godfrey Harold Hardy (1877-1947) and John Edensor Littlewood (1885-1977), Frigyes Riesz (1880-1956) and Marcel Riesz (1886-1969), William Feller (1906-1970), Wilhem Grünwald (1909-1989) and other scientists.

In ***1819*** S-F. Lacroix became the first to extensively discuss of fractional derivative in his 700-page book under the title "*Traité du Calcul Différentiel et du Calcul Intégral*" [33] where he expressed the notion of fractional derivative as follows: $\frac{d^{1/2}}{dx^{1/2}}x = 2\sqrt{x}/\sqrt{\pi}$. Then, in ***1822***, J. Fourier published his book titled "*Théorie Analytique de la Chaleur*" [23] in which he mentioned fractional derivative, however, did not provide an application. Until year ***1823*** the initial application of fractional calculus in physical problem appeared by N. H. Abel in his paper [3], when he used a derivative of order 1/2 to the *Tautochrone* problem.

Then, through the years ***1847-1939*** numerous scientists have focused on introducing the definition of derivatives and integrals of arbitrary order like Liouville, Riemann, Grünwald, Hadamard, weyl, Hardy, all definition are different except in certain specific cases. Furthermore, and in ***1949*** M. Riesz start learning about fractional differential equation, in ***1974*** Keith B. Oldham and Jerome Spanier published their first book titled: "*The fractional calculus*"[36] and it focused only to the topic of fractional calculus. Additionally, the initial Publishing work deal with fractional differential equation (FDEs) and their application was in ***1999*** by Igor Podlubny title of his work is "*Fractional differential equation*"[40], also he published in ***2002*** work under the title: "*Geometric and Physical Interpretation of Fractional Integration and fractional Differentiation*"[41]. Since then, fractional calculus has developed further and still of great interest to mathematicians and scientists.
for more detail about the history of fraction calculus see [54, 55].

## ■ Fractional derivatives & operator

The notion of fractional derivative is as olde as calculus, the ordinary derivative can be prolonged and generalized to a fractional derivative by substituting the integer order with a non-integer order of derivative. Unlike the ordinary order, there exists numerous definitions of fractional derivatives and multiple papers have been published on various kinds of fractional derivative, including: the Riemann-Liouville derivative, the Caputo derivative, the Riesz derivative, the Hilfer derivative, the Grünwald derivative and the AtanganaBaleanu derivative, and other formulations of fractional derivative. Each of these derivative has a variety of disadvantages and advantages which are different from one another. Basically, these different and various definition of fractional derivative have been employed in modeling a wide variety of diverse application among of the valuable applications include: image processing, economic and finance, control system, electro-chemistry, material science, bioengineering we mention to earlier work [25, 35, 6, 39] and the references therein.

Our focuses in the current thesis is to solve fractional partial differential equations using fractional derivative, particularly the Riesz fractional derivative, which plays a basics role in non-linear elliptic fractional problems, and the Riemann-Liouville time fractional derivative along with fractional Laplacian operator to solve non-linear evolution partial differential equations.

In ***2015*** T-T. Shieh and D. Spector [48] focused on presenting a novel type of fractional partial differential equations depending on the distributional Riesz gradient or s-gradient $D^s$. This derivative will be the basic element focus in our work. Moreover, this derivative highlights diverse and distinctive properties, among them:

- where the fractional parameter $s$ converge to 1, the distributional Riesz gradient converge to the classical gradient. Additionally, as $s$ converge to 0 the Riesz kernel approximates the identity.
- This gradient can be expressed as a convolution between the Riesz kernel and the classical gradient for sufficiently smooth functions.
- In the weak formulation of partial differential equations the Riesz fractional gradient manifest as a fundamental component.
- In the introduction of Campos's paper [11], this fractional gradient shows continuous dependence on the parameter $s$, implying that small variations in the parameter $s$ lead to continuous and small modifications in the operator $D^s$.
- The s-gradient is the only operator with regard to element requirements that fulfills the homogeneity of parameter $s$ and rotational and translational invariance as shown in Šilhavý paper [47] . Furthermore, J. C. Bellido et all [7] concentrated on this derivative for a given nonlocal model in continuum mechanics in their study.

Throughout the recent years, the fractional Laplacian denoted by $(-\Delta)^s u$ has been a gereralization and extension of the Laplacian operator $\Delta u$, drawing the interest of a growing number of scientists studying nonlinear partial differential equations to model various physical phenomena. In contras, the fractional operator has a different equivalent definition on the whole-space $\mathbb{R}^d$ as a Fourier multiplier with the symbol $-|\xi|^s$, as a singular integral operator, as the inverse of the Riesz potential operator, as a fractional power in the sense of Bochner's or Balakrishnan's definition and other definition. Conversely, M. Kwaśnicki in his paper [29] present ten equivalent definitions of the fractional Laplacian in the Lebesgue space. On the other hand, in the paper of A.A Kilbas et all [30] and in article by M. Cai et all [10] it is widely regarded to the Riesz definition.

Furthermore, the fractional Laplacian operator indeed possesses numerous interesting properties, including a non-locality feature, positivity and continuity, it is a self-adjoint operator with a compact inverse and it is also symmetric. These properties highlight its powerful tool for studying various mathematical problems and modeling in diverse fields see previous publications [15, 10, 29, 12] and the references therein.

Throughout our work, we intend to focus on the definition of fractional Laplacian according to an integral in sense of the Cauchy principle value in the real space

$$(-\Delta)^s u(x) = c(d,s)p.v.\int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\mathrm{d}y \quad \forall x \in \mathbb{R}^d, \tag{1}$$

where the concept *p.v.* is named the Cauchy principal value and the element $c(d,s)$ is a normalization constant. Further, we can define the fractional Laplacian in deferent type using eigenfunctions and eigenvalue of Laplacian operator, these eigenvalue possess a finite multiplicity and constitute a diverging sequence (see[2])

$$0 < \lambda_1^s(\Omega) \leq \lambda_2^s(\Omega) \leq \lambda_3^s(\Omega) \leq \cdots \to +\infty.$$

This non-local operator will be the basic element together with Riemann-Liouville time fractional derivative in our evolution equation discussed in chapter 05.

# THESIS GOALS & OBJECTIVE

The main goal of this thesis is to study certain types of fractional partial differential equations (FDPEs), both elliptic and parabolic. Basically, Our thesis relies around answering the following three inquiries: do our problems have at least one solutions ?, if a solution exists, is it unique ?, are classical methods suitable for studying our problems ?. Our objective is to bridge the current gaps and provide comprehensive answers to these questions. In this context, we are going to apply classical techniques and procedures to demonstrate their usefulness in dealing with fractional partial differential equations. Specifically, we will emphasize fixed point theory, the topological degree technique, and the Galerkin approach, illustrating how these methods are effective in solving FPDEs. We will focus on five types of fractional partial differential equations, three of which are elliptic and two parabolic. The first equation is semi-linear fractional equation, and the second is semi-linear fractional system. We will demonstrate how the combination of topological degree theory and fixed point theory is instrumental in establishing the existence and uniqueness of solutions for these two problems. In the third equation, we will explore how fixed point theory is also advantageous for examining the existence of solutions for some nonlinear fractional problem.

The forth and five problems involve time fractional equations, where we will employ the Galerkin method along with topological degree method to prove the existence and uniqueness of solutions.

# OUTLINES OF THE THESIS

The basic propose of this work is to examine the existence and uniqueness of weak solutions for certain type of fractional partial differential equations. The thesis is structured with an introduction, six main chapters, and a conclusion, a compilation of our published articles, and a bibliography. The introduction provides background informations, rationale, motivations, and the objective of the study. It includes a comprehensive summary of the thesis, outlining research aims, motivations, methodology, and a historical overview of development of fractional calculus over time. The initial chapter serves as a preliminary section, introducing key theories and fundamental principles that will be referenced throughout the subsequent chapters.

--→ The second chapter deals with the existence, uniqueness of weak solution for the following semi-linear fractional problem involving the distributional Risz fractional derivative with Dirichlet condition

$$\begin{cases} -D^s.D^s u(x) + g(x, u(x)) = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded open domain, with a Lipschitz boundary, $0 < s < 1$ satisfying $2s < d$, $f \in L^2(\Omega)$ and $g(x, u)$ is semi-linear term defined from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ verifying certain hypothesis. The study of the aforementioned problem relied on topological degree approach to establish the existence of solutions. Then by applying the Banach contraction principle theorem, we demonstrate the uniqueness of solution.

--→ The third chapter presented the existence and uniqueness of weak solution for the semilinear fractional system. Where the idea of this chapter is generalized the equation in the second chapter into the following system

$$\begin{cases} -D^s.D^s u(x) + f_1(x, u(x), v(x)) = \psi_1(x), & \text{in } \Omega, \\ -D^s.D^s v(x) + f_2(x, u(x), v(x)) = \psi_2(x), & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded open domain, with a Lipshitz boundary, $s \in (0, 1)$ and $2s < d$, $(\psi_1, \psi_2) \in (L^2(\Omega) \times L^2(\Omega))$ and $f_1(x, k, p), f_2(x, k, p) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

measurable on $x \in \Omega$ and continuous on $k, p \in \mathbb{R}$ with suitable assumptions. The technique used in this chapter to establish the existence and uniqueness of solutions is the same as in the second chapter, which involves topological degree method together with the application of Banach fixed point theorem.

⇢ The fourth chapter, our attention is drawn to study the existence of weak solution for the following nonlinear fractional problem involving a nonlocal operator with fractional transport term

$$\begin{cases} -D^s.D^s u(x) - div^s(b\varphi(u(x))) = f(x, u(x), D^s u(x)), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

Where $\Omega \subset \mathbb{R}^d$ is bounded open domain, with a Lipschitz boundary, $b \in [L^2(\Omega)]^d$, $s \in (0,1)$, and $f(x,p,q) : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we intend to demonstrate, using the Schauder fixed point theorem and under specific assumptions regarding on the nonlinear part, that the posed problem is guaranteed to have at least one weak solutions.

The nonlocal operator $-D^s \cdot (D^s u)$ that we proposed as an essential component in the above three equations has captured the interest of numerous researchers. This attributed to its notable abilities, including the ability to display versatility in its applicability to multiple physical setting, which makes it ideal for a large range of system with multiple dimensional properties. Subsequently, we will mention to some references when the research work on this operator [48, 49, 47, 42, 31, 11, 1, 10, 32] and the references therein. The result of three problem above in Bessel potential space this functional space is essential for the results, it is one space associate to Riesz fractional gradient see [48] for more detail.

⇢ Within the scope of the five chapter, our interest in studying the of nonlinear evolution equation, which is divided into two parts. In the first part, we will focus on studying the following linear classifications of partial differential equations with Riemann-Liouville time fractional of order $s \in (0,1)$ and fractional Laplacian operator

$$\begin{cases} \text{Find } \mathscr{U} : [0, \mathcal{T}] \times \Omega \to \mathbb{R}, \\ {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}(t,x) + (-\Delta)^s \mathscr{U}(t,x) = h(t,x), & \text{on } [0,\mathcal{T}] \times \Omega, \\ \mathscr{U} = 0, & \text{on } [0,\mathcal{T}] \times \mathbb{R}^d \setminus \Omega, \\ (g_{1-s} * \mathscr{U})(0) = w, & \text{on } \mathbb{R}^d \setminus \Omega, \end{cases}$$

we employed the Galerkin method to prove the above problem has a unique weak solution. In the latter part of this chapter, we are interested in studying the existence of weak solutions using the Leray-Schauder degree method for the flowing semilinear fractional problem

$$\begin{cases} \text{Find } \mathscr{U} \,:\, [0,T]\times\Omega \to \mathbb{R}, & \\ {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}(t,x) + (-\Delta)^{s}\mathscr{U}(t,x) + h(\mathscr{U}(t,x)) = 0, & \text{on } [0,T]\times\Omega, \\ \mathscr{U} = 0, & \text{on } [0,T]\times\mathbb{R}^{d}\setminus\Omega \\ (g_{1-s} * \mathscr{U})(0,.) = w, & \text{on } \mathbb{R}^{d}\setminus\Omega. \end{cases}$$

--→ The last chapter, we propose the applications of Leray-Schauder degree theory to investigate the existence of at least one weak solutions for the following semilinear fractional problem

$$\begin{cases} \text{Find } \vartheta \,:\, [0,T]\times\Omega \to \mathbb{R}, & \text{such that,} \\ {}^{R}_{0}D^{\beta}_{t}\vartheta(t,x) - \Delta\vartheta(t,x) + \varphi(\vartheta(t,x)) = 0, & \text{on } [0,T]\times\Omega \\ \vartheta = 0, & \text{on}[0,T]\times\partial\Omega \\ (g_{1-\beta} * \vartheta)(0) = z, & \text{on } \partial\Omega. \end{cases}$$

Then, in the classical way we prove the uniqueness results.

# Chapter 1

# Preliminaries

In this chapter, we introduce and recall the necessary mathematical tools, notations, basic concepts, and results that will be needed in the following chapters. Specifically, definition, theorems and propositions on fractional analysis, some basic concept of fractional Sobolev space and fractional operators. Additionally, we include information on the space $L^p(0,T;\mathbb{X})$ along with some propositions and theorems. In this chapter, we present this information without proofs.

**Contents**

## 1.1 Element of functional Analysis

**Definition 1.1.1.** *(see[8]) Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty\},$$

*with the norm*

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left[\int_\Omega |f(x)|^p dx\right]^{1/p}.$$

*We pose*

$L^\infty(\Omega) = \{f\colon \Omega \to \mathbb{R} \text{ measurable, and exists } C, \text{ such that } |f(x)| \leq C \quad \text{a.e. on } \Omega\}.$

*with the norm*

$$\|f\|_{L^\infty(\Omega)} = inf\,\{C\colon |f(x)| \le C \quad a.e.\ on\ \Omega\}.$$

**Definition 1.1.2.** *(see[8]) Recall that $L^p_{loc}(\Omega)$ denotes the set of locally integrable function on $\Omega$, i.e.*
$L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R}$ *measurable, such that for all* $K_{\ compact} \subset \Omega$,
$\int_K |f(x)|^p dx < \infty\}.$

**Remark 1.1.1.** *The following statement is hold (see[8])*

$$L^p(\Omega) \subset L^p_{loc}(\Omega).$$

**Proposition 1.1.1.** *(see [8])*

> *(i) For $1 \le p \le \infty$, the space $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space.*
>
> *(ii) For $1 \le p < \infty$, the space $(L^p(\Omega), \|\cdot\|_p)$ is a separable space.*
>
> *(iii) For $1 < p < \infty$, the space $(L^p(\Omega), \|\cdot\|_p)$ is a reflexive space.*

**Theorem 1.1.1.** *(see[8])( Dominated Convergence Theorem, Lebegue) Let $f_n$ be a sequence of functions in $L^1(\Omega)$ that satisfy*

*(a) $f_n(x) \to f(x)$ a.e. on $\Omega$,*

*(b) there is a function $g \in L^1(\Omega)$ such that for all $n$, $|f_n(x)| \le g(x)$ a.e. on $\Omega$.*

*Then $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1(\Omega)} \to 0$.*

**Theorem 1.1.2.** *(see[8]) Let $(f_n)$ be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$ be such that $\|f_n - f\|_p \to 0$.*
*Then, there exist a subsequence $(f_{n_k})$ and a function $h \in L^p(\Omega)$ such that*

*(a) $f_{n_k}(x) \to f(x)$ a.e. on $\Omega$.*

*(b) $|f_{n_k}(x)| \le h(x)$ for all $k$, a.e. on $\Omega$.*

**Theorem 1.1.3.** *(see[8]). (Lax-Milgram theorem). Let $b$ continuous and coercive bilinear form and $l$ is a continuous linear form on a Hilbert space $H$, then there is a unique function $u$ belong $\in H$ such that:*

$$b(u, v) = l(v), \quad for\ all\ v \in H.$$

**Remark 1.1.2.** *(see[8]). If the bilinear form a from above theorem 1.1.3 is symmetric, then the function $u$ is only element of $H$ that minimize the map $J : H \to \mathbb{R}$ which define as follows*

$$J(v) = \frac{1}{2} b(v, v) - l(v),\ for\ all\ v \in H,$$

*that is*

$$J(u) = \min_{v \in H} J(v)\ and\ J(u) < J(v),\ if\ u \ne v.$$

### 1.1.1 Some inequalities and equalities

**Theorem 1.1.4.** *(see[8])(Hölder's inequality) Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$[1] *with* $1 \leq p \leq \infty$. *Then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f(x)g(x)|\mathrm{d}x \leq \|f\|_p \|g\|_{p'}.$$

*In the particular case* $p = p' = 2$ *we get the* ***Cauchy-Schwarz*** *inequality:*

$$\int_\Omega |f(x)g(x)|\mathrm{d}x \leq \|f\|_2 \|g\|_2.$$

▶ *The Cauchy-Schwarz inequality is broadly generalized by Hölder's inequality.*

**Theorem 1.1.5.** *(see[4])(Minkowski's inequality) If* $1 \leq p \leq \infty$, *then*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

**Theorem 1.1.6.** *(see[24]) (almost everywhere equality in* $L^1(\Omega)$*) Let* $\Omega$ *be an open set of* $\mathbb{R}^{\mathbb{N}}$, $N \geq 1$, *and let* $f, g \in L^1_{loc}(\Omega)$ *then*

$$\left(\forall \varphi \in C_c^\infty(\Omega), \int_\Omega f(x)\varphi(x)\mathrm{d}x = \int_\Omega g(x)\varphi(x)\mathrm{d}x\right) \Leftrightarrow (f = g, \ a.e \ ).$$

### 1.1.2 Some results on convolution

**Theorem 1.1.7.** *(see[8]) Let* $f \in L^1(\mathbb{R}^d)$ *and* $g \in L^p(\mathbb{R}^d)$, *with* $1 \leq p \leq \infty$. *Hence, for almost everything* $x \in \mathbb{R}^d$, *the function* $y \mapsto f(x - y)g(y)$ *is integrable on* $\mathbb{R}^d$, *we pose*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\mathrm{d}y.$$

*Therefore,* $f * g \in L^p(\mathbb{R}^d)$ *and* $\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}$.

**Proposition 1.1.2.** *(see[8]) Let* $f \in C_c^k(\mathbb{R}^d)$ $(k \geq 1)$ *and* $g \in L^1_{loc}(\mathbb{R}^d)$. *Then*

- $f * g \in C^k(\mathbb{R}^d)$.

*In particular, if* $f \in C_c^\infty(\mathbb{R}^d)$ *and* $g \in L^1_{loc}(\mathbb{R}^d)$, *then*

- $f * g \in C^\infty(\mathbb{R}^d)$.

## 1.2 Fractional Sobolev space

Fractional Sobolev spaces constitute a significant field of study that generalizes many classical Sobolev space results to the realm of non-local and fractional setting.

This section related the definition of some fractional Sobolev space and fractional operator and some basic properties.

---

[1] Let $1 \leq p \leq \infty$; we denote by $p'$ the conjugate exponent, $\frac{1}{P} + \frac{1}{p'} = 1$.

### 1.2.1 The fractional space $W^{s,p}(\Omega)$ space

**Definition 1.2.1.** *(see [18]) For $0 < s < 1$ and any $1 \leq p < +\infty$, we define $W^{s,p}(\Omega)$ as follows*

$$W^{s,p}(\Omega) : = \left\{ u \in L^p(\Omega); \quad \frac{|u(x) - u(y)|}{|x-y|^{\frac{d}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

*possessed of the norm*

$$\|u\|_{W^{s,p}(\Omega)} : = \left( \int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} dxdy \right)^{\frac{1}{p}},$$

*where the term*

$$[u]_{W^{s,p}(\Omega)} : = \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

*is the Gagliardo semi-norm of $u$.*

**Proposition 1.2.1.** *(see [18]) Let $\Omega$ be an open subset of $\mathbb{R}^d$, $0 < s < 1$, then we have*

> *(a) For $1 \leq p < \infty$, $W^{s,p}(\Omega)$ is a Banach and separable space.*
>
> *(b) For $1 < p < \infty$, $W^{s,p}(\Omega)$ is a reflexive space.*

**Theorem 1.2.1.** *(see [18]) For any $s > 0$, the space $C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^d)$.*

**Remark 1.2.1.** *(see [18]) We note $W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|.\|_{W^{s,p}}}$[2], according to above theorem 1.2.1, we have*

$$W_0^{s,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d).$$

*In fact, when $\Omega$ is a subset of $\mathbb{R}^d$, $W_0^{s,p}(\Omega) \neq W^{s,p}(\Omega)$, i.e. $C_c^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$.*

**Proposition 1.2.2.** *(see [18])(Continue embedding). Let $\Omega$ be a Lipschitz open subset of $\mathbb{R}$, $0 < s < 1$ and $p > 1$, then*

> *(1) If $sp < d$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leq \dfrac{dp}{d-sp}$.*
>
> *(2) If $d = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.*

---

[2] $W_0^{s,p}(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ in the norm $\|.\|_{W^{s,p}}$

(3) *If* $sp > d$ *then* $W^{s,p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$, *more precisely:*
$W^{s,p}(\Omega) \hookrightarrow C^{0;s-d/p}(\Omega)$.

**Theorem 1.2.2.** *(see [18])(Compact embeddings) Let* $\Omega$ *be a bounded Lipschitz open subset of* $\mathbb{R}$, $0 < s < 1$, $p > 1$ *and* $d \geq 1$, *then*

(1) *If* $sp < d$, *then the embedding of* $W^{s,p}(\Omega)$ *into* $L^q(\Omega)$ *is compact for every* $q < \dfrac{dp}{d-sp}$.

(2) *If* $d = sp$, *then the embedding of* $W^{s,p}(\Omega)$ *into* $L^q(\Omega)$ *is compact for every* $q < \infty$.

(3) *If* $sp > d$ *then the embedding of* $W^{s,p}(\Omega)$ *into* $C^{0,\lambda}$ [a] *is compact for every* $\lambda < \dfrac{s-d}{p}$.

---

[a] (see[20]) The $C^{0,\lambda}(\Omega) = \{u \in C^0(\Omega) : [u]_\lambda < \infty, \}$ such that $[u]_\lambda = \sup\limits_{x,y\in\Omega|x\neq y} \left\{ \dfrac{|u(x)-u(y)|}{|x-y|^\lambda} \right\}$.

- $C^{0,1}$ = space of Lipschitz-continuous functions.

Now, we looks at the case (p=2) in the space $W^{s,p}(\mathbb{R})$ this is a very special cases when the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^d)$ and $W_0^{s,2}(\mathbb{R}^d)$ which are Hilbert spaces, they are noted $H^s(\mathbb{R}^d)$ and $H_0^s(\mathbb{R}^d)$ respectively, for $s \in (0,1)$.
✍ We define the space $H^s(\mathbb{R}^d)$ as follows

$$H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{d}{2}+s}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^d)} = \left( \|u\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dxdy \right)^{1/2}.$$

The space $H^s(\mathbb{R}^d)$ equipped with the inner product $\langle .,. \rangle_{H^s(\mathbb{R}^d)}$ is a Hilbert space .

**Proposition 1.2.3.** *(see [18]) Let* $\Omega$ *be a Lipschitz bounded open subset of* $\mathbb{R}^d$ *and* $s \in ]0,1[$ *such that* $d > 2s$. *Let* $u : \Omega \to \mathbb{R}$ *be a measurable function compactly supported. Then, there exists a positive constant* $C_{emb} > 0$ *depending on d, s and* $\Omega$ *(*$C_{emb} = C_{emb}(d,s,\Omega)$*) such that*

$$\|u\|_{L^2(\Omega)} \leq C_{emb}\|u\|_{H_0^s(\Omega)}. \tag{1.1}$$

**Remark 1.2.2.** *If* $\Omega$ *is bounded then* $\|\cdot\|_{H_0^s(\Omega)}$ *is a norm of* $H_0^s(\Omega)$ *equivalent to* $\|\cdot\|_{H^s(\Omega)}$ *with*

$$\|u\|_{H_0^s(\Omega)} = \left( \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dxdy \right)^{1/2}.$$

### 1.2.2 Fractional operator

In this subsection, we focus to introduce the definition of fractional Laplacian operator $(-\Delta)^s$ and the distributional Riesz fractional derivative $D^s$ and some properties that will be used in the following chapter.

#### Fractional Laplacian operator

The fractional Laplacian, denoted by $(-\Delta)^s$ reduces to the classical Laplacian operator $(-\Delta)$ when the fractional order $s$ equals 1. Therefor, the fractional Laplacian serves as a generalization of the classical Laplacian operator. Various definitions of the fractional Laplacian exist in the whole space $\mathbb{R}^d$. In this thesis, we utilize the singular integral definition of fractional Laplacian (for more information see [15, 18, 29]).

**Definition 1.2.2.** *[15, 18, 29] The fractional Laplacian operator is given by the Cauchy principal value integral in the real space, for all* $s \in (0,1)$ *and every* $u \in \mathscr{S}$[3]

$$(-\Delta)^s u(x) = c(d,s) p.v. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy \quad \forall x \in \mathbb{R}^d,$$

*where*

$$c(d,s) = \left( \int_{\mathbb{R}^d} \frac{1-\cos y}{|y|^{d+2s}} dy \right)^{-1}, \qquad \forall y \in \mathbb{R}^d,$$

*is a normalization constant.*
*In other side, p.v. denotes the Cauchy principal value defined by*

$$p.v. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy, = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy.$$

**Remark 1.2.3.** *[15] The Schwarz space* $\mathscr{S}(\mathbb{R}^d)$ *is dense in* $H^s(\mathbb{R}^d)$*. So, we can extended the fractional Laplacian operator* $(-\Delta)^s$ *by density to* $u \in H^s(\mathbb{R}^d)$*.*

**Proposition 1.2.4.** *(see [15]). Let* $u \in C_0^\infty(\mathbb{R}^d)$*,* $d > 1$*, then the following statements holds*

*(a)* $\lim_{s \to 0^+} (-\Delta)^s u(x) = u(x)$,

*(b)* $\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x)$.

---

[3] $\mathscr{S}$ is the Schwarz space of rapidly decreasing functions defined in [9] as

$$\mathscr{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha, \beta \in \mathbf{N}_0^d, \quad \sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| < \infty \right\}.$$

### Distributional Riesz fractional derivative

we will define the distributional Riesz fractional gradient or s-fractional gradient denoted by $(D^s u)$ or $(\nabla^s u)$ as defined by Shieh and Spector (see [48]). Additionally, the s-gradient $(D^s u)$ converges strongly to the classical gradient $(Du)$ as $s \nearrow 1$.[For more information and properties about this derivative see [13, 31, 42, 47, 48, 49]].

▷ First, recall the generalized Riesz potentials of order $s$ (called Riesz potentials) for $s \in (0, 1)$ given by the formula (see [48])

$$(I_s * u)(x) = \mathcal{C}_{d,1-s} \int_{\mathbb{R}^d} \frac{u(y)}{|x-y|^{d-s}} dy,$$

$$\text{with} \qquad I_{1-s}(x) = \frac{\mathcal{C}_{d,1-s}}{|x|^{d-1+s}} \text{ and } \qquad \mathcal{C}_{d,s} = 2^{-s}\pi^{-\frac{d}{2}} \frac{\Gamma\left(\frac{d+s-1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}.$$

**Definition 1.2.3.** *(see [Definition 1.1 of [48]]). Let $0 < s < 1$ and $p \in (1, \infty)$. If $u \in L^p(\mathbb{R}^d)$ we define s-gradient as follow*

$$D_j^s u := \frac{\partial^s u}{\partial x_j^s} := \frac{\partial}{\partial x_j} \left(I_{1-s} * u\right), \quad j = \overline{1, \dots, d},$$

*where $\dfrac{\partial}{\partial x_j}$ is in the sense that*

$$\begin{aligned}\left\langle \frac{\partial^s u}{\partial x_j^s}, v \right\rangle &= \left\langle \frac{\partial}{\partial x_j} \left(I_{1-s} * u\right), v \right\rangle, \\ &= (-1) \left\langle \left(I_{1-s} * u\right), \frac{\partial v}{\partial x_j} \right\rangle, \\ &= -\int_{\mathbb{R}^d} \left(I_{1-s} * u\right) \frac{\partial v}{\partial x_j} dx,\end{aligned}$$

*for every $v \in C_c^\infty(\mathbb{R}^d)$.*

**Definition 1.2.4.** *(see [Theorem 1.2 of [48]]). If $u \in C_c^\infty(\mathbb{R}^d)$ then*

$$D^s u = I_{1-s} * Du.$$

**Definition 1.2.5.** *(see [13, 32, 48]). We define the Riesz fractional s-gradient $(D^s)$ and the s-divergence $(D^s.)$ by integral form for sufficiently regular functions $u$ and vector $\varphi$ as follow*

$$D^s u(x) = \mathcal{C}_{d,s} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+s+1}} (x-y) dy,$$

*and*

$$D^s.\varphi(x) = \mathcal{C}_{d,s} \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+s+1}} (x-y) dy.$$

**Proposition 1.2.5.** *(see [31, 32, 42]). For $u \in C_c^\infty(\mathbb{R}^d)$ in distributional sense the distributional Riesz fractional derivative verifies the following property*

$$-D^s.D^s u = (-\Delta)^s u = -\sum_{j=1}^{d} \frac{\partial^s}{\partial x_j^s}\frac{\partial^s}{\partial x_j^s} u.$$

*Furthermore, for $Du \in L^p(\mathbb{R}^d)^d \cap L^q(\mathbb{R}^d)^d$, $1 < q < p$, we have the following convergence*

$$D^s u \to Du \text{ in } L^p(\mathbb{R}^d)^d, \text{ as } s \nearrow 1,$$

*where $Du$ is classical derivative.*

✍ The s-gradient differential operator exhibit properties parallel to the classical gradient, we mention to the property of the duality between s-fractional gradient and the s-fractional divergence

$$D^s \cdot \varphi(x) = \mathrm{div}^s \varphi(x) = \sum_{j=1}^{d} \frac{\partial^s \varphi_j}{\partial x_j^s}.$$

**Proposition 1.2.6.** *(see [Proposition 2.3 of [11]]).(Duality between the s-gradient and the s-divergence). Let $s \in [0,1)$, $v \in C_c^\infty(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} v(x) D^s \cdot \varphi(x) \mathrm{d}x = -\int_{\mathbb{R}^d} \varphi(x) \cdot D^s v(x) \mathrm{d}x.$$

The following notion of weak fractional gradient allows a definition of function spaces, similarly to the classical Sobolev space, suitable to study problems involving s-fractional gradients.

**Definition 1.2.6.** *(see [[11]]).(Weak s-fractional gradient). Let $0 \geq s < 1$ and consider $1 \geq p \geq \infty$ if $s > 0$ or $1 < p < \infty$ if $s = 0$. We define the weak s-fractional gradient of a function $f \in L^p(\mathbb{R}^d)$ the function $\mathscr{T} \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ that satisfies*

$$\int_{\mathbb{R}^d} f D^s.\Phi dx = -\int_{\mathbb{R}^d} \mathscr{T}.\Phi dx \quad \forall \Phi \in C_c^\infty(\mathbb{R}^{\mathbb{R}^d,\mathbb{R}^d}).$$

*To simplify the notation, we write $D^s f = \mathscr{T}$.*

✍ It is important to note that this definition only makes sense because $D^s. = \mathrm{div}^s : C_c^\infty(\mathbb{R}^d, \mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ continuously, see [[13], Corollary 2.3] for the case $s > 0$ and [[27], Corollary 5.2.8] for the case $s = 0$.

### 1.2.3 The fractional spaces $X^{s,p}$ and $L^{s,p}$

In this subsection we will focus to introduce the fractional space $X^{s,p}(\mathbb{R}^d)$ and the Bessel potential space $L^{s,p}(\mathbb{R}^d)$, $L_0^{s,p}(\Omega)$. Additionally, some basic necessaries and proprieties that will be needed.

**Definition 1.2.7.** *[48] Let $p \in (1, \infty)$ we define the space $X^{s,p}(\mathbb{R}^d)$ if $u \in C_c^\infty(\mathbb{R}^d)$ as follows*

$$X^{s,p}(\mathbb{R}^d) = \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{X^{s,p}(\mathbb{R}^d)}},$$

*when the norm $\|\cdot\|_{X^{s,p}(\mathbb{R}^d)}$ define as:*

$$\|u\|_{X^{s,p}(\mathbb{R}^d)} = \|u\|^p_{L^p(\mathbb{R}^d)} + \|D^s u\|^p_{L^p(\mathbb{R}^d)}.$$

**Definition 1.2.8.** *[48, 49] Let $p \in (1, \infty)$ we define the Bessel Potential space $L^{s,p}(\mathbb{R}^d)$ as the following*

$$L^{s,p}(\mathbb{R}^d) \, : \, = h_s(L^p(\mathbb{R}^d)),$$

*in the sense that every $u \in L^{s,p}(\mathbb{R}^d)$ can be written as follows*

$$u = h_s * f, \text{ for some } f \in L^p(\mathbb{R}^d),$$

*where $h_s$ is the Bessel potentials define for $s \in \mathbb{R}$ by (see [48])*

$$h_s(x) \, : \, = \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_0^\infty \exp\left(\frac{-\pi|x|^2}{\delta}\right) \exp\left(\frac{-\delta}{4\pi}\right) \delta^{\frac{s-d}{2}} \frac{\mathrm{d}\delta}{\delta}.$$

**Proposition 1.2.7.** *(see [48])(Proposition of the space $L^{s,p}(\mathbb{R}^d)$ ).*

> *(a) If $s$ is a non-negative integer and $p \in (1, \infty)$, then $L^{s,p}(\mathbb{R}^d)$ coincides with the space $W^{s,p}(\mathbb{R}^d)$, the norm in the two spaces being equivalent.*
>
> ▶ *This deduce is true for any $s$ if $p = 2$.*
>
> *(b) For $0 < s < 1$, according the theorem 1.7 pp.05 and remark 2.3 pp. 09 in [48] that*
>
> $$X^{s,2}(\mathbb{R}^d) = L^{s,2}(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d),$$
>
> *with the norm*
>
> $$\|u\|^2_{L^{s,2}(\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|D^s u\|^2_{L^2(\mathbb{R}^d)}.$$

**Definition 1.2.9.** *[48] Let $s \in (0, 1)$, we consider the $L_0^{s,2}(\Omega)$ space as follows*

$$L_0^{s,2}(\Omega) = \overline{\left\{u \in C_c^\infty(\mathbb{R}^d), supp(u) \subset \Omega\right\}}^{\|\cdot\|_{L^{s,2}(\mathbb{R}^d)}},$$

*also,if $\Omega$ subset $\mathbb{R}^d$, we have*

$$L_0^{s,2}(\Omega) = \left\{u \in L^{s,2}(\mathbb{R}^d; u = 0 \in \mathbb{R}^d \setminus \Omega)\right\}.$$

**Proposition 1.2.8.** *(see [31])(Fractional Poincaré inequality) Let $s \in ]0, 1[$, then there exists a constant $C_{emb} = C(\Omega, d, s) > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C_{emb} \|D^s u\|_{L^2(\mathbb{R}^d)},$$

*for every $u \in L_0^{s,2}(\Omega)$.*

**Remark 1.2.4.** *We note from the above proposition 1.2.8 that the norms* $\|u\|^2_{L^{s,2}(\mathbb{R}^d)}$ *and* $\|D^s u\|^2_{L^2(\mathbb{R}^d)}$ *are equivalent norms in* $L^{s,2}_0(\Omega)$.
*So, we consider the space* $L^{s,2}_0(\Omega)$ *with the norm*

$$\|u\|^2_{L^{s,2}_0(\Omega)} = \|D^s u\|^2_{L^2(\mathbb{R}^d)} := \frac{C^2_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y,$$

*and with the inner product*

$$\langle u, v\rangle_{L^{s,2}_0(\Omega)} = \int_{\mathbb{R}^d} D^s u(x)\cdot D^s v(x)dx.$$

*It is a Hilbert space.*

**Proposition 1.2.9.** *(see [31]).(embeddings continuous and compact) Let* $s\in]0,1]$ *and* $1\leq q\leq 2^*$ *where* $2^* = \dfrac{2d}{d-2s}$ *and* $2s<d$ *then, according the Sobolev Poincaré inequality, we have the following embedding*

$$L^{s,2}_0(\Omega)\hookrightarrow L^q(\Omega),$$

*the above embedding is compact for* $1\leq q<2^*$.

## 1.3 The space $L^p(0,T;\mathbb{X})$

✍ In the following denote $\mathbb{X}$ be a Banach space endowed with the norm $\|.\|_{\mathbb{X}}$ and $T$ be a strictly positive real. We define the following spaces $1\leq p\leq\infty$:

$$C([0,T];\mathbb{X}) = \{u:\,[0,T]\to\mathbb{X}\text{ continue }\},$$

$$L^p(0,T;\mathbb{X}) = \{u:\,(0,T)\to\mathbb{X}\text{ mesurable; }\int_0^T\|u(t)\|^p_{\mathbb{X}}dt<\infty\},$$

with the norm

$$\|u\|_{L^p(0,T;\mathbb{X})} = \left(\int_0^T\|u(t)\|^p_{\mathbb{X}}dt\right)^{\frac{1}{p}}.$$

► for every $1\leq p\leq\infty$, the space $L^p(0,T;\mathbb{X})$ is a Banach space.
Recall that

$$L^1_{Loc}(\mathbb{X};E) = \{f:\,f\in L^1(K)\text{ for every compact }K\text{ of }\mathbb{X}\}.$$

**Proposition 1.3.1.** *[50] For* $1\leq p\leq\infty$ *we have:*

*(i) If* $\mathbb{X}$ *is separable then* $L^p(0,T;\mathbb{X})$ *is also separable.*

*(ii) If* $\mathbb{X}$ *is reflexive then* $L^p(0,T;\mathbb{X})$ *is reflexive* $(1<p<\infty)$.

**Remark 1.3.1.** *[50] Let $\mathbb{X}$ and $\mathbb{Y}$ two Banach spaces such that $\mathbb{X} \hookrightarrow \mathbb{Y}$. Then, it is clear that $L^p(0,T;\mathbb{X}) \hookrightarrow L^p(0,T;\mathbb{Y})$, $1 \leq p \leq \infty$.*

**Theorem 1.3.1.** *[50, 21] Let $1 < p \leq \infty$ and $1 \leq q \leq \infty$. Let $\mathbb{X}, \mathbb{Y}$ and $\mathbb{Z}$ Banach spaces such as $\mathbb{X} \overset{c}{\hookrightarrow} \mathbb{Y} \hookrightarrow \mathbb{Z}$. if $A$ is a bounded subset in $W^{1,p}(0,T;\mathbb{Z})$[4] and in $L^p(0,T;\mathbb{X})$, then $A$ is relatively compact in $C([0,T];\mathbb{Z})$ and in $L^q(0,T;\mathbb{Y})$.*

## 1.4 Fixed point theorems

There are several important results on different cases of fixed point theorems, and it is an essential to prove existence and uniqueness of solution of various mathematical models including ordinary and partial differential equations, integral equations, and variational inequalities. In this part we will mention two fixed point theorems. Firstly, the Brouwer fixed point theorem which said " *every continuous mapping of the closd unit in $\mathbb{R}^d$ posses a fixed point (**1910**)*" and prove it in ***1912***. Secondly, the Schauder fixed point theorem, extended by Schauder in ***1930*** from Brouwer's theorem asserts that any compact convex set in a Banach space has at least a fixed point. Finally, we mention The Banach contraction principle where Stefan Banach introduced the notion of a Banach space and developed a fixed point theorem for a contraction mapping in ***1922***, to show that the mapping has a unique fixed point. We refer to [14, 22, 26, 38, 52] for further information about evolution of the fixed point theorem over time.

**Theorem 1.4.1.** *[14, 52](Schauder's fixed point theorem) Let $F$ be a closed bounded, convex, and non-empty subset of a Banach space $(\mathbb{Y}, \|.\|)$. Then any continuous compact mapping $\mathcal{B} : F \to F$ has at least one fixed point.*

**Definition 1.4.1.** *(see [pp. 2, Definition 1.2.1 of [52]]). Let $\mathcal{B}$ be a mapping of a metric space $\mathbb{X}$ into $\mathbb{X}$. We say that $\mathcal{B}$ is a contraction mapping if there exists a number $k$ such that $0 < k < 1$ and*

$$d(\mathcal{B}x, \mathcal{B}y) \leq kd(x,y) \qquad \textit{for all } x, y \in \mathbb{X}.$$

**Theorem 1.4.2.** *(see [pp. 2, Theorem 1.2.2 of [52]]) (Banach Contraction principle, 1922) Any contraction mapping of a complete non-empty metric space $\mathbb{X}$ into $\mathbb{X}$ has a unique fixed-point in $\mathbb{X}$.*

## 1.5 Leray Schauder degree theory

Jean Leray and Juliusz Schauder in ***1934*** developed the concept of degree also known as Leray-Schauder degree to a class of transformations on Banach spaces

---

[4] [21] $W^{1,p}(0,T;\mathbb{Z}) = \{u \in L^p(0,T;\mathbb{Z}) | u' \in L^p(0,T;\mathbb{Z}).\}$

associated with certain type of mapping (see [5, 16]). Leray-Schauder degree theory is an important topological tool to establish the existence of solutions for certain types of nonlinear equations.
In the following lines, we will present the definitions of compact mapping and Leray-Schauder degree for a compact operator in Banach space.

**Definition 1.5.1.** *(see [pp. 87, Definition 36 of [24] ]). Let* $\mathbb{F}$ *be a real Banach space. We denote by* $\mathsf{B}$ *the set of triplets* $(Id-g,\Omega,z)$ *where* $\Omega$ *is a bounded open set in* $\mathbb{F}$. $g:\overline{\Omega}\to\mathbb{F}$ *is a* *compact mapping* *(equivalent,* $g$ *is continuous and the set* $\{g(x), x\in\overline{\Omega}\}$ *is relatively compact subset of* $\mathbb{F}$*) and* $z\in\mathbb{F}$ *such that* $z\notin\{x-g(x), x\in\partial\Omega\}$.

**Theorem 1.5.1.** *(see [pp. 87, Theorem 3.7 of [24]]).(Leray-Schauder degree). Let* $\mathbb{F}$ *be a Banach space and* $\mathsf{B}$ *the set of triples* $(Id-g,\Omega,y)$ *where* $\Omega$ *is a bounded open subset of* $\mathbb{F}$*, we have* $z\in\mathbb{F}$ *and* $g:\overline{\Omega}\to\mathbb{R}^d$ *is continuous and compact function with* $z\notin(Id-g)(\partial\Omega)$*, then there exists a map* $d:\mathsf{B}\to\mathbb{Z}$ *satisty the following properties:*

> *(1)* ***Normality:*** *if* $z\in\Omega$ *therefor* $d(Id,\Omega,z)=1$.
>
> *(2)* ***Additivity:*** *if* $\Omega_1\cup\Omega_2\subset\Omega$ *with* $\Omega_1\cap\Omega_2=\phi$ *and* $z\notin\{x-g(x), x\in\overline{\Omega}/\Omega_1\cup\Omega_2\}$*, then*
>
> $$d(Id-g,\Omega,z)=d(Id-g,\Omega_1,z)+d(Id-g,\Omega_2,z).$$
>
> *(3)* ***Homotopy invariance:*** *if* $h$ *is a compact map from* $[0,1]\times\overline{\Omega}$ *into* $\mathbb{F}$*,* $z$ *is a continous map from* $[0,1]$ *into* $\mathbb{F}$ *and for every* $t\in[0,1]$ *we have* $z(t)\notin\{h(t,x), x\in\partial\Omega\}$ *then*
>
> $$d(Id-h(0,.),\Omega,z(0))=d(Id-h(1,.),\Omega,z(1)).$$

*d is named the Leray-Schauder degree.*

**Proposition 1.5.1.** *(see [24]). Leray-Schauder degree verifies the following property:*

> *(a) if* $d(Id-g,\Omega,z)\neq 0$ *then there exists* $x\in\Omega$ *such that* $x-g(x)=z$.
>
> *(b) For all* $y\in\mathbb{F}$*,* $d(Id-g,\Omega,z)=d(Id-g-y,\Omega,z-y)$.
>
> *(c) For all* $y\in\mathbb{F}$*,* $d(Id-g,\Omega,z)=d((Id-g)(.-y),y+\Omega,z)$.
>
> *(d)* $d(Id-g,\Omega,.)$ *is constant on connected component of* $E-(Id-g)(\partial\Omega)$.
>
> *(e) Let* $d(Id-g,\Omega,z)\in\mathbb{F}$ *and* $R=dis(z,Id-g)(\partial\Omega)>0$*. If* $h$*:* $\overline{\Omega}\to\mathbb{R}^d$

*compact and* $y \in \mathbb{R}^d$ *are such as* $\sup\limits_{\partial\Omega}\big(\|g-h\| + \|z-y\| < R\big)$, *then*

$$d(Id - g, \Omega, z) = d(Id - g, \Omega, y).$$

# Chapter 2

# Existence and uniqueness of weak solutions for semilinear fractional equation involving the distributional Riesz fractional derivative

In this chapter, we will study the existence and uniqueness of weak solutions for semilinear fractional problem in fractional Sobolev space. We use Leray-Schauder degree to establish the existence of our problem under certain hypothesis on the semilinear term. In additional to hypothesis on the semilinear term, we prove the uniqueness of our results using the Banach contraction principle theorem.

**Contents**

***References***

- ❒ [43] Saadi C, Lakhal H, Slimani K, Dob S. Existence and uniqueness of distributional solution for semilinear fractional elliptic equation involving new operator and some numerical results. *Math Methods Appl Sci.* 2022;45(7):3843-3854. `doi:10.1002/mma.8019`.

## 2.1 Introduction and proposed model

In the last decade, fractional calculus has been the focus of attention of many researchers and scientists in various fieds due to its numerous applications. Fractional derivative include a variety of definitions and concepts such as: Atangan-aBaleanu derivative, Katugampola derivative, Caputo Fabrizio derivative, Riesz derivative, Riemann-Liouville derivative, etc..

This chapter present theoretical results for fractional partial differential equation related to the distributional Riesz fractional derivative. In fact, these types of problems is interesting and have been extensive where many researches and scientists attracted with distributional Riesz fractional derivative and presented important results in various fields. We recommend reviewing some notable works on this topic see ([1, 31, 42, 43, 45, 47, 48, 49, 51]) and the references therein.
In this current chapter, we are interested in the Riesz fractional derivative where we are going to study the existence and uniqueness results for the semilinear fractional problem involving this derivative under certain hypothesis. We present our problem as follows:

$$\begin{cases} -D^s.D^s u(x) + g(x,u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{2.1}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded open domain with a Lipschitz boundary, $s \in (0,1)$ satisfying $2s < d$.
In fact, where $s \to 1$ our problem is similar to classical problem

$$\begin{cases} -\Delta u + g(x,u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Where $-\Delta u$ is the classical Laplacian.
The weak formulation for the problem (2.1) is as follows

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^d} D^s u(x).D^s w(x)dx + \int_\Omega g(x,u)w(x)dx = \int_\Omega f(x)w(x)dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall w \in L_0^{s,2}(\Omega), \end{cases} \tag{2.2}$$

the fractional derivative $D^s$ is named the distributional Riesz fractional derivative which defined as (see Definition 1.2.3). Additionally, $f \in L^2(\Omega)$ and the map $g(x,k)$:$\Omega \times \mathbb{R} \to \mathbb{R}$ measurable on $x \in \Omega$ and continuous on $k \in \mathbb{R}$ satisfying the following hypothesis:

(H1) Growth hypothesis
$|g(x,k)| \leq a(x) + h|k| \quad \forall k \in \mathbb{R}, \quad \text{a.e. } x \in \Omega,$
where $a \in L^2(\Omega)$, and $h \in \mathbb{R}^+$.

(H2) Sign hypothesis:
$g(x,k)k \geq 0 \quad \forall k \in \mathbb{R} \quad \text{a.e} \quad x \in \Omega.$

The first left side $-D^s.(D^s u)$ is nonlocal operator define in the duality sense [31] that is mean

$$< -D^s.(D^s u), v >= \int_{\mathbb{R}^d} D^s u.D^s v dx, \text{ with } D^s u \in [L^2(\mathbb{R}^d)].$$

Recently, problems involving the distributional Riesz fractional derivative have been thoroughly investigated. Shieh and Spector [48] were the first to consider a new class of fractional partial differential equations involving the nonlocal operator $-D^s.(D^s u)$ and they studied the existence and uniqueness of linear fractional partial differential equation and using Lax-Milgram theorem. Our study extends their work to semilinear cases, proposing the Leray-Schauder degree method to deal with it.

This chapter is structured as follows: in the first section we prove the existence results using the Leray-Schauder degree theory with some hypothesis in the semilinear term, the third section we will using Banach fixed point theorem, along with additional to hypothesis to prove the uniqueness results.

## 2.2 Existence results

We consider the following linear problem to illustrate a specific application for expressing a fixed point problem.

For $u \in L^2(\Omega)$, we define the following linear problem

$$\begin{cases} -D^s.D^s v(x) = tf(x) - tg(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^d \setminus \Omega, \end{cases} \tag{2.3}$$

where $f \in L^2(\Omega)$.

**Theorem 2.2.1.** *Under the hypothesis (H1), the problem (2.3) has a unique weak solution $v \in L_0^{s,2}(\Omega)$.*

***Proof.*** We will use Lax-Milgram theorem for proving theorem 2.2.1.

For all $u \in L^2(\Omega)$, we have $g(.,u) \in L^2(\Omega)$. The weak formulation of the problem (2.3) is as follows:

$$\begin{cases} \text{Find } v \in L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^d} D^s v(x).D^s w(x) dx = t\int_\Omega f(x)w(x)dx - t\int_\Omega g(x,u)w(x)dx, \\ \qquad\qquad\qquad\qquad \forall w \in L_0^{s,2}(\Omega). \end{cases} \tag{2.4}$$

With the bilinear form define as

$$a(v,w) : L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega) \to \mathbb{R},$$
$$(v,w) \mapsto a(v,w) = \int_{\mathbb{R}^d} D^s v(x).D^s w(x) dx,$$

and the linear form define as follow

$$
\begin{aligned}
l(w) :\ & L_0^{s,2}(\Omega) \to \mathbb{R}, \\
& w \mapsto l(w) = t\int_\Omega f(x)w(x)dx - t\int_\Omega g(x,u)w(x)dx \quad \forall t \in [0,1], \\
& \qquad\qquad = l_f(w) - l_u(w),
\end{aligned}
$$

when $l_f(w) = t\int_\Omega f(x)w(x)dx$ and $l_u(w) = t\int_\Omega g(x,u)w(x)dx$.

**Step 01:** We prove that the bilinear form $a(.,.)$ is coercive.
For all $v \in L_0^{s,2}(\Omega)$

$$
\begin{aligned}
a(v,v) &= \int_{\mathbb{R}^d} |D^s v(x)|^2 dx, \\
&= \|v\|^2_{L_0^{s,2}(\Omega)},
\end{aligned}
$$

thus, a(.,.) is coercive.

**Step 02:** We prove that the bilinear form $a(.,.)$ is continuous.
For all $w, v \in L_0^{s,2}(\Omega)$, we have

$$
|a(w,v)| = |\int_{\mathbb{R}^d} D^s v(x).D^s w(x)dx|,
$$

then, applying the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
|a(w,v)| &\leq \|D^s v\|_{L^2(\mathbb{R}^d)}\|D^s w\|_{L^2(\mathbb{R}^d)}, \\
&= \|v\|_{L_0^{s,2}(\Omega)}\|w\|_{L_0^{s,2}(\Omega)}.
\end{aligned}
$$

Therefore $a(.,.)$ is continuous.

**Step 03:** We prove that the linear form $l(.)$ is continuous.
For all $w \in L_0^{s,2}(\Omega)$, we have

$$
\begin{aligned}
|l(w)| &= |l_f(w) - l_u(w)|, \\
&\leq |l_f(w)| + |l_u(w)|. \qquad (2.5)
\end{aligned}
$$

▷ We will simplify the right hand side of the above inequality.

- We start by simplify the $|l_f(w)|$.

Hence, from proposition 1.2.8 and the Hölder inequality, hypothesis (H1), we obtain

$$
\begin{aligned}
|l_f(w)| &= |t\int_\Omega f(x)w(x)dx|, \\
&\leq \|f\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}, \\
&\leq C_{emb}|f\|_{L^2(\Omega)}\|w\|_{L_0^{s,2}(\Omega)},
\end{aligned}
$$

on the other hand, we will simplify the term $|l_u(w)|$, we have

$$\begin{aligned} |l_u(w)| &= |t\int_\Omega g(x,u)w(x)dx|, \\ &\leq \int_\Omega |g(x,u)||w(x)|dx, \end{aligned}$$

from the hypothesis (H1) and the Cauchy-Schwarz inequality, we obtain

$$|l_u(w)| \leq C_{emb}\left(\|a\|_{L^2(\Omega)} + h\|u\|_{L^2(\Omega)}\right)\|w\|_{L_0^{s,2}(\Omega)},$$

when we came back to (2.5), we find

$$|l(w)| \leq C_{emb}\left(\|a\|_{L^2(\Omega)} + h\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)\|w\|_{L_0^{s,2}(\Omega)}.$$

Hence, $L(.)$ is continuous.

- As a result, we may apply the Lax-Milgram theorem and conclude that the problem (2.3) has a unique weak solution $v \in L_0^{s,2}(\Omega)$. □

✍ Now, we consider The following operator which is well defined

$$\begin{aligned} T\,:\,[0,1]\times L^2(\Omega) &\to L_0^{s,2}(\Omega), \\ (t,u) &\mapsto T(t,u) = v, \end{aligned}$$

where $v$ is the solution of (2.3).

**Remark 2.2.1.** *We remark that our problem (2.1) is equivalent to the following problem*

$$\begin{cases} u \in L^2(\Omega), \\ T(1,u) = u, \end{cases} \tag{2.6}$$

*and we will prove that by the Leray-Schauder degree.*

For prove the equation (2.6), we need the following lemmas about the conditions of Leray-Schauder degree theory

**Lemma 2.2.1.** **(Priori estimate)** *We will show that*
$\exists R > 0$ *such that*

$$\begin{cases} T(t,u) = u, \\ t \in [0,1], u \in L^2(\Omega), \end{cases} \Rightarrow \|u\|_{L^2(\Omega)} < R + 1.$$

✒ ***Proof.*** Let $T(t,u) = u$, for every $t \in [0,1]$, we have

$$\int_{\mathbb{R}^d} D^s u(x) D^s w(x)dx = \int_\Omega tf(x)w(x)dx - \int_\Omega tg(x,u)wdx,$$

we take $w(x) = u(x)$ from (2.4) and according the hypothesis (H2), we find

$$\begin{aligned}\int_{\mathbb{R}^d} D^s u(x) D^s u(x) dx &= \int_\Omega t f(x) u(x) dx - \int_\Omega t g(x,u) u dx, \\ &\leq \int_\Omega t f(x) u(x) dx,\end{aligned}$$

then, by Cauchy-Schwarz inequality implies that

$$\int_{\mathbb{R}^d} |D^s u(x)|^2 dx \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)},$$

moreover, from the proposition 1.2.8, we obtain

$$\frac{1}{C_{emb}^2} \|u\|_{L^2(\Omega)}^2 \leq \int_{\mathbb{R}^d} |D^s u(x)|^2 dx \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)},$$

then,

$$\|u\|_{L^2(\Omega)} \leq C_{emb}^2 \|f\|_{L^2(\Omega)} = R,$$

so,

$$\|u\|_{L^2(\Omega)} < R + 1.$$

We can deduce from the above inequality that for all $t \in [0,1]$ there are no solutions to the equation $T(t,u) = u$ in the boundary of the ball
$B(0, R+1) = \{v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} < R+1\}$. □

**Lemma 2.2.2.** *Under the hypothesis (H1), $T$: $[0,1] \times L^2(\Omega) \to L^2(\Omega)$ is continuous.*

***Proof.*** Let $\{t_n, u_n\}_{n\in\mathbb{N}} \subset [0,1] \times L^2(\Omega)$ which converges to $(t,u)$ in $[0,1] \times L^2(\Omega)$ when $n \to +\infty$. We will show that if $T(t_n, u_n)$ converges to $T(t,u)$, putting $T(t_n, u_n) = v_n$ and $T(t,u) = v$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} D^s v_n(x) . D^s w(x) dx = &\int_\Omega t_n f(x) w(x) dx \\ &- \int_\Omega t_n g(x, u_n) w(x) dx, \quad \forall w \in L_0^{s,2}(\Omega),\end{aligned} \tag{2.7}$$

on other hand, we have

$$\begin{aligned}\int_{\mathbb{R}^d} D^s v(x) . D^s w(x) dx = &\int_\Omega t f(x) w(x) dx \\ &- \int_\Omega t g(x,u) w(x) dx, \quad \forall w \in L_0^{s,2}(\Omega),\end{aligned} \tag{2.8}$$

taking the difference of the two formulas (2.7) and (2.8), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \left(D^s v_n - D^s v\right) . D^s w dx = &\int_\Omega (t_n - t) f w dx \\ &- \int_\Omega \left(t g(x,u) - t_n g(x, u_n)\right) w dx, \forall w \in L_0^{s,2}(\Omega),\end{aligned}$$

then, we take $w(x) = v_n(x) - v(x)$ and apply the Cauchy-Schwarz inequality, we find

$$\|v_n - v\|^2_{L_0^{s,2}(\Omega)} \le \left(|t_n - t|\|f\|_{L^2(\Omega)} + \|tg(.,u) - t_n g(.,u_n)\|_{L^2(\Omega)}\right)\|v_n - v\|^2_{L^2(\Omega)},$$

according the proposition 1.2.8 , we get

$$\|v_n - v\|_{L^2(\Omega)} \le C^2_{emb}\left(|t_n - t|\|f\|_{L^2(\Omega)} + \|tg(.,u) - t_n g(.,u_n)\|_{L^2(\Omega)}\right), \tag{2.9}$$

we have $u_n \to u$ in $L^2(\Omega)$ implies that

$$\begin{cases} u_{n_k} \to u, & \text{a.e on } \Omega, \\ |u_{n_k}| < H, & \text{for all } k \text{ a.e on } \Omega, \text{ or } H \in L^2(\Omega), \end{cases}$$

hence, from the hypothesis (H1), we obtain

$$\begin{cases} g(x, u_{n_k}) & \to g(x,u), & a.e \quad \text{on } \Omega, \\ |g(x, u_{n_k}| & < a(x) + hH & \in L^2(\Omega) \qquad \text{for all } k \text{ a.e on } \Omega, \end{cases}$$

then, from Lebesgue convergence theorem, we find that $g(x,u_n) \to g(x,u)$ in $L^2(\Omega)$, we have $(t_n)_{n\in\mathbb{N}}$ converges to $t$ when $n \to +\infty$. Therfore, when we came back to the iniquality (2.9), we find $v_n$ converges to $v$ in $L^2(\Omega)$.
So $T$ is continuous from $[0,1] \times L^2(\Omega)$ into $L^2(\Omega)$. □

**Lemma 2.2.3.** *Under the hypothesis (H1)* $\{T(t,u), t \in [0,1], u \in \overline{B}_{R+1}\}$ *is relatively compact in* $L^2(\Omega)$.

***Proof.*** Let $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ and $(u_n)_{n\in\mathbb{N}} \subset \overline{B}_{R+1}$, we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^d} D^s v_n(x).D^s v_n(x)dx\Big| &= \Big|\int_\Omega t_n f(x)v_n(x)dx - \int_\Omega t_n g(x,u_n)v_n(x)dx\Big|, \\ &\le \Big|\int_\Omega t_n f(x)v_n(x)dx\Big| + \Big|\int_\Omega t_n g(x,u_n)v_n(x)dx\Big|, \end{aligned}$$

then, using Cauchy-Schwarz inequality, we find

$$\|v_n\|^2_{L_0^{s,2}(\Omega)} \le \left(\|f\|_{L^2(\Omega)} + \|g(.,u_n)\|_{L^2(\Omega)}\right)\|v_n\|_{L^2(\Omega)},$$

from the proposition 1.2.8, we obtain

$$\|v_n\|_{L_0^{s,2}(\Omega)} \le C_{emb}\left(\|f\|_{L^2(\Omega)} + \|g(.,u_n)\|_{L^2(\Omega)}\right), \tag{2.10}$$

using the hypothesis (H1) the sequence $\{g(x,u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$.
Then by (2.10), we obtain

$$\|v_n\|_{L_0^{s,2}(\Omega)} \le M,$$

where $M = C_{emb}\left(\|f\|_{L^2(\Omega)} + \|g(.,u_n)\|_{L^2(\Omega)}\right)$. Consequently $(v_n)_{n\in\mathbb{N}}$ is bounded in $L_0^{s,2}(\Omega)$ so $v_n \rightharpoonup v$ in $L_0^{s,2}(\Omega)$, according to proposition 1.2.9 we conclude that there is a subsequence of $\{v_{n_k}\}_{k\in\mathbb{N}}$ which converges to $v$ in $L^2(\Omega)$. □

✍ Our first main result is the following theorem.

**Theorem 2.2.2.** *Under the hypothesis (H1) and (H2), the problem (2.1) has a weak solutions* $u \in L_0^{s,2}(\Omega)$.

***Proof.*** From the previous lemmas 2.2.1,2.2.2 and 2.2.3 we concluded that $d(I_d - T(t,.), \bar{B}_{R+1}, 0)$ is well defined, by invariance of the topological degree, for $t \in [0,1]$, we get $\deg(T(t,\cdot), B(0,R), 0)$ is constant.
By the homotopy invariance property, we have

$$\begin{aligned} d(I_d - T(1,.), \overline{B}_{R+1}, 0) &= d(I_d - T(0,.), \overline{B}_{R+1}, 0), \\ &= d(I_d, \overline{B}_{R+1}, 0) = 1 \neq 0, \end{aligned}$$

therefore

$$I_d(u) - T(1,u) = 0 \Leftrightarrow u = T(1,u).$$

Hence we have showed that $u$ is a solution of (2.1).
This completes of the existence results. ☐

## 2.3 Uniqueness results

In this section, we will focus on proving the existence and uniqueness of solution using the Banach contraction principle theorem.
Let assume that $g$ is Lipschitz continuous with respect to the scened variable, that is means there exists a constant $C > 0$ such that for almost every $x \in \Omega$ and for any $k_1, k_2 \in \mathbb{R}$

$$\|g(x,k_1) - g(x,k_2)\|_{L^2(\Omega)} \leq C\|k_1 - k_2\|_{L^2(\Omega)}. \tag{2.11}$$

Thus, in order to prove that $T$ admit a unique fixed point, it is sufficient to prove that $T$ is contraction.
We take $B_t(u) = T(t,u)$ for all $t \in [0,1]$.

**Lemma 2.3.1.** *The operator* $B_t$ *is contraction from* $L^2(\Omega)$ *to* $L^2(\Omega)$ *for all* $t \in [0,1]$.

***Proof.*** Let $u_1, u_2 \in L^2(\Omega)$ and for all $t \in [0,1]$, we have

$$\int_{\mathbb{R}^d} D^s v_1(x) D^s w(x)dx = t\int_\Omega f(x)w(x)dx - t\int_\Omega g(x,u_1)w(x)dx, \tag{2.12}$$

on other hand side, we have

$$\int_{\mathbb{R}^d} D^s v_2(x) D^s w(x)dx = t\int_\Omega f(x)w(x)dx - t\int_\Omega g(x,u_2)w(x)dx, \tag{2.13}$$

by taking the difference between the equations (2.12) and (2.13), we get

$$\int_{\mathbb{R}^d} D^s(v_1(x) - v_2(x)) D^s w(x) dx = t \int_{\Omega} (g(x, u_2) - g(x, u_1)) w(x) dx,$$

then, we take $w(x) = v_1(x) - v_2(x)$, we find

$$\int_{\mathbb{R}^d} D^s(v_1(x) - v_2(x))^2 dx = t \int_{\Omega} (g(x, u_2) - g(x, u_1))(v_1(x) - v_2(x)) dx,$$

then, using Cauchy-Schwarz inequality, we obtain

$$\|v_1 - v_2\|^2_{L_0^{s,2}(\Omega)} \leq \|g(., u_2) - g(., u_1)\|_{L^2(\Omega)} \|v_1 - v_2\|_{L^2(\Omega)},$$

from the proposition 1.2.8, and hypothesis (2.11), we get

$$\|v_1 - v_2\|_{L^2(\Omega)} \leq C C^2_{emb} \|u_1 - u_2\|_{L^2(\Omega)}.$$

Consequently $B_t$ is a contraction if $CC^2_{emb} < 1$.
As result, we have $B_t$ is a contraction if $CC^2_{emb} < 1$, so we apply the Banach contraction principle theorem and we conclude that $T$ admit a unique fixed point $u \in L^2(\Omega)$, hence the problem (2.1) admit a unique weak solution. □

# Chapter 3

# Study the existence and uniqueness results for semilinear fractional system involving a nonlocal operator

In this chapter, we are interested in the semilinear fractional system involving a non-local operator. We use Leray-Schauder degree method to deal with the existence results under some conditions on the semilinear term. In additional to conditions, we prove the uniqueness results and using the Banach contraction principle theorem.

**Contents**

***References***

- ❐ [45] Saadi C, Lakhal H, Slimani K, Dob S. *Distributional solution for semilinear system involving fractional gradient and a numerical example. Int. J. Nonlinear Anal. Appl.* **14**(2023), 1175-1185. http://dx.doi.org/10.22075/ijnaa.2022.27420.3595.

## 3.1 Introduction and proposed model

In the present chapter, our objective is to establish the existence and uniqueness of weak solutions in a Bessel Potential space for the following semilinear fractional system

$$\begin{cases} -D^s.D^s u(x) + f_1(x,u,v) = \psi_1(x) & \text{in } \Omega, \\ -D^s.D^s v(x) + f_2(x,u,v) = \psi_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{3.1}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded open domain, with Lipschitz boundary, $s \in (0,1)$ with $2s < d$, $(\psi_1, \psi_2) \in (L^2(\Omega) \times L^2(\Omega))$ and $f_1(x,k,p), f_2(x,k,p) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ measurable on $x \in \Omega$ and continuous on $k, p \in \mathbb{R}$.
This system was inspired by The previous chapter , where we transformed an equation into system.
To achieve our goal, we need certain conditions on the semilinear term. Therefore, let us assume that the semilinear functions $f_1(x,k,p)$ and $f_2(x,k,p)$ satisfy the following hypothesis

(C1) Growth condition

$$\begin{aligned} |f_1(x,k,p)| &\leq a_1(x) + h_1|k| + r_1|p| \quad \forall k,p \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \\ |f_2(x,k,p)| &\leq a_2(x) + h_2|k| + r_2|p| \quad \forall k,p \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \end{aligned}$$

where $a_1, a_2 \in L^2(\Omega)$, and $h_1, h_2, r_1, r_2 \in \mathbb{R}^+$.

(C2) Sign condition:

$$\begin{aligned} f_1(x,k,p)k \geq 0 \quad \forall k,p \in \mathbb{R} \quad \text{a.e} \quad x \in \Omega. \\ f_2(x,k,p)p \geq 0 \quad \forall k,p \in \mathbb{R} \quad \text{a.e} \quad x \in \Omega. \end{aligned}$$

The weak solutions of our system (3.1) is a solutions of the the following weak formulations

$$\begin{cases} \text{Find } (u,v) \in L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^d} D^s u(x).D^s\phi_1(x)dx + \int_\Omega f_1(x,u,v)\phi_1(x)dx = \int_\Omega \psi_1(x)\phi_1(x)dx, \quad \forall \phi_1 \in L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^d} D^s v(x).D^s\phi_2(x)dx + \int_\Omega f_2(x,u,v)\phi_2(x)dx = \int_\Omega \psi_2(x)\phi_2(x)dx, \quad \forall \phi_2 \in L_0^{s,2}(\Omega), \end{cases}$$

we get the previous weak formulations by applying the fractional integration par parts formulation associated with the distributional Riesz fractional gradient see ([13, 47]).

## 3.2 Existence results

In this section, we discuss the existence of our system (3.1) using the Leray Schauder degree theory to establish our results. First, we reformulate equivalent of our problem, and then we prove that the equivalent problem has a fixed point which a solution of our system.

### Fixed point formulation of the problem (3.1) and Statement results

For $\overline{u}, \overline{v} \in L^2(\Omega)$, we define the following linear problem

$$\begin{cases} -D^s.D^s u(x) = t\psi_1(x) - tf_1(x, \overline{u}, \overline{v}) & \text{in } \Omega, \\ -D^s.D^s v(x) = t\psi_2(x) - tf_2(x, \overline{u}, \overline{v}) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \tag{3.2}$$

where $\psi_1, \psi_2 \in L^2(\Omega)$ and $t \in [0, 1]$.

✍ The following theorem give us the existence and uniqueness of the above linear system (3.2).

**Theorem 3.2.1.** *Thanks to conditions (C1), the problem* (3.2) *has a unique weak solutions* $(u, v) \in L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega)$.

***Proof.*** Using the Lax-Milgram theorem for proving the previous theorem 3.2.1.
For all $(\overline{u}, \overline{v}) \in L^2(\Omega) \times L^2(\Omega)$, we have $f_1(., \overline{u}, \overline{v}), f_2(., \overline{u}, \overline{v}) \in L^2(\Omega)$.
The weak formulation of the problem (3.2) is given as follows:

$$\begin{cases} \text{Find } (u, v) \in L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega), & \\ a_1(u, \phi_1) = l(\phi_1), & \forall \phi_1 \in L_0^{s,2}(\Omega), \\ a_2(v, \phi_2) = l(\phi_2), & \forall \phi_2 \in L_0^{s,2}(\Omega), \end{cases} \tag{3.3}$$

with

$$\begin{aligned} a_1(.,.) :\ & L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega) \to \mathbb{R}, \\ & \qquad (u, \phi_1) \mapsto a_1(u, \phi_1) = \int_{\mathbb{R}^d} D^s u(x).D^s \phi_1(x) dx, \\ l_1(.) :\ & L_0^{s,2}(\Omega) \to \mathbb{R}, \\ & \qquad w \mapsto l(\phi_1) = t \int_\Omega \psi_1(x)\phi_1(x) dx \\ & \qquad\qquad - t \int_\Omega f_1(x, \overline{u}, \overline{v})\phi_1(x) dx \quad \forall t \in [0, 1], \end{aligned}$$

and

$$\begin{aligned} a_2(.,.) :\ & L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega) \to \mathbb{R}, \\ & \qquad (v, \phi_2) \mapsto a_2(v, \phi_2) = \int_{\mathbb{R}^d} D^s v(x).D^s \phi_2(x) dx, \\ l_2(.) :\ & L_0^{s,2}(\Omega) \to \mathbb{R}, \\ & \qquad w \mapsto l(\phi_2) = t \int_\Omega \psi_2(x)\phi_2(x) dx \\ & \qquad\qquad - t \int_\Omega f_2(x, \overline{u}, \overline{v})\phi_2(x) dx \quad \forall t \in [0, 1]. \end{aligned}$$

***Step 01***: We will prove the bilinear forms $a_1(.,.)$ and $a_2(.,.)$ are coercive.
For all $\phi_1 \in L_0^{s,2}(\Omega)$, we have

$$\begin{aligned} a_1(\phi_1,\phi_1) &= \int_{\mathbb{R}^d} D^s\phi_1(x).D^s\phi_1(x)dx, \\ &= \int_{\mathbb{R}^d} |D^s\phi_1(x)|^2 dx = \|\phi_1\|^2_{L_0^{s,2}(\Omega)}. \end{aligned}$$

Thus, $a_1(.,.)$ is coercive.
In the other hand side, for all $\phi_2 \in L_0^{s,2}(\Omega)$, we have

$$\begin{aligned} a_2(\phi_2,\phi_2) &= \int_{\mathbb{R}^d} D^s\phi_2(x).D^s\phi_2(x)dx, \\ &= \int_{\mathbb{R}^d} |D^s\phi_2(x)|^2 dx = \|\phi_2\|^2_{L_0^{s,2}(\Omega)}. \end{aligned}$$

Therefore, $a_2(.,.)$ is coercive.
✍ We deduce that the bilinear forms $a_1(.,.), a_2(.,.)$ are coercive.

***Step 02:*** We will prove in this step that the bilinear form and the linear form are continuous.
For prove the bilinear form $a_1(.,.), a_2(.,.)$ and the linear form $l_1(.), l_2(.)$ are continuous.
- **First:** we prove the bilinear forms are continuous.

for every $\phi_1$ belongs $L_0^{s,2}(\Omega)$. Using the Cauchy-Schwarz inequality, for all $u, \phi_1 \in L_0^{s,2}(\Omega)$, we have

$$\begin{aligned} |a_1(u,\phi_1)| &= |\int_{\mathbb{R}^d} D^s u(x).D^s\phi_1(x)dx|, \\ &\leq \|D^s u\|_{L^2(\mathbb{R}^d)}\|D^s\phi_1\|_{L^2(\mathbb{R}^d)}, \\ &= \|u\|_{L_0^{s,2}(\Omega)}\|\phi_1\|_{L_0^{s,2}(\Omega)}, \end{aligned}$$

therefore $a_1(.,.)$ is continuous.
In other hand side, we have for every $\phi_2$ belongs $L_0^{s,2}(\Omega)$, we have

$$\begin{aligned} |a_2(v,\phi_2)| &= |\int_{\mathbb{R}^d} D^s v(x).D^s\phi_2(x)dx|, \\ &\leq \|D^s v\|_{L^2(\mathbb{R}^d)}\|D^s\phi_2\|_{L^2(\mathbb{R}^d)}, \\ &= \|v\|_{L_0^{s,2}(\Omega)}\|\phi_2\|_{L_0^{s,2}(\Omega)}, \end{aligned}$$

thus, $a_2(.,.)$ is continuous.
✍ So, we conclude that the bilinear forms $a_1(.,.), a_2(.,.)$ are continuous.
- **Second:** we prove the linear form are continuous.

For every $\phi_1 \in L_0^{s,2}(\Omega)$, according to proposition 1.2.8 and the Cauchy-Schwarz inequality and condition (C1), we find

$$\begin{aligned}
|l_1(\phi_1)| &= \Big|t\int_\Omega \psi_1(x)\phi_1(x)dx - t\int_\Omega f_1(x,\overline{u},\overline{v})\phi_1(x)dx\Big|,\\
&\le \Big|\int_\Omega \psi_1(x)\phi_1(x)dx\Big| + \Big|\int_\Omega f_1(x,\overline{u},\overline{v})\phi_1(x)dx\Big|,\\
&\le \|\psi_1\|_{L^2(\Omega)}\|\phi_1\|_{L^2(\Omega)} + \int_\Omega \left(a_1(x) + h_1|\overline{u}| + r_1|\overline{v}|\right)|\phi_1(x)|dx,\\
&\le C_{emb}\|\psi_1\|_{L^2(\Omega)}\|\phi_1\|_{L_0^{s,2}(\Omega)}dx\\
&\quad + C_{emb}\left(\|a_1\|_{L^2(\Omega)} + h_1\|\overline{u}\|_{L^2(\Omega)} + r_1\|\overline{v}\|_{L^2(\Omega)}\right)\|\phi_1\|_{L_0^{s,2}(\Omega)},
\end{aligned}$$

so,

$$\begin{aligned}
&|l_1(\phi_1)|\\
&\le C_{emb}\left(\|a_1\|_{L^2(\Omega)} + h_1\|\overline{u}\|_{L^2(\Omega)} + r_1\|\overline{v}\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\right)\|\phi_1\|_{L_0^{s,2}(\Omega)}.
\end{aligned} \tag{3.4}$$

Hence, $l_1(.)$ is continuous.
In another hand sid, we have for all $\phi_2 \in L_0^{s,2}(\Omega)$. According the proposition 1.2.8 and the Cauchy-Schwarz inequality and condition (C1), we obtain

$$\begin{aligned}
|l_2(\phi_2)| &= \Big|t\int_\Omega \psi_2(x)\phi_2(x)dx - t\int_\Omega f_2(x,\overline{u},\overline{v})\phi_2(x)dx\Big|,\\
&\le \Big|\int_\Omega \psi_2(x)\phi_2(x)dx\Big| + \Big|\int_\Omega f_2(x,\overline{u},\overline{v})\phi_2(x)dx\Big|,\\
&\le \|\psi_2\|_{L^2(\Omega)}\|\phi_2\|_{L^2(\Omega)} + \int_\Omega \left(a_2(x) + h_2|\overline{u}| + r_2|\overline{v}|\right)|\phi_2(x)|dx,\\
&\le C_{emb}\|\psi_2\|_{L^2(\Omega)}\|\phi_2\|_{L_0^{s,2}(\Omega)}\\
&\quad + C_{emb}\left(\|a_2\|_{L^2(\Omega)} + h_2\|\overline{u}\|_{L^2(\Omega)} + r_2\|\overline{v}\|_{L^2(\Omega)}\right)\|\phi_2\|_{L_0^{s,2}(\Omega)}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&|l_2(\phi_2)|\\
&\le C_{emb}\left(\|a_2\|_{L^2(\Omega)} + h_2\|\overline{u}\|_{L^2(\Omega)} + r_2\|\overline{v}\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\right)\|\phi_1\|_{L_0^{s,2}(\Omega)},
\end{aligned} \tag{3.5}$$

so, $l_2(.)$ is continuous.
✍ In the end, we prove from (3.4) et (3.5) that the linear forms $l_1(.)$ and $l_2(.)$ are continuous.
As a result, we can apply the Lax-Milgram theorem and conclude that the problem (3.2) has a unique weak solution $(u,v) \in L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega)$. □

■ Now, we will consider the following operator

$$\begin{aligned}
H\,:\,[0,1]\times L^2(\Omega)\times L^2(\Omega) &\to L_0^{s,2}(\Omega)\times L_0^{s,2}(\Omega),\\
(t,\overline{u},\overline{v}) &\mapsto H(t,\overline{u},\overline{v}) = (u,v),
\end{aligned}$$

where $(u,v)$ is a solutions of the system (3.2). We showed in the previous proof that the map $H$ is well-defined because the value $(u,v)$ is uniquely by the elements $t,\overline{u}$ and $\overline{v}$.

**Remark 3.2.1.** *We remark that our problem* (3.1) *is equivalent to the following fixed point problem*

$$\begin{cases} (u,v) \in L^2(\Omega) \times L^2(\Omega), \\ H(1,\overline{u},\overline{v}) = (\overline{u},\overline{v}), \end{cases} \tag{3.6}$$

*and we will prove the problem* (3.6) *by Leray-Schauder degree theory.*

□ For establishing our results, we need to considering Some several lemmas about the condition of the Leray-Schauder degree theory.

### 3.2.1 Several Lemmas

In the present section, we will use the Leray-Shauder degree theory to obtain existence result of the our system (3.1).

**Lemma 3.2.1.** *(Priori estimate) We will show that*
$\exists R > 0$, $\forall (\overline{u},\overline{v}) \in L^2(\Omega) \times L^2(\Omega)$ *such that*

$$\begin{cases} H(t,\overline{u},\overline{v}) = (\overline{u},\overline{v}), \\ t \in [0,1], (\overline{u},\overline{v}) \in L^2(\Omega) \times L^2(\Omega), \end{cases} \quad \Rightarrow \|(\overline{u},\overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} < R+1.$$

***Proof.*** Let $H(t,\overline{u},\overline{v}) = (u,v) = (\overline{u},\overline{v})$, for all $t \in [0,1]$, that is mean

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} D^s\overline{u}(x).D^s\phi_1(x)dx = t\int_\Omega \psi_1(x)\phi_1(x)dx - t\int_\Omega \Phi_1(x,\overline{u},\overline{v})\phi_1(x)dx, \\ \hfill \forall \phi_1 \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^d} D^s\overline{v}(x).D^s\phi_2(x)dx = t\int_\Omega \psi_2(x)\phi_2(x)dx - t\int_\Omega \Phi_2(x,\overline{u},\overline{v})\phi_2(x)dx, \\ \hfill \forall \phi_2 \in L_0^{s,2}(\Omega), \end{cases} \tag{3.7}$$

then, we take $\phi_1(x) = \overline{u}(x)$ and $\phi_2(x) = \overline{v}(x)$, according the condition (C2) and Cauchy-Schwarz inequality, we get

$$\begin{cases} \int_{\mathbb{R}} |D^s\overline{u}(x)|^2dx & \leq t\int_\Omega \psi_1(x)\overline{u}(x)dx, \\ \int_{\mathbb{R}} |D^s\overline{v}(x)|^2dx & \leq t\int_\Omega \psi_2(x)\overline{v}(x)dx, \end{cases} \Rightarrow \begin{cases} \int_{\mathbb{R}} |D^s\overline{u}(x)|^2dx & \leq \|\psi_1\|_{L^2(\Omega)}\|\overline{u}\|_{L^2(\Omega)}, \\ \int_{\mathbb{R}} |D^s\overline{v}(x)|^2dx & \leq \|\psi_2\|_{L^2(\Omega)}\|\overline{v}\|_{L^2(\Omega)}, \end{cases}$$

moreover, from the proposition 1.2.8, we find

$$\begin{cases} \|\overline{u}\|_{L^2(\Omega)} & \leq C_{emb}^2\|\psi_1\|_{L^2(\Omega)}, \\ \|\overline{v}\|_{L^2(\Omega)} & \leq C_{emb}^2\|\psi_2\|_{L^2(\Omega)}, \end{cases} \tag{3.8}$$

hence, taking the sum between two inequalities of (3.8), we find

$$\|(\overline{u},\overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_{emb}^2\|\psi_1\|_{L^2(\Omega)} + C_{emb}^2\|\psi_2\|_{L^2(\Omega)}, \tag{3.9}$$

thus, we put $R = C_{emb}^2\left(\|\psi_1\|_{L^2(\Omega)} + \|\psi_2\|_{L^2(\Omega)}\right)$, implies that

$$\|(\overline{u}, \overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} < R + 1. \tag{3.10}$$

✍ Finally, we can deduce from equation (3.10) that for every $t \in [0, 1]$ there are no solutions to the equation $H(t, \overline{u}, \overline{v}) = (\overline{u}, \overline{v})$ in the boundary of the sphere $B(0, R+1) = \{v \in L^2(\Omega) : \|(\overline{u}, \overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} < R + 1.\}$. □

**Lemma 3.2.2.** *Under the condition (C1), $H$: $[0, 1] \times L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is continuous.*

***Proof.*** Let $\{t_n, \overline{u}_n, \overline{u}_n\}_{n\in\mathbb{N}} \subset [0, 1] \times L^2(\Omega) \times L^2(\Omega)$ which converges to $(t, \overline{u}, \overline{v})$ in $[0, 1]\times L^2(\Omega)\times L^2(\Omega)$ when $n \to +\infty$. We will show that $H(t_n, \overline{u}_n, \overline{v}_n)$ converges to $H(t, \overline{u}, \overline{v})$, we pose $H(t_n, \overline{u}_n, \overline{v}_n) = (u_n, v_n)$ and $H(t, \overline{u}, \overline{v}) = (u, v)$, we obtain

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} D^s u_n(x).D^s\phi_1(x)dx = \int_\Omega t_n\psi_1(x)\phi_1(x)dx - \int_\Omega t_n f_1(x, \overline{u}_n, \overline{v}_n)\phi_1(x)dx, \\ \hfill \forall\phi_1 \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^d} D^s v_n(x).D^s\phi_2(x)dx = \int_\Omega t_n\psi_2(x)\phi_2(x)dx - \int_\Omega t_n f_2(x, \overline{u}_n, \overline{v}_n)\phi_2(x)dx, \\ \hfill \forall\phi_2 \in L_0^{s,2}(\Omega), \end{cases} \tag{3.11}$$

and

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} D^s u(x).D^s\phi_1(x)dx = \int_\Omega t\psi_1(x)\phi_1(x)dx - \int_\Omega t f_1(x, \overline{u}, \overline{v})\phi_1(x)dx, \\ \hfill \forall\phi_1 \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^d} D^s v(x).D^s\phi_2(x)dx = \int_\Omega t\psi_2(x)\phi_2(x)dx - \int_\Omega t f_2(x, \overline{u}, \overline{v})\phi_2(x)dx, \\ \hfill \forall\phi_2 \in L_0^{s,2}(\Omega), \end{cases} \tag{3.12}$$

then, we take the difference between (3.11) and (3.12), we obtain

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} \left(D^s u_n - D^s u\right).D^s\phi_1 dx = \\ \qquad \displaystyle\int_\Omega (t_n - t)\psi_1\phi_1 dx - \int_\Omega \left(t f_1(x, \overline{u}, \overline{v}) - t_n f_1(x, \overline{u}_n, \overline{v}_n)\right)\phi_1 dx, \\ \hfill \forall\phi_1 \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^d} \left(D^s v_n - D^s v\right).D^s\phi_2 dx = \\ \quad \displaystyle\int_\Omega (t_n - t)\psi_2\phi_2 dx - \int_\Omega \left(t f_2(x, \overline{u}, \overline{v}) - t_n f_2(x, \overline{u}_n, \overline{v}_n)\right)\phi_2 dx, \\ \hfill \forall\phi_2 \in L_0^{s,2}(\Omega), \end{cases} \tag{3.13}$$

thus, taking $\phi_1(x) = u_n(x) - u(x)$, $\phi_2(x) = v_n(x) - v(x)$ and by Cauchy-Schwarz inequality and proposition 1.2.8, we find

$$\begin{cases} \|u_n - u\|_{L^2(\Omega)} \leq \\ \quad C_{emb}^2\left(|t_n - t|\|\psi_1\|_{L^2(\Omega)} + \|t f_1(., \overline{u}, \overline{v}) - t_n f_1(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\right), \\ \|v_n - v\|_{L^2(\Omega)} \leq \\ \quad C_{emb}^2\left(|t_n - t|\|\psi_2\|_{L^2(\Omega)} + \|t f_2(., \overline{u}, \overline{v}) - tn f_2(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\right), \end{cases} \tag{3.14}$$

in the other hand side, we have $(\overline{u}_n, \overline{v}_n)$ converge to $(\overline{u}, \overline{v})$ in $L^2(\Omega) \times L^2(\Omega)$, implies that

$$\begin{cases} \overline{u}_{n_k} \to \overline{u}, & \text{a.e on } \Omega,\\ |\overline{u}_{n_k}| < G, & \text{a.e on } \Omega,\\ \text{or } G \in L^2(\Omega), \text{ for all } k \ , & \end{cases} \quad \& \quad \begin{cases} \overline{v}_{n_k} \to \overline{v}, & \text{a.e on } \Omega,\\ |\overline{v}_{n_k}| < K, & \text{a.e on } \Omega,\\ \text{or } K \in L^2(\Omega), \text{ for all } k \ , & \end{cases}$$

moreover, from the condition (C1), we obtain

$$\begin{cases} f_1(x, \overline{u}_{n_k}, \overline{v}_{n_k}) & \to f_1(x, \overline{u}, \overline{v}), \qquad a.e\text{on } \Omega,\\ |f_1(x, \overline{u_{kn}}, \overline{v}_{n_k})| & < a(x) + h_1 G + r_1 K \quad \in L^2(\Omega), \qquad \text{for all } k \ a.e \text{ on } \Omega, \end{cases}$$

and

$$\begin{cases} f_2(x, \overline{u}_{n_k}, \overline{v}_{n_k}) & \to f_2(x, \overline{u}, \overline{v}), \qquad a.e\text{on } \Omega,\\ |f_2(x, \overline{u}_{n_k}, \overline{v}_{n_k})| & < a(x) + h_2 G + r_2 K \quad \in L^2(\Omega) \qquad \text{for all } k \ a.e \text{ on } \Omega, \end{cases}$$

then, from Lebesgue convergence theorem, $f_1(x, \overline{u}_n, \overline{v}_n)$ converges to $f_1(x, \overline{u}, \overline{v})$ in $L^2(\Omega)$ and we have $(t_n)_{n\in\mathbb{N}} \to t$ when $n \to +\infty$.
Therefore, $(u_n, v_n) \to (u, v)$ in $L^2(\Omega)\times L^2(\Omega)$ implies that $H(t_n, u_n, v_n) \to H(t, u, v)$ in $L^2(\Omega) \times L^2(\Omega)$ .
So, $H$ is continuous from $[0, 1] \times L^2(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$. □

**Lemma 3.2.3.** *Under the condition (C1),* $\{H(t, \overline{u}, \overline{v}), t \in [0, 1], (\overline{u}, \overline{v}) \in \overline{B}_{R+1}\}$ *is relatively compact in* $L^2(\Omega) \times L^2(\Omega)$.

***Proof.*** Let $(t_n)_{n\in\mathbb{N}} \subset [0, 1]$ and $(\overline{u}_n, \overline{v}_n)_{n\in\mathbb{N}} \subset \overline{B}_{R+1}$, we have

$$\begin{cases} \int_{\mathbb{R}^d} D^s u_n(x).D^s\phi_1(x)dx & = \int_\Omega t_n\psi_1(x)\phi_1(x)dx - \int_\Omega t_n f_1(x, \overline{u}_n, \overline{v}_n)\phi_1(x)dx,\\ \int_{\mathbb{R}^d} D^s v_n(x).D^s\phi_2(x)dx & = \int_\Omega t_n\psi_2(x)\phi_2(x)dx - \int_\Omega t_n f_2(x, \overline{u}_n, \overline{v}_n)\phi_2(x)dx, \end{cases}$$

then, we take $\phi_1(x) = u_n(x)$, $\phi_2(x) = v_n(x)$ and by Cauchy-Schwarz inequality, we obtain

$$\begin{cases} \int_{\mathbb{R}^d} |D^s u_n(x)|^2 dx & \leq \|\psi_1\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} + \|f_1(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)},\\ \int_{\mathbb{R}^d} |D^s v_n(x)|^2 dx & \leq \|\psi_2\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)} + \|f_2(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}, \end{cases}$$

thus, according the proposition 1.2.8, we obtain

$$\begin{cases} \|u_n\|^2_{L_0^{s,2}(\Omega)} & \leq C_{emb}\left(\|\psi_1\|_{L^2(\Omega)} + \|f_1(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\right)\|u_n\|_{L_0^{s,2}(\Omega)},\\ \|v_n\|^2_{L^2(\Omega)} & \leq C_{emb}\left(\|\psi_2\|_{L_0^{s,2}(\Omega)} + \|f_2(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\right)\|v_n\|_{L_0^{s,2}(\Omega)}, \end{cases} \tag{3.15}$$

hence, by the condition (C1) the sequences $\{f_1(x,\overline{u}_n,\overline{v}_n)\}_{n\in\mathbb{N}}$ and $\{f_2(x,\overline{u}_n,\overline{v}_n)\}_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$ that is mean there exists $M_1, M_2$ strictly positive such that $\|f_1(.,\overline{u}_n,\overline{v}_n)\|_{L^2(\Omega)} \leq M_1$ and $\|f_2(.,\overline{u}_n,\overline{v}_n)\|_{L^2(\Omega)} \leq M_2$,
so, when we came back the equation (3.15), we find

$$\begin{cases} \|u_n\|_{L_0^{s,2}(\Omega)} \leq C_{emb}\left(\|\psi_1\|_{L^2(\Omega)} + M_1\right), \\ \|v_n\|_{L_0^{s,2}(\Omega)} \leq C_{emb}\left(\|\psi_2\|_{L^2(\Omega)} + M_2\right), \end{cases} \tag{3.16}$$

by the combination between two inequalities of the equation (3.16), we obtain

$$\|(u_n,v_n)\|_{L_0^{s,2}(\Omega)\times L_0^{s,2}(\Omega)} \leq M,$$

where $M = C_{emb}\Big(\left(\|\psi_1\|_{L^2(\Omega)} + M_1\right) + \left(\|\psi_2\|_{L^2(\Omega)} + M_2\right)\Big)$.
Consequently $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $L_0^{s,2}(\Omega)\times L_0^{s,2}(\Omega)$ so $(u_n,v_n) \rightharpoonup (u,v)$ in $L_0^{s,2}(\Omega)\times L_0^{s,2}(\Omega)$, according to proposition 1.2.9 we conclude that there is a subsequence of $\{(u_{n_k},v_{n_k})\}_{k\in\mathbb{N}}$ which converges to $(u,v)$ in $L^2(\Omega)\times L^2(\Omega)$. ☐

✍ Our first results is the following Theorem.

**Theorem 3.2.2.** *Under the condition (C1) and (C2), the system* (3.1) *has at least one weak solution* $(u,v) \in L_0^{s,2}(\Omega)\times L_0^{s,2}(\Omega)$.

- Now, we will show the proof of the existence theorem 3.2.2.

***Proof.*** Thanks to the previous lemmas 3.2.1,3.2.2 and 3.2.3, we concluded that $d(I_d - H(t,.,.),\bar{B}_{R+1},0)$ is well defined and by the homotopy invariance property of the Leray-Schauder degree, we find

$$\begin{aligned} d(I_d - H(1,.,.),\overline{B}_{R+1},0) &= d(I_d - H(0,.,.),\overline{B}_{R+1},0), \\ &= d(I_d,\overline{B}_{R+1},0) = 1 \neq 0, \end{aligned}$$

therefore

$$I_d(\overline{u},\overline{v}) - H(1,\overline{u},\overline{v}) = 0 \Leftrightarrow (\overline{u},\overline{v}) = H(1,\overline{u},\overline{v}).$$

Hence, we have showed that $(\overline{u},\overline{v})$ is a solution of (3.1). ☐

## 3.3 Uniqueness of distributional solution

In this section, we will make some assumption about the functions $f_1$ and $f_2$ to prove the uniqueness of weak solution of the problem (3.1).
Let assume that the functions $f_1$ and $f_2$ are a Lipschitz continuous functions with respect to the second variable, that is means there exists constants $c_1, c_2 \in \mathbb{R}^+$ for

almost every $x \in \Omega$ and for any $k = (k_1, k_2), \tilde{k} = (\tilde{k}_1, \tilde{k}_2), l = (l_1, l_2), \tilde{l} = (\tilde{l}_1, \tilde{l}_2) \in L^2(\Omega) \times L^2(\Omega)$,

$$\begin{cases} \|f_1(x,k) - f_1(x,\tilde{k})\|_{L^2(\Omega)} \leq c_1 \|k - \tilde{k}\|_{L^2(\Omega)\times L^2(\Omega)}, \\ \|f_2(x,l) - f_2(x,\tilde{l})\|_{L^2(\Omega)} \leq c_1 \|l - \tilde{l}\|_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{3.17}$$

✍ As a result, proving that $H$ is contraction is sufficient to show that $H$ admit a unique fixed point.

**Lemma 3.3.1.** *The operator $H$ is contraction from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$ for all $t \in [0, 1]$.*

***Proof.*** Let $(\overline{u}_1, \overline{u}_2), (\overline{v}_1, \overline{v}_2)$ belongs $L^2(\Omega) \times L^2(\Omega)$ and for every $t \in [0, 1]$, we have

$$\begin{cases} \int_{\mathbb{R}^d} D^s u_1(x).D^s \phi_1(x) dx = t \int_\Omega \psi_1(x)\phi_1(x) dx - t \int_\Omega f_1(x, \overline{u}_1, \overline{v}_1)\phi_1(x) dx, \\ \int_{\mathbb{R}^d} D^s v_1(x).D^s \phi_2(x) dx = t \int_\Omega \psi_2(x)\phi_2(x) dx - t \int_\Omega f_2(x, \overline{u}_1, \overline{v}_1)\phi_2(x) dx, \end{cases}$$

and

$$\begin{cases} \int_{\mathbb{R}^d} D^s u_2(x).D^s \phi_1(x) dx = t \int_\Omega \psi_1(x)\phi_1(x) dx - t \int_\Omega f_1(x, \overline{u}_2, \overline{v}_2)\phi_1(x) dx, \\ \int_{\mathbb{R}^d} D^s v_2(x).D^s \phi_2(x) dx = t \int_\Omega \psi_2(x)\phi_2(x) dx - t \int_\Omega f_2(x, \overline{u}_2, \overline{v}_2)\phi_2(x) dx, \end{cases}$$

then, we make the difference between the two previous systems and we take $\phi_1(x) = u_1(x) - u_2(x)$, $\phi_2(x) = v_1(x) - v_2(x)$, we obtain

$$\begin{cases} \int_{\mathbb{R}^d} |D^s u_1(x) - D^s u_2(x)|^2 dx = t \int_\Omega \left(f_1(x, \overline{u}_2, \overline{v}_2) - f_1(x, \overline{u}_1, \overline{v}_1)\right)(u_1(x) - u_2(x)) dx, \\ \int_{\mathbb{R}^d} |D^s v_1(x) - D^s v_2(x)|^2 dx = t \int_\Omega \left(f_2(x, \overline{u}_2, \overline{v}_2) - f_2(x, \overline{u}_1, \overline{v}_1)\right)(v_1(x) - v_2(x)) dx, \end{cases}$$

thus, we apply Cauchy-Schwarz inequality, we find

$$\begin{cases} \|u_1 - u_2\|^2_{L_0^{s,2}(\Omega)} \leq \|f_1(., \overline{u}_1, \overline{v}_1) - f_1(., \overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)} \|u_1 - u_2\|_{L^2(\Omega)}, \\ \|v_1 - v_2\|^2_{L_0^{s,2}(\Omega)} \leq \|f_2(., \overline{u}_1, \overline{v}_1) - f_2(., \overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)} \|v_1 - v_2\|_{L^2(\Omega)}, \end{cases}$$

thanks to proposition 1.2.8, and condition (3.17), we get

$$\begin{cases} \frac{1}{C_{emb}^2} \|u_1 - u_2\|^2_{L^2(\Omega)} & \leq \|u_1 - u_2\|_{L_0^{s,2}(\Omega)}, \\ & \leq c_1 \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)} \|u_1 - u_2\|_{L^2(\Omega)}, \\ \frac{1}{C_{emb}^2} \|v_1 - v_2\|^2_{L^2(\Omega)} & \leq \|v_1 - v_2\|_{L_0^{s,2}(\Omega)}, \\ & \leq c_2 \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)} \|v_1 - v_2\|_{L^2(\Omega)}, \end{cases}$$

so,

$$\begin{cases}\|u_1 - u_2\|_{L^2(\Omega)} \leq C_{emb}^2 c_1 \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)},\\ \|v_1 - v_2\|_{L^2(\Omega)} \leq C_{emb}^2 c_2 \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)},\end{cases} \tag{3.18}$$

by adding between two inequalities of (3.18), we arrived to

$$\|(u_1, v_1) - (u_2, v_2)\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_{emb}(c_1 + c_2)\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)},$$

that is means

$$\|H(t, \overline{u}_1, \overline{v}_1) - H(t, \overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_{emb}(c_1 + c_2)\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)}.$$

Finally, we conclude that if $C_{emb}^2(c_1 + c_2) < 1$ then $H$ is a contraction.
And by Banach contraction principle theorem our results is that: $H$ admits a unique fixed point $(\overline{u}, \overline{v}) \in L^2(\Omega) \times L^2(\Omega)$,
hence, the problem (3.1) admit a unique weak solution.

□

Chapter 4

# Existence results for non-linear fractional problem involving the distributional Riesz gradient

Our primary objective in this chapter is to investigate the existence of weak solutions for non-linear fractional problem involving by distributional Riesz gradient. The results of this problem will be examined in the fractional Sobolev space. To achieve this, we apply Schauder fixed point theory under specific nonlinearity assumptions.

**Contents**

***References***

❐ [51] Slimani K, Saadi C, Lakhal H, *Existence results for convection-reaction fractional problem involving the distributional Riesz derivative. Math Meth Appl Sci.* **45**(2022), 10247-10255. `doi:10.1002/mma.8365`.

## 4.1 Introduction and Motivation

In this chapter, we consider the following non-linear fractional problem involving a non-local operator

$$\begin{cases} -D^s.D^s u(x) - div^s(b\varphi(u)) = f\Big(x, u(x), D^s u(x)\Big), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{4.1}$$

Where $\Omega \subset \mathbb{R}^d$ is bounded open domain with a Lipschitz boundary, $b \in [L^2(\Omega)]^d$, $s \in (0,1)$ and $f(x,p,q) : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$.
With the specific following assumptions

- (A1) $f(x,p_1,p_2)$ measurable on $x \in \mathbb{R}$ and continuous on $(p_1,p_2) \in \mathbb{R}\times\mathbb{R}^d$.
- (A2) Grow assumption: $\exists \xi \in L^2(\Omega)$ such that

$$|f(x,p_1,p_2)| \leq K\left(\xi(x) + |p_1|^\delta + |p_2|^\delta\right), \quad a.e. \quad p_1,p_2 \in\in \mathbb{R}\times\mathbb{R}^d, \delta \in (0,1).$$

- (A3) $\varphi \in C(\mathbb{R},\mathbb{R})$ and exists $K1 \geq 0$ such as

$$|\varphi(p)| \leq K1|p|^\delta, \forall p \in \mathbb{R}, \delta \in (0,1).$$

The weak formulation of the problem (4.1) is as follow

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^d} D^s u(x) D^s v(x) dx + \quad \int_{\mathbb{R}^d} b\varphi(u(x)) D^s v(x) = \\ \qquad\qquad\qquad\qquad \int_{\mathbb{R}^d} f(x,u(x),D^s u(x)) v(x) dx, \forall v \in L_0^{s,2}(\Omega), \end{cases} \tag{4.2}$$

Our work focuses on generalizing and extending a set of results from classical Sobolev spaces to fractional Sobolev spaces. Specifically, our problem (4.1) constitutes a fractional version of (see pp 82, [24]), where the focus was on finding the existence of weak solution in classical Sobolev spaces. In fact, the second term of the left hand side on the equation (4.1) is a fractional convection term. This term $div^s(b\varphi(u))$ resembles the standard $div(b\varphi(u))$ as $s \to 1$.
This chapter is structured as follows: In the following section we present the main results and suggest the applications of the Schauder fixed point theorem.

## 4.2 Existence results

This section focuses on demonstrating the existence of weak solutions for our problem (4.1). We propose the Schauder fixed point theorem to achive this. We formulate the fixed point problem to deduce an equivalent to our problem (4.1). Additionally, we present a variety of auxiliary lemmas concerning the conditions of the Schauder fixed point theorem.

### 4.2.1 Fixed point formulation

In this subsection, we will focus to conclude the fixed point problem that is equivalent to the our problem (4.1).
▷ Let us starting by consider the map $T$ as follow

$$\begin{aligned} T : L_0^{s,2}(\Omega) &\longrightarrow L_0^{s,2}(\Omega), \\ \overline{u} &\longmapsto T(\overline{u}) = u, \end{aligned}$$

where $u$ is solution to the following linear problem

$$\begin{cases} -D^s.D^s u(x) - div^s(b\varphi(\overline{u}(x))) = f(x,\overline{u}(x),D^s\overline{u}(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{4.3}$$

**Theorem 4.2.1.** *Thanks to assumption (A2) and (A3), the problem (4.3) has unique weak solution $u \in L_0^{s,2}(\Omega)$.*

**Remark 4.2.1.** *The map $T$ is well define because the linear problem (4.3) admit unique weak solution. So, the value $u$ is uniquely by the element $\overline{u}$.*

***Proof.*** We are going to applying the Lax-Milgram theorem for proving the theorem (4.2.1).
For all $\overline{u} \in L_0^{s,2}(\Omega)$, we have $f(.,\overline{u},D^s\overline{u}) \in L^2(\mathbb{R}^d)$.
The following problem is the weak formulation of the problem (4.3)

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^d} D^s u(x) D^s v(x) dx \quad + \int_{\mathbb{R}^d} b\varphi(\overline{u}(x)) D^s v(x) \\ \qquad\qquad\qquad\qquad\qquad = \displaystyle\int_{\mathbb{R}^d} f(x,\overline{u}(x),D^s\overline{u}(x))v(x)dx, \quad \forall v \in L_0^{s,2}(\Omega), \end{cases} \tag{4.4}$$

with the bilinear form consider as follows

$$\begin{aligned} a(u,v) \,:\, L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega) &\to \mathbb{R}, \\ (u,v) \mapsto a(u,v) &= \int_{\mathbb{R}^d} D^s u(x) D^s v(x) dx, \end{aligned}$$

and the linear form consider as follow

$$\begin{aligned} & l(v) \,:\, L_0^{s,2}(\Omega) \to \mathbb{R}, \\ & \qquad v \mapsto l(v) = -\int_{\mathbb{R}^d} b\varphi(\overline{u}(x)) D^s v(x) dx + \int_{\mathbb{R}^d} f(x,\overline{u}(x),D^s\overline{u}(x))v(x)dx. \end{aligned}$$

▷ Now, we will verify the conditions of the Lax-Milgram theorem.

***Step 01:*** We proving that the bilinear form $a(.,.)$ is coercive.
for all $v \in L_0^{s,2}(\Omega)$ , we have

$$\begin{aligned} a(v,v) &= \int_{\mathbb{R}^d} |D^s v(x)|^2 dx, \\ &= \|v\|^2_{L_0^{s,2}(\Omega)}, \end{aligned}$$

so, we deduce from above $a(.,.)$ is coercive.

***Step 02:*** Now, we are going to prove that the bilinear form and the linear form are continuous.

- Firstly, we start by proving the bilinear form $a(.,.)$ is continuous.

By Cauchy-Schwarz inequality, we find

$$\begin{aligned} |a(u,v)| &= |\int_{\mathbb{R}^d} D^s u(x)\cdot D^s v(x)dx|, \\ &\leq \Big(\int_{\mathbb{R}^d} |D^s u(x)|^2\Big)^{1/2}\Big(\int_{\mathbb{R}^d} |D^s v(x)|^2\Big)^{1/2}, \\ &= \|u\|_{L_0^{s,2}(\Omega)}\|v\|_{L_0^{s,2}(\Omega)}, \end{aligned}$$

hence, $a(.,.)$ is continuous.

- Secondly, we will prove that the linear form $l(.)$ is continuous.

For all $v \in L_0^{s,2}(\Omega)$ , we have

$$\begin{aligned} |l(v)| &= \left| -\int_{\mathbb{R}^d} b\varphi(\overline{u}(x))D^s v(x)dx + \int_{\mathbb{R}^d} f(x,\overline{u}(x),D^s\overline{u}(x))v(x)dx\right|, \\ &= \left|\int_{\mathbb{R}^d} b\varphi(\overline{u}(x))D^s v(x)dx\right| + \left|\int_{\mathbb{R}^d} f(x,\overline{u}(x),D^s\overline{u}(x))v(x)dx\right|, \end{aligned}$$

then, using the assumption $(A2)$, $(A3)$ and Holder inequality , we get to

$$\begin{aligned} |l(v)| &\leq \int_{\mathbb{R}^d} |b|K_1|\overline{u}|^{\delta}|D^s v(x)|dx + \int_{\mathbb{R}^d} K(|\xi(x)| + |\overline{u}|^{\delta} + |D^s\overline{u}|^{\delta})|v(x)|dx, \\ &\leq K_1\big(\int_{\mathbb{R}^d} |b|^2dx\big)^{1/2}\big(\int_{\mathbb{R}^d} |\overline{u}(x)|^{2\delta}dx\big)^{1/2}\big(\int_{\mathbb{R}^d} |D^s v(x)|^2dx\big)^{1/2} \\ &+ K\Bigg(\big(\int_{\mathbb{R}^d} |\xi(x)|^2dx\big)^{1/2} + \big(\int_{\mathbb{R}^d} |\overline{u}(x)|^{2\delta}dx\big)^{1/2} \\ &+ \big(\int_{\mathbb{R}^d} |D^s\overline{u}(x)|^{2\delta}dx\big)^{1/2}\Bigg)\big(\int_{\mathbb{R}^d} |v(x)|^2dx\big)^{1/2}, \end{aligned}$$

according to proposition 1.2.8, we derive

$$\begin{aligned} &|l(v)| \\ &\leq \Bigg(K1\|b\|_{L^2(\Omega)^d}\||\overline{u}|^{\delta}\|_{L^2(\Omega)} + C_{emb}K\big(\|\xi\|_{L^2(\Omega)} + \||\overline{u}|^{\delta}\|_{L^2(\Omega)} \\ &\quad + \||D^s\overline{u}|^{\delta}\|_{L^2(\mathbb{R}^d)}\big)\Bigg)\|D^s v\|_{L^2(\mathbb{R}^d)}, \\ &\leq \Bigg(C_{emb}K\|\xi\|_{L^2(\Omega)} + \Bigg(K1\|b\|_{L^2(\Omega)^d} \\ &\quad + C_{emb}K\Bigg)|\Omega|^{1-\delta/2}\|\overline{u}\|^{\delta}_{L^2(\Omega)} + C_{emb}K\|D^s\overline{u}\|^{\delta}_{L^2(\mathbb{R}^d)}\Bigg)\|v\|_{L_0^{s,2}(\Omega)}, \end{aligned}$$

so

$$|l(v)| \leq E\|v\|_{L_0^{s,2}(\Omega)},$$

where

$$E = \left(C_{emb}K\|\xi\|_{L^2(\Omega)} + \left(\left(K1\|b\|_{L^2(\Omega)^d} + C_{emb}K\right)|\Omega|^{1-\delta/2}C_{emb}^{\delta} + C_{emb}K\right)\|\overline{u}\|_{L_0^{s,2}(\Omega)}^{\delta}\right).$$

Hence, the linear form $L(.)$ is continuous.

$\triangleright$ As a result, we can apply the Lax-Milgram theorem and conclude that the problem (4.2.1) has unique weak solution $u \in L_0^{s,2}(\Omega)$. □

**Remark 4.2.2.** *We remark that the problem (4.1) is equivalent to the following fixed point problem*

$$u \in L_0^{s,2}(\Omega), \quad T(u) = u.$$

### 4.2.2 Several auxiliary Lemmas

In this subsection, we will present several auxiliary lemmas about the conditions the Schauder fixed point theorem.

**Lemma 4.2.1.** *Thanks to assumption $(A1)$, the operator $T$ map identifier from the ball $B_R$ into the ball $B_R$.*

***Proof.*** For all $\overline{u} \in B_R$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} D^s u(x) D^s v(x) dx \\ &= -\int_{\mathbb{R}^d} b\varphi(\overline{u}) D^s v dx + \int_{\mathbb{R}^d} f(x, \overline{u}(x), D^s\overline{u}(x)) v(x) dx, \text{ for all } v \in L_0^{s,2}(\Omega), \end{aligned} \tag{4.5}$$

then, Let us choose $v = u$ in the above equation, we obtain

$$\int_{\mathbb{R}^d} D^s u(x) D^s u(x) dx = -\int_{\mathbb{R}^d} b\varphi(\overline{u}(x)) D^s u(x) dx + \int_{\mathbb{R}^d} f(x, \overline{u}(x), D^s\overline{u}(x)) u(x) dx,$$

therefor, using the assumptions $(C2)$, $(C3)$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} |D^s u(x)|^2 dx &= \left| -\int_{\mathbb{R}^d} b\varphi(\overline{u}(x)) D^s u(x) dx + \int_{\mathbb{R}^d} f(x, \overline{u}(x), D^s\overline{u}(x)) u(x) dx \right|, \\ &\leq \left| \int_{\mathbb{R}^d} b\varphi(\overline{u}(x)) D^s u(x) dx \right| + \left| \int_{\mathbb{R}^d} f(x, \overline{u}(x), D^s\overline{u}(x)) u(x) dx \right|, \\ &\leq \int_{\mathbb{R}^d} K_1 |b| |\overline{u}(x)|^{\delta} |D^s u(x)| dx \\ &\quad + \int_{\mathbb{R}^d} K(|\xi(x)| + |\overline{u}|^{\delta} + |D^s\overline{u}|^{\delta}) |u(x)| dx,\end{aligned}$$

by Cauchy-Schwarz inequality, we find

$$\begin{aligned}\|u\|^2_{L_0^{s,2}} \leq{}& K1\|b\|_{L^2(\Omega)^d}\||\overline{u}|^\delta\|_{L^2(\Omega)}\|D^s u\|_{L^2(\mathbb{R}^d)} \\ &+ K\Bigg(\|\xi\|_{L^2(\Omega)} + \||\overline{u}|^\delta\|_{L^2(\Omega)} + \||D^s\overline{u}|^\delta\|_{L^2(\mathbb{R}^d)}\Bigg)\|u\|_{L^2(\Omega)},\end{aligned}$$

then, from the Hölder inequality and proposition 1.2.8, we obtain

$$\begin{aligned}\|u\|_{L_0^{s,2}(\Omega)} \leq{}& |\Omega|^{1-\delta/2}K1\|b\|_{L^2(\Omega)^d}\|\overline{u}\|^\delta_{L^2(\Omega)} \\ &+ C_{emb}K\Bigg(\|\xi\|_{L^2(\Omega)} + (|\Omega|^{1-\delta/2}C^\delta_{emb} + 1)\|\overline{u}\|^\delta_{L_0^{s,2}(\Omega)}\Bigg), \\ &\leq C_{emb}K\|\xi\|_{L^2(\Omega)} + \Bigg(K(|\Omega|^{1-\delta/2}C^\delta_{emb} + 1) + |\Omega|^{1-\delta/2}K1\|b\|_{L^2(\Omega)^d}\Bigg)R^\delta,\end{aligned}$$

it follows that, for every $\overline{u} \in B_R$,

$$\begin{aligned}&\|u\|_{L_0^{s,2}(\Omega)} \leq R, \quad \text{with} \\ &R > C_{emb}K\|\xi\|_{L^2(\Omega)} + \Bigg(K(|\Omega|^{1-\delta/2}C^\delta_{emb} + 1) + |\Omega|^{1-\delta/2}K1\|b\|_{L^2(\Omega)^d}\Bigg)R^\delta. \qquad (4.6)\end{aligned}$$

From the previous equation (4.6), we remark that if $R$ is large the operator $T$ is map define from the ball $B_R\{u \in L_0^{s,2}, \|u\|_{L_0^{s,2}} \leq R\}$ into the ball $B_R\{u \in L_0^{s,2}, \|u\|_{L_0^{s,2}} \leq R\}$. □

**Lemma 4.2.2.** *Thanks to assumption* (A1),(A2) *and* (A3) $T : L_0^{s,2}(\Omega) \to L_0^{s,2}(\Omega)$ *is continuous.*

***Proof.*** Let $\{\overline{u}_n\}_{n\in\mathbb{N}} \subset L_0^{s,2}(\Omega)$ which converge to $\overline{u}$ in $L_0^{s,2}(\Omega)$ when $n \to +\infty$. We want to prove that $T(\overline{u}_n)$ converge to $T(\overline{u})$ in $L_0^{s,2}(\Omega)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} D^s u_n(x)D^s v(x)dx \\ &\quad= -\int_{\mathbb{R}^d} b\varphi(\overline{u}_n(x))D^s v(x)dx + \int_{\mathbb{R}^d} f(x,\overline{u}_n(x),D^s\overline{u}_n(x))v(x)dx, \qquad (4.7)\end{aligned}$$

in the other hand, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} D^s u(x)D^s v(x)dx \\ &\quad= -\int_{\mathbb{R}^d} b\varphi(\overline{u}(x))D^s v(x)dx + \int_{\mathbb{R}^d} f(x,\overline{u}(x),D^s\overline{u}(x))v(x)dx, \qquad (4.8)\end{aligned}$$

then, taking the difference between two formulas (4.7) and (4.8), we get

$$\begin{aligned}&\int_{\mathbb{R}^d}(D^s u_n(x) - D^s u(x))D^s v(x)dx\\&= -\int_{\mathbb{R}^d} b(\varphi(\overline{u}_n(x) - \varphi(\overline{u}(x))))D^s v(x)dx\\&+\int_{\mathbb{R}^d}\left(f(x,\overline{u}_n(x),D^s\overline{u}_n(x)) - f(x,\overline{u}(x),D^s\overline{u}(x))\right)v(x)dx,\end{aligned}$$

hence, choosing $v(x) = u_n(x) - u(x)$, we arrived to

$$\begin{aligned}&\int_{\mathbb{R}^d}|(D^s u_n(x) - D^s u(x))|^2 dx\\&= \Bigg| -\int_{\mathbb{R}^d} b\big(\varphi(\overline{u}_n(x)) - \varphi(\overline{u}(x))\big)\big(D^s u_n(x) - D^s u(x)\big)dx\\&+\int_{\mathbb{R}^d}\left(f(x,\overline{u}_n(x),D^s\overline{u}_n(x)) - f(x,\overline{u}(x),D^s\overline{u}(x))\right)\big(u_n(x) - u(x)\big)\Bigg|dx,\\&\leq \Bigg|\int_{\mathbb{R}^d} b\big(\varphi(\overline{u}_n(x)) - \varphi(\overline{u}(x))\big)\big(D^s u_n(x) - D^s u(x)\big)dx\Bigg|\\&+\Bigg|\int_{\mathbb{R}^d}\left(f(x,\overline{u}_n(x),D^s\overline{u}_n(x)) - f(x,\overline{u}(x),D^s\overline{u}(x))\right)\big(u_n(x) - u(x)\big)\Bigg|dx,\end{aligned}$$

therefor, we use Cauchy-Schwarz inequality and proposition 1.2.8, we obtain

$$\begin{aligned}\|u_n - u\|_{L_0^{s,2}(\Omega)} &\leq \|b\|_{L^2(\Omega)^d}\|\varphi(\overline{u}_n) - \varphi(\overline{u})\|_{L^2(\Omega)}\\&+ C_{emb}\|f(.,\overline{u}_n,D^s\overline{u}_n) - f(.,\overline{u},D^s\overline{u})\|_{L^2(\mathbb{R}^d)},\end{aligned}\tag{4.9}$$

we have $\overline{u}_n$converges to $\overline{u}$ in $L_0^{s,2}(\Omega)$, so

$$\begin{cases}\overline{u}_{n_k} \to \overline{u}, & a.e \text{ on } \Omega,\\ |\overline{u}_{n_k}| < H_1 & \text{for all } k\ a.e, \text{ or } H_1 \in L^2(\Omega),\end{cases}$$

and

$$\begin{cases}D^s\overline{u}_{n_k} \to D^s\overline{u}, & a.e \text{ on } \mathbb{R}^d,\\ |D^s\overline{u}_{n_k}| < H_2 & \text{for all } k\ a.e \text{ on } \mathbb{R}^d, \text{ or } H_2 \in L^2(\mathbb{R}^d),\end{cases}$$

thus, from the assumption $(A1)$ and $(A2)$, we get to

$$\begin{cases}f(x,\overline{u}_{n_k},D^s\overline{u}_{n_k}) \to f(x,\overline{u},D^s\overline{u}), & a.e \text{ on } \mathbb{R}^d,\\ |f(x,\overline{u}_{n_k},D^s\overline{u}_{n_k})| < K(\xi(x) + |H1|^\delta + |H2|^\delta) \in L^2(\mathbb{R}^d) & \text{for all } k\ a.e \text{ on } \mathbb{R}^d,\end{cases}$$

furthermore, applying the assumption $(A3)$, we arrived to

$$\begin{cases}\varphi(\overline{u}_{n_k}) \to \varphi(\overline{u}), & a.e \text{ on } \Omega,\\ |\varphi(\overline{u}_{n_k})| < K1|H1|^\delta \in L^2(\Omega), & \text{for all } k\ a.e \text{ on } \Omega,\end{cases}$$

from the Lebesgue convergence theorem, we obtain

$$f(x, \overline{u}_n, D^s\overline{u}_n) \to f(x, \overline{u}, D^s\overline{u}) \text{ in } L^2(\mathbb{R}^d),$$
$$\text{and}$$
$$\varphi(\overline{u}_n) \to \varphi(\overline{u}) \text{ in } L^2(\Omega).$$

Finally, when we go back to equation (**??**), we find $u_n \to u$ in $L_0^{s,2}(\Omega)$. So the map $T$ is continuous from $L_0^{s,2}(\Omega)$ into $L_0^{s,2}(\Omega)$. □

**Lemma 4.2.3.** *Thanks to assumption A1 and (A3), the set $\{T(\overline{u}), \overline{u} \in \overline{B}_R\}$ is relatively compact in $L_0^{s,2}(\Omega)$.*

***Proof.*** Let the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ subset $\overline{B}_R$, this implies that the sequence $\overline{u}_n$ is bounded in $L_0^{s,2}(\Omega)$, so $\overline{u}_n \rightharpoonup \overline{u}$ in $L_0^{s,2}(\Omega)$-weak. Then, according the proposition 1.2.8, we conclude that there is a subsequence of $\{\overline{u}_{n_k}\}_{k\in\mathbb{N}}$ which is converges to $\overline{u}$ in $L^2(\Omega)$ (that is mean: relatively compact in $L^2(\Omega)$) and implies that the sequence $(f(., \overline{u}_n, D^s\overline{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^d)$ and $(\varphi(\overline{u}_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$,
thanks to assumptions $(A2)$, $(A3)$, we obtain

$$\begin{aligned} f(x, \overline{u}_n, D^s\overline{u}_n) \rightharpoonup f(x, \overline{u}_n, D^s\overline{u}_n) &\quad \text{in } L^2(\mathbb{R}^d) - weak, \\ &and \\ \varphi(\overline{u}_n) \rightharpoonup \varphi(\overline{u}) &\quad \text{in } L^2(\Omega) - weak, \end{aligned}$$

we have

$$\begin{aligned} \int_{\mathbb{R}^d} D^s u_n(x) D^s v(x) dx = &- \int_{\mathbb{R}^d} b\varphi(\overline{u}_n(x)) D^s v(x) dx \\ &+ \int_{\mathbb{R}^d} f(x, \overline{u}_n(x), D^s\overline{u}_n(x)) v(x) dx, \end{aligned} \tag{4.10}$$

in the other hand side, we have

$$\begin{aligned} \int_{\mathbb{R}^d} D^s u(x) D^s v(x) dx = &- \int_{\mathbb{R}^d} b\varphi(\overline{u}(x)) D^s v(x) dx \\ &+ \int_{\mathbb{R}^d} f(x, \overline{u}(x), D^s\overline{u}(x)) v(x) dx, \end{aligned} \tag{4.11}$$

taking the difference between two formulas (4.10) and (4.11), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} (D^s u_n(x) - D^s u(x)) D^s v(x) dx \\ &= - \int_{\mathbb{R}^d} b(\varphi(\overline{u}_n(x) - \varphi(\overline{u}(x)))) D^s v(x) dx \\ &+ \int_{\mathbb{R}^d} \left( f(x, \overline{u}_n(x), D^s\overline{u}_n(x)) - f(x, \overline{u}(x), D^s\overline{u}(x)) \right) v(x) dx, \end{aligned}$$

than, we choose $v(x) = \overline{u}_n(x) - \overline{u}(x)$, we get

$$\begin{aligned}\|u_n - u\|^2_{L_0^{s,2}(\Omega)} = &- \int_{\mathbb{R}^d} b\left(\varphi(\overline{u}_n(x)) - \varphi(\overline{u}(x))\right)(D^s u_n - D^s u)dx \\ &+ \int_{\mathbb{R}^d} \left(f(x, \overline{u}_n, D^s\overline{u}_n) - f(x, \overline{u}, D^s\overline{u})\right)(u_n - u)dx,\end{aligned}$$

when passing to the limit when $n \to \infty$, we arrived to

$$\|u_n - u\|_{L_0^{s,2}(\Omega)} \to 0.$$

Finally, we deduce that the set $\{T(\overline{u}), \overline{u} \in \overline{B}_R\}$ is relatively compact in $L_0^{s,2}(\Omega)$. ☐

✍ The main our results is the following theorem

**Theorem 4.2.2.** *Thanks to assumption* $(A1)$, $(A2)$, *and* $(A3)$ *the problem* (4.1) *has at least one weak solution* $u \in L_0^{s,2}(\Omega)$.

***Proof***. From the previous lemmas 4.2.1, 4.2.2 and Lemma 4.2.3, the conditions of the Schauder fixed point theorem is true. So

$$T(u) = u.$$

Therefore, we have showed that $u$ is solution of problem (4.1).
Hence, there exists at least one fixed point $u \in L_0^{s,2}(\Omega)$, which prove the main result. ☐

Chapter 5

# Study the existence results for time fractional semilinear equation

In this chapter, we are interesting in studying the existence results for time fractional equations. Our work is divided into two parts. In the first part. We study the existence and uniqueness of weak solutions for time fractional linear equations using the Galerkin approach. In the second part, we investigate the existence results of the time fractional semilinear equation. We use the Leray-Schauder degree theory under some conditions on the semilinear term.

**Contents**

***References***

- ❒ [46] saadi C, Lakhal H, Slimani K., *On study the existence results for time fractional equation using Topological degree method. Int. Nonlinear Anal. Appl.* In Press, 1-12. `http://dx.doi.org/10.22075/ijnaa.2023.2023.28891.4017`

## 5.1 Introduction and Motivation

The main focuses of this chapter is to establish the existence, uniqueness of time-fractional problem that introduced with fractional Laplacian.
In the first part, we are interesting in studying the flowing time fractional linear

problem

$$\begin{cases} \text{Find } \mathscr{U} : [0,T]\times\Omega \to \mathbb{R}, \\ {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}(t,x) + (-\Delta)^{s}\mathscr{U}(t,x) = h(t,x), & \text{on } [0,T]\times\Omega, \\ \mathscr{U} = 0, & \text{on } [0,T]\times\mathbb{R}^{d}\setminus\Omega, \\ (g_{1-s}*\mathscr{U})(0) = w, & \text{on } \mathbb{R}^{d}\setminus\Omega, \end{cases} \tag{5.1}$$

and using the Galerkin approach to deal with the existence and uniqueness for the above problem. In the second part, we are interesting in studying the flowing fractional semilinear problem

$$\begin{cases} \text{Find } \mathscr{U} : [0,T]\times\Omega \to \mathbb{R}, \\ {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U} + (-\Delta)^{s}\mathscr{U} + h(\mathscr{U}) = 0, & \text{on } [0,T]\times\Omega, \\ \mathscr{U} = 0, & \text{on } [0,T]\times\mathbb{R}^{d}\setminus\Omega, \\ (g_{1-s}*\mathscr{U})(0,.) = w, & \text{on } \mathbb{R}^{d}\setminus\Omega. \end{cases} \tag{5.2}$$

We suggest the topological method to deal with the existence results.
Let us beginning with the definition of the fractional Laplacian as an integral in the sens of the Cauchy principle value in the real space for all $z\in\mathscr{S}, \forall s\in(0,1)$, as follows

$$(-\Delta)^{s}\mathscr{U}(x) = c(d,s)p.v.\int_{\mathbb{R}^{d}} \frac{\mathscr{U}(x)-\mathscr{U}(y)}{|x-y|^{d+2s}}dy, \quad x\in\mathbb{R}^{d},$$

with $c(d,s) = \pi^{-(2s+d/2)}\dfrac{\Gamma(s+d/2)}{\Gamma(-s)}$, and $\mathscr{S}$ is the Schwarz space.
This nonlocal operator have according to spectral theory an eigenvalues. These values a finite threat and form a diverging sequence (see[2])

$$0<\lambda^{s}_{1}(\Omega)\le\lambda^{s}_{2}(\Omega)\le\lambda^{s}_{3}(\Omega)\le\cdots\to+\infty.$$

In the other hand side, the authors in [17] define the Riemann-Liouville time fractional derivatives ${}^{RL}\mathscr{D}^{s}_{0,t}$ as follow
▷ For $z\in L^{2}(0,T;E)$, if $g_{1-s}*z\in H^{1}(0,T;E)$ then

$${}^{RL}\mathscr{D}^{s}_{0,t}z = \frac{d}{dt}\{g_{1-s}*z\},$$

▷ the adjoint of Riemann-Liouville derivatives denoted as ${}^{RL}\mathscr{D}^{s}_{t,T}$ is define as the following

$${}^{RL}\mathscr{D}^{s}_{t,T}\phi(t) = \int_{t}^{T} g_{1-s}(y-t)\frac{d}{dt}\phi(y)dy, \quad \text{for all } t\in[0,T].$$

with $g_{1-s}$ we denote the kernel of order $1-s$, and the convolution of $g_{1-s}*z$ defined as follows

$$g_{1-s} * z(t) = \int_0^t g_{1-s}(t-y)z(y)dy, \text{ a.e. } t \in [0,T],$$

and

$$g_s(t) = \frac{1}{\Gamma(s)} t^{s-1} \in L^1_{loc}([0,+\infty]).$$

The arrangement of this chapter is as follows: In the next section, we will mention certain results and characteristics that will be used in this chapter. In section 03, we suggest the Galerkin approach for proving the existence and uniqueness results of time-fractional linear equation. In section 04, thanks to certain assumption, we will prove the existence of weak solutions for time-fractional semilinear equation. Applying the topological degree methods.

## 5.2 Preliminaries

In this section, we will define some characteristics and results which we will use to reach our goal.
Let $(E, \|.\|)$ be a real Banach space, and $T$ be a positive number, and $\Omega$ subset of $\mathbb{R}^d$ with a Lipschitz boundary.
Let introduce the following spaces, $p, q \in [1, \infty)$

$$W^s_{p,q}(0,T;X,Y) = \{u \in L^p(0,T;X), \quad {}^{RL}\mathscr{D}^s_{0,t}u \in L^p(0,T;Y)\},$$

and

$${}_0W^s_{p,q}(0,T;X,Y) = \{u \in W^s_{p,q}(0,T;X,Y), \quad (g_{1-s} * \mathscr{U})(0) = 0 \text{ in } Y\}.$$

For more information about it see [37].

**Theorem 5.2.1.** *[17] If $p \in L^2(0,T;E)$ and $q \in L^1(0,T)$ then*

$$v * u \in L^2(0,T;E) \quad \text{and } \|v * u\|_{L^2(0,T;E)} \le \|v\|_{L^1(0,T)} \|u\|_{L^2(0,T;E)}. \tag{5.3}$$

**Theorem 5.2.2.** *[17] Let $(H, (.,.))$ be real Hilbert space, $v \in L^2(0,T;H)$ and $s \in (0,1)$. Then*

$$\int_0^T \big(v(t), g_s * v(t)\big)\, dt \ge 0.$$

**Proposition 5.2.1.** *[17] Let $s \in (0,1)$ and $v \in L^2(0,T;E)$. If $v$ admits a derivative of order $s$ in $L^2(0,T;E)$, then*

$$v = (g_{1-s} * v)(0) g_s + g_s * {}^{RL}\mathscr{D}^s_{0,t} v \quad \text{in } L^1(0,T;E). \tag{5.4}$$

**Proposition 5.2.2.** *[17] Let $s \in (0,1)$, $v \in L^2(0,T;E)$ and $\phi \in H^1(0,T)$. Assume that $v$ admits a derivative of order $s$ in $L^2(0,T,E)$. Then*

$$\int_0^T {}^{RL}\mathscr{D}^s_{0,t}v(t)\phi(t)dt = -\int_0^T v(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt + [g_{1-s} * v\phi]_0^T \quad \text{in } E, \tag{5.5}$$

$\triangleright$ *if $\phi \in C_c^\infty(0,T)$ then*

$$\Big\| \int_0^T v(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt \Big\| \leq \sqrt{T} g_{2-s}(T)\|v\|_{L^2(0,T;E)}\|\phi^{'}\|_{L^\infty(0,T)}. \tag{5.6}$$

**Remark 5.2.1.** *[17] We remark that from the Proposition 5.2.2, we can define the fractional derivative in the sense of distributions. and we sees the following linear map*

$$\mathcal{D}(0,T) \to E, \tag{5.7}$$

$$\phi \mapsto -\int_0^T v(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt. \tag{5.8}$$

*The equation (5.8) is a distribution of order (at most) 1. Denote by $\mathcal{D}^{'}(0,T;E)$ the set of distributions with values in $E$.*

The following definition defines the weak derivation

**Definition 5.2.1.** *[17] Let $s \in (0,1)$ and $v \in L^2(0,T;E)$. Then the weak derivative of order $s$ of $v$ is the vector valued distribution, denoted by ${}^{RL}\mathscr{D}^s_{0,t}$, and defined, for all $\phi \in \mathcal{D}(0,T)$, as follows*

$$\left\langle {}^{RL}\mathscr{D}^s_{0,t}v, \phi \right\rangle = -\int_0^T v(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt.$$

*If we want to emphasize the duality occurring in the bracket above, we will write $\left\langle {}^{RL}\mathscr{D}^s_{0,t}v, \phi \right\rangle_{\mathcal{D}^{'}(0,T;E),\mathcal{D}(0,T)}$, instead of $\left\langle {}^{RL}\mathscr{D}^s_{0,t}v, \phi \right\rangle$.*

**Proposition 5.2.3.** *[17] Let $s \in (0,1)$, $E$ be a real Banach space and $v \in L^2(0,T,E^{'})$. We assume that $v$ admits a derivative of order $s$ in $L^2(0,T,E^{'})$. Then, for each $w$ in $E$, $\langle v, w\rangle_{E^{'},E}$ admits a derivative o order $s$ in $L^2(0,T)$ and*

$$\left\langle {}^{RL}\mathscr{D}^s_{0,t}v, w \right\rangle_{E^{'},E} = {}^{RL}\mathscr{D}^s_{0,t}\{\langle v, w\rangle_{E^{'},E}\}, \quad \text{in} \quad L^2(0,T). \tag{5.9}$$

**Corollaire 5.2.1.** *[37] For $E$ be a real Banach space densely and continuously embedded into a real Hilbert space $H$. Assume that $v \in {}_0W^s_{2,2}(0,T;E,E^{'})$, then, for every $\tau \in [0,T]$,*

$$\frac{1}{2} g_{1-s} * \|v(.)\|_H^2(t) \leq \int_0^\tau \left\langle {}^{RL}\mathscr{D}^s_{0,t}v(x), v(x) \right\rangle dx. \tag{5.10}$$

**Theorem 5.2.3.** *[37] Let $E$ be a real Banach space densely and continuously embedded into a real Hilbert space $H$, $s \in (0,1)$ and $p \geq 2$ be such that $s > 1/p^{'}$. Assume $v \in W^s_{p,p^{'}}(0,T,E,E^{'})$, and $(g_{1-s} * v)(0) \in E$. Then*

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}v(t), v(t) - (g_{1-s} * v)(0)g_s(t) \right\rangle_{E,E^{'}} dt \geq 0.$$

**Definition 5.2.2.** *[17]*
*Let $T > 0$ and $s \in (0,1)$, we mention by*

$$H^s(0,T;H_0^s(\Omega),H^{-s}(\Omega)) = \{v \in L^2(0,T;H_0^s(\Omega)) \text{ whose } {}^{RL}\mathscr{D}_{0,t}^s v \in L^2(0,T;H^{-s}(\Omega))\}.$$

*Where ${}^{RL}\mathscr{D}_{0,t}^s v$ is weak fractional derivative.*

▷ Putting (just a notation)

$$H_0^s(\Omega) = Y \quad \text{and} \quad H^{-s}(\Omega) = Y^{'}.$$

▷ Throughout this chapter, we have assumed that $s > 1/2$ and $d > 2s$. Also, we assume $\Omega$ be a Lipschitz bounded open subset of $\mathbb{R}^d$.

## 5.3 Galerkin method for Time-fractional linear equation

In this Section, considering the following time fractional linear problem

$$\begin{cases} \text{Find } \mathscr{U} \in H^s(0,T;Y,Y^{'}), \\ {}^{RL}\mathscr{D}_{0,t}^s \mathscr{U} + (-\Delta)^s \mathscr{U} = h, & \text{in } L^2(0,T;Y^{'}), \\ (g_{1-s} * \mathscr{U})(0) = w, & \text{in } L^2(\Omega), \end{cases} \tag{5.11}$$

and we utilize the Galerkin approach to demonstrate the existence and uniqueness of weak solutions for the above problem.

**Lemma 5.3.1.** *The problem (5.11) has unique weak solution, $\mathscr{U} \in H^s(0,T;Y,Y^{'})$.*

***Proof.*** **<u>Part 01</u>.** ***Existence of a weak solution.*** This part is divided into fourth steps.
<u>First</u>, taking the space $E_n$ the vector space generated by $\varphi_1, \dots, \varphi_n$, implies that $E_n = vect\{\varphi_1, \dots, \varphi_n\}$ and $(\varphi_k)_{k\leq 1}$ forms an Hilbertian basis of $L^2(\Omega)$.
▷ we can see that $((\lambda_k^s)^{-1/2}\varphi_k)$ is a Hilbertian basic of $Y$, where $\lambda_k^s \in (0,+\infty)$ is k$^{th}$ eigenvalues of the operator $(-\Delta)^s$, $k = 1, 2, \dots$
<u>Secondly</u>, let decompose the initial condition $w$. Since $Y$ is a Hilbert space the we writing $w$ as follows

$$w = \sum_{k\geq 1} a_k(t)\varphi_k(x) \quad \text{in } Y, \tag{5.12}$$

we have $E_n$ a space of finite dimension then

$$w_n = \sum_{k=1}^{n} a_k(t)\varphi_k(x) \quad \text{in } E_n, \tag{5.13}$$

the following property is true implies that $w_n \to w$ in $Y$. In the last, we define our approximated problem for every integer $n \geq 1$, as the following form

$$\begin{cases} \text{Find } \mathscr{U}_n \in L^2(0,T;E_n) \text{ such that } {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n \in L^2(0,T;Y^{'}), \\ \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \varphi \right\rangle_{Y',Y} + c(d,s)\iint_{\mathbb{R}^{2d}} \dfrac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}} dydx = \langle h, \varphi \rangle_{Y',Y}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } L^2(0,T), \forall \varphi \in E_n, \\ (g_{1-s} * \mathscr{U}_n)(0) = w_n. \end{cases} \tag{5.14}$$

**Step 01:** ***Solvability of the approximated problem.*** We suppose The following decomposition

$$\mathscr{U}_n = \sum_{k=1}^{n} y_k(t)\varphi_k(x), \Rightarrow {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n = \sum_{k=1}^{n} {}^{RL}\mathscr{D}^s_{0,t} y_k(t)\varphi_k(x),$$

$$h_k = \left\langle h(t), \varphi_k \right\rangle_{Y',Y}.$$

So, when we substitute the above decomposition in equation (5.14), it become as follow

$$\int_0^T \left\langle \sum_{k=1}^{n} {}^{RL}\mathscr{D}^s_{0,t} y_k(t)\varphi_k, \varphi \right\rangle_{Y',Y} dt + \int_0^T \int_{\mathbb{R}^d} (-\Delta)^s (\sum_{k=1}^{n} y_k(t)\varphi_k)\varphi dxdt = \int_0^T \int_\Omega h\varphi_n dxdt,$$

then, taking $\varphi = \varphi_n$ belong $E_n$, we find

$$\sum_{k=1}^{n} \int_0^T {}^{RL}\mathscr{D}^s_{0,t} y_k(t) \left\langle \varphi_k, \varphi_n \right\rangle_{Y',Y} dt + \sum_{k=1}^{n} \int_0^T y_k(t) \int_{\mathbb{R}^d} \lambda_k^s \varphi_k \varphi_n dxdt = \int_0^T \int_\Omega h\varphi_n dxdt.$$

Hence, we have $\varphi_n$ orthonormal system, implies that

$$\int_0^T \left( {}^{RL}\mathscr{D}^s_{0,t} y_k(t) + \lambda_k^s y_k(t) - h_k \right) dt = 0,$$

then,

$${}^{RL}\mathscr{D}^s_{0,t} y_k(t) + \lambda_k^s y_k(t) = h_k,$$

in other hand side, we have

$$(g_{1-s} * \mathscr{U}_n)(0) = w_n. \to (g_{1-s} * y_k)(0) = a_k.$$

Finally, we arrived to

$$\begin{cases} {}^{RL}\mathscr{D}^s_{0,t} y_k(t) + \lambda_k^s y_k(t) = h_k \quad \text{in } L^2(0,T), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall k = 1, \dots, n, \\ (g_{1-s} * y_k)(0) = a_k. \end{cases} \tag{5.15}$$

The local result for equation (5.15) belong to $L^2(0,\tau)$, for small positive $\imath$, it is solvable (see [19, chap 5]). We are now going to prove that there is a global result for equation (5.15). We know that if it is blow up, then the global solution does not exist, and for that, we use a proof by contradiction.
Let assume that $T_m$ is finite. Then, for all $\imath \in (0,T)$, we apply the convolution by $g_s$ on the equation (5.15) and according proposition 5.2.1, we obtain

$$\begin{aligned} g_s * {}^{RL}\mathscr{D}^s_{0,t} y_k(t) + \lambda^s_k g_s * y_k(t) &= g_s * h_k, \\ y_k(t) - a_k g_s + \lambda^s_k g_s * y_k(t) &= g_s * h_k, \end{aligned}$$

then, we find

$$y_k(t) + \lambda^s_k g_s * y_k(t) = a_k g_s + g_s * h_k \quad \text{in } L^2(0,\imath). \tag{5.16}$$

Hence, multiplying the equation (5.16) by $y_k$ and integrate on $(0,\imath)$, we arrived to

$$\int_0^\imath |y_k(t)|^2 dt + \lambda^s_k \int_0^\imath y_k(t) g_s * y_k(t) dt = \int_0^\imath a_k g_s y_k(t) dt + \int_0^\imath g_s * h_k y_k(t) dt,$$

according theorem 5.2.2, since $\lambda^s_k \geq 0$ and $s > 1/2$ we get to

$$\int_0^\imath |y_k(t)|^2 dt \leq \int_0^\imath a_k g_s y_k(t) dt + \int_0^\imath g_s * h_k y_k(t) dt,$$

then, according to Cauchy-Schwarz inequality, we get

$$\|y_k\|^2_{L^2(0,\imath)} \leq |a_k| \|g_s\|_{L^2(0,T_m)} \|y_k\|_{L^2(0,\imath)} + \|g_s * h_k\|_{L^2(0,T_m)} \|y_k\|_{L^2(0,\imath)},$$

therefor,

$$\|y_k\|_{L^2(0,\imath)} \leq |a_k| \|g_s\|_{L^2(0,T_m)} + \|g_s * h_k\|_{L^2(0,T_m)}. \tag{5.17}$$

We conclude that $y_k$ bounded in $L^2(0,\tau)$ as $\tau$ approaches $T_m$. That contradiction with the condition of blow up, so that $T_m = +\infty$. Our deduce here is for all time $T \geq 0$ the (5.14) has only one solution.

**Step 02:** ***Priori estimates.*** In the present step, we are going to prove that $\mathscr{U}_n$ is bounded in $L^2(0,T;Y)$ and ${}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n$ is bounded in $L^2(0,T;Y')$.
For that we use $g_s \in L^2(0,T)$, we have

$$\begin{aligned} \int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \varphi \right\rangle_{Y',Y} dt + c(d,s) \int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x) - \mathscr{U}_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{d+2s}} dydxdt \\ = \int_0^T \int_\Omega h\varphi dxdt, \end{aligned}$$

then, we take $\varphi(x) = \mathscr{U}_n - g_s w_n$, on the above equation, we obtain

$$\begin{aligned}&\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \mathscr{U}_n - g_s w_n\right\rangle_{Y',Y} dt\\&\quad + c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))\big((\mathscr{U}_n(x)-\mathscr{U}_n(y)-g_s(w_n(x)-w_n(y))\big)}{|x-y|^{d+2s}}dydxdt\\&\qquad\qquad = \int_0^T \langle h, \mathscr{U}_n - g_s w_n\rangle\, dxdt,\end{aligned}$$

hence, thanks the theorem 5.2.3, we arrived to

$$\begin{aligned}&c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}} \frac{|\mathscr{U}_n(x)-\mathscr{U}_n(y)|^2}{|x-y|^{d+2s}}dydxdt\\&\quad\le c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(w_n(x)-w_n(y))}{|x-y|^{d+2s}}g_s dydxdt\\&\quad + \int_0^T\int_\Omega h\mathscr{U}_n dxdt - \int_0^T\int_\Omega h w_n g_s dxdt,\end{aligned}$$

then, we use Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&c(d,s)\|\mathscr{U}_n\|^2_{L^2(0,T;Y)}\\&\quad\le c(d,s)\int_0^T \|\mathscr{U}_n\|_Y\|w_n\|_Y g_s dt + \int_0^T \|h\|_{Y'}\|\mathscr{U}_n\|_y dt + \int_0^T \|h\|_{Y'}\|w_n\|_Y g_s dt,\\&\quad\le \Bigg[c(d,s)\|w_n\|_Y\|g_s\|_{L^2(0,T)} + \|h\|_{L^2(0,T;Y')}\Bigg]\|\mathscr{U}_n\|_{L^2(0,T;Y)}\\&\quad + \|w_n\|_Y\|g_s\|_{L^2(0,T)}\|h\|_{L^2(0,T;Y')},\end{aligned}$$

for proving $\mathscr{U}_n$ is bounded, we solve the quadratic equation with a variable $\|\mathscr{U}_n\|$, as follows

$$a\|\mathscr{U}_n\|^2_{L^2(0,\mathcal{T};Y)} + b\|\mathscr{U}_n\|_{L^2(0,\mathcal{T};Y)} + c = 0, \tag{5.18}$$

such as

$$\begin{aligned}a &= c(d,s),\\ b &= \Bigg[c(d,s)\|w_n\|_Y\|g_s\|_{L^2(0,T)} + \|h\|_{L^2(0,T;Y')}\Bigg],\\ c &= \|w_n\|_Y\|g_s\|_{L^2(0,T)}\|h\|_{L^2(0,T;Y')},\end{aligned}$$

we find the equation (5.18) have two different solutions, and we have $w_n \to w$ in $Y$, finally, we deduce the following estimation

$$\|\mathscr{U}_n\|_{L^2(0,T;Y)} \le K_1. \tag{5.19}$$

Where $K_1$ is constant independent of $n$.

▷ Our next goal is to prove that ${}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n$ is bounded in $L^2(0,T;Y^{'})$.
Indeed, we have

$$\left|\left\langle {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n, \varphi\right\rangle_{Y',Y}\right| \leq c(d,s)\left|\iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}}dydx\right| + \left|\langle h, \varphi\rangle_{Y',Y}\right|, \quad \forall \varphi \in Y.$$

Then, according to Cauchy-Schwarz inequality, we find

$$\left|\left\langle {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n, \varphi\right\rangle_{Y',Y}\right| \leq c(d,s)\|\mathscr{U}_n\|_Y\|\varphi\|_Y + \|h\|_{Y'}\|\varphi\|_Y,$$

therefore, we obtain

$$\int_0^T \|{}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n\|^2_{Y'}dt \leq 2c^2(d,s)\int_0^T \|\mathscr{U}_n\|^2_Y dt + 2\int_0^T \|h\|^2_{Y'}dt.$$

In the last, we arrived to

$$\|{}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n\|_{L^2((0,T;Y'))} \leq k_2, \tag{5.20}$$

where $k_2 = \left(2c^2(d,s)k_1^2 + 2\|h\|^2_{L^2(0,T,Y')}\right)^{1/2}$.
Thus, we have from (5.19) and (5.20) that there exists $\mathscr{U} \in L^2(0,T;Y)$ such as

$$\mathscr{U}_n \rightharpoonup \mathscr{U} \quad \text{in } L^2(0,T;Y) - \text{weak},$$

and

$${}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n \rightharpoonup {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U} \quad \text{in } L^2(0,T;Y^{'}) - \text{weak}.$$

**Step 03:** ***Passage to the limit.***
Our focuses in this step is to return from the approximation problem to the exact problem.
we have

$${}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n + (-\Delta)^s\mathscr{U}_n = h, \tag{5.21}$$

we multiply the equation (5.21) by $\psi \in \mathcal{D}(0,T)$ and integrate on 0, $T$, then, we are multiplying it again by $\varphi_k$ and integrate, for $k \geq 1$ be fixed and $n \geq k$, we get to

$$\left\langle \int_0^T {}^{RL}\mathscr{D}^{s}_{0,t}\mathscr{U}_n(t)\psi(t)dt, \varphi_k\right\rangle_{Y',Y} + \int_{\mathbb{R}^d}\int_0^T (-\Delta)^s\mathscr{U}_n\psi dt\varphi_k dx = \int_\Omega \int_0^T h(t)\psi(t)dt\varphi_k dx,$$

moreover, from proposition 5.2.2, and passing to the limit in $n$, we get to

$$\left\langle \int_0^T {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}(t)\psi(t)dt, \varphi_k \right\rangle_{Y',Y} + \int_{\mathbb{R}^d}\int_0^T (-\Delta)^s\mathscr{U}\psi dt\varphi_k dx = \int_\Omega\int_0^T h(t)\psi(t)dt\varphi_k dx,$$

then,

$$^{RL}\mathscr{D}^s_{0,t}\mathscr{U} + (-\Delta)^s\mathscr{U} - h = 0 \quad \text{in } \mathcal{D}^{'}(0,T;Y^{'}).$$

Moreover, we have $h \in L^2(0,T;Y^{'})$ and $(-\Delta)^s \in L^2(0,T;Y^{'})$, implies that ${}^{RL}\mathscr{D}^s_{0,t}\mathscr{U} \in L^2(0,T;Y^{'})$ and $\mathscr{U} \in L^2(0,T;Y)$, we deduce that $\mathscr{U} \in W^s_{2,2}(0,T;Y,Y^{'})$ and

$$^{RL}\mathscr{D}^s_{0,t}\mathscr{U} + (-\Delta)^s\mathscr{U} - h = 0 \quad \text{in } L^2(0,T;Y^{'}).$$

**Step 04:** ***Initial condition.*** Let $\psi$ in $H^s_0(0,T)$, such as $\psi(T)=0$, we have

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}, \varphi_k \right\rangle \psi(t)dt + \int_0^T\int_{\mathbb{R}^d}(-\Delta)^s\mathscr{U}\varphi_k dx\psi(t)dt = \int_0^T \langle h, \varphi_k\rangle \psi(t)dt,$$

then, thanks to proposition 5.2.3, we obtain

$$\int_0^{\mathscr{U}T} {}^{RL}\mathscr{D}^s_{0,t}\langle \mathscr{U}, \varphi_k\rangle \psi(t)dt + \int_0^T\int_{\mathbb{R}^d}(-\Delta)^s\mathscr{U}\varphi_k dx\psi(t)dt = \int_0^T \langle h, \varphi_k\rangle \psi(t)dt,$$

moreover, according to proposition and 5.2.2, we find

$$\begin{aligned} -\int_0^T \langle \mathscr{U}, \varphi_k\rangle {}^{RL}\mathscr{D}^s_{t,T}\psi(t)dt + \left\langle (g_{1-s} * \mathscr{U})(0), \varphi_k \right\rangle \psi(0) \\ + \int_0^T\int_{\mathbb{R}^d}(-\Delta)^s\mathscr{U}\varphi_k dx\psi(t)dt = \int_0^T \langle h, \varphi_k\rangle \psi(t)dt. \end{aligned} \tag{5.22}$$

In the other hand side, we have

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \varphi_k \right\rangle \psi(t)dt + \int_0^T\int_{\mathbb{R}^d}(-\Delta)^s\mathscr{U}_n\varphi_k dx\psi(t)dt = \int_0^T \langle h, \varphi_k\rangle \psi(t)dt,$$

therefore, according to propositions 5.2.2 and 5.2.3, we arrived to

$$\begin{aligned} -\int_0^T \langle \mathscr{U}_n, \varphi_k\rangle {}^{RL}\mathscr{D}^s_{t,T}\psi(t)dt + \langle w_n, \varphi_k\rangle \psi(0) + \int_0^T\int_{\mathbb{R}^d}(-\Delta)^s\mathscr{U}_n\varphi_k dx\psi(t)dt \\ = \int_0^T \langle h, \varphi_k\rangle \psi(t)dt, \end{aligned}$$

thus, when we passing to the limit, we get to

$$\begin{aligned} -\int_0^T \langle \mathscr{U}, \varphi_k\rangle {}^{RL}\mathscr{D}^s_{t,T}\psi(t)dt + \langle w, \varphi_k\rangle \psi(0) + \int_0^T\int_{\mathbb{R}^d}(-\Delta)^s\mathscr{U}\varphi_k dx\psi(t)dt \\ = \int_0^T \langle h, \varphi_k\rangle \psi(t)dt. \end{aligned} \tag{5.23}$$

In conclusion, according the uniqueness of limit, we obtainment equation (5.22) is equal to (5.23).
Finally, our deduce is

$$(g_{1-s} * \mathscr{U})(0) = w \quad a.e. \text{ in } \Omega.$$

This implies that completes the proof of existence result.
**Part 02.** ***Uniqueness of the solution*** Let $\mathscr{U}$ and $\hat{\mathscr{U}}$ two solution of the problem (5.11), then we have

$$\begin{cases} {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U} + (-\Delta)^s\mathscr{U} = h, \\ (g_{1-s} * \mathscr{U})(0) = w, \end{cases} \tag{5.24}$$

and

$$\begin{cases} {}^{RL}\mathscr{D}^s_{0,t}\hat{\mathscr{U}} + (-\Delta)^s\hat{\mathscr{U}} = h, \\ (g_{1-s} * \hat{\mathscr{U}})(0) = w. \end{cases} \tag{5.25}$$

Thus, we take the deference between equations (5.24) and (5.25), we arrived to

$$\begin{cases} {}^{RL}\mathscr{D}^s_{0,t}(\mathscr{U} - \hat{\mathscr{U}}) + (-\Delta)^s(\mathscr{U} - \hat{\mathscr{U}}) = 0, \\ (g_{1-s} * (\mathscr{U} - \hat{\mathscr{U}}))(0) = 0. \end{cases} \tag{5.26}$$

Moreover, thanks to the initial condition we have $(\mathscr{U} - \hat{\mathscr{U}}) \in {}_0W^s_{2,2}(0, T; Y, Y')$. Then, we multiplier equation (5.26) by $(\mathscr{U} - \hat{\mathscr{U}})$ and integer on $(0, \tau)$ , $\tau \in (0, T]$, we get

$$\begin{aligned} &\int_0^\tau \left\langle {}^{RL}\mathscr{D}^s_{0,t}(\mathscr{U} - \hat{\mathscr{U}}), (\mathscr{U} - \hat{\mathscr{U}}) \right\rangle dt \\ &\qquad + \int_0^\tau \int_{\mathbb{R}^d} (-\Delta)^s(\mathscr{U} - \hat{\mathscr{U}})(\mathscr{U} - \hat{\mathscr{U}})dxdt = 0. \end{aligned} \tag{5.27}$$

Therefore, according to corollary 5.2.1 , we arrived to

$$g_{1-s} * \|(\mathscr{U} - \hat{\mathscr{U}})(.)\|^2_{L^2(\Omega)}(s) + \int_0^\tau \|\mathscr{U} - \hat{\mathscr{U}}\|^2_Y \leq 0,$$

and thus

$$\int_0^\tau g_{1-s}(s-t)\|(\mathscr{U} - \hat{\mathscr{U}})(t)\|^2_{L^2(\Omega)}dt \leq 0.$$

In the other hand side, we have $g_{1-s}$ is decreasing function, then

$$g_{1-s}(T)\int_0^\tau \|(\mathscr{U} - \hat{\mathscr{U}})(t)\|^2_{L^2(\Omega)}dt \leq 0.$$

In the last, we deduce that $(\mathscr{U} - \hat{\mathscr{U}})(t) = 0$. So, the problem (5.11) has a unique solution. □

## 5.4 Topological degree for a time-fractional semilinear equation

In this section, we are investigate to prove the existence of weak solution for the following fractional semilinear problem

$$\begin{cases} \text{Find } \mathscr{U} \in H^s(0,T;Y,Y^{'}), & \\ {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U} + (-\Delta)^s\mathscr{U} + h(\mathscr{U}) = 0, & \text{in } L^2(0,T;Y^{'}), \\ (g_{1-s} * \mathscr{U})(0,.) = w, & \text{in } L^2(\Omega). \end{cases} \tag{5.28}$$

Where the function $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous map satisfy, for some positive constant $r$ , $h$ the following assumptions

$(c_1)$ Growth assumption:

$$|h(\mathscr{U})| \leq r + r|\mathscr{U}|, \quad \forall \mathscr{U} \in \mathbb{R}.$$

$(c_2)$ Sing assumption:

$$h(\mathscr{U})\mathscr{U} \geq -r,$$

and we suggest the application of Leray-Schauder degree theory to demonstrate it.

▷ The following theorem give us the existence of weak solution

**Theorem 5.4.1.** *Under the hypothesis $(c_1)$ and $(c_2)$ the problem (5.28) has at least one weak solution $\mathscr{U} \in H^s(0,T;Y,Y^{'})$.*

### 5.4.1 New formulation of problem (5.28)

In this subsection, we are going to present a fixed point problem which equivalent to our problem (5.28).

In the beginning, we are going to define the following homotopy $H$ by

$$\begin{aligned} H : [0,1] \times L^2(0,T;L^2(\Omega)) &\to L^2(0,T;Y), \\ (\lambda, \overline{\mathscr{U}}) &\mapsto H(\lambda, \overline{\mathscr{U}}) = \mathscr{U}, \end{aligned}$$

where $\mathscr{U}$ is a weak solution to the following linear problem

$$\begin{cases} \text{Find } \mathscr{U} : [0,T] \times \Omega \to \mathbb{R}, & \text{such that,} \\ {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U} + (-\Delta)^s\mathscr{U} + \lambda h(\overline{\mathscr{U}}) = 0, & \text{on } [0,\mathcal{T}] \times \Omega, \\ \mathscr{U} = 0, & \text{on} [0,T] \times \mathbb{R}^d/\Omega, \\ (g_{1-s} * \mathscr{U})(0,.) = \lambda w, & \text{on } \mathbb{R}^d/\Omega. \end{cases} \tag{5.29}$$

**Lemma 5.4.1.** *We see in the section 03 our problem (5.29) has a unique solution* $\mathscr{U} \in H^s(0,T;Y,Y^{'})$.

We conclude that our problem (5.28) is equivalent to the following fixed point problem

$$\begin{cases} \mathscr{U} \in L^2(0,T;L^2(\Omega)), \\ H(1,\mathscr{U}) = \mathscr{U}. \end{cases} \tag{5.30}$$

Our goal here is prove the above problem, and utilizing the Leray-Schauder degree theory.

### 5.4.2 Several auxiliary Lemmas

In this subsection, we will present several auxiliary lemmas about the conditions of the Leray-Schauder degree method.

**Lemma 5.4.2.** *(Priori estimate). According to assumptions* $(c_1)$, $(c_2)$ *there exists* $R > 0$, *for all* $\mathscr{U} \in L^2(0,T,L^2(\Omega))$ *such that*

$$\begin{cases} H(\lambda,\mathscr{U}) = \mathscr{U}, \\ \lambda \in [0,1], \mathscr{U} \in L^2(0,T,L^2(\Omega)), \end{cases} \quad \Rightarrow \|\mathscr{U}\|_{L^2(0,T,L^2(\Omega))} < R+1.$$

***Proof***. Let $H(\lambda,\mathscr{U}) = \mathscr{U}$, for every $\lambda \in [0,1]$, we have

$$\begin{cases} \text{Find } \mathscr{U} \in L^2(0,T;Y) \text{ such that } {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U} \in L^2(0,T;Y^{'}), \\ \displaystyle\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}, \varphi \right\rangle_{Y',Y} dt + c(d,s)\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}(x)-\mathscr{U}(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}} dydxdt \\ \qquad\qquad\qquad\qquad\qquad\qquad + \lambda \displaystyle\int_0^T \int_\Omega h(\mathscr{U})\varphi dxdt = 0, \quad \forall \varphi \in Y, \end{cases}$$

then, we take $\varphi = \mathscr{U} - \lambda w g_s$, we obtain

$$\begin{aligned} &\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}, \mathscr{U} - \tau w g_s \right\rangle_{Y',Y} dt \\ &+ c(d,s)\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}(x)-\mathscr{U}(y))\big((\mathscr{U}(x)-\mathscr{U}(y)) - \lambda g_s(w(x)-w(y))\big)}{|x-y|^{d+2s}} dydxdt \\ &\qquad + \lambda \int_0^T \int_\Omega h(\mathscr{U})(\mathscr{U} - \tau w g_s) dxdt = 0. \end{aligned}$$

In fact, according the theorem 5.2.3, we deduce that the first integral above is not negative.

Hence, we get

$$\begin{aligned}
&c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}}\frac{|\mathscr{U}(x)-\mathscr{U}(y)|^2}{|x-y|^{d+2s}}dxdt\\
&\leq\left|\int_0^T\int_\Omega -h(\mathscr{U})\mathscr{U}\,dxdt\right|+\left|\int_0^T\int_\Omega h(\mathscr{U})wg_s dxdt\right|\\
&+\left|c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}}\frac{(\mathscr{U}(x)-\mathscr{U}(y))(w(x)-w(y))}{|x-y|^{d+2s}}g_s dydxdt\right|,
\end{aligned}$$

therefore, thanks to assumption $(c_2)$, we find

$$\begin{aligned}
&c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}}\frac{|\mathscr{U}(x)-\mathscr{U}(y)|^2}{|x-y|^{d+2s}}dxdt\\
&\leq r|\Omega|T+\left|\int_0^T\int_\Omega h(\mathscr{U})wg_s dxdt\right|\\
&+c(d,s)\left|\int_0^T\iint_{\mathbb{R}^{2d}}\frac{(\mathscr{U}(x)-\mathscr{U}(y))(w(x)-w(y))}{|x-y|^{d+2s}}g_s dydxdt\right|,
\end{aligned}$$

then, we find

$$\|\mathscr{U}\|^2_{L^2(0,T;H^{1=0}(\Omega))}\leq r|\Omega|T+I_1+I_2. \tag{5.31}$$

When

$$I_1=\left|\int_0^T\int_\Omega h(\mathscr{U})wg_s dxdt\right|,$$

and

$$I_2=c(d,s)\left|\int_0^T\iint_{\mathbb{R}^{2d}}\frac{(\mathscr{U}(x)-\mathscr{U}(y))(w(x)-w(y))}{|x-y|^{d+2s}}g_s dydxdt\right|,$$

by the assumption $(c_1)$, Cauchy-Schwarz inequality and according the proposition 1.2.8, we arrived to

- $$\begin{aligned}I_1&\leq r\int_0^T\int_\Omega|w||g_s|dxdt+r\int_0^T\int_\Omega|\mathscr{U}||w||g_s|dxdt,\\ &\leq rC_{emb}\sqrt{T|\Omega|}\|w\|_Y\|g_s\|_{L^2(0,T)}+rC^2_{emb}\|w\|_Y\|g_s\|_{L^2(0,T)}\|\mathscr{U}\|_{L^2(0,T;Y)},\end{aligned}$$

thus, we use Hölder inequality on $I_2$, we arrived to

$$I_2\leq c(d,s)\|w\|_Y\|g_s\|_{L^2(0,T)}\|\mathscr{U}\|_{L^2(0,T;Y)}.$$

In the last, we return to equation (5.31), we find

$$\begin{aligned}c(d,s)\|\mathscr{U}\|^2_{L^2(0,T;Y)}\leq&\left[\Big(rC^2_{emb}+c(d,s)\Big)\|w\|_Y\|g_s\|_{L^2(0,T)}\right]\|\mathscr{U}\|_{L^2(0,T;Y)}\\ &+rC_{emb}\sqrt{T|\Omega|}\|g_s\|_{L^2(0,T)}\|w\|_Y+r|\Omega|T.\end{aligned}$$

The solving of the above inequality, which is solved the second-degree equation, we arrived to

$$\|\mathscr{U}\|_{L^2(0,T;L^2(\Omega))} \leq L = R,$$

hence, we get

$$\|\mathscr{U}\|_{L^2(0,T;L^2(\Omega))} < R+1. \tag{5.32}$$

Then, from the equation (5.32) our result is that, there are no solution of the $H(\lambda,\mathscr{U}) = \mathscr{U}$ on the edge of $B_{R+1} = \{\mathscr{U} \in L^2(0,T;L^2(\Omega)) : \|\mathscr{U}\|_{L^2(0,T;L^2(\Omega))} < R+1\}$, and this is for every $\lambda \in [0,1]$. □

**Lemma 5.4.3.** *Under the assumption* ($c_1$)*, the homotopy* $\{H(\lambda,\overline{\mathscr{U}}); \lambda \in [0,1], \overline{\mathscr{U}} \in B_{R+1}\}$ *is relatively compact in* $L^2(0,T;L^2(\Omega))$.

***Proof.*** Let $(\lambda_n, \overline{\mathscr{U}}_n)_{n\in\mathbb{N}} \subset [0,1] \times \overline{B}(0,R+1)$, we have

$$\begin{aligned}&\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \mathscr{U}_n - \lambda_n w g_s\right\rangle_{Y',Y} dt\\ &\quad + c(d,s)\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))\big((\mathscr{U}_n(x)-\mathscr{U}_n(y)) - \lambda_n g_s(w(x)-w(y))\big)}{|x-y|^{d+2s}} dydxdt\\ &\qquad + \lambda_n \int_0^T \int_\Omega h(\overline{\mathscr{U}}_n)(\mathscr{U}_n - \lambda_n w g_s) dxdt = 0,\end{aligned}$$

according to theorem 5.2.3 , we arrived to

$$\begin{aligned}&c(d,s)\int_0^T \iint_{\mathbb{R}^{2d}} \frac{|\mathscr{U}_n(x)-\mathscr{U}_n(y)|^2}{|x-y|^{d+2s}} dydxdt\\ &\leq \left| c(d,s)\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(w(x)-w(y))}{|x-y|^{d+2s}} g_s dydxdt\right| + \left|\int_0^T\int_\Omega h(\overline{\mathscr{U}}_n)\mathscr{U}_n dxdt\right|\\ &\quad + \left|\int_0^T\int_\Omega h(\overline{\mathscr{U}}_n) w g_s dxdt\right|,\end{aligned}$$

then, putting

$$\|\mathscr{U}\|^2_{L^2(0,T;Y)} \leq I_1^{'} + I_2^{'} + I_3^{'}, \tag{5.33}$$

where

$$I_1^{'} = c(d,s)\left|\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(w(x)-w(y))}{|x-y|^{d+2s}} g_s dydxdt\right|,$$

and $I_2^{'} = \left|\int_0^T\int_\Omega h(\overline{\mathscr{U}}_n)\mathscr{U}_n dxdt\right|$, with $I_3^{'} = \left|\int_0^T\int_\Omega h(\overline{\mathscr{U}}_n) w g_s dxdt\right|$.

Under the hypothesis ($c_1$), Cauchy-Schwarz inequality and Proposition 1.2.8, we find

- $$I_2^{'} \leq r\int_0^T\int_\Omega |\mathscr{U}_n|dxdt + r\int_0^T\int_\Omega |\overline{\mathscr{U}}_n||\mathscr{U}_n|dxdt, \\ \leq rC_{emb}\left[\sqrt{T|\Omega|} + \|\overline{\mathscr{U}}_n\|_{L^2(0,T;L^2(\Omega))}\right]\|\mathscr{U}_n\|_{L^2(0,T;Y)},$$

and

- $$I_3^{'} \leq r\int_0^T\int_\Omega |w_n||g_s|dxdt + r\int_0^T\int_\Omega |\overline{\mathscr{U}}_n||w||g_s|dxdt, \\ \leq rC_{emb}\left[\sqrt{T|\Omega|} + \|\overline{\mathscr{U}}_n\|_{L^2(0,T;L^2(\Omega))}\right]\|w\|_Y\|g_s\|_{L^2(0,T)},$$

hence, using the Hölder inequality to $I_1^{'}$, we obtain

- $$I_1^{'} \leq c(d,s)\|w\|_Y\|g_s\|_{L^2(0,s)}\|\mathscr{U}_n\|_{L^2(0,T;Y)},$$

then, when we return to (5.33), we find

$$\begin{aligned} c(d,s)\|\mathscr{U}_n\|^2_{L^2(0,T;Y)} \\ \leq \Big[rC_{emb}\big(\sqrt{T|\Omega|} + \|\overline{\mathscr{U}}_n\|_{L^2(0,T;L^2(\Omega))}\big) + c(d,s)\|w\|_Y\|g_s\|_{L^2(0,T)}\Big]\|\mathscr{U}_n\|_{L^2(0,T;Y)} \\ + rC_{emb}\Big(\sqrt{T|\Omega|} + \|\overline{\mathscr{U}}_n\|_{L^2(0,T;L^2(\Omega))}\Big)\|w\|_Y\|g_s\|_{L^2(\Omega)}, \end{aligned}$$

hence, we arrived to

$$\|\mathscr{U}_n\|_{L^2(0,T;Y)} \leq \hat{M}_1,$$

where $\hat{M}_1$ is constant independent of $n$.
▷ We are going now to prove that ${}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n$ is bounded in $L^2(0,T;Y^{'})$.
Indeed, we have

$$\begin{aligned} &\left|\left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \varphi\right\rangle_{Y',Y}\right| \\ &\leq c(d,s)\left|\iint_{\mathbb{R}^{2d}} \frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}}dydx\right| + \int_\Omega |h(\overline{\mathscr{U}}_n)||\varphi|dx, \quad \forall\varphi\in Y. \end{aligned}$$

Therefore, according to Cauchy-Schwarz inequality, proposition 1.2.8 and the assumption $(c_1)$, we find

$$\left|\left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n, \varphi\right\rangle_{Y',Y}\right| \leq c(d,s)\|\mathscr{U}_n\|_Y\|\varphi\|_Y + rC_{emb}\sqrt{\Omega}\|\varphi\|_Y + rC_{emb}\|\overline{\mathscr{U}}_n\|_{L^2(\Omega)}\|\varphi\|_Y,$$

thus, we obtain

$$\int_0^T \|{}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n\|^2_{Y'}dt \le 2c^2(d,s)\int_0^T \|\mathscr{U}_n\|^2_Y dt + 4r^2C^2_{emb}\left(|\Omega|T + \int_0^T \|\overline{\mathscr{U}}_n\|^2_{L^2(\Omega)}dt\right),$$

in the last, we find

$$\|{}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n\|_{L^2((0,T;Y'))} \le \hat{M}_2, \tag{5.34}$$

where $\hat{M}_2 = \left(2c^2(d,s)\hat{M}_1^2 + 4r^2C^2_{emb}\left(T|\Omega| + (R+1)^2\right)\right)^{1/2}$.

In conclusion, we arrived to

$$(\mathscr{U}_n)_{n\in\mathbb{N}}\text{is bounded in } L^2(0,T;Y),$$
$$\text{and}$$
$$({}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n)_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,T;Y').$$

Finally, thanks to Aubin-Simon theorem we deduce that the homotopy $\{H(\lambda,\overline{\mathscr{U}});\lambda\in[0,1],\overline{\mathscr{U}}\in B_{R+1}\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. ☐

**Lemma 5.4.4.** *Under to hypothesis ($c_1$), the map* $H:[0,1]\times L^2(0,T;L^2(\Omega))\to L^2(0,T;L^2(\Omega))$ *is continuous.*

***Proof.*** Let $\{(\lambda_n,\overline{\mathscr{U}}_n)\}_{n\in\mathbb{N}}$ subset $[0,1]\times L^2(0,T;L^2(\Omega))$ which converge to $(\lambda,\overline{\mathscr{U}})$ in $[0,1]\times L^2(0,T;L^2(\Omega))$ when $n\to+\infty$. Our goal is to prove that $H(\lambda_n,\overline{\mathscr{U}}_n)\to H(\lambda,\overline{\mathscr{U}})$ in $L^2(0,T;L^2(\Omega))$, we pose for all $n\in\mathbb{N}$ that $H(\lambda_n,\overline{\mathscr{U}}_n)=\mathscr{U}_n$ and $H(\lambda,\overline{\mathscr{U}})=\mathscr{U}$, we have

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U}_n,\varphi\right\rangle_{Y',Y}dt + c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}}\frac{(\mathscr{U}_n(x)-\mathscr{U}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}}dydxdt + \lambda_n\int_0^{\mathscr{U}T}\int_\Omega h(\overline{\mathscr{U}}_n)\varphi dxdt = 0 \quad \forall\varphi\in Y, \tag{5.35}$$

and

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\mathscr{U},\varphi\right\rangle_{Y',Y}dt + c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}}\frac{(\mathscr{U}(x)-\mathscr{U}(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}}dydxdt + \lambda\int_0^T\int_\Omega h(\overline{\mathscr{U}})\varphi dxdt = 0 \quad \forall\varphi\in Y. \tag{5.36}$$

Hence, taking the subtraction between two equations (5.35) and (5.36), we obtain

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}(\mathscr{U}_n-\mathscr{U}),\varphi\right\rangle_{Y',Y}dt + c(d,s)\int_0^T\iint_{\mathbb{R}^{2d}}\frac{\big((\mathscr{U}_n(x)-\mathscr{U}_n(y))-(\mathscr{U}(x)-\mathscr{U}(y))\big)(\varphi(x)-\varphi(y))}{|x-y|^{d+2s}}dydxdt + \int_0^T\int_\Omega(\lambda h(\overline{\mathscr{U}})-\lambda_n h(\overline{\mathscr{U}}_n))\varphi dxdt = 0 \quad \forall\varphi\in Y,$$

then, we take $\varphi = (\mathscr{U}_n - \mathscr{U}) - (\lambda_n - \lambda) w g_s$ and thanks to theorem 5.2.3, we find

$$\begin{aligned}
&c(d,s)\int_0^T \|\mathscr{U}_n - \mathscr{U}\|_Y^2 dt \\
&\leq c(d,s)\int_0^T \iint_{\mathbb{R}^{2d}} \frac{\big((\mathscr{U}_n(x)-\mathscr{U}_n(y)) - (\mathscr{U}(x)-\mathscr{U}(y))\big)\big(\lambda_n-\lambda)(w(x)-w(y))\big)}{|x-y|^{d+2s}} g_s dydxdt \\
&+ \int_0^T \int_\Omega (\lambda h(\overline{\mathscr{U}}) - \lambda_n h(\overline{\mathscr{U}}_n))(\mathscr{U}_n - \mathscr{U}) dxdt \\
&+ \int_0^T \int_\Omega (\lambda h(\overline{\mathscr{U}}) - \lambda_n h(\overline{\mathscr{U}}_n))(\lambda_n - \lambda) w g_s dxdt.
\end{aligned}$$

Therefor, utilizing the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
&c(d,s)\|\mathscr{U}_n - \mathscr{U}\|_{L^2(0,T;Y)}^2 \\
&\leq \left[ c(d,s)\|g_s\|_{L^2(0,T)}|\lambda_n - \lambda|\|w\|_Y + \|\lambda h(\overline{\mathscr{U}}) - \lambda_n h(\overline{\mathscr{U}}_n)\|_{L^2(0,T;Y')} \right] \|\mathscr{U}_n - \mathscr{U}\|_{L^2(0,T;Y)} \\
&\quad + \|\lambda h(\overline{\mathscr{U}}) - \lambda_n h(\overline{\mathscr{U}}_n)\|_{L^2(0,T;Y')}|\lambda_n - \lambda|\|w\|_Y\|g_s\|_{L^2(0,T)}.
\end{aligned}$$

Then, we have $h$ Lipschitz map, produces that $h(\overline{\mathscr{U}}_n) \to h(\overline{\mathscr{U}})$ in $L^2(0,T;L^2(\Omega))$ and we have $\lambda_n \to \lambda$ in $[0,1]$ when $n \to +\infty$, and according to Poincaré inequality, we arrived to

$$\|\mathscr{U}_n - \mathscr{U}\|_{L^2(0,T;L^2(\Omega))} \to 0 \quad \text{when } n \to +\infty.$$

Consequently, $H$ is continuous from $[0,1] \times L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$.

□

***Proof of Theorem 5.4.1.*** We deduce from the proof of lemmas 5.4.2, 5.4.3 and 5.4.4 the degree $d(I_d - H(\tau,.), B(0,R+1), 0)$ is well define, and according the homotopy invariance property, we get to

$$\begin{aligned}
d(I_d - H(1,.), B(0,R+1), 0) &= d(I_d - H(0,.), B(0,R+1), 0), \\
&= d(I_d, B(0,R+1), 0), \\
&= 1 \neq 0,
\end{aligned}$$

hence,

$$\mathscr{U} - H(1,\mathscr{U}) = 0 \Rightarrow \mathscr{U} = H(1,\mathscr{U}).$$

So, our deduce is that our problem (5.28) has of weak solutions for $\mathscr{U} \in H^s(0,T;Y,Y')$.

□

Chapter 6

# Theoretical results for time fractional semilinear equation

In this chapter, we investigate the existence and uniqueness of weak solutions for the Riemann-Liouville time fractional semilinear equation. We proved the existence results using the Leray-Schauder degree method. Then, the uniqueness of the solution is showed in the classical way.

**Contents**

***References***

❐ [44] Saadi C, Lakhal H, Slimani K, *Theoretical results for time fractional semilinear equation.* Submitted for publication in *International journal.*

## 6.1 Introduction and Motivation

In this work, we study the existence of weak solution for the following problem

$$\begin{cases} \text{Find } \vartheta : [0,T]\times\Omega \to \mathbb{R}, & \text{such that,} \\ {}^{RL}\mathscr{D}^{s}_{0,t}\vartheta - \Delta\vartheta + \varphi(\vartheta) = 0, & \text{on } [0,T]\times\Omega, \\ \vartheta = 0, & \text{on}[0,T]\times\partial\Omega, \\ (g_{1-s} * \vartheta)(0) = z, & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

where $T$ be a positive number, with $\Omega$ be a bounded open subset of $\mathbb{R}^d$, and $0 < s < 1$. $\{{}^{RL}\mathscr{D}^{s}_{0,t}\}$ designate the Riemann-Liouville fractional derivatives of order $s$.

The following is a summary of our work. In the next section, we present some preliminaries that we will use in our forthcoming discussions. In the third section, we study the existence of a weak solution and use Leray-Schauder degree theory. In the fourth section, we prove the uniqueness of solution in the classical way.

## 6.2 Preliminaries

In this section, we introduce the convolution of function, with fractional kernel and fractional derivatives, weak fractional derivatives and some basic definition which are used throughout the paper. for more details (see [40]).
Let $(E, \|.\|)$ be a real Banach space, and $T > 0$. Let's start our work by defining convolution of functions

**Definition 6.2.1.** *[37] Let $s \in (0,1)$, $T > 0$ and $\vartheta \in L^2(0,T;E)$. We say that $\vartheta$ admits a (forward) derivative of order $s$ in $L^2(0,T;E)$ if*

$$g_{1-s} * \vartheta \in H^1(0,T;E).$$

*This cas, its (forward) derivative of order $s$ is the fonction of $L^2(0,T;E)$ defined as follows*

$$^{RL}\mathscr{D}^s_{0,t}\vartheta = \frac{d}{dt}\{g_{1-s} * \vartheta\}.$$

**Proposition 6.2.1.** *[17] Let $s \in (0,1)$, $\vartheta \in L^2(0,T,E)$ and $\phi \in H^1(0,T)$. Assume that $\vartheta$ admits a derivative of order $s$ in $L^2(0,T,E)$. Then*

$$\int_0^T {}^{RL}\mathscr{D}^s_{0,t}\vartheta(t)\phi(t)dt = -\int_0^T \vartheta(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt + [g_{1-s} * \vartheta\phi]_0^T \quad in\ E, \tag{6.2}$$

*where $^{RL}\mathscr{D}^s_{t,T}$ is related to the adjoint of forward derivatives defined by*

$$^{RL}\mathscr{D}^s_{t,T}\phi(t) = \int_t^T g_{1-s}(y-t)\frac{d}{dt}\phi(y)dy, \quad for\ all\ t \in [0,T].$$

*In addition, if $\phi \in C_c^\infty(0,T)$ then*

$$\|\int_0^T \vartheta(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt\| \le \sqrt{T}g_{2-s}(T)\|\vartheta\|_{L^2(0,T;E)}\|\phi^{'}\|_{L^\infty(0,T)}. \tag{6.3}$$

Above, this property allows defining the fractional derivative in the sense of distributions. Actually (6.3) see that the linear map

$$\begin{aligned} \mathscr{D}(0,T) &\to E, \\ \phi &\mapsto -\int_0^T \vartheta(t)^{RL}\mathscr{D}^s_{t,T}\phi(t)dt, \end{aligned}$$

is a distribution, of order is 1. Denote by $\mathscr{D}^{'}(0,T;E)$ the set of distributions in $E$.

**Definition 6.2.2.** *Let $T > 0$ and $0 < s < 1$. We mention by*

$$H^s(0,T;H_0^1(\Omega),H^{-1}(\Omega)) = \{\vartheta \in L^2(0,T;H_0^1(\Omega)) \text{ whose } {}^{RL}\mathscr{D}_{0,t}^s\vartheta \in L^2(0,T;H^{-1}(\Omega))\}.$$

*Where ${}^{RL}\mathscr{D}_{0,t}^s\vartheta$ is weak fractional derivative.*

▷ Now, we give the weak formulation of the problem (6.1)

$$\begin{cases}\text{Find } \vartheta \in L^2(0,T;H_0^1(\Omega)) \text{ such that } {}^{RL}\mathscr{D}_{0,t}^s\vartheta \in L^2(0,T;H^{-1}(\Omega))\\ \int_0^T \left\langle {}^{RL}\mathscr{D}_{0,t}^s\vartheta, w\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt + \int_0^T \int_\Omega \nabla\vartheta\nabla w dxdt + \int_0^T \int_\Omega \varphi(\vartheta) w dxdt = 0,\\ \quad \forall w \in H_0^1(\Omega).\end{cases}$$

When $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous map satisfy, for some positive constant $C$

(h1) Growth hypothesis:

$$|\varphi(\vartheta)| \leq C + C|\vartheta|, \quad \forall \vartheta \in \mathbb{R}.$$

▷ The main result of our effort is the following theorem

**Theorem 6.2.1.** *Assume $z \in H_0^1(\Omega)$*

*(1) If $s \in (1/2, 1)$ then (6.1) has a unique solution.*

*(2) If $0 < s \leq \frac{1}{2}$ then*

  *(2-1) if $z = 0$ then (6.1)has a unique solution.*

  *(2-2) if $z \neq 0$ then (6.1) has no solution;*

**Remark 6.2.1.** *If $s \leq 1/2$ and $z \neq 0$, then from the Proposition 5.2.1, we conclude that (6.1) does not have a solution. On the other side, if $z = 0$ then the solvability of (6.1) can be achieved as in the case $s \in (\frac{1}{2}, 1)$. Thus we will only consider in the sequel the case where $s > \frac{1}{2}$.*

## 6.3 Existence results by Topological degree

In this section, we are going to prove the existence of weak solution to the following problem

$$\begin{cases}\text{Find } \vartheta \in H^s(0,T;H_0^1(\Omega),H^{-1}(\Omega)),\\ {}^{RL}\mathscr{D}_{0,t}^s\vartheta - \Delta\vartheta + \varphi(\vartheta) = 0, & \text{in } L^2(0,T;H^{-1}(\Omega)),\\ (g_{1-s} * \vartheta)(0,.) = z, & \text{in } L^2(\Omega),\end{cases} \tag{6.4}$$

using the application of Leray-Schauder degree theory.
▷ The incoming theorem giving us the existence of weak solution of our problem (6.1).

**Theorem 6.3.1.** *Under the hypothesis (h1) our problem* (6.1) *has at least one weak solutions*
$\vartheta \in H^s(0,T; H_0^1(\Omega), H^{-1}(\Omega))$.

### 6.3.1 Fixed point formulation

In this subsection, we give a fixed point problem which is equivalent to the our problem (6.1).
In the beginning, we define the following homotopy $A$ as follow

$$\begin{aligned} A :\ [0,1] \times L^2(0,T; L^2(\Omega)) &\to L^2(0,T; H_0^1(\Omega)), \\ (\lambda, \overline{\vartheta}) &\mapsto A(\lambda, \overline{\vartheta}) = \vartheta, \end{aligned}$$

where $\vartheta$ is a solution to the following linear problem

$$\begin{cases} \text{for } \overline{\vartheta} \in L^2(0,T; H_0^1(\Omega)), & \\ {}^{RL}\mathscr{D}_{0,t}^{s}\vartheta - \Delta\vartheta + \lambda\varphi(\overline{\vartheta}) = 0, & \text{in } L^2(0,T; H^{-1}(\Omega)), \\ (g_{1-s} * \vartheta)(0,.) = \lambda z, & \text{in } L^2(\Omega), \end{cases} \tag{6.5}$$

**Lemma 6.3.1.** *The problem* (6.5) *has a unique solution* $\vartheta \in H^s(0,T; H_0^1(\Omega), H^{-1}(\Omega))$.

***Proof.*** We can shown in [17] the authors study the existence and uniqueness of weak solution for the problem (6.5) and they used the Galerkin method. □

We conclude that our problem (6.1) is equivalent to the following problem

$$\begin{cases} \vartheta \in L^2(0,T; L^2(\Omega)), \\ A(1,\vartheta) = \vartheta. \end{cases} \tag{6.6}$$

### 6.3.2 Statement Lemmas

In this subsection, we will present statement lemmas about the conditions of Leray-Schauder degree theory.

**Lemma 6.3.2.** *(Priori estimate). Thanks to hypothesis (h1), there exists* $R > 0$, *for all* $\vartheta \in L^2(0,T, L^2(\Omega))$ *such that*

$$\begin{cases} A(\lambda,\vartheta)=\vartheta, \\ \lambda\in[0,1], \vartheta\in L^2(0,T,L^2(\Omega)), \end{cases} \quad \Rightarrow \|\vartheta\|_{L^2(0,T,L^2(\Omega))} < R+1.$$

***Proof***. Let $\lambda$ in $[0,1]$ and $A(\lambda,\overline{\vartheta})=\overline{\vartheta}=\vartheta$, that is means

$$\begin{cases} \text{Find } \vartheta\in L^2(0,T;H_0^1(\Omega)) \text{ such that } {}^{RL}\mathscr{D}_{0,t}^s\vartheta\in L^2(0,T;H^{-1}(\Omega)), \\ \int_0^T \left\langle {}^{RL}\mathscr{D}_{0,t}^s\vartheta, w\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt \\ +\int_0^T\int_\Omega \nabla\vartheta\nabla w dxdt+\lambda\int_0^T\int_\Omega \varphi(\vartheta)wdxdt=0, \quad \forall w\in H_0^1(\Omega), \end{cases}$$

we have $g_s\in L^2(0,T)$ and we take $w=\vartheta-g_s\tau z$, we obtain

$$\int_0^T \left\langle {}^{RL}\mathscr{D}_{0,t}^s\vartheta, \lambda-g_s\lambda z\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt+\int_0^T\int_\Omega \nabla\vartheta\nabla(\vartheta-g_s\tau z)dxdt$$
$$+\lambda\int_0^T\int_\Omega \varphi(\vartheta)(\lambda-g_s\lambda z)dxdt=0. \qquad (6.7)$$

The key point of our prove is to show that, the first integral above is not negative From the proposition 5.2.1 , we have

$$\vartheta-(g_{1-s}*\vartheta)(0)g_s=g_s*{}^{RL}\mathscr{D}_{0,t}^s\vartheta \quad \text{in } L^2(0,T;H_0^1(\Omega)),$$

according the theorem 5.2.3 we deduce that the first integral in equation (6.7) non-negative. Then,

$$\int_0^T \left\langle {}^{RL}\mathscr{D}_{0,t}^s\vartheta, \vartheta-(g_{1-s}*\vartheta)(0)g_s\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt\geq 0.$$

When we go back to equation (6.7), we get to

$$\int_0^T\int_\Omega |\nabla\vartheta|^2dxdt\leq|\int_0^T\int_\Omega \varphi(\vartheta)\vartheta dxdt|+|\int_0^T\int_\Omega \varphi(\vartheta)zg_sdxdt|+|\int_0^T\int_\Omega \nabla\vartheta\nabla zg_sdxdt|,$$

then, we can write the above inequality as follows

$$\|\vartheta\|^2_{L^2(0,T;H^{1=0}(\Omega))}\leq I_1+I_2+I_3, \qquad (6.8)$$

where

$$I_1=|\int_0^T\int_\Omega \varphi(\vartheta)\vartheta dxdt|,$$

and

$$I_2=|\int_0^T\int_\Omega \varphi(\vartheta)zg_sdxdt|,$$

with

$$I_3=|\int_0^T\int_\Omega \nabla\vartheta\nabla zg_sdxdt|.$$

Hence, according the hypothesis (h1), using the Cauchy-Schwarz and Poincaré inequalities, we get to

- $\begin{aligned} I_1 &\leq C\int_0^T\int_\Omega |\vartheta| dxdt + C\int_0^T\int_\Omega |\vartheta|^2 dxdt, \\ &\leq C_{emb}C\sqrt{T|\Omega|}\|\vartheta\|_{L^2(0,T,H_0^1(\Omega))} + C_{emb}^2 C\|\vartheta\|^2_{L^2(0,T;H_0^1(\Omega))}, \end{aligned}$

and

- $\begin{aligned} I_2 &\leq C\int_0^T\int_\Omega |z||g_s| dxdt + C\int_0^T\int_\Omega |\vartheta||z||g_s| dxdt, \\ &\leq C_{emb}C\sqrt{T|\Omega|}\|z\|_{H_0^1}\|g_s\|_{L^2(0,T)} + C_{emb}^2 C\|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(0,T)}\|\vartheta\|_{L^2(0,T;H_0^1(\Omega))}. \end{aligned}$

we use the Hölder inequality to $I_3$, we find

$$I_3 \leq \|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(0,T)}\|\vartheta\|_{L^2(0,T;H_0^1(\Omega))}.$$

Then, we go back to equation (6.8), we obtain

$$\begin{aligned} &(1 - C_{emb}^2 C)\|\vartheta\|^2_{L^2(0,T;H_0^1(\Omega))} \\ &\leq \left[\sqrt{T|\Omega|}C_{emb}C + (C_{emb}^2 C + 1)\|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(0,T)}\right]\|\vartheta\|_{L^2(0,T;H_0^1(\Omega))} \\ &+ C_{emb}C\sqrt{T|\Omega|}\|g_s\|_{L^2(0,T)}\|z\|_{H_0^1(\Omega)}. \end{aligned}$$

Therefore, when we solve inequality, which is solved by solving a second-degree equation, we arrived to

$$\|\vartheta\|_{L^2(0,T;L^2(\Omega))} \leq M = R, \quad \text{with} \quad (1 - C_{emb}^2 C) > 0,$$

then, we obtain

$$\|\vartheta\|_{L^2(0,T;L^2(\Omega))} < R + 1. \tag{6.9}$$

Consequently, from the equation (6.9) we conclude that there are no solution to equation $A(\lambda,\vartheta) = \vartheta$ in the edge of ball $B(0, R+1) = \{\vartheta \in L^2(0,T;L^2(\Omega)) : \|\vartheta\|_{L^2(0,T;L^2(\Omega))} < R+1\}$, and this is for all $\lambda \in [0,1]$. □

**Lemma 6.3.3.** *Thanks to hypothesis (h1),* $\{A(\lambda,\overline{\vartheta}); \lambda \in [0,1], \overline{\vartheta} \in \overline{B}(0,R+1)\}$ *is relatively compact in* $L^2(0,T;L^2(\Omega))$.

***Proof.*** Let $(\lambda_n)_{n\in\mathbb{N}}$ subset $[0,1]$ and $(\overline{\vartheta}_n)_{n\in\mathbb{N}}$ subset $\overline{B}(0,R+1)$

**Step 01** We are going to prove that $\vartheta_n$ is bounded in $L^2(0,T;H_0^1(\Omega))$, we have

$$\int_0^T \left\langle {}^{RL}\mathscr{D}_{0,t}^s\vartheta_n, \vartheta_n - g_s\lambda_n z\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt + \int_0^T\int_\Omega \nabla\vartheta_n\nabla(\vartheta_n - g_s s_n z)dxdt$$
$$+ \lambda_n\int_0^T\int_\Omega \varphi(\overline{\vartheta}_n)(\vartheta_n - g_s\lambda_n z)dxdt = 0.$$

Thanks the Theorem 5.2.3 , we arrived to

$$\int_0^T\int_\Omega |\nabla\vartheta_n|^2dxdt$$
$$\leq |\int_0^T\int_\Omega \varphi(\overline{\vartheta}_n)\vartheta_n dxdt| + |\int_0^T\int_\Omega \varphi(\overline{\vartheta}_n)zg_s dxdt| + |\int_0^T\int_\Omega \nabla\vartheta_n\nabla zg_s dxdt|.$$

Then, we can write the above equation as

$$\|\vartheta\|^2_{L^2(0,T;H_0^1(\Omega))} \leq I_1^{'} + I_2^{'} + I_3^{'}, \tag{6.10}$$

where

$$I_1^{'} = |\int_0^T\int_\Omega \varphi(\overline{\vartheta}_n)\vartheta_n dxdt|,$$

with

$$I_2^{'} = |\int_0^T\int_\Omega \varphi(\overline{\vartheta}_n)zg_s dxdt|,$$

and

$$I_3^{'} = |\int_0^T\int_\Omega \nabla\vartheta_n\nabla zg_s dxdt|.$$

By the hypothesis (h1), Cauchy-Schwarz inequality and poincaré inequality, we find

- $$I_1^{'} \leq C\int_0^T\int_\Omega |\vartheta_n|dxdt + C\int_0^T\int_\Omega |\overline{\vartheta}_n||\vartheta_n|dxdt,$$
$$\leq C_{emb}C\left[\sqrt{T|\Omega|} + \|\overline{\vartheta}_n\|_{L^2(0,T;L^2(\Omega))}\right]\|\vartheta_n\|_{L^2(0,T;H_0^1(\Omega))},$$

and

- $$I_2^{'} \leq C\int_0^T\int_\Omega |z||g_s|dxdt + C\int_0^T\int_\Omega |\overline{\vartheta}_n||z||g_\beta|dxdt,$$
$$\leq C_{emb}C\left[\sqrt{T|\Omega|} + \|\overline{\vartheta}\|_{L^2(0,T;L^2(\Omega))}\right]\|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(0,T)}.$$

Then, we apply the Hölder inequality to $I_3^{'}$, we derive to

- $$I_3^{'} \leq \|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(0,T)}\|\vartheta_n\|_{L^2(0,T;H_0^1(\Omega))},$$

when go-back to equation (6.3.2), we find

$$
\begin{aligned}
&\|\vartheta_n\|^2_{L^2(0,T;H_0^1(\Omega))}\\
&\leq \left[C_{emb}C\big(\sqrt{T|\Omega|}+\|\overline{\vartheta}_n\|_{L^2(0,T;L^2(\Omega))}\big)+\|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(0,T)}\right]\|\vartheta_n\|_{L^2(0,T;H_0^1(\Omega))}\\
&+CC_{emb}\left(\sqrt{T|\Omega|}+\|\overline{\vartheta}_n\|_{L^2(0,T;L^2(\Omega))}\right)\|z\|_{H_0^1(\Omega)}\|g_s\|_{L^2(\Omega)}.
\end{aligned}
$$

In the last, we arrived to

$$\|\vartheta_n\|_{L^2(0,T;H_0^1(\Omega))}\leq K_1, \tag{6.11}$$

where the constant $K_1$ does not relate to $n$.

**Step 02** we are going to prove that ${}^{RL}\mathscr{D}^s_{0,t}\vartheta_n$ is bounded in $L^2(0,T;H^{-1}(\Omega)$, we have

$$ {}^{RL}\mathscr{D}^s_{0,t}\vartheta_n-\Delta\vartheta_n+\lambda_n\varphi(\overline{\vartheta}_n)=0, \tag{6.12}$$

therefore, multiplying the equation (6.12) by a function $w\in H_0^1(\Omega)$, and integrating from $\Omega$, we find

$$\left|\left\langle {}^{RL}\mathscr{D}^s_{0,t}\vartheta_n,w\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right|\leq\int_\Omega|\nabla\vartheta_n||\nabla w|dx+\lambda_n\int_\Omega|\varphi(\overline{\vartheta}_n)||w|dx,$$

hence, according the Cauchy-Schwarz, Poincaré inequalities and hypothesis (h1), and integrating to $(0,T)$ we get to

$$\|{}^{RL}\mathscr{D}^s_{0,t}\vartheta_n\|_{L^2(0,T;H^{-1}(\Omega))}\leq K_2, \tag{6.13}$$

where the constant $K_2$ does not depend to $n$.
Then, from **Step 01 & 02**, we obtain

$$
\begin{gathered}
(\vartheta_n)_{n\in\mathbb{N}}\text{ is bounded in }L^2(0,T;H_0^1(\Omega)),\\
\text{and}\\
({}^{RL}\mathscr{D}^s_{0,t}\vartheta_n)_{n\in\mathbb{N}}\text{ is bounded in }L^2(0,T;H^{-1}(\Omega)).
\end{gathered}
$$

Finally, thanks to Aubin-Simon theorem as a result that $\exists\{\vartheta_{n_k}\}_{k\in\mathbb{N}}$ converge to $\vartheta$ strongly in $L^2(0,T;L^2(\Omega))$. □

**Lemma 6.3.4.** *Thanks to hypothesis (h1),*
$A:[0,1]\times L^2(0,T;L^2(\Omega))\to L^2(0,T;L^2(\Omega))$ *is continuous.*

***Proof***. Let $\{\tau_n\}_{n\in\mathbb{N}} \subset [0,1]$ is converge to $\tau$ in $[0,1]$, and $\{\overline{\vartheta}_n\}_{n\in\mathbb{N}} \subset L^2(0,T;L^2(\Omega))$ is converge to $\overline{\vartheta}$ in $L^2(0,T;L^2(\Omega))$ when $n \to +\infty$. We are going to prove that $A(\lambda_n, \overline{\vartheta}_n) \to A(\lambda, \overline{\vartheta})$ in $L^2(0,T;L^2(\Omega))$, putting $A(\lambda_n, \overline{\vartheta}_n) = \vartheta_n$ for all $n \in \mathbb{N}$ and $A(\lambda, \overline{\vartheta}) = \vartheta$, we have

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\vartheta_n, w\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt + \int_0^T\int_\Omega \nabla\vartheta_n\nabla w\,dxdt + \lambda_n \int_0^T\int_\Omega \varphi(\overline{\vartheta}_n) w\,dxdt = 0 \qquad \forall w \in H_0^1(\Omega), \tag{6.14}$$

and

$$\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}\vartheta, w\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt + \int_0^T\int_\Omega \nabla\vartheta\nabla w\,dxdt + \lambda \int_0^T\int_\Omega \varphi(\overline{\vartheta}) w\,dxdt = 0, \qquad \forall w \in H_0^1(\Omega). \tag{6.15}$$

Then, taking subtraction between the two equations (6.14) and (6.15), we get

$$\begin{aligned}&\int_0^T \left\langle {}^{RL}\mathscr{D}^s_{0,t}(\vartheta_n-\vartheta), w\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt \\ &+ \int_0^T\int_\Omega \nabla(\vartheta_n-\vartheta)\nabla w\,dxdt + \int_0^T\int_\Omega (\lambda\varphi(\overline{\vartheta}) - \lambda_n\varphi(\overline{\vartheta}_n)) w\,dxdt = 0, \quad \forall w \in H_0^1(\Omega),\end{aligned}$$

taking $w = (\vartheta_n - \vartheta) - (\lambda_n - \lambda) z g_s$ and thanks the theorem 5.2.3, we find

$$\begin{aligned}&\int_0^T \|\vartheta_n - \vartheta\|^2_{H_0^1(\Omega)} dt \\ &\leq \int_0^T\int_\Omega \nabla(\vartheta_n-\vartheta)\nabla z(\lambda_n-\lambda) g_s\,dxdt + \int_0^T\int_\Omega (\lambda\varphi(\overline{\vartheta}) - \lambda_n\varphi(\overline{\vartheta}_n))(\vartheta_n-\vartheta)\,dxdt \\ &\quad + \int_0^T\int_\Omega (\lambda\varphi(\overline{\vartheta}) - \lambda_n\varphi(\overline{\vartheta}_n))(\lambda_n-\lambda) z g_s\,dxdt.\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\|\vartheta_n - \vartheta\|^2_{L^2(0,T;H_0^1(\Omega))} \\ &\leq \left[ \|g_s\|_{L^2(0,T)} |\tau_n - \lambda| \|z\|_{H_0^1(\Omega)} + \|\lambda\varphi(\overline{\vartheta}) - \lambda_n\varphi(\overline{\vartheta}_n)\|_{L^2(0,T;H^{-1}(\Omega))} \right] \|\vartheta_n - \vartheta\|_{L^2(0,T;H_0^1(\Omega))} \\ &+ \|\lambda\varphi(\overline{\vartheta}) - \lambda_n\varphi(\overline{\vartheta}_n)\|_{L^2(0,T;H^{-1}(\Omega))} |\lambda_n - \lambda| \|z\|_{H_0^1(\Omega)} \|g_s\|_{L^2(0,T)}.\end{aligned}$$

From $\varphi$ Lipschitz map, produces that $\varphi(\overline{\vartheta}_n) \to \varphi(\overline{\vartheta})$ in $L^2(0,T;L^2(\Omega))$ and we have $\tau_n \to s$ in $[0,1]$ when $n \to +\infty$, and by Poincaré inequality, we get

$$\|\vartheta_n - \vartheta\|_{L^2(0,T;L^2(\Omega))} \to 0 \quad \text{when } n \to +\infty, \tag{6.16}$$

as a result that $A$ is continuous from $[0,1] \times L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$. □

***Proof of main results (theorem 6.3.1)***. We have from lemma 6.3.2 there are no solution the equation $I_d - A(\lambda, \vartheta) = 0$ in the edge of the ball $B(0, R+1)$ and from lemma 6.3.3 shows that for $s \in [0,1]$, and $\overline{\vartheta} \in \overline{B}(0, R+1)$ the set $\{A(\lambda, \overline{\vartheta}),\}$ is relatively compact in $L^2(0,T; L^2(\Omega))$. Moreover, lemma 6.3.4 show that the homotopy $A$ is continuous from $[0,1] \times L^2(0,T; L^2(\Omega))$ into $L^2(0,T; L^2(\Omega))$.
So the degree $d(Id - A(\lambda, .), B(0, R+1), 0)$ is well define, from the property of homotopy invariance , we derive

$$\begin{aligned} d(Id - A(1,.), B(0,R+1), 0) &= d(Id - A(0,.), B(0,R+1), 0), \\ &= d(Id, B(0,R+1), 0) = 1 \neq 0, \end{aligned}$$

subsequently

$$\vartheta - A(1, \vartheta) = 0 \Rightarrow \vartheta = A(1, \vartheta).$$

Our result is that we prove the existence of weak solution to problem (6.1), $\vartheta \in H^s(0,T; H_0^1(\Omega), H^{-1}(\Omega))$. □

## 6.4 Uniqueness of solution

The uniqueness of weak solution of problem (6.1) is demonstrated in this section. Let $\vartheta_1, \vartheta_2$ be two solution to (6.1), then

$$\begin{cases} {}^{RL}\mathscr{D}^s_{0,t}\vartheta_1 - \Delta\vartheta_1 + \varphi(\vartheta_1) = 0 \quad \text{in } L^2(0,T; H^{-1}(\Omega)), \\ (g_{1-s} * \vartheta_1)(0) = z, \end{cases} \tag{6.17}$$

and

$$\begin{cases} {}^{RL}\mathscr{D}^s_{0,t}\vartheta_2 - \Delta\vartheta_2 + \varphi(\vartheta_2) = 0, \quad \text{in } L^2(0,T; H^{-1}(\Omega)), \\ (g_{1-s} * \vartheta_2)(0) = z. \end{cases} \tag{6.18}$$

Taking the deference between (6.17) and (6.18), we obtain

$${}^{RL}\mathscr{D}^s_{0,t}\big(\vartheta_1 - \vartheta_2\big) - \Delta\big(\vartheta_1 - \vartheta_2\big) + \varphi(\vartheta_1) - \varphi(\vartheta_2) = 0, \tag{6.19}$$

$$(g_{1-s} * (\vartheta_1 - \vartheta_2))(0) = 0. \tag{6.20}$$

Let $\tau \in (0,T]$. Testing (6.19) with $\vartheta_1 - \vartheta_2$, we get

$$\begin{aligned} &\int_0^\tau \big\langle {}^{RL}\mathscr{D}^s_{0,t}\big(\vartheta_1 - \vartheta_2\big), \vartheta_1 - \vartheta_2 \big\rangle dt \\ &+ \int_0^\tau \|\vartheta_1 - \vartheta_2\|^2_{H_0^1(\Omega)} dt + \int_0^\tau \int_\Omega \big(\varphi(\vartheta_1) - \varphi(\vartheta_2)\big)(\vartheta_1 - \vartheta_2) dx dt = 0, \end{aligned} \tag{6.21}$$

we have, $\int_0^\tau \|\vartheta_1-\vartheta_2\|^2_{H_0^1(\Omega)}dt \geq 0$ and with the Lipschitz assumption, we get

$$\int_0^\tau \left\langle {}^{RL}\mathscr{D}^s_{0,t}(\vartheta_1-\vartheta_2), \vartheta_1-\vartheta_2\right\rangle dt \leq C\int_0^\tau \|\vartheta_1-\vartheta_2\|^2_{L^2(\Omega)}dt,$$

from (6.20), we have $(\vartheta_1-\vartheta_2)$ lies in ${}_0W^s_{2,2}(0,T;H_0^1(\Omega),H^{-1}(\Omega))$, and from corollary 5.2.1, we arrived

$$g_{1-s}*\|(\vartheta_1-\vartheta_2)(.)\|^2_{L^2(\Omega)}(\tau) \leq 2C\int_0^\tau \|\vartheta_1-\vartheta_2\|^2_{L^2(\Omega)}dt. \tag{6.22}$$

If $\exists\tau\in(0,T]$ such that $\vartheta_1-\vartheta_2=0$ almost every on $[0,\tau]$ then we set

$$t_0 = sup\{\tau\in(0,T];\vartheta_1-\vartheta_2=0 \text{ almost every on } [0,\tau]\}.$$

Otherwise, $t=0$ is used. It is now sufficient to demonstrate that $t_0=T$ to obtain uniqueness. By using contradiction to make a point. Assume that $t_0\in[0,T)$, after that, for each of the $\tau\in(t_0,T]$, we have

$$\int_{t_0}^\tau \|\vartheta_1-\vartheta_2\|^2_{L^2(\Omega)} = \int_0^\tau \|\vartheta_1-\vartheta_2\|^2_{L^2(\Omega)}dt \neq 0, \tag{6.23}$$

then (6.22) it becomes

$$g_{1-s}(\tau-t_0)\int_{t_0}^\tau \|\vartheta_1-\vartheta_2\|^2_{L^2(\Omega)}dt \leq 2C\int_{t_0}^\tau \|\vartheta_1-\vartheta_2\|^2_{L^2(\Omega)}dt,$$

from (6.4), it yields

$$g_{1-s}(\tau-t_0)\leq 2C \quad \text{ on } (t_0,T].$$

That impossibility show that $t_0=T$.

***Proof of main result.*** We conclude from the third section that there is at least one weak solution, and we used the Leray-Schauder theory to prove this. In the fourth section, we proved the uniqueness of weak solution.
Our conclusion is that the problem (6.1) has a unique solution
$\vartheta\in H^s(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. □

# CONCLUSION

In conclusion, this thesis is devoted to investigating five categories of nonlinear fractional partial differential equations of elliptic and parabolic type with Dirichlet boundary conditions. The prominent fractional operators featured in this thesis include the fractional Laplacian, the distributional Riesz fractional gradient and the Riemann-Liouville time fractional derivative. These fractional problems represent extensions and generalizations of numerous classical partial differential equations, emphasizing the non-local characteristics of fractional operators. In the first chapter, we introduce some concepts and preliminary results. This thesis provides as a general and specialized overview, given us a clearer understanding of the remaining chapters.

Through the powerful approach adopted by certain methods, some sufficient conditions have been devised on the nonlinear terms, establishing existence and uniqueness of weak solutions. These five categories of problems are formulated in the weak formulation and then utilizing tree techniques, which are: Leray-Schauder degree theory, fixed point theorem and the Galerkin method.

we will proceed to outline key points discussed in this thesis as follows:

(a) First, the steps to solve our non-linear fractional problems by the technique of Leray-Schauder degree theory and the fixed point theorem are as follows:

  (a-i) for each our nonlinear fractional problem, we formulate linear fractional problem associate with it. Then, we study the existence and uniqueness of weak solutions for linear problem by Lax-Milgram theorem.

  (a-ii) We define an operator that is well-defined thanks to results of step *(a-i)*. additionally, it enables us to observe the equivalent solutions for our non-linear fractional problem.

  (a-iii) Under certain sufficient condition on the semilinear or nonlinear term, we are satisfying the conditions of the Leray-Schauder degree theory or fixed point theorem to prove the existence results.

  We have obtained the results by topological degree method in the first, second, forth and five class of problem. Additionally, by fixed point theorem in the third class of problem.

(b) Second, we solve our time fractional linear equation by the technique of the Galerkin method.

The techniques used in this thesis are effective and powerful in studying a wide range of fractional partial differential equations. These methods are utilizing in numerous contexts to establish the existence of solutions.
In conclusion, this research can be expanded, and the techniques used can be extensively utilized to solving numerous nonlinear fractional partial differential equations.

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