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# Thesis

In order to obtain the diploma of

**3rd cycle doctorate (LMD) in Mathematics**
**Option: Applied functional analysis**

# Study of certain partial differential equations of fractional order

**Presented by:**

***Tahar Belhadi***

**Publicly supported : October 26, 2023 at 10:00 a.m.**

**In front of the jury composed of :**

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جامعة 20 أوت 1955 -سكيكدة
كلية العـلوم
قسم الرياضيات
**المرجع:…………**

# Thèse

En vue de l'obtention du diplôme de

**Doctorat de 3° cycle (LMD) en Mathématiques**
**Option:** *Analyse fonctionnelle appliquée*

# Etude de certaines équations aux dérivées partielles d'ordre fractionnaire

**Présentée par :**

***Tahar Belhadi***

**Soutenue publiquement : 26 Octobre 2023 à 10 :00 h**

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# Dedication

I dedicate this work:
To my parents, brothers and sisters.
To my friend "Ali Slimani".
Finally, I dedicate this work to the entire Belhadi family.

# ABSTRACT

In this thesis, we study some nonlinear elliptic problems involving rational operators. The difficulty lies in introducing non-classical functional spaces. In the first part, we studied the Dirichlet boundary value problem involving the rational operator $p(x, y)-$Laplacian. We base the outcome of existence under certain conditions on nonlinearity. Our goal is to apply Berkovits degree theorem to quasi-continuous operators of type $(S_+)$ generalized to real reflective Banach spaces. Through this theory, we prove the existence of weak, non-intuitive solutions to this problem. We then study another problem involving $D^s$, which is the distributive fractional Riesz gradient for $0 < s < 1$, as a nonlocal operator generated by an Lèvy operation. Based on the invariance result for pseudotone operators, we have proven that there is at least one weak solution to such a problem. Moreover, we obtain the uniqueness of the solution to the problem under some considerations, and all of this from the theoretical side. From the numerical side, we have used a technique that is a combination of the Elzaki transformation method and the variational iteration method called the Elzaki variational iteration method (EVIM) on nonlinear fractional partial differential equations.

**Keywords:** Fractional $p(x, y)$-Laplacian problems, Generalized fractional Sobolev spaces, Topological degree, Weak solution, $s$-gradient, Bessel potential spaces, Nonlinear time-fractional partial differential equations, Variational iteration method, Elzaki transform, Elzaki variational iteration method (EVIM).

# RÉSUMÉ

Dans cette thèse, nous étudions quelques problèmes elliptiques non linéaires impliquant des opérateurs fractionnaires. Les difficultés sont l'introduction d'espaces fonctionnels non classiques. Dans la première partie, nous étudions le problème aux limites de Dirichlet impliquant l'opérateur fractionnaire $p(x, y)$-Laplacien. Nous établissons un résultat d'existence sous certaines conditions sur la non linéarité. Notre objectif est d'appliquer la théorie des degrés de Berkovits pour les opérateurs semi-continus de type $(S_+)$ généralisés dans des espaces de Banach réflexifs réels. Au moyen de ce théorème, nous prouvons l'existence de solutions faibles non triviales pour ce problème. Puis nous étudions un autre problème impliquant $D^s$ est le gradient fractionnaire distributionnel de Riesz pour $0 < s < 1$, en tant qu'opérateur non local généré par un processus de Lévy. Récemment, les processus de Lévy stables, qui donnent lieu à des équations avec le Laplacien fractionnaire, ont suscité beaucoup d'intérêt. Sur la base du résultat de surjectivité pour les opérateurs pseudomonotones, nous prouvons l'existence d'au moins une solution faible à un tel problème. De plus, nous obtenons l'unicité de la solution du problème sous certaines considérations, , et tout cela du côté théorique. Du côté numérique, nous avons utilisé une technique qui est une combinaison de la méthode de transformation d'Elzaki et de la méthode d'itération variationnelle appelée méthode d'itération variationnelle d'Elzaki (EVIM) sur les équations aux dérivées partielles fractionnaires non linéaires.

**Mots-clé:** $p(x, y)$-problèmes Laplacien fractionnaires, Espaces de Sobolev fractionnaires généraliséés, Degré topologique, Solution faible, s-gradient, Espaces potentiels de Bessel, Equations aux dérivées partielles fractionnaires non linéaires, Méthode d'itération variationnelle, Transformation d'Elzaki, Méthode d'itération variationnelle d'Elzaki (EVIM).

## ملخـــــص

الغرض من هذا العمل هو دراسة بعض المسائل الإهليجية غير الخطية التي تتضمن عوامل كسرية. تتمثل الصعوبات في إدخال فضاءات وظيفية غير كلاسيكية. في الجزء الأول، قمنا بدراسة مسألة حدود ديريكلي التي تتضمن العامل الكسري p(x,y)-Laplacien. نثبت نتيجة الوجود في ظل فرضيات معينة على اللاخطية. هدفنا هو تطبيق نظرية الدرجة الطوبولوجية لـ Berkovits على العوامل شبه المستمرة والتي تفي بالشرط $(S_+)_T$ في فضاءات باناخ الانعكاسية الحقيقية. ومن خلال هذه النظرية نثبت وجود حلول ضعيفة غير بديهية لهذه المشكلة. ثم قمنا بدراسة مشكلة أخرى تتضمن $D^s$. و هو التدرج الكسري التوزيعي لـريس من أجل $0<s<1$، كعامل غير محلي تم إنشاؤه بواسطة عملية ليفي . واستنادًا إلى نتيجة الغمور لعامل التوتر الرتابى، فإننا نثبت وجود حل ضعيف واحد على الأقل لمثل هذه المشكلة. علاوة على ذلك، فإننا نحصل على وحدانية حل المشكلة في ظل شروط معينة، وكل هذا من الجانب النظري. وأما من الجانب العددي، استخدمنا تقنية وهي عبارة عن مزيج من طريقة التحويل الزكي وطريقة التكرار التغايرية تسمى طريقة الزاكي التكرارية التغايرية (EVIM) في المعادلات التفاضلية الجزئية الكسرية غير الخطية.

**الكلمات المفتاحية**: كسور p(x,y)-Laplacien، فضاءات سوبوليف الكسرية المعممة، درجة طوبولوجية، حل ضعيف، فضاءات بيسل، المعادلات التفاضلية الجزئية الكسرية غير الخطية، طريقة التكرار المتغير،تحويل الزاكي، طريقة تكرار التباين لالزاكي (EVIM).

# Notations

- $\mathbb{R}^n$ n-dimensional Euclidean space.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain $\Omega$.
- $\complement\Omega$ the complement of $\Omega$.
- $C^\infty(\mathit{\Omega})$ space of infinitely differentiable functions on $\mathit{\Omega}$.
- $C_0^\infty(\mathit{\Omega})$ space of infinitely differentiable functions on $\mathit{\Omega}$ whose supports are compact.
- $p^-$ ess inf of $p(.)$ on $\mathit{\Omega}$.
- $p^+$ ess sup of $p(.)$ on $\mathit{\Omega}$.
- $\mathrm{L}^p(\Omega)$ Lebesgue space on $\mathit{\Omega}$ with constant exponent $p$.
- $\mathrm{L}^\infty(\Omega)$ Lebesgue space on $\mathit{\Omega}$ where $p = \infty$.
- $\mathrm{W}^{s,p}(\Omega)$ fractional Sobolev space on $\mathit{\Omega}$ with constant exponent $p$ and $s \in (0, 1)$.
- $\mathrm{W}_0^{s,p}(\Omega)$ closure of $C_0^\infty(\mathit{\Omega})$ in $\mathrm{W}^{s,p}(\Omega)$.
- $\mathrm{L}^{p(x)}(\Omega)$ Lebesgue space on $\mathit{\Omega}$ with variable exponent $p(x)$.

- $\rho_{p(.)}(u)$ modular functional on $\mathrm{L}^{p(x)}(\Omega)$.

- $\mathrm{W}^{g}$ generalized fractional Sobolev space on $\mathit{\Omega}$ with variable exponents with $s \in (0,1)$.

- $\rho_{g}(u)$ modular on $\mathrm{W}^{g}$.

- $\mathrm{W}_{0}^{g}$ space of zero $W^{g}$ functions p.p on $\mathbb{R}^{n} \backslash \mathit{\Omega}$.

- $\Delta_{p}$ $p$-Laplace operator.

- $(-\Delta_{p})^{s}$ fractional p-laplacian.

- $\Delta_{p}(.)$ $p(.)$-Laplace operator.

- $(-\Delta)^{s}$ fractional Laplace operator of order $s$.

- $\Delta_{p(.)}^{K} u(x)$ $p(.)$-generalized fractional Laplacian applied to $u$.
- $\mathbb{F}$ is the Fourier transform

- $\mathcal{S}'$ is the Schwartz space

- $< .,. >$ denotes the scalar product.

- $W^{s,2}(\mathbb{R}^{n}) = H^{s}(\mathbb{R}^{n})$, $W_{0}^{s,2}(\mathbb{R}^{n}) = H_{0}^{s}(\mathbb{R}^{n})$.

- $B_{R}$ open ball centered at the point 0 with the radius $R$.

- $\partial B_{R}$ sphere centered at the point 0 with the radius $R$.

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# INTRODUCTION

The study of partial differential equations dates from the 18th century. There is a wide variety of physical models introduced in the works of Cauchy, d'Alembert, Euler, Hamilton, Jacobi, Lagrange, and Laplace. The middle of the 19th century is enriched by the work of Riemann, Poincaré, and Hilbert. From there, the EDPs receive their title of nobility because they answer many questions that scientists ask themselves.

Fractional partial differential equations (FPDEs) are for many authors the generalization of partial differential equations (PDEs). More precisely, they are the limit cases of the PDEs because of their non-local property. In fact, the differential operators at a given point x depend on the values of the applied function in a small ball about $x$. But the Fractional Laplacian depends on all values of the function in $\mathbb{R}^n$.

Non-local equations, in particular, fractional differential equations can be elliptic, parabolic, or hyperbolic. This is a vague classification since each type of these equations has its specific characterizations. The elliptic equations are the one that doesn't depends on time, they are called the time-independent equations.

One of the reasons for the huge development of the theory of classical Lebesgue and fractional Sobolev spaces $L^p$ and $W^{s,p}$ (where $1 \leqslant p \leqslant \infty$) is the description of many phenomena arising in applied sciences. For instance, many materials can be modeled with sufficient accuracy using the function spaces $L^p$ and $W^{s,p}$, where $p$ is a fixed constant. For some materials with inhomogeneities, for instance, electrorheological fluids (sometimes referred to as "smart fluids"), this approach is not adequate, but rather the exponent p should be able to vary. This leads us to the study of variable exponent Lebesgue and fractional Sobolev spaces, $L^{p(x)}$ and $W^{s,p(x)}$, where p is a real-valued function. Historically, Lebesgue spaces with variable exponent were introduced by Orlicz in 1931 [50]. He defined the space $L^{p(x)}(\mathbb{R}^n)$ and established the Hölder inequality. During the years that followed, these spaces were considered a class of Musielak-Orlicz spaces, themselves a special case of modular spaces. On the other hand, Lebesgue spaces with variable exponents were discovered independently by the Russian mathematician Tsenov, and extensively developed first by Sharapudinov and then by Zhikov who [74] started a new direction of the investigation, which created the relationship between spaces with variable exponent and variational integrals with nonstandard growth conditions.

It is important to note that $(-\Delta_{p(x)})^s$ is a generalized operator of the fractional $p$-Laplacian operator $(-\Delta_p)^s$ (that is when $p(x,y) = p =$ constant ) and it is the fractional version of the $p(x)-$Laplacian $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}u)$ that is connected with the variable exponent Sobolev space. Over the last few years, interest in these operators, as well as pseudo-differential operators in general, has steadily increased. Nonlocal operators such as $(-\Delta)^s$ and its generalization $\mathcal{L}_K$ naturally arise in continuum mechanics, phase transition phenomena, population dynamics, and game theory, see, for example, [16, 42, 46] for examples of stochastical stabilization of Lévy processes. The applications of non-local integrodifferential sparked interest in understanding them. Indeed, they have a wide range of applications, including the thin obstacle issue, optimization, finance, stratified materials, anomalous diffusion, crystal dislocation, picture deblurring and denoising, and so on. We refer to [15, 17, 18, 19, 61, 65] and the references therein for more information. The study of these equations is a difficult task because there are no general methods for their resolutions. Each problem requires an appropriate approach depending on the type of linearity.

The study of fractional differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. We give in what follows two relevant examples [53] that justify the mathematical study of models involving variable exponents.

**Example 0.1** (Bingham fluids). In 1920, E. Bingham was surprised that some paints do not run, like honey. He studied such a behavior and described a strange phenomenon. There are fluids that flow then stop spontaneously (Bingham fluids). Within them, the forces that create flow reach a first threshold. As this threshold is not reached, the fluid flows without deforming as a solid. Invented in the 17th century, the "Flemish medium" makes painting oil thixotropic: it fluidizes under the pressure of the brush but freezes as soon as you leave the rest. While the exact composition of the medium Flemish remains unknown, it is known that the bonds form gradually between its components, which is why the picture freezes in a few minutes. Thanks to this wonderful medium, Rubens painted La Kermesse in 24 h.

**Example 0.2** (Electrorheological Fluids). The constitutive equation for the motion of an electrorheological fluid is

$$u_t + divA(u) + (u.\nabla)u + \nabla\omega = f, \tag{1}$$

where $u : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}^3$ is the velocity of the fluid at a point in space-time, $\omega : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}$ is the pressure, $f : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}^3$ represents external forces, and the stress tensor $A : W^{1,1}_{loc} \longrightarrow \mathbb{R}^{3,3}$ is of the form

$$A(u)(x) = \eta(x)(1 + |Du(x)|^2)^{(p(x)-2)/p(x)} Du(x),$$

where $Du = (\nabla u + \nabla u^T)/2$ is the symmetric part of the gradient of $u$.

We observe that the highest order differential term in (1) is

$$div\left((1 + (1 + |Du(x)|^2)^{(p(x)-2)/p(x)} Du(x)\right).$$

The degenerate case corresponds to the Laplace operator with variable exponent.

While Uriel Kaufmann and others with him were the first to extend the Sobolev spaces with variable exponents to include the fractional case in 2017 [36], and they prove a compact embedding theorem of these spaces into variable exponent Lebesgue spaces.

Our approach to solving this type of problem is based on the theory of topological degree and the method of monotone operators. First, we study a non-linear fractional elliptic equation with one unknown $u$ of the form

$$\begin{cases} \mathcal{L}^{p(x)}_K u + |u|^{q(x)-2}u + g(u) = f(x,u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a bounded regular open domain, $q \in C(\overline{\Omega})$ and the nonlocal integro-differential operator of elliptic type $\mathcal{L}^{p(x)}_K$,

$$\begin{aligned} \mathcal{L}^{p(x)}_K u &= p.v. \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))K(x,y)dy \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))K(x,y)dy, \quad x \in \mathbb{R}^N, \end{aligned}$$

where $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1, +\infty)$ is a continuous bounded function. The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, +\infty)$ is a measurable function. In the fourth chapter, we are interested in studying the existence and uniqueness of weak solutions for the fractional Laplacian problem of the unknown function $u$ of the form

$$\begin{cases} -div_s(D^s u) = -D^s.(D^s u) = f(x,u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$, $s$ is a fixed number between 0, 1 and $div_s(D^s u)$ is the fractional version of a 2-Laplacian and defined by (see [55])

$$div_s(D^s u) = D^s.(D^s u) = \sum_{i=1}^{N} \frac{\partial^s}{\partial x_i^s}\left(\frac{\partial^s u}{\partial x_i^s}\right)$$

Finding the analytical solution for fractional differential equations has a lot of attention to mathematicians's interest in recent years. Several methods are proposed to achieve this goal. The variational iteration method plays an important role in recent research in this field. This method was proposed by the Chinese mathematician He [33]. It has been shown that this procedure is a powerful tool for solving various kinds of problems. For example, this scheme is used for solving delay differential equations in [33]. Application of the variational iteration method to the Helmholtz equation is investigated in [48]. This technique is used in [1] for solving Burger's and coupled Burger's equations.

With the rapid development of nonlinear sciences, there appears an ever-increasing interest of scientists and engineers in the analytic asymptotic methods for nonlinear problems. Though it is very easy for us now to find the solutions of linear systems by means of computers, it is, however, still very difficult to solve nonlinear problems either numerically or theoretically. This is possibly due to the fact that the various discredited methods or numerical simulations apply iteration methods to find their numerical solutions to nonlinear problems, and nearly all iterative techniques are sensitive to initial solutions. So it is very difficult to obtain converged results in case of strong nonlinearity. Perturbation techniques provide the most versatile tools available in the nonlinear analysis of engineering problems.

As an advantage of the variational iteration method over the decomposition procedure of Adomian, the former method provides the solution to the problem without calculating Adomian's polynomials. This technique solves the problem without any need for discretization of the variables. Therefore, it is not affected by computation round-off errors, and one is not faced with the necessity of large computer memory and time.

Different transformations are characterized by their simplicity and speed of use in FPDEs, such as Laplace transform, Somodo transform, natural transformation, and Elsack transformation, which led various researchers in this field to combine these transformations and the aforementioned methods. Among these methods: Laplace homotopy analysis method [27], Laplace decomposition method [35], Laplace variational iteration method [70], homotopy perturbation Sumudu transform method [71], homotopy analysis Sumudu transform method [51], variational iteration Sumudu transform method [2], natural transform homotopy perturbation method [45], natural decomposition method [52], homotopy analysis natural transform method [54] and natural variational iteration method (NVIM) [38].

### Plan of the thesis

This thesis is divided into five chapters divided as follows:

### Chapter 1

In this chapter, we give more details about the properties of Bessel's potential spaces and fractional Lebesgue-Sobolev spaces with variable exponent. We study precisely that Poincaré and Hölder-type inequalities and certain Sobolev-type immersions are retained. We also provide some definitions and results about fractional calculus.

### Chapter 2:

In this chapter, We introduce a topological degree theory for a class of demicontinuous operators of generalized $(S_+)$ type in real reflexive Banach spaces, based on the recent Berkovits degree, we also study the method of monotone operators. We also show the important method in numerical approximations, which is the variable iteration method (VIM).

### Chapter 3:

deals with solving an elliptical fractional equation of the form

$$\begin{cases} \mathcal{L}_K^{p(x)} u + |u|^{q(x)-2}u + g(u) = f(x,u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4}$$

Here we show that if $f$ and $g$ satisfy certain conditions, then we can apply the Berkovits degree theorem to the problem (4), hence the existence of a non-trivial solution.

### Chapter 4:

This chapter is devoted to studying the existence and uniqueness of weak solutions for the fractional Laplacian problem

$$\begin{cases} -div_s(D^s u) = -D^s.(D^s u) = f(x,u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

By checking bounded, hemi-continuous, coercive, and monotone of the operator which is formed from the weak form of Problem (5) we prove as a result the existence and uniqueness of the weak solutions.

### Chapter 5:

In this chapter, we study in general the nonlinear time-fractional equations, we present their exact solutions using a new technique that is presented in reference [38], but by using the Elzaki transform, so this technique becomes a combination of the Elzaki transform

and the variational iteration method It is called the Elzaki variational iteration method (EVIM), this method was dropped on three numerical examples to prove the accuracy and efficiency of this technique.
Fanily we have a conclusion for all the works its study in this thesis.

# CHAPTER 1

# PRELIMINARIES

In this chapter we present some definitions and reminders of the results necessary for the continuation of this work. After recalling some basic results on functional spaces, we also present an introduction to the theory of Elzaki transform.

## 1.1 Generalized Lebesgue spaces

In this part, we recall some definitions and properties of fractional Lebesgue-Sobolev spaces with variable exponent commonly called generalized fractional Sobolev spaces. (see for example [24],[26] and [22]).

**Definition 1.1** Let $E$ be a $\mathbb{K}$-vector space. Each function $\rho : E \longrightarrow [0, \infty]$ satisfies the following properties:

(a) $\rho(x) = 0 \Leftrightarrow x = 0$.

(b) $\rho(\lambda x) = \rho(x)$ for all $x \in E$, $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.

(c) $\rho(tx + (1-t)y) \leq t\rho(x) + (1-t)\rho(y)$ for all $x, y \in E$, $t \in [0, 1]$.

(d) $\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x)$ for every $x \in E$.

(e) $\rho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$ .

is called a modular on $E$.

And now we present the following proposition, which despite its simplicity, we need later.

**Proposition 1.1** *Let $\rho$ be modular on $E$. Then the mapping $\lambda \longmapsto \rho(\lambda x)$ is non-decreasing for every $x \in E$. Moreover,*

$$\rho(\lambda x) = \rho(|\lambda| x) \leq |\lambda| \rho(x), \forall |\lambda| \leq 1,$$

$$\rho(\lambda x) = \rho(|\lambda| x) \geq |\lambda| \rho(x), \forall |\lambda| \geq 1.$$

**Definition 1.2** Let $\rho$ be a modular on $E$. The set $E_\rho = \{x \in E : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$ is called a modular space.

By definition the limit and the Proposition 1.1, we conclude that $E_\rho$ coincids with the space

$$\{x \in E : \rho(\lambda x) < \infty \ for \ some \ \lambda > 0\}.$$

**Theorem 1.1** *Let $\rho$ be a modular on $E$. Then $E_\rho$ is a normed $\mathbb{K}$ vector space. The norm, called the Luxemburg norm, is defined by*

$$\|x\|_\rho := \inf \left\{ \lambda > 0 : \rho(\frac{1}{\lambda} x) \leq 1 \right\}.$$

Consider the set

$$\mathrm{C}_+(\overline{\Omega}) = \left\{ p \in \mathrm{C}(\overline{\Omega}) : p(x) \geq 1 \ for \ all \ x \in \overline{\Omega} \right\},$$

where $\Omega$ is a bounded open set in $\mathbb{R}^n$. For all $p \in \mathrm{C}_+(\overline{\Omega})$, we define

$$p^- = \inf_{x \in \overline{\Omega}} p(x) \quad and \quad p^+ = \sup_{x \in \overline{\Omega}} p(x)$$

such that

$$1 < p^- \leqslant p(x) \leqslant p^+ < +\infty. \tag{1.1}$$

A very important role in manipulating the generalized Lebesgue spaces with a variable exponent is played by the modular $\rho_{p(.)} : \mathrm{L}^{p(x)}(\Omega) \longrightarrow \mathbb{R}$ which is defined by

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)} dx.$$

**Proposition 1.2** *Given $p(.) \in \mathrm{C}_+(\overline{\Omega})$, if $\lambda \geq 1$, then*

$$\rho_{p(.)}(u) \leq \lambda \rho_{p(.)}(u) \leq \lambda^{p^-} \rho_{p(.)}(u) \leq \rho_{p(.)}(\lambda u) \leq \lambda^{p^+} \rho_{p(.)}(u). \tag{1.2}$$

*And if $0 < \lambda < 1$, then*

$$\lambda^{p^+} \rho_{p(.)}(u) \leq \rho_{p(.)}(\lambda u) \leq \lambda^{p^-} \rho_{p(.)}(u) \leq \lambda \rho_{p(.)}(u) \leq \rho_{p(.)}(u). \tag{1.3}$$

**Proof.** If $\lambda \geq 1$, then

$$|u(x)|^{p(x)} \leq \lambda |u(x)|^{p(x)} \leq \lambda^{p^-} |u(x)|^{p(x)} \leq |\lambda u(x)|^{p(x)} \leq \lambda^{p^+} |u(x)|^{p(x)}, \ in\ \Omega.$$

And if $0 < \lambda < 1$, we have

$$\lambda^{p^+} |u(x)|^{p(x)} \leq |\lambda u(x)|^{p(x)} \leq \lambda^{p^-} |u(x)|^{p(x)} \leq \lambda |u(x)|^{p(x)} \leq |u(x)|^{p(x)}, \ in\ \Omega.$$

By integrating, in both cases, the inequalities above we obtain (1.2) and (1.3). ■

**Proposition 1.3** *The modular $\rho_{p(.)}$ check all the properties mentioned in the Definition 1.1.*

**Proof.** Let $u, v \in E = \mathrm{L}^0(\Omega)$.

(a) We have

$$\begin{aligned} \rho_{p(.)}(u) = 0 \ &\Leftrightarrow \ |u(x)|^{p(x)} = 0 \\ &\Leftrightarrow \ |u(x)| = 0 \\ &\Leftrightarrow \ u(x) = 0 \ in\ \Omega. \end{aligned}$$

(b) Obvious, from the definition of $\rho_{p(.)}$.

(c) We consider the function $\varphi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ of expression $\varphi(x, s) = |s|^{p(x)}$. It is clear that $\varphi$ is convex with respect to $s$. From the property of the Lebesgue integral, we conclude that $\rho_{p(.)}$ is convex.

(d) Since $|\lambda|^{p^+} \rho_{p(.)}(u) \leq \rho_{p(.)}(\lambda u) \leq |\lambda|^{p^-} \rho_{p(.)}(u)$, then

$$\lim_{\lambda \to 1^-} \rho_{p(.)}(\lambda u) = \rho_{p(.)}(u).$$

(e) Obvious, from the definition of $\rho_{p(.)}$.

■

**Proposition 1.4** *Given $p(.) \in \mathrm{C}_+(\overline{\Omega})$, then*

$$\rho_{p(.)}(u) < \infty \Leftrightarrow \exists \lambda_0 > 0 \ such\ that\ \rho_{p(.)}(\lambda_0 u) < \infty.$$

**Proof.** Clearly, if $\rho_{p(.)}(u) < \infty$, then $\exists \lambda_0 = 1$ such that $\rho_{p(.)}(\lambda_0 u) < \infty$.
Conversely, if $\exists \lambda_0 > 0$ such that $\rho_{p(.)}(\lambda_0 u) < \infty$, then by Proposition 1.2 we obtian

$$\rho_{p(.)}(\lambda_0 u) < \infty \ \ for\ some \ \ 0 < \lambda_0 < 1.$$

But then

$$\rho_{p(.)}(u) = \int_{\Omega} \left( \frac{|u(x)|\lambda_0}{\lambda_0} \right)^{p(x)} dx \leq \lambda_0^{-p^+} \rho_{p(.)}(\lambda_0 u) < \infty.$$

■

According to Definition 1.2, Proposition 1.3 and Proposition 1.4, we define the generalized Lebesgue space $\mathrm{L}^{p(x)}(\Omega)$ by

$$\mathrm{L}^{p(x)}(\Omega) = E_{\rho_{p(.)}} = \left\{ u : \Omega \longrightarrow \mathbb{R} \textit{ measurable } : \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

**Remark 1.1** The modular does not satisfy the triangle inequality, i.e.,

$$\rho_{p(.)}(u+v) \leq \rho_{p(.)}(u) + \rho_{p(.)}(v).$$

However, there is a substitute that is sometimes useful. For $1 \leq p < \infty$ and $a, b \geq 0$, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Therefore, for almost every $x \in \Omega$,

$$|u(x) + v(x)|^{p(x)} \leq 2^{p(x)-1}(|u(x)|^{p(x)} + |v(x)|^{p(x)}),$$

in particular,

$$\rho_{p(.)}(u+v) \leq 2^{p^+-1}(\rho_{p(.)}(u) + \rho_{p(.)}(v)).$$

We will refer to this as the modular triangle inequality.

Let be the quantity defined on the space $\mathrm{L}^{p(x)}(\Omega)$ by

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0, \quad \rho_{p(.)}(\frac{u}{\lambda}) = \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \leq 1 \right\}.$$

**Proposition 1.5** *The space* $\mathrm{L}^{p(x)}(\Omega)$ *is a vector space.*

**Proof.** It is a direct proof based on proof of Theorem 1.1. ■

**Proposition 1.6** *The expression* $\|.\|_{p(.)}$ *is a norm on* $\mathrm{L}^{p(x)}(\Omega)$*, so-called Luxemburg norm.*

**Proof.** Let $u, v \in \mathrm{L}^{p(x)}(\Omega)$ and $\alpha \in \mathbb{R}$, then we have:

(i) $\|u\|_{p(.)} \geq 0$.

(ii) $\|u\|_{p(.)} = 0 \Leftrightarrow u = 0$.

(iii) $\|\alpha u\|_{p(.)} = |\alpha| \|u\|_{p(.)}$.

(iv) $\|u+v\|_{p(.)} \leq \|u\|_{p(.)} + \|v\|_{p(.)}$.

It is a direct proof based on proof of Theorem 1.1. ■

**Example 1.1** If the function $p(x) = p \in [1, \infty[$, then $\|.\|_p$ is the usual norm of the space $\mathrm{L}^p(\Omega)$.

Indeed, as is obvious for $u = 0$, we assume that $u \neq 0$. By definition, we have:

$$\begin{aligned} \|u\|_{p(.)} &= \inf \left\{ \lambda > 0 : \quad \rho_{p(.)}(\frac{u}{\lambda}) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \quad \int_{\Omega} |\frac{u}{\lambda}|^{p(x)} dx \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \quad \frac{1}{\lambda^p} \|u\|_p^p \leq 1 \right\} \\ &= \inf \{ \lambda > 0 : \quad \|u\|_p \leq \lambda \} \\ &= \|u\|_p. \end{aligned}$$

**Proposition 1.7** *If* $u \in \mathrm{L}^{p(x)}(\Omega) \backslash \{0\}$*, then*

$$\|u\|_{p(.)} = a \ \ if \ and \ only \ if \ \rho_{p(.)}(\frac{u}{a}) = 1.$$

**Proof.** ($\Rightarrow$) Suppose that $\|u\|_{p(.)} = a$, then

$$\rho_{p(.)}(\frac{u}{a}) \leq 1.$$

Suppose that

$$\rho_{p(.)}(\frac{u}{a}) < 1.$$

For all $t > 0$, the modular $\rho_{p(.)}(\frac{u}{t})$ is continuous, convex and decreasing.
Then there exists $\delta > 0$, such that

$$\rho_{p(.)}(\frac{u}{t}) < 1, \ for \ all \ t \in (a - \delta, a + \delta).$$

Thus, we will have $a > a - \frac{\delta}{2} \in (a - \delta, a + \delta)$, which is absurd.
Consequently

$$\rho_{p(.)}(\frac{u}{a}) = 1.$$

($\Leftarrow$) We suppose that $\rho_{p(.)}(\frac{u}{a}) = 1$. Hence $\|u\|_{p(.)} \leq a$.
Suppose that $\|u\|_{p(.)} < a$. Then there exists $\lambda_0 > 0$ satisfying

$$\|u\|_{p(.)} \leq \lambda_0 < a.$$

In other words

$$\rho_{p(.)}(\frac{u}{a}) < \rho_{p(.)}(\frac{u}{\lambda_0}) \leq 1,$$

which is obviously absurd, so $\|u\|_{p(.)} = a$. This completes the proof. ■

### 1.1.1 Properties of spaces $\mathrm{L}^{p(x)}(\Omega)=\mathrm{W}^{0,p(x)}(\Omega)$

In this section, we recall some useful properties of the variable exponent spaces. We will insist on the fact that all the properties already encountered in classical Lebesgue spaces remain true, in particular the Poincaré inequality, Hölder inequality.

**Proposition 1.8** *([26]) If $u \in \mathrm{L}^{p(x)}(\Omega)$, then we have,*

*(1)* $\|u\|_{p(.)} < 1(=1;>1)$ *if and only if* $\rho_{p(.)}(u) < 1(=1;>1)$.

*(2) If* $\|u\|_{p(.)} > 1$, *then* $\|u\|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p^+}$.

*(3) If* $\|u\|_{p(.)} < 1$, *then* $\|u\|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p^-}$.

**Proof.** The case $u=0$ is trivial. We therefore assume that $u \neq 0$.

According to the above proposition $\|u\|_{p(.)}=1 \Leftrightarrow \rho_{p(.)}(u)=1$.
Now if $\|u\|_{p(.)}=a<1$, then $1<\frac{1}{a}$. By virtue of the increasing function $\rho_{p(.)}(\lambda u)$, we get

$$\rho_{p(.)}(u) < \rho_{p(.)}(\frac{u}{a}) = 1.$$

Conversely if $\rho_{p(.)}(u)<1$, then $\|u\|_{p(.)} \leq 1$. We deduce from Proposition 1.7 that

$$\|u\|_{p(.)} < 1.$$

The other inequalities are proved in a similar way.

(2) If we assume that $\|u\|_{p(.)}=a>1$, then

$$\rho_{p(.)}(\frac{u}{a}) = 1.$$

Since $\frac{1}{a}<1$, we get by Proposition 1.2,

$$\frac{1}{a^{p^+}}\rho_{p(.)}(u) \leq \rho_{p(.)}(\frac{u}{a}) = 1 \leq \frac{1}{a^{p^-}}\rho_{p(.)}(u),$$

i.e.

$$\|u\|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p^+}.$$

For point (3), we adapt the approach of (2) in a similar way. ■

**Proposition 1.9** *([26]) Given the sequence $(u_n) \in \mathrm{L}^{p(x)}(\Omega)$ and given $u \in \mathrm{L}^{p(x)}(\Omega)$, then the following statements are equivalent:*

*(1)* $\lim_{n\to\infty}\|u_n-u\|_{p(.)}=0$.

*(2)* $\lim_{n\to\infty}\rho_{p(.)}(u_n-u)=0$.

**Proof.** (1) $\Rightarrow$ (2). We suppose that $\lim_{n\to\infty}\|u_n-u\|_{p(.)}=0$. By definition of the limit, for all $\epsilon\in]0,1[$, there exists $n_0\in\mathbb{N}$ such that if $n\geq n_0$, we have

$$\|u_n-u\|_{p(.)}<\epsilon<1.$$

It follows that

$$\rho_{p(.)}(u_n-u)\leq\|u_n-u\|_{p(x)}^{p^-},$$

and consequently

$$\lim_{n\to\infty}\rho_{p(.)}(u_n-u)=0.$$

(2) $\Rightarrow$ (1). The converse is just as obvious because if $\lim_{n\to\infty}\rho_{p(.)}(u_n-u)=0$, then

$$\rho_{p(.)}(u_n-u)<\epsilon^{p^+}\ \textit{for n large enough.}$$

Considering Proposition 1.8, (1) , we obtain

$$\|u_n-u\|_{p(.)}<1.$$

It comes by virtue of the results of Proposition 1.8, (3),

$$\|u_n-u\|_{p(.)}^{p^+}\leq\rho_{p(.)}(u_n-u)<\epsilon^{p^+}.$$

or

$$\lim_{n\to\infty}\|u_n-u\|_{p(.)}=0.$$

This completes the proof. ■

Let $\hat{p}(.)\in\mathrm{C}_+(\overline{\Omega})$ be the conjugate exponent of $p(.)$, that is, $\dfrac{1}{p(x)}+\dfrac{1}{\hat{p}(x)}=1$. Then we have the following Hölder-type inequality

**Proposition 1.10** *([26])(Hölder's inequality). If* $u\in\mathrm{L}^{p(x)}(\Omega)$ *and* $v\in\mathrm{L}^{\hat{p}(x)}(\Omega)$*, so*

$$\left|\int_\Omega u(x)v(x)dx\right|\leq\left(\frac{1}{p^-}+\frac{1}{\hat{p}^-}\right)\|u\|_{p(.)}\|v\|_{\hat{p}(.)}\leqslant 2\|u\|_{p(.)}\|v\|_{\hat{p}(.)}.$$

**Proof.** Let's pose $\|u\|_{p(.)}=a$ and $\|v\|_{\hat{p}(.)}=b$. According to Young's inequality, we have

$$\begin{aligned}\left|\int_\Omega \frac{u(x)v(x)}{ab}dx\right| &\leq \int_\Omega \frac{|u(x)|}{a}\frac{|v(x)|}{b}dx\\ &\leq \int_\Omega \frac{1}{p(x)}\left|\frac{u(x)}{a}\right|^{p(x)}dx+\int_\Omega\frac{1}{\hat{p}(x)}\left|\frac{v(x)}{b}\right|^{\hat{p}(x)}dx\\ &\leq \frac{1}{p^-}\int_\Omega\left|\frac{u(x)}{a}\right|^{p(x)}dx+\frac{1}{\hat{p}^-}\int_\Omega\left|\frac{v(x)}{b}\right|^{\hat{p}(x)}dx\\ &\leq \frac{1}{p^-}+\frac{1}{\hat{p}^-}\end{aligned}$$

hence the result. ■

**Theorem 1.2** *([22]) Let the sequence* $(u_n) \in \mathrm{L}^{p(x)}(\Omega)$ *converge to* $u$*, then there exists a subsequence* $(u_{n_k})$ *such that*

*(a)* $u_{n_k}(x) \longrightarrow u(x)$ *in* $\Omega$*,*

*(b)* $|u_{n_k}(x)| \leq h(x)$ *in* $\Omega$*, with* $h \in \mathrm{L}^{\hat{p}(x)}(\Omega)$*.*

**Theorem 1.3** *([22]) Suppose that (1.1) is satisfied. If* $\Omega$ *is a bounded open domain, then* $(\mathrm{L}^{p(x)}(\Omega), \|u\|_{p(.)})$ *is a reflexive and separable Banach space.*

**Theorem 1.4** *([22]) Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set.*

*(1) The space* $C_0(\Omega)$ *is dense in* $\mathrm{L}^{p(x)}(\Omega)$*.*

*(2) The space* $C_0^\infty(\Omega)$ *is dense in* $\mathrm{L}^{p(x)}(\Omega)$*.*

### 1.1.2 Nemytskii operator

In this part, we define the Nemytskii operator in the space $\mathrm{L}^{p(x)}(\Omega)$. This operator is intimately linked to the Caratheodory function. In the following $\Omega$ denotes an open set of $\mathbb{R}^n$.

**Definition 1.3** (Caratheodory function). A function $f$ from $\Omega \times \mathbb{R}$ in $\mathbb{R}$ is called Caratheodory, if it satisfies:

1. The map: $t \mapsto f(x,t)$ is continuous p.p. $x \in \Omega$.
2. The map: $x \mapsto f(x,t)$ is measurable for all $t \in \mathbb{R}$.

**Theorem 1.5** *Let* $u : \Omega \longrightarrow \mathbb{R}$ *and* $f \in C(\overline{\Omega}, \mathbb{R})$ *. The operator* $N_f$ *defined by* $(N_f u)(x) = f(x, u(x))$ *is called the Nemytskii operator relative to* $f$*.*
*If* $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$*, a function of Caratheodory verifying*

$$|f(x,s)| \leq a(x) + b|s|^{\frac{p(x)}{q(x)}}, \text{ for all } x \in \Omega \text{ and } s \in \mathbb{R}.$$

$p(.), q(.) \in C_+(\overline{\Omega})$*,* $a \in \mathrm{L}^{q(x)}(\Omega)$*,* $a(x) \geq 0$ *and* $b \geq 0$*, then the Nemytskii operator* $N_f$ *of* $\mathrm{L}^{p(x)}(\Omega)$ *in* $\mathrm{L}^{q(x)}(\Omega)$ *is a continuous and bounded operator.*

## 1.2 Fractional Sobolev spaces with variable exponents

The purpose of this paragraph is to define generalized fractional Sobolev spaces with variable exponents and to study the topological properties of these spaces. These spaces will represent the functional framework in which we will seek solutions.

Let $p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (1, +\infty)$ be a bounded continuous function satisfying:

$$1 < p^- \leqslant p(x, y) \leqslant p^+ < +\infty. \tag{1.4}$$

$$p \textit{ is symmetric } \ (p(x, y) = p(y, x)). \tag{1.5}$$

$$p((x, y) - (z, z)) = p(x, y), \ \ \forall (x, y), (z, z) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.6}$$

Consider the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (1, +\infty)$ which will be a measurable function such that:

$$K(x, y) = K(y, x), \ \ \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.7}$$

$$(\exists k_0 > 0) : K(x, y) \geq k_0 |x - y|^{-(n+sp(x,y))}, \ \ \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \textit{ and } x \neq y. \tag{1.8}$$

$$mK \in \mathrm{L}^1(\mathbb{R}^n \times \mathbb{R}^n), \qquad \textit{where } m(x, y) = \min\{1, |x - y|^{p(x,y)}\}. \tag{1.9}$$

Let $\Omega$ be a Lipschitz open bounded subset of $\mathbb{R}^N$ and $s \in (0, 1)$ fixed such that $sp^+ < n$. We denote by:

$$\overline{p}(x) = p(x, x), \quad \forall x \in \overline{\Omega}.$$

$Q$ the set:

$$Q := (\mathbb{R}^n \times \mathbb{R}^n) \backslash (\complement\Omega \times \complement\Omega), \quad \textit{where } \complement\Omega = \mathbb{R}^n \backslash \Omega.$$

Two typical examples of kernel $K$ are:

. the kernel: $K_1(x, y) = |x - y|^{-(n+sp(x,y))}$.

. the kernel $K_2(x, y) = |x - y|^{-(n+sp(x,y))} a(x, y)$ with $a : \mathbb{R}^n \longrightarrow [1, +\infty)$ a bounded function ($a \in \mathrm{L}^\infty(\mathbb{R}^n)$) and $a(x - y) = a(y - x)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Let us now introduce the generalized fractional Sobolev spaces with variable exponent defined by:

$$\begin{aligned} W^g &= W^{K,p(x,y)} \\ &= \left\{ \begin{array}{c} u : \mathbb{R}^n \longrightarrow \mathbb{R} \textit{ such that } u_{|\Omega} \in \mathrm{L}^{\overline{p}(x)}(\Omega) \textit{ with} \\ \int_Q \dfrac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x, y) dx dy < \infty, \ \textit{for some } \lambda > 0 \end{array} \right\} \end{aligned}$$

We define on $W^g$ the functional:

$$\rho_g(u) = \rho_{K,p(x,y)}(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x, y) dx dy + \int_\Omega |u(x)|^{\overline{p}(x)} dx,$$

$\rho_g$ is a convex modular on $W^g$ with associated norm:

$$\|u\|_{\rho_g} = \inf \left\{ \lambda > 0 : \rho_g\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

We also define the closed subspace of $W^g$ by:

$$W_0^g = W_0^{K,p(x,y)} = \{u \in W^g : \ u = 0 \ p.p \textit{ on } \mathbb{R}^n \backslash \Omega\}.$$

For $u \in W_0^g$, we define the functional:

$$\rho_g^0(u) = \rho_{K,p(x,y)}^0(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x,y)dxdy,$$

which defines a convex modular on $W_0^g$ with associated norm:

$$\begin{aligned} \|u\|_{\rho_g^0} &= \inf\left\{\lambda > 0 : \rho_g^0(\frac{u}{\lambda}) \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)dxdy \leq 1\right\} = [u]_{K,p(x,y)}. \end{aligned}$$

**Remark 1.2** The modular $\rho_g^0$ does not satisfy the triangular inequality but it is replaced by another sometimes useful

$$\rho_g^0(u+v) \leq 2^{p^+}(\rho_g^0(u) + \rho_g^0(v)),$$

we will refer to this as the modular triangle inequality.

### 1.2.1 Properties of spaces $W^g$

**Lemma 1.1** *([6]) Let $p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (1, +\infty)$ be a continuous variable exponent and $K : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (0, +\infty)$ is a measurable function satisfying (1.7) and (1.9). Then for any $u \in W_0^g$, we have*

$$(i) \quad 1 \leq [u]_{K,p(x,y)} \quad \Rightarrow \quad [u]_{K,p(x,y)}^{p^-} \leq \rho_g^0(u) \leq [u]_{K,p(x,y)}^{p^+}. \tag{1.10}$$

$$(ii) \quad 1 \geq [u]_{K,p(x,y)} \quad \Rightarrow \quad [u]_{K,p(x,y)}^{p^+} \leq \rho_g^0(u) \leq [u]_{K,p(x,y)}^{p^-}. \tag{1.11}$$

**Proof.** $(i)$. Let $u \in W_0^g$ and $\lambda \in (0,1)$, we have

$$\lambda^{p^+} \leq \lambda^{p(x,y)} \leq \lambda^{p^-},$$

or

$$\rho_g^0(\lambda u) = \int_Q \lambda^{p(x,y)} |u(x) - u(y)|^{p(x,y)} K(x,y)dxdy,$$

so

$$\lambda^{p^+} \rho_g^0(u) \leq \rho_g^0(\lambda u) \leq \lambda^{p^-} \rho_g^0(u).$$

Now if $1 < [u]_{K,p(x,y)}$, then for $\lambda = \dfrac{1}{[u]_{K,p(x,y)}} \in (0,1)$ we obtian:

$$\frac{\rho_g^0(u)}{[u]_{K,p(x,y)}^{p^+}} \leq \rho_g^0(\frac{u}{[u]_{K,p(x,y)}}) \leq \frac{\rho_g^0(u)}{[u]_{K,p(x,y)}^{p^-}}.$$

Since $\rho_g^0(\dfrac{u}{[u]_{K,p(x,y)}}) = 1$. Then the desired result follows.

$(ii)$. Let $u \in W_0^g$ and $\lambda \geq 1$, we have

$$\lambda^{p^-} \leq \lambda^{p(x,y)} \leq \lambda^{p^+},$$

so

$$\lambda^{p^-} \rho_g^0(u) \leq \rho_g^0(\lambda u) \leq \lambda^{p^+} \rho_g^0(u).$$

Now if $1 \geq [u]_{K,p(x,y)}$, then for $\lambda = \dfrac{1}{[u]_{K,p(x,y)}} \geq 1$ we get:

$$\frac{\rho_g^0(u)}{[u]_{K,p(x,y)}^{p^-}} \leq \rho_g^0(\frac{u}{[u]_{K,p(x,y)}}) \leq \frac{\rho_g^0(u)}{[u]_{K,p(x,y)}^{p^+}}.$$

Since $\rho_g^0(\dfrac{u}{[u]_{K,p(x,y)}}) = 1$. Then the desired result follows. ■

**Proposition 1.11** *([6]) Let $p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (1, +\infty)$ be a continuous variable exponent and $K : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (0, +\infty)$ is a measurable function satisfying (1.7) and (1.9). Given the sequence $(u_n) \in \mathrm{W}_0^g$ and given $u \in \mathrm{W}_0^g$, then the following statements are equivalent:*

*(1)* $\lim_{n\to\infty} \|u_n - u\|_{\rho_g^0} = 0$.

*(2)* $\lim_{n\to\infty} \rho_g^0(u_n - u) = 0$.

**Proof.** (1) $\Rightarrow$ (2).
We suppose that $\lim_{n\to\infty} \|u_n - u\|_{\rho_g^0} = 0$.
By definition of the limit, for all $\epsilon \in ]0, 1[$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, we have

$$\|u_n - u\|_{\rho_g^0} < \epsilon < 1.$$

According to (1.11), we get:

$$\rho_g^0(u_n - u) \leq \|u_n - u\|_{\rho_g^0}^{p^-},$$

and consequently

$$\lim_{n\to\infty} \rho_g^0(u_n - u) = 0.$$

(2) $\Rightarrow$ (1).
The converse is just as obvious because if $\lim_{n\to\infty} \rho_g^0(u_n - u) = 0$, then

$$\rho_g^0(u_n - u) < \epsilon^{p^+} \; for \; n \; large \; enough.$$

Considering Proposition 1.7, we obtian

$$\|u_n - u\|_{\rho_g^0} < 1.$$

According to (1.11), we get:

$$\|u_n - u\|_{\rho_g^0}^{p^+} \leq \rho_g^0(u_n - u) < \epsilon^{p^+}.$$

or

$$\lim_{n\to\infty} \|u_n - u\|_{\rho_g^0} = 0.$$

This completes the proof. ■

**Proposition 1.12** *([6])* $\|.\|_{W^g} = \|.\|_{W^{K,p(x,y)}}$ *is a norm on* $W^g$*, with:*

$$\|u\|_{W^g} = \|u\|_{\overline{p}(.)} + [u]_{K,p(x,y)}.$$

Now we have the tools to prove a very important theorem:

**Theorem 1.6** *([6]) Let* $\Omega$ *be a Lipschitz bounded domain in* $\mathbb{R}^n$ *and* $s \in (0,1)$*. Let* $p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (1,+\infty)$ *be a continuous variable exponent such that* $sp(x,y) < n$ *for all* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$*. We assume that (1.4) and (1.5) hold. Let* $r : \overline{\Omega} \longrightarrow (1,+\infty)$ *be a continuous bounded variable exponent such that:*

$$1 < r^- \leqslant r(x) < p_s^*(x) = \frac{n\overline{p}(x)}{n - s\overline{p}(x)}, \quad \forall x \in \overline{\Omega}.$$

*Let* $K : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (0,+\infty)$ *be a measurable function satisfying (1.7)-(1.9). Then*

*(a) There exists a constant* $C = C(n,p,r,s,\Omega) > 0$ *such that for any function* $u \in W^g$ *we have:*

$$\|u\|_{r(.)} \leq C\|u\|_{s,p(x,y)} \leq C\max\{1,\tilde{k_0}\}\|u\|_{K,p(x,y)}.$$

*(b) There exists a positive constant* $C_0 = C_0(n,p,s,\tilde{k_0},\Omega)$ *such that:*

$$[u]_{K,p(x,y)} \leq \|u\|_{K,p(x,y)} \leq C_0[u]_{K,p(x,y)}.$$

**Proof.**

(a). Let $u \in W^g$, in view of ([6], Lemma 7), we have $u \in W^{s,p(x,y)}(\Omega)$ and

$$\|u\|_{s,p(x,y)} \leq \max\{1,\tilde{k_0}\}\|u\|_{W^g}. \tag{1.12}$$

When we combine (1.12) with ([6], Theorem 1), we get

$$\|u\|_{r(x)} \leq C\|u\|_{s,p(x,y)} \leq C\max\{1,\tilde{k_0}\}\|u\|_{W^g}.$$

(b). The definition of $\|u\|_{W^g}$, the statement (a), assumptions (1.7)-(1.9), and

$$1 < p^- \leqslant \overline{p}(x) < p_s^*(x),$$

can all be combined to simply prove this assertion.

■

**Remark 1.3** (1). The assertion $(a)$ implies that:

$$W^g \hookrightarrow W^{s,p(x,y)}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^{r(x)}(\Omega)$$

$$W_0^g \hookrightarrow \mathrm{L}^{r(x)}(\Omega),$$

where $1 < r^- \leqslant r(x) < p_s^*(x), \quad \forall x \in \overline{\Omega}$.

(2). Since the embedding $W^{s,p(x,y)}(\Omega) \hookrightarrow \mathrm{L}^{r(x)}(\Omega)$ is compact, then the following embeddings

$$W^g \hookrightarrow\hookrightarrow \mathrm{L}^{r(x)}(\Omega)$$

$$W_0^g \hookrightarrow\hookrightarrow \mathrm{L}^{r(x)}(\Omega).$$

(3). As a consequence of assertion (b), $[.]_{K,p(x,y)}$ is an equivalent norm of $\|.\|_{W^g}$ on $W_0^g$.

**Lemma 1.2** *([6]) $(W_0^g, [.]_{K,p(x,y)})$ is a separable, reflexive Banach space and uniformly convex.*

We prove a convergence property for a bounded sequence in $W_0^g$ in the following lemma.

**Lemma 1.3** *([6]) Under the same assumptions of Theorem 1.6. Let $\{u_j\}$ be a bounded sequence in $W_0^g$. Then there exists $u \in \mathrm{L}^{r(x)}(\mathbb{R}^n)$, with $u = 0$ a.e in $\mathbb{R}^n \backslash \Omega$ such that up to a subsequence*

$$u_l \longrightarrow u \text{ strongly in } \mathrm{L}^{r(x)}(\Omega) \text{ when } l \longrightarrow +\infty.$$

A sequence of smooth functions with compact support can be used to approximate any function in the fractional Sobolev space $W^g$.

**Lemma 1.4** *([6]) Suppose (1.4)-(1.6) be satisfied. Then the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^g$.*

## 1.3 Distributional Riesz fractional gradient

We recall in what follows some definitions and basic properties of the Bessel potential spaces $\mathtt{L}^{s,2}(\mathbb{R}^N)$ and spaces $X^{s,p}(\mathbb{R}^N)$. In that context, we refer to the book of E. Stein [63] and the papers of [59] and [60].

### 1.3.1 Bessel potential spaces $\mathtt{L}^{s,2}(\mathbb{R}^N)$

We start with the Bessel potentials $\varrho_s$, for $s \in \mathbb{R}^N_+$ . The Bessel potentials $\varrho_s$ are defined by

$$\varrho_s(x) := \frac{1}{(4\pi)^{\frac{s}{2}}\Gamma(\frac{s}{2})}\int_0^\infty e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{s-N}{2}} \frac{dt}{t}.$$

And can be shown to satisfy, for $t, s > 0$ :

1) $\varrho_s * \varrho_t = \varrho_{s+t}$.

2) $\widehat{\varrho_s}(\zeta) = (1 + 4\pi^2|\zeta|^2)^{-\frac{s}{2}}$.

3) $\|\varrho_s\|_{\mathtt{L}^1(\mathbb{R}^N)}$.

Then the Bessel potential spaces $\mathtt{L}^{s,2}(\mathbb{R}^N)$ are defined as follows.

**Definition 1.4** For $s \in (0, 1)$, we define $\mathtt{L}^{s,2}(\mathbb{R}^N)$ by

$$\mathtt{L}^{s,2}(\mathbb{R}^N) := \varrho_s(\mathtt{L}^2(\mathbb{R}^N)) = \left\{\varrho_s * f : f \in \mathtt{L}^2(\mathbb{R}^N)\right\},$$

with the norm

$$\|u\|_{\mathtt{L}^{s,2}(\mathbb{R}^N)} = \|f\|_{\mathtt{L}^2(\mathbb{R}^N)}.$$

**Theorem 1.7** *[59] The following statements hold.*

1) *If* $s \geqslant 0 \Rightarrow \overline{C_0^\infty(\mathbb{R}^N)}^{\mathtt{L}^{s,2}(\mathbb{R}^N)} = \mathtt{L}^{s,2}(\mathbb{R}^N)$.

2) *If* $s \geqslant 0 \Rightarrow \left[\mathtt{L}^{s,2}(\mathbb{R}^N)\right]' = \mathtt{L}^{-s,2}(\mathbb{R}^N)$.

3) *If* $t < s \Rightarrow \mathtt{L}^{s,2}(\mathbb{R}^N) \hookrightarrow \mathtt{L}^{t,2}(\mathbb{R}^N)$.

4) *If* $s \in (0, 1) \Rightarrow \mathtt{L}^{s,2}(\mathbb{R}^N)$ *coincides with the space* $W^{s,2}(\mathbb{R}^N)$,

   *where*

$$W^{s,2}(\mathbb{R}^N) := \left\{u \in \mathtt{L}^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy < \infty\right\}.$$

### 1.3.2 Distributional Riesz fractional gradient

**Definition 1.5** Let $s \in (0, 1)$. If $u \in \mathtt{L}^p(\mathbb{R}^N)$ for some $1 < p < \infty$ such that $I_{1-s} * u$ is well-defined, we define the distributional Riesz fractional gradient

$$(D^s u)_j := \frac{\partial^s u}{\partial x_j^s}, \qquad j = 1, \ldots, N,$$

where

$$\frac{\partial^s u}{\partial x_j^s} := \frac{\partial}{\partial x_j} I_{1-s} u,$$

in the sense that

$$\langle\frac{\partial^s u}{\partial x_j^s}, v\rangle = (-1)\langle I_{1-s}u, \frac{\partial v}{\partial x_j}\rangle = -\int_{\mathbb{R}^N}(I_{1-s}*u)\frac{\partial v}{\partial x_j}dx,$$

for every $v \in C_c^\infty(\mathbb{R}^N)$.

**Example 1.2** Let $s \in (0,1)$. If $u \in C_0^\infty(\mathbb{R}^N)$, then

$$D^s u = I_{1-s} * Du.$$

**Proof.** Let $u, v \in C_c^\infty(\mathbb{R}^N)$, define $K_v := supp\ v$ and let $R > 0$ such that for all $x \in K_v$,

$$supp\ u(x-.) \subset B(0,R).$$

Then

$$\begin{aligned}
\langle\frac{\partial^s u}{\partial x_j^s}, v\rangle &= (-1)\langle I_{1-s}u, \frac{\partial v}{\partial x_j}\rangle \\
&= -\int_{\mathbb{R}^N}(I_{1-s}*u)\frac{\partial v}{\partial x_j}dx \\
&= -\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_{1-s}*u)\frac{v(x+h_ne_j)-v(x)}{h_n}dx \\
&= -\lim_{n\to\infty}\int_{K_v}\int_{B(0,R)}\frac{1}{h_n}\frac{u(x-h_ne_j-y)-u(x-y)}{|y|^{N-1+s}}dyv(x)dx.
\end{aligned}$$

Now, for $x \in K_v$ and $y \in B(0,R)$ we have

$$\left|\frac{1}{h_n}\frac{u(x-h_ne_j-y)-u(x-y)}{|y|^{N-1+s}}\right| \le \frac{L}{|y|^{N-1+s}} \in \mathtt{L}^1(B(0,R)),$$

which implies that for every $x \in K_v$,

$$\left|\int_{B(0,R)}\frac{1}{h_n}\frac{u(x-h_ne_j-y)-u(x-y)}{|y|^{N-1+s}}dy\right| \le C(L,R).$$

Therefore, the pointwise almost everywhere convergence

$$\frac{1}{h_n}\frac{u(x-h_ne_j-y)-u(x-y)}{|y|^{N-1+s}} \longrightarrow -\frac{\frac{\partial u(x-y)}{\partial x_j}}{|y|^{N-1+s}}$$

and Lebesgue's dominated convergence theorem implies

$$\langle\frac{\partial^s u}{\partial x_j^s}, v\rangle = \int_{\mathbb{R}^N}(I_{1-s}*\frac{\partial u}{\partial x_j})vdx,$$

and the desired result follows by localizing in $v$. ■

**Theorem 1.8** *Let $s \in (0,1)$. Then*

$$\widehat{\frac{\partial u}{\partial x_j}} = -(2\pi)^s i\zeta_j |\zeta|^{-1+s}\widehat{u} \tag{1.13}$$

*for all $u \in \Im(\mathbb{R}^N)$.*

**Proof.** Suppose that $C_c^\infty(\mathbb{R}^N)$. Then by the definition of the Fourier transform of a tempered distribution and the definition of the fractional partial derivatives we have

$$\langle \widehat{\frac{\partial^s u}{\partial x_j^s}}, v\rangle := \langle \frac{\partial}{\partial x_j}(I_{1-s} * u), \widehat{v}\rangle = -\langle (I_{1-s} * u), \frac{\partial \widehat{v}}{\partial x_j}\rangle \tag{1.14}$$

However, since $I_{1-s} \in \Im(\mathbb{R}^N)'$ and $u \in \Im(\mathbb{R}^N)$, we have that the convolution is well-defined as a tempered distribution and thus the Fourier transform of the convolution is the product of the Fourier transforms. Therefore, since $\widehat{I_{1-s}} = (2\pi|\zeta|)^{-1+s}$ and $\frac{\partial \widehat{v}}{\partial x_j} = (2\pi i\zeta_j v)\hat{}$, we conclude that

$$\langle \widehat{\frac{\partial^s u}{\partial x_j^s}}, v\rangle = -\langle (2\pi)^s i\zeta_j |\zeta|^{-1+s}\widehat{u}, v\rangle,$$

which is the desired result. ■

**Definition 1.6** For $s \in (0,1)$. If $u \in C_c^\infty(\mathbb{R}^N)$, we define

$$X^{s,2}(\mathbb{R}^N) := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{X^{s,2}(\mathbb{R}^N)}}$$

with the norm

$$\|u\|^2_{X^{s,2}(\mathbb{R}^N)} = \|u\|^2_{\mathrm{L}^2(\mathbb{R}^N)} + \|D^s u\|^2_{\mathrm{L}^2(\mathbb{R}^N)}.$$

For $u \in X^{s,p}(\mathbb{R}^N)$, with an abuse of notation, we use $D^s u$ to denote the limit of the Cauchy sequence $\{D^s u_n\}_{n\in\mathbb{N}}$.

**Proposition 1.13** *[59] If $s \in (0,1)$, then*

$$X^{s,p}(\mathbb{R}^N) = \mathrm{L}^{s,p}(\mathbb{R}^N).$$

By $\mathrm{L}_0^{s,p}(\Omega)$, we denote the subspace of $\mathrm{L}^{s,p}(\mathbb{R}^N)$ i.e.

$$\mathrm{L}_0^{s,p}(\Omega) := \left\{u \in \mathrm{L}^{s,p}(\mathbb{R}^N) : u = 0, \ \mathit{on}\ \mathbb{R}^N \setminus \Omega\right\}.$$

**Theorem 1.9** *[59] Let $s \in (0,1)$, and $1 \le q < \frac{2N}{N-2s}$, then there exists a constant $C = C(\Omega, N, s) > 0$ such that*

$$\left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}} \le C\|D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)}$$

*for all $u \in \mathrm{L}^{s,2}(\mathbb{R}^N)$.*

Using the Theorem 1.9, we remark that the norm $\|u\|_{\mathrm{L}^2(\mathbb{R}^N)}+\|D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)}$ is equivalent to $|D^s u|_{\mathrm{L}^2(\mathbb{R}^N)}$ in $\mathrm{L}_0^{s,2}(\Omega)$. The space $\mathrm{L}_0^{s,2}(\Omega)$ with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} D^s u.D^s v dx,$$

is a Hilbert space.

Next, we recall some embedding results

**Theorem 1.10** *[59](Fractional Sobolev inequality). Let $s \in (0,1)$ be such that $2s < N$. Then there exists a constant $C = C(N,s) > 0$ such that*

$$\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^N)} \leqslant C\|D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)}$$

*for all $u \in \mathrm{L}^{s,2}(\mathbb{R}^N)$, where $2^* = \frac{2N}{N-2s}$.*

**Proposition 1.14** *[44] Let $s \in (0,1)$. Then the embedding*

$$\mathrm{L}_0^{s,2}(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$$

*is compact for $1 \leq q < 2^*$.*

## 1.4 Gamma and Mittag-Leffler functions

The Gamma and Mittag-Leffler functions, which will be utilized later, are introduced in this paragraph. In the theory of fractional calculus, these two functions are a very important role.

### 1.4.1 Gamma function

The Swiss mathematician Leonhard Euler (1707-1783) developed the gamma function in an effort to generalize the factorial of non-integer values.

**Definition 1.7** Euler's Gamma function is a function that naturally extends the factorial to real numbers, and even to complex numbers. For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, we define the Gamma function by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{1.15}$$

We can prove by integration by parts in (1.15) that

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{(z+1)-1} dt = \left[-t^z e^{-t}\right]_{t=0}^{t=+\infty} + z\int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z). \tag{1.16}$$

Property (1.16) allows us to establish that

$$\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}.$$

### 1.4.2 Mittag-Leffler function

In 1903, Mittag-Leffler introduced the exponential function's generalization, the one-parameter Mittag-Leffler function.

**Definition 1.8** Let $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, we define the well-known Mittag-Leffler function as follows

$$E_\alpha(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(n\alpha + 1)}. \tag{1.17}$$

In the theory of fractional calculus, the two-parameter Mittag-Leffler function plays a very important role. Agarwal and Erdelyi introduced this function in 1953-1954, the following series expansion defines it:

**Definition 1.9** As a generalization of the function $E_\alpha$ is the function $E_{\alpha,\beta}$ defined by

$$E_{\alpha,\beta}(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, (Re(\alpha), Re(\beta)) > 0. \tag{1.18}$$

From the definition of $E_{\alpha,\beta}$ (1.18), we find the following relations:

$$E_{\alpha,1}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(\alpha k + 1)} = E_\alpha(r). \tag{1.19}$$

$$E_{1,1}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{r^k}{k!} = e^r. \tag{1.20}$$

$$E_{1,2}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(k + 2)} = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)!} = \frac{1}{r} \sum_{k=0}^{\infty} \frac{r^k}{(k+1)!} = \frac{e^r - 1}{r}, \tag{1.21}$$

and in general

$$E_{1,m}(r) = \frac{1}{r^{m-1}} \left[ e^r - \sum_{k=0}^{m-2} \frac{r^k}{k!} \right]. \tag{1.22}$$

## 1.5 Functional spaces

### 1.5.1 Spaces of absolutely continuous functions

Now let $\Omega = [0, T] (0 < T < +\infty)$ be a finite interval of $\mathbb{R}$ and $n \in \mathbb{N}^*$.

**Definition 1.10** ([40]) We denote by $AC(\Omega)$ the space of absolutely continuous functions on $\Omega$ consisting of functions $f$ which are primitives of Lebesgue-summable functions i.e:

$$AC(\Omega) = \left\{ f / \exists \varphi \in L^1(\Omega) : f(t) = c + \int_0^t \varphi(s) ds \right\}.$$

Thus, any absolutely continuous function $f$ has a summable derivative $f' = \varphi$, almost everywhere on $\Omega$, and therefore $c = f(0)$.

**Definition 1.11** ([40]) We denote by $AC^n(\Omega)$, the space of functions $f$ defined on $\Omega$ with values in $\mathbb{C}$ which have continuous derivatives on $\Omega$ up to order $n-1$ and such that $f^{(n-1)} \in AC(\Omega)$ i.e.

$$AC^n(\Omega) = \Big\{ f : \Omega \longrightarrow \mathbb{C}, f^{(k)} \in C(\Omega), k \in 0, \ldots, n-1, f^{(n-1)} \in AC(\Omega) \Big\}.$$

In particular $AC^1(\Omega) = AC(\Omega)$.

A characterization of the functions of this space is given by the following lemma:

**Lemma 1.5** *([40]) A function $f \in AC^n(\Omega)$, $n \in \mathbb{N}^*$, if and only if it is represented in the form:*

$$f(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f^{(n)}(s) ds + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

### 1.5.2 Spaces of continuous functions with weights

**Definition 1.12** ([40]) Let $\Omega = [0,T] (0 < T < +\infty)$ be a finite interval of $\mathbb{R}$ and $\mu \in \mathbb{C} (0 \leqslant Re(\mu) < 1)$. We denote by $C_\mu(\Omega)$ the space of functions $f$ defined on $\Omega$ such that the function $t^\mu f(t) \in C(\Omega)$ i.e.:

$$C_\mu(\Omega) = \Big\{ f : \Omega \longrightarrow \mathbb{C}, (.)^\mu f(.) \in C(\Omega) \Big\},$$

fitted with the norm:

$$\|f\|_{C_\mu(\Omega)} = \|(.)^\mu f(.)\|_{C(\Omega)} = \max_{t \in \Omega} |t^\mu f(t)|.$$

The space $C_\mu(\Omega)$ is called the space of continuous functions with weights.
In particular, $C_0(\Omega) = C(\Omega)$.

**Definition 1.13** ([40]) For $n \in \mathbb{N}^*$ we denote by $C_\mu^n(\Omega)$ the space of functions $f$ which have continuous derivatives on $\Omega$ up to the order $(n-1)$, such that $f^{(n)} \in C_\mu(\Omega)$ i.e.:

$$C_\mu^n(\Omega) = \Big\{ f : \Omega \longrightarrow \mathbb{C}, f^{(k)} \in C(\Omega), k \in 0, \ldots, n-1, f^{(n)} \in C_\mu(\Omega) \Big\},$$

fitted with the norm:

$$\|f\|_{C_\mu^n(\Omega)} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{C(\Omega)} + \|f^{(n)}\|_{C_\mu(\Omega)}.$$

In particular, $C_\mu^0(\Omega) = C_\mu(\Omega)$.

## 1.6 Integrals and fractional derivatives

The purpose of this part is to introduce some definitions and results of fractional calculus in the sense of Caputo. We will start with the definition of the Riemann-Liouville integral (see [40]).

### 1.6.1 Fractional integral in the sense of Riemann-Liouville

**Definition 1.14** The Riemann-Liouville fractional integral of order $\alpha$ of a function $f \in L^1([0,T]), T > 0$ is given by

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s)ds, & t > 0, \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{1.23}$$

**Theorem 1.11** *([34]) If $f \in L^1([0,T]), T > 0$, then $I^\alpha f$, $\alpha > 0$, exists for almost all $t \in [0,T]$ and moreover $I^\alpha f \in L^1([0,T])$ i.e:*

$$\|I^\alpha f\|_{L^1([0,T])} \leqslant \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^1([0,T])}, \quad \forall f \in L^1([0,T]).$$

An important property of the fractional integration operator $I^\alpha f$, called the semi-group property, is given by the following lemma:

**Lemma 1.6** *([34]) Let $\alpha, \beta > 0$ for any function $f \in L^1([0,T]), T > 0$ we have:*

$$I^\alpha \left(I^\beta f(t)\right) = I^{\alpha+\beta} f(t) = I^\beta \left(I^\alpha f(t)\right),$$

*for almost any $t \in [0,T]$. If moreover $f \in C([0,T])$, then this identity is true $\forall t \in [0,T]$.*

### 1.6.2 Fractional derivative in the sense of Caputo

In this part we give the definition of the fractional derivative in the sense of Caputo as well as some essential properties.

**Definition 1.15** The Caputo fractional derivative of order $\alpha > 0$ of a function $f$ defined on $[0,T]$ is given by

$${}^cD^\alpha f(t) = \begin{cases} I^{n-\alpha} D^n f(t) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, & t > 0, n-1 < \alpha < n, \\ f^{(n)}(t), & \alpha = n. \end{cases} \tag{1.24}$$

In particular, when $0 < \alpha < 1$, then:

$$\begin{aligned} {}^cD^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds, t > 0 \\ &= I^{1-\alpha} f'(t). \end{aligned} \tag{1.25}$$

Among the relations that combine the Riemann-Liouville fractional integral and Liouville-Caputo fractional derivative, we distinguish the following relation

$$I^{\alpha}\, {}^{c}D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) \frac{t^{k}}{k!}, t > 0, n-1 < \alpha < n. \tag{1.26}$$

**Theorem 1.12** *([40]) Let $\alpha > 0$ and let $n = [\alpha] + 1$. If $f \in AC^{n}([0,T])$, then the fractional Caputo derivative ${}^{c}D^{\alpha} f(t)$ exists almost everywhere on $[0,T]$.*

*1) If $\alpha \notin \mathbb{N}$, then ${}^{c}D^{\alpha} f(t)$ is given by (1.24). In particular, it takes the form (1.25) for $0 < \alpha < 1$.*

*2) If $\alpha \in \mathbb{N}$, then:*

$${}^{c}D^{\alpha} f(t) = f^{(n)}(t).$$

**Theorem 1.13** *([40]) Let $\alpha > 0$ and let $n = [\alpha]+1$. If $f \in C^{n}([0,T])$, then the fractional Caputo derivative ${}^{c}D^{\alpha} f(t)$ is continuous on $[0,T]$, $T > 0$.*

*1) If $\alpha \notin \mathbb{N}$, then ${}^{c}D^{\alpha} f(t)$ is given by (1.24). In particular, it takes the form (1.25) for $0 < \alpha < 1$.*

*2) If $\alpha \in \mathbb{N}$, then:*

$${}^{c}D^{\alpha} f(t) = f^{(n)}(t).$$

In this work, we consider the time fractional derivative in the sense of Caputo.

**Definition 1.16** ([37]) Let $u(x,t) \in C^{n}_{-1}(I \times \Omega), n \in \mathbb{N}^{*}, x \in I \subset \mathbb{R}$. The fractional derivative temporal in the sense of Caputo of $u$ is defined for $t > 0$ by:

$${}^{c}D^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{0}^{t} (t-s)^{n-\alpha-1} \frac{\partial^{n} u(x,s)}{\partial s^{n}} ds, & n-1 < \alpha < n, \\ \dfrac{\partial^{n} u(x,t)}{\partial t^{n}}, & \alpha = n. \end{cases} \tag{1.27}$$

## 1.7 Elzaki transform

It was Oliver Heaviside and Joseph Fourier who first developed integral transformation techniques in the 19th century. The core concept is to express a function $f(t)$ in terms of a transformation $F(z)$

$$F(z) := \int_{-\infty}^{+\infty} K(z,t) f(t) dt,$$

where the functions $K(z,t)$ are called kernels.

There are a number of important integral transformations like that of Fourier, Laplace, Sumudu, Hankel, Laguerre, Hermite, Mellin and the natural transformation. They are defined by choosing different kernels.

### 1.7.1 Definitions and properties

In this part, we present the definitions and properties of the Elzaki transformation. Tarig. M. Elzaki [25, 47] recently introduced a new integral transform "Elzaki transform" to solve ordinary and partial differential equations in the time domain.

Given a function $f(t)$, $t \in \mathbb{R}$, then the general integral transformation is defined by:

$$\mathcal{T}[f(t)](s) = \int_{-\infty}^{+\infty} K(s,t) f(t) dt, \tag{1.28}$$

where $K(s,t)$ represents the kernel of the transformation, and $s$ is a real or complex number independent of $t$. Note that when $K(s,t)$ is $e^{-st}$, and $tJ_n(st)$, then equation (1.28) gives respectively the Laplace transformation, and the Hankel transformation.

Now, for $f(t)$, $t \in (-\infty, +\infty)$, consider the integral transforms defined by (see [39], 2008):

$$\mathcal{T}[f(t)](\rho) = \int_{-\infty}^{+\infty} K(t) f(\rho t) dt, \tag{1.29}$$

and

$$\mathcal{T}[f(t)](s,\rho) = \int_{-\infty}^{+\infty} K(s,t) f(\rho t) dt. \tag{1.30}$$

The kernel $K(t) = e^{-t}$ in equation (1.29) gives the integral Sumudu transform where the parameter $s$ is replaced by $\rho$. Equation (1.30) is the combination of the integral transforms (1.28) and (1.29) with the kernels already defined. Moreover, for any value of $n$, the generalized Laplace and Sumudu transforms are defined respectively by:

$$\mathcal{L}[f(t)] = U(s) = s^n \int_0^{+\infty} e^{-s^{n+1}t} f(s^n t) dt, \tag{1.31}$$

and

$$\mathcal{S}[f(t)] = V(\rho) = \rho^n \int_0^{+\infty} e^{-\rho^n t} f(\rho^{n+1} t) dt. \tag{1.32}$$

Note that when $n = 0$, equation (1.31) and equation (1.32) are respectively the Laplace and Sumudu transforms.

Before we define the Elzaki transform, we need the following set of functions

$$A = \left\{ h(t) : \exists M, k_1, k_2 > 0, \quad |h(t)| < M e^{|t|/k_j}, \quad if\ t \in (-1)^j \times [0, \infty[ \right\},$$

the Elzaki transformation of the function $f(t) \in A$ for $t \in \mathbb{R}$ is defined by [25, 47]:

$$E[f(t)] = \rho \int_{-\infty}^{+\infty} e^{-t/\rho} f(t) dt = El(\rho), \quad -k_1 < \rho < k_2, \tag{1.33}$$

where $E[f(t)]$ is the Elzaki transformation of the function $f(t)$ and the variable $\rho$ is the Elzaki transformation variable.

Moreover, if the function $f(t)H(t)$ which is defined on the positive real axis, where $H(.)$ is the Heaviside function, $t \in (0, +\infty)$ then, the Elzaki transformation is defined as follows:

$$\begin{aligned} E[f(t)H(t)] &= E^{+}[f(t)] = EL^{+}(\rho) = \rho \int_{0}^{+\infty} e^{-t/\rho} f(t)dt \qquad (1.34) \\ &= \rho^{2} \int_{0}^{+\infty} e^{-t} f(\rho t)dt. \qquad (1.35) \end{aligned}$$

Note that, equation (1.34) is reduced to the Laplace transformation $EL^{+}(\rho) = \rho U(\frac{1}{\rho})$ and equation (1.35) is reduced to the Sumudu transformation $EL^{+}(\rho) = \rho^{2} V(\rho)$.

### properties

Some basic properties of the Elzaki transformation are given as follows [25, 47]:

**property 1:** The Elzaki transformation is a linear operator. In other words, for each $\lambda, \beta \in \mathbb{R}$, we have:

$$E^{+}[\lambda f + \beta g] = \lambda E^{+}[f(t)] + \beta E^{+}[g(t)].$$

**property 2:** If $f^{(n)}(t)$ is the nth derivative of the function $f(t)$ with respect to $t$ then its Elzaki transformation is given by:

$$E^{+}[f^{(n)}(t)] = \frac{E^{+}[f(t)]}{\rho^{n}} - \sum_{k=0}^{n-1} \rho^{2-n+k} f^{(k)}(0).$$

**property 3:** Suppose that $EL_{f}^{+}(\rho)$ and $EL_{g}^{+}(\rho)$ are the Elzaki transforms of $f(t)$ and $g(t)$ respectively, defined on the set $A$, then the Elzaki transform of their convolution is given by:

$$E^{+}[(f * g)(t)] = \frac{1}{\rho} EL_{f}^{+}(\rho) EL_{g}^{+}(\rho),$$

where the convolution of two functions is defined by:

$$(f * g)(t) = \int_{0}^{t} f(\xi) g(t-\xi) d\xi = \int_{0}^{t} f(t-\xi) g(\xi) d\xi.$$

**property 4:** Some elementary functions and their transformations.

$$E^{+}[1] = \rho^{2},$$

$$E^{+}[t] = \rho^{3},$$

$$E^{+}[t^{n-1}/(n-1)!] = \rho^{n+1}, \quad n = 1, 2, \ldots$$

**property 5:** If $\alpha$ is a fractional number, then the Elzaki transform of $t^{\alpha}$ is given by

$$E^{+}[t^{\alpha}] = \Gamma(\alpha + 1) \rho^{\alpha+2}.$$

**Theorem 1.14** ***(Elzaki transform of fractional derivatives)*** *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leqslant n$ and $EL^+(\rho)$ be the Elzaki transform of the function $f(t)$, then the Elzaki transform of the Liouville-Caputo fractional derivative of $f(t)$ of order $\alpha$, is given by*

$$E^+[{}^cD^\alpha f(t)] = \frac{EL^+(\rho)}{\rho^\alpha} - \sum_{k=0}^{n-1} \rho^{k-\alpha+2}[D^k f(t)]_{t=0}. \tag{1.36}$$

# CHAPTER 2

# METHODS FOR SOLVING NONLINEAR FRACTIONAL FPDES

In nonlinear functional analysis, many mathematical problems can be reduced to a study of the set of solutions of an equation of form $F(x) = y$, where $F$ is a map between appropriate spaces $X$ and $Y$, and $y$ is an element of $Y$. Topological methods are an effective, widely used standard tool in the study of nonlinear equations.

## 2.1 Topological degree

The notion of the degree was first introduced by Brouwer in 1912 [13] in finite-dimensional spaces. Leray and Schauder generalized 1934 [43] the degree theory for compact perturbations of identity in infinite-dimensional Banach spaces. In 1972 and 1973 Fuhrer and Amann and Weiss [4], independently, proved that the Brouwer degree is uniquely determined by only a few conditions. These conditions, called normalization, additivity, and invariance under homotopy, provide a natural basis for the formal definition of classical topological degree theory.

### 2.1.1 Leray-Schauder topological degree

Let $X$ and $Y$ be two real Banach spaces. Given a nonempty subset $\Omega$ of $X$, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of $\Omega$ in $X$, respectively.

**Definition 2.1** An operator $A : \Omega \subset X \longrightarrow Y$ is said to be

(1) bounded, if it takes any bounded set into a bounded set.

(2) locally bounded, if for each $u \in \Omega$ there exists a neighborhood $U$ of $u$ such that the set $A(U)$ is bounded.

### Leray-Schauder map

**Definition 2.2** An operator $A : \Omega \subset X \longrightarrow Y$ is said to be

(a) compact, if continuous and $A(\overline{\Omega})$ is compact. It is said to be completely continuous if the image of any bound set is relatively compact.

(b) demicontinuous, if for any sequence $(u_n) \subset \Omega$ and for each $u \in \Omega$, $u_n \to u$ implies $A(u_n) \rightharpoonup A(u)$.

We will see that the compact perturbations of the identity have very interesting topological properties.

**Definition 2.3** (Leray-Schauder map) A map of the form $A = I - K$, where $I$ is the identity map of $\overline{\Omega}$ into $X$ and $K : \overline{\Omega} \longrightarrow X$ is a compact map is called compact perturbation of the identity.

We consider the following class of operators:

$$\mathcal{F}_{LS}(X) := \{A : \overline{G} \subset X \to X \mid G \in \mathcal{O},\ A = I + K,\ \textit{where } K \textit{ is compact}\},$$

where $\mathcal{O}$ denotes the collection of all bounded open sets in $X$.

**Theorem 2.1** *([43])(Leray-Schauder degree). There exists a unique degree function*

$$d_{LS} : \{(F, G, h) : G \in \mathcal{O}, F \in \mathcal{F}_{LS}(\overline{G}), h \notin F(\partial G)\} \longrightarrow \mathbb{Z}$$

*that satisfies the following properties:*

*(a) (Normalization).*

$$d_{LS}(I, G, h) = +1, \quad if\ h \in I(G).$$

*(b) (Additivity).*
*Let $F \in \mathcal{F}_{LS}(\overline{G})$. If $G_1$ and $G_2$ are two disjoint open subsets of $G$ such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then we have*

$$d_{LS}(F, G, h) = d_{LS}(F, G_1, h) + d_{LS}(F, G_2, h).$$

*(c) (Homotopy).*
*Let $M \in C([0,1] \times \overline{G}, X)$ be a compact map and define $H(t, u) = u - M(t, u)$. If $h : [0,1] \longrightarrow X$ is continuous and $h(t) \notin H([0,1] \times \partial G)$, then*

$$d_{LS}(H, G, h(t)) = const, \quad \forall t \in [0,1].$$

*(d) (Existence).*
*If $d_{LS}(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G$.*

### 2.1.2 Berkovits topological degree

Let $X$ be a real reflexive separable Banach space, such that both $X$ and its dual space $X^*$ are locally uniformly convex.

**Definition 2.4** A mapping $A : \Omega \subset X \longrightarrow X^*$ is said to be of class $(S_+)$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup \langle A(u_n), u_n - u \rangle \leqslant 0$, we have $u_n \to u$.

#### Duality map

**Definition 2.5** The duality map $J : X \longrightarrow X^*$ is determined via the Hahn-Banach theorem by the conditions

$$\|J(u)\| = \|u\|, \qquad \langle J(u), u \rangle = \|u\|^2 \qquad for\ all\ u \in X. \tag{2.1}$$

The duality map has many important properties listed in the following lemma.

**Lemma 2.1** *([10]) The duality map $J$ satisfies:*

*(a) $J$ is strictly monotone,*

*(b) $J$ is of class $(S_+)$,*

*(c) $J$ is homogeneous,*

*(d) $J$ is homeomorphism,*

*(d) $J^{-1}$ is the duality map from $X^*$ to $X \simeq X^{**}$, and thus of class $(S_+)$.*

#### Approximation of mappings

**Proposition 2.1** *([11]) Let $X$ be a real reflexive separable Banach space. Then there exist a real separable Hilbert space $W$ and a compact linear injection $\psi : W \longrightarrow X$ such that $\psi(W)$ is dense in $X$.*

Using Proposition 2.1 and the injection $\psi$, we define a further "adjoint" mapping $\widehat{\psi} : X^* \longrightarrow W$ by setting

$$(\widehat{\psi}(w), v) = \langle w, \psi(v) \rangle \qquad for\ all\ v \in W\ and\ w \in X^*,$$

where $(.,.)_W$ stands for the inner product of $W$. Then $\widehat{\psi} : X^* \longrightarrow W$:

(a) $\widehat{\psi}$ is compact linear mapping.

(b) Since $\psi(W)$ is dense in $X$, it is easily seen that also $\widehat{\psi}$ is injective.

We consider the following class of operators:

$$\mathcal{F}_1(\Omega) := \{F : \Omega \subset X \to X^* \mid \ F \, is \, bounded, \, demicontinuous \, and \, satisfies \, condition (S_+)\},$$

$$\mathcal{F}_{S_+}(X) := \{F \in \mathcal{F}_1(\overline{G}) \mid G \in \mathcal{O}\}.$$

We associate to $F \in \mathcal{F}_{S_+}(X)$ a family $F_\varepsilon \in \mathcal{F}_{LS}(X)$ of mappings given by

$$F_\varepsilon = I + \frac{1}{\varepsilon}\psi\widehat{\psi}F : \overline{G} \subset X \longrightarrow X, \qquad for \; any \; \varepsilon > 0.$$

For fixed $\varepsilon > 0$ we find that $F_\varepsilon$ is of type $I - g$, $g$ compact, a map for which the Leray-Schauder degree is defined.

**Lemma 2.2** *([11]) Let $F \in \mathcal{F}_{S_+}(X)$, with $0 \notin F(\partial G)$. Then there exists $\varepsilon' > 0$ such that $0 \notin F_\varepsilon(\partial G)$ for all $0 < \varepsilon < \varepsilon'$. Moreover*

$$d_{LS}(F_\varepsilon, G, 0) = constant, \quad for \; all \quad 0 < \varepsilon < \varepsilon'.$$

**Definition 2.6** ([11]) Let

$$M_{S_+} = \{(F, G, 0) : G \in \mathcal{O}, F \in \mathcal{F}_{S_+}(X), 0 \notin F(\partial G)\}.$$

Then we define a degree function $d_S : M_{S_+} \longrightarrow \mathbb{Z}$ as follows:

$$d_S(F, G, 0) := d_{LS}(F_\varepsilon, G, 0), \qquad for \; all \; 0 < \varepsilon < \varepsilon', \tag{2.2}$$

where $\varepsilon' > 0$ is given by Lemma 2.2.

**Remark 2.1** Since $F - h$ is also of class $(S_+)$, for any $h$ in $X^*$, we can define

$$d_S(F, G, h) = d_S(F - h, G, 0), \qquad for \; any \; h \notin F(\partial G). \tag{2.3}$$

**Theorem 2.2** *([11]) The integer-valued function $d_S$ defined by (2.2) and (2.3) for all $F \in \mathcal{F}_{S_+}(X)$ satisfies the properties (a)-(d) of Theorem 2.1 with the duality map $J$ as the normalizing map*

$$d_S(J, G, h) = \begin{cases} +1, & h \in J(G), \\ 0, & h \notin J(\overline{G}). \end{cases} \tag{2.4}$$

### 2.1.3 Extension of the Berkovits topological degree for mappings that satisfy the $(S_+)_T$ condition

Let $X$ be a real reflexive Banach space with dual space $X^*$. The symbol $\langle ., . \rangle$ denotes the usual dual paring between $X^*$ and $X$ in this order. In the reflexive case where the bidual space $X^{**}$ is identified with $X$, we sometimes write $\langle y, x \rangle$ for $\langle x, y \rangle_{X^*}$ for $x \in X$ and $y \in X^*$.

**Definition 2.7** A mapping $F : \Omega \subset X \longrightarrow X^*$ is said to be quasimonotone, if for any $(u_n) \subset \Omega$, with $u_n \rightharpoonup u$ implies $\limsup \langle F(u_n), u_n - u\rangle \geqslant 0$.

**Definition 2.8** For any operator $F : \Omega \subset X \longrightarrow X$ and any bounded operator $T : \Omega_1 \subset X \longrightarrow X^*$ such that $\Omega \subset \Omega_1$, we say that:

(1) $F$ satisfies condition $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u, y_n := T(u_n) \rightharpoonup y$ and $\limsup\langle F(u_n), y_n - y\rangle \leqslant 0$, we have $u_n \to u$.

(2) $F$ has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u, y_n := T(u_n) \rightharpoonup y$ , we have $\limsup\langle F(u_n), y_n - y\rangle \geqslant 0$.

We consider the following classes of operators:

$$\begin{aligned} \mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \subset X \to X \mid \ F \ is \ bounded, \ demicontinuous \ and \ satisfies \ condition (S_+)_T\}, \\ \mathcal{F}_{T}(\Omega) &:= \{F : \Omega \subset X \to X \mid \ F \ is \ demicontinuous \ and \ satisfies \ condition (S_+)_T\}, \end{aligned}$$

for any $\Omega \subset D_F$ and any $T \in \mathcal{F}_1(\Omega)$, where $D_F$ denotes the domain of $F$.
Let

$$\begin{aligned} \mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\overline{G}) \mid G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\}, \\ \mathcal{F}(X) &:= \{F \in \mathcal{F}_{T}(\overline{G}) \mid G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\}, \end{aligned}$$

here, $T \in \mathcal{F}_1(\overline{G})$ is called an essential inner map to $F$.

### Approximation of mappings

Let $Y$ be a real separable reflexive Banach space with dual space $Y^*$ such that $Y$ and its dual space $Y^*$ are locally uniformly convex. Taking $X = Y^*$ in Proposition 2.1 we conclude the existence of a real separable Hilbert space $W$ and a compact linear injection

$$\phi : W \longrightarrow Y^*$$

such that

$$\overline{\phi(W)} = Y^*.$$

We define a further "adjoint" mapping $\widehat{\phi} : Y \longrightarrow W$ by setting

$$(\widehat{\phi}(u), v)_W = \langle u, \phi(v)\rangle \quad for\ all\ v \in W\ and\ v \in Y,$$

where $(.,.)_W$ stands for the inner product of $W$. Then $\widehat{\phi} : Y \longrightarrow W$:

(i) $\widehat{\phi}$ is compact linear mapping.

(ii) Since $\phi(W)$ is dense in $Y^*$, it is easily seen that also $\widehat{\phi}$ is injective.

We associate to $F \in \mathcal{F}_B(X)$ a family $F_\mu \in \mathcal{F}_{S_+}(X)$ of mappings given by

$$F_\mu = T + \mu\phi\widehat{\phi}F, \qquad \textit{for all } \mu > 0.$$

Clearly $F_\mu$ maps $G \subset X$ into $X^*$ and $F_\mu \in \mathcal{F}_{S_+}(X)$.

**Lemma 2.3** *([10]) Let $F \in \mathcal{F}_B(X)$ with $G \in \mathcal{O}$ such that $0 \notin F(\partial G)$. Then there exists $\mu_0 > 0$ such that $0 \notin F_\mu(\partial G)$ and the value of $d_S(F_\mu, G, 0)$ is constant for all $\mu > \mu_0$.*

**Theorem 2.3** *([10]) There exists a unique degree function*

$$d_B : \{(F, G, 0) : G \in \mathcal{O}, F \in \mathcal{F}_B(X), 0 \notin F(\partial G)\} \longrightarrow \mathbb{Z}$$

*where*

$$d_B(F, G, 0) = \lim_{\mu\to\infty} d_S(F_\mu, G, 0),$$

*which achieve the properties mentioned in Theorem 2.1 above.*

**Remark 2.2** For any $h \notin F(\partial G)$ we set

$$d_B(F, G, h) = d_B(F - h, G, 0).$$

### 2.1.4 Extension of the Leray-Schauder degree for mappings that satisfy the $(S_+)_T$ condition

First, we establish the relationship between operators satisfying $(S_+)_T$ and $(Q\mathcal{M})_T$.

**Lemma 2.4** *([41]) Let $T : \overline{G} \longrightarrow X^*$ be a bounded operator, where $G$ is a bounded open set in a real reflexive Banach space $X$. Then it has the following properties:*

*(a) If $F : \overline{G} \longrightarrow X$ has the property $(Q\mathcal{M})_T$, then for any sequence $(u_k)$ in $\overline{G}$ with $u_k \rightharpoonup u$ and $y_k := Tu_k \rightharpoonup y$, we have*

$$\liminf \langle Fu_k, y_k - y\rangle \geq 0.$$

*(b) If $F_1$, $F_2 : \overline{G} \longrightarrow X$ have the property $(Q\mathcal{M})_T$, then so do $F_1 + F_2$ and $\alpha F_1$ for any positive number $\alpha$.*

The following result shows that the operator of the form $I + S \circ T$ belongs to the class $\mathcal{F}(X)$.

**Lemma 2.5 (see, ([41]))** *Lets* $T \in \mathcal{F}_1(\overline{G})$ *be continuous and* $S : D_S \subset X^* \longrightarrow X$ *be demicontinuous such that* $T(\overline{G}) \subset D_S$, *where* $G$ *is a bounded open set in a real reflexive Banach space* $X$. *Then the following statements are true :*

*(i) If* $S$ *is quasimonotone, then* $F = I + S \circ T \in \mathcal{F}_T(\overline{G})$, *where* $I$ *denotes the identity operator.*

*(ii) If* $S$ *is of class* $(S_+)$, *then* $SoT \in \mathcal{F}_T(\overline{G})$.

**Proof.**

(i) Let $(u_n)$ be any sequence in $\overline{G}$ such that

$$\begin{cases} u_n \rightharpoonup u, \\ y_n := Tu_n \rightharpoonup y, \\ \limsup\limits_{n\to\infty} \langle Fu_n, y_n - y\rangle \leqslant 0. \end{cases} \tag{2.5}$$

Since the sequence $(\langle Tu_n, u_n - u\rangle)$ is bounded in $\mathbb{R}$ i.e.,

$$\exists R > 0, \quad \textit{such that } \ |\langle Tu_n, u_n - u\rangle| \leqslant R, \quad \forall n \in \mathbb{N},$$

implies that there is a subsequence $(u_k)$ of $(u_n)$ such that

$$\lim_{n\to\infty} \langle Tu_k, u_k - u\rangle = l, \quad (l \in \mathbb{R}).$$

We know that

$$\lim_{n\to\infty} \langle u_k, y_k - y\rangle = \lim_{n\to\infty} \langle u_k - u, y_k\rangle + \lim_{n\to\infty} \langle u, y_k - y\rangle,$$

in view of $X^{**} \cong X$, we have that

$$\lim \langle u_k, y_k - y\rangle_{X^*} = \lim \langle Tu_k, u_k - u\rangle_X. \tag{2.6}$$

By the quasimonotonicity of $S$, (2.5), and (2.6), we get

$$\begin{aligned} 0 &\leq \limsup \langle Sy_k, y_k - y\rangle \\ &= \limsup \langle u_k + Sy_k, y_k - y\rangle - \lim \langle u_k, y_k - y\rangle \\ &\leq -\lim \langle Tu_k, u_k - u\rangle. \end{aligned}$$

Since $T \in \mathcal{F}_1(\overline{G})$ satisfies condition $(S_+)$, we have $u_n \longrightarrow u$. Thus, the operator $F$ satisfies condition $(S_+)_T$. Since $F$ is demicontinuous on $\overline{G}$, we conclude that $F \in \mathcal{F}_T(\overline{G})$.

(ii) Let $(u_n)$ be any sequence in $\overline{G}$ such that

$$\begin{cases} u_n \rightharpoonup u, \\ y_n := Tu_n \rightharpoonup y, \\ \limsup\limits_{n\to\infty} \langle S\circ Tu_n, y_n - y\rangle \leqslant 0. \end{cases} \tag{2.7}$$

Since $S$ satisfies condition $(S_+)$, it follows that

$$\begin{cases} y_n \longrightarrow y, \\ \lim\limits_{n\to\infty} \langle Tu_n, u_n - u\rangle = \lim\limits_{n\to\infty} \langle Tu_n - y, u_n - u\rangle = 0. \end{cases} \tag{2.8}$$

Since $T$ satisfies condition $(S_+)$ and by (2.8), we have

$$u_k \longrightarrow u.$$

Consequently, we obtain that $S\circ T \in \mathcal{F}_T(\overline{G})$. This completes the proof.

■

**Definition 2.9** For a bounded operator $T : \overline{G} \subset X \longrightarrow X^*$, we say that a homotopy $H : [0,1]\times\overline{G} \longrightarrow X$ satisfies condition $(S_+)_T$ if for any sequence $(t_n, u_n)$ in $[0,1]\times\overline{G}$ such that

$$u_n \rightharpoonup u, \quad t_n \longrightarrow t, \quad y_n := Tu_n \rightharpoonup y, \quad and \quad \limsup_{n\to\infty}\langle H(t_n,u_n), y_n - y\rangle \leq 0,$$

we have $u_n \longrightarrow u$.

We will see that each affine homotopy with a common essential inner map satisfies condition $(S_+)_T$.

**Lemma 2.6** *([41]) Let $G$ be a bounded open subset of a real reflexive Banach space $X$, and let, $T \in \mathcal{F}_1(\overline{G})$ be continuous. If $F, S \in \mathcal{F}_T(\overline{G})$, then an affine homotopy $H : [0,1]\times\overline{G} \longrightarrow X$ defined by*

$$H(t,u) := (1-t)F(u) + tS(u) \quad for \quad (t,u) \in [0,1]\times\overline{G}$$

*satisfies condition $(S_+)_T$.*

*In this case, the homotopy is called an admissible affine homotopy with the common essential inner map $T$.*

**Proof.** Let $(u_n)$ be any sequence in $\overline{G}$ and $(t_n)$ any sequence in $[0,1]$ such that

$$u_n \rightharpoonup u, \quad t_n \longrightarrow t, \quad y_n := Tu_n \rightharpoonup y, \quad and \quad \limsup_{n\to\infty}\langle H(t_n,u_n), y_n - y\rangle \leq 0.$$

If $t = 1$, then it follows from $S \in \mathcal{F}_T(\overline{G})$ that

$$\limsup_{n\to\infty}\langle Su_n, y_n - y\rangle = \limsup_{n\to\infty}\langle H(t_n, u_n), y_n - y\rangle \leq 0,$$

and

$$u_n \longrightarrow u.$$

If $t \in [0, 1)$, then the property $(QM)_T$ of $S$, in view of Lemma 2.4, implies that

$$\begin{aligned}(1-t)\limsup_{n\to\infty}\langle Fu_n, y_n - y\rangle &\leq (1-t)\limsup_{n\to\infty}\langle Fu_n, y_n - y\rangle + t\liminf_{n\to\infty}\langle Su_n, y_n - y\rangle \\ &\leq \limsup_{n\to\infty}\langle H(t_n, u_n), y_n - y\rangle \\ &\leq 0.\end{aligned}$$

Since $F$ satisfies condition $(S_+)_T$, we have $u_n \longrightarrow u$.
In both cases, we have shown that $u_n \longrightarrow u$. This completes the proof. ■

**Theorem 2.4** *([41]) Let $G$ be an open subset of a real reflexive Banach space $X$, and let $Y$ be a real normed space. Suppose that $H : [0, 1] \times \overline{G} \longrightarrow Y$ is a demicontinuous homotopy. If $S \subset G$ is a nonempty compact set, then there exists an open set $G_0$ and a positive constant $R$ such that*

*(a) $S \subset G_0 \subset G$ and*

*(b) $\|H(t, u)\| \leq R$ for all $t \in [0, 1]$ and all $u \in \overline{G_0}$.*

We demonstrate that any demicontinuous operator on closed bounded sets that satisfies condition $(S_+)_T$ is proper.

**Lemma 2.7** *([41]) Let $G$ be a bounded open set in a real reflexive Banach space $X$, and let $H : [0, 1] \times \overline{G} \longrightarrow X$ be a demicontinuous homotopy satisfying condition $(S_+)_T$, where $T : \overline{G} \longrightarrow X^*$ is bounded. For any compact set $A \subset X$,*

$$K := \{u \in \overline{G} | H(t, u) \in A \ for \ some \ t \in [0, 1]\}$$

*is a compact subset of $X$.*

**Corollary 2.1** *Suppose that $F : \overline{G} \longrightarrow X$ is demicontinuous and satisfies condition $(S_+)_T$, where $G$ is a bounded open subset of $X$ and $T$ is bounded on $\overline{G}$. For every $h \notin F(\partial G)$, there exists an open set $G_0$ such that $F^{-1}(h) \subset G_0 \subset G$ and $F$ is bounded on $\overline{G_0}$.*

**Proof.** By Lemma 2.7, we obtain

$$F^{-1}(h) = \{u \in \overline{G} | H(0,u) = Fu \in A = \{h\}\}$$

is a compact subset of $X$. And since $h \notin F(\partial G)$, this implies

$$F^{-1}(h) \subset G.$$

Applying Theorem 2.4 with the constant homotopy $H(0,u) = Fu$ and $S = F^{-1}(h)$, there exists an open set $G_0$ and a positive constant $R$ such that

$$F^{-1}(h) \subset G_0 \subset G,$$

and $\|H(0,u)\| = \|Fu\| \leqslant R$, i.e. $F$ is bounded on $\overline{G}_0$. ■

**Definition 2.10**

$$M = \{(F,G,h) : G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_T(\overline{G}), h \notin F(\partial G)\}.$$

Then we define a degree function $d_T : M \longrightarrow \mathbb{Z}$ as follows:

$$d_T(F,G,h) := d_B(F|_{\overline{G_0}}, G_0, h),$$

where $G_0$ is any open subset of $G$ with $F^{-1}(h) \subset G_0$ and $F$ is bounded on $\overline{G_0}$, according to Corollary 2.1.

**Theorem 2.5** *([41]) The above degree $d_T$ for the class $\mathcal{F}(X)$ has the following properties:*

*(a).* ***Existence property.***
*If $d_T(F,G,h) \neq 0$, then there exists $u \in G$ such that $F(u) = h$.*

*(b).* ***Additivity property.***
*Assume that $G_1$ and $G_2$ are two disjoint open subsets of $G$. If $h \notin F(\overline{G} \setminus (G_1 \cup G_2)$, then*

$$d_T(F,G,h) = d_T(F,G_1,h) + d_T(F,G_2,h).$$

*(c).* ***Normalization property.***
*For any $h \in G$, we have*

$$d_T(I,G,h) = 1.$$

*(d).* ***Homotopy property.***
*Suppose that $H : [0,1] \times \overline{G} \longrightarrow X$ is an admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \longrightarrow X$ is a continuous path in $X$ such that $h(t) \notin H(t,\partial G)$ for all $t \in [0,1]$. Then*

$$d_T(H(t,.),G,h(t)) = \textit{constant} \quad \textit{for all } t \in [0,1].$$

*(e). **Boundary dependence property.***
*If $F, S \in \mathcal{F}_T(\overline{G})$ coincide on $\partial G$ and $h \notin F(\partial G)$, then*

$$d_T(F, G, h) = d_T(S, G, h).$$

**Proof.**

(a). If $d_T(F, G, h) \neq 0$, then we have by Definition 2.10

$$d_B(F, G_0, h) \neq 0$$

for a suitable open set $G_0 \subset G$. By Theorem 2.3, the equation $Fu = h$ has a solution in $G_0$ which also belongs to $G$.

(b). Let $K = \{u \in G : Fu = h\}$ and $K_i = \{u \in G_i : Fu = h\}$ for $i = 1, 2$.
Note that

$$K = K_1 \cup K_2 \quad \textit{and} \quad K_1 \cap K_2 = \emptyset.$$

As $h \notin F(\overline{G} \setminus (G_1 \cup G_2)$, implies

$$h \notin F(\partial G_1) \quad \textit{and} \quad h \notin F(\partial G_2).$$

By Corollary 2.1, there exist open sets $G_{01}$ and $G_{02}$ such that

$$\begin{cases} K_1 \subset G_{01} \subset G_1, \ \textit{and } F \textit{ is bounded on } \overline{G_{01}}, \\ K_2 \subset G_{02} \subset G_2, \ \textit{and } F \textit{ is bounded on } \overline{G_{02}}. \end{cases} \tag{2.9}$$

Set $G_0 := G_{01} \cup G_{02}$. Obviously, $F$ is bounded on $\overline{G_0}$ and $K \subset G_0 \subset G$, and so

$$h \notin F(\overline{G_0} \backslash (G_{01} \cup G_{02})).$$

Hence it follows from Definition 2.10 and Theorem 2.3 that

$$\begin{aligned} d_T(F, G, h) &= d_B(F|_{\overline{G_0}}, G_0, h) \\ &= d_B(F|_{\overline{G_{01}}}, G_{01}, h) + d_B(F|_{\overline{G_{02}}}, G_{02}, h) \\ &= d_T(F, G_1, h) + d_T(F, G_2, h). \end{aligned}$$

(c). Since the identity operator $I$ is bounded, this results from the Theorem 2.3 as follows:

$$d_T(I, G, h) = d_B(I, G, h) = 1.$$

Actually, the identity operator

$$I = J^{-1} \circ J$$

belongs to $\mathcal{F}_J(\overline{G})$. In view of Lemma 2.5, where $J$ denotes the duality operator. It is known in, e.g., [10] that

$$\begin{cases} J : X \longrightarrow X^* \text{ } is \text{ } bounded, \text{ } continuous \text{ } and \text{ } satisfies \text{ } condition \text{ } (S_+), \\ J^{-1} : X^* \longrightarrow X \text{ } is \text{ } continuous \text{ } and \text{ } satisfies \text{ } condition \text{ } (S_+). \end{cases} \tag{2.10}$$

(d). Note that $A = \{h(t) \in X | t \in [0,1]\}$ is a compact subset of $X$. Set

$$S = \{u \in \overline{G} | H(t,u) \in A \text{ } for \text{ } some \text{ } t \in [0,1]\}$$

by Lemma 2.7, $S$ is a compact subset of $X$. In particular, we have $S \subset G$. Indeed,
let $u_0 \in S \Rightarrow \exists t_0 \in [0,1]$ such that

$$h(t_0) = H(t_0, u_0) \in X.$$

Since $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, we obtain

$$H(t_0, u_0) \notin H(t_0, \partial G)$$

i.e.,

$$u_0 \notin \partial G \Rightarrow u_0 \in G.$$

According to Theorem 2.4, there exists an open set $G_0$ such that

$$\begin{cases} S \subset G_0 \subset G, \\ and \text{ } H \text{ } is \text{ } bounded \text{ } on \text{ } [0,1] \times \overline{G_0}. \end{cases} \tag{2.11}$$

This implies that

$$h(t) \notin H(t, \partial G_0),$$

and

$$d_T(H(t,.), G, h(t)) = d_B(H(t,.), G_0, h(t)) \quad for \text{ } each \text{ } t \in [0,1].$$

By Theorem 2.3, we conclude that the value of $d_T(H(t,.), G, h(t))$ is constant for all $t \in [0,1]$.

(e). Consider an affine homotopy $H : [0,1] \times \overline{G} \longrightarrow X$ given by

$$H(t,u) = (1-t)Fu + tSu \quad for \text{ } (t,u) \in [0,1] \times \overline{G}.$$

As $h \notin F(\partial G) = H(t, \partial G)$ for all $t \in [0,1]$, the homotopy invariance property (d) of the degree $d_T$ implies that

$$d_T(F, G, h) = d_T(S, G, h).$$

This completes the proof.

■

In view of following lemma, the degree $d_T$ does not depend on the choice of the set $G_0$.

**Lemma 2.8** *([41]) Let $F \in \mathcal{F}_T(\overline{G})$ be an operator, where $G$ is a bounded open set in $X$ and $T \in \mathcal{F}_1(\overline{G})$. Suppose that for $i = 1, 2$, $G_i$ is an open subset of $G$ such that*

$$F^{-1}(h) \subset G_i \subset G \quad \textit{and} \quad F \textit{ is bounded on } \overline{G_i}.$$

*Then the degree $d_B(F, G_i, h)$ is well defined for $i = 1, 2$ and*

$$d_B(F, G_1, h) = d_B(F, G_2, h).$$

**Proof.** For $i = 1, 2$, since $F \in \mathcal{F}_{T,B}(\overline{G_i})$ and $h \notin F(\partial G_i)$, the degree $d_B(F, G_i, h)$ is well defined and

$$F^{-1}(h) \subset G_1 \cap G_2 \subset G_i$$

implies

$$h \notin F(\overline{G_i} \backslash (G_1 \cap G_2)).$$

Applying Theorem 2.3, we get

$$d_B(F, G_1, h) = d_B(F, G_1 \cap G_2, h) = d_B(F, G_2, h).$$

■

**Remark 2.3** If $F$ is bounded on $\overline{G}$, then we may take $G_0 = G$ and $d_T(F, G, h) = d_B(F, G, h)$, which means that $d_T$ and $d_B$ coincide on $\mathcal{F}_{T,B}(\overline{G})$.

## 2.2 Method of monotone operators

The main theorem on pseudomonotone operators is the basic mathematical tool that helps to establish the existence of solutions to various kinds of equations. This method consists in transforming a given problem into a problem of the form:

$$Au = f^*, u \in X,$$

where $A : X \longrightarrow X^*$ is a pseudomonotone operator on the real B-space $X$.

In this section we recall the main theorem on pseudomonotonic operators due to Brezis that we will use to obtain various existence results. We start with the definition of monotone operator.

### 2.2.1 Monotone operators

**Definition 2.11** An operator $A : X \longrightarrow X^*$ is said:

(a) Monotonous if $\langle Au - Av, u - v\rangle \geqslant 0$, $\forall u, v \in X$.

(b) Strictly monotonous if $\langle Au - Av, u - v\rangle > 0$, $\forall u, v \in X$, $u \neq v$.

### 2.2.2 Hemicontinuous operators

**Definition 2.12** Let $A : X \longrightarrow X^*$ be an operator on the real B-space $X$.

(a) $A$ is said to be demicontinuous iff

$$\lim_{n\to\infty} u_n \longrightarrow u \quad \textit{implies} \quad Au_n \rightharpoonup Au.$$

(b) $A$ is said to be hemicontinuous iff the real function

$$t \longmapsto \langle A(u + tv), w\rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in X$.

### 2.2.3 Pseudomonotone operators

**Definition 2.13** The operator $A : X \longrightarrow X^*$ is pseudomonotone iff $u_n \rightharpoonup u$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leqslant 0$$

implies

$$\langle Au, u - \psi\rangle \leqslant \liminf_{n\to\infty} \langle Au_n, u_n - \psi\rangle, \quad \textit{for all } \psi \in X.$$

**Lemma 2.9** *([72]) Let $A, B : X \longrightarrow X^*$ be operators on the real reflexive Banach space $X$. Then the following implications hold:*

*(a) If $A$ is monotone and hemicontinuous, then $A$ is pseudomonotone.*

*(b) If $A$ and $B$ are pseudomonotone, then $A + B$ is pseudomonotone.*

**Proof.** $(a)$. Let

$$\begin{cases} u_n \rightharpoonup u \quad as\ n \to +\infty, \\ \limsup\limits_{n\to+\infty} \langle Au_n, u_n - u\rangle \leqslant 0. \end{cases} \tag{2.12}$$

Since the operator $A$ is monotone, we obtain that

$$\langle Au_n, u_n - u\rangle \geqslant \langle Au, u_n - u\rangle, \qquad \forall n \in \mathbb{N},$$

and since $u_n \rightharpoonup u$, and $Au \in X^*$, we have

$$\langle Au, u_n - u\rangle \longrightarrow 0, \quad as\ n \to +\infty,$$

hence

$$0 \leqslant \lim_{n\to\infty} \langle Au_n, u_n - u\rangle \leqslant \limsup_{n\to+\infty} \langle Au_n, u_n - u\rangle \leqslant 0.$$

This implies

$$\lim_{n\to\infty} \langle Au_n, u_n - u\rangle = 0.$$

We pose now $v = u + t(w - u)$ with $t > 0$. By the monotonicity of $A$, we find

$$\langle Au_n - Av, u_n - v\rangle \geqslant 0$$

and hence

$$t\langle Av, u - w\rangle + \langle Av, u_n - u\rangle - \langle Au_n, u_n - u\rangle \leqslant t\langle Au_n, u - w\rangle,$$

and as

$$\begin{cases} \lim\limits_{n\to\infty} \langle Au_n, u_n - u\rangle = 0, \\ \lim\limits_{n\to\infty} \langle Av, u_n - u\rangle = 0, \\ \lim\limits_{n\to\infty} \langle Au_n, u - w\rangle = \lim\limits_{n\to\infty} \langle Au_n, u_n - w\rangle + \lim\limits_{n\to\infty} \langle Au_n, u - u_n\rangle. \end{cases} \tag{2.13}$$

This implies

$$\langle Av, u - w\rangle \leqslant \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle, \quad \forall w \in X.$$

The operator $A$ is hemicontinuous. Thus, letting $t \longrightarrow 0$, we obtain

$$\langle Au, u - w\rangle \leqslant \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle, \quad \forall w \in X,$$

i.e., $A$ is pseudomonotone.
$(b)$. Let

$$\begin{cases} u_n \rightharpoonup u \ \ as\ n \to +\infty, \\ \limsup\limits_{n\to+\infty} \langle Au_n + Bu_n, u_n - u\rangle \leqslant 0. \end{cases} \tag{2.14}$$

First, we have to prove that

$$\limsup_{n\to+\infty} \langle Au_n, u_n - u\rangle \leqslant 0 \quad and \quad \limsup_{n\to+\infty} \langle Bu_n, u_n - u\rangle \leqslant 0.$$

We assume that there exists a subsequence, again denoted by $(u_n)$, such that

$$\lim_{n\to\infty} \langle Au_n, u_n - u\rangle = l > 0,$$

and hence

$$\lim_{n\to\infty} \langle Bu_n, u_n - u\rangle \leqslant -l.$$

Since $B$ is pseudomonotone, this implies

$$\langle Bu, u - w\rangle \leqslant \liminf_{n\to\infty}\langle Bu_n, u_n - w\rangle, \quad \forall w \in X.$$

Letting $w = u$, we obtain the contradiction $0 \leqslant -l$.
In the same way, we prove that

$$\limsup_{n\to+\infty}\langle Bu_n, u_n - u\rangle \leqslant 0, \quad \forall w \in X.$$

Since $A$ and $B$ are pseudomonotone, it follows from above that,

$$\begin{cases} \langle Au, u - w\rangle \leqslant \liminf\limits_{n\to\infty}\langle Au_n, u_n - w\rangle, & \forall w \in X \\ \langle Bu, u - w\rangle \leqslant \liminf\limits_{n\to\infty}\langle Bu_n, u_n - w\rangle, & \forall w \in X \end{cases} \tag{2.15}$$

and hence

$$\langle Au + Bu, u - w\rangle \leqslant \liminf_{n\to\infty}\langle Au_n + Bu_n, u_n - w\rangle, \quad \forall w \in X,$$

i.e., $A + B$ is pseudomonotone. ■

The main theorem on pseudomonotone operators due to BrÃ©zis is given by the next theorem.

**Theorem 2.6** *([72]) Let $X$ be a real, reflexive Banach space, and let $T : X \longrightarrow X^*$ an operator that satisfies the following conditions*

*(a) $T$ is bounded.*

*(b) $T$ is pseudomonotone.*

*(c) $T$ is coercive.*

*Then, the equation*

$$Tu = f^* \tag{2.16}$$

*admits at least one solution $u \in X$ for all $f^* \in X^*$.*

## 2.3 Variational iteration method (VIM)

The variational iteration method (VIM) was proposed and developed by the Chinese mathematician Je-Haun-He in the early 1990s, it was proposed for the first time to solve problems in mechanics. This method has been used to solve a wide variety of linear and non-linear problems with successive approximations rapidly converging to the exact solution if it exists. The method is based on determining the Lagrange multiplier optimally through variational theory (see, [30, 31, 32]).

### 2.3.1 Description of the method

To illustrate the basic idea of the method, we consider the following general nonlinear equation:

$$L(u(t)) + N(u(t)) = h(t) \tag{2.17}$$

where $L$ is a linear differential operator, $N$ is a nonlinear operator, and $h(t)$ is a given continuous function.

The basic character of the method is to construct a correction function for the system (2.17), which reads

$$\begin{aligned} u_{n+1}(t) &= u_n(t) \\ &+ \int_0^t \lambda(s)\{L(u_n(s)) + N(\tilde{u}_n(s)) - h(s)\}ds, \end{aligned} \tag{2.18}$$

where $\lambda(s)$ is a general Lagrange multiplier. The index $n$ represents the $n^{th}$ approximation and $\tilde{u}_n(s)$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n(s) = 0$. To solve equation (2.17) by the VIM method, one must first determine the Lagrange multiplier $\lambda$ which will be identified in an optimal way via integration by parts. Then the successive approximations $u_n$ of the solution $u(t)$ will be obtained by using the Lagrange multiplier and a well-chosen function $u_0$ (which must at least satisfy the initial conditions), consequently, the exact solution will be the limit:

$$\lim_{n\to\infty} u_n(t) = u(t).$$

### 2.3.2 Alternative approach to VIM

In this section, we present an alternative approach to the VIM.
Let the following non-linear differential equation:

$$L(u(t)) + N(u(t)) = h(t), t > 0. \tag{2.19}$$

With the initial conditions

$$u^{(k)}(0) = c_k, \quad k = 1, 2, \ldots, m-1, \tag{2.20}$$

where $L$ is a linear differential operator defined by ($L = \frac{d^m}{dt^m}, m \in \mathbb{N}^*$), $N$ is a non-linear operator, $h$ a known function and $c_k$ are real numbers.

According to the variational iteration method, we can construct a functional correction formula for (2.19) as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\{\frac{d^m u_n(s)}{ds^m} + N(\tilde{u}_n(s)) - h(s)\}ds \tag{2.21}$$

where $\lambda(s)$ is a general Lagrange multiplier which can be identified optimally via variational theory, $u_n$ is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

So, we first determine the Lagrange multiplier $\lambda(s)$ after obtaining the latter, it is substituted in the correction function expression (2.18) to obtain a sequence of $u_n(t), n > 0$ approximations to solution $u(t)$.

Making the above correct functional stationary with respect to $u_n$

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(s)\{\frac{d^m u_n(s)}{ds^m} + N(\tilde{u}_n(s)) - h(s)\}ds = 0.$$

Here we apply restricted variation to the non-linear term $Nu$, in which case we can easily define the multiplier so equality above takes the form

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(s)\{\frac{d^m u_n(s)}{ds^m} - h(s)\}ds = 0.$$

**case 1.** $m = 1$. noticing that $\delta u(0) = 0$

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(s)\{\frac{du_n(s)}{ds} - h(s)\}ds \\ &= \delta u_n(t) + \lambda(s)\delta u_n(s)|_{s=t} - \int_0^t \lambda^{'}(s)\delta u_n(s)ds \\ &= 0. \end{aligned}$$

For arbitrary $\delta u$, from the above relation , we obtain the Euler-Lagrange equation

$$-\lambda^{'}(s) = 0,$$

and the following stationary condition

$$1 + \lambda(t) = 0.$$

The Lagrange multiplier, therefore, can be readily identified,

$$\lambda(s,t) = -1.$$

Substituting the identified Lagrange multiplier into (2.18) results in the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \{L(u_n(s)) + N(u_n(s)) - h(s)\}ds.$$

**case 2.** $m = 2$. noticing that $\delta u(0) = 0$

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(s)\{\frac{d^2 u_n(s)}{ds^2} - h(s)\}ds \\ &= \delta u_n(t) + \lambda(s)\delta u_n^{'}(s)|_{s=t} - \lambda^{'}(s)\delta u_n(s)|_{s=t} + \int_0^t \lambda^{''}(s)\delta u_n(s)ds = 0.\end{aligned}$$

We, therefore, have the following stationary conditions:
$\lambda^{''}(s) = 0$,
$\lambda(s)|_{s=t} = 0$,
$1 - \lambda^{'}(s)|_{s=t} = 0$.
The Lagrange multiplier, therefore, can be readily identified

$$\lambda(s,t) = (s-t).$$

Substituting the identified Lagrange multiplier into (2.18) results in the following iteration formulation:

$$u_{n+1}(t) = u_n(t) + \int_0^t \{(s-t)L(u_n(s)) + N(u_n(s)) - h(s)\}ds.$$

and in general:

$$\lambda(s,t) = \frac{(-1)^m}{(m-1)!}(s-t)^{m-1} \tag{2.22}$$

Therefore, by substituting (2.22) in the functional (2.18), we obtain the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{(-1)^m}{(m-1)!}(s-t)^{m-1}\{L(u_n(s)) + N(u_n(s)) - h(s)\}ds. \tag{2.23}$$

Now, define the operator $T\langle u\rangle$ as,

$$T\langle u\rangle = \int_0^t \frac{(-1)^m}{(m-1)!}(s-t)^{m-1}\{L(u_k(s)) + N(u_k(s)) - h(s)\}ds. \tag{2.24}$$

and define the components $\phi_k,\ k = 0, 1, \ldots,$ as,

$$\begin{cases} \phi_0 = u_0 \\ \phi_1 = T\langle\phi_0\rangle \\ \phi_2 = T\langle\phi_0 + \phi_1\rangle \\ \cdots \\ \cdots \\ \phi_{k+1} = T\langle\phi_0 + \phi_1 + \cdots + \phi_k\rangle. \end{cases} \tag{2.25}$$

So, we have

$$u(t) = \lim_{k\to+\infty} u_k(t) = \sum_{k=0}^{\infty} \phi_k(t). \tag{2.26}$$

Finally, the solution to problem (2.17) can be deduced using (2.24) and (2.25), in series form:

$$u(t)=\sum_{k=0}^{\infty}\phi_k(t). \tag{2.27}$$

The initial approximation $\phi_0=u_0$ can be chosen freely if it satisfies the initial conditions of the problem. The success of the method depends on the correct choice of the initial approximation $\phi_0$. In this alternative approach, we choose the initial approximation as follows:

$$\phi_0(t)=\sum_{k=0}^{m-1}\frac{c_k}{k!}t^k. \tag{2.28}$$

### 2.3.3 Convergence analysis

In this paragraph, we study the convergence of the variational iteration method, according to the alternative approach of the VIM method presented in the previous paragraph (see, [66, 49]).

**Theorem 2.7** *Let $H$ be a Hilbert space and $T:H\longrightarrow H$, an operator defined by (2.24). If $\exists 0<\mu<1$ such that*

$$\|T\langle\phi_0+\phi_1+\ldots+\phi_{k+1}\rangle\|\leq\mu\|T\langle\phi_0+\phi_1+\ldots+\phi_k\rangle\| \qquad \forall k\in\mathbb{N}\cup\{0\}$$

*then, the series solution $u(t)=\sum_{k=0}^{\infty}\phi_k(t)$, defined in (2.27), converges.*

**Proof.** Let $(S_n)_{n\geq 0}$ be a sequence of general term defined by:

$$\begin{cases} S_0=\phi_0, \\ S_n=\sum_{k=0}^{n}\phi_k. \end{cases} \tag{2.29}$$

and we show that $(S_n)_{n\geq 0}$ is a Cauchy sequence in the Hilbert space $H$. For this purpose, consider,

$$\|S_{n+1}-S_n\|=\|\phi_{n+1}\|\leqslant\mu\|\phi_n\|\leqslant\mu^2\|\phi_{n-1}\|\leqslant\cdots\leqslant\mu^{n+1}\|\phi_0\|. \tag{2.30}$$

For every $n,m\in\mathbb{N}$, $n\geqslant m$, we have,

$$\begin{aligned} \|S_n-S_m\| &\leqslant \|S_n-S_{n-1}\|+\|S_{n-1}-S_{n-2}\|+\cdots+\|S_{m+1}-S_m\| \\ &\leqslant \mu^n\|\phi_0\|+\mu^{n-1}\|\phi_0\|+\cdots+\mu^{m+1}\|\phi_0\| \\ &\leqslant (\mu^{n-m-1}+\mu^{n-m-2}+\cdots+\mu+1)\mu^{m+1}\|\phi_0\|, \end{aligned} \tag{2.31}$$

and as $(\mu^{n-m-1}+\mu^{n-m-2}+\cdots+\mu+1)(1-\mu)=(1-\mu^{n-m})$, we have

$$\|S_n-S_m\|\leqslant\frac{1-\mu^{n-m}}{1-\mu}\mu^{m+1}\|\phi_0\|\leqslant\frac{1}{1-\mu}\mu^{m+1}\|\phi_0\|,$$

and as $0<\mu<1$, we get:

$$\lim_{n,m\to+\infty}\|S_n-S_m\|=0.$$

Therefore, $(S_n)_{n\geqslant 0}$ is a Cauchy sequence in the Hilbert space $H$, and this implies that the series solution $u(t)=\sum_{k=0}^{\infty}\phi_k(t)$ converges. ■

**Theorem 2.8** *If the series solution $u(t)=\sum_{k=0}^{\infty}\phi_k(t)$, defined in (2.27), converges then it is an exact solution of the nonlinear problem (2.17).*

**Proof.** Suppose that the series solution $u(t)=\sum_{k=0}^{\infty}\phi_k(t)$ converges, then we have:

$$\lim_{k\to+\infty}\phi_k=0,$$

$$\sum_{k=0}^{n}(\phi_{k+1}-\phi_k)=\phi_{n+1}-\phi_0,$$

so

$$\sum_{k=0}^{\infty}(\phi_{k+1}-\phi_k)=\lim_{n\to+\infty}(\phi_{n+1}-\phi_0)=-\phi_0. \tag{2.32}$$

By applying the operator $L=\frac{d^m}{dt^m}$ , $m\in\mathbb{N}$ , on both sides of equation (2.32), we obtain:

$$\sum_{j=0}^{\infty}L(\phi_{j+1}-\phi_j)=-L(\phi_0)=0. \tag{2.33}$$

On the other hand, form (2.25), we have

$$L(\phi_{j+1}-\phi_j)=L(T\langle\phi_0+\phi_1+\cdots+\phi_j\rangle-T\langle\phi_0+\phi_1+\cdots+\phi_{j-1}\rangle),\quad k\geqslant 1. \tag{2.34}$$

Using the definition of the operator $T\langle u\rangle$ defined by (2.24), we obtain:

$$\begin{aligned} L(\phi_{j+1}-\phi_j) &= L\left(\int_0^t\frac{(-1)^m}{(m-1)!}(s-t)^{m-1}\{L(\sum_{i=0}^{j}\phi_i)-L(\sum_{i=0}^{j-1}\phi_i)\}ds\right) \\ &\quad +L\left(\int_0^t\frac{(-1)^m}{(m-1)!}(s-t)^{m-1}\{N(\sum_{i=0}^{j}\phi_i)-N(\sum_{i=0}^{j-1}\phi_i)\}ds\right),\ j\geqslant 1. \end{aligned} \tag{2.35}$$

Now, the operator $T\langle u\rangle$, defined by (2.24), gives the integral of the mth time of $Lu(t)+Nu(t)-h(t)$. Since the differential operator $L=\frac{d^m}{dt^m}$ of order $m$ is the inverse of the integral operator mth time, then equation (2.35) becomes:

$$L(\phi_{j+1}-\phi_j)=L(\phi_j)+N(\sum_{i=0}^{j}\phi_i)-N(\sum_{i=0}^{j-1}\phi_i),\ j\geqslant 1. \tag{2.36}$$

Therefore, we have:

$$\begin{aligned}\sum_{j=0}^{n} L(\phi_{j+1}-\phi_j) &= L(\phi_0)+N(\phi_0)-h(t)\\ &+L(\phi_1)+N(\phi_0+\phi_1)-N(\phi_0)\\ &+L(\phi_2)+N(\phi_0+\phi_1+\phi_2)-N(\phi_0+\phi_1)\\ &\dots\\ &\dots\\ &+L(\phi_n)+N(\sum_{i=0}^{n}\phi_i)-N(\sum_{i=0}^{n-1}\phi_i). \qquad (2.37)\end{aligned}$$

Therefore,

$$\sum_{j=0}^{\infty} L(\phi_{j+1}-\phi_j) = L(\sum_{j=0}^{\infty}\phi_j)+N(\sum_{j=0}^{\infty}\phi_j)-h(t). \qquad (2.38)$$

From (2.33) and (2.38), we can observe that $u(t) = \sum_{k=0}^{\infty}\phi_k(t)$ is an exact solution of problem (2.17). ■

CHAPTER 3

# EXISTENCE OF WEAK SOLUTIONS OF THE FRACTIONAL $P(X,Y)$-LAPLACIAN PROBLEM BY TOPOLOGICAL DEGREE

## 3.1 Introduction

This chapter is devoted to the study of the existence of weak solutions for the following problem involving the fractional $p(x,y)$-Laplacian with Dirichlet boundary condition:

$$\begin{cases} \mathcal{L}_K^{p(x)}u + |u|^{q(x)-2}u + g(u) = f(x,u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ is a bounded regular open domain, $q \in C_+(\overline{\Omega})$ and the nonlocal integro-differential operator of elliptic type $\mathcal{L}_K^{p(x)}$,

$$\begin{aligned} \mathcal{L}_K^{p(x)}u &= p.v. \int_{\mathbb{R}^N} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy, \quad x \in \mathbb{R}^N, \end{aligned}$$

where $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1,+\infty)$ is a continuous bounded function that satisfies (1.4)-(1.6). The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is a measurable function that has the properties (1.7)-(1.9).

Recently, there has been a lot of interest in investigating problems involving nonlocal fractional integrodifferential operators. In [58], the authors studied the following fractional elliptic problem

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u + f(x,u) = 0, & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.2}$$

where $\lambda > 0$ is a real paramater, $2^* = \dfrac{2n}{n-2s}$ and $\mathcal{L}_K$ is an integro-differential operators of nonlocal fractional type defined as follows:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^N.$$

They used variational approaches to prove the existence of a result in this situation. H. Hajaiej, G. Molica Bisci and L. Vilasi [28] studied the following problem

$$\begin{cases} \mathcal{L}_K u = \mu |u|^{2^*-2}u + \lambda g(u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.3}$$

They demonstrated the existence of one weak solution to (4.1) by minimizing the energy in a tiny ball in a fractional Sobolev space.
In [20] the authors were interested in studying the following problem:

$$\begin{cases} (-\Delta_{p(x)})^s u + |u|^{q(x)-2}u = \lambda V(x)|u|^{r(x)-2}u, & in \ \ \Omega, \\ u = 0, & on \ \partial\Omega, \end{cases} \tag{3.4}$$

where $(-\Delta_{p(x)})^s$ is the fractional $p(x,y)-$Laplace operator given by

$$(-\Delta_{p(x)})^s u = p.v. \int_{\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy, \quad s \in (0,1).$$

S. Taarabti studied in [64] the problem (3.1) in the case, $q(x) = \overline{p}(x)$, $g(u) = 0$ and $f(x,u) = \lambda V(x)|u|^{r(x)-2}u$:

$$\begin{cases} \mathcal{L}_K^{p(x)} u + |u|^{\overline{p}(x)-2}u = \lambda V(x)|u|^{q(x)-2}u, & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.5}$$

It is important to note that $(-\Delta_{p(x)})^s$ is a generalized operator of the fractional $p$-Laplacian operator $(-\Delta_p)^s$ (that is when $p(x,y) = p =$ constant ) and it is the fractional version of the $p(x)-$Laplacian $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}u)$ that is connected with the variable exponent Sobolev space. Over the last few years, interest in these operators, as well as pseudo-differential operators in general, has steadily increased. Nonlocal operators such as $(-\Delta)^s$ and its generalization $\mathcal{L}_K$ naturally arise in continuum mechanics, phase transition phenomena, population dynamics, and game theory, See, for example, [16, 42, 46] for examples of stochastical stabilization of Lévy processes. The applications of non-local integrodifferential sparked interest in understanding them. Indeed, they have a wide range of applications, including the thin obstacle issue, optimization, finance, stratified materials, anomalous diffusion, crystal dislocation, picture deblurring and denoising, and so on. We refer to [15, 17, 18, 19, 61, 65] and the references therein for more information.

## 3.2 Hypotheses

**Definition 3.1** ([8]) We say that $u \in E_0 = W_0^g$ is a weak solution of (3.1), if

$$\begin{aligned} &\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\eta(x)-\eta(y))K(x,y)dxdy \\ +\ &\int_\Omega |u(x)|^{q(x)-2}u(x)\eta(x)dx + \int_\Omega g(u)\eta(x)dx = \int_\Omega f(x,u)\eta(x)dx, \end{aligned}$$

for any $\eta \in E_0$.

(H1) $f$ is a Carathéodory function such that there exists $b \in \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega)$, $r \in \mathrm{C}_+(\overline{\Omega})$ with $1 < r^- \leqslant r(x) \leqslant r^+ < (\overline{p})^-$ and $C > 0$:

$$|f(x,y)| \leqslant C(b(x) + |y|^{r(x)-1})$$

for a.a. $x \in \Omega$ and all $y \in \mathbb{R}$.

(H2) $g$ is a continuous real function defined on $\mathbb{R}$, such that there exists $\lambda > 0$ and $\theta \in \mathrm{C}_+(\overline{\Omega})$ with $1 < \theta^- \leqslant \theta(x) \leqslant \theta^+ < (\overline{p})^-$ such that

$$|g(t)| \leqslant \lambda |t|^{\theta(x)-1}$$

for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$.

(H3) $q \in \mathrm{C}_+(\overline{\Omega})$ with $1 < q^- \leqslant q(x) \leqslant q^+ < (\overline{p})^-$.

## 3.3 Main result and proof

Now we present our main result in this work

**Theorem 3.1** *([8]) Under hypotheses (H1), (H2) and (H3), there is a weak solution $u \in E_0$ to problem (3.1).*

Let's consider the nonlineare operator $V : E_0 \longrightarrow \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega)$ defined by

$$V(u) = \psi u + \Gamma u + \varphi u,$$

where

$$\begin{aligned} \psi u(x) &= |u(x)|^{q(x)-2}u(x), \\ \Gamma u(x) &= g(u(x)), \\ \varphi u(x) &= -f(x,u(x)). \end{aligned}$$

Let (1.4), (1.5), (1.7), (1.8) and (1.9) be satisfied. Then the operator

$$\mathcal{L}_K^{p(x)} : E_0 \longrightarrow E_0^*,$$

defined by

$$\langle \mathcal{L}_K^{p(x)} u, v\rangle = \int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy,$$

where $E_0^*$ is the dual space of $E_0$.

**Theorem 3.2 (see [64] )** *Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is a measurable function satisfy (1.7), (1.8) and (1.9) and suppose that (1.4) and (1.5) be satisfied. Then, the following properties hold:*

*1) $\mathcal{L}_K^{p(x)} : E_0 \longrightarrow E_0^*$ is a bounded operator.*

*2) $\mathcal{L}_K^{p(x)} : E_0 \longrightarrow E_0^*$ is a homeomorphism.*

*3) $\mathcal{L}_K^{p(x)} : E_0 \longrightarrow E_0^*$ is a mapping of type $(S_+)$.*

**Lemma 3.1** *([8]) $\psi, \Gamma, \varphi$ are bounded.*

**Proof.** We use the inequalities (1.8), and hypothesis $(H_1)$, we have for any $u \in E_0$,

$$\begin{aligned}
\|\varphi u\|_{\hat{\overline{p}}(x)} &\leq \rho_{\hat{\overline{p}}(x)}(\varphi u)+1,\\
&= \int_\Omega |f(x,u(x)|^{\hat{\overline{p}}(x)}dx+1,\\
&\leqslant C\left(\rho_{\hat{\overline{p}}(x)}(b)+\rho_{m(x)}(u)\right)+1,\\
&\leqslant C\left(\rho_{\hat{\overline{p}}(x)}(b)+\|u\|_{m(x)}^{m^+}+\|u\|_{m(x)}^{m^-}\right)+1,
\end{aligned}$$

since $m(x)=(r(x)-1)\hat{\overline{p}}(x) < p_s^*(x)$, then by the continuous embedding $E_0 \hookrightarrow \mathsf{L}^{m(x)}(\Omega)$,

$$\|\varphi u\|_{\hat{\overline{p}}(x)} \leq C\max\{1, c_0^{m^+}, c_0^{m^-}\}\left(\rho_{\hat{\overline{p}}(x)}(b)+[u]_{K,p(x,y)}^{m^+}+[u]_{K,p(x,y)}^{m^-}\right)+1,$$

where $c_0$ is the continuous embeding constant given in the Theorem 1.6. This implies that $\varphi$ is bounded on $E_0$.

We prove now that $\psi$ is bounded. Let $u \in E_0$, we have

$$\begin{aligned}
\|\psi u\|_{\hat{\overline{p}}(x)} &\leq \rho_{\hat{\overline{p}}(x)}(\psi u)+1,\\
&= \int_\Omega |u|^{(q(x)-1)\hat{\overline{p}}(x)}dx+1,\\
&\leqslant \|u\|_{z(x)}^{z^+}+\|u\|_{z(x)}^{z^-}+1,
\end{aligned}$$

where $z(x) = (q(x)-1)\hat{\overline{p}}(x)$.

Since $p_s^*(x) - z(x) = \dfrac{N\overline{p}(x)(\overline{p}(x)-q(x)) + s\overline{p}^2(x)(q(x)-1)}{(N-s\overline{p}(x))(\overline{p}(x)-1)} > 0$, then by continuous embedding $E_0 \hookrightarrow \mathtt{L}^{z(x)}(\Omega)$,

$$\|\psi u\|_{\hat{\overline{p}}(x)} \leq \max\{c_1^{z^+}, c_1^{z^-}\}\left([u]_{K,p(x,y)}^{z^+} + [u]_{K,p(x,y)}^{z^-}\right) + 1,$$

where $c_1$ is the continuous embiding constant. This imblies that $\psi$ is bounded in $E_0$. We prove that $\Gamma$ is bounded. Using the inequalities (1.8), and the hypothesis of $(H_2)$, we get for any $u \in E_0$,

$$\begin{aligned}
\|\Gamma u\|_{\hat{\overline{p}}(x)} &\leq \rho_{\hat{\overline{p}}(x)}(\Gamma u) + 1, \\
&= \int_\Omega |g(u(x))|^{\hat{\overline{p}}(x)} dx + 1, \\
&\leqslant (\lambda+1)^{\overline{p}^+} \rho_{w(x)}(u) + 1, \\
&\leqslant (\lambda+1)^{\overline{p}^+}\left(\|u\|_{w(x)}^{w^+} + \|u\|_{w(x)}^{w^-}\right) + 1,
\end{aligned}$$

since $w(x) = (\theta(x)-1)\hat{\overline{p}}(x) < p_s^*(x)$, then by the continuous embedding $E_0 \hookrightarrow \mathtt{L}^{w(x)}(\Omega)$,

$$\|\Gamma u\|_{\hat{\overline{p}}(x)} \leq (\lambda+1)^{\overline{p}^+} \max\{c_2^{w^+}, c_2^{w^-}\}\left([u]_{K,p(x,y)}^{w^+} + [u]_{K,p(x,y)}^{w^-}\right) + 1,$$

where $c_2$ is the continuous embedding constant. This implies $\Gamma$ is bounded on $E_0$.

■

**Lemma 3.2** *([8]) $\psi, \Gamma, \varphi$ are continuous operators.*

**Proof.** Let $u_n \to u$ in $E_0$. Then $u_n \to u$ in $\mathtt{L}^{\overline{p}(x)}(\Omega)$. Hence there exist a subsequence $(u_k)$ of $(u_n)$ and measurable function $h \in \mathtt{L}^{\overline{p}(x)}(\Omega)$ such that

$$u_k(x) \longrightarrow u(x),$$

and

$$|u_k(x)| \leq h(x),$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. From the $(H1)-(H3)$,

$$\begin{aligned}
\psi u_k(x) &\longrightarrow \psi u(x) \text{ a.e. } x \in \Omega, \\
\Gamma u_k(x) &\longrightarrow \Gamma u(x) \text{ a.e. } x \in \Omega, \\
\varphi u_k(x) &\longrightarrow \varphi u(x) \text{ a.e. } x \in \Omega,
\end{aligned}$$

and

$$\begin{aligned}
|\psi u_k(x)| &\leqslant |h(x)|^{q(x)-1}, \\
|\Gamma u_k(x)| &\leqslant \lambda|h(x)|^{\theta(x)-1}, \\
|\varphi u_k(x)| &\leqslant C(b(x) + |h(x)|^{r(x)-1}).
\end{aligned}$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Since

$$\begin{aligned}
&|h(x)|^{q(x)-1} \in \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega),\\
&\lambda |h(x)|^{\theta(x)-1} \in \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega),\\
&C(b(x) + |h(x)|^{r(x)-1}) \in \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega),
\end{aligned}$$

and taking into account the equals

$$\begin{aligned}
\rho_{\hat{\overline{p}}(x)}(\varphi u_k - \varphi u) &= \int_\Omega |f(x,u_k(x)) - f(x,u(x))|^{\hat{\overline{p}}(x)} dx,\\
\rho_{\hat{\overline{p}}(x)}(\psi u_k - \psi u) &= \int_\Omega ||u_k(x)|^{q(x)-2} u_k(x) - |u(x)|^{q(x)-2} u(x)|^{\hat{\overline{p}}(x)} dx,\\
\rho_{\hat{\overline{p}}(x)}(\Gamma u_k - \Gamma u) &= \int_\Omega |g(u_k(x)) - g(u(x))|^{\hat{\overline{p}}(x)} dx,
\end{aligned}$$

the dominated convergence theorem and the equivalence in the Proposition 1.9 implies that

$$\begin{array}{llll}
\psi u_k & \longrightarrow & \psi u & in \quad \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega),\\
\Gamma u_k & \longrightarrow & \Gamma u & in \quad \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega),\\
\varphi u_k & \longrightarrow & \varphi u & in \quad \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega).
\end{array}$$

So, the operators $\psi, \Gamma$ and $\varphi$ are continuous. ■

**Lemma 3.3** *([8]) The operator $F : E_0 \longrightarrow E_0^*$ defined by*

$$\langle F(u), \eta \rangle = \int_\Omega \left( -f(x,u) + g(u) + |u|^{q(x)-2} u \right) \eta dx,$$

*is compact and quasimonotone.*

**Proof.** Since the operator $I : E_0 \longrightarrow \mathrm{L}^{\overline{p}(x)}(\Omega)$ is compact, then the adjoint operator $I^* : \mathrm{L}^{\hat{\overline{p}}(x)}(\Omega) \longrightarrow E_0^*$ is also compact. Therefore, the compositions $I^* \circ \psi, I^* \circ \Gamma, I^* \circ \varphi : E_0 \longrightarrow E_0^*$ are compact by Lemma 3.2. We deduce that $F = I^* \circ \psi + I^* \circ \Gamma + I^* \circ \varphi$ is compact.
To show that $F$ is quasimonotone, let $(u_n)$ be a sequence in $E_0$ such that $u_n \rightharpoonup u$.
Since $u_n \rightharpoonup u$ and $F$ is compact, then

$$(u_n) \ is \ bounded \ in \ E_0 \ and \ F(u_n) \to F(u) \ in \ E_0^*.$$

And so we find

$$\lim_{n\to\infty} \langle F(u_n), u_n - u \rangle = \lim_{n\to\infty} \langle F(u_n) - F(u), u_n - u \rangle = 0$$

So $F$ is quasimonotone.

■

**Proof.** Proof of Theorem 3.1. Let $F : E_0 \longrightarrow E_0^*$ be as in Lemma 3.3 and $\mathcal{L}_K^{p(x)} : E_0 \longrightarrow E_0^*$, as in section 3.3,

$$\langle \mathcal{L}_K^{p(x)} u, \eta \rangle = \int_Q |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\eta(x) - \eta(y))K(x,y)dxdy.$$

It is clear from the above that $u \in E_0$ is a weak solution of (3.1) if and only if:

$$\mathcal{L}_K^{p(x)} u = -Fu \tag{3.6}$$

Thanks to the properties of the operator $\mathcal{L}_K^{p(x)}$ seen in Theorem 3.2, that the operator $\mathcal{L}_K^{p(x)} : E_0 \longrightarrow E_0^*$ is bounded, continuous, and satisfies condition $(S_+)$. Now let $X = E_0^*$ and identify $X^*$ with $E_0$. By the main theorem on monotone operators due to Browder and Minty in (see [72], Theorem 26A), the inverse operator $T := (\mathcal{L}_K^{p(x)})^{-1} : X \longrightarrow X^*$ is bounded, continuous and satisfies condition $(S_+)$. Furthermore, Lemma 3.3 shows that operator $F : X^* \longrightarrow X$ is bounded, continuous and quasimonotone.
Consequently, equation (3.6) is equivalent to

$$u = Tv \quad and \quad v + F \circ Tv = 0. \tag{3.7}$$

■

To solve equation (3.7), we need the following lemma

**Lemma 3.4** *([8]) The set*

$$B_K = \{v \in X | v + tF \circ Tv = 0 \quad for\ some\ t \in [0,1]\}$$

*is bounded.*

**Proof.** Let $v \in B_K$ and take $u = Tv$, this implies that

$$[u]_{K,p(x,y)} = [Tv]_{K,p(x,y)}.$$

- If $[u]_{K,p(x,y)} \leq 1 \Rightarrow \{Tv : v \in B_K\}$ is bounded.
- If $[u]_{K,p(x,y)} \geq 1$, then by Lemma 1.1, the conditions $(H_1) - (H_3)$ and the Hölder

inequality, we get

$$\begin{aligned}
[T\upsilon]^{(\overline{p})^-}_{K,p(x,y)} &= [u]^{(\overline{p})^-}_{K,p(x,y)} \leq \rho^0_g(u),\\
&= \langle \mathcal{L}^{p(x)}_K u, u\rangle = \langle \upsilon, T\upsilon\rangle,\\
&= -t\langle F\circ T\upsilon, T\upsilon\rangle,\\
&= t\int_\Omega \left(f(x,u) - g(u) - |u|^{q(x)-2}u\right) u dx,\\
&\leqslant Ct\left(\int_\Omega |b(x)u(x)|dx + \rho_{r(x)}(u)\right) + \lambda t\rho_{\theta(x)}(u) + t\rho_{q(x)}(u),\\
&\leqslant Ct\left(2\|b\|_{\hat{\overline{p}}(x)}\|u\|_{\overline{p}(x)} + \|u\|^{r^+}_{r(x)} + \|u\|^{r^-}_{r(x)}\right) + \lambda t\left(\|u\|^{\theta^+}_{\theta(x)} + \|u\|^{\theta^-}_{\theta(x)}\right)\\
&\quad + t\left(\|u\|^{q^+}_{q(x)} + \|u\|^{q^-}_{q(x)}\right).
\end{aligned}$$

From the continuous embedding $E_0 \hookrightarrow \mathrm{L}^{\overline{p}(x)}(\Omega), \mathrm{L}^{r(x)}(\Omega), \mathrm{L}^{\theta(x)}(\Omega), \mathrm{L}^{q(x)}(\Omega)$, we have

$$\begin{aligned}
[T\upsilon]^{(\overline{p})^-}_{K,p(x,y)} &\leqslant \mathit{const}\left([T\upsilon]_{K,p(x,y)} + [T\upsilon]^{r^+}_{K,p(x,y)} + [T\upsilon]^{r^-}_{K,p(x,y)} + [T\upsilon]^{\theta^+}_{K,p(x,y)}\right)\\
&\quad + \mathit{const}\left([T\upsilon]^{\theta^-}_{K,p(x,y)} + [T\upsilon]^{q^+}_{K,p(x,y)} + [T\upsilon]^{q^-}_{K,p(x,y)}\right).
\end{aligned}$$

Since $(\overline{p})^- > r^+, \theta^+, q^+$ and $F$ is bounded, we then conclude from the above inequality that $B_K$ is bounded in $X$. ■

We are now in a position to complete the proof of Theorem 3.1.

**Proof.** From Lemma 3.4, there exists $R > 0$ such that

$$\|\upsilon\|_X < R \quad \textit{for all} \quad \upsilon \in B_K.$$

It follows that

$$\upsilon + tF\circ T \neq 0 \quad \textit{for all} \quad \upsilon \in \partial B_R \textit{ and } \quad t \in [0,1].$$

From Lemma 2.5 it follows that

$$I + F\circ T \in \mathcal{F}_T(\overline{B_R(0)}) \quad \textit{and} \quad I = \mathcal{L}^{p(x)}_K \circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

consider a homotopy $\Theta : [0,1]\times\overline{B_R(0)} \longrightarrow X$ given by

$$\Theta(t,\upsilon) := \upsilon + tF\circ T\upsilon \quad \textit{for} \quad (t,\upsilon)\in[0,1]\times\overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree $d_T$ stated in Theorem 2.5, we obtian

$$d_T(I + F\circ T, B_R(0), 0) = d_T(I, B_R(0), 0) = 1$$

then, there exists $\upsilon \in B_R(0)$ such that

$$\upsilon + F\circ T\upsilon = 0.$$

Which says that $u = T\upsilon$ is a weak solution of (3.1). This completes the proof. ■

# CHAPTER 4

# EXISTENCE RESULT WITH THE MAIN THEOREM ON PSEUDOMONOTONE OPERATORS

## 4.1 Introduction

This chapter is devoted to studying the existence and uniqueness of weak solutions for the fractional Laplacian problem

$$\begin{cases} -div_s(D^s u) = -D^s.(D^s u) = f(x,u), & in \ \ \Omega, \\ u = 0, & in \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$, $s$ is a fixed number between 0, 1 and $div_s(D^s u)$ is the fractional version of a 2-Laplacian and defined by (see [55])

$$div_s(D^s u) = D^s.(D^s u) = \sum_{i=1}^{N} \frac{\partial^s}{\partial x_i^s}(\frac{\partial^s u}{\partial x_i^s})$$

Shieh and Spector have recently studied a novel class of fractional partial differential equations based on the distributional Riesz fractional derivatives in a pair of two fascinating works [59] and [60]. Instead of using the well-known fractional Laplacian, their starting concept is the distributional Riesz fractional gradient of order $s \in (0,1)$, which will be called here the $s$-gradient $D^s$, for brevity: for $u \in \mathrm{L}^p(\mathbb{R}^N)$, $p \in (1,\infty)$, we set

$$D_j^s u = \frac{\partial^s u}{\partial x_j^s} = \frac{\partial}{\partial x_j}(I_{1-s} * u), \quad 0 < s < 1, \quad j = 1, \cdots, N,$$

where $\frac{\partial}{\partial x_j}$ is taken in the distributional sense, for every $v \in C_c^\infty(\mathbb{R}^N)$,

$$\langle \frac{\partial^s u}{\partial x_j^s}, v \rangle = (-1)\langle (I_{1-s} * u), \frac{\partial v}{\partial x_j} \rangle = -\int_{\mathbb{R}^N} (I_{1-s} * u)\frac{\partial v}{\partial x_j} dx,$$

with $I_s$ denoting the Riesz potential of order $s$, $0 < s < 1$:

$$I_s(x) := \frac{\gamma(N, 1-s)}{|x|^{N-s}},$$

where

$$\gamma(N, s) := \frac{2^s \Gamma(\frac{N+s+1}{2})}{\pi^{\frac{N}{2}} \Gamma(\frac{1-s}{2})}.$$

Thus, we can write the $s$-gradient ($D^s$) and the $s$-divergence ($D^s.$) for sufficiently regular functions $u$ and vector $\varphi$ ([21], [59], [60]) in integral form, respectively, by

$$D^s u(x) := \gamma(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{z u(x+z)}{|z|^{N+s+1}} \chi_\epsilon(0, z) dz = \gamma(N, s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+s}} \frac{x-y}{|x-y|} dy$$

and

$$D^s.\varphi(x) := \gamma(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{z.\varphi(x+z)}{|z|^{N+s+1}} \chi_\epsilon(0, z) dz = \gamma(N, s) \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+s}} . \frac{x-y}{|x-y|} dy,$$

where $\chi_\epsilon(x, z)$ is the characteristic function of the set $\{(x, z) : |z - x| > \epsilon\}$ for $\epsilon > 0$. As it was shown in [59], $D^s$ has nice properties for $u \in C_c^\infty(\mathbb{R}^N)$, namely it coincides with the fractional Laplacian as follows:

$$(-\Delta)^s u(x) = -D^s.D^s u,$$

where, $0 < s < 1$,

$$\begin{aligned} (-\Delta)^s u(x) &= \gamma^2(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \chi_\epsilon(x, y) dy \\ &= \frac{1}{2} \gamma^2(N, s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2S}} dy. \end{aligned}$$

The study of elliptic equations involving fractional operators is an exciting field of nonlinear analysis. These problems have recently received a lot of attention, both for pure mathematical research and for practical applications in the real world. Indeed, this sort of operator appears in a variety of contexts, including the representation of a variety of physical processes, optimization, population dynamics, and mathematics nance. We have already mentioned that the $-D^s.D^s u$ operator coincides with the $(-\Delta)^s u$ operator in the case $u \in C_c^\infty(\mathbb{R}^N)$, and the last operator has many applications in various fields, for example, the fractional Laplacian operator $(-\Delta)^s, 0 < s < 1$, provides a simple model to describe some jump Levy processes in probability theory (see for example [5, 14, 16, 23, 12] and the references therein). As examples of applications of the problem (4.1) ($u \in C_c^\infty(\mathbb{R}^N)$), we state the following two models:

The existence and uniqueness of weak solutions to problems involving the fractional Laplacian $(-\Delta)^s$ have been studied in many articles, for example, [56, 57]

## 4.2 Existence of solutions

We will present the idea of weak solutions for problem (4.1) in this part, as well as the existence and uniqueness results for these solutions [9]. Firstly, we cite the following assumptions:

$(h_1)$ $s \in (0,1)$ with $2s < n$.

$(h_2)$ $f \in \mathrm{L}^{\alpha}(\Omega)$ with $\alpha > \dfrac{2n}{n+2s}$.

$(h_3)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition and

$$|f(x,t)| \leqslant a(x) + b|t| \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $a \in \mathrm{L}^2(\Omega)$ and $b \in \mathbb{R}$.

$(h_4)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a decreasing function with respect to the second variable.

$(h_5)$ There exists $c_0 \geqslant 0$ such that

$$(f(x,t) - f(x,s))(t-s) \leqslant c_0 |t-s|^2$$

for a.a. $x \in \Omega$ and all $(t,s) \in \mathbb{R} \times \mathbb{R}$.

**Definition 4.1** ([9]) We say that $u \in \mathrm{L}_0^{s,2}(\Omega)$ is a weak solution of problem (4.1), if

$$\int_{\mathbb{R}^N} D^s u . D^s v dx = \int_{\Omega} f(x,u) v(x) dx, \qquad \forall v \in X := \mathrm{L}_0^{s,2}(\Omega). \tag{4.2}$$

### 4.2.1 $f$ is independent of $u$

If $f$ is independent of $u$, we have one of our main result of this work is the following Theorem:

**Theorem 4.1** *([9]) If $f(x,u) = f(x)$ and if hypotheses $(h_1)$, $(h_2)$ hold, then, the problem (4.1) has a unique weak solution.*

**Proof. Existence part.** Let the operators

$$L_1 : \mathrm{L}_0^{s,2}(\Omega) \longrightarrow (\mathrm{L}_0^{s,2}(\Omega))',$$

$$L_2 : \mathrm{L}_0^{s,2}(\Omega) \longrightarrow \mathbb{R},$$

such that

$$\langle L_1(u), \eta \rangle = \int_{\mathbb{R}^N} D^s u . D^s \eta dx,$$

and

$$\langle L_2, \eta \rangle = \int_{\Omega} f(x) \eta(x) dx.$$

The proof of existence part of Theorem 4.1 is divided into several steps.

- **Step 1. The operators $L_1$ and $L_2$ are bounded.**

We use Hölder-type inequality, we have for any $u, \eta \in \mathrm{L}_0^{s,2}(\Omega)$,

$$\begin{aligned} |\langle L_1(u), \eta\rangle| &\leq \int_{\mathbb{R}^N} |D^s u . D^s \eta| dx \\ &\leq \|D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)} \|D^s \eta\|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq |u|_{\mathrm{L}_0^{s,2}(\Omega)} |\eta|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

This implies that $L_1$ is bounded.

On the other hand, using again Hölder-type inequality, hypothesis $(h_1)$ and $(h_2)$, we get

$$\begin{aligned} |\langle L_2(u), \eta\rangle| &\leq \int_{\Omega} |f(x)\eta| dx \\ &\leq \|f\|_{\mathrm{L}^{\alpha}(\Omega)} \|\eta\|_{\mathrm{L}^{\alpha'}(\Omega)} \\ &\leq M \|f\|_{\mathrm{L}^{\alpha}(\Omega)}) |\eta|_{\mathrm{L}_0^{s,2}(\Omega)} \end{aligned}$$

where $M$ is constant of continuous embedding given by Theorem 1.9. Hence, the operator $L_2$ is bounded.

- **Step 2. The operator $L_1$ is hemi-continuous and the operator $L_2$ is continuous.**

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0^{s,2}(\Omega)$ and $u \in \mathrm{L}_0^{s,2}(\Omega)$ such that $u_n$ converges strongly to $u$ in $\mathrm{L}_0^{s,2}(\Omega)$. Firstly, we will prove that $L_1$ and $L_2$ are continuous in $\mathrm{L}_0^{s,2}(\Omega)$, indeed,

$$\begin{aligned} |\langle L_1(u_n) - L_1(u), \eta\rangle| &= \left| \int_{\mathbb{R}^N} (D^s u_n - D^s u) . D^s \eta dx \right| \\ &\leq \|D^s u_n - D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)} \|D^s \eta\|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq |u_n - u|_{\mathrm{L}_0^{s,2}(\Omega)} |\eta|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

And

$$\begin{aligned} |\langle L_2, u_n - u\rangle| &= \left| \int_{\Omega} f(x)(u_n(x) - u(x) dx \right| \\ &\leq \|f\|_{\mathrm{L}^{\alpha}(\Omega)} \|u_n - u\|_{\mathrm{L}^{\alpha'}(\Omega)} \\ &\leq M \|f\|_{\mathrm{L}^{\alpha}(\Omega)}) |u_n - u|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

Consequently

$$L_1(u_n) \longrightarrow L_1(u) \qquad in \ (\mathrm{L}_0^{s,2}(\Omega))'.$$

$$L_2(u_n) \longrightarrow L_2(u) \qquad in \ \mathbb{R}.$$

This implies that the operators $L_1$ and $L_2$ are continuous in $\mathrm{L}_0^{s,2}(\Omega)$, in $\mathbb{R}$ respectively. Therefore, $L_1$ is hemi-continuous in $\mathrm{L}_0^{s,2}(\Omega)$.

- **Step 3 . The operator $L_1$ is coercive.**

For any $u \in \mathrm{L}_0^{s,2}(\Omega)$, we have

$$\begin{aligned} \langle L_1(u), u\rangle &= \int_{\mathbb{R}^N} |D^s u|^2 dx \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

Therefore

$$\frac{\langle L(u), u\rangle}{|u|_{\mathrm{L}_0^{s,2}(\Omega)}} \longrightarrow +\infty \quad as \ |u|_{\mathrm{L}_0^{s,2}(\Omega)} \longrightarrow +\infty.$$

Hence, the operator $L_1$ is coercive.

- **Step 4. The operator $L_1$ is monotone.**

For that, it suffices to prove that $L_1$ is monotone

$$\langle L_1(u) - L_1(v), u - v\rangle = \int_{\mathbb{R}^N} |D^s u - D^s v|^2 dx \geqslant 0 \qquad for\ all\ u, v \in \mathrm{L}_0^{s,2}(\Omega).$$

Therefore $L_1$ is monotone. Hence, the existence of weak solution for problem (4.1) follows from Theorem 2.6.

**Uniqueness part.** Let $u$ and $w$ be two weak solutions of problem (4.1). As a test function for the solution $u$, we take $v = u - w$ in equality (4.2) and for the solution $w$, we take $v = w - u$ as a test function in (4.2), we have

$$\int_{\mathbb{R}^N} D^s u.D^s(u-w)dx = \int_{\Omega} f(u-w)dx$$

and

$$\int_{\mathbb{R}^N} D^s w.D^s(w-u)dx = \int_{\Omega} f(w-u)dx.$$

By summing up the two above equalities, we get

$$\int_{\mathbb{R}^N} |D^s u - D^s w|^2 dx = 0.$$

This implies that

$$u = w \qquad a.e\ in\ \Omega.$$

■

### 4.2.2 $f$ is dependent of $u$

If f is dependent of $u$, we have

**Theorem 4.2** *([9]) If hypotheses $(h_1), (h_3) - (h_5)$ hold, then, the problem (4.1) has a unique weak solution.*

**Proof. Existence part.** Let the operator

$$T : \mathrm{L}_0^{s,2}(\Omega) \longrightarrow (\mathrm{L}_0^{s,2}(\Omega))'$$

such that

$$\begin{aligned} \langle T(u), \eta\rangle &= \int_{\mathbb{R}^N} D^s u . D^s \eta dx - \int_\Omega f(x,u)\eta(x)dx \\ &= \langle \psi(u), \eta\rangle - \langle \Phi(u), \eta\rangle \end{aligned}$$

The proof of existence part of Theorem 4.2 is divided into several steps.

• **Step 1. The operator $T$ is bounded.**

From step 1 in the proof of Theorem 4.1 we can see that the operator $\psi$ is bounded.

On the other hand, using again Hölder-type inequality, hypotheses $(h_1)$, $(h_3)-(h_4)$, we get

$$\begin{aligned} |\langle \Phi(u), \eta\rangle| &\leq \int_\Omega |f(x,u)\eta| dx \\ &\leq \|a(x) + b|u|\|_{\mathrm{L}^2(\Omega)} \|\eta\|_{\mathrm{L}^2(\Omega)} \\ &\leq M(\|a\|_{\mathrm{L}^2(\Omega)} + |b|\|u\|_{\mathrm{L}^2(\Omega)})|\eta|_{\mathrm{L}_0^{s,2}(\Omega)}, \end{aligned}$$

where $M$ is constant of continuous embedding given by Theorem 1.9. Hence, the operator $T$ is bounded.

• **Step 2. The operator $T$ is hemi-continuous.**

Let $\{u_n\}_{n\in\mathbb{N}} \in \mathrm{L}_0^{s,2}(\Omega)$ and $u \in \mathrm{L}_0^{s,2}(\Omega)$ such that $u_n$ converges strongly to $u$ in $\mathrm{L}_0^{s,2}(\Omega)$.

We conclude by proving step 2 in proving Theorem 4.1 that the operator $\psi$ is continuous in space $\mathrm{L}_0^{s,2}(\Omega)$. We then suffice to prove the continuity of the operator $\Phi$

$$\begin{aligned} |\langle \Phi(u_n) - \Phi(u), \eta\rangle| &= \left| \int_\Omega (f(x,u_n) - f(x,u))\eta dx \right| \\ &\leq \|f(x,u_n) - f(x,u)\|_{\mathrm{L}^2(\Omega)} \|\eta\|_{\mathrm{L}^2(\Omega)} \\ &\leq M\|f(x,u_n) - f(x,u)\|_{\mathrm{L}^2(\Omega)} |\eta|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

By Theorem 1.9,

$$u_n \longrightarrow u \ in \ \mathrm{L}_0^{s,2}(\Omega) \Rightarrow u_n \longrightarrow u \ in \ \mathrm{L}^2(\Omega). \tag{4.3}$$

Using Lebesgue's convergence inverse theorem, we get

$$\begin{cases} u_n(.) \longrightarrow u(.) \qquad a.e. \ in \ \Omega \\ \exists g(.) \in \mathrm{L}^2(\Omega) : |u_n(x)| \leqslant g(x) \ \ \forall n, \ a.e. \ in \ \Omega, \end{cases} \tag{4.4}$$

Now, using Lebesgue's convergence theorem and hypothesis $(h_3)$, we derive

$$f(x,u_n) \longrightarrow f(x,u_n) \ in \ \mathrm{L}^2(\Omega). \tag{4.5}$$

So, $\Phi(u_n) \longrightarrow \Phi(u)$ *in* $(\mathrm{L}_0^{s,2}(\Omega))'$. Then $\Phi$ is continuous. Therefore, $T$ is hemi-continuous in $\mathrm{L}_0^{s,2}(\Omega)$.

- **Step 3. The operator $T$ is coercive.**

For any $u \in \mathrm{L}_0^{s,2}(\Omega)$, we have

$$\begin{aligned}\langle T(u), u\rangle &= \int_{\mathbb{R}^N} |D^s u|^2 dx - \int_\Omega f(x,u) u dx \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)} - (\|a\|_{\mathrm{L}^2(\Omega)} + |b| \|u\|_{\mathrm{L}^2(\Omega)}) \|u\|_{\mathrm{L}^2(\Omega)} \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)} - M(\|a\|_{\mathrm{L}^2(\Omega)} + |b| \|u\|_{\mathrm{L}^2(\Omega)}) |u|_{\mathrm{L}_0^{s,2}(\Omega)}.\end{aligned}$$

Therefore

$$\frac{\langle T(u), u\rangle}{|u|_{\mathrm{L}_0^{s,2}(\Omega)}} \longrightarrow +\infty \quad as \ |u|_{\mathrm{L}_0^{s,2}(\Omega)} \longrightarrow +\infty.$$

Hence, the operator $T$ is coercive.

- **Step 4. The operator $T$ is monotone.**

Applying hypothesis $(h_4)$, it is for each $u, v \in \mathrm{L}_0^{s,2}(\Omega)$,

$$\langle T(u) - T(v), u - v\rangle = \int_{\mathbb{R}^N} |D^s u - D^s v|^2 dx - \int_\Omega (f(x,v) - f(x,u))(v-u) dx \geqslant 0.$$

Therefore $T$ is monotone. Hence, the existence of weak solution for problem (4.1) follows from Theorem 2.6.

**Uniqueness part.** Let $u, v \in \mathrm{L}_0^{s,2}(\Omega)$ be two weak solutions of (4.1). Considering the weak formulation of $u$ and $v$, by choosing $w = u - v$ as a test function, we have

$$\int_{\mathbb{R}^N} D^s u . D^s (u-v) dx = \int_\Omega f(x,u)(u-v) dx$$

and

$$\int_{\mathbb{R}^N} D^s v . D^s (u-v) dx = \int_\Omega f(x,v)(u-v) dx.$$

Subtracting the above two equations, we have

$$\int_{\mathbb{R}^N} (D^s u - D^s v) . D^s (u-v) dx = \int_\Omega (f(x,u) - f(x,v))(u-v) dx. \tag{4.6}$$

For all $x \in \mathbb{R}^N$, then we have the following inequality:

$$(D^s u - D^s v) . D^s (u-v) \geqslant \frac{1}{4} |D^s (u-v)|^2. \tag{4.7}$$

Using assumption $(h_5)$ in (4.7), then it follows that

$$\begin{aligned}\frac{1}{4} \int_{\mathbb{R}^N} |D^s(u-v)|^2 dx &\leqslant \int_{\mathbb{R}^N} (D^s u - D^s v) . D^s (u-v) dx \\ &= \int_\Omega (f(x,u) - f(x,v))(u-v) dx \\ &\leqslant c_0 \int_\Omega |u-v|^2 dx \\ &\leqslant c_0 M \int_{\mathbb{R}^N} |D^s(u-v)|^2 dx.\end{aligned}$$

Consequently, when $4c_0M < 1$, it follows from the above inequality that $u = v$ and so the solution of (4.1) is unique. The proof is complete. $\blacksquare$

## CHAPTER 5

# ELZAKI VARIATIONAL ITERATION METHOD (EVIM) FOR SOLVING NONLINEAR TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

## 5.1 Introduction

Nonlinear PDEs represent an important and interesting topic due to their paramount importance in modeling many phenomena in various fields such as quantum mechanics, image processing, environment, economic system, epidemiology, wave scattering and propagation, magnetic resonance imaging, computational fluid dynamics, magnetic hydrodynamics, tube jumping, faster flow phenomena than Existing, sound, disturbances, acoustic transmission, traffic, demographic models, medical imaging, the correct distribution of oxygen to healing tissues and electrical signals to nerves, etc., and what has strengthened its position in scientific research is what was recently published in conjunction with what the world witnessed due to the Coronavirus, where several papers in which PDEs have been used to describe cases of COVID or make a model of the shape of COVID-19 [69, 67, 3]. However, in some intractable and complex problems, fractional PDEs are resorted to because their use gives greater accuracy and flexibility compared to PDEs, which helped them to appear in various fields such as fluid mechanics, electrical circuits, diffusion, damping laws, relaxation processes, mathematical biology, and so on. Therefore, linear and nonlinear fractional partial differential equations have attracted the attention of a large number of researchers to delve into finding exact and approximate solutions to such equations. They developed many methods for this purpose, such as Adomian decomposition method (ADM) [62], variational iteration method (VIM) [73] homotopy analysis method (HAM) [68] and homotopy perturbation method (HPM).

One of the numerical methods in which we do not need to field the discretization of the domain and linearization of given differential equations is the variational iteration method (VIM). This method became applied in a large number of articles after they overcame the calculate the Lagrange multiplier of the correction function of the given differential equation by restricting non-linear terms and this made this method popular compared to other available methods that were used for the first time to FPDEs by He in [29].

Different transformations are characterized by their simplicity and speed of use in FPDEs, such as Laplace transform, Somodo transform, natural transformation, and Elsack transformation, which led various researchers in this field to combine these transformations and the aforementioned methods. Among these methods: Laplace homotopy analysis method [27], Laplace decomposition method [35], Laplace variational iteration method [70], homotopy perturbation Sumudu transform method [71], homotopy analysis Sumudu transform method [51], variational iteration Sumudu transform method [2], natural transform homotopy perturbation method [45], natural decomposition method [52], homotopy analysis natural transform method [54] and natural variational iteration method (NVIM) [38].

In this chapter, we are working on that new technique mentioned in reference [38], but we preferred to replace the natural transform with the Elzaki transform because the latter has clearer features and properties as it changes with one variable instead of two variables in the natural transform, which makes it easier to calculate, so our new technique is a mixture of two important methods in the for solving nonlinear time-fractional partial differential equations analytically and numerically, Elzaki transform method and variational iteration method, called Elzaki variational iteration method (EVIM).

## 5.2 Solution of nonlinear time-fractional equations by the Elzaki variational iteration method (EVIM)

To illustrate the basic ideas of the Elzaki variational iteration method (EVIM), consider the following non-linear non-homogeneous partial differential equation of temporal fractional order:

$$^{c}D_t^{\alpha}u(x,y,t) = Lu(x,y,t) + Nu(x,y,t) + H(x,y,t), \quad t > 0, \ (x,y) \in \mathbb{R}^2, \tag{5.1}$$

with the initial conditions

$$\frac{\partial^i u}{\partial t^i}(x,y,0) = f_i(x,y), \qquad i = 0,1,2,\ldots,n-1, \tag{5.2}$$

where $^{c}D_t^{\alpha}$ is the Liouville-Caputo fractional derivative operator of order $n-1 < \alpha \leqslant n, n \in \mathbb{N}^*$, $L$ is a linear function, $N$ is a nonlinear function and $H$ is a given function.

By applying the Elzaki transform to both sides of equation (5.1), we obtain:

$$E^{+}[{}^{c}D_{t}^{\alpha}u(x,y,t)] = E^{+}[Lu(x,y,t)] + E^{+}[Nu(x,y,t)] + E^{+}[H(x,y,t)].$$

Using Theorem 1.14 and the initial conditions (5.2), we have:

$$\begin{aligned} E^{+}[u(x,y,t)] &= \rho^{\alpha}\sum_{k=0}^{n-1}\rho^{k-\alpha+2}[D^{k}u(x,y,t)]_{t=0} + \rho^{\alpha}E^{+}[H(x,y,t)] + \\ &\quad +\rho^{\alpha}E^{+}\left[Lu(x,y,t) + Nu(x,y,t)\right]. \end{aligned} \tag{5.3}$$

Now we apply the inverse Elzaki transform to both sides of equation (5.3), to find the following

$$u(x,y,t) = K(x,y,t) + E^{-1}\left(\rho^{\alpha}E^{+}[Lu(x,y,t) + Nu(x,y,t)]\right), \tag{5.4}$$

where $K(x,y,t)$ represents the term resulting from the source term and the prescribed initial conditions.

In order to apply EVIM we have to take the first partial derivative with respect to $t$ of equation (5.4), we get

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t}(x,y,t) - \frac{\partial K}{\partial t}(x,y,t) - \\ &\quad -\frac{\partial}{\partial t}E^{-1}\left(\rho^{\alpha}E^{+}\left[Lu(x,y,t) + Nu(x,y,t)\right]\right). \end{aligned} \tag{5.5}$$

Therefore, by application the variational iteration method (see Chapter 2), on equation (5.5), we get a correct function as follows

$$\begin{aligned} u_{n+1}(x,y,t) &= u_{n}(x,y,t) \\ &\quad - \int_{0}^{t}\left(\frac{\partial u_{n}(x,y,s)}{\partial s} - \frac{\partial}{\partial s}E^{-1}\left(\rho^{\alpha}E^{+}[Lu_{n}(x,y,s) + Nu_{n}(x,y,s)]\right) - \frac{\partial K(x,y,s)}{\partial s}\right)ds. \end{aligned} \tag{5.6}$$

It can simplify (5.6) using the Property 1.7.1, and (5.4) for Elzaki transform to become the correct function as follows

$$\begin{aligned} u_{n+1}(x,y,t) &= K(x,y,t) \\ &\quad +E^{-1}\left(\rho^{\alpha}E^{+}[Lu_{n}(x,y,t) + Nu_{n}(x,y,t)]\right). \end{aligned} \tag{5.7}$$

Let's remember that

$$u(x,y,t) = \lim_{n\to\infty} u_{n}(x,y,t).$$

According to the previous limit, we can obtain the exact solution if it exists or obtain an approximate solution for the considered equation (5.1).

## 5.3 Applications of the EVIM method

In this part, we use the EVIM ([7]) to solve three numerical examples of nonlinear time-fractional partial differential equations.

**Example 5.1** ([7]) Let's start with the following example, which we consider the 2-dimensional nonlinear time-fractional partial differential equation

$$^{c}D_t^{\alpha}u = u_{xxx}u_y - u, \tag{5.8}$$

with the initial conditions

$$u(x,y,0) = x^2+y^2, \quad \frac{\partial u}{\partial t}(x,y,0) = x^2+y^2, \tag{5.9}$$

where $^{c}D_t^{\alpha}$ is the Liouville-Caputo fractional derivative operator of order $\alpha (1 < \alpha \leqslant 2)$, and $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Using (5.7) we get the following correct function

$$u_{n+1}(x,y,t) = (1+t)(x^2+y^2) + E^{-1}\left(\rho^{\alpha}E^{+}[u_{nxxx}u_{ny} - u_n]\right). \tag{5.10}$$

According to the iteration formula (5.10), we have

$u_0 = (1+t)(x^2+y^2)$,

$u_1 = \left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)(x^2+y^2)$,

$u_2 = \left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)(x^2+y^2)$,

$\vdots$

In general, the general term is given by the following relationship

$$u_n(x,y,t) = \sum_{k=0}^{n}\left(\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)(x^2+y^2). \tag{5.11}$$

So, the exact solution of equations (5.8) and (5.9) is the limit of the general term, or in other words

$$\begin{aligned} u(x,y,t) &= \lim_{n\to\infty} u_n(x,y,t) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)(x^2+y^2). \\ &= (E_{\alpha}(-t^{\alpha}) + tE_{\alpha,2}(-t^{\alpha}))(x^2+y^2). \end{aligned}$$

For $\alpha = 2$, then

$$u(x,y,t) = (E_2(-t^2)+tE_{2,2}(-t^2))(x^2+y^2) = (\cos t + \sin t)(x^2+y^2).$$

The surface and behavior of the solution for this example is graphically presented in Figures 5.1 and 5.2 for different fractional orders of $\alpha$.

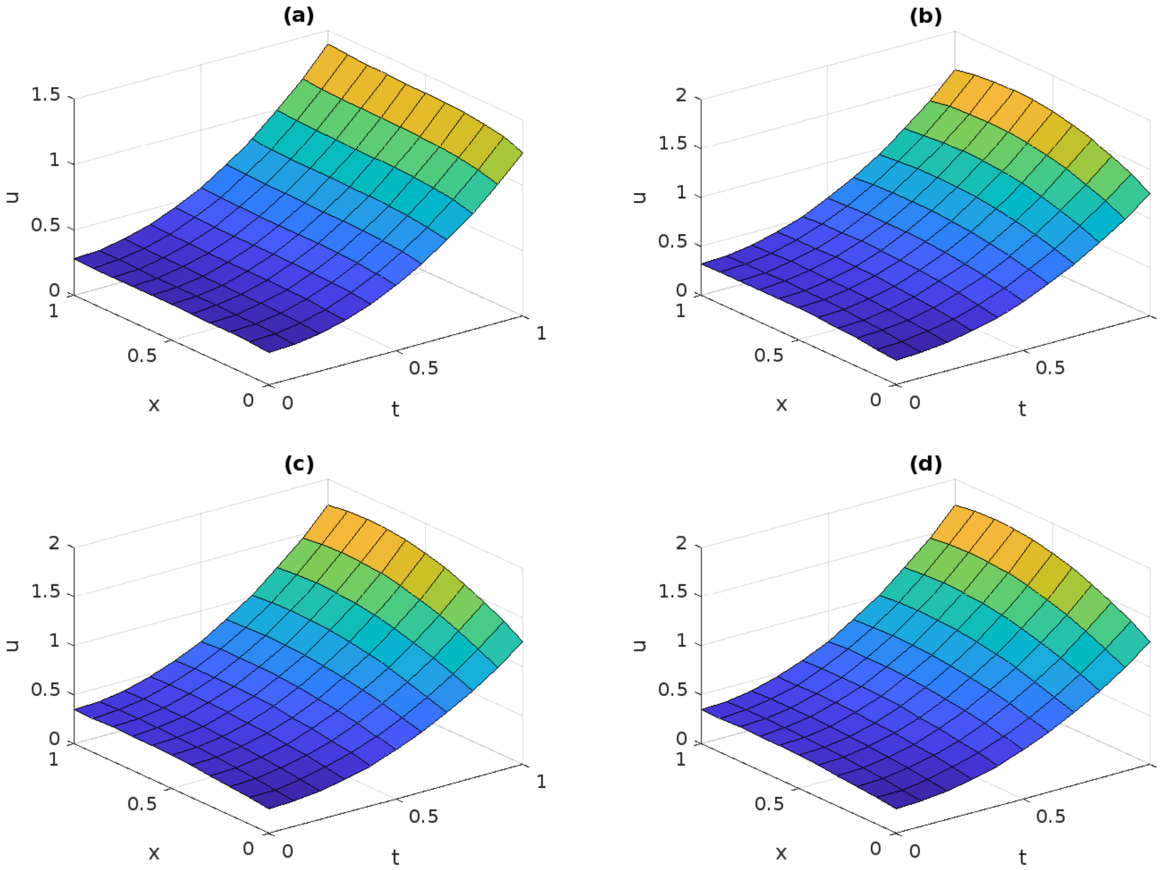

Figure 5.1: The surface graph of the $3^{th}$ order approximate solution by EVIM and the exact solution of Example 5.1 when $y = 0.5$: $(a)$ $u$ when $\alpha = 1.2$, $(b)$ $u$ when $\alpha = 1.8$, $(c)$ $u$ when $\alpha = 2$, and $(d)$ $u$ is exact.

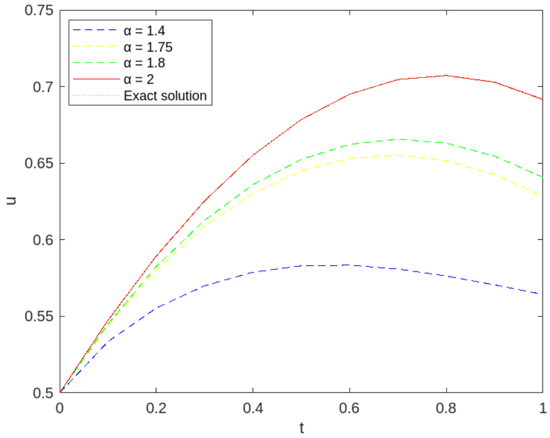

Figure 5.2: The behavior of the exact solution and the $3^{th}$ order approximate solution by EVIM of $u$ for different values of $\alpha$ for Example 5.1 when $x = y = 0.5$.

| $t$ | $\alpha = 1.4$ | $\alpha = 1.75$ | $\alpha = 1.8$ | $\alpha = 2$ | exact solution | $\lvert u_{exact} - u_{EVIM}\rvert$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.5335 | 0.5443 | 0.5451 | 0.5474 | 0.5474 | $7.0424 \times 10^{-10}$ |
| 0.3 | 0.5700 | 0.6087 | 0.6127 | 0.6254 | 0.6254 | $5.2711 \times 10^{-7}$ |
| 0.5 | 0.5830 | 0.6450 | 0.6525 | 0.6785 | 0.6785 | $1.1575 \times 10^{-5}$ |
| 0.7 | 0.5809 | 0.6553 | 0.6657 | 0.7046 | 0.7045 | $8.9104 \times 10^{-5}$ |
| 0.9 | 0.5705 | 0.6425 | 0.6546 | 0.7029 | 0.7025 | $4.1069 \times 10^{-4}$ |

Table 5.1: The numerical values of the $3^{th}$ order approximate solution for Example 5.1 when $x = y = 0.5$.

**Example 5.2** ([7]) Consider the following 2-dimensional nonlinear time-fractional partial differential equation

$$ {}^{c}D_t^{\alpha} u = u_x^2 \frac{\partial^2}{\partial x \partial y}(x u_y) - \frac{1}{4} u_{yy} \frac{\partial^2}{\partial x^2}(u_x u_y) - 2u_x^2 u_{yy} + u, \tag{5.12} $$

with the initial conditions

$$ u(x,y,0) = \exp(xy), \quad \frac{\partial u}{\partial t}(x,y,0) = \exp(xy), \tag{5.13} $$

where ${}^{c}D_t^{\alpha}$ is the Liouville-Caputo fractional derivative operator of order $\alpha (1 < \alpha \leqslant 2)$, and $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Using (5.7) we get the following correct function

$$ \begin{aligned} u_{n+1}(x,y,t) &= (1+t)\exp(xy) + E^{-1}\left(\rho^{\alpha} E^{+}[u_{nx}^2 \frac{\partial^2}{\partial x \partial y}(x u_{ny}) - \frac{1}{4} u_{nyy} \frac{\partial^2}{\partial x^2}(u_{nx} u_{ny})]\right) \\ & + E^{-1}\left(\rho^{\alpha} E^{+}[-2u_{nx}^2 u_{nyy} + u_n]\right). \end{aligned} \tag{5.14} $$

According to the iteration formula 5.14, we have

$u_0 = (1+t)\exp(xy)$,

$u_1 = \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\exp(xy)$,

$u_2 = \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)\exp(xy)$,

$\vdots$

In general, the general term is given by the following relationship

$$ u_n(x,y,t) = \sum_{k=0}^{n} \left(\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)\exp(xy). \tag{5.15} $$

So, the exact solution of equations (5.12) and (5.13) is the limit of the general term, or in other words

$$ \begin{aligned} u(x,y,t) &= \lim_{n\to\infty} u_n(x,y,t) = \sum_{k=0}^{\infty} \left(\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)\exp(xy) \\ &= (E_{\alpha}(t^{\alpha}) + tE_{\alpha,2}(t^{\alpha}))\exp(xy). \end{aligned} $$

For $\alpha = 2$, then

$$u(x,y,t) = (E_2(t^2) + tE_{2,2}(t^2))\exp(xy) = \exp(xy+t).$$

The surface and behavior of the solution for this example is graphically presented in Figures 5.3 and 5.4 for different fractional orders of $\alpha$.

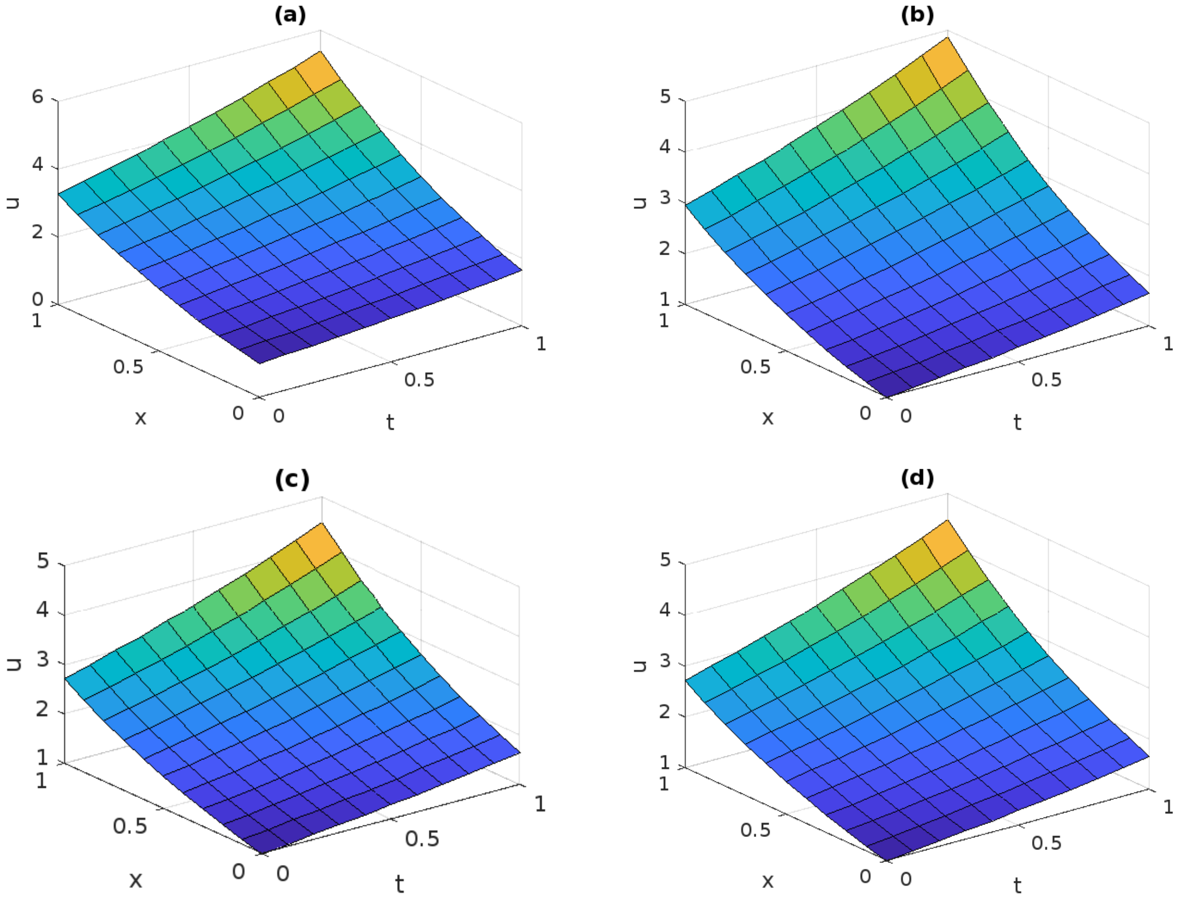

Figure 5.3: The surface graph of the $3^{th}$ order approximate solution by EVIM and the exact solution of Example 5.2 when $y = 0.5$: $(a)$ $u$ when $\alpha = 1.5$, $(b)$ $u$ when $\alpha = 1.75$, $(c)$ $u$ when $\alpha = 2$, and $(d)$ $u$ is exact.

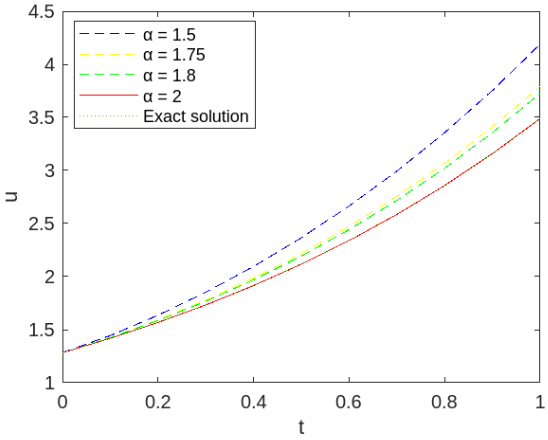

Figure 5.4: The behavior of the exact solution and the $3^{th}$ order approximate solution by EVIM of $u$ for different values of $\alpha$ for Example 5.2 when $x = y = 0.5$.

| $t$ | $\alpha = 1.5$ | $\alpha = 1.75$ | $\alpha = 1.8$ | $\alpha = 2$ | exact solution | $\lvert u_{exact} - u_{EVIM}\rvert$ |
|---|---|---|---|---|---|---|
| 0.1 | 1.4444 | 1.4272 | 1.4250 | 1.4191 | 1.4191 | $1.8092 \times 10^{-9}$ |
| 0.3 | 1.8532 | 1.7787 | 1.7677 | 1.7333 | 1.7333 | $1.3580 \times 10^{-6}$ |
| 0.5 | 2.3659 | 2.2174 | 2.1940 | 2.1170 | 2.1170 | $2.9987 \times 10^{-5}$ |
| 0.7 | 2.9932 | 2.7560 | 2.7173 | 2.5855 | 2.5857 | $2.3278 \times 10^{-4}$ |
| 0.9 | 3.7524 | 3.4125 | 3.3554 | 3.1471 | 3.1582 | $1.0848 \times 10^{-3}$ |

Table 5.2: The numerical values of the $3^{th}$ order approximate solution for Example 5.2 when $x = y = 0.5$.

**Example 5.3** ([7]) Consider the following example, which we consider the 2-dimensional nonlinear time-fractional partial equation

$$ {}^{c}D_t^{\alpha} u = y^2 \frac{\partial^2}{\partial x \partial y}(x^2 u_x u_y) - 3xyu_x u_y - u, \tag{5.16}$$

with the initial conditions

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = xy, \tag{5.17}$$

where ${}^{c}D_t^{\alpha}$ is the Liouville-Caputo fractional derivative operator of order $\alpha(1 < \alpha \leqslant 2)$, and $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Using (5.7) we get the following correct function

$$u_{n+1}(x, y, t) = txy + E^{-1}\left(\rho^{\alpha} E^{+}[y^2 \frac{\partial^2}{\partial x \partial y}(x^2 u_{nx} u_{ny}) - 3xyu_{nx}u_{ny} - u_n]\right). \tag{5.18}$$

According to the iteration formula (5.18), we have

$u_0 = txy,$

$u_1 = \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) xy,$

$u_2 = \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) xy,$

$\vdots$

In general, the general term is given by the following relationship

$$u_n(x, y, t) = \sum_{k=0}^{n} \left(\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)(xy). \tag{5.19}$$

So, the exact solution of equations (5.16) and (5.17) is the limit of the general term, or in other words

$$\begin{aligned} u(x, y, t) &= \lim_{n\to\infty} u_n(x, y, t) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)(xy) \\ &= (tE_{\alpha,2}(-t^{\alpha}))(xy). \end{aligned}$$

For $\alpha = 2$, then

$$u(x, y, t) = (tE_{2,2}(-t^2))(xy) = xy\sin(t).$$

The surface and behavior of the solution for this example is graphically presented in Figures 5.5 and 5.6 for different fractional orders of $\alpha$.

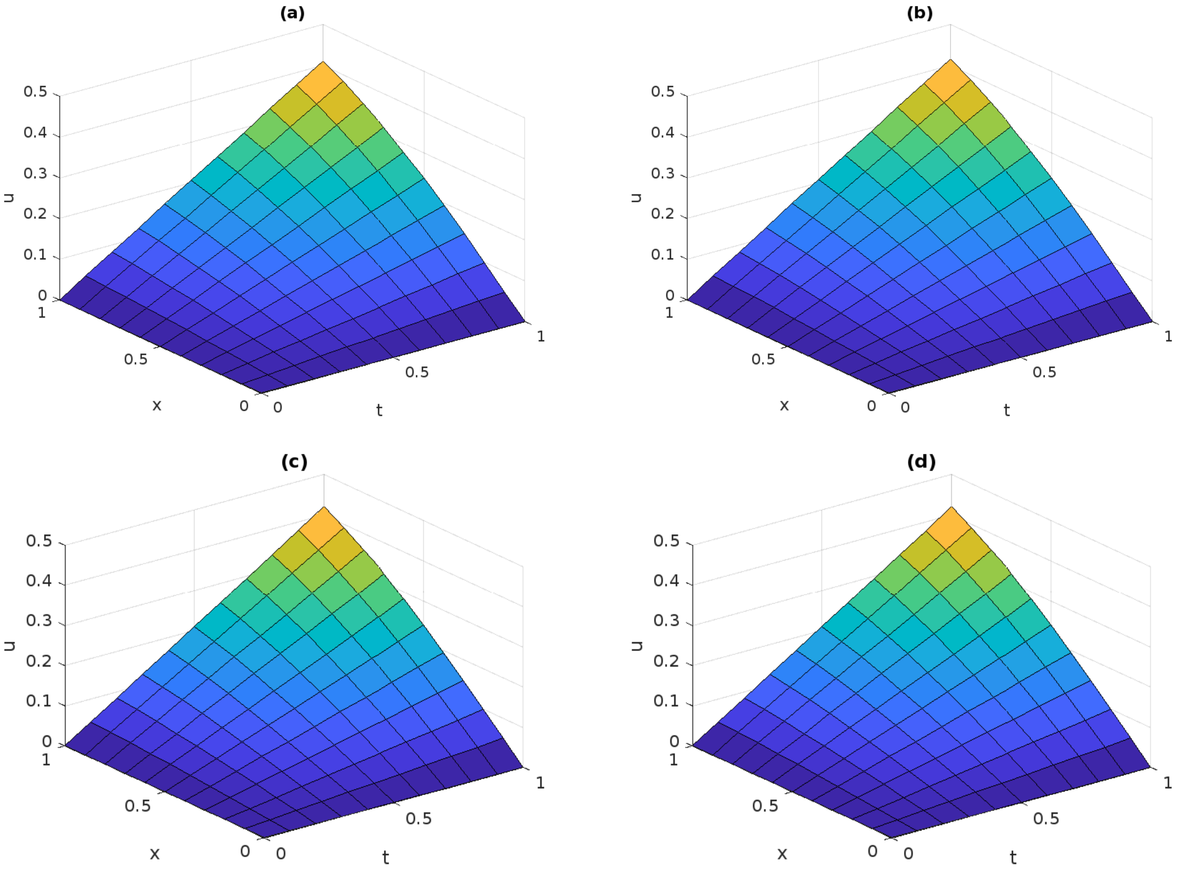

Figure 5.5: The surface graph of the $3^{th}$ order approximate solution by EVIM and the exact solution of Example 5.3 when $y = 0.5$: $(a)$ $u$ when $\alpha = 1.9$, $(b)$ $u$ when $\alpha = 1.95$, $(c)$ $u$ when $\alpha = 2$, and $(d)$ $u$ is exact.

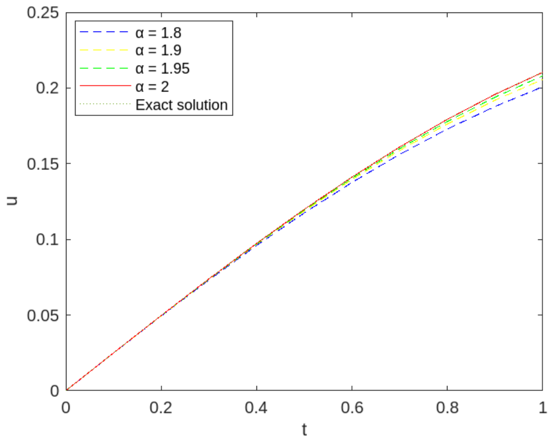

Figure 5.6: The behavior of the exact solution and the $3^{th}$ order approximate solution by EVIM of $u$ for different values of $\alpha$ for Example 5.3 when $x = y = 0.5$.

| $t$ | $\alpha = 1.8$ | $\alpha = 1.9$ | $\alpha = 1.95$ | $\alpha = 2$ | exact solution | $\lvert u_{exact} - u_{EVIM} \rvert$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.0249 | 0.0249 | 0.0250 | 0.0250 | 0.0250 | $4.9596 \times 10^{-12}$ |
| 0.3 | 0.0732 | 0.0736 | 0.0737 | 0.0739 | 0.0739 | $1.0835 \times 10^{-8}$ |
| 0.5 | 0.1175 | 0.1188 | 0.1193 | 0.1199 | 0.1199 | $3.8618 \times 10^{-7}$ |
| 0.7 | 0.1562 | 0.1588 | 0.1599 | 0.1611 | 0.1611 | $4.0574 \times 10^{-6}$ |
| 0.9 | 0.1878 | 0.1920 | 0.1940 | 0.1959 | 0.1958 | $2.3460 \times 10^{-5}$ |

Table 5.3: The numerical values of the $3^{th}$ order approximate solution for Example 5.3 when $x = y = 0.5$.

## 5.4 Conclusion

In this chapter, the Elzaki transform method and the variational iteration method are combined to form a new technique called the Elzaki variational iteration method (EVIM) that helps in finding the exact solutions to nonlinear time-fractional partial differential equations. Looking at the technique mentioned in reference [38], it turns out that this new technique is direct and fast, and to confirm that, this technique was dropped on three various numerical examples. The numerical results shown above confirm the accuracy and ease of this technique. This technique can be extended to apply to other fractional partial differential equations.

# CONCLUSION

In this thesis, we have studied two nonlinear non-local fractional elliptic problems one involving the fractional Laplacian operator and the other involving the fractional p(x,y)-Laplacian with Dirichlet boundary condition. We established existence theorems using the theory of topological degree and the method of monotone operators. The difficulties encountered are variable exponent equations. This led us to work on non-classical fractional Sobolev spaces.

In perspective, we plan to study problems of the same nature by applying variational methods such as the mountain pass theorem and Ekeland's variational principle and we also seek to compensate the second member $f(x,u)$ with what is called the convection term $f(x,u,\nabla u)$.

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