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Université 20 Août 1955 -Skikda
Faculté des Sciences
Département des Mathématiques
Réf: ………………

جامعة 20 أوت 1955 -سكيكدة
كلية العـلوم
قسم الرياضيات
المرجع:..........

Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques

Filière : ***Mathématiques***
Spécialité: ***Mathématiques appliquées***

Existence et comportement asymptotique de la solution d'un problème de type hyperbolique à retard

Présentée par:

BOULKHELOUA Chaima

Soutenue publiquement : ../../…. à …h

Directeur de thèse : Khochemane Houssem Eddine, MCA, ENSET, Azzaba, Skikda.

Co-Directeur de thèse : Bouzettouta Lamine, MCA, Université 20 Août 1955, Skikda.

Devant les Jury:

Lallouche Abdllah	MCA	Université 20 Août 1955, Skikda	**Président**
Hebhoub Fahima	MCA	Université 20 Août 1955, Skikda	**Examinatrice**
Boucenna Djalal	MCA	ENSET, Azzaba, Skikda	**Examinateur**
Meziani Mohamed Sief Eddine	MCA	ENSET, Azzaba, Skikda	**Examinateur**

Année Universitaire: 2025/2026

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جامعة 20 أوت 1955 -سكيكدة
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قسم الرياضيات
المرجع:………..

Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: ***Mathematics***
Speciality: ***Applied Mathematics***

Existence and asymptotic behavior for the solution of a delayed hyperbolic problem

Presented by:

BOULKHELOUA Chaima

Publicly discussed : ….…….

Supervisor of thesis : Khochemane Houssem Eddine, MCA, ENSET, Azzaba, Skikda.

Co- Supervisor of thesis : Bouzettouta Lamine, MCA, 20 August 1955 University of Skikda.

In front of the Jury:

Lallouche Abdllah	MCA	20 August 1955 University of Skikda	**President**
Hebhoub Fahima	MCA	20 August 1955 University of Skikda	**Examiner**
Boucenna Djalal	MCA	ENSET, Azzaba, Skikda	**Examiner**
Meziani Mohamed Sief Eddine	MCA	ENSET, Azzaba, Skikda	**Examiner**

University year: 2025/2026

Dedicate

To the one who brought me up young and sheperded me big.. To those whose prayers were the secret of my success.. To the source of tenderness " my mother and my father ".

To the one who has accompanied me since we carried small bags and with her I walked the path step by step and is still accompanying me until now " my only sister Asma ".

To my teachers and everyone who taught me during my academic career.

To all my friends and to everyone who helped me and stood with me and remembered me in the difficulties of my family and relatives.

To those mentioned by my heart and forgotten by my pen.

To who the heart refuses to forget, and the pen refuses to mention them.

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PUBLICATIONS RESULTING FROM THIS THESIS

❒ Boulkheloua Chaima, Khochemane Houssem Eddine, Bouzettouta Lamine

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ABSTRACT IN ENGLISH

This thesis focuses on the study of the existence and asymptotic behavior of the solution of hyperbolic problems in delay with various heat conduction equations. We are interested in proving the existence of the solutions using the semigroup theory, and injectivity to prove uniqueness. We demonstrate the exponential stability using the energy method by constructing a suitable Lyapunov functional equivalent to the system's energy. In addition, one of the systems was studied numerically. In this regard, we study the following three problems: The first concerned with a delayed swelling porous thermoelastic soils model, the second is related to a heat porous thermoelastic system in the presence of distributed delay without frictional damping, and the last one is devoted to the study of Lord Shulman thermoelastic system with porous damping and distributed delay term.

Key words : *Hyperbolic problems, Delay, Swelling porous system, Porous system, Heat conduction equations, Semigroup theory, Lyapunov functional, Exponential stability*.

ABSTRACT IN FRENCH

Cette thèse porte sur l'étude de l'existence et du comportement asymptotique de la solution de problèmes hyperboliques à retard avec diverses équations de conduction thermique. Nous cherchons à démontrer l'existence des solutions à l'aide de la théorie des semi-groupes et l'injectivity pour démontrer l'unicité. On démontre la stabilité exponentielle par la méthode énergétique en construisant une fonctionnelle de Lyapunov appropriée équivalente à l'énergie du système. De plus, l'un des systèmes a été étudié numériquement. À cet effet, nous étudions les trois problèmes suivants : le premier concerne un modèle de sols thermoélastiques poreux à gonflement retardé ; le deuxième concerne un système thermoélastique poreux à la chaleur en présence d'un retard distribué sans amortissement par frottement ; et le dernier est consacré à l'étude du système thermoélastique de Lord Shulman avec amortissement poreux et terme de retard distribué.

Mots-clés : *Problèmes hyperboliques, Retard, Système poreux gonflant, Système poreux, Équations de conduction thermique, Théorie des semi-groupes, Fonctionnelle de Lyapunov, Stabilité exponentielle*.

ملخص

تركز هذه الأطروحة على دراسة الوجود والسلوك التقاربي للحلول لمسائل من النوع الزائدي بتأخير، وذلك تحت تأثير قوانين مختلفة للتوصيل الحراري. يهدف هذا العمل إلى اثبات وجود الحلول باستخدام نظرية أنصاف الزمر ووحدانيتها باستعمال التباين والتحقق من الاستقرار الأسي لها بالاعتماد على طريقة الطاقة وذلك من خلال بناء دالة ليابونوف المكافئة لدالة الطاقة المرتبطة بالجمل المدروسة. إضافة إلى ذلك، تم اجراء دراسة عددية لاحدى الجمل قيد البحث بهدف توضيح السلوك الديناميكي للحل والتحقق من النتائج النظرية المتحصل عليها. وفي هذا الصدد، تناولت الأطروحة ثلاث مسائل رئيسية: الأولى تتعلق بنموذج تربة- حرارية- مرنة مسامية منتفخة بتأخير، أما الثانية فتعالج وسطا مساميا ذو مرونة حرارية مع تأخير موزع في غياب تأثير التخميد الاحتكاكي، في حين تعنى الأخيرة بدراسة جملة لورد شولمان المرنة الحرارية في ظل تأثير التخميد المسامي والتأخير الموزع.

كلمات مفتاحية: جمل معادلات تفاضلية جزئية زائدية، تأخير، وسط مسامي منتفخ، وسط مسامي، توصيل حراري، نظرية أنصاف الزمر، دالة الطاقة، دالة ليابونوف، استقرار أسي.

NOTATIONS

- PDEs: the abbreviation for " partial differential equations".
- $\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).
- $\mathbb{R}^d$: the real space of dimension d or d-dimensional real Euclidean space.
- Ω: the open domain in $\mathbb{R}^N$
- $L^p(\Omega)$: Lebesgue space.
- $L^p_{loc}(\Omega)$: set of locally integrable functions.
- $W^{k,p}(\Omega)$: the Sobolev space.
- $C^k(\Omega)$: the space of k times continously differentiable functions.
- $C^\infty(\Omega) = \cap_{k\in\mathbb{N}} C^k(\Omega)$.
- $C^\infty_c(\Omega)$: the space of $C^\infty(\Omega)$ functions with compact support.
- $< .,. >$: the scalar product.
- $\|.\|$: the norm.
- C_0: the abbreviation for "Cesàro summable of order 0".
- c_0: a generic positive constant.

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GENERAL INTRODUCTION

Mathematics serves as a fundamental tool across diverse scientific disciplines, providing a universal language to describe and understand the complex phenomena observed in nature. From physics and engineering to biology and economics, mathematics plays a crucial role in modeling and analyzing a wide range of scientific phenomena. This interdisciplinary reliance on mathematics highlights its versatility and power as a tool for scientific inquiry.

In the realm of physics and engineering, mathematics enable the formulation of precise models to describe the behavior of physical systems. Differential equations, for example, are used to model the dynamics of particles, fluids, and electromagnetic fields, allowing scientists and engineers to predict the outcomes of complex interactions and design innovative technologies.

Furthermore, mathematics serves as a bridge between different scientific disciplines, facilitating interdisciplinary collaborations and the integration of knowledge from diverse fields. Through mathematical modeling, scientists can explore connections between seemingly unrelated phenomena and uncover the underlying principles that govern complex systems.

In summary, the need for mathematics in modeling various scientific phenomena transcends disciplinary boundaries, underscoring its importance as a unifying tool in the pursuit of scientific understanding. By harnessing the power of mathematics, researchers across different fields can address complex challenges, make informed predictions, and drive innovation in science and technology.

Partial Differential Equations (PDEs) play a fundamental role in modeling a wide array of phenomena across various fields of science and engineering. At their core, PDEs are mathematical equations that involve multiple independent variables, such as time and space, and partial derivatives of functions. They are employed to describe the behavior of systems that evolve over time and space, making them indispensable tools in understanding natural processes, physical phenomena, and engineering systems.

The significance of PDEs lies in their ability to capture the complex dynamics and intricate relationships present in diverse phenomena. From fluid dynamics to heat transfer, from electromagnetic fields to quantum mechanics, PDEs provide a rigorous framework for quantifying and predicting the behavior of these systems. By expressing physical laws and principles in mathematical terms, PDEs enable researchers and engineers to simulate, analyze, and optimize real-world systems and processes.

One of the key advantages of PDEs is their generality and versatility. They can be tailored to model a vast range of phenomena with varying degrees of complexity, allowing for the investigation of systems at different scales and levels of detail. Whether studying the propagation of waves in the ocean, the diffusion of pollutants in the atmosphere, or the spread of diseases in populations, PDEs offer a unified language to formulate and solve problems across disciplines.

Moreover, the development and analysis of numerical methods for solving PDEs have led to significant advancements in computational science and engineering. These methods enable researchers to approximate solutions of complex PDEs, providing insights into phenomena that may be challenging or impossible to study analytically. Additionally, the advent of high-performance computing has facilitated the simulation of large-scale systems, leading to breakthroughs in fields such as climate modeling, materials science, and astrophysics.

The delay differential equations (DDEs)

In the context of partial differential equations (PDEs), a delay term refers to a mathematical expression that involves the value of a function at some previous time(s) or spatial location(s).

In equations with delay terms, the evolution of the system at any given time or location depends not only on its current state but also in its past states. This introduces a temporal or spatial delay into the equation, which can significantly affect the behavior and stability of the system.

The presence of delay may be a source of instability, as demonstrated in studies [15], where it was proved that an arbitrarily small delay may destabilize a system, which is asymptotically stable in the absence of a delay. Delay terms are often encountered in systems where there is a time lag between cause and effect, such as in models of biological processes, chemical reactions, or systems with feedback loops. These delays can arise due to various reasons, such as the time it takes for signals to propagate through a medium or the time required for processes to occur.

Analyzing and solving PDEs with delay terms often involves numerical methods tailored to handle the additional complexity introduced by the delays. Mathematically, we mention some type of delay for simple differential equation where the time delay is denoted by τ :

Constant time delay

A constant time delay refers to a fixed period of time that elapses between the occurrence of an event and the subsequent response or action. It's denoted by a constant τ, its equation is written as follows:

$$\frac{d}{dt}x\left(t\right)=f\left(t,x\left(t-\tau\right)\right)\ ,$$

Distributed time delay

A distributed time delay refers to a delay term in which the effect of past states of inputs is distributed over a certain spatial region rather than occurring at a single point. This concept is particularly relevant in systems where the propagation of signals or phenomena occurs with a finite speed, resulting in a spatial distribution of time delays. Its equation is written as follows:

$$\frac{d}{dt}x\left(t\right)=f\left(t,\int_{0}^{\tau}\mu\left(s\right)x\left(t-s\right)ds\right)\ ,$$

Varying time delay

A varying time delay refers to a situation where the time between the occurrence of an event and the subsequent response or action fluctuates over time. Unlike a constant time delay, which remains fixed, a varying time delay can change in duration, either gradually or abruptly, due to fluctuations or changes in the system or environment. These fluctuations may arise from factors such as system dynamics, external disturbances, or nonlinear interactions, and they can impact the temporal behavior and stability of the system. It is often represented as $\tau\left(t\right)$, its equation is written as follows:

$$\frac{d}{dt}x\left(t\right)=f\left(t,x\left(t-\tau\left(t\right)\right)\right)\ ,$$

In 1990 , V. Komornik and E. Zuazua [49] proved that the wave equation system

$$\begin{cases} u_{tt}\left(x,t\right)-\Delta u\left(x,t\right)=0\ , & in\ \Omega\times\left(0,\infty\right)\\ \dfrac{\partial u}{\partial v}\left(x,t\right)=F\left(x,u_{t}\right)\ , & in\ \Gamma_{0}\times\left(0,\infty\right)\\ u=0\ , & on\ \Gamma_{1}\times\left(0,\infty\right)\\ u\left(x,0\right)=u^{0}\ ,\ \ u_{t}\left(x,0\right)=u^{1}\ , & in\ \Omega \end{cases} \tag{0.0.1}$$

is exponentially stable.

Where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary Γ, (Γ_0, Γ_1) being partition on Γ, and v is the unit normal vector of Γ pointing towards the exterior of Ω.

In 2006, S. Nicaise and C. Pignotti [66] proposed system (0.0.1) with a delay term in the boundary condition

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \ , \quad in\ \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial v}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau) \ , \quad in\ \Gamma_0 \times (0,\infty) \\ u = 0 \ , \quad on\ \ \Gamma_1 \times (0,\infty) \\ u(x,0) = u^0 \ , \quad u_t(x,0) = u^1 \ , \quad in\ \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau) \quad , \quad in\ \Gamma_0 \times (0,\tau) \end{cases} \tag{0.0.2}$$

They also considered the problem for the wave equation with a delayed velocity term and mixed Dirichlet–Neumann boundary condition

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a(x)\left(\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\right) = 0 \ , \quad in\ \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial v}(x,t) = 0 \ , \quad in\ \Gamma_0 \times (0,\infty) \\ u = 0 \ , \quad on\ \ \Gamma_1 \times (0,\infty) \\ u(x,0) = u^0 \ , \quad u_t(x,0) = u^1 \ , \quad in\ \Omega \\ u_t(x,t-\tau) = g_0(x,t-\tau) \quad , \quad in\ \Gamma_0 \times (0,\tau) \end{cases} \tag{0.0.3}$$

where $a \in L^\infty(\Omega)$ is a function such that

$$a(x) \geq 0 \ \ in\ \Omega \quad and \quad a(x) > a_0 > 0 \ \ in\ \ \omega\,,$$

where $\omega \subset \Omega$ is an open neighborhood of Γ_0 .

In both cases, based on the hypothesis that $\mu_2 < \mu_1$, they prove exponential stability of the solution. If one of the above assumptions is not satisfied, some instability results are also given by constructing some sequences of delays for which the energy of some solutions does not tend to zero. In 2008, [67] they obtained exponential stability results in the case where the distributed delay replaces the constant delay in (0.0.2) - (0.0.3)

$$\int_{\tau_1}^{\tau_2} \mu_2(s)\, u_t(x,t-s)\, ds\,,$$

provided that

$$\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1 .$$

They also provided an example of instability when the following assumption was not verified

$$\|\alpha\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1 .$$

On the other hand, in collaboration with J. Valein [69], they obtained a similar result when we replace the constant delay term in (0.0.2) on the boundary with a varying delay term of the form:

$$\mu_2\, u_t\,(x, t - \tau(t)) \ ,$$

assuming that

$$\mu_2 < \sqrt{1-d}\, \mu_1 \,,$$

where d is a constant such that

$$\tau'(t) < d < 1 \quad , \quad t > 0 \,.$$

Another type of delay called neutral delay is referred to in earlier research [47, 85, 22]

For more examples studying the stability of PDEs with the time delay, the reader can refer to references [48, 68, 70]

Porous Elastic theory

The theory of porous resilience is concerned with studying the behavior of materials containing spaces (pores) filled with liquid or gas, such as soil, sedimentary rock, concrete, sponges, and many biological materials. These blanks significantly affect the properties of mechanical material, such as strength, toughness, and compression.

In the theory of porous resilience, the relationship between flexibility and porosity is essential to understand how materials are deformed and respond to external forces, such as fluid pressure in pores, distribution and size of pores, fluid interaction with porous walls, and the interaction of these factors with mechanical loads. This theory allows modeling the behaviour of natural and geometric materials in more precise ways, helping to improve the design and construction of engineering facilities and analysis of natural phenomena. This theory is widely applied in fields such as soil mechanics, geosciences, and geotechnical engineering.

The theory relies on partial differential equations that describe the equilibrium of forces and displacements in porous materials. The elasticity of the solid framework and the effects of pressures within the voids are taken into account.

We consider the following two basic evolution equations of the one-dimensional porous materials theory:

$$\rho\, u_{tt} = T_x \quad , \quad J\, \varphi_{tt} = H_x + D\ , \tag{0.0.4}$$

where T , H and D represent, respectively, the stress tensor, the equilibrated stress vector and the equilibrated body force with the following constitutive equations

$$T = \mu\, u_x + b\, \varphi \quad , \quad D = -b\, u_x - \xi\, \varphi \quad , \quad H = \delta\, \varphi_x\ , \tag{0.0.5}$$

where $u = u\,(x,t)$ is the displacement of the solid elastic material, $\varphi = \varphi\,(x,t)$ is the volume fraction, ρ is the mass density, J equals to product of the equilibrated inertia by the mass density. The coefficients μ , b , ξ , $\delta$are positive constitutive constants such that

$$\mu\ \xi\ > b^2\,. \tag{0.0.6}$$

Swelling Porous Elastic theory

The theory of swelling porous elasticity is a mathematical and physical model used to describe the behavior of porous materials that swell when exposed to liquids or gases. This theory is particularly applied to materials such as hydrogels, clay soils, polymer membranes, and other substances that can interact with surrounding fluids.

The theory is based on the idea that a porous material consists of a solid framework containing pores that can be filled with fluid. When the fluid is absorbed, the solid framework swells, leading to changes in shape, volume, and mechanical stresses within the material.

The main components of the swelling porous elasticity theory include:

1. **Solid Elasticity:** Describes the mechanical response of the solid framework without considering

the fluid.

2. **Pore Swelling:** Describes how the volume of the pores changes when the fluid is absorbed.

3. **Interaction Between Solid and Fluid:** Describes the mutual forces between the solid framework and the fluid in the pores and how this affects changes in volume and shape.

This theory is used in various applications, including environmental engineering, materials science, and biological sciences, where it can help design new materials and analyze the behavior of natural and industrial systems that interact with fluids.

To describe the relationship between stresses, deformations, and fluid pressure within the pores, we consider the following two basic evolution equations:

$$\rho_u\, u_{tt} = T_x + P_1 + F_1 \quad , \quad \rho_z\, z_{tt} = H_x - P_2 + F_2 \ , \tag{0.0.7}$$

where the constituents u and z represent the displacement of the fluid and elastic solid material. The parameters ρ_u and ρ_z are the densities of each constituent which are assumed to be strictly positive constants. T , H are the partial tensions, F_1 , F_2 are the external forces, P_1 , P_2 are internal body forces associated with the dependent variables u , z . We assume that the constitutive equations of partial tensions are given as in [42] by

$$\begin{pmatrix} T \\ H \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} u_x \\ z_x \end{pmatrix}, \tag{0.0.8}$$

where $\mathcal{M}$ is a positive definite symmetric array, i.e.,

$$\alpha_1\, \alpha_3 \, > \, \alpha_2^2 \, . \tag{0.0.9}$$

Thermoelasticity theory

According to the thermoelasticity theory, a body's deformation is connected to a change in enthalpy and, consequently, a change in temperature. In other words, thermoelasticity is the study of the relationship between a material's thermal conductivity and pressure, as well as its elastic qualities and temperature.

Fourier's law of heat conduction (1822)

In the classical model of heat diffusion, heat flow is governed by Fourier's law of thermal conductivity [34], which is considered as the best foundational equation for modeling heat conduction and states that heat flow is proportional to the temperature gradient. The thermal conductivity equation is given by Fourier's law as

$$q = -\kappa\,\theta_x\ , \tag{0.0.10}$$

where $\kappa > 0$ represents the coefcient of thermal conductivity of the material.

However, this model has a major drawback, as its use leads to physical inconsistency For the infnite speed of heat diffusion, in other words, any thermal disturbance at one point will instantly transfer to other parts of the body. To overcome this paradox, other theories of thermoelasticity have emerged.

Cattaneo Maxwell's heat conduction (1948)

The Cataneo Maxwell Thermal Conductivity Equation [12, 13] is an adjustment to the classic Furier model of thermal conductivity. This equation aims to address some limitations in the Ferrer model, especially in cases that require a more accurate description of thermal conductivity in short-time materials or small distances, such as nanomaterials or when dealing with thermal pulses. Cataneo Maxwell's equation is to add a time limit to describe the time response to thermal flow, and appears as follows:

$$\tau_0\,q_t + q + \kappa\,\theta_x\ = 0\,, \tag{0.0.11}$$

where τ_0 is a positive constant represents the time lag in the response of the heat flux to the temperature gradient and is referred to as the thermal relaxation time.

When the thermal relaxation period τ_0 is not too small, the Cataneo Maxwell's equation can show it wavelength solutions. This means that heat can spread as a wave rather than a normal spread, and this is the theoretical basis of the second sound phenomenon. This phenomenon usually appears at very low temperatures, such as helium II and high purity crystal forms of solids.

Green and Naghdi Theories

Green and Naghdi [36, 37] proposed a way to eliminate the paradox of infnite velocities, they used an analogy between the concepts and equations of purely thermal theories and purely mechanical theories and came up with three types of constitutive equations for heat flow in a fxed solid cohesive material classifed as type *I* , type *II* and type *III* , where Type *I* leads to the usual thermal conductivity according to Fourier's law. In type *II* and type *III* theories, the constitutive equations for the heat flux are given by

$$q = -f_1\,\psi_x \quad , \quad q = -f_1\,\psi_x - f_2\,\theta_x \ ,$$

respectively, where

$$\psi = \theta_0\,(x) + \int_0^t \theta\,(x, \tau)\,d\tau \ ,$$

is the thermal displacement and f_1 , f_2 are positive constants.

These theories represent an attempt to understand how thermal changes affect materials and composites at the atomic and molecular levels.

Lord and Shulman Theory

This theory [53] is based on the study of a system of four hyperbolic equations with heat dissipation; in this case, the heat equation is also hyperbolic as opposed to the parabolic one obtained from Fourier's law.

$$\rho_1 u_{tt} = t_x \quad , \quad J\varphi_{tt} = h_x + g \quad , \quad \rho_1 T_0 \eta_t = q_x \quad , \quad \rho_1 e_t = P_x + q - Q \ , \tag{0.0.12}$$

Where ρ_1 is the mass density, J is the product of the mass density by the equilibrated inertia, T_0 is the reference temperature at the equilibrium state (which we assume to be equal to one for simplicity), t is the stress tensor, η is the entropy, q is the heat flux vector, h is the equilibrated stress, g is the equilibrated body force, P is the first heat flux moment, Q is the mean heat flux and e is the first moment of energy.

The variables u and φ are, respectively, the displacement of the solid elastic material and the volume fraction.

The constitutive equations are:

$$\begin{aligned}
t &= (\lambda + 2\mu)\,u_x + \mu_0\varphi - \beta_0\,(\tau\theta_t + \theta) & h &= a_0\varphi_x - \mu_2\,(\tau T_t + T)\,, \\
g &= -\mu_0 u_x - \xi\varphi + \beta_1\,(\tau\theta_t + \theta) & \rho_1\eta &= \beta_0 u_x + \beta_1\varphi + a\,(\tau\theta_t + \theta)\,, \\
q &= \delta\theta_x + k_1 T & P &= -k_2 T_x, \\
Q &= (\delta - k_3)\,\theta_x + (k_1 - k_2)\,T & \rho_1 e &= -b\,(\tau T_t + T) - \mu_2\varphi_x.
\end{aligned} \tag{0.0.13}$$

The variable θ denotes the temperature, T denotes the microtemperature, which represents the variation of the temperature inside the microelement, λ and μ are the usual parameters and τ is the relaxation parameter, which is assumed to be small but strictly positive. The coefficients β_0 , β_1 , μ_0 , δ , a_0 are positive constants, denotes respectively, the coupling between the displacement and the temperature, the coupling between the displacement and the volume fraction, the coupling between the displacement and the porosity, the thermal conductivity and the thermal capacity.
The rest of the constituent parameters k_1 , k_2 , k_3 , ξ and μ_2 define the characteristics of the material and, in particular, they define the couplings and satisfy the inequalities:

$$\mu_0{}^2 < \mu_1\xi, \tag{0.0.14}$$

and

$$k_2{}^2 < \delta k_1, \tag{0.0.15}$$

Where $\mu_1 = \lambda + 2\mu$, by substituting (0.0.13) into (0.0.12) , we obtain the following system:

$$\begin{cases} \rho_1 u_{tt} = \mu_1 u_{xx} + \mu_0\varphi_x - \beta_0\left(\tau\,\theta_{tx} + \theta_x\right), & \text{in } (0,1)\times(0,\infty) \\ J\,\varphi_{tt} = a_0\varphi_{xx} - \mu_2\left(\tau\,T_{tx} + T_x\right) - \mu_0 u_x - \xi\varphi + \beta_1\left(\tau\,\theta_t + \theta\right), & \text{in } (0,1)\times(0,\infty) \\ a(\tau\,\theta_t + \theta)_t = -\beta_0\,u_{tx} - \beta_1\varphi_t + \delta\theta_{xx} + k_1 T_x, & \text{in } (0,1)\times(0,\infty) \\ b(\tau\,T_t + T)_t = k_2 T_{xx} - \mu_2\varphi_{tx} - k_2 T - k_3\theta_x, & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{0.0.16}$$

Gurtin and Pipkin Theory

It is important to note that the second sound and type *III* theories cannot adequately explain the memory effect that predominates in specific materials, especially at low temperatures. As a result, a more general fundamental assumption about heat flow to thermal memory is required. In [39] Gurtin and Pipkin prosed that heat flux depends on the integrated history of the weighted temperature gradient against a relaxation function called the heat flux kernel. They developed a general nonlinear theory in which thermal disturbances propagate with a finite speed. According to this theory, the linear constitutive equation for q is given as follows

$$q = -\int_{-\infty}^{t} k\,(t-s)\,\theta_x\,(x,s)\,ds\,, \tag{0.0.17}$$

where $k(s)$ is the heat conductivity relaxation kernal. The presence of the convolution term (0.0.17) renders the porous system coupled with the heat equation into a fully hyperbolic system, this allows the heat to propagate with finite speed and admits to describe the memory effect of heat conduction. We note that many different constitutive models arise from different choices for $k(s)$, in particular, if we take $k(s) = \kappa\,\delta(s)$, where δ is the Dirac mass weighted at 0, then (0.0.17) reduced to the Fourier's law (0.0.10), and if we choose

$$k(s) = \frac{\kappa}{\tau_0} e^{-\frac{s}{\tau_0}} \ , \quad \tau_0 > 0 \, ,$$

we obtained Cattaneo's law (0.0.11). So (0.0.17) is a generalized from Fourier's and Cattaneo's law.

Results Description

The aim of this dissertation is to investigate the well posedness as well as the asymptotic behavior of solutions of some systems for the one-dimensional case.

Our contributions start from the second chapter , based on (0.0.7) , (0.0.8) and the Cattaneo Maxwell's heat conduction (0.0.11) we investigate, under the assumption (0.0.9), the asymptotic stability of solution of the following problem:

$$\begin{cases} \rho_z z_{tt} = \alpha_1 z_{xx} + \alpha_2 u_{xx} \ , & in \ (0,L) \times (0,\infty) \\ \rho_u u_{tt} = \alpha_3 u_{xx} + \alpha_2 z_{xx} + \beta\,\theta_x - \mu\, u_t(x,t-\tau) \ , & in \ (0,L) \times (0,\infty) \\ c\,\theta_t = -q_x + \beta\, u_{tx} \ , & in \ (0,L) \times (0,\infty) \\ \tau_0\, q_t = -q - k\,\theta_x \ , & in \ (0,L) \times (0,\infty) \end{cases} \tag{0.0.18}$$

with the boundary conditions

$$z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = \theta_x(0,t) = \theta(L,t) = q(0,t) = 0 \ , \ t > 0 \tag{0.0.19}$$

and the initial data

$$\begin{cases} z(x,0) = z_0(x) \ , \ z_t(x,0) = z_1(x) \ , \ \theta(x,0) = \theta_0(x) \ , \ x \in (0,L) \\ u(x,0) = u_0(x) \ , \ u_t(x,0) = u_1(x) \ , \ q(x,0) = q_0(x) \ , \ x \in (0,L) \\ u_t(x,-t) = f_0(x,t) \quad in \ (0,L) \times (0,\tau) \end{cases} \tag{0.0.20}$$

where $\theta = \theta(x,t)$ represents the temperature difference, $q = q(x,t)$ is the heat flux and the coefficients

$\alpha_1, \alpha_2, \alpha_3, c, \beta, \tau_0, k$ are positive constants represent the constitutive parameters defining the coupling among the different components of the materials, μ is a real number, $\tau > 0$ represents the time delay. The initial data $z_0, z_1, u_0, u_1, \theta_0, q_0$ belongs to a suitable functional space and f_0 is a history function. We'll prove that the unique dissipation given by the second sound is strong enough to stabilize exponentially the system regardless of the wave speeds of the system or any other condition on coefficients in spite of the existence of the delay without the damping mechanism to control the undesirable delay effects that may be a source of instability of systems which are uniformly asymptotically stable in the absence of delay unless additional control terms have been used (see[16, 15, 38, 87]). Also, the introducing of delay may lead to well-posedness, as shown in many works such as [15, 79] and the references therein.

In chapter three, according to (0.0.4) and (0.0.5), we study the well posedness and exponenetial stability of the following porous thermoelastic system with distributed delay time

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x - \gamma_1 u_t - \int\limits_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, u_t(x, t-\sigma)\, d\sigma, & in\ (0,\pi)\times(0,\infty) \\ J\,\varphi_{tt} = \alpha\,\varphi_{xx} - b\,u_x - \xi\,\varphi + \delta\,\theta - \tau\,\varphi_t, & in\ (0,\pi)\times(0,\infty) \\ c\,\theta_t = -q_x - \beta\,u_{xt} - \delta\,\varphi_t, & in\ (0,\pi)\times(0,\infty) \end{cases} \tag{0.0.21}$$

with the boundary conditions and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \ , \ t > 0, \\ u(x,0) = u_0(x) \ , \ \varphi(x,0) = \varphi_0(x) \ , \ \theta(x,0) = \theta_0(x) \ , \ x \in (0,\pi), \\ u_t(x,0) = u_1(x) \ , \ \varphi_t(x,0) = \varphi_1(x) \ , \ x \in (0,\pi), \\ u_t(x,-t) = f_0(x,t) \ , \ x \in (0,\pi) \ , \ t \in (0,\tau_2), \end{cases} \tag{0.0.22}$$

where $u = u(x,t)$ is the transversal displacement, $\varphi = \varphi(x,t)$ is the volume fraction, $\theta = \theta(x,t)$ is the temperature variation from an equilibrium reference value and $q = q(x,t)$ is the heat flux. The coefficients $\rho_1,\ J,\ c,\ \mu,\ \alpha,\ b,\ \xi,\ \tau,\ \gamma_1$ are positive constitutive constants such that (0.0.6) holds.

The coefficient β and δ are the coupling constants that are different from zero but their signs does not matter in the analysis. The term $\int\limits_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, u_t(x, t-\sigma)\, d\sigma$ is a distributed delay that acting only on the porous equation and $\gamma_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. The initial data $u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ f_0$ belongs to a suitable functional space.

And the heat conduction equation is given by Gurtin Pipkin thermal law (0.0.17). We'll establish an exponential stability result without any restriction on the coefficients.

In the last chapter we consider the problem (0.0.16) with distributed delay time acting on the porous equation in the absence of micro-temperature effect, taking into account the parameters μ_2, k_1, k_2, k_3 equal to zero. We get the following system

$$\begin{cases} \rho_1 u_{tt} = \mu_1 u_{xx} + \mu_0 \varphi_x - \beta_0 (\tau \theta_{tx} + \theta_x), \\ J \varphi_{tt} = a_0 \varphi_{xx} - \mu_0 u_x - \xi \varphi + \beta_1 (\tau \theta_t + \theta) - \gamma_1 \varphi_t - \int\limits_{\tau_1}^{\tau_2} \gamma_2 (s) \varphi_t (x, t-s) ds, \\ a(\tau \theta_t + \theta)_t = -\beta_0 u_{tx} - \beta_1 \varphi_t + \delta \theta_{xx} = 0, \end{cases} \tag{0.0.23}$$

where $(x,t) \in (0,1) \times (0, \infty)$.

With the initial data:

$$\begin{array}{llll} u(x,0) = u_0(x) & \varphi(x,0) = \varphi_0(x), & \theta(x,0) = \theta_0(x) & \\ u_t(x,0) = u_1(x) & \varphi_t(x,0) = \varphi_1(x), & \theta_t(x,0) = \theta_1(x) & x \in (0,1) \\ \varphi_t(x,-t) = f_0(x,t) & x \in (0,1) \quad , \quad t \in (0,\tau_2), & & \end{array}$$

and Neumann-Dirichlet boundary conditions

$$u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = \theta(0,t) = \theta(1,t) = 0 \quad , \quad t \geq 0.$$

The initial data $u_0, u_1, \varphi_0, \varphi_1, \theta_0, \theta_1$ belongs to the suitable functional space, the term $\int\limits_{\tau_1}^{\tau_2} \gamma_2(s) \varphi_t(x, t-s) ds$ is a distributed delay that acts only on the porous equation and $\gamma_2 : [\tau_1, \tau_2] \to IR$ is a bounded function, where τ_1, τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$.

We'll show that the dissipations due to thermal effects with porous damping are strong enough to stabilize the system exponentially, independently of the wave speeds of the system.

Methodology

We use the multiplier method to establish the desired stability results of the systems. Multiplier method relies mostly on the construction of an appropriate Lyapunov functional $\mathcal{L}$ equivalent to the energy of the solution E. By equivalence $\mathcal{L} \sim E$, we mean

$$a\,E\,(t) \leq \mathcal{L}\,(t) \leq b\,E\,(t) \quad , \quad \forall t \geq 0\,, \tag{0.0.24}$$

for two positive constants a and b. To prove the exponential stability, we show that $\mathcal{L}$ satisfies

$$\mathcal{L}'\,(t) \leq -\lambda\,\mathcal{L}\,(t)\ ,\ \forall t \geq 0\,, \tag{0.0.25}$$

for some $\lambda > 0$. A simple integration of (0.0.25) over $(0, t)$ together with (0.0.24) gives the desired exponential stability result.

In the numerical part, we discretize the problem using an Euler scheme combined with the finite difference method in order to approximate the temporal and spatial derivatives, respectively. We then apply a fixed-point iterative algorithm to solve the resulting discrete system. Furthermore, we establish a sufficient condition that guarantees the convergence of this iterative process. The obtained numerical results confirm the theoretical analysis and highlight that the system exhibits a pronounced sensitivity to even small time delays, emphasizing the crucial role of delay parameters in the system's dynamic behavior.

Chapter 1

The Conceptual And Theoretical Framework Of The Thesis

In this chapter, we recall some basic knowledge in functional analysis and semigroup theory, most of which will be used in the subsequent chapters. The reader can easily find the detailed proofs in the related literature. see, G. Allaire [1], H. Brezis [10], T. Cazenave and A. Hareaux [14], L. Debnath and P. Mikusinski [17], I. Ellouze [23], L.C. Evans [25], H. K. Khalil [44], A. Pazy [76] and W. Rudin [82].

1.1 Concept of functional analysis

In this section we shall introduce and state some necessary materials needed in the proof of our results, and shortly the basic results which concerning the Banach spaces, the L^P space, Sobolev spaces and some important inequalities.

1.1.1 Banach spaces

Let $(E; \|\|_E)$ be a normed space and (x_n) sequence of E .

Definition 1.1.1. *[17]*
A sequence (x_n) is called a Cauchy sequence if for any given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\forall p\,,\, q \geq N_\varepsilon \;;\; \left\|x_p - x_q\right\| < \varepsilon\,.$$

Definition 1.1.2. *(**Complete space**)[17]*
A normed space E is called complete if every Cauchy sequence in E converges to an element of E.

Definition 1.1.3. *(**Banach space**)[17]*
A complete normed space is called a Banach space.

Definition 1.1.4. *(**Banach algebra**)[82]*
A complex algebra is a vector space E over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies

1. $x\,(yz) = (xy)\,z$,
2. $(x+y)\,z = xz + yz, \qquad x\,(y+z) = xy + xz$,
3. $\alpha\,(xy) = (\alpha x)\,y = x\,(\alpha y)$,

for all x, y and z in E and for all scalars α.

If, in addition, E is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \, \|y\| \quad (x \in E, \ y \in E),$$

and if E contains a unit element e such that

$$xe = ex = x \quad (x \in E),$$

and

$$\|e\| = 1,$$

then E is called a Banach algebra.

1.1.2 Hilbert spaces

Definition 1.1.5. ***(Linear form)***

A linear form $\varphi(.)$ is a linear application of a real vector space E in $\mathbb{R}$ such that for all $\alpha \in \mathbb{R}$ and all $u, v \in E$

1. $\varphi(\alpha u) = \alpha \varphi(u)$,
2. $\varphi(u + v) = \varphi(u) + \varphi(v)$.

Definition 1.1.6. ***(Bilinear form)***

A bilinear form $\varphi(.,.)$ on a real vector space E is an application of $E \times E$ in $\mathbb{R}$ such that for all $\alpha, \beta \in \mathbb{R}$ and all $u, v, u_1, u_2, v_1, v_2 \in E$

1. $\varphi(\alpha u_1 + \beta u_2, v) = \alpha \varphi(u_1, v) + \beta \varphi(u_2, v)$,
2. $\varphi(u, \alpha v_1 + \beta v_2) = \alpha \varphi(u, v_1) + \beta \varphi(u, v_2)$.

That is to say φ is linear with respect to each variable.

Remark 1.1.1. ***(Symmetrical)***

The bilinear form $\varphi(.,.)$ is said to be symmetrical if for all $u, v \in E$

$$\varphi(u, v) = \varphi(v, u) .$$

Definition 1.1.7. ***(Inner product)***

Let E be a vector space on the field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{C}$). A inner product on E is an application $\phi : E \times E \to \mathbb{K}$ such that for all u, u_1, u_2, $v \in E$ and $\lambda \in \mathbb{K}$ we have

1. $\phi(u,u) \geq 0$ *and* $\phi(u,u) = 0$ *if and only if* $u = 0$,
2. $\phi(u,v) = \overline{\phi(v,u)}$,
3. $\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v)$,
4. $\phi(\lambda u, v) = \lambda \phi(u,v)$.

Definition 1.1.8. *(**Semi norm, norm**)*

Let E be a vector space on the field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{C}$). We call semi norm on E the application $N : E \to \mathbb{R}_+$ that verifies $\forall u$, $v \in E$, $\forall \alpha \in \mathbb{K}$

1. $N(\alpha u) = |\alpha| N(u)$, *where* $|\,.\,|$ *refers to the absolute value if* $\mathbb{K} = \mathbb{R}$ *or module if* $\mathbb{K} = \mathbb{C}$,
2. $N(u+v) \leq N(u) + N(v)$,
3. *Also if N verifies: $N(u) = 0 \Leftrightarrow u = 0$, we said that N is a norm on E and we note N by $\|\,.\,\|_E$.*

Definition 1.1.9. *(**Prehilbertian space, Hilbert space**)[10]*

Let E be a vector space on the field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{C}$).

We say that E is a prehilbertian space if it equipped with a scalar product $\langle\,.\,,\,.\,\rangle_E$.

We say that the space E equipped with a scalar product is a Hilbert space if it is complete for the associated norm $\|\,.\,\|_E$ such that

$$\|u\|_E = \sqrt{(u,u)_E} \ , \quad \forall u \in E\,.$$

1.1.3 Fuctional spaces

1.1.3.1 The $L^p(\Omega)$ spaces

Definition 1.1.10. *[10]*

Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^N$, $n \in \mathbb{N}$. Define the standard Lebesgue space; $L^p(\Omega)$, by

$$L^p(\Omega) = \left\{ f : \Omega \to R \ \text{ is measurable and } \int_\Omega |f(x)|^p dx < +\infty \right\}.$$

Notation 1.1.1. *For $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^{\infty}(\Omega) = \Big\{ f : \Omega \to R \text{ is measurable, } \exists C \in R \text{ such that, } |f(x)| \leq C \text{ a.e in } \Omega \Big\} .$$

Also, we denot by

$$\|f\|_{\infty} = inf \{C, |f(x)| \leq C \text{ a.e in } \Omega\} .$$

Theorem 1.1.1. ***(Fischer–Riesz)****[10]*
It is well known that $L^{p}(\Omega)$ supplied with the norm $\|f\|_{p}$ is a Banach space, for all $1 \leq p \leq \infty$.

Remark 1.1.2. *In particularly, when $p = 2$, $L^{2}(\Omega)$ equipped with the inner product*

$$(f, g)_{L^{2}(\Omega)} = \int_{\Omega} f(x) g(x) dx ,$$

is a Hilbert space.

1.1.3.2 Distribution spaces

Definition 1.1.11. *Let Ω be a bounded open of $\mathbb{R}^{N}$, a test function φ is a function defined on Ω infinitely differentiable with compact support.*

We denote by $D(\Omega)$ or $C_{c}^{\infty}(\Omega)$ the test functions space, such that

$$D(\Omega) = \{\varphi \in C^{\infty}(\Omega) \ ; \ supp(\varphi) \subset \Omega \text{ compact}\} ,$$

where

$$supp(\varphi) = \overline{\{x \in \Omega \ ; \ \varphi(x) \neq 0\}} .$$

We call distribution any continuous linear function on vector space $D(\Omega)$.
The set of distributions is a vector space noted $D'(\Omega)$ such that

$$\langle u , \varphi \rangle_{D \times D'} = \langle u_i , \varphi_i \rangle \quad , \quad \forall u \in D'(\Omega) \ , \ \varphi \in D(\Omega) .$$

Definition 1.1.12. *Let v be a function of $L^{2}(\Omega)$. we said that v is derivable in the weak sense in $L^{2}(\Omega)$*

if there are functions $w_i \in L^2(\Omega)$ *for* $i \in \{1, ..., N\}$ *such that*

$$\int_{\Omega} v(x) \frac{\partial \phi}{\partial x_i}(x)\, dx = -\int_{\Omega} w_i(x)\, \phi(x)\, dx \quad , \quad \forall \phi \in D(\Omega) \; .$$

Each w_i *is called the weak partial derived of* v *and noted* $\frac{\partial v}{\partial x_i}$.

Lemma 1.1.1. *Let* $f \in L^2(\Omega)$, *if for any function* $\phi \in D(\Omega)$ *we have*

$$\int_{\Omega} f(x)\, \phi(x)\, dx = 0 \,,$$

then $f(x) = 0$ *almost every where in* Ω.

1.1.3.3 The Sobolev space $W^{m,p}(\Omega)$

Sobolev spaces play a fundamental role in variational computation. They owe their name to the Russian mathematician Sergei Lvovich Sobolev (1908-1989).

Sobolev space $W^{1,p}(\Omega)$

Definition 1.1.13. *[10]*
Let Ω *be a bounded open of* $\mathbb{R}^N$, *and* $p \in \mathbb{R}$ *where* $1 \leq p < \infty$, *we define the Sobolev space* $W^{1,p}(\Omega)$ *by*

$$W^{1,p}(\Omega) = \left\{ \begin{array}{l} u \in L^p(\Omega) \,,\, \exists v \in L^p(\Omega) \;; \\ \int_{\Omega} u(x)\, \varphi'(x)\, dx = -\int_{\Omega} v(x)\, \varphi(x)\, dx \;;\; \forall \varphi \in C_c^1(\Omega) \end{array} \right\} .$$

For $u \in W^{1,p}(\Omega)$ we denote $u' = v$ (in the weak sense).

Sobolev space $W^{m,p}(\Omega)$

Definition 1.1.14. *[10]*
Let Ω *be a bounded open of* $\mathbb{R}^N$, $m \geq 2$ *and a real* p, $1 \leq p < \infty$, *we define the Sobolev space* $W^{m,p}(\Omega)$ *by*

$$W^{m,p}(\Omega) = \left\{ \begin{array}{l} u \in L^p(\Omega) \quad , \;\forall \alpha \;\; where \;\; |\alpha| \leq m \,,\; \exists v_\alpha \in L^p(\Omega) \;; \\ \int_{\Omega} u\, D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_{\Omega} v_\alpha\, \varphi\, dx \quad ; \quad \forall \varphi \in C_c^\infty(\Omega) \end{array} \right\} ,$$

where $\alpha \in N^n$, $|\alpha| = \sum_{i=1}^{N} \alpha_i$ *and* $D^\alpha \varphi = \dfrac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}}$.

Theorem 1.1.2. *[14]*
$W^{m,p}(\Omega)$ *is is a Banach space with their usual norm*

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0\, <\, |\alpha|\, \leq m} \|D^\alpha u\|_{L^p(\Omega)}\,.$$

Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.

Definition 1.1.15. *When* $p = 2$*, we prefer to denote by* $W^{m,2}(\Omega) = H^m(\Omega)$ *and* $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ *supplied with the norm*

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\, \leq m} \|D^\alpha u\|_2^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ is a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha|\, \leq m} \int_\Omega D^\alpha u\, D^\alpha v\, dx\,.$$

1.1.4 Elliptic equations

We consider the following problem

$$\begin{cases} Lu = f & in\, \Omega, \\ u = 0 & on\, \partial\Omega. \end{cases} \tag{1.1.1}$$

Where Ω is an open , bounded subset of $\mathbb{R}^n$, and $u : \overline{\Omega} \to \mathbb{R}$ is the unknown, $u = u(x,t)$. Here $f : \Omega \to \mathbb{R}$ is given, and L denotes a second-order partial differential operator.

Theorem 1.1.3. *(**Green formula**)[1]*
For all $u \in H^2(\Omega)$, $v \in H^1(\Omega)$ *we have*

$$-\int_\Omega \Delta u\, v\, dx = \int_\Omega \nabla u\, \nabla v\, dx - \int_\Gamma \frac{\partial u}{\partial \eta} v\, d\sigma\,,$$

where $\dfrac{\partial u}{\partial \eta}$ *is a normal derivation of* u *at* Γ.

Theorem 1.1.4. *(**Lax Milgram theorem**)[1]*
Assume that $a(u,v)$ is a continuous coercive bilinear form on H. Then, given any $\varphi \in H^*$, there exists a unique element $u \in H$ such that

$$a(u,v) = \langle \varphi, v\rangle \quad \forall v \in H\,.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H \;\; and \;\; \frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}.$$

Definition 1.1.16. *(**Weak solution**)[25]*
We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.1.1) *if*

$$a(u,v) = \langle f, v\rangle \;\; , \quad \forall v \in H_0^1(\Omega)\,. \tag{1.1.2}$$

Remark 1.1.3. *[25]. We note that a problem with prescribed, nonzero boundary values can easily be stransformed into zero boundary conditions. We spell this out by supposing now that $\partial\Omega \in C^1$ and $u \in H^1(\Omega)$ is a weak solution of*

$$\begin{cases} Lu = f \;\; in\,\Omega, \\ u = g \quad\; on\,\partial\Omega. \end{cases}$$

This means that $u = g$ on $\partial\Omega$ in the trace sense, and furthermore that the bilinear form (1.1.2) *holds for all $u \in H_0^1(\Omega)$. That is be possible, it is necessary for g to be the trace of some H^1 function, say w. But then $\tilde{u} := u - w$ belongs to $H_0^1(\Omega)$, and is a weak solution of the boundary-value problem*

$$\begin{cases} L\tilde{u} = \tilde{f} \;\; in\,\Omega, \\ \tilde{u} = 0 \quad\; on\,\partial\Omega, \end{cases}$$

where $\tilde{f} := f\,,\, Lw \in H^{-1}(\Omega)\,.$

1.1.4.1 Regularity of Weak Solutions

Definition 1.1.17. *[10]*
We say that an open set Ω is of class $C^m, m \geq 1$ an integer, if for every $x \in \Gamma$ there exist a neighborhood U

of $x \in \mathbb{R}^N$ and a bijective mapping $H : Q \to U$ such that

$$H \in C^m\left(\overline{Q}\right) , \quad H^{-1} C^m\left(\overline{U}\right) , \; H\left(Q_+\right) = U \cap \Omega , \quad H\left(Q_0\right) = U \cap \Gamma \, .$$

We say that Ω is of class C^∞ if it is of class C^m for all m.

The main regularity results are the following.

Theorem 1.1.5. *(Regularity for the Dirichlet problem)[10]*
Let Ω be an open set of class C^2 with Γ bounded (or else $\Omega = \mathbb{R}^N_+$). Let $f \in L^2\left(\Omega\right)$ and let $u \in H^1_0\left(\Omega\right)$ satisfy

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\,\varphi = \int_\Omega f\,\varphi \, , \; \forall \varphi \in H^1_0\left(\Omega\right) .$$

Then $u \in H^2\left(\Omega\right)$ and $\|u\|_{H^2} \leq C\|f\|_{L^2}$, where C is a constant depending only on Ω. Furthermore, if Ω is of class C^{m+2} and $f \in H^m\left(\Omega\right)$, then

$$u \in H^{m+2}\left(\Omega\right) \quad \textit{and} \quad \|u\|_{H^{m+2}} \leq C\|f\|_{H^m} \, .$$

In particular, if $f \in H^m\left(\Omega\right)$ with $m > \dfrac{N}{2}$, then $u \in C^2\left(\overline{\Omega}\right)$. Finally, if Ω is of class C^∞ and if $f \in C^\infty\left(\overline{\Omega}\right)$, then $u \in C^\infty\left(\overline{\Omega}\right)$.

Theorem 1.1.6. *(Regularity for the Neumann problem)[10]*
With the same assumptions as in Theorem 1.1.5 one obtains the same conclusions for the solution of the Neumann problem, i.e., for $u \in H^1\left(\Omega\right)$ such that

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\,\varphi = \int_\Omega f\,\varphi \, , \; \forall \varphi \in H^1\left(\Omega\right) .$$

1.1.5 Functional inequalities

Theorem 1.1.7. *(**Poincaré's inequality**)*
Let Ω be a connected and Lipschitzian bounded open of $\mathbb{R}^n$ and $1 \leq p < \infty$. Then there exists a constant C , depending only on Ω and p such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad , \qquad \forall u \in W^{1,p}_0\left(\Omega\right) \, .$$

Theorem 1.1.8. *(**Poincaré-Wirtinger's inequality**)*

Let Ω be a connected and Lipschitzian bounded open of $\mathbb{R}^n$ and $1 \leq p < \infty$. Then there exists a constant C , depending only on Ω and p such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad , \qquad \forall u \in W^{1,p}(\Omega) \; ,$$

where

$$u_\Omega = \frac{1}{|\Omega|}\int\limits_\Omega u\,(y)\,dy\,,$$

is the average value of u over Ω, with $|\Omega|$ standing for the Lebesgue measure of the domain Ω .

Notation 1.1.2. *Let $1 \leq p \leq \infty$, we denote by q the conjugate of p .i.e $\frac{1}{p} + \frac{1}{q} = 1$*

Theorem 1.1.9. *(**Hölder's inequality**)[10]*
Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p\,,\, q \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\Omega)$ and

$$\int\limits_\Omega |fg|\,dx \leq \|f\|_p.\|g\|_q\,.$$

Remark 1.1.4. *(**Cauchy-Schwarz's inequality**)*
The Cauchy-Schwarz inequality is a special case of the Hölder inequality with $p = q = 2$,

$$\int\limits_\Omega |fg|\,dx \leq \|f\|_2.\|g\|_2\,.$$

Theorem 1.1.10. *(**Young's inequality**)[10]*
If $a \geq 0$ and $b \geq 0$ are nonnegative real numbers and if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$a\,b \leq \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Equality holds if and only if $a^p = b^q$.

Remark 1.1.5. *If $p = q = 2$ we get*

$$a\,b \leq \frac{a^2}{2} + \frac{b^2}{2}\,,$$

which also gives rise to the so-called Young's inequality with ε (valid for every $\varepsilon > 0$)

$$a\,b \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\,\varepsilon}b^2 \quad \text{or} \quad a\,b \leq \varepsilon\,a^2 + \frac{1}{4\,\varepsilon}b^2\,.$$

1.2 The semigroup theory

In this section, we recall some basic knowledge in semigroups, most of which will be used in the subsequent chapters.

1.2.1 Some notions about linear operators

In the following, we consider that X and Y are Banach spaces on the field of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$.

Definition 1.2.1. *[10] A linear operator on X is a linear map A defined on a vector subspace $D(A) \subset X$ with value in X.*

$D(A)$ is called the domain of operator A.

Notation 1.2.1. *We denote such an operator by $(A, D(A))$.*

Definition 1.2.2. *(**Bounded linear operator**)[10]*
Let $(A, D(A))$ be a linear operator in X. We say that $(A, D(A))$ is bounded if

1. *$D(A) = X$.*
2. *A is continuous, i.e., there exists $C \geq 0$ such that*

$$\|Ax\|_X \leq C\|x\|_X \quad , \quad \forall x \in D(A) \,.$$

Otherwise, we say that A is unbounded.

Definition 1.2.3. *(**Closed linear operator**)[10]*

- *The graph of a linear operator $(A, D(A))$ in X is the subspace of $X \times X$ defined by*

$$G(A) = \{(x,\ Ax)\,; x \in D(A)\} \subset X \times X \,.$$

- *We say that $(A, D(A))$ is closed if its graph $G(A)$ is a closed one of $X \times X$.*

Theorem 1.2.1. *(**Closed graph theorem**)[10]*
Let A be a linear operator from X into Y. Assume that the graph of A , $G(A)$ is closed in $X \times Y$. Then A is continuous.

1.2.2 Maximal Monotone Operators in a Hilbert space

In the following, we assume that X is a Hilbert space (real or complex) for the inner product $(\,.\,,\,.\,)$.

Definition 1.2.4. *[10]*

- *An unbounded linear operator $A : D(A) \subset X \to X$ is said to be monotone if it satisfies*

$$(Au\,,\,u) \geq 0\;,\quad \forall u \in D(A)\;.$$

- *It is called maximal monotone if, in addition, for every $\lambda > 0$, $R(\lambda I + A) = X$, i.e.,*

$$\forall \lambda > 0\,, \forall f \in X\,, \exists u \in D(A) \;\; such\,that \;\; \lambda u + Au = f\,.$$

Remark 1.2.1. *[10] Some authors say that A is accretive or that $-A$ is dissipative.*

Proposition 1.2.1. *[10]*
Let A be a maximal monotone operator. Then

- *$D(A)$ is dense in X,*
- *A is a closed operator,*
- *For every $\lambda > 0$, $(\lambda I + A)$ is bijective from $D(A)$ into X, $(\lambda I + A)^{-1}$ is a bounded operator, and*

$$\left\|(\lambda I + A)^{-1}\right\| \leq \frac{1}{\lambda}\,.$$

Theorem 1.2.2. *[76]*
Let $A : D(A) \subset X \to X$ be a monotone operator. A is maximal monotone if and only if

$$\exists \lambda_0 > 0 \; such\,that \; \forall f \in X\,, \exists u \in D(A) \;:\; \lambda_0 u + Au = f\,.$$

1.2.3 Basic theory of semigroups

Definition 1.2.5. *(**Semigroups**)[76]*
Let X be a Banach space or a closed subset of a Banach space. A one parameter family $S(t)$, $0 \leq t < +\infty$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

1. $S(0) = Id$ (I is the identity operator on X).

2. $S(t+s) = S(t)\, S(s)$ for every $t\,,\ s \geq 0$.

Definition 1.2.6. *(**Contraction Semigroups**)[76]*
The semigroup $S(t)$ is a contraction semigroup if there exists a constant $0 < \alpha < 1$ such that for all $t > 0$,

$$\|S(t)\,x - S(t)\,y\| \leq \alpha\,\|x-y\| \quad for\ all\ x,y \in X\,.$$

Definition 1.2.7. *(**Uniformly Continuous Semigroups**)[76]*
A semigroup of bounded linear operators, $S(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|S(t) - I\| = 0\,.$$

Definition 1.2.8. *(C_0 **- Semigroups**)[76]*
A semigroup $S(t)$, $0 \leq t < +\infty$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t\to 0} S(t)\,x = x \quad or \quad \lim_{t\to 0} \|S(t)\,x - x\| = 0 \quad for\ all\ x \in X\,.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

Theorem 1.2.3. *[76]*
Let $S(t)$, $0 \leq t < +\infty$, be a C_0 semigroup on a Banach space X. Thene there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq M\,e^{\omega\,t} \ \ for\ \ t \in 0 \leq t < +\infty\,.$$

Obviously, if $M = 1$ and $\omega = 0$, then we obtain a C_0 - semigroup of nonexpansions or contractions.

Definition 1.2.9. *(**The Infinitesimal Generator**)[76]*
The linear operator A defined by

$$D(A) = \left\{ x \in X \ :\ \lim_{t\to 0} \frac{S(t)\,x - x}{t}\ exists \right\},$$

and

$$Ax = \lim_{t\to 0} \frac{S(t)\,x - x}{t} = \frac{d\,S(t)}{dt}_{t=0} \quad for\ all\ x \in D(A)\,.$$

is the infinitesimal generator of the semigroup $S(t)$, $D(A)$ is the domain of A.

Corollary 1.2.1. *[76]*
If A is the infinitesimal generator of a C_0 semigroup $S(t)$. Then $D(A)$, the domain of A, is dense in X and A is a closed linear operator.

Theorem 1.2.4. *[76]*
Let $S(t)$ and $T(t)$ be C_0 semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then $S(t) = T(t)$ for $t \geq 0$.

Theorem 1.2.5. *[76]*
Let $S(t)$ be a C_0 semigroup and let A be its infinitesimal generator. Then

1. For $x \in X$, $\lim\limits_{h\to 0} \dfrac{1}{h} \displaystyle\int_t^{t+h} S(s)\, x\, ds = S(t)\, x$.

2. For $x \in X$, $\displaystyle\int_0^t S(s)\, x\, ds \in D(A)$ *and* $A\left(\displaystyle\int_0^t S(s)\, x\, ds\right) = S(t)\, x - x$.

3. For $x \in D(A)$, $S(t)\, x \in D(A)$ *and* $\dfrac{d}{dt} S(t)\, x = AS(t)\, x = S(t)\, Ax$.

4. For $x \in D(A)$, $S(t)\, x - S(s)\, x = \displaystyle\int_s^t S(\tau)\, Ax\, d\tau = \int_s^t AS(\tau)\, x\, d\tau$.

Theorem 1.2.6. *(**Hille-Yosida**)[76]*
A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions $S(t)$, $t \geq 0$ if and only If

1. *A is closed and $\overline{D(A)} = X$.*

2. *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}.$$

Theorem 1.2.7. *(**Lumer-Phillips**)[76]*
Let A be a linear operator with dense domain $D(A)$ in X

1. *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $(\lambda_0 I - A)$ is X, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

2. *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $Re\langle Ax, x^*\rangle \leq 0$.*

1.3 The Abstract Cauchy Problem

In this section we consider the initial value problem

$$\begin{cases} \dfrac{d\,u(t)}{dt} + A\,u(t) = K \quad , \quad t > 0 \quad , \\ u(0) = u_0 \quad , \end{cases} \tag{1.3.3}$$

where A is the infinitesimal generator of a C_0 semigroup $S(t)$, defined in a dense subset $D(A)$ of a Banach space X.

Lemma 1.3.1. *[76]*
Suppose that $K = 0$ and $u_0 \in D(A)$. Then problem (3.3.46) *has a unique classical solution u such that*

$$u \in C^1(([0, +\infty[, X) \cap C([0, +\infty[), D(A)) ,$$

and

$$u(t) = S(t)\,u_0 .$$

Lemma 1.3.2. *[76]*
Suppose that $K = K(t) \in C^1([0, +\infty[, X)$ and $u_0 \in D(A)$. Then problem (3.3.46) *admits a unique global classical solution u such that*

$$u \in C^1(([0, +\infty[, X) \cap C([0, +\infty[), D(A)) ,$$

which can be described as

$$u(t) = S(t)\,u_0 + \int_0^t S(t-\tau)\,K(\tau)\,d\tau .$$

Lemma 1.3.3. *[76]*

Suppose that $K = K(t) \in C([0,+\infty[\,, D(A))$ *and* $u_0 \in D(A)$. *Then problem* (3.3.46) *admits a unique global classical solution* u .

Lemma 1.3.4. *[76]*
Suppose that $K = K(t) \in C([0,+\infty[\,, X)$, $K_t \in L^1([0,T], X)$ *for any* $T > 0$ *and* $u_0 \in D(A)$. *Then problem* (3.3.46) *admits a unique global classical solution* u .

Lemma 1.3.5. *[76]*
When $K = K(u)$ *satisfies the global Lipschitz condition, i.e., there is a positive constant* L *such that for all* $u, v \in X$,

$$\|K(u) - K(v)\|_X \leq L\|u - v\|_X$$

. *Furthermore, suppose that* $u_0 \in X$. *Then problem* (3.3.46) *admits a global mild solution* $u(t)$ *such that* $u(t)$ *belongs to* $C([0,+\infty[\,, X)$ *and satisfies the following integral equation:*

$$u(t) = S(t)\,u_0 + \int_0^t S(t-\tau)\,K(u(\tau))\,d\tau\,.$$

Moreover, let $u(t)$, $\tilde{u}(t)$ *be the global mild solutions corresponding to* u_0 *and* $\tilde{u}_0$. *Then for all* $t \geq 0$, *the following estimates holds:*

$$\|u(t) - \tilde{u}(t)\|_X \leq e^{Lt}\|u_0 - \tilde{u}_0\|_X.$$

1.4 Stability

Theorem 1.4.1. *[23]*
Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *be a function satisfying the following condition: For all* $T > 0$, $\exists\, k$, C *such that:*

$$|f(t,x) - f(t,y)| \leq k\,|x-y|\ ,\ \forall x,y \in \mathbb{R}^n\,,\ \forall t \in [0,T]\ ,$$

$$|f(t,x_0)| \leq C\,,\ \ \forall t \in [0,T]\ ,$$

then the system

$$\begin{cases} x_t(t) = f(t,x)\ ,\ \forall t > 0 \\ x(t_0) = x_0\,. \end{cases} \tag{1.4.4}$$

admits a unique solution $x(t, t_0, x_0)$*, satisfait :*

$$x(t, t_1, x_1) = x(t, x_0) \ \ , \forall t \geq t_1 \geq t_0$$

, where, $x_1 = x(t_1)$ *.*

Definition 1.4.1. *(**Balance point**)[23]*
A point $x_0 \in \mathbb{R}^n$ *is an equilibrium point of the system* (1.4.4) *if*

$$\forall t \geq 0\,,\ f(t, x_0) = 0$$

. Lyapunov theory is interested in the behavior of a solution $x(t, t_0, \overline{x_0})$ *which begins at a point* $\overline{x_0} \neq x_0$ *and* $\|\overline{x_0} - x_0\| < \delta$ *for a fairly small* $\delta > 0$*. To simplify things we take* $x_0 = 0$ *and we note* $\overline{x_0}$ *by* x_0*.*

Definition 1.4.2. *(**Lyapunov Stability**)[44]*
An equilibrium point $x = 0$ *is said to be Lyapunov stable if for every* $\varepsilon > 0$*,* $t_0 > 0$*, there exists a* $\delta = \delta(t_0, \varepsilon) > 0$*, such that if* $\|x_0\| < \delta$*, then* $\|x(t, t_0, x_0)\| < \varepsilon$ *for all* $t \geq t_0$*. This means that if the system starts close to the equilibrium point* $x = 0$*, it remains close to* $x = 0$ *for all future times.*

Definition 1.4.3. *(**Asymptotic Stability**)[44]*
An equilibrium point $x = 0$ *is said to be asymptotically stable if it is Lyapunov stable, and in addition, there exists a* $\delta > 0$*, such that if* $\|x_0\| < \delta$*, then* $\|x(t, t_0, x_0)\| \to 0$ *as* $t \to \infty$*. This means that the system not only remains close to the equilibrium point but also eventually returns to it over time.*

Definition 1.4.4. *(**Exponential Stability**)[44]*
An equilibrium point $x = 0$ *is said to be exponentially stable if it is asymptotically stable and the convergence to the equilibrium point occurs at an exponential rate. Formally, there exist positive constants* C *and* α *such that* $\|x(t)\| \leq C\|x_0\|\, e^{-\alpha(t-t_0)}$ *for all* $t \geq t_0 \geq 0$*. This indicates that the system returns to the equilibrium point exponentially fast after a perturbation, and the constant* α *is called the decay rate or also the speed of convergence.*

Definition 1.4.5. *(**Lyapunov's Direct Method**)[44]*
Lyapunov's direct method is a technique used to assess the stability of an equilibrium point $x = 0$ *without solving the differential equations of the system. The method involves constructing a Lyapunov function* $\mathcal{L}$*, which is a scalar function that has specific properties:*

1. *$\mathcal{L}$ is positive definite:* $\mathcal{L}(.,x) > 0$ *for all* $x \neq 0$ *and* $\mathcal{L}(0) = 0$

2. $\mathcal{L}_t(.,x) < 0$ *for all* $x \neq 0$

CHAPTER 2

Theoretical Analysis and Convergence of a Fixed-Point Scheme For A Delayed Swelling Porous Thermoelastic Soils Model

In this chapter, we consider a one-dimensional swelling problem in porous elastic soils with second sound and a constant internal delay, where the heat conduction is given by Cattaneo's law (0.0.18). We show that the system is well-posed using the semigroup approach. Then, based on the energy method as well as by constructing a suitable Lyapunov functional, we prove that the unique dissipation given only by the second sound is strong enough to provoke an exponential decay of the solution without any relationship between the system parameters. For the numerical study, we discretize the continuous problem by performing a temporal discretization using Euler scheme and the classical finite difference method for spatial discretization. To solve the discretized problem, we propose to introduce a fixed point algorithm and derive the condition for which the proposed algorithm converges. Finally, we present some numerical test to illustrate the theoretical results by taking different delay weights and show that the studied system is highly reactive to small delays [7].

2.1 Well-posedness

In this section, we give the existence and uniqueness of solutions for the system (0.0.18) using semigroup theory.

First, we introduce as in [66], new dependent variable

$$\varphi(x,\rho,t) = u_t(x,t-\rho\,\tau) \qquad in\ (0,L)\times(0,1)\times(0,\infty) \tag{2.1.1}$$

A simple differentiation shows that φ satisfies

$$\tau\,\varphi_t(x,\rho,t) + \varphi_\rho(x,\rho,t) = 0 \qquad in\ (0,L)\times(0,1)\times(0,\infty) \tag{2.1.2}$$

Hence problem (0.0.18) takes the form:

$$\begin{cases} \rho_z\, z_{tt} = \alpha_1\, z_{xx} + \alpha_2\, u_{xx} & in\ (0,L)\times(0,\infty) \\ \rho_u\, u_{tt} = \alpha_3\, u_{xx} + \alpha_2\, z_{xx} + \beta\,\theta_x - \mu\,\varphi(x,1,t) & in\ (0,L)\times(0,\infty) \\ c\,\theta_t = -q_x + \beta\, u_{tx} & in\ (0,L)\times(0,\infty) \\ \tau_0\, q_t = -q - k\,\theta_x & in\ (0,L)\times(0,\infty) \\ \tau\,\varphi_t = -\varphi_\rho & in\ (0,L)\times(0,1)\times(0,\infty) \end{cases} \tag{2.1.3}$$

with the boundary and the initial data

$$
\begin{cases}
z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = \theta_x(0,t) = \theta(L,t) = q(0,t) = 0 \ , \ t > 0 \\
z(x,0) = z_0(x) \quad , \quad z_t(x,0) = z_1(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,L) \\
u(x,0) = u_0(x) \quad , \quad u_t(x,0) = u_1(x) \quad , \quad q(x,0) = q_0(x) \quad , \quad x \in (0,L) \\
\varphi(x,\rho,0) = f_0(x,\rho\tau) \qquad in\ (0,L) \times (0,\tau)
\end{cases}
\tag{2.1.4}
$$

Second, we introduce the vector function $U = (z, v, u, \psi, \theta, q, \varphi)^T$, with $v = z_t$, and $\psi = u_t$.

We consider the following spaces:

$$
\begin{aligned}
\tilde{H}^1(0,L) &= \left\{ f \in H^1(0,L);\ f(0) = 0 \right\}, \\
\tilde{H}^1_*(0,L) &= \left\{ f \in H^1(0,L);\ f(L) = 0 \right\}, \\
\tilde{H}^2(0,L) &= H^2(0,L) \cap \tilde{H}^1(0,L),
\end{aligned}
$$

and

$$
\mathcal{H} = \tilde{H}^1(0,L) \times L^2(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2((0,L) \times (0,1)).
$$

Then $\mathcal{H}$, along with the inner product

$$
\begin{aligned}
\langle U, \tilde{U} \rangle_{\mathcal{H}} = {} & \rho_z \int_0^L v\tilde{v}dx + \rho_u \int_0^L \psi\tilde{\psi}dx + \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L u_x \tilde{u}_x dx \\
& + \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right) \left(\frac{\alpha_2}{\sqrt{\alpha_1}} \tilde{u}_x + \sqrt{\alpha_1} \tilde{z}_x \right) dx + c \int_0^L \theta\tilde{\theta}dx \\
& + \frac{\tau_0}{k} \int_0^L q\tilde{q}dx + |\mu| \tau \int_0^L \int_0^1 \varphi(x,\rho,t)\, \tilde{\varphi}(x,\rho,t)\, d\rho\, dx
\end{aligned}
\tag{2.1.5}
$$

is a Hilbert space for any $U = (z, v, u, \psi, \theta, q, \varphi)^T \in \mathcal{H}$ and $\tilde{U} = (\tilde{z}, \tilde{v}, \tilde{u}, \tilde{\psi}, \tilde{\theta}, \tilde{q}, \tilde{\varphi})^T \in \mathcal{H}$.

The system (2.1.3) can be rewritten as follows:

$$
\begin{cases}
U_t + (\mathcal{A} + \mathcal{B})U = 0, \ t > 0, \\
U(x,0) = U_0(x) = (z_0, z_1, u_0, u_1, \theta_0, q_0, f_0)^T,
\end{cases}
$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\dfrac{\alpha_1}{\rho_z} z_{xx} - \dfrac{\alpha_2}{\rho_z} u_{xx} \\ -\psi \\ -\dfrac{\alpha_3}{\rho_u} u_{xx} - \dfrac{\alpha_2}{\rho_u} z_{xx} - \dfrac{\beta}{\rho_u}\theta_x + \dfrac{|\mu|}{\rho_u}\psi + \dfrac{\mu}{\rho_u}\varphi(x,1,t) \\ \dfrac{1}{c} q_x - \dfrac{\beta}{c}\psi_x \\ \dfrac{1}{\tau_0} q + \dfrac{k}{\tau_0}\theta_x \\ \dfrac{1}{\tau}\varphi_\rho \end{pmatrix}.$$

The domain of $\mathcal{A}$ is given by

$$\begin{aligned} D(\mathcal{A}) = \Big\{ & U \in \mathcal{H} \mid z, u \in \tilde{H}^2(0,L);\ v, \psi, q \in \tilde{H}^1(0,L); \\ & \theta \in \tilde{H}^1_*(0,L); \varphi, \varphi_\rho \in L^2((0,L)\times(0,1)) \\ & z_x(L) = u_x(L) = \theta_x(0) = 0 \Big\}. \end{aligned}$$

and the operator $\mathcal{B} : D(\mathcal{B}) = \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{B}U = -\frac{|\mu|}{\rho_u} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now we have the following existence and uniqueness result

Theorem 2.1.1. *Let $U_0 \in \mathcal{H}$ and assume that* (0.0.9) *holds. Then, there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ for problem* (2.1.3)-(2.1.4). *Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. We use the semi-group approach. So we prove that $\mathcal{A}$ is a maximal monotone operator and that $\mathcal{B}$ is a Lipschitz continuous operator.

First, we prove that $\mathcal{A}$ is monotone. Let $U \in D(\mathcal{A})$, then we have

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = |\mu| \int_0^L \psi^2 dx + \mu \int_0^L \psi\, \varphi(x,1,t)\, dx + \frac{1}{k}\int_0^L q^2 dx + |\mu| \int_0^L \int_0^1 \varphi_\rho\, \varphi\, d\rho\, dx. \tag{2.1.6}$$

Using integration by parts and the fact that $\varphi(x,0,t) = \psi(x,t)$, the last term in the right-hand side of (2.1.6) gives

$$\int_0^L \int_0^1 \varphi_\rho\, \varphi\, d\rho\, dx = \frac{1}{2}\int_0^L \varphi^2(x,1,t)\, dx - \frac{1}{2}\int_0^L \psi^2 dx.$$

Also, using Young inequality we get

$$-\mu \int_0^L \psi\, \varphi(x,1,t)\, dx \leq \frac{|\mu|}{2}\int_0^L \psi^2 dx + \frac{|\mu|}{2}\int_0^L \varphi^2(x,1,t)\, dx.$$

Consequently, (2.1.6) yields

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq \frac{1}{k}\int_0^L q^2 dx \geq 0.$$

Therefore, the operator $\mathcal{A}$ is monotone. Next, we prove that the operator $\mathcal{I} + \mathcal{A}$ is surjective. For any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathcal{H}$, we prove that there exist $U \in D(\mathcal{A})$ such that

$$(\mathcal{I} + \mathcal{A})\, U = F. \tag{2.1.7}$$

The problem (2.1.7) , leads to solve the following system

$$\begin{cases} z - v = f_1 \in \tilde{H}^1(0,L), \\ \rho_z\, v - \alpha_1\, z_{xx} - \alpha_2\, u_{xx} = \rho_z\, f_2 \in L^2(0,L), \\ u - \psi = f_3 \in \tilde{H}^1(0,L), \\ (\rho_u + |\mu|)\, \psi - \alpha_3\, u_{xx} - \alpha_2\, z_{xx} - \beta\, \theta_x + \mu\, \varphi(x,1,t) = \rho_u\, f_4 \in L^2(0,L), \\ c\,\theta + q_x - \beta\, \psi_x = c\, f_5 \in L^2(0,L), \\ (\tau_0 + 1)\, q + k\, \theta_x = \tau_0\, f_6 \in L^2(0,L), \\ \tau\, \varphi + \varphi_\rho = \tau\, f_7 \in L^2((0,L)\times(0,1)), \end{cases} \tag{2.1.8}$$

The last equation in (2.1.8) with $\varphi(x,0,t)=\psi(x,t)$ has a unique solution given by

$$\varphi(x,\rho,t)=\psi(x,t)\,e^{-\tau\rho}+\tau\,e^{-\tau\rho}\int_0^{\rho}e^{\tau\sigma}f_7(x,\sigma,t)\,d\sigma. \tag{2.1.9}$$

From the sixth equation in (2.1.8) , we define

$$\theta(x,t)=\frac{\tau_0+1}{k}\int_x^L q(y)\,dy-\frac{\tau_0}{k}\int_x^L f_6(y)\,dy. \tag{2.1.10}$$

Inserting $v=z-f_1$, $\psi=u-f_3$, (2.1.9) , (2.1.10) in $(2.1.8)_2$, $(2.1.8)_4$, $(2.1.8)_5$, we get

$$\begin{cases} -\alpha_1 z_{xx}+\rho_z z-\alpha_2 u_{xx}=g_1\in L^2(0,L)\,, \\ -\alpha_3 u_{xx}+\lambda u-\alpha_2 z_{xx}+\dfrac{\beta(\tau_0+1)}{k}q=g_2\in L^2(0,L)\,, \\ q_x+\dfrac{c(\tau_0+1)}{k}\displaystyle\int_x^L q(y)\,dy-\beta u_x=g_3\in L^2(0,L)\,, \end{cases} \tag{2.1.11}$$

where

$$\begin{aligned} & g_1=\rho_z(f_1+f_2)\in L^2(0,L)\,, \\ & g_2=\rho_u f_4+\lambda f_3+\frac{\beta\tau_0}{k}f_6-\tau\mu e^{-\tau}\int_0^1 e^{\tau\sigma}f_7(x,\sigma,t)\,d\sigma\in L^2(0,L)\,, \\ & g_3=c f_5+\frac{c\tau_0}{k}\int_x^L f_6(y)\,dy-\beta f_{3_x}\in L^2(0,L)\,, \\ & \lambda=\rho_u+|\mu|+\mu e^{-\tau}. \end{aligned}$$

To solve (2.1.11), we consider

$$B((z,u,q);(\tilde{z},\tilde{u},\tilde{q}))=\mathcal{G}(\tilde{z},\tilde{u},\tilde{q})\,, \tag{2.1.12}$$

where $B:\left[\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)\times L^2(0,L)\right]^2\to\mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}B\left((z,u,q);(\tilde{z},\tilde{u},\tilde{q})\right) &= \rho_z\int_0^L z\,\tilde{z}\,dx+\lambda\int_0^L u\,\tilde{u}\,dx+\frac{\tau_0+1}{k}\int_0^L q\,\tilde{q}\,dx+\alpha_1\int_0^L z_x\,\tilde{z}_x\,dx\\&+\alpha_3\int_0^L u_x\,\tilde{u}_x\,dx+\alpha_2\int_0^L u_x\,\tilde{z}_x\,dx+\alpha_2\int_0^L z_x\,\tilde{u}_x\,dx\\&+\frac{c(\tau_0+1)^2}{k^2}\int_0^L\left(\int_x^L q(y)\,dy\right)\left(\int_x^L \tilde{q}(y)\,dy\right)dx\\&+\frac{\beta(\tau_0+1)}{k}\left(\int_0^L q\,\tilde{u}\,dx-\int_0^L u\,\tilde{q}\,dx\right),\end{aligned}$$

and $\mathcal{G}:\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)\times L^2(0,L)\to\mathbb{R}$ is the linear form given by

$$\mathcal{G}(\tilde{z},\tilde{u},\tilde{q})=\int_0^L g_1\,\tilde{z}\,dx+\int_0^L g_2\,\tilde{u}\,dx+\frac{\tau_0+1}{k}\int_0^L g_3\left(\int_x^L\tilde{q}(y)\,dy\right)dx.$$

Let $\mathcal{V}=\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)\times L^2(0,L)$ equipped with the norm

$$\|(z,u,q)\|_V^2=\|z\|_2^2+\|u\|_2^2+\|q\|_2^2+\|u_x\|_2^2+\|z_x\|_2^2,$$

then, using Young's inequality, we can easily prove that

$$\begin{aligned}\left|B\left((z,u,q);(z,u,q)\right)\right| &\geq \rho_z\int_0^L z^2dx+\lambda\int_0^L u^2dx+\frac{\tau_0+1}{k}\int_0^L q^2dx+\left(\alpha_1-\frac{\alpha_2}{\varepsilon}\right)\int_0^L z_x^2dx\\&+(\alpha_3-\alpha_2\,\varepsilon)\int_0^L {u_x}^2dx+\frac{c(\tau_0+1)^2}{k^2}\int_0^L\left(\int_x^L q(y)\,dy\right)^2dx,\end{aligned}$$

if we chose $\varepsilon\in\left]\frac{\alpha_2}{\alpha_1},\frac{\alpha_3}{\alpha_2}\right[$, we get

$$\left|B\left((z,u,q);(z,u,q)\right)\right|\geq M_0\,\|(z,u,q)\|_V^2,$$

where $M_0 = min\left\{\rho_z\,,\,\lambda,\dfrac{\tau_0+1}{k},\alpha_1-\dfrac{\alpha_2}{\varepsilon},\alpha_3-\alpha_2\,\varepsilon\right\}$. Thus, B is coercive. Moreover, we can easily see that B and $\mathcal{G}$ are bounded. Consequently, by Lax-Milgram Lemma, system (2.1.11) has a unique solution $(z,u,q)\in\mathcal{V}$ satisfying (2.1.12).

Substituting z and u in $(2.1.8)_1$ and $(2.1.8)_3$, respectively, we obtain

$$v,\psi\in\tilde{H}^1(0,L)\,,$$

then, inserting q in (2.1.10) and $(2.1.8)_6$ we get

$$\theta\in\tilde{H}^1_*(0,L)\,.$$

Similarly, inserting ψ in (2.1.9) and bearing in mind $(2.1.8)_7$, we obtain

$$\varphi,\varphi_\rho\in L^2\left((0,L)\times(0,1)\right).$$

Moreover, if we take $(\tilde{z},\tilde{u})\equiv(0,0)\in\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)$, then (2.1.12) reduces to

$$\begin{aligned}&\int_0^L q\,\tilde{q}\,dx+\frac{c\,(\tau_0+1)}{k}\int_0^L\left(\int_x^L q\,(y)\,dy\right)\left(\int_x^L\tilde{q}\,(y)\,dy\right)dx\\&-\beta\int_0^L u\,\tilde{q}\,dx=\int_0^L g_3\left(\int_x^L\tilde{q}\,(y)\,dy\right)dx,\forall\tilde{q}\in L^2(0,L)\,.\end{aligned}\tag{2.1.13}$$

That is

$$\begin{aligned}&-\int_0^L(q-\beta\,u)\left(\int_x^L\tilde{q}\,(y)\,dy\right)_x dx\\&=\int_0^L\left[-\frac{c\,(\tau_0+1)}{k}\left(\int_x^L q\,(y)\,dy\right)+g_3\right]\left(\int_x^L\tilde{q}\,(y)\,dy\right)dx,\forall\tilde{q}\in L^2(0,L)\,,\end{aligned}\tag{2.1.14}$$

which implies

$$q_x=\beta\,u_x-\frac{c\,(\tau_0+1)}{k}\left(\int_x^L q\,(y)\,dy\right)+g_3,\in L^2(0,L)\,.$$

So,

$$q \in H^1(0,L),$$

and the other hand, we have (2.1.14). Thus

$$q(0)\int_0^L \tilde{q}(y)\,dy = 0,\ \forall \tilde{q} \in L^2(0,L).$$

Since $\tilde{q} \in L^2(0,L)$ is arbitrary. Then,

$$q(0) = 0.$$

Consequently

$$q \in \tilde{H}^1(0,L).$$

If we choose $(\tilde{u}, \tilde{q}) \equiv (0,0) \in H^1(0,L) \times L^2(0,L)$ in (2.1.12) we have

$$\int_0^L (\alpha_1 z_x + \alpha_2 u_x)\,\tilde{z}_x dx = \int_0^L (g_1 - \rho_z z)\,\tilde{z}\,dx, \forall \tilde{z} \in \tilde{H}^1(0,L). \tag{2.1.15}$$

This last is also true for any function $\phi \in C^1(0,L)$, $\phi(0) = 0$ which is in $\tilde{H}^1(0,L)$, thus

$$-\alpha_1 z_{xx} - \alpha_2 u_{xx} = g_1 - \rho_z z, \in L^2(0,L).$$

Similarly, if we select $(\tilde{z}, \tilde{q}) \equiv (0,0) \in H^1(0,L) \times L^2(0,L)$ in (2.1.12), we find

$$\int_0^L (\alpha_3 u_x + \alpha_2 z_x)\,\tilde{u}_x dx = \int_0^L \left(g_2 - \lambda\, u - \frac{\beta(\tau_0+1)}{k} q\right) \tilde{u}\,dx, \forall \tilde{u} \in \tilde{H}^1(0,L). \tag{2.1.16}$$

This last is also true for any function $\phi \in C^1(0,L)$, $\phi(0) = 0$ which is in $\tilde{H}^1(0,L)$, thus

$$-\alpha_3 u_{xx} - \alpha_2 z_{xx} = g_2 - \lambda\, u - \frac{\beta(\tau_0+1)}{k} q, \in L^2(0,L).$$

Therefore,

$$u_{xx}, z_{xx} \in L^2(0,L).$$

So,

$$z, u \in \tilde{H}^2(0, L).$$

Finally, from $(2.1.8)_6$, we get

$$\theta_x(0) = 0,$$

and from (2.1.15), (2.1.16), we find

$$\begin{aligned} \tilde{z}(L)[\alpha_1 z_x(L) + \alpha_2 u_x(L)] &= 0, \\ \tilde{u}(L)[\alpha_3 u_x(L) + \alpha_2 z_x(L)] &= 0. \end{aligned}$$

Since $\tilde{z}, \tilde{u} \in \tilde{H}^1(0, L)$ are arbitrary. Then

$$z_x(L) = u_x(L) = 0.$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (2.1.7) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal. With this, we conclude that $\mathcal{A}$ is a maximal monotone operator. On the other hand, it is obvious that operator $\mathcal{B}$ is Lipschitz continuous. Consequently, $\mathcal{A} + \mathcal{B}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{H}$. Therefore, the well-posedness result follows from the Lummer Phillips theorem (see [76]). $\square$

2.2 Exponential decay

In this section, we state and prove technical lemmas needed for the proof of our stability result.

Lemma 2.2.1. *Let $(z, u, \theta, q, \varphi)$ be the solution of (2.1.3)-(2.1.4). Then, the energy functional $E(t)$, defined by*

$$\begin{aligned} E(t) = & \frac{\rho_z}{2}\int_0^L z_t^2 dx + \frac{\rho_u}{2}\int_0^L u_t^2 dx + \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \frac{1}{2}\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx \\ & + \frac{c}{2}\int_0^L \theta^2 dx + \frac{\tau_0}{2k}\int_0^L q^2 dx + \frac{|\mu|\tau}{2}\int_0^L\int_0^1 \varphi^2(x, \rho, t)\, d\rho\, dx, \end{aligned} \tag{2.2.17}$$

satisfies

$$E'(t) \leq |\mu| \int_0^L u_t^2 dx - \frac{1}{k} \int_0^L q^2 dx. \tag{2.2.18}$$

Proof. Multiplying $(2.1.3)_1$, $(2.1.3)_2$, $(2.1.3)_3$, $(2.1.3)_4$ by z_t , u_t , θ , $\frac{1}{k}q$ respectively, integrating over $(0, L)$, and Multiplying$(2.1.3)_5$ by $|\mu|\ \varphi$, integrating over $(0, L) \times (0, 1)$ then,using integration by part and taking into account the boundary conditions and summing them up, we obtain

$$\begin{aligned} &\frac{d}{2dt} \int_0^L \left(\rho_u u_t^2 + \rho_z z_t^2 + c\theta^2 + \alpha_3 u_x^2 + \alpha_1 z_x^2 + 2\alpha_2 z_x u_x + \frac{\tau_0}{k} q^2 \right) dx \\ &+ \frac{d}{dt} \frac{|\mu|\, \tau}{2} \int_0^L \int_0^1 \varphi^2 (x, \rho, t)\, d\rho\, dx \\ &= \frac{|\mu|}{2} \int_0^L u_t^2 dx - \frac{1}{k} \int_0^L q^2 dx - \frac{|\mu|}{2} \int_0^L \varphi^2 (x, 1, t)\, dx - \mu \int_0^L u_t\, \varphi (x, 1, t)\, dx. \end{aligned} \tag{2.2.19}$$

Using the fact that

$$\begin{aligned} &\alpha_3 u_x^2 + \alpha_1 z_x^2 + 2\alpha_2 z_x u_x \\ &= \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L u_x^2 dx + \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 dx. \end{aligned} \tag{2.2.20}$$

Then, using Young's inequality on the last term in (2.2.19) we have

$$-\mu \int_0^L u_t\, \varphi (x, 1, t)\, dx \leq \frac{|\mu|}{2} \int_0^L u_t^2 dx + \frac{|\mu|}{2} \int_0^L \varphi^2 (x, 1, t)\, dx. \tag{2.2.21}$$

Inserting (2.2.20) and (2.2.21) in (2.2.19), we get (2.2.17) and (2.2.18). □

Lemma 2.2.2. *Let* $(z, u, \theta, q, \varphi)$ *be the solution of* (2.1.3) - (2.1.4). *Then, the functional*

$$I_1(t) = \rho_u \int_0^L u_t u dx - \frac{\alpha_2}{\alpha_1} \rho_z \int_0^L z_t u dx,\ t \geq 0,$$

satisfies for any $\varepsilon_1 > 0$,

$$I_1'(t) \le -\frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \left(\rho_u + \frac{\alpha_2^2\rho_z^2}{4\varepsilon_1\alpha_1^2}\right)\int_0^L u_t^2 dx + \varepsilon_1 \int_0^L z_t^2 dx$$
$$+ C_0 \int_0^L \left(\theta^2 + \varphi^2(x,1,t)\right) dx, \ \forall t \ge 0. \tag{2.2.22}$$

Proof. By differentiating $I_1(t)$, using $(2.1.3)_1$,$(2.1.3)_2$ and integrating by parts together with the boundary conditions, we obtain

$$I_1'(t) = -\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \rho_u \int_0^L u_t^2 dx - \beta \int_0^L u_x\theta dx - \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L u_t z_t dx - \mu \int_0^L u\,\varphi(x,1,t)\,dx. \tag{2.2.23}$$

Young's and Poincaré's inequalities lead to

$$-\mu \int_0^L u\,\varphi(x,1,t)\,dx \le \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + C_0 \int_0^L \varphi^2(x,1,t)\,dx, \tag{2.2.24}$$

$$-\beta \int_0^L u_x\theta dx \le \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + C_0 \int_0^L \theta^2 dx, \tag{2.2.25}$$

and

$$-\frac{\alpha_2}{\alpha_1}\rho_z \int_0^L u_t z_t dx \le \varepsilon_1 \int_0^L z_t^2 dx + \frac{\alpha_2^2\rho_z^2}{4\varepsilon_1\alpha_1^2}\int_0^L u_t^2 dx. \tag{2.2.26}$$

Substituting (2.2.24) , (2.2.25) and (2.2.26) in (2.2.23), we get (2.2.22). □

Lemma 2.2.3. *Let $(z, u, \theta, q, \varphi)$ be the solution of* (2.1.3)-(2.1.4). *Then, the functional*

$$I_2(t) = \rho_u\alpha_2 \int_0^L u_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx - \frac{\alpha_2^2}{\alpha_1}\rho_z \int_0^L z_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx, \ t \ge 0,$$

satisfies, for any $\varepsilon_2 > 0$,

$$I_2'(t) \le -\frac{\alpha_2^2\rho_z}{2\sqrt{\alpha_1}}\int_0^L z_t^2 dx + C_1 \int_0^L u_t^2 dx + C_{\varepsilon_2}\int_0^L \left(\theta^2 + u_x^2 + \varphi^2(x,1,t)\right)dx + \varepsilon_2 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx. \tag{2.2.27}$$

Proof. By differentiating $I_2(t)$, using $(2.1.3)_1$,$(2.1.3)_2$ and integrating by parts together with the boundary conditions, we obtain

$$
\begin{aligned}
I_2'(t) = & -\frac{\alpha_2^2\rho_z}{\sqrt{\alpha_1}}\int_0^L z_t^2 dx + \alpha_2\left(\rho_u\sqrt{\alpha_1} - \frac{\alpha_2^2\rho_z}{\alpha_1\sqrt{\alpha_1}}\right)\int_0^L u_t z_t dx \\
& - \alpha_2\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx + \frac{\alpha_2^2\rho_u}{\sqrt{\alpha_1}}\int_0^L u_t^2 dx \\
& - \alpha_2\beta\int_0^L \theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx - \mu\,\alpha_2\int_0^L \varphi(x,1,t)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}\,z\right)dx. \qquad (2.2.28)
\end{aligned}
$$

Using Young's and Poincaré's inequalities, we get

$$
\alpha_2\left(\rho_u\sqrt{\alpha_1} - \frac{\alpha_2^2\rho_z}{\alpha_1\sqrt{\alpha_1}}\right)\int_0^L u_t z_t dx \leq \frac{\alpha_2^2\rho_z}{2\sqrt{\alpha_1}}\int_0^L z_t^2 dx + C_1\int_0^L u_t^2 dx, \qquad (2.2.29)
$$

$$
-\mu\,\alpha_2\int_0^L \varphi(x,1,t)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}\,z\right)dx \leq \frac{\varepsilon_2}{3}\int_0^L\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}\,z_x\right)^2 dx + C_{\varepsilon_2}\int_0^L \varphi^2(x,1,t)\,dx, \qquad (2.2.30)
$$

$$
-\alpha_2\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx \leq \frac{\varepsilon_2}{3}\int_0^L\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx + C_{\varepsilon_2}\int_0^L u_x^2 dx, \qquad (2.2.31)
$$

and

$$
-\alpha_2\beta\int_0^L \theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx \leq \frac{\varepsilon_2}{3}\int_0^L\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx + C_{\varepsilon_2}\int_0^L \theta^2 dx. \qquad (2.2.32)
$$

Inserting (2.2.29)-(2.2.32) in (2.2.28), we obtain (2.2.27). □

Lemma 2.2.4. *Let* $(z, u, \theta, q, \varphi)$ *be the solution of* (2.1.3) - (2.1.4). *Then, the functional*

$$
I_3(t) = \rho_u\int_0^L u u_t dx + \rho_z\int_0^L z z_t dx,\ t \geq 0,
$$

satisfies

$$I_3'(t) \leq -\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx - \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx + C_2 \int_0^L \left(\theta^2 + \varphi^2(x,1,t)\right) dx. \tag{2.2.33}$$

Proof. Differentiating the functional $I_3(t)$ using $(2.1.3)_1$, $(2.1.3)_2$ and integrating by parts, we obtain

$$I_3'(t) = \rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx - \alpha_3 \int_0^L u_x^2 dx - 2\alpha_2 \int_0^L u_x z_x dx - \alpha_1 \int_0^L z_x^2 dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L u\, \varphi(x,1,t)\, dx. \tag{2.2.34}$$

Note that

$$-\alpha_3 \int_0^L u_x^2 dx - 2\alpha_2 \int_0^L u_x z_x dx - \alpha_1 \int_0^L z_x^2 dx = -\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx - \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx. \tag{2.2.35}$$

So, (2.2.34) becomes

$$I_3'(t) = \rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L u\, \varphi(x,1,t)\, dx - \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx - \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx. \tag{2.2.36}$$

Using Young's and Poincaré's inequalities

$$-\beta \int_0^L \theta u_x dx \leq \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + C_2 \int_0^L \theta^2 dx. \tag{2.2.37}$$

$$-\mu \int_0^L u\, \varphi(x,1,t)\, dx \leq \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2\, dx + C_2 \int_0^L \varphi^2(x,1,t)\, dx. \tag{2.2.38}$$

Substituting (2.2.37) and (2.2.38) into (2.2.36), we get (2.2.33). □

Lemma 2.2.5. *Let* $(z, u, \theta, q, \varphi)$ *be the solution of* (2.1.3)-(2.1.4). *Then, the functional*

$$I_4(t) = -c\rho_u \int_0^L \theta \left(\int_x^L u_t(y)\, dy \right) dx, \; \forall t \geq 0,$$

satisfies, for any $\varepsilon_3, \varepsilon_4, \varepsilon_5 > 0$, *the following estimate*

$$\begin{aligned} I_4'(t) \leq & -\frac{\beta\rho_u}{2} \int_0^L u_t^2 dx + \varepsilon_3 \int_0^L u_x^2 dx + \varepsilon_4 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 dx \\ & + \frac{\rho_u}{2\beta} \int_0^L q^2 dx + C_3 \left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4} + \frac{1}{\varepsilon_5} \right) \int_0^L \theta^2 dx + \varepsilon_5 \int_0^L \varphi^2(x,1,t)\, dx. \end{aligned} \quad (2.2.39)$$

Proof. By differentiating $I_4(t)$, using $(2.1.3)_2$, $(2.1.3)_3$ and integrating by parts, we obtain

$$\begin{aligned} I_4'(t) = & \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx + c\alpha_3 \int_0^L \theta u_x dx + c\alpha_2 \int_0^L \theta z_x dx + c\beta \int_0^L \theta^2 dx \\ & + c\,\mu \int_0^L \theta \int_x^L \varphi(y,1,t)\, dy\, dx. \end{aligned} \quad (2.2.40)$$

Using the fact that

$$c\alpha_3 \int_0^L \theta u_x dx + c\alpha_2 \int_0^L \theta z_x dx = c\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L \theta u_x dx + \frac{c\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right) dx.$$

Then, (2.2.40) can be rewritten as follows

$$\begin{aligned} I_4'(t) = & \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx + c\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L \theta u_x dx + c\,\mu \int_0^L \theta \int_x^L \varphi(y,1,t)\, dy\, dx. \\ & + \frac{c\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right) dx + c\beta \int_0^L \theta^2 dx \end{aligned} \quad (2.2.41)$$

Young's inequality leads to

$$c\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L \theta u_x dx \leq \varepsilon_3 \int_0^L u_x^2 dx + \frac{C_3}{\varepsilon_3} \int_0^L \theta^2 dx, \quad (2.2.42)$$

$$\frac{c\alpha_2}{\sqrt{\alpha_1}}\int_0^L \theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)dx \le \varepsilon_4\int_0^L\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)^2dx+\frac{C_3}{\varepsilon_4}\int_0^L \theta^2dx, \tag{2.2.43}$$

and

$$\rho_u\int_0^L qu_tdx \le \frac{\beta\rho_u}{2}\int_0^L u_t^2dx+\frac{\rho_u}{2\beta}\int_0^L q^2dx. \tag{2.2.44}$$

Young's and Cauchy Schwarz inequalities lead to

$$c\,\mu\int_0^L \theta\int_x^L \varphi\left(y,1,t\right)dy\,dx \le \varepsilon_5\int_0^L \varphi^2\left(x,1,t\right)dx+\frac{C_3}{\varepsilon_5}\int_0^L \theta^2dx. \tag{2.2.45}$$

Estimate (2.2.39) follows by substituting (2.2.42)-(2.2.45) into (2.2.41). □

Lemma 2.2.6. *Let* (z,u,θ,q,φ) *be the solution of* (2.1.3) - (2.1.4). *Then, the functional*

$$I_5(t)=-c\tau_0\int_0^L \theta\left(\int_x^L q\left(y\right)dy\right)dx,\ \forall t\ge 0,$$

satisfies, for any $\varepsilon_6>0$, *the following estimate*

$$I_5'(t)\le -\frac{ck}{2}\int_0^L \theta^2dx+\varepsilon_6\int_0^L u_t^2dx+C_4\left(1+\frac{1}{\varepsilon_6}\right)\int_0^L q^2dx. \tag{2.2.46}$$

Proof. By differentiating $I_5(t)$, using $(2.1.3)_3$, $(2.1.3)_4$ and integrating by parts, we obtain

$$\begin{aligned} I_5'(t)=&-ck\int_0^L \theta^2dx+\tau_0\int_0^L q^2dx-\tau_0\beta\int_0^L u_tqdx\\ &+c\int_0^L \theta\left(\int_x^L q\left(y\right)dy\right)dx. \end{aligned} \tag{2.2.47}$$

Using Young's inequality, we get

$$-\tau_0\beta\int_0^L u_tqdx \le \varepsilon_6\int_0^L u_t^2dx+\frac{C_4}{\varepsilon_6}\int_0^L q^2dx. \tag{2.2.48}$$

Young's and Cauchy Schwarz inequalities leads to

$$c\int_0^L \theta\left(\int_x^L q\left(y\right)dy\right)dx \le \frac{ck}{2}\int_0^L \theta^2dx+C_4\int_0^L q^2dx. \tag{2.2.49}$$

Inserting (2.2.48)-(2.2.49) in (2.2.47), we obtain (2.2.46). □

Lemma 2.2.7. *Let* (z,u,θ,q,φ) *be the solution of* (2.1.3)-(2.1.4)*. Then, the functional*

$$I_6(t)=\tau\int_0^L\int_0^1 e^{-\tau\rho}\varphi^2(x,\rho,t)\,d\rho\,dx, \tag{2.2.50}$$

satisfies

$$I_6'(t)\le\int_0^L u_t^2\,dx-e^{-\tau}\left(\int_0^L\varphi^2(x,1,t)\,dx+\tau\int_0^L\int_0^1\varphi^2(x,\rho,t)\,d\rho\,dx\right). \tag{2.2.51}$$

Proof. By differentiating $I_6(t)$ and using $(2.1.3)_5$, we obtain

$$I_6'(t)=\int_0^L u_t^2\,dx-e^{-\tau}\int_0^L\varphi^2(x,1,t)\,dx-\tau\int_0^L\int_0^1 e^{-\tau\rho}\varphi^2(x,\rho,t)\,d\rho\,dx.$$

Using that fact that $\varphi(x,0,t)=\psi(x,t)$ and $e^{-\tau}\le e^{-\tau\rho}\le 1$ for all $\rho\in[0,1]$, we get (2.2.51). □

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t)=NE(t)+N_1I_1(t)+N_2I_2(t)+2(I_3+I_4)(t)+N_3I_5(t)+N_4I_6(t), \tag{2.2.52}$$

where N, N_1, N_2, N_3, N_4 are positive constants.

Theorem 2.2.1. *Let* (z,u,θ,q,φ) *be the solution of* (2.1.3)-(2.1.4)*. Then, there exist two positive constants* κ_1 *and* κ_2 *such that the Lyapunov functional* (2.2.52) *satisfies*

$$\kappa_1E(t)\le\mathcal{L}(t)\le\kappa_2E(t),\ \forall t\ge 0, \tag{2.2.53}$$

and

$$\mathcal{L}'(t)\le-\beta_1E(t),\ \beta_1>0. \tag{2.2.54}$$

Proof. From (2.2.52), we have

$$
\begin{aligned}
|\mathcal{L}(t)-NE(t)| \leq{}& N_1\rho_u\int_0^L |u_t u|\,dx + N_1\frac{\alpha_2}{\alpha_1}\rho_z\int_0^L |z_t u|\,dx + N_2\rho_u\alpha_2\int_0^L \left|u_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u+\sqrt{\alpha_1}z\right)\right| dx\\
&+ N_2\frac{\alpha_2^2}{\alpha_1}\rho_z\int_0^L \left|z_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u+\sqrt{\alpha_1}z\right)\right| dx + 2\rho_u\int_0^L |uu_t|\,dx + 2\rho_z\int_0^L |zz_t|\,dx\\
&+ 2c\rho_u\int_0^L \left|\theta\left(\int_x^L u_t(y)\,dy\right)\right| dx + N_3c\tau_0\int_0^L \left|\theta\left(\int_x^L q(y)\,dy\right)\right| dx\\
&+ N_4\tau\int_0^L\int_0^1 e^{-\tau\rho}\varphi^2(x,\rho,t)\,d\rho\,dx.
\end{aligned}
$$

By using the Young's, Poincaré's, Cauchy-Schwarz inequalities, we obtain

$$
|\mathcal{L}(t)-NE(t)| \leq \gamma E(t),\ \gamma>0,
$$

which yields

$$
(N-\gamma)E(t) \leq \mathcal{L}(t) \leq (N+\gamma)E(t),
$$

by choosing N (depending on N_1, N_2, N_3 and N_4) sufficiently large we obtain (2.2.53).

Now, By differentiating $\mathcal{L}(t)$, exploiting (2.2.18), (2.2.22), (2.2.27), (2.2.33), (2.2.39), (2.2.46), (2.2.51) and setting $\varepsilon_1=\frac{1}{N_1}$, $\varepsilon_2=\frac{1}{N_2}$, $\varepsilon_3=\varepsilon_5=\frac{1}{2}$, $\varepsilon_4=\frac{1}{4}$, $\varepsilon_6=\frac{1}{N_3}$, we get

$$
\begin{aligned}
\mathcal{L}'(t) \leq{}& -\left[\rho_u\beta-|\mu|N-N_1\left(\rho_u+\frac{\alpha_2^2\rho_z^2N_1}{4\alpha_1^2}\right)-C_1N_2-2\rho_u-1-N_4\right]\int_0^L u_t^2dx\\
&-\left[\frac{N}{k}-\frac{\rho_u}{\beta}-N_3C_4(1+N_3)\right]\int_0^L q^2dx-\left[\left(\alpha_3-\frac{\alpha_2^2}{\alpha_1}\right)\left(\frac{N_1}{2}+1\right)-C_{\varepsilon_2}N_2-1\right]\int_0^L u_x^2dx\\
&-\left[\frac{\alpha_2^2\rho_zN_2}{2\sqrt{\alpha_1}}-1-2\rho_z\right]\int_0^L z_t^2dx-\left[\frac{ckN_3}{2}-18C_3-2C_2-C_{\varepsilon_2}N_2-C_0N_1\right]\int_0^L \theta^2dx\\
&-\frac{1}{2}\int_0^L\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x+\sqrt{\alpha_1}z_x\right)^2dx-\left[e^{-\tau}N_4-1-2C_2-N_2C_{\varepsilon_2}-N_1C_0\right]\int_0^L\varphi^2(x,1,t)\,dx\\
&-\tau e^{-\tau}N_4\int_0^L\int_0^1\varphi^2(x,\rho,t)\,d\rho\,dx. \qquad (2.2.55)
\end{aligned}
$$

Now, we select our parameters appropriately as follows:

First, we choose N_2 large enough so that

$$\frac{\alpha_2^2 \rho_z}{2\sqrt{\alpha_1}} N_2 - 1 - 2\rho_z > 0.$$

Next, we select N_1 large enough so that

$$\frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)(N_1 + 2) - C_{\varepsilon_2} N_2 - 1 > 0.$$

We take N_3 large such that

$$\frac{ckN_3}{2} - 18C_3 - 2C_2 - C_{\varepsilon_2} N_2 - C_0 N_1 > 0.$$

We pick N_4 large so that

$$e^{-\tau} N_4 - 1 - 2C_2 - N_2 C_{\varepsilon_2} - N_1 C_0 > 0.$$

We choose N large enough so that (2.2.53) remains valid, further

$$\frac{N}{k} - \frac{\rho_u}{\beta} - N_3 C_4 (1 + N_3) > 0.$$

Finally, by taking $|\mu|$ so small that All these choices with the relation (2.2.55) leads to

$$\mathcal{L}'(t) \leq -\alpha_1 \int_0^L \left(z_t^2 + u_x^2 + u_t^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t) \, d\rho \right) dx. \quad (2.2.56)$$

On the other hand, from Eq. (2.2.17), we obtain (2.2.54). □

Now, we can state and prove the following stability result

Lemma 2.2.8. *Let* $(z, u, \theta, q, \varphi)$ *be the solution of* (2.1.3) - (2.1.4). *Then, for any* $U_0 \in D(\mathcal{A})$, *there exist two positive constants* λ_1 *and* λ_2 *such that*

$$E(t) \leq \lambda_2 e^{-\lambda_1 t}, \ \forall t \geq 0. \quad (2.2.57)$$

Proof. By using the estimation (2.2.54), we get

$$\mathcal{L}'(t) \leq -\beta_1 E(t),\ t \geq 0,$$

having in mind the equivalence of $E(t)$ and $\mathcal{L}(t)$ we infer that

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}(t),\ t \geq 0, \tag{2.2.58}$$

where $\lambda_1 = \dfrac{\beta_1}{\kappa_2}$. A simple integration of (2.2.58) gives

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\lambda_1 t},\ t \geq 0,$$

which yields the serial result (2.2.57) with $\lambda_2 = \dfrac{\mathcal{L}(0)}{\kappa_1}$ and by using the other side of the equivalence relation (2.2.53) again. The proof is complete. □

2.3 Study of Convergence

In the second section, we established the existence and uniqueness of the solution to system (2.1.3) - (2.1.4) , but it is challenging to determine the exact value of the solution. To address this, in the following, we will numerically solve the system in the one-dimension domain Ω of length L., allowing us to determine approximations of the solution. To this end, we will employ the Euler scheme for discretizing temporal variable and the classical finite differences method for discretizing space. To solve the descretized problem, we use a fixed point algorithm with study of their convergence. In addition, we provide an example where the numerical experiment demonstrates that the discrete energy E^n decays exponentially for different choices of the system parameters, supporting the asymptotic behavior of the discretized issue solution.

Let us introduce the functions $\widehat{z} = z_t$, $\widehat{u} = u_t$ and for any $N, m, M \in \mathbb{N}$, we introduce the nets

$$\Omega_N = \{x_i = \rho_i = ih,\ i = 0, 1, ..., N+1\ ,\ \text{where } h = \frac{L}{N+1}\ \},$$
$$\Gamma_M = \{t_n = n\Delta t,\ n = 0, 1, ..., M+1\ ,\ \text{where } \Delta t = \frac{T}{M+1}\ \},$$
$$\Upsilon_M = \{t_n = n\Delta t,\ n = -M', -M'+1, ..., 0\ ,\ \text{with } 0 < M' < M\ \}.$$

such that the width of delay mesh is $\tau = M'\Delta t$.

Now, using a backward Euler scheme in time and finite differences in space, we define the following approximation of the derivatives:

$$\phi_{xx}(x_i,t_n) = \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{h^2},\ \phi_x(x_i,t_n) = \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2h},\ \phi_t(x_i,t_n) = \frac{\phi_i^n - \phi_i^{n-1}}{\Delta t},$$
$$\phi_t(x_i,t_{n-M'}) = \frac{\phi_i^{n-M'} - \phi_i^{n-M'-1}}{\Delta t},\ 1 \le i \le N,\ 1 \le n \le M.$$

where $\phi = \phi(x,t)$ be a function with second order partial derivatives. As a result, our problem consists to find $(\hat{z}, \hat{u}, \theta, q, \varphi)$ satisfying the discrete formulation of the system (2.1.3) - (2.1.4) presented by the following numerical scheme

$$\begin{cases} \dfrac{\rho_z}{\Delta t}(\widehat{z}_i^n - \widehat{z}_i^{n-1}) = \dfrac{\alpha_1}{h^2}(z_{i+1}^n - 2z_i^n + z_{i-1}^n) + \dfrac{\alpha_2}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \\[2ex] \dfrac{\rho_u}{\Delta t}(\widehat{u}_i^n - \widehat{u}_i^{n-1}) = \dfrac{\alpha_3}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \dfrac{\alpha_2}{h^2}(z_{i+1}^n - 2z_i^n + z_{i-1}^n) \\ +\dfrac{\beta}{2h}(\theta_{i+1}^n - \theta_{i-1}^n) - \dfrac{\mu}{\Delta t}\left(u_i^{n-M'} - u_i^{n-M'-1}\right), \\[2ex] \dfrac{c}{\Delta t}(\theta_i^n - \theta_i^{n-1}) = \dfrac{-1}{2h}(q_{i+1}^n - q_{i-1}^n) + \dfrac{\beta}{2h}(\widehat{u}_{i+1}^n - \widehat{u}_{i-1}^n), \\[2ex] \dfrac{\tau_0}{\Delta t}(q_i^n - q_i^{n-1}) = -q_i^n - \dfrac{k}{2h}(\theta_{i+1}^n - \theta_{i-1}^n), \\[2ex] \dfrac{\tau}{\Delta t}\left(\varphi_{i,j}^n - \varphi_{i,j}^{n-1}\right) = -\dfrac{1}{2h}\left(\varphi_{i,j+1}^n - \varphi_{i,j-1}^n\right), \end{cases} \tag{2.3.59}$$

where $z_i^n = z(x_i,t_n)$, $\hat{z}_i^n = z_t(x_i,t_n)$, $u_i^n = u(x_i,t_n)$, $\hat{u}_i^n = u_t(x_i,t_n)$, $\theta_i^n = \theta(x_i,t_n)$, $q_i^n = q(x_i,t_n)$, $u_i^{n-M'} = u(x_i,t_{n-M'})$, $\varphi_{i,j}^n = \varphi(x_i,\rho_j,t_n)$ for all $i = 1,2,...,N$, $j = 1,2,...,m$ and $n = 1,2,...,M$, with $\varphi(x_i,\rho_{m+1},t_n) = u_i^{n-M'}$ according to (2.1.1). To simplicity our numerical calculations in our scheme, we consider the discrete boundary conditions given by

$$z_0^n = u_0^n = \theta_{N+1}^n = q_0^n = 0,\ z_{N+1}^n = z_N^n,\ u_{N+1}^n = u_N^n,\ \theta_1^n = \theta_0^n, \tag{2.3.60}$$

and initial conditions

$$\begin{cases} z_i^0 = z_0\left(x_i\right), u_i^0 = u_0\left(x_i\right), \theta_i^0 = \theta_0\left(x_i\right), q_i^0 = q_0\left(x_i\right) \\ \widehat{z}_i^0 = z_1\left(x_i\right) \text{ and } \widehat{u}_i^0 = u_1\left(x_i\right) \\ \varphi_{i,0}^n = \widehat{u}_i^n,\ u_i^{n-M'} = f_0\left(x_i, t_{n-M'}\right),\ \varphi_{i,j}^0 = f_0\left(x_i, \rho_j.t_{-M'}\right),\ t_{n-M'} \in \Upsilon_M, \\ \varphi_{N+1,N+1}^n = \hat{u}_{N+1}^{n-M'} \end{cases} \tag{2.3.61}$$

where

$$z_i^n = z_i^{n-1} + \Delta t \widehat{z}_i^n \text{ and } u_i^n = u_i^{n-1} + \Delta t \widehat{u}_i^n$$

for all $i = 1, 2, ..., N$ and $n = 1, 2, ..., M$.

Note that to find $\{\widehat{z}^n, \widehat{u}^n, \theta^n, q^n, \varphi^n\}$, we need to solve five coupled systems of algebraic equations. So, to solve the problem (2.3.59)-(2.3.61) at each time step we propose to consider the following fixed-point algorithm:

$$\begin{cases} \widehat{z}_i^{n,l} = \dfrac{\alpha_1 \Delta t}{\rho_z h^2}(z_{i+1}^{n,l-1} - 2z_i^{n,l-1} + z_{i-1}^{n,l-1}) + \dfrac{\alpha_2 \Delta t}{\rho_z h^2}(u_{i+1}^{n,l-1} - 2u_i^{n,l-1} + u_{i-1}^{n,l-1}) + \widehat{z}_i^{n-1}, \\ \widehat{u}_i^{n,l} = \dfrac{\alpha_3 \Delta t}{\rho_u h^2}(u_{i+1}^{n,l-1} - 2u_i^{n,l-1} + u_{i-1}^{n,l-1}) + \dfrac{\alpha_2 \Delta t}{\rho_u h^2}(z_{i+1}^{n,l-1} - 2z_i^{n,l-1} + z_{i-1}^{n,l-1}) \\ + \dfrac{\beta \Delta t}{2\rho_u h}\left(\theta_{i+1}^{n,l-1} - \theta_{i-1}^{n,l-1}\right) + \widehat{u}_i^{n-1} - \dfrac{\mu}{\rho_u}\left(u_i^{n-M'} - u_i^{n-M'-1}\right), \\ \theta_i^{n,l} = \theta_i^{n-1} - \dfrac{\Delta t}{2ch}\left(q_{i+1}^{n,l-1} - q_{i-1}^{n,l-1}\right) + \dfrac{\beta \Delta t}{2ch}(\widehat{u}_{i+1}^{n,l} - \widehat{u}_{i-1}^{n,l}), \\ (1 + \dfrac{\Delta t}{\tau_0})q_i^{n,l} = -\dfrac{k\Delta t}{2h\tau_0}\left(\theta_{i+1}^{n,l} - \theta_{i-1}^{n,l}\right) + q_i^{n-1}, \\ \varphi_{i,j}^{n,l} = \varphi_{i,j}^{n-1,l-1} - \dfrac{\Delta t}{2h\tau}\left(\varphi_{i,j+1}^{n,l-1} - \varphi_{i,j-1}^{n,l-1}\right) \end{cases} \tag{2.3.62}$$

with

$$\begin{cases} z_i^{n,0} = z_i^{n-1},\ u_i^{n,0} = u_i^{n-1},\ \theta_i^{n,0} = \theta_i^{n-1},\ q_i^{n,0} = q_i^{n-1},\ z_i^{n,l} = z_i^{n-1,l} + \Delta t \widehat{z}_i^{n,l}, \\ u_i^{n,l} = u_i^{n-1,l} + \Delta t \widehat{u}_i^{n,l},\ \varphi_{i,j}^{n,0} = \hat{u}_{i,j}^{n-M'}, \end{cases} \tag{2.3.63}$$

for all $i, j = 1, 2, ..., N$, $n = 1, 2, ..., M$ and $l = 1, 2, ...$

At each time step, we solve the scheme (2.3.62) by an iterative procedure that was stopped when the difference between two successive iterations became smaller than a given tolerance ε.

2.3.1 Convergence of the proposed point fixed algorithm

Let $\widehat{Z}^{n,l} = \left(\widehat{z}_i^{n,l}\right)_{1\leq i\leq N}$, $\widehat{U}^{n,l} = \left(\widehat{u}_i^{n,l}\right)_{1\leq i\leq N}$, $\widehat{\Theta}^{n,l} = \left(\widehat{\theta}_i^{n,l}\right)_{1\leq i\leq N}$, $\widehat{Q}^{n,l} = \left(\widehat{q}_i^{n,l}\right)_{1\leq i\leq N}$, $\vartheta^{n,l} = \left(\varphi_{i,j}^{n,l}\right)_{1\leq i,j\leq N}$. Then, the system (2.3.62)-(2.3.63) can be rewritten as follows

$$\begin{cases} \widehat{Z}^{n,l} = \dfrac{\alpha_1\,\Delta t}{\rho_z h^2} A\, Z^{n,l-1} + \dfrac{\alpha_2\,\Delta t}{\rho_z h^2} A\, U^{n,l-1} + \widehat{Z}^{n-1} \\[2ex] \widehat{U}^{n,l} = \dfrac{\alpha_3\,\Delta t}{\rho_u h^2} A\, U^{n,l-1} + \dfrac{\alpha_2\,\Delta t}{\rho_u h^2} A\, Z^{n,l} + \dfrac{\beta\,\Delta t}{2\rho_u h} B\,\Theta^{n,l-1} + \widehat{U}^{n-1} \\ -\dfrac{\mu}{\rho_u}\left(U^{n-M'} - U^{n-M'-1}\right) \\[2ex] \Theta^{n,l} = \Theta^{n-1} - \dfrac{\Delta t}{2c\,h} C\, Q^{n,l-1} + \dfrac{\beta\,\Delta t}{2c\,h} D\,\widehat{U}^{n,l} \\[2ex] \left(1 + \dfrac{\Delta t}{\tau_0}\right) Q^{n,l} = -\dfrac{k\,\Delta t}{2h\,\tau_0} B\,\Theta^{n,l} + Q^{n-1} \\[2ex] \vartheta^{n,l} = \vartheta^{n-1,l-1} - \dfrac{\Delta t}{2h\,\tau} E\,\vartheta^{n,l-1} \end{cases} \tag{2.3.64}$$

with

$$\begin{cases} Z^{n,0} = Z^{n-1},\ U^{n,0} = U^{n-1},\ \Theta^{n,0} = \Theta^{n-1},\ Q^{n,0} = Q^{n-1},\ Z^{n,l} = Z^{n-1,l} + \Delta t \widehat{Z}^{n,l}, \\ U^{n,l} = U^{n-1,l} + \Delta t \widehat{U}_i^{n,l},\ \vartheta^{n,0} = \widehat{U}^{n-M'} \end{cases} \tag{2.3.65}$$

where A, B, C, D are real matrices of dimensions $(n \times n)$ and E is a real matrix of dimension $\left(n^2 \times n^2\right)$ defined as follows

$$A = diag\,(1,-2,1)\,,\ D = diag(-1,0,1),$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} D & E_1 & \cdots & E_1 \\ E_1 & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & E_1 \\ E_1 & \cdots & E_1 & D \end{pmatrix},\ E_1 = 0_{\mathbb{R}^n\times\mathbb{R}^n}$$

Now, we are ready to state and prove the following convergence result

Lemma 2.3.1. *Let $\left(\widehat{Z}^{n,l}, \widehat{U}^{n,l}, \widehat{\Theta}^{n,l}, \widehat{Q}^{n,l}, \vartheta^{n,l}\right)$ be the solution of the system (2.3.64)-(2.3.65). Then, the fixed point algorithm proposed to solve the system (2.3.64)-(2.3.65) converges if and only if the following condition is satisfied*

$$\Delta t < min\left\{\frac{2h\,\tau}{]|E|[}, h\sqrt{\frac{\rho_u}{\alpha_3\,]|A|[}}, h\sqrt{\frac{\rho_u}{\alpha_1\,]|A|[}}, h\sqrt{\frac{\rho_z}{\alpha_1\,]|A|[}}, (\Delta t)_2\right\}, \tag{2.3.66}$$

where $(\Delta t)_2 = \dfrac{1-\sqrt{1+4r\tau_0}}{2r}$ *with* $r = \dfrac{k}{2c\,h^2}\,]|C|[\,]|B|[$ *and* $]|.|[$ *represent a subordinate matrix norm to the vector norm* $\|.\|_V$ *.*

Proof. From $(2.3.64)_1$ we can write

$$\left\|\widehat{Z}^{n,l+1} - \widehat{Z}^{n,l}\right\|_V \leq \frac{\alpha_1\Delta t}{\rho_z h^2}\ ]|A|[\ \left\|Z^{n,l} - Z^{n,l-1}\right\|_V + \frac{\alpha_2\Delta t}{\rho_z h^2}\ ]|A|[\ \left\|U^{n,l} - U^{n,l-1}\right\|_V. \tag{2.3.67}$$

On the other hand, we use the fact that

$$\begin{cases} Z^{n,l} = Z^{n-1} + \Delta t\,\widehat{Z}^{n,l}, \\ Z^{n,l-1} = Z^{n-1} + \Delta t\,\widehat{Z}^{n,l-1}, \end{cases}$$

and

$$\begin{cases} U^{n,l} = U^{n-1} + \Delta t\, \widehat{U}^{n,l}, \\ U^{n,l-1} = U^{n-1} + \Delta t\, \widehat{U}^{n,l-1}. \end{cases}$$

Then, (2.3.67) can be rewritten as follows

$$\left\| \widehat{Z}^{n,l+1} - \widehat{Z}^{n,l} \right\|_V \leq \frac{\alpha_1 (\Delta t)^2}{\rho_z h^2} \ ]|A|[\ \left\| \widehat{Z}^{n,l} - \widehat{Z}^{n,l-1} \right\|_V + \frac{\alpha_2 (\Delta t)^2}{\rho_z h^2} \ ]|A|[\ \left\| \widehat{U}^{n,l} - \widehat{U}^{n,l-1} \right\|_V .$$

After $l$iterations, we get

$$\begin{aligned} \left\| \widehat{Z}^{n,l+1} - \widehat{Z}^{n,l} \right\|_V \leq & \left(\frac{\alpha_1 (\Delta t)^2}{\rho_z h^2} \ ]|A|[\right)^l \left\| \widehat{Z}^{n,1} - \widehat{Z}^{n,0} \right\|_V \\ & + \alpha_2 \sum_{k=0}^{k=l-1} \alpha_1^k \left(\frac{(\Delta t)^2}{\rho_z h^2} \ ]|A|[\right)^{k+1} \left\| \widehat{U}^{n,l-k} - \widehat{U}^{n,l-k-1} \right\|_V . \end{aligned} \tag{2.3.68}$$

By using with $(2.3.64)_2$, we can easily obtain

$$\begin{aligned} \left\| \widehat{U}^{n,l+1} - \widehat{U}^{n,l} \right\|_V \leq & \left(\frac{\alpha_3 (\Delta t)^2}{\rho_u h^2} \ ]|A|[\right)^l \left\| \widehat{U}^{n,1} - \widehat{U}^{n,0} \right\|_V \\ & + \frac{\alpha_2 (\Delta t)^2}{\rho_u h^2} \ ]|A|[\ \left\| \widehat{Z}^{n,l+1} - \widehat{Z}^{n,l} \right\|_V \\ & + \frac{\beta \Delta t}{2\rho_u\, h}]|B|[\ \left\| \Theta^{n,l} - \Theta^{n,l-1} \right\|_V . \end{aligned} \tag{2.3.69}$$

Also, from $(2.3.64)_3$ and $(2.3.64)_4$, we have

$$\begin{aligned} \left\| \Theta^{n,l+1} - \Theta^{n,l} \right\|_V \leq & \frac{\Delta t}{2c\, h}]|C|[\ \left\| Q^{n,l} - Q^{n,l-1} \right\|_V \\ & + \frac{\beta\, \Delta t}{2c\, h}]|D|[\ \left\| \widehat{U}^{n,l+1} - \widehat{U}^{n,l} \right\|_V , \end{aligned} \tag{2.3.70}$$

and

$$\left\| Q^{n,l+1} - Q^{n,l} \right\|_V \leq \frac{\mu_1\, k\, \Delta t}{2\tau_0\, h}]|B|[\ \left\| \Theta^{n,l+1} - \Theta^{n,l} \right\|_V , \ \mu_1 = \frac{\tau_0}{1 + \Delta t}. \tag{2.3.71}$$

From $(2.3.64)_5$, we can write

$$\left\| \vartheta^{n,l+1} - \vartheta^{n,l} \right\|_V \leq \left\| \vartheta^{n-1,l} - \vartheta^{n-1,l-1} \right\|_V + \frac{\Delta t}{2h\, \tau}]|E|[\ \left\| \vartheta^{n,l} - \vartheta^{n,l-1} \right\|_V , \tag{2.3.72}$$

we estimate the last term of (2.3.72), every time, and replacing the result in (2.3.72), we obtain

$$\left\|\vartheta^{n,l+1}-\vartheta^{n,l}\right\|_V \leq \sum_{i=0}^{n_0}\sum_i+\sum_{i=1}^{n_0}\Gamma_i,\ n_0\geq 1, \tag{2.3.73}$$

with

$$\sum_0=\mu_2^l\left\|\vartheta^{n,1}-\vartheta^{n,0}\right\|_V,\ \mu_2=\frac{\Delta t}{2h\,\tau}]|E|[,$$

and

$$\sum_i=\sum_{\sigma_1=0}^{l-1}\sum_{\sigma_2=0}^{l-\sigma_1-2}\sum_{\sigma_3=0}^{l-(\sigma_1+\sigma_2)-3}\cdots\sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j-i}\mu_2^{l-i}\left\|\vartheta^{n-i,1}-\vartheta^{n-i,0}\right\|_V,\ i\geq 1,$$

where

$$\varsigma_j=\begin{cases}0,\ if\ j=0,\\ \sigma_j,\ Otherwise,\end{cases} \tag{2.3.74}$$

and for all $i\geq 1$

$$\Gamma_i=\sum_{\sigma_1=0}^{l-1}\sum_{\sigma_2=0}^{l-\sigma_1-2}\sum_{\sigma_3=0}^{l-(\sigma_1+\sigma_2)-3}\cdots\sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j-i}\mu_2^{\sum_{k=1}^{i}\sigma_k}\left\|\vartheta^{n-i,l-\sum_{k=1}^{i}\sigma_k-(i-1)}-\vartheta^{n-i,l-\sum_{k=1}^{i}\sigma_k-i}\right\|_V.$$

For n_0 sufficiently large such that $\sum_{k=1}^{i}\sigma_k=l-i$, the estimate (2.3.73) becomes

$$\left\|\vartheta^{n,l+1}-\vartheta^{n,l}\right\|_V\leq\sum_{i=0}^{n_0}\sum_i+\sum_{i=0}^{n_0}\sum_{\sigma_1=0}^{l-1}\sum_{\sigma_2=0}^{l-\sigma_1-2}\cdots\sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j-i}\mu_2^{l-i}\left\|\vartheta^{n-i,1}-\vartheta^{n-i,0}\right\|_V.$$

In order to $\left(\vartheta^{n,l}\right)_{l\geq 0}$ converge for $l\to\infty$, it is necessary that $\mu_2<1$, which leads to

$$\Delta t<\frac{2h\,\tau}{]|E|[}. \tag{2.3.75}$$

Now, by replacing (2.3.68) into (2.3.69), we have

$$\begin{aligned}\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V \leq & \left(\frac{\alpha_3(\Delta t)^2}{\rho_u h^2}\,]|A|[\right)^l \left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V \\ & + \alpha_2\alpha_1^l\left(\frac{(\Delta t)^2}{\rho_z h^2}\,]|A|[\right)^{l+1}\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V \\ & + \alpha_2^2\sum_{k=0}^{l-1}\alpha_1^k\left(\frac{(\Delta t)^2}{\rho_z h^2}\,]|A|[\right)^{k+2}\left\|\widehat{U}^{n,l-k}-\widehat{U}^{n,l-k-1}\right\|_V \\ & + \frac{\beta\,\Delta t}{2\rho_u h}\,]|B|[\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V . \end{aligned} \tag{2.3.76}$$

From (2.3.70), we get

$$\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V \leq \frac{\Delta t}{2c\,h}\,]|C|[\left\|Q^{n,l-1}-Q^{n,l-2}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\,]|D|[\left\|\widehat{U}^{n,l}-\widehat{U}^{n,l-1}\right\|_V .$$

From (2.3.71), we arrive at

$$\left\|Q^{n,l-1}-Q^{n,l-2}\right\|_V \leq \frac{\mu_1\,k\,\Delta t}{2\tau_0\,h}\,]|B|[\left\|\Theta^{n,l-1}-\Theta^{n,l-2}\right\|_V ,$$

then,

$$\begin{aligned}\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V & \leq \gamma_0\left\|\Theta^{n,l-1}-\Theta^{n,l-2}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\,]|D|[\left\|\widehat{U}^{n,l}-\widehat{U}^{n,l-1}\right\|_V \\ & \ \vdots \\ & \leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\,]|D|[\sum_{k=0}^{l-2}\gamma_0^k\left\|\widehat{U}^{n,l-k}-\widehat{U}^{n,l-k-1}\right\|_V ,\end{aligned}$$

where

$$\gamma_0 = \frac{k\mu_1(\Delta t)^2}{2c\,h^2\tau_0}\,]|C|[\,]|B|[.$$

Therefore, the estimate (2.3.76) becomes

$$\begin{aligned}
&\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V\\
&\leq \left(\left(\frac{\alpha_3(\Delta t)^2}{\rho_u h^2}]|A|[\right)^l+\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{l+1}\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V\\
&+\frac{\alpha_2}{\alpha_1}\left(\frac{\alpha_1(\Delta t)^2}{\rho_u h^2}]|A|[\right)^{l+1}\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V+\frac{\beta\,\Delta t}{2\rho_u h}]|B|[\ \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V\\
&+\sum_{k=0}^{l-2}\left(\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{k+2}+\frac{1}{c\,\rho_u}\left(\frac{\beta\,\Delta t}{2h}\right)^2]|B|[]|D|[\gamma_0^k\right)\left\|\widehat{U}^{n,l-k}-\widehat{U}^{n,l-k-1}\right\|_V.
\end{aligned}$$

To reduce this last estimation, we use the following notation

$$\begin{aligned}
\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V &\leq \chi_{1,l}\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V+\chi_{2,l}\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V\\
&\quad+\chi_{3,l}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{\sigma_1=0}^{l-2}\xi_{\sigma_1}\left\|\widehat{U}^{n,l-\sigma_1}-\widehat{U}^{n,l-\sigma_1-1}\right\|_V,
\end{aligned}\tag{2.3.77}$$

where

$$\chi_{1,l}=\left(\left(\frac{\alpha_3(\Delta t)^2}{\rho_u h^2}]|A|[\right)^l+\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{l+1}\right),$$

$$\chi_{2,l}=\frac{\alpha_2}{\alpha_1}\left(\frac{\alpha_1(\Delta t)^2}{\rho_u h^2}]|A|[\right)^{l+1},$$

$$\chi_{3,l}=\frac{\beta\,\Delta t}{2\rho_u h}]|B|[\ \gamma_0^{l-1},$$

$$\xi_{\sigma_1}=\left(\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{\sigma_1+2}+\frac{1}{c\,\rho_u}\left(\frac{\beta\,\Delta t}{2h}\right)^2]|B|[]|D|[\gamma_0^{\sigma_1}\right).$$

By using the same iterative process as in the estimation of (2.3.72), we can estimate the last term of (2.3.77) as follows

$$\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V \leq \left(\chi_{1,l}+\sum_{i=1}^{n_0-1}\Upsilon_i\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V+\left(\chi_{2,l}+\sum_{i=1}^{n_0-1}\Psi_i\right)\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V$$
$$+\left(\chi_{3,l}+\sum_{i=1}^{n_0-1}\Phi_i\right)\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{i=1}^{n_0}\Lambda_i,$$

with

$$\Upsilon_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{1,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi_{\sigma_k}$$

$$\Psi_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{2,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi_{\sigma_k},$$

$$\Phi_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{3,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi_{\sigma_k},$$

$$\Lambda_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i}\varsigma_j-(i+1)}\prod_{i=1}^{n_0}\xi_{\sigma_i}\left\|U^{n,l-\sum\limits_{i=1}^{n_0}\sigma_i-(n_0-1)}-U^{n,l-\sum\limits_{i=1}^{n_0}\sigma_i-n_0}\right\|_V,$$
$$\xi_{\sigma_k}=\left(\left(\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)\right)^{\sigma_k+2}+\frac{1}{c\,\rho_u}\left(\frac{\beta\,\Delta t}{2h}\right)^2]|B|[]|D|[\,\gamma_0^{\sigma_k}\right),\ k=\overline{1,i},$$

and ς_j is defined by (2.3.74).

Until $\sum\limits_{i=0}^{n_0}\sigma_i=l-n_0-1,\ (l>n_0+1)$,

$$\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V \leq \left(\chi_{1,l}+\sum_{i=1}^{n_0-1}\Upsilon_i\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V+\left(\chi_{2,l}+\sum_{i=1}^{n_0-1}\Psi_i\right)\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V$$
$$+\left(\chi_{3,l}+\sum_{i=1}^{n_0-1}\Phi_i\right)\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{i=0}^{n_0}\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i}\varsigma_j-(i+1)}\prod_{i=1}^{n_0}\xi_{\sigma_i}\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V.$$

In order to $\left(\widehat{U}^{n,l}\right)_{l\geq 0}$ converge, it's necessary and sufficient to the following conditions hold

$$\left\{\begin{array}{l}\dfrac{\alpha_3(\Delta t)^2}{\rho_u h^2}\,]|A|[<1\\ \dfrac{\alpha_1(\Delta t)^2}{\rho_u h^2}\,]|A|[<1\\ \dfrac{\alpha_1(\Delta t)^2}{\rho_z h^2}\,]|A|[<1\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\Delta t<h\sqrt{\dfrac{\rho_u}{\alpha_3\,]|A|[}}\\ \Delta t<h\sqrt{\dfrac{\rho_u}{\alpha_1\,]|A|[}}\\ \Delta t<h\sqrt{\dfrac{\rho_z}{\alpha_1\,]|A|[}}\end{array}\right. \tag{2.3.78}$$

and

$$\gamma_0<1.$$

This last condition is equivalent to

$$r\frac{(\Delta t)^2}{(\tau_0+\Delta t)}<1, \tag{2.3.79}$$

with

$$r=\frac{k}{2c\,h^2}\,]|C|[\,]|B|[,$$

the inequation (2.3.79) holds if and only if

$$\Delta t\in\,](\Delta t)_1\,,(\Delta t)_2[,$$

where

$$(\Delta t)_1=\frac{1-\sqrt{1+4r\tau_0}}{2r}<0,\quad (\Delta t)_2=\frac{1+\sqrt{1+4r\tau_0}}{2r}.$$

Because $\Delta t>0$, Therefore, (2.3.79) holds if and only if

$$\Delta t\in\,]0,(\Delta t)_2[\,. \tag{2.3.80}$$

By the same technique, we can easily obtain

$$\begin{aligned}\left\|\widehat{Z}^{n,l+1}-\widehat{Z}^{n,l}\right\|_V &\leq \left(\chi_{4,l}+\sum_{i=1}^{n_0-1}\Psi_i^{'}\right)\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V+\left(\chi_{5,l}+\sum_{i=1}^{n_0-1}\sum\nolimits_i^{'}\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V\\ &+\left(\sum_{i=1}^{n_0-1}\Phi_i^{'}\right)\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{i=0}^{n_0}\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum_{j=0}^{i}\varsigma_j-(i+1)}\prod_{i=1}^{n_0}\acute{\xi}_{\sigma_i}\left\|\widehat{U}^{n,2}-\widehat{U}^{n,1}\right\|_V,\end{aligned}$$

with

$$\acute{\xi}_{\sigma_k} = \left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}\,]|A|[\right)^{\sigma_k+1}, k=\overline{1,i},$$

$$\sum\nolimits_i' = \sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)} \chi_{1,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\acute{\xi}_{\sigma_k},$$

$$\Psi_i' = \sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)} \chi_{2,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\acute{\xi}_{\sigma_k},$$

$$\Phi_i' = \sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)} \chi_{3,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\acute{\xi}_{\sigma_k},$$

we deduce that under the same conditions (2.3.78) on Δt , the sequence $\left(\widehat{Z}^{n,l}\right)_{l\geq 0}$ converges. From (2.3.70)

$$\begin{aligned}\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V &\leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\,]|D|[\sum_{\sigma_1=0}^{l-2}\gamma_0^{\sigma_1}\left\|\widehat{U}^{n,l-\sigma_1}-\widehat{U}^{n,l-\sigma_1-1}\right\|_V\\ &\leq\leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V + \sum_{\sigma_1=0}^{l-2}\xi''_{\sigma_1}\left\|\widehat{U}^{n,l-\sigma_1}-\widehat{U}^{n,l-\sigma_1-1}\right\|_V,\end{aligned}$$

where

$$\xi''_{\sigma_1} = \gamma_0^{\sigma_1}\frac{\beta\,\Delta t}{2c\,h}\,]|D|[.$$

By using the same technique as in the estimation of the last term of (2.3.77), we deduce that

$$\begin{aligned}\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V &\leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V\\ &+\sum_{i=1}^{n_0-1}\left(\sum\nolimits_i''\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V+\Psi_i''\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V+\Phi_i''\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V\right)\\ &+\sum_{i=1}^{n_0}\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\delta_1-3}\cdots\sum_{\sigma_i=0}^{n_0+1-\sum\limits_{j=i-1}^{n_0}\sigma_j-(i+1)}\prod_{i=1}^{n_0}\xi''_{\sigma_i}\left\|\widehat{U}^{n,2}-\widehat{U}^{n,1}\right\|_V\end{aligned}$$

with

$$\xi''_{\sigma_k} = \gamma_0^{\sigma_k} \frac{\beta\,\Delta t}{2c\,h}]|D|[\, , \; k = \overline{1,i} \, , \; \sum_i'' = \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1} \varsigma_j-(i+1)} \chi_{1,l-\sum_{k=1}^{i}\sigma_k-i} \cdot \prod_{k=1}^{i} \xi''_{\sigma_k},$$

$$\Psi_i'' = \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1} \varsigma_j-(i+1)} \chi_{2,l-\sum_{k=1}^{i}\sigma_k-i} \cdot \prod_{k=1}^{i} \xi''_{\sigma_k}, ,$$

$$\Phi_i'' = \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1} \varsigma_j-(i+1)} \chi_{3,l-\sum_{k=1}^{i}\sigma_k-i} \cdot \prod_{k=1}^{i} \xi''_{\sigma_k} \, .$$

In order to the sequence $\left(\Theta^{n,l}\right)_{l\geq 0}$ converges, it's necessary and sufficient that $\gamma_0 < 1$. From (2.3.71), if $\left(\Theta^{n,l}\right)_{l\geq 0}$ converges, then, $\left(Q^{n,l}\right)_{l\geq 0}$ converges under the same condition on $\left(\Theta^{n,l}\right)_{l\geq 0}$. Therefore, from (2.3.75), (2.3.78) and (2.3.80), the fixed point iterative scheme converges if and only if Δt verifies (2.3.66). □

2.3.2 Approximation of the discrete energy

To approximate the continuous energy (2.2.17), we use the trapezoidal quadrature formula to compute the integral $I = \int_0^L f(x)dx$

$$I_N = \sum_{i=0}^{N+1} a_i f(x_i) \simeq I,$$

where the weights $\{a_i\}_{i=1}^{N}$ are given by $a_0 = a_{N+1} = \frac{h}{2}$ and for $i = 1, 2, ..., N$, $a_i = h$. Concerning the trapezoidal quadrature formula in tow dimensional case to compute the last part of (2.2.17), we use the following approximation

$$\int_0^L \int_0^1 f(x,y)dydx \simeq \frac{h^2}{4}\left(f(x_0,y_0) + f(x_0,y_{N+1}) + f(x_{N+1},y_0) + f(x_{N+1},y_{N+1})\right) + h^2 \sum_{i=1}^{N}\sum_{j=1}^{N} f(x_i,y_j)$$
$$+ \frac{h^2}{2}\sum_{i=1}^{N}\left(f(x_i,y_0) + f(x_i,y_{N+1})\right) + \frac{h^2}{2}\sum_{j=1}^{N}\left(f(x_0,y_j) + f(x_{N+1},y_j)\right)$$

Therefore, the discrete energy at the time step t_n of system (2.3.59)-(2.3.61) is given by

$$E^n = \frac{1}{2}\sum_{i=0}^{N+1} a_i[\rho_z\left(\hat{z}_i^n\right)^2 + \rho_u\left(\hat{u}_i^n\right)^2 + c\left(\theta_i^n\right)^2 + \frac{\tau_0}{k}\left(q_i^n\right)^2 + \alpha_1\left((z_x)_i^n\right)^2 + \alpha_3\left((u_x)_i^n\right)^2 + 2b\left(u_x\right)_i^n\left(z_x\right)_i^n]$$
$$+\frac{|\mu|\,\tau}{2}[\frac{h^2}{4}\left(\left(\varphi_{0,0}^n\right)^2 + \left(\varphi_{0,N+1}^n\right)^2 + \left(\varphi_{N+1,0}^n\right)^2 + \left(\varphi_{N+1,N+1}^n\right)^2\right)$$
$$+\frac{h^2}{2}\sum_{j=1}^{N}\left(\varphi_{0,j}^n\right)^2 + \frac{h^2}{2}\sum_{j=1}^{N}\left(\varphi_{N+1,j}^n\right)^2 + \frac{h^2}{2}\sum_{i=1}^{N}\left(\varphi_{i,0}^n\right)^2 + \frac{h^2}{2}\sum_{i=1}^{N}\left(\varphi_{i,N+1}^n\right)^2 + h^2\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\varphi_{i,j}^n\right)^2], \tag{2.3.81}$$

with

$$\hat{u}_i^n = u_t\left(x_i, t_n\right)\;,\quad \left(u_x\right)_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h}\quad,\quad \hat{z}_i^n = z_t\left(x_i, t_n\right)\;,\quad \left(z_x\right)_i^n = \frac{z_{i+1}^n - z_{i-1}^n}{2h}.$$

which is a discretization of the continuous energy (2.2.17).

2.3.3 Numerical illustration

In the table below, we consider five different choices of delay's weight and we note that the asymptotic behavior was reached in the last case even if in the first four cases the weights of delay are not considerable. This shows the great influence of the delay on the stability of this type of system throughout time.

Case	Weight of delay	Iterations time	Asymptotic behavior
1	$\mu = 0.9$	200	Unreached
2	$\mu = 0.03$	200	Unreached
3	$\mu = 0.005$	200	Unreached
4	$\mu = 0.0004$	200	Unreached
5	$\mu = 0.00008$	200	Reached

Table 2.1: Asymptotic behavior for different cases of delay's weight.

In the next, we describe the following numerical example where the asymptotic behavior was reached, that is the case when $\mu_0 = 0.00008$ and for different choices of the system parameters with the condition (0.0.9) holds.

Test 1 For this numerical test, we choose the following different values for the coefficients of the system

$$\rho_z = 3.5,\ \alpha_3 = 0.2,\ \mu = 0.8,\ \alpha_2 = 0.1,\ \beta = 0.05,\ \tau = 0.1$$
$$\rho_u = 1.5,\ \alpha_1 = 0.1,\ c = 4,\ \tau_0 = 0.06,\ c = 1.5,\ k = 10^{-5}$$

We run our code for the following discretizations parameters: $N = 100$, $m = 150$, $M = 200$, $L = 1$, $T = 1$ and take $\varepsilon = 10^{-5}$. With the following initial conditions

$$z_0(x) = 10^{-2}\left(x^3 - \frac{3}{2}x^2\right),\ z_1(x) = \frac{1}{8}\left(2x^2 - 4x\right),\ u_0(x) = \frac{1}{8}\left(2x^2 - 2x\right),$$
$$u_1(x) = 0,\ \theta_0(x) = 0,\ q_0(x) = \frac{1}{5}x^3 e^{-\frac{3}{2}x^2},\ f_0(x, t-\tau) = \frac{1}{2}.10^{-3}\cos(x).\cos(\frac{1}{10\pi}(t-\tau))$$

with the above parameters choice, we deduce that

$$\frac{2h\tau}{]|E|[} = 4.901960,\ h\sqrt{\frac{\rho_u}{\alpha_3\,]|A|[}} = 0.013424,\ h\sqrt{\frac{\rho_u}{\alpha_1\,]|A|[}} = 0.018985,$$
$$h\sqrt{\frac{\rho_z}{\alpha_1\,]|A|[}} = 0.029000,\ (\Delta t)_2 = 34.766946.$$

with

$$]|\mathcal{M}|[= \max_i \sum_j |\mathcal{M}_{ij}|.$$

Note that

$$dt = 0.005 < 0.013424 = min\left\{\frac{2h\tau}{]|E|[}, h\sqrt{\frac{\rho_u}{\alpha_3\,]|A|[}}, h\sqrt{\frac{\rho_u}{\alpha_1\,]|A|[}}, h\sqrt{\frac{\rho_z}{\alpha_1\,]|A|[}}, (\Delta t)_2\right\}.$$

which confirms the convergence of the algorithm.

Here are the evolution in time of the solutions $z, u, \theta, q, \varphi(x, 1, t)$ and of the discrete energy.

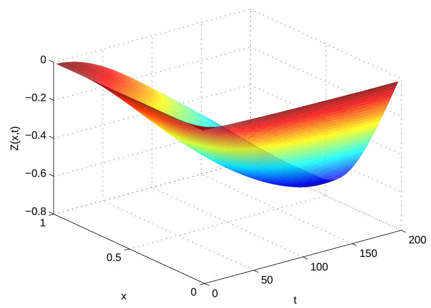

Figure 2.1: Evolution in time of the function z

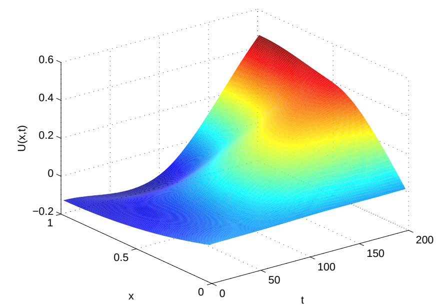

Figure 2.2: Evolution in time of the function u

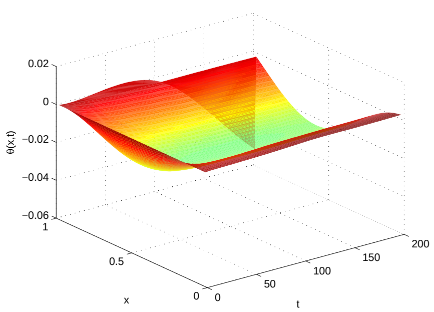

Figure 2.3: Evolution in time of the function θ

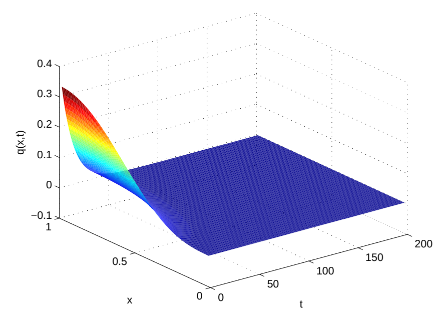

Figure 2.4: Evolution in time of the function q

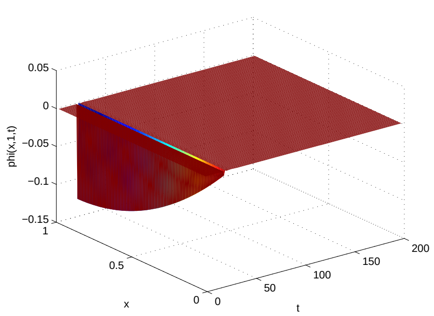

Figure 2.5: Evolution in time of the function $\varphi(x,1,t)$

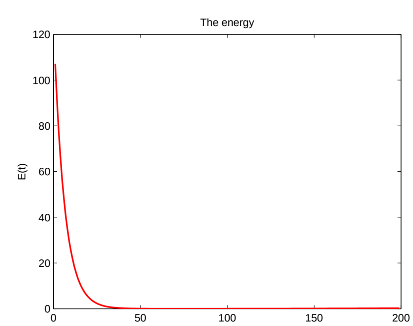

Figure 2.6: Evolution in time of the discrete energy

In above numerical test, the condition (2.3.66) holds and graphics presented in the Figures 1,5 show the evolution in time of the approximations solutions z, u, θ, q and $\varphi(x,1,t)$ on the interval $[0,T]$, for different choices of the system parameters and of the initial data. Furthermore, the Figure 6 shows that the approximate energy (2.3.81) decays in an exponential manner which confirms the main theoretical result obtained.

Chapter 3

Exponential Decay Of A Heat Porous Thermoelastic System In The Presence Of Distributed Delay Without Frictional Damping

In this chapter, we consider a one-dimensional porous thermoelastic system with herditary heat conduction and a distributed delay (0.0.21)- (0.0.22) without frictional damping, where the heat conduction is given by Gurtin Pipkin law (0.0.17). The existence and uniqueness of the solution are obtained by the use of Hille-Yosida theorem. Then, based on the energy method and by constructing a suitable Lyapunov functional, we prove under some assumptions on the derivative of the heat-flux kernel, that the solution of the system decays exponentially without any assumptions on the wave speeds.

3.1 Preliminaries

We note that the presence of the convolution term in the constitutive equation for q renders the family operators mapping the initial value $(u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ f_0)$ into the solution (u, φ, θ) not match the semigroup properties. This is due to the fact that the solution value of θ at time t depends on the whole function up to time t. In order to overcome this difficulty we introduce the new variables

$$\theta^t(x,s) = \theta(x, t-s) \quad , \quad s \geq 0, \tag{3.1.1}$$

and

$$\eta(x,s) = \eta^t(x,s) = \int_0^s \theta^t(x,\lambda)\, d\lambda \quad , \quad s \geq 0, \tag{3.1.2}$$

which denote the past history and the summed past history of θ up to t, respectively.

Clearly, $\eta^t(x,s)$ satisfies the following conditions

$$\eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \ , \ t > 0,$$

$$\eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0,$$

$$\eta(x,0) = \lim_{s\to 0} \eta^t(x,s) = 0 \quad , \quad x \in (0,\pi) \ , \ t > 0,$$

it is easy to demonstrate that

$$\eta_t(x,s) = \theta - \eta_s(x,s) \qquad in \ \ (0,\pi) \times (0,\infty) \times (0,\infty). \tag{3.1.3}$$

Furthermore, assuming that $\lim\limits_{s\to\infty} k(s)=0$, then, a simple computations leads to

$$q=-\int_{-\infty}^{t} k(t-s)\,\theta_x(x,s)\,ds=\int_{0}^{\infty} k'(s)\,\eta_x^t(x,s)\,ds,$$

setting $\kappa(s)=-k'(s)$, system (0.0.21)-(0.0.22) and equation (3.1.3) takes the form

$$\begin{cases}
\rho_1\, u_{tt}=\mu\, u_{xx}+b\,\varphi_x-\beta\,\theta_x-\displaystyle\int_{\tau_1}^{\tau_2}\gamma(\sigma)\,u_t(x,t-\sigma)\,d\sigma & in\ (0,\pi)\times(0,\infty)\,,\\
J\,\varphi_{tt}=\alpha\,\varphi_{xx}-b\,u_x-\xi\,\varphi+\delta\,\theta-\tau\,\varphi_t & in\ (0,\pi)\times(0,\infty)\,,\\
c\,\theta_t=\displaystyle\int_{0}^{\infty}\kappa(s)\,\eta_{xx}^t(x,s)\,ds-\beta\,u_{xt}-\delta\,\varphi_t & in\ (0,\pi)\times(0,\infty)\,,\\
\eta_t(x,s)=\theta-\eta_s(x,s) & in\ (0,\pi)\times(0,\infty)\times(0,\infty)\,,
\end{cases}\tag{3.1.4}$$

with the boundary conditions and the initial data

$$\begin{cases}
u(0,t)=u(\pi,t)=\varphi_x(0,t)=\varphi_x(\pi,t)=\theta_x(0,t)=\theta_x(\pi,t)=0\ ,\ t>0,\\
\eta_x(0,s)=\eta_x(\pi,s)=0\ ,\quad s\geq 0\ ,\ t>0,\\
u(x,0)=u_0(x)\quad,\quad \varphi(x,0)=\varphi_0(x)\quad,\quad \theta(x,0)=\theta_0(x)\quad,\quad x\in(0,\pi)\,,\\
u_t(x,0)=u_1(x)\quad,\quad \varphi_t(x,0)=\varphi_1(x)\quad,\quad x\in(0,\pi)\,,\\
\eta^0(x,s)=\eta_0(x,s)\quad,\quad x\in(0,\pi)\quad,\quad s\geq 0,\\
\eta(x,0)=0\quad,\quad x\in(0,\pi)\ \ ,\ t>0,\\
u_t(x,-t)=f_0(x,t)\quad,\quad x\in(0,\pi)\quad,\quad t\in(0,\tau_2)\,.
\end{cases}\tag{3.1.5}$$

Concerning the memory kernel κ, we assume the following set of hypotheses:

$(H1):\kappa\in C\left(IR^+\right)\cap L^1\left(IR^+\right)$,

$(H2):\kappa(s)\geq 0\ ,\ \kappa'(s)\leq 0\ ,\ \ \forall s\geq 0\ ,\ \ \kappa(0)>0$,

$(H3):\displaystyle\int_0^\infty s\,\kappa(s)\,ds=1$,

$(H4):\exists r>0\ \ ;\ \ \kappa'(s)\leq -r\,\kappa(s)\quad,\quad\forall s\geq 0$, Concerning the delay, we only assume that

$$|\gamma(\sigma)| \in C([\tau_1, \tau_2]). \tag{3.1.6}$$

In view of the boundary conditions, our system can have solutions (uniform in variable x) which do not decay. In other words, it is known that for the problem determined by (3.1.4)-(3.1.5) we can always take solutions where φ and θ are constant, for this purpose, we impose that

$$\int_0^\pi \varphi_0(x)\,dx = \int_0^\pi \varphi_1(x)\,dx = \int_0^\pi \theta_0(x)\,dx = 0. \tag{3.1.7}$$

It should be noted that condition (3.1.7) is imposed to guarantee that the solution decays. Thus, if we want to avoid this behavior, we need to impose the condition (3.1.7). In addition, as in [2], to be able to use Poincare's inequality for φ and θ, we perform the following transformation.
From $(3.1.4)_2$ and $(3.1.4)_3$ respectively we have

$$\begin{cases} J\displaystyle\int_0^\pi \varphi_{tt}\,dx + \tau \int_0^\pi \varphi_t\,dx + \xi \int_0^\pi \varphi\ dx - \delta \int_0^\pi \theta\,dx = 0, \\ c\displaystyle\int_0^\pi \theta_t\,dx + \delta \int_0^\pi \varphi_t\,dx = 0. \end{cases} \tag{3.1.8}$$

Taking $\psi(t) = \int_0^\pi \varphi\ dx$ and $\vartheta(t) = \int_0^\pi \theta\,dx$, we can observe that $\psi(0) = \int_0^\pi \varphi_0\ dx$, $\psi'(0) = \int_0^\pi \varphi_1\ dx$ and $\vartheta(0) = \int_0^\pi \theta_0\,dx$. Moreover, (ψ, ϑ) is a solution of the following initial value system

$$\begin{cases} J\,\psi'' + \tau\,\psi' + \xi\,\psi - \delta\,\vartheta = 0 \quad , \quad \forall t \geq 0, \\ c\,\vartheta' + \delta\,\psi' = 0 \qquad\qquad , \qquad\quad \forall t \geq 0, \\ \psi\,(0) = \int\limits_0^\pi \varphi_0\; dx = 0, \\ \psi'\,(0) = \int\limits_0^\pi \varphi_1\; dx = 0, \\ \vartheta\,(0) = \int\limits_0^\pi \theta_0\, dx = 0. \end{cases}$$

The solution of the system is $\psi\,(t) = \vartheta\,(t) = 0 \quad , \quad \forall t \geq 0.$

As a result

$$\int\limits_0^\pi \varphi\,(x,t)\,dx = \int\limits_0^\pi \theta\,(x,t)\;dx = 0 \quad , \quad \forall t \geq 0.$$

In addition, from (3.1.2) we get

$$\int\limits_0^\pi \eta\,(x,s)\;dx = 0 \quad , \quad \forall t \geq 0 \quad , \quad \forall s \geq 0.$$

In the following, we put

$$\int\limits_0^\infty \kappa\,(s)\,ds = \kappa_0 = k\,(0)$$

3.2 Well-Posedness

In this section, we give the existence and uniqueness of solutions for the system (3.1.4)-(3.1.5) using semigroup theory.

First, we introduce as in [66], new dependent variable

$$z\,(x,\rho,\sigma,t) = u_t\,(x,t-\rho\,\sigma) \qquad in\ (0,\pi)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty)\,. \tag{3.2.9}$$

A simple differentiation shows that z satisfies

$$\sigma\,z_t\,(x,\rho,\sigma,t) + z_\rho\,(x,\rho,\sigma,t) = 0 \quad in\ (0,\pi)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty)\,. \tag{3.2.10}$$

Hence problem (3.1.4) becomes

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b \varphi_x - \beta \theta_x - \int\limits_{\tau_1}^{\tau_2} \gamma(\sigma) z(x,1,\sigma,t) d\sigma & in \ (0,\pi) \times (0,\infty), \\ J \varphi_{tt} = \alpha \varphi_{xx} - b u_x - \xi \varphi + \delta \theta - \tau \varphi_t & in \ (0,\pi) \times (0,\infty), \\ c \theta_t = \int\limits_0^\infty \kappa(s) \eta^t_{xx}(x,s) ds - \beta u_{xt} - \delta \varphi_t & in \ (0,\pi) \times (0,\infty), \\ \sigma z_t = -z_\rho & in \ (0,\pi) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty), \\ \eta_t(x,s) = \theta - \eta_s(x,s) & in \ (0,\pi) \times (0,\infty) \times (0,\infty), \end{cases} \tag{3.2.11}$$

with the boundary and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \ , \ t > 0, \\ \eta_x(0,s) = \eta_x(\pi,s) = 0 \ , \quad s \geq 0 \ , \ t > 0, \\ u(x,0) = u_0(x) \ , \ \varphi(x,0) = \varphi_0(x) \ , \ \theta(x,0) = \theta_0(x) \ , \ x \in (0,\pi), \\ u_t(x,0) = u_1(x) \ , \ \varphi_t(x,0) = \varphi_1(x) \ , \ x \in (0,\pi), \\ \eta^0(x,s) = \eta_0(x,s) \ , \ x \in (0,\pi) \ , \ s \geq 0, \\ \eta(x,0) = 0 \ , \ x \in (0,\pi) \ , \ t > 0, \\ z(x,\rho,\sigma,0) = f_0(x,\rho\sigma) & in \ (0,\pi) \times (0,1) \times (0,\tau_2). \end{cases} \tag{3.2.12}$$

Second, we introduce the vector function $U = (u, v, \varphi, \phi, \theta, z, \eta)^T$, with $v = u_t$, and $\phi = \varphi_t$.
We consider the following Hilbert spaces:

$$\begin{aligned} L^2_*(0,\pi) &= \left\{ w \in L^2(0,\pi) \ , \ \int\limits_0^\pi w(x) dx = 0 \right\}, \\ H^1_*(0,\pi) &= H^1(0,\pi) \cap L^2_*(0,\pi), \\ H^2_*(0,\pi) &= \left\{ w \in H^2(0,\pi) \ ; \ w_x(0) = w_x(\pi) = 0 \right\}. \end{aligned}$$

Furthermore, we introduce the weight Hilbert spaces

$$\mathcal{M}_1 = L^2_\kappa\left((0,\infty); H^1_*(0,\pi)\right) = \left\{ w : R_+ \to H^1_*(0,\pi) \ ; \ \int\limits_0^\infty \kappa(s) \|w_x(s)\|_2^2 ds < \infty \right\},$$

and

$$\mathcal{H} = H_{\kappa}^{1}\left((0,\infty)\ ,\ H_{*}^{1}(0,\pi)\right) = \{\eta\ /\ \eta, \eta_s \in \mathcal{M}_1\}.$$

We define the enegy space by

$$\begin{aligned}\mathbb{H} =& H_0^1(0,\pi) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times H_*^1(0,\pi) \\ &\times L^2((0,\pi)\times(0,1)\times(\tau_1,\tau_2)) \times \mathcal{M}_1.\end{aligned}$$

Then $\mathbb{H}$, along with the inner product

$$\begin{aligned}\langle U, \tilde{U}\rangle_{\mathbb{H}} =& \rho_1 \int_0^\pi v\tilde{v}dx + J\int_0^\pi \phi\tilde{\phi}dx + c\int_0^\pi \theta\tilde{\theta}dx + \alpha\int_0^\pi \varphi_x\tilde{\varphi}_x dx + \frac{\mu}{2}\int_0^\pi \left(u_x + \frac{b}{\mu}\varphi\right)\left(\tilde{u}_x + \frac{b}{\mu}\tilde{\varphi}\right)dx \\ &+ \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)\int_0^\pi u_x\,\tilde{u}_x\,dx + \frac{\xi}{2}\int_0^\pi \left(\varphi + \frac{b}{\xi}u_x\right)\left(\tilde{\varphi} + \frac{b}{\xi}\tilde{u}_x\right)dx + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)\int_0^\pi \varphi\,\tilde{\varphi}\,dx \\ &+ \int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} \sigma\,|\gamma_2(\sigma)|\,z\,\tilde{z}\,d\sigma\,d\rho\,dx + \int_0^\infty \kappa(s)\int_0^\pi \eta_x\,\tilde{\eta}_x\,dx\,ds, \end{aligned} \tag{3.2.13}$$

is a Hilbert space for any $U = (u, v, \varphi, \phi, \theta, z, \eta)^T \in \mathbb{H}$ and $U(\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\phi}, \tilde{\theta}, \tilde{z}, \tilde{\eta})^T \in \mathbb{H}$.
The system (3.2.11)-(3.2.12) can be rewritten as follows:

$$\begin{cases} U'(t) = (\mathcal{A} + \mathcal{B})\,U(t)\ ,\ t > 0, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, f_0, \eta_0)^T, \end{cases}$$

where $: D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is a linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{\mu}{\rho_1}u_{xx} + \dfrac{b}{\rho_1}\varphi_x - \dfrac{\beta}{\rho_1}\theta_x - \dfrac{\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,d\sigma}{\rho_1}v - \dfrac{1}{\rho_1}\int_{\tau_1}^{\tau_2}\gamma(\sigma)\, z(x,\ 1,\ \sigma,\ t)\ d\sigma \\ \phi \\ \dfrac{\alpha}{J}\varphi_{xx} - \dfrac{b}{J}u_x - \dfrac{\xi}{J}\varphi + \dfrac{\delta}{J}\theta - \dfrac{\tau}{J}\phi \\ \dfrac{1}{c}\int_0^{\infty}\kappa(s)\ n_{xx}(x,\ s)\ ds - \dfrac{\beta}{c}v_x - \dfrac{\delta}{c}\phi \\ -\dfrac{1}{\sigma}z_\rho \\ \theta - \eta_s \end{pmatrix}.$$

and the domain of $\mathcal{A}$ is given by

$$\begin{aligned} D(\mathcal{A}) = \Big\{ & U \in \mathbb{H}/\ u \in H^2(0,\pi) \cap H_0^1(0,\pi)\ \ ;\ \varphi, \theta \in H_*^2(0,\pi) \cap H_*^1(0,\pi)\ \ ; \\ & v \in H_0^1(0,\pi)\ \ ;\ \phi \in H_*^1(0,\pi)\ \ ;\ z, z_\rho \in L^2((0,\pi)\times(0,1)\times(\tau_1,\tau_2))\ \ ; \\ & \eta \in \mathcal{H}\ ;\ \int_0^{\infty}\kappa(s)\ n_{xx}(x,\ s)\ ds \in L^2(0,\pi)\ \ ;\ \eta(x,0) = 0\}, \end{aligned}$$

and the operator $\mathcal{B} : D(\mathcal{B}) = \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathcal{B}U = \frac{\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,d\sigma}{\rho_1}\begin{pmatrix} 0 \\ v \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we have the following existence and uniqueness result

Theorem 3.2.1. *Let* $U_0 \in \mathbb{H}$ *and assume that* (0.0.6) *holds. Then, there exists a unique solution*

$U \in C(\mathbb{R}_+, \mathbb{H})$ *for problem* (3.2.11)-(3.2.12). *Moreover, if* $U_0 \in D(\mathcal{A})$, *then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathbb{H}).$$

Proof. We use the semi-group approach. So we prove that $\mathcal{A}$ is a maximal dissipative operator and $\mathcal{B}$ is a Lipschitz continuous operator.

First, we prove that $\mathcal{A}$ is dissipative. Let $U \in D(\mathcal{A})$, then we have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathbb{H}} = &-\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right)\int_0^\pi v^2\, dx - \tau\int_0^\pi \phi^2\, dx + \int_0^\pi\int_0^\infty \kappa(s)\ \eta_s\, \eta_{xx}\, ds\, dx \\ &-\int_0^\pi v\int_{\tau_1}^{\tau_2}\gamma(\sigma)\, z(x, 1, \sigma, t)\, d\sigma\, dx - \int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\ z_\rho\, z\, d\sigma\, d\rho\, dx. \end{aligned} \tag{3.2.14}$$

Integrating by part and using the fact that $z(x, 0, t) = v(x, t)$, the last term in the right-hand side of (3.2.14) leads to

$$\begin{aligned} -\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\ z_\rho\, z\, d\sigma\, d\rho\, dx = &-\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x, 1, \sigma, t)\, d\sigma\, dx \\ &+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right)\int_0^\pi v^2\, dx. \end{aligned}$$

Also, using Young's inequality, we find

$$\begin{aligned} &-\int_0^\pi v\int_{\tau_1}^{\tau_2}\gamma(\sigma)\, z(x, 1, \sigma, t)\, d\sigma\, dx \\ &\leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right)\int_0^\pi v^2\, dx + \frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x, 1, \sigma, t)\, d\sigma\, dx. \end{aligned}$$

In addition, integrating by part and using the fact that $\underset{s\to\infty}{lim}\ \kappa(s) = 0$ we get

$$\int_0^\pi\int_0^\infty \kappa(s)\ \eta_s\, \eta_{xx}\, ds\, dx = \frac{1}{2}\int_0^\infty \kappa'(s)\int_0^\pi n_x^2\, dx\, ds.$$

Consequently, by the use of $(H2)$,(3.2.14) becomes

$$\langle \mathcal{A}U, U\rangle_{\mathbb{H}} \leq \frac{1}{2}\int_0^\infty \kappa'(s)\int_0^\pi n_x^2\,dx\,ds - \tau\int_0^\pi \phi^2\,dx \leq 0.$$

Hence, the operator $\mathcal{A}$ is dissipative.

Second, we prove that the operator $\lambda I - \mathcal{A}$ is surjective.
For any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathbb{H}$, we prove that there exist $U \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})\, U = F. \tag{3.2.15}$$

The problem (3.2.15) , leads to solve the following system

$$\begin{cases} \lambda u - v = f_1 \in H_0^1(0,\pi), \\ \left(\lambda \rho_1 + \displaystyle\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\,d\sigma\right) v - \mu\, u_{xx} - b\, \varphi_x + \beta\, \theta_x \\ + \displaystyle\int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(x,\ 1,\ \sigma,\ t)\ d\sigma = \rho_1 f_2 \in L^2(0,\pi), \\ \lambda \varphi - \phi = f_3 \in H_*^1(0,\pi), \\ (\lambda J + \tau)\, \phi - \alpha\, \varphi_{xx} + b\, u_x + \xi\, \varphi - \delta\, \theta = J f_4 \in L_*^2(0,\pi), \\ \lambda c\theta - \displaystyle\int_0^\infty \kappa(s)\, \eta_{xx}(x,s)\,ds + \beta v_x + \delta \phi = c f_5 \in H_*^1(0,\pi), \\ \lambda \sigma z + z_\rho = \sigma f_6 \in L^2((0,\pi)\times(0,1)\times(0,\infty)), \\ \lambda \eta - \theta + \eta_s = f_7 \in \mathcal{M}_1. \end{cases} \tag{3.2.16}$$

Suppose that u and φ are given with the appropriate regularity. Then, the first and the third equations in (3.2.16) yield

$$v = \lambda u - f_1 \in H_0^1(0,\pi), \tag{3.2.17}$$

and

$$\phi = \lambda \varphi - f_3 \in H_*^1(0,\pi), \tag{3.2.18}$$

respectively.

The sixth equation in (3.2.16) together with (3.2.17) and the fact that $z(x,0)=v(x,t)$ gives

$$z(x,\rho,\sigma,t)=\lambda u(x,t)\,e^{-\lambda\sigma\rho}-f_1\,e^{-\lambda\sigma\rho}+\sigma\,e^{-\lambda\sigma\rho}\int_0^{\rho}e^{\lambda\sigma y}\,f_6(x,y,\sigma,t)\,dy. \tag{3.2.19}$$

The last equation in (3.2.16) under the condition $\eta(0)=0$ gives

$$\eta(x,s)=\frac{1}{\lambda}\theta(x,t)\left(1-e^{-\lambda s}\right)+\int_0^{s}e^{\lambda(w-s)}f_7(w)\,dw. \tag{3.2.20}$$

By the use of integration by parts, it can be easily shown that the second, fourth and fifth equations in (3.2.16) satisfy the following:

$$\begin{cases} \left(\lambda\rho_1+\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,d\sigma\right)\int_0^{\pi}v\,\tilde{u}\,dx+\mu\int_0^{\pi}u_x\,\tilde{u}_x\,dx+b\int_0^{\pi}\varphi\,\tilde{u}_x\,dx-\beta\int_0^{\pi}\theta\,\tilde{u}_x\,dx \\ +\int_0^{\pi}\tilde{u}\int_{\tau_1}^{\tau_2}\gamma(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx=\rho_1\int_0^{\pi}f_2\,\tilde{u}\,dx, \\ \\ (\lambda J+\tau)\int_0^{\pi}\phi\,\tilde{\varphi}\,dx+\alpha\int_0^{\pi}\varphi_x\,\tilde{\varphi}_x\,dx+b\int_0^{\pi}u_x\,\tilde{\varphi}\,dx+\xi\int_0^{\pi}\varphi\,\tilde{\varphi}\,dx-\delta\int_0^{\pi}\theta\,\tilde{\varphi}\,dx \\ =J\int_0^{\pi}f_4\,\tilde{\varphi}\,dx, \\ \\ c\int_0^{\pi}\theta\,\tilde{\theta}\,dx+\frac{1}{\lambda}\int_0^{\pi}\tilde{\theta}_x\int_0^{\infty}\kappa(s)\,\eta_x\,ds\,dx+\frac{\beta}{\lambda}\int_0^{\pi}v_x\,\tilde{\theta}\,dx+\frac{\delta}{\lambda}\int_0^{\pi}\phi\,\tilde{\theta}\,dx=\frac{c}{\lambda}\int_0^{\pi}f_5\,\tilde{\theta}\,dx. \end{cases} \tag{3.2.21}$$

Moreover, by using (3.2.17)-(3.2.20) , we have the following corresponding weak formulation for the second, fourth and fifth equation in (3.2.16):
Finding $(u,\varphi,\theta)\in H_0^1(0,\pi)\times H_*^1(0,\pi)\times H_*^1(0,\pi)$ such that
for all $\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right)\in H_0^1(0,\pi)\times H_*^1(0,\pi)\times H_*^1(0,\pi)$ we have:

$$B\left((u,\varphi,\theta);\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right)\right)=l\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right), \tag{3.2.22}$$

where $B : \left[H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B\left((u,\varphi,\theta);(\tilde{u},\tilde{\varphi},\tilde{\theta})\right) = {} & \mu_0 \int_0^\pi u\,\tilde{u}\,dx + \mu \int_0^\pi u_x\,\tilde{u}_x\,dx + \mu_1 \int_0^\pi \varphi\,\tilde{\varphi}\,dx + \alpha \int_0^\pi \varphi_x\,\tilde{\varphi}_x\,dx \\ & + c \int_0^\pi \theta\,\tilde{\theta}\,dx + c_\kappa \int_0^\pi \theta_x\,\tilde{\theta}_x\,dx + b \int_0^\pi (\varphi\,\tilde{u}_x + u_x\,\tilde{\varphi})\,dx \\ & + \beta \int_0^\pi \left(u_x\tilde{\theta} - \theta\,\tilde{u}_x\right)\,dx + \delta \int_0^\pi \left(\varphi\,\tilde{\theta} - \theta\,\tilde{\varphi}\right)\,dx, \end{aligned}$$

and $l : H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi) \to \mathbb{R}$ is the linear form given by

$$l\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right) = \int_0^\pi g_1\,\tilde{u}\,dx + \int_0^\pi g_2\,\tilde{\varphi}\,dx + \int_0^\pi g_3\,\tilde{\theta}\,dx + \int_0^\pi g_4\,\tilde{\theta}_x\,dx,$$

where

$$\begin{aligned} & \mu_0 = \lambda^2\rho_1 + \lambda \int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\,d\sigma + \lambda \int_{\tau_1}^{\tau_2} \gamma(\sigma)\,e^{-\lambda\sigma}d\sigma \;\; > 0 \;,\;\; \mu_1 = \lambda^2 J + \xi + \lambda\,\tau \;\; > 0, \\ & c_\kappa = \frac{1}{\lambda^2} \int_0^\infty \kappa(s)\left(1 - e^{-\lambda s}\right)ds \;\; > 0, \\ & g_1 = \frac{\mu_0}{\lambda} f_1 + \rho_1 f_2 - \int_{\tau_1}^{\tau_2} \sigma\,\gamma(\sigma)\,e^{-\lambda\sigma} \int_0^1 e^{\lambda\sigma y} f_6(x,y,\sigma,t)\,dy\,d\sigma \;\in L^2(0,\pi), \\ & g_2 = (\lambda\,J + \tau)\,f_3 + J\,f_4 \;\in L^2(0,\pi) \;\;,\;\; g_3 = \frac{\beta}{\lambda} f_{1_x} + \frac{\delta}{\lambda} f_3 + \frac{c}{\lambda} f_5 \;\in L^2(0,\pi), \\ & g_4 = -\frac{1}{\lambda} \int_0^\infty \kappa(s) \int_0^s e^{\lambda(w-s)} f_{7_x}(w)\,dw\,ds \;\in L^2(0,\pi). \end{aligned}$$

Now, for $\mathcal{V} = H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ equipped with the norm

$$\|(u,\varphi,\theta)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2,$$

we have

$$|B((u,\varphi,\theta);(u,\varphi,\theta))| = \mu_0 \int_0^{\pi} u^2\,dx + \mu \int_0^{\pi} u_x^2\,dx + \mu_1 \int_0^{\pi} \varphi^2\,dx + \alpha \int_0^{\pi} \varphi_x^2\,dx$$
$$+ c \int_0^{\pi} \theta^2\,dx + c_\kappa \int_0^{\pi} \theta_x^2\,dx + 2b \int_0^{\pi} u_x\,\varphi\,dx.$$

On the other hand , we can write

$$\mu\,u_x^2 + \mu_1\,\varphi^2 + 2b\,u_x\,\varphi = \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + \mu_1\left(\varphi + \frac{b}{\mu_1}u_x\right)^2\right] + \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_1}\right)u_x^2 + \left(\mu_1 - \frac{b^2}{\mu}\right)\varphi^2\right],$$

then, using (0.0.6) we deduce that

$$\mu\,u_x^2 + \mu_1\,\varphi^2 + 2b\,u_x\,\varphi \geq \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_1}\right)u_x^2 + \left(\mu_1 - \frac{b^2}{\mu}\right)\varphi^2\right],$$

As a result

$$|B((u,\varphi,\theta);(u,\varphi,\theta))| \geq M\,\|(u,\varphi,\theta)\|_V^2,$$

where $M = min\left\{\frac{1}{2}\left(\mu - \frac{b^2}{\mu_1}\right)\ ;\ \frac{1}{2}\left(\mu_1 - \frac{b^2}{\mu}\right)\ ;\ \alpha\ ;\ c\ ;\ \mu_0\ ;\ c_\kappa\right\}$. Thus, B is coercive. Moreover, we can easily see that B and l are bounded. Consequently, by Lax-Milgram Lemmam we conclude that there exists a unique $(u,\varphi,\theta) \in \mathcal{V}$ which satisfies (3.2.22).
Substituting u in (3.2.17) and (3.2.19), respectively, we get

$$v \in\ H_0^1(0,\pi) \quad , \quad z \in L^2((0,\pi)\times(0,1)\times(\tau_1,\tau_2)),$$

and z in $(3.2.16)_6$, we obtain

$$z_\rho \in L^2((0,\pi)\times(0,1)\times(\tau_1,\tau_2)),$$

then, inserting φ in (3.2.18), we find

$$\phi \in\ H_*^1(0,\pi).$$

Similarly, inserting θ in (3.2.20) and bearing in mind (3.2.16) $_7$, we obtain

$$\eta \in \mathcal{H} \quad , \quad \eta\left(x,0\right)=0.$$

Moreover, if we take $\left(\tilde{\varphi},\tilde{\theta}\right)\equiv(0,0)\in H_*^1\left(0,\pi\right)\times H_*^1\left(0,\pi\right)$, then (3.2.22) reduces to

$$\mu\int_0^\pi u_x\,\tilde{u}_x\,dx+b\int_0^\pi \varphi\tilde{u}_x\,dx-\beta\int_0^\pi \theta\tilde{u}_x\,dx=-\int_0^\pi\left(-g_1+\mu_0\,u\right)\tilde{u}\,dx \quad , \ \forall\tilde{u}\in H_0^1\left(0,\pi\right),$$

that is,

$$\mu\,u_{xx}=-g_1+\mu_0\,u-b\,\varphi_x+\beta\,\theta_x \quad , \quad in\ L^2\left(0,\pi\right),$$

which implies,

$$u\in H^2\left(0,\pi\right)\cap H_0^1\left(0,\pi\right).$$

Then, we choose $\left(\tilde{u},\tilde{\theta}\right)\equiv(0,0)\in H_0^1\left(0,\pi\right)\times H_*^1\left(0,\pi\right)$, then (3.2.22) become

$$\alpha\int_0^\pi \varphi_x\,\tilde{\varphi}_x\,dx=-\int_0^\pi\left(\mu_1\,\varphi+b\,u_x-\delta\,\theta-g_2\right)\,\tilde{\varphi}\,dx \quad , \ \forall\tilde{\varphi}\in H_*^1\left(0,\pi\right). \tag{3.2.23}$$

Here, we can not use the regularity theorem, because $\tilde{\varphi}\in H_*^1\left(0,\pi\right)$. So, we take $\psi\in H_0^1\left(0,\pi\right)$ and we set

$$\tilde{\varphi}\left(x\right)=\psi\left(x\right)-\int_0^\pi\psi\left(x\right)\,dx.$$

It's clear that $\tilde{\varphi}\in H_*^1\left(0,\pi\right)$. Then, a substitution in (3.2.23) leads to

$$\alpha\int_0^\pi \varphi_x\,\psi_x\,dx=-\int_0^\pi r\,\psi\,dx \quad , \ \forall\psi\in H_0^1\left(0,\pi\right),$$

where,

$$r=\mu_1\,\varphi+b\,u_x-\delta\,\theta-g_2.$$

That is,

$$\alpha\,\varphi_{xx}=\mu_1\,\varphi+b\,u_x-\delta\,\theta-g_2 \ , \ in\,L^2\left(0,\pi\right), \tag{3.2.24}$$

which implies,

$$\varphi \in H^2(0,\pi).$$

On the other hand, from (3.2.23) and integrating by parts, we get

$$\alpha\left[\varphi_x\,\tilde{\varphi}\right]_0^\pi - \alpha\int_0^\pi \varphi_{xx}\,\tilde{\varphi}\,dx + \int_0^\pi \left(\mu_1\,\varphi + b\,u_x - \delta\,\theta - g_2\right)\,\tilde{\varphi}\,dx = 0\ ,\ \ \forall\tilde{\varphi}\in H_*^1(0,\pi),$$

and from (3.2.24), we obtain

$$\varphi_x(\pi)\ \varphi(\pi) - \varphi_x(0)\ \varphi(0) = 0.$$

Since $\tilde{\varphi} \in H_*^1(0,\pi)$ is arbitrary, then

$$\varphi_x(\pi)\ = \varphi_x(0)\ = 0.$$

Consequently,

$$\varphi \in H_*^2(0,\pi) \cap H_*^1(0,\pi).$$

Similary if we take $(\tilde{u},\tilde{\varphi}) \equiv (0,0) \in H_0^1(0,\pi) \times H_*^1(0,\pi)$, we find

$$\theta \in H_*^2(0,\pi) \cap H_*^1(0,\pi).$$

Finally, from $(3.2.16)_5$, we get

$$\int_0^\infty \kappa(s)\,\eta_{xx}(x,s)\,ds \in L^2(0,\pi).$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (3.2.15) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal. Thus, we conclude that $\mathcal{A}$ is a maximal dissipative operator. On the other hand, it is obvious that operator $\mathcal{B}$ is Lipschitz continuous. Consequently, $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathbb{H}$. Therefore, the well-posedness result follows from the Hill-Yosida theorem. (see [76]) □

3.3 Exponential decay

In this section, we state and prove the technical lemmas needed for the proof of our stability result.

Lemma 3.3.1. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.2.11) - (3.2.12) . *Then, the energy functional* $E(t)$, *defined by*

$$E(t)=\frac{1}{2}\int_0^{\pi}\left(\rho_1 u_t^2+J\varphi_t^2+c\theta^2+\mu u_x^2+\alpha\varphi_x^2+\xi\varphi^2+2b\varphi u_x\right)dx$$
$$+\frac{1}{2}\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\left|\gamma(\sigma)\right| z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx+\frac{1}{2}\int_0^{\infty}\kappa(s)\int_0^{\pi}\eta_x^2(x,s)\,dx\,ds, \tag{3.3.25}$$

satisfies

$$E'(t)\le\left(\int_{\tau_1}^{\tau_2}\left|\gamma(\sigma)\right|d\sigma\right)\int_0^{\pi}u_t^2\,dx+\frac{1}{2}\int_0^{\infty}\kappa'(s)\int_0^{\pi}\eta_x^2(x,s)\,dx\,ds-\tau\int_0^{\pi}\varphi_t^2\,dx. \tag{3.3.26}$$

Proof. Multiplying $(3.2.11)_1$, $(3.2.11)_2$, $(3.2.11)_3$ by u_t , φ_t , θ respectively, integrating over $(0,\pi)$, and Multiplying$(3.2.11)_4$ by $|\gamma_2(\sigma)|\ z$, integrating over $(0,\pi)\times(0,1)\times(\tau_1,\tau_2)$ then,using integration by part and taking into account the boundary conditions and summing them up, we obtain

$$\frac{d}{2\,dt}\left\{\int_0^{\pi}\left(\rho_1 u_t^2+J\varphi_t^2+c\theta^2+\mu u_x^2+\alpha\varphi_x^2+\xi\varphi^2+2b\varphi u_x\right)dx+\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\left|\gamma(\sigma)\right| z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\right\}$$
$$=-\tau\int_0^{\pi}\varphi_t^2\,dx-\int_0^{\pi}u_t\int_{\tau_1}^{\tau_2}\gamma(\sigma)\ z(x,1,\sigma,t)\,d\sigma\,dx+\int_0^{\pi}\theta\int_0^{\infty}\kappa(s)\ \eta_{xx}(x,s)\,ds\,dx$$
$$-\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\left|\gamma(\sigma)\right| z_\rho z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx.$$

Using $(3.2.11)_5$, we get

$$\frac{d}{2\,dt}\left\{\int_0^{\pi}\left(\rho_1 u_t^2+J\varphi_t^2+c\theta^2+\mu u_x^2+\alpha\varphi_x^2+\xi\varphi^2+2b\varphi u_x\right)dx+\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\left|\gamma(\sigma)\right| z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\right\}$$
$$=-\int_0^{\pi}u_t\int_{\tau_1}^{\tau_2}\gamma(\sigma)\ z(x,1,\sigma,t)\,d\sigma\,dx-\tau\int_0^{\pi}\varphi_t^2\,dx-\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\left|\gamma(\sigma)\right| z_\rho z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$
$$+\int_0^{\pi}\int_0^{\infty}\kappa(s)\ \eta_t\eta_{xx}(x,s)\,ds\,dx+\int_0^{\pi}\int_0^{\infty}\kappa(s)\ \eta_s\eta_{xx}(x,s)\,ds\,dx, \tag{3.3.27}$$

integrating by part the last two terms of (3.3.27), we obtain

$$E(t) = \frac{1}{2}\int_0^\pi \left(\rho_1 u_t^2 + J\varphi_t^2 + c\,\theta^2 + \mu u_x^2 + \alpha\varphi_x^2 + \xi\varphi^2 + 2b\,\varphi u_x\right) dx$$
$$+ \frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} \sigma\,|\gamma(\sigma)|\, z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx + \frac{1}{2}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2(x,s)\, dx\, ds, \tag{3.3.28}$$

and

$$E'(t) = -\int_0^\pi u_t \int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(x,1,\sigma,t)\, d\sigma\, dx - \int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z_\rho z(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx$$
$$-\tau\int_0^\pi \varphi_t^2\, dx - \frac{1}{2}\int_0^\infty \kappa(s)\frac{\partial}{\partial s}\int_0^\pi \eta_x^2(x,s)\, dx\, ds. \tag{3.3.29}$$

On the other hand, we have $z(x,0,\sigma,t) = u_t(x,t)$, then

$$-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z_\rho z(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx = -\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x,1,\sigma,t)\, d\sigma\, dx \tag{3.3.30}$$
$$+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right)\int_0^\pi u_t^2\, dx. \tag{3.3.31}$$

By applying integration by parts and bringing in mind that $\lim\limits_{s\to\infty} \kappa(s) = 0$, we find

$$-\frac{1}{2}\int_0^\infty \kappa(s)\frac{\partial}{\partial s}\int_0^\pi \eta_x^2(x,s)\, dx\, ds = -\frac{1}{2}\lim_{s\to b}\left[\kappa(s)\int_0^\pi \eta_x^2(x,s)\, dx\right]_{b=0}^{b=\infty} + \frac{1}{2}\int_0^\infty \kappa'(s)\int_0^\pi \eta_x^2(x,s)\, dx\, ds$$
$$= \frac{1}{2}\int_0^\infty \kappa'(s)\int_0^\pi \eta_x^2(x,s)\, dx\, ds. \tag{3.3.32}$$

Then, using Young's inequality on the first term in (3.3.29) leads to

$$-\int_0^{\pi} u_t \int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(x,1,\sigma,t)\, d\sigma\, dx \leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right)\int_0^{\pi} u_t^2\, dx + \frac{1}{2}\int_0^{\pi}\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x,1,\sigma,t)\, d\sigma\, dx. \tag{3.3.33}$$

Inserting (3.3.31) , (3.3.32) and (3.3.33) in (3.3.29), we get (3.3.26). □

Remark 3.3.1. *The energy function $E(t)$ defined in* (3.3.25) *is nonnegative. In fact,*

$$\mu\, u_x^2 + \xi\, \varphi^2 + 2b\, u_x\, \varphi = \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + \xi\left(\varphi + \frac{b}{\xi}u_x\right)^2\right] + \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right],$$

we conclude from (0.0.6) *that*

$$\mu\, u_x^2 + \xi\, \varphi^2 + 2b\, u_x\, \varphi \geq \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right],$$

which leads to

$$\begin{aligned} E(t) \;&>\; \frac{1}{2}\int_0^{\pi}\left(\rho_1\, u_t^2 + J\,\varphi_t^2 + c\,\theta^2 + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \alpha\,\varphi_x^2 + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right)dx \\ &+ \frac{1}{2}\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2} \sigma\, |\gamma(\sigma)|\, z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx + \frac{1}{2}\int_0^{\infty}\kappa(s)\int_0^{\pi}\eta_x^2(x,s)\, dx\, ds, \end{aligned}$$

then, $E(t)$ is nonnegative.

Lemma 3.3.2. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.2.11) - (3.2.12). *Then, the functional*

$$I_1(t) = \rho_1 \int_0^{\pi} u_t u\ dx\ ,\ t \geq 0,$$

satisfies

$$I_1{}'(t) \leq -\frac{\mu}{2}\int_0^{\pi} u_x^2\, dx + \rho_1\int_0^{\pi} u_t^2\, dx + c_0\int_0^{\pi}\left(\varphi^2 + \theta^2\right)dx + c_0\int_0^{\pi}\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\ z^2(x,1,\sigma,t)\ d\sigma\, dx. \tag{3.3.34}$$

Proof. By differentiating $I_1(t)$, using $(3.2.11)_1$ and integrating by parts together with the boundary conditions, we get

$$I_1{}'(t) = -\mu \int_0^\pi u_x^2\,dx - b\int_0^\pi \varphi\, u_x\,dx + \beta\int_0^\pi \theta\, u_x\,dx + \rho_1\int_0^\pi u_t^2\,dx - \int_0^\pi u\int_{\tau_1}^{\tau_2} \gamma(\sigma)\; z(x,1,\sigma,t)\;d\sigma\,dx. \tag{3.3.35}$$

The application of Young's , Poincaré and Cauchy Schwarz inequalities lead to

$$-b\int_0^\pi \varphi\, u_x\,dx \le \frac{\mu}{6}\int_0^\pi u_x^2\,dx + \frac{3\,b^2}{2\,\mu}\int_0^\pi \varphi^2 dx, \tag{3.3.36}$$

$$\beta\int_0^\pi \varphi\, u_x\,dx \le \frac{\mu}{6}\int_0^\pi u_x^2\,dx + \frac{3\,\beta^2}{2\,\mu}\int_0^\pi \theta^2 dx, \tag{3.3.37}$$

and

$$-\int_0^\pi u\int_{\tau_1}^{\tau_2}\gamma(\sigma)\; z(x,1,\sigma,t)\;d\sigma\,dx \le \frac{\mu}{6}\int_0^\pi u_x^2\,dx + \frac{3}{2\,\mu}\left(\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\;d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\;z^2(x,1,\sigma,t)\;d\sigma\,dx. \tag{3.3.38}$$

Substituting (3.3.36) , (3.3.37) and (3.3.38) in (3.3.35), we get (3.3.34). □

Lemma 3.3.3. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.2.11) - (3.2.12). *Then, the functional*

$$I_2(t) = J\int_0^\pi \varphi\,\varphi_t\,dx + \frac{\tau}{2}\int_0^\pi \varphi^2 dx + \frac{b\,\rho_1}{\mu}\int_0^\pi \varphi\int_0^x u_t(y)\,dy\,dx\;,\;\; t\ge 0,$$

satisfies

$$I_2{}'(t) \le -\alpha\int_0^\pi \varphi_x^2 dx - \frac{\chi}{2}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\delta - \frac{b\,\beta}{\mu}\right)^2\int_0^\pi \theta^2 dx + \varepsilon_0\int_0^\pi u_t^2 dx \tag{3.3.39}$$

$$+\left(J + \frac{b^2\rho_1^2}{4\mu^2\varepsilon_0}\right)\int_0^\pi \varphi_t^2 dx + c_0\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,z^2(x,1,\sigma,t)\,d\sigma\,dx, \tag{3.3.40}$$

where $\chi = \xi - \dfrac{b^2}{\mu}$.

Proof. By differentiating $I_2(t)$, using $(3.2.11)_2$ and integrating by parts together with the boundary conditions, we obtain

$$I_2{}'(t) = -\alpha \int_0^\pi \varphi_x^2 dx - \chi \int_0^\pi \varphi^2 dx + J \int_0^\pi \varphi_t^2 dx + \left(\delta - \frac{b\beta}{\mu}\right) \int_0^\pi \varphi\,\theta\,dx$$
$$+ \frac{b\rho_1}{\mu} \int_0^\pi \varphi_t \int_0^x u_t(y)\,dy\,dx - \frac{b}{\mu} \int_0^\pi \varphi \int_0^x \int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(y,1,\sigma,t)\,d\sigma\,dy\,dx. \tag{3.3.41}$$

By the use of Young's inequality, we get

$$\delta - \frac{b\beta}{\mu} \int_0^\pi \varphi\,\theta\,dx \leq \frac{\chi}{4} \int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\delta - \frac{b\beta}{\mu}\right)^2 \int_0^\pi \theta^2 dx, \tag{3.3.42}$$

Cauchy Schwarz and Young's inequalities, give us

$$\frac{b\rho_1}{\mu} \int_0^\pi \varphi_t \int_0^x u_t(y)\,dy\,dx \leq \varepsilon_0 \int_0^\pi u_t^2 dx + \frac{b^2\rho_1^2}{4\mu^2\varepsilon_0} \int_0^\pi \varphi_t^2 dx, \tag{3.3.43}$$

and

$$-\frac{b}{\mu} \int_0^\pi \varphi \int_0^x \int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(y,1,\sigma,t)\,d\sigma\,dy\,dx$$
$$\leq \frac{\chi}{4} \int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\frac{b\pi}{\mu}\right)^2 \left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\,d\sigma\right) \int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, z^2(x,1,\sigma,t)\,d\sigma\,dx, \tag{3.3.44}$$

Inserting (3.3.42)-(3.3.44) in (3.3.41), we obtain (3.3.40). □

Lemma 3.3.4. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.2.11) - (3.2.12). *Then, the functional*

$$I_3(t) = -c\rho_1 \int_0^\pi \theta \int_0^x u_t(y)\,dy\,dx\ ,\ t \geq 0,$$

satisfies, for any $\varepsilon_1 > 0$, the following estimate

$$I_3{}'(t) \leq -\frac{\rho_1\,|\beta|}{2}\int_0^\pi u_t^2\,dx + \varepsilon_1\int_0^\pi \left(u_x^2+\varphi^2\right)\,dx + c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^\pi \theta^2\,dx + c_0\int_0^\pi \varphi_t^2 dx$$
$$+\frac{\rho_1\,\kappa_0}{|\beta|}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds + \pi^2\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\;z^2(x,1,\sigma,t)\,d\sigma\,dx. \tag{3.3.45}$$

Proof. Differentiating the functional $I_3(t)$ using $(3.2.11)_1$, $(3.2.11)_3$ and integrating by parts together with the boundary conditions, we obtain

$$I_3{}'(t) = -\rho_1\,\beta\int_0^\pi u_t^2\,dx - \mu\,c\int_0^\pi \theta\,u_x\,dx - b\,c\int_0^\pi \theta\,\varphi\,dx + \beta\,c\int_0^\pi \theta^2\,dx + \rho_1\delta\int_0^\pi \varphi\int_0^x u_t(y)\,dy\,dx$$
$$+c\int_0^\pi \theta\int_0^x\int_{\tau_1}^{\tau_2}\gamma(\sigma)\;z(y,1,\sigma,t)\,d\sigma\,dy\,dx + \rho_1\int_0^\pi u_t\int_0^\infty \kappa(s)\,\eta_x(x,s)\,ds\,dx. \tag{3.3.46}$$

Young's and Cauchy Schwarz inequalities leads to

$$-\mu\,c\int_0^\pi \theta\,u_x\,dx \leq \varepsilon_1\int_0^\pi u_x^2\,dx + \frac{\mu^2c^2}{4\varepsilon_1}\int_0^\pi \theta^2\,dx, \tag{3.3.47}$$

$$-b\,c\int_0^\pi \theta\,\varphi\,dx \leq \varepsilon_1\int_0^\pi \varphi^2\,dx + \frac{b^2c^2}{4\varepsilon_1}\int_0^\pi \theta^2\,dx, \tag{3.3.48}$$

$$\int_0^\pi \theta\int_0^x\int_{\tau_1}^{\tau_2}\gamma(\sigma)\;z(y,1,\sigma,t)\,d\sigma\,dy\,dx$$
$$\leq \frac{c^2}{4\varepsilon_1}\int_0^\pi \theta^2\,dx + \pi^2\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\;z^2(x,1,\sigma,t)\,d\sigma\,dx, \tag{3.3.49}$$

$$\rho_1\int_0^\pi u_t\int_0^\infty \kappa(s)\,\eta_x(x,s)\,ds\,dx \leq \frac{\rho_1\,|\beta|}{4}\int_0^\pi u_t^2\,dx + \frac{\rho_1\,\kappa_0}{|\beta|}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds, \tag{3.3.50}$$

$$\rho_1\delta\int_0^\pi \varphi\int_0^x u_t\left(y\right)dy\,dx \le \frac{\rho_1\left|\beta\right|}{4}\int_0^\pi u_t^2dx+\frac{\rho_1\delta^2}{\left|\beta\right|}\int_0^\pi \varphi_t^2dx. \tag{3.3.51}$$

Substituting (3.3.47)-(3.3.51) in (3.3.46), we get (3.3.45). □

Lemma 3.3.5. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.2.11) - (3.2.12). *Then, the functional*

$$I_4\left(t\right)=-\frac{c}{\kappa_0}\int_0^\pi \theta\int_0^\infty \kappa\left(s\right)\,\eta\left(x,s\right)ds\,dx\ ,\ t\ge 0,$$

satisfies, for any $\varepsilon_2,\varepsilon_3>0$*, the following estimate*

$$\begin{aligned}I_4'\left(t\right)\le{}&-\frac{c}{2}\int_0^\pi \theta^2\,dx+\varepsilon_2\int_0^\pi \varphi_t^2\,dx+\varepsilon_3\int_0^\pi u_t^2\,dx-\frac{c\,\kappa\left(0\right)}{2\kappa_0^2}\int_0^\infty \kappa'\left(s\right)\int_0^\pi \eta_x^2\,dx\,ds\\&+c_0\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)\int_0^\infty \kappa\left(s\right)\int_0^\pi \eta_x^2\,dx\,ds.\end{aligned} \tag{3.3.52}$$

Proof. By differentiating $I_4(t)$, using $(3.2.11)_3$, $(3.2.11)_5$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned}I_4'\left(t\right)={}&-c\int_0^\pi \theta^2\,dx+\frac{c}{\kappa_0}\int_0^\pi \theta\int_0^\infty \kappa\left(s\right)\,\eta_s\left(x,s\right)ds\,dx+\frac{1}{\kappa_0}\int_0^\pi\left(\int_0^\infty \kappa\left(s\right)\,\eta_x\left(x,s\right)ds\right)^2dx\\&-\frac{\beta}{\kappa_0}\int_0^\pi u_t\int_0^\infty \kappa\left(s\right)\,\eta_x\left(x,s\right)ds\,dx+\frac{\delta}{\kappa_0}\int_0^\pi \varphi_t\int_0^\infty \kappa\left(s\right)\,\eta\left(x,s\right)ds\,dx.\end{aligned} \tag{3.3.53}$$

Using Young's , Poincaré and Cauchy Schwarz inequalities, we find

$$\frac{\delta}{\kappa_0}\int_0^\pi \varphi_t\int_0^\infty \kappa\left(s\right)\,\eta\left(x,s\right)ds\,dx\le\varepsilon_2\int_0^\pi \varphi_t^2\,dx+\frac{\delta^2}{4\varepsilon_2\,\kappa_0}\int_0^\infty \kappa\left(s\right)\int_0^\pi \eta_x^2\,dx\,ds, \tag{3.3.54}$$

$$-\frac{\beta}{\kappa_0}\int_0^\pi u_t\int_0^\infty \kappa\left(s\right)\,\eta_x\left(x,s\right)ds\,dx\le\varepsilon_3\int_0^\pi u_t^2\,dx+\frac{\beta^2}{4\varepsilon_3\,\kappa_0}\int_0^\infty \kappa\left(s\right)\int_0^\pi \eta_x^2\,dx\,ds, \tag{3.3.55}$$

$$\frac{1}{\kappa_0}\int_0^\pi\left(\int_0^\infty \kappa\left(s\right)\,\eta_x\left(x,s\right)ds\right)^2dx\le\int_0^\infty \kappa\left(s\right)\int_0^\pi \eta_x^2\,dx\,ds, \tag{3.3.56}$$

and

$$\frac{c}{\kappa_0}\int\limits_0^{\pi}\theta\int\limits_0^{\infty}\kappa\left(s\right)\,\eta_s\left(x,s\right)ds\,dx\leq\frac{c}{2}\int\limits_0^{\pi}\theta^2\,dx-\frac{c\,\kappa\left(0\right)}{2\kappa_0^2}\int\limits_0^{\infty}\kappa'\left(s\right)\int\limits_0^{\pi}\eta_x^2\,dx\,ds. \tag{3.3.57}$$

Estimate (3.3.52) follows by substituting (3.3.54)-(3.3.57) into (3.3.53). □

Lemma 3.3.6. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.2.11) - (3.2.12). *Then, the functional*

$$I_5\left(t\right)=\int\limits_0^{\pi}\int\limits_0^{1}\int\limits_{\tau_1}^{\tau_2}\sigma\,e^{-\sigma\rho}\left|\gamma\left(\sigma\right)\right|z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx\quad,\quad t\geq 0,$$

satisfies the estimate

$$\begin{aligned}I_5{}'\left(t\right)\leq&-m_1\int\limits_0^{\pi}\int\limits_{\tau_1}^{\tau_2}\left|\gamma\left(\sigma\right)\right|z^2\left(x,1,\sigma,t\right)d\sigma\,dx+\left(\int\limits_{\tau_1}^{\tau_2}\left|\gamma\left(\sigma\right)\right|d\sigma\right)\int\limits_0^{\pi}u_t^2dx\\&-m_1\int\limits_0^{\pi}\int\limits_0^{1}\int\limits_{\tau_1}^{\tau_2}\sigma\left|\gamma\left(\sigma\right)\right|z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx\quad,\quad t\geq 0.\end{aligned} \tag{3.3.58}$$

Proof. By differentiating $I_5(t)$, using $(3.2.11)_4$, integrating by parts and using the fact that $z\left(x,0,\sigma,t\right)=u_t\left(x,t\right)$ gives, we obtain

$$\begin{aligned}I_5{}'\left(t\right)=&-\int\limits_0^{\pi}\int\limits_{\tau_1}^{\tau_2}e^{-\sigma}\left|\gamma\left(\sigma\right)\right|z^2\left(x,1,\sigma,t\right)d\sigma\,dx+\left(\int\limits_{\tau_1}^{\tau_2}\left|\gamma\left(\sigma\right)\right|d\sigma\right)\int\limits_0^{\pi}u_t^2\,dx\\&-\int\limits_0^{\pi}\int\limits_0^{1}\int\limits_{\tau_1}^{\tau_2}\sigma\,e^{-\sigma\rho}\left|\gamma\left(\sigma\right)\right|z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx,\end{aligned}$$

using the fact that $e^{-\sigma}\leq e^{-\sigma\rho}\leq 1$ we get for all $\rho\in[0,1]$

$$\begin{aligned}I_5{}'\left(t\right)\leq&-\int\limits_0^{\pi}\int\limits_{\tau_1}^{\tau_2}e^{-\sigma}\left|\gamma\left(\sigma\right)\right|z^2\left(x,1,\sigma,t\right)d\sigma\,dx+\left(\int\limits_{\tau_1}^{\tau_2}\left|\gamma\left(\sigma\right)\right|d\sigma\right)\int\limits_0^{\pi}u_t^2\,dx\\&-\int\limits_0^{\pi}\int\limits_0^{1}\int\limits_{\tau_1}^{\tau_2}\sigma\,e^{-\sigma}\left|\gamma\left(\sigma\right)\right|z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx.\end{aligned}$$

Since $-e^{-\sigma}$ is an increasing function, we have $-e^{-\sigma} \leq -e^{-\tau_2}$ for all $\sigma \in [\tau_1, \tau_2]$. Finally , setting $m_1 = e^{-\tau_2}$, we get (3.3.58) □

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = N\,E\,(t) + I_1\,(t) + N_1 I_2\,(t) + \frac{2}{|\beta|\;\rho_1} I_3\,(t) + N_2 I_4\,(t) + N_3 I_5\,(t)\,, \tag{3.3.59}$$

where N, N_1, N_2, N_3 are positive constants.

Lemma 3.3.7. *Let* $(u, \varphi, \theta, z, \eta)$ *be a solution of* (3.2.11) - (3.2.12). *Then, there exist two positive constants* λ_1 *and* λ_2 *such that the Lyapunov functional* (3.3.59) *satisfies*

$$\lambda_1 E\,(t) \leq \mathcal{L}(t) \leq \lambda_2 E\,(t)\,,\; \forall t \geq 0, \tag{3.3.60}$$

and

$$\mathcal{L}'(t) \leq -\varsigma_1 E(t) + \varsigma_2 \int_0^\infty \kappa\,(s)\,\|\eta_x\|^2 ds \;\;;\;\; \varsigma_1\;,\;\varsigma_2 > 0. \tag{3.3.61}$$

Proof. From (3.3.59), we have

$$\begin{aligned}
|\mathcal{L}(t) - NE\,(t)| \leq{}& \rho_1 \int_0^\pi |u_t u|\,dx + N_1 J \int_0^\pi |\varphi_t \varphi|\,dx + N_1 \frac{\tau}{2} \int_0^\pi \varphi^2 dx + N_1 \frac{b\,\rho_1}{\mu} \int_0^\pi \varphi \int_0^x u_t\,(y)\,dy\,dx \\
&+ \frac{2c}{|\beta|} \int_0^\pi \left| \theta \int_0^x u_t\,(y)\,dy \right| dx + \frac{N_2\,c}{\kappa_0} \int_0^\pi \left| \theta \int_0^\infty \kappa\,(s)\,\eta\,(x,s)\,ds \right| dx \\
&+ N_3 \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho}\,|\gamma\,(\sigma)|\,z^2\,(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx.
\end{aligned}$$

By applying Young's, Poincaré and Cauchy-Schwarz inequalities, we get

$$|\mathcal{L}(t) - NE\,(t)| \leq \varsigma E\,(t)\,,\; \varsigma > 0,$$

which yields

$$(N - \varsigma)\,E\,(t) \leq \mathcal{L}\,(t) \leq (N + \varsigma)\,E\,(t)\,,$$

by choosing N (depending on N_1, N_2, and N_3) sufficiently large we obtain (3.3.60).

Now, By differentiating $\mathcal{L}\,(t)$, exploiting (3.3.26), (3.3.34), (3.3.40), (3.3.45), (3.3.52), (3.3.58) and setting $\varepsilon_0 = \dfrac{1}{2N_1}$, $\varepsilon_1 = \dfrac{\mu\rho_1|\beta|}{8}$, $\varepsilon_2 = \dfrac{1}{N_2}$, $\varepsilon_3 = \dfrac{\rho_1}{N_2}$, we get

$$\begin{aligned}
\mathcal{L}'(t) \le & -\left(\frac{1}{2} - 2\rho_1 - \gamma(\tau_2 - \tau_1)(N + N_3)\right)\int_0^\pi u_t^2 dx - \left(\frac{N_1\chi}{2} - c_0 - \frac{\mu}{4}\right)\int_0^\pi \varphi^2 dx \\
& - \frac{\mu}{4}\int_0^\pi u_x^2 dx - m_1 N_3 \int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} \sigma|\gamma(\sigma)| z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx - \alpha N_1 \int_0^\pi \varphi_x^2 dx \\
& - \left(m_1 N_3 - c_0 - c_0 N_1 - \frac{\mu\pi^2}{4}\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\, d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma(\sigma)| z^2(x,1,\sigma,t)\, d\sigma\, dx \\
& - \left(\frac{c N_2}{2} - c_0 - \frac{1}{\chi}\left(\delta - \frac{b\beta}{\mu}\right)^2 N_1 - \frac{2c_0}{\rho_1|\beta|}\left(1 + \frac{8}{\mu\rho_1|\beta|}\right)\right)\int_0^\pi \theta^2 dx \\
& - \left(N\tau - \left(J + \frac{b^2\rho_1^2 N_1}{2}\right)N_1 - \frac{2c_0}{|\beta|\rho_1} - 1\right)\int_0^\pi \varphi_t^2 dx \\
& + \left(\frac{2\kappa_0}{\beta^2} + c_0 N_2\left(1 + N_2 + \frac{N_2}{\rho_1}\right)\right)\int_0^\infty \kappa(s)\|\eta_x\|^2 ds \\
& + \left(\frac{N}{2} - \frac{c\,\kappa(0)}{2\kappa_0^2}N_2\right)\int_0^\infty \kappa'(s)\|\eta_x\|^2 ds. \qquad (3.3.62)
\end{aligned}$$

where $\gamma = \sup\limits_{\sigma\in[\tau_1,\tau_2]} |\gamma(\sigma)|$

Now, we select our parameters appropriately as follows:

First, we choose N_1 large enough so that

$$\alpha_1 = \frac{N_1\chi}{2} - c_0 - \frac{\mu}{4} > 0.$$

Next, we select N_2 large enough so that

$$\alpha_2 = \frac{c N_2}{2} - c_0 - \frac{1}{\chi}\left(\delta - \frac{b\beta}{\mu}\right)^2 N_1 - \frac{2c_0}{\rho_1|\beta|}\left(1 + \frac{8}{\mu\rho_1|\beta|}\right) > 0.$$

We take N_3 large such that

$$m_1 N_3 - c_0 - c_0 N_1 - \frac{\mu \pi^2}{4} \int_{\tau_1}^{\tau_2} |\gamma(\sigma)| \, d\sigma > 0.$$

We choose N large enough so that (3.3.60) remains valid, further

$$\alpha_3 = N\tau - \left(J + \frac{b^2 \rho_1^2 N_1}{2} \right) N_1 - \frac{2c_0}{|\beta| \rho_1} - 1 > 0 and \quad \frac{N}{2} - \frac{c\,\kappa(0)}{2\kappa_0^2} N_2 > 0.$$

Finally, by taking γ so small that all these choices with the relation (3.3.60) leads to

$$\alpha_4 = \frac{1}{2} - 2\rho_1 - \gamma(\tau_2 - \tau_1)(N + N_3) > 0.$$

Let $\alpha_5 = \frac{\mu}{4}$, $\alpha_6 = \alpha N_1$, $\alpha_7 = m_1 N_3$, $\alpha_8 = \frac{2\kappa_0}{\beta^2} + c_0 N_2 \left(1 + N_2 + \frac{N_2}{\rho_1} \right)$

Ultimately, (3.3.62) turns out to be

$$\mathcal{L}'(t) \leq -\omega \left[\int_0^{\pi} \left(u_t^2 + \varphi_t^2 + \theta^2 + u_x^2 + \varphi_x^2 + \varphi^2 \right) dx \right] - \omega \int_0^{\pi} \int_0^{1} \int_{\tau_1}^{\tau_2} \sigma |\gamma(\sigma)| z^2(x, \rho, \sigma, t) \, d\sigma \, d\rho \, dx$$
$$+ \alpha_8 \int_0^{\infty} \kappa(s) \|\eta_x\|^2 ds.$$

Meanwhile, by revisiting the energy functional (3.3.25) and utilizing Young's inequality we find (3.3.61) □

Now, we can state and prove the following stability result.

Theorem 3.3.1. *Assume that* (0.0.6) *holds and* κ *satisfies* $(H1)$ - $(H4)$. *Then system* (3.2.11)-(3.2.12) *is exponentially stable. In other words there exist two positive constants* v_1 *and* v_2 *such that*

$$E(t) \leq v_2 e^{-v_1 t} \ , \ \forall t \geq 0. \tag{3.3.63}$$

Proof. Multiplying (3.3.25) by r, using $(H4)$, we end up with

$$\mathcal{Y}'(t) \leq -r\varsigma_1 E(t) \ , \ \forall t \geq 0, \tag{3.3.64}$$

where $\mathcal{Y}(t) = r\mathcal{L}(t) + 2\varsigma_2 E(t)$.Using(3.3.60),it's readily follows,for some $a_0, b_0 > 0$

$$a_0 E(t) \leq \mathcal{Y}(t) \leq b_0 E(t) \quad , \quad \forall t \geq 0. \tag{3.3.65}$$

Consequently, inequality (3.3.64) becomes

$$\mathcal{Y}'(t) \leq -v_1 \mathcal{Y}(t) \quad , \quad \forall t \geq 0, \tag{3.3.66}$$

where $v_1 = \dfrac{r\varsigma_1}{b_0}$. A simple integration of (3.3.66) over $(0, t)$ induces

$$Y(t) \leq \mathcal{Y}(0) e^{-v_1 t} \quad , \quad \forall t \geq 0. \tag{3.3.67}$$

Thus, by combining (3.3.65) and (3.3.67), we obtain (3.3.63), which ultimately leads to the conclusion of our stability result. $\square$

Chapter 4

Exponential Stability Of Lord Shulman Thermoelastic System With Porous Damping And Distributed Delay Term

In this chapter, we consider a one-dimensional Lord-Shulman thermoelastic system with porous damping and distributed delay term acting on the porous equation (0.0.23). Under suitable assumptions on the weight of distributed delay, we establish the well-posedness of the system by using semigroup theory and we show that the dissipations due to thermal effects with porous damping are strong enough to stabilise the system exponentially, independently of the wave speeds of the system [8].

4.1 Preliminaries

As in [66] we introduce the following new dependent variable

$$z(x,\rho,s,t) = \varphi_t(x,t-\rho s), \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty) \tag{4.1.1}$$

Is is easy to check that z satisfies

$$s\, z_t(x,\rho,s,t) = -z_\rho(x,\rho,s,t). \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty) \tag{4.1.2}$$

Consequently, problem (0.0.23) is equivalent to

$$\begin{cases} \rho_1 u_{tt} = \mu_1 u_{xx} + \mu_0 \varphi_x - \beta_0(\tau\theta_{tx} + \theta_x), \\ J\varphi_{tt} = a_0\varphi_{xx} - \mu_0 u_x - \xi\varphi + \beta_1(\tau\theta_t + \theta) - \gamma_1\varphi_t - \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(s)\; z(x,1,s,t)\, ds, \\ a(\tau\theta_t + \theta)_t = -\beta_0 u_{tx} - \beta_1\varphi_t + \delta\theta_{xx}, \\ s\, z_t = -z_\rho, \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty) \end{cases} \tag{4.1.3}$$

where $(x,t) \in (0,1)\times(0,\infty)$.

With the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \;\; \varphi(x,0) = \varphi_0(x), \;\; \theta(x,0) = \theta_0(x) \quad in\,(0,1), \\ u_t(x,0) = u_1(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; \theta_t(x,0) = \theta_1(x) \quad in\,(0,1), \\ z(x,\rho,s,0) = f_0(x,\rho s) \quad in\,(0,1)\times(0,1)\times(0,\tau_2), \\ u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \geq 0 \\ z(x,0,s,t) = \varphi_t(x,t) \quad in\,(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \end{cases} \tag{4.1.4}$$

concerning the weight of the delay, we only assume that

$$\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds < \gamma_1. \tag{4.1.5}$$

Meanwhile, using $(2,3)_1$ and the boundary conditions, we get

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\, dx = 0. \qquad \forall t \geq 0 \tag{4.1.6}$$

So, by solving (4.1.5) and using the initial data of u, we obtain

$$\int_0^1 u(x,t)\, dx = t\int_0^1 u_1(x)\, dx + \int_0^1 u_0(x)\, dx\ . \qquad \forall t \geq 0$$

Consequently, if we set

$$\overline{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)\, dx - \int_0^1 u_0(x)\, dx \quad t \geq 0\ , \ \ x \in (0,1)$$

We end up with

$$\int_0^1 \overline{u}(x,t)\, dx = 0, \quad t \geq 0$$

Thus, Poincaré's inequality can be applied on $\overline{u}$. In addition, simple substitution shows that $(\overline{u}, \varphi, \theta)$ is the solution of problem (4.1.3) with initial data for $\overline{u}$ given as

$$\overline{u}(x,0) = \overline{u}_0(x) = u_0(x) - \int_0^1 u_0(x)\, dx,$$

and

$$\overline{u}_t(x,0) = \overline{u}_1(x) = u_1(x) - \int_0^1 u_1(x)\, dx.$$

In what follows, we will work with $\overline{u}$ but, for convenience, we write u instead of $\overline{u}$.

4.2 Well-Posedness

In this section, we give the existence and uniqueness result for problem (4.1.3)-(4.1.4) using the semigroupe theory.

First, we introduce the vector function

$$U = (u, v, \varphi, \phi, \theta, \psi, z)^T.$$

and the new dependent variables

$$v = u_t \quad , \quad \phi = \varphi_t \quad , \quad \psi = \theta_t.$$

Then the system (4.1.3)-(4.1.4) can be written as follows:

$$\begin{cases} U_t = AU, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \theta_1, f_0)^T, \end{cases} \tag{4.2.7}$$

Where $A : D(A) \subset H \rightarrow H$ is a linear operator defined by

$$AU = \begin{pmatrix} v \\ \frac{\mu_1}{\rho_1} u_{xx} + \frac{\mu_0}{\rho_1} \varphi_x - \frac{\beta_0}{\rho_1} (\tau \psi_x + \theta_x) \\ \phi \\ \frac{a_0}{J} \varphi_{xx} - \frac{\mu_0}{J} u_x - \frac{\xi}{J} \varphi + \frac{\beta_1}{J} (\tau \psi + \theta) - \frac{\gamma_1}{J} \phi - \frac{1}{J} \int\limits_{\tau_1}^{\tau_2} \gamma_2(s) z(x,1,s,t)\, ds \\ \psi \\ -\frac{1}{\tau} \psi - \frac{\beta_0}{a\tau} v_x - \frac{\beta_1}{a\tau} \phi + \frac{\delta}{a\tau} \theta_{xx} \\ -\frac{1}{s} z_\rho(x, \rho, s, t) \end{pmatrix}$$

and H is the energy space given by

$$H = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1) \times (\tau_1, \tau_2))$$

such that

$$H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1)$$
$$L_*^2(0,1) = \left\{ w \in L^2(0,1) : \int_0^1 w(x)\,dx = 0 \right\}$$

For any $U = (u, v, \varphi, \phi, \theta, \psi, z)^T \in H$ and $\widetilde{U} = \left(\widetilde{u}, \widetilde{v}, \widetilde{\varphi}, \widetilde{\phi}, \widetilde{\theta}, \widetilde{\psi}, \widetilde{z}\right)^T \in H$ we equip H with the inner product defined by

$$\begin{aligned}
\left(U, \widetilde{U}\right)_H &= \rho_1 \int_0^1 v\,\widetilde{v}\,dx + J \int_0^1 \phi\,\widetilde{\phi}\,dx + a_0 \int_0^1 \varphi_x\,\widetilde{\varphi}_x\,dx + \delta\tau \int_0^1 \theta_x\,\widetilde{\theta}_x\,dx + a \int_0^1 (\tau\,\psi + \theta)\left(\tau\,\widetilde{\psi} + \widetilde{\theta}\right)dx \\
&+ \mu_1 \int_0^1 \left(u_x + \frac{\mu_0}{\mu_1}\varphi\right)\left(\widetilde{u}_x + \frac{\mu_0}{\mu_1}\widetilde{\varphi}\right)dx + \left(\xi - \frac{\mu_0^2}{\mu_1}\right)\int_0^1 \varphi\,\widetilde{\varphi}dx \\
&+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z(x,\rho,s,t)\,\widetilde{z}(x,\rho,s,t)\,ds\,d\rho\,dx.
\end{aligned}$$

The domain of A is given by

$$\begin{aligned}
D(A) = \Big\{ & U \in H \;/\; u \in H_*^2(0,1) \cap H_*^1(0,1) \;;\; v \in H_*^1(0,1) \;;\; \varphi, \theta \in H^2(0,1) \cap H_0^1(0,1) \\
& \phi, \psi \in H_0^1(0,1) \;;\; z, z_\rho \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \Big\}.
\end{aligned}$$

Where

$$H_*^2(0,1) = \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\}.$$

Clearly, $D(A)$ is dense in H .

Now, we can give the following existence result.

Theorem 4.2.1. *Let $U_0 \in H$, Problem (4.2.7) has a unique solution $U \in C(\mathbb{R}_+, H)$. Moreover, if $U_0 \in D(A)$, then $U \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, H)$*

Proof. The result follows from Lumer-phillips theorem provided we prove that A is a maximal dissipative operator. Firstly, we prove that A is dissipative. For any $U \in D(A)$, and using the inner product, we get

$$(A\,U,U)_H = -\left(\gamma_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\ ds\right)\int_0^1 \phi^2 dx - \delta\int_0^1 \theta_x^2\,dx$$
$$-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\ z^2(x,1,s,t)\,ds\,dx - \int_0^1 \phi\int_{\tau_1}^{\tau_2} \gamma_2(s)\,z(x,1,s,t)\,ds\,dx \tag{4.2.8}$$

using Young's inequality on the last term in (4.2.8) we obtain

$$-\int_0^1 \phi\int_{\tau_1}^{\tau_2} \gamma_2(s)\,z(x,1,s,t)\,ds\,dx \le \left(\frac{1}{2}\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,ds\right)\int_0^1 \phi^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\ z^2(x,1,s,t)\,ds\,dx \tag{4.2.9}$$

Substituting (4.2.9) into (4.2.8) and using (4.1.5) we get

$$(AU,U)_H = -\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,ds\right)\int_0^1 \phi^2 dx - \delta\int_0^1 \theta_x^2\,dx \le 0$$

Hence, the operator A is dissipative. Next, we prove that the operator A is maximal. It is sufficient to show that the operator $I - A$ is surjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in H$, we prove that there exist $U \in D(A)$ such that

$$(I - A)\,U = F \tag{4.2.10}$$

That is

$$\begin{cases} u - v = f_1 \in H_*^1(0,1) \\ v - \dfrac{\mu_1}{\rho_1} u_{xx} - \dfrac{\mu_0}{\rho_1}\varphi_x + \dfrac{\beta_0}{\rho_1}(\tau\psi_x + \theta_x) = f_2 \in L_*^2(0,1) \\ \varphi - \phi = f_3 \in H_0^1(0,1) \\ \phi - \dfrac{a_0}{J}\varphi_{xx} + \dfrac{\mu_0}{J}u_x + \dfrac{\xi}{J}\varphi - \dfrac{\beta_1}{J}(\tau\psi + \theta) + \dfrac{\gamma_1}{J}\phi + \dfrac{1}{J}\displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(s)\, z(x,1,s,t)\, ds = f_4 \in L^2(0,1) \\ \theta - \psi = f_5 \in H_0^1(0,1) \\ \psi + \dfrac{\beta_0}{a\,\tau}v_x + \dfrac{\beta_1}{a\,\tau}\phi - \dfrac{\delta}{a\,\tau}\theta_{xx} + \dfrac{1}{\tau}\psi = f_6 \in L^2(0,1) \\ z + \dfrac{1}{s}z_\rho = f_7 \in L^2((0,1),(0,1),(\tau_1,\tau_2)) \end{cases} \tag{4.2.11}$$

We note that the last equation in (4.2.11) with $z(x,0,s,t) = \phi(x,t)$ has a unique solution given by

$$z(x,\rho,s,t) = \phi(x,t)\, e^{-s\rho} + s\, e^{-s\rho} \int_0^{\rho} e^{s\,\sigma} f_7(x,\sigma,s,t)\, d\sigma \tag{4.2.12}$$

Inserting $v = u - f_1$, $\phi = \varphi - f_3$, $\psi = \theta - f_5$ and (4.2.12) in $(3,5)_2$, $(3,5)_4$ and $(3,5)_6$,we get

$$\begin{cases} \rho_1 u - \mu_1 u_{xx} - \mu_0 \varphi_x + \beta_0(\tau+1)\theta_x = h_1 \in L^2(0,1) \\ \mu_4 \varphi - a_0 \varphi_{xx} + \mu_0 u_x - \beta_1(\tau+1)\theta = h_2 \in L^2(0,1) \\ a(\tau+1)\theta - \delta\theta_{xx} + \beta_0 u_x + \beta_1 \varphi = h_3 \in L^2(0,1) \end{cases} \tag{4.2.13}$$

where

$$h_1 = \rho_1(f_1 + f_2) + \beta_0 \tau f_{5\,x}$$

$$h_2 = \left(J + \gamma_1 + \int_{\tau_1}^{\tau_2} e^{-s}\gamma_2(s)\, ds\right) f_3 + J f_4 - \beta_1 \tau f_5 + \int_{\tau_1}^{\tau_2} s\, e^{-s}\gamma_2(s) \int_0^1 e^{s\,\sigma} f_7(x,\sigma,s,t)\, d\sigma\, ds$$

$$h_3 = a\ (\tau+1)\, f_5 + a\,\tau f_6 + \beta_0 f_{1x} + \beta_1 f_3$$

$$\mu_4 = J + \xi + \gamma_1 + \int_{\tau_1}^{\tau_2} e^{-s}\gamma_2\,(s)\ ds$$

To solve (4.2.13), we consider the following variational formulation:

$$B\left((u,\varphi,\theta)\,;\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)\right) = L\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right) \tag{4.2.14}$$

where $B : \left[H_*^1\,(0,1)\times H_0^1\,(0,1)\times H_0^1\,(0,1)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}
B\left((u,\varphi,\theta)\,;\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)\right) &= \rho_1 \int_0^1 u\,\widetilde{u}\,dx + \mu_4 \int_0^1 \varphi\,\widetilde{\varphi}\,dx + a(\tau+1)^2 \int_0^1 \theta\,\widetilde{\theta}\,dx \\
&+ \mu_1 \int_0^1 u_x\,\widetilde{u}_x\,dx + a_0 \int_0^1 \varphi_x\,\widetilde{\varphi}_x\,dx + \delta\,(\tau+1) \int_0^1 \theta_x\,\widetilde{\theta}_x\,dx \\
&+ \mu_0 \int_0^1 \left(u_x\,\widetilde{\varphi} + \varphi\,\widetilde{u}_x\right) dx + \beta_0\,(\tau+1) \int_0^1 \left(u_x\,\widetilde{\theta} - \theta\,\widetilde{u}_x\right) dx \\
&+ \beta_1\,(\tau+1) \int_0^1 \left(\varphi\,\widetilde{\theta} - \theta\,\widetilde{\varphi}\right) dx
\end{aligned}$$

and $L : H_*^1\,(0,1)\times H_0^1\,(0,1)\times H_0^1\,(0,1) \to \mathbb{R}$ is the linear form given by

$$L\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right) = \int_0^1 h_1\,\widetilde{u}\,dx + \int_0^1 h_2\,\widetilde{\varphi}\,dx + (\tau+1) \int_0^1 h_3\,\widetilde{\theta}\,dx$$

Now, for $V = H_*^1\,(0,1)\times H_0^1\,(0,1)\times H_0^1\,(0,1)$ equipped with the norm

$$\|(u,\varphi,\theta)\|_V^2 = \|u\|_{L^2(0,1)}^2 + \|u_x\|_{L^2(0,1)}^2 + \|\varphi\|_{L^2(0,1)}^2 + \|\varphi_x\|_{L^2(0,1)}^2 + \|\theta\|_{L^2(0,1)}^2 + \|\theta_x\|_{L^2(0,1)}^2$$

we have

$$B\left((u,\varphi,\theta);(u,\varphi,\theta)\right)=\rho_1\int_0^1 u^2dx+\mu_4\int_0^1 \varphi^2dx+a(\tau+1)^2\int_0^1 \theta^2dx+\mu_1\int_0^1 u_x^2dx$$
$$+a_0\int_0^1 \varphi_x^2dx+\delta\left(\tau+1\right)\int_0^1 \theta_x^2dx+2\mu_0\int_0^1 u_x\,\varphi\,dx$$

on the other hand, we can write

$$\mu_1\,u_x^2+2\mu_0\,u_x\,\varphi\;+\mu_4\,\varphi^2=\frac{1}{2}\left[\mu_1\left(u_x+\frac{\mu_0}{\mu_1}\varphi\right)^2+\mu_4\left(\varphi+\frac{\mu_0}{\mu_4}u_x\right)^2+\left(\mu_1-\frac{\mu_0^2}{\mu_4}\right)u_x^2+\left(\mu_4-\frac{\mu_0^2}{\mu_1}\right)\varphi^2\right]$$

using (0.0.14), we deduce

$$\mu_1\,u_x^2+2\mu_0\,u_x\,\varphi\;+\mu_4\,\varphi^2>\frac{1}{2}\left[\left(\mu_1-\frac{\mu_0^2}{\mu_4}\right)u_x^2+\left(\mu_4-\frac{\mu_0^2}{\mu_1}\right)\varphi^2\right]$$

then, for some $M_0>0$

$$B\left((u,\varphi,\theta);(u,\varphi,\theta)\right)\geq M_0\;\|(u,\varphi,\theta)\|_V^2$$

Thus B is coercive. Similarly, we can easily prove that the bilinear and linear forms B and L are continue.

Consequently, by the Lax-Miligram Lemma the variational problrm (4.2.14) has aunique solution

$$(u,\varphi,\theta)\in H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$$

satifying

$$B\left((u,\varphi,\theta);\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)\right)=L\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)\qquad\forall\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)\in V$$

The subsitition of u, φ and θ into $(3,5)_1$, $(3,5)_3$ and $(3,5)_5$,respectively, we obtain

$$(v,\phi,\psi)\in H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$$

Similarly, inserting ϕ in (4.2.12) and bearning in mind $(3,5)_7$, we find

$$z\,,\;z_\rho\;\in L^2\left((0,1)\,,(0,1)\,,(\tau_1,\tau_2)\right)$$

Now, if $(\varphi, \theta) \equiv (0,0) \in H_0^1(0,1) \times H_0^1(0,1)$, then (4.2.14) reduces to

$$\rho_1 \int_0^1 u\,\widetilde{u}\,dx + \mu_1 \int_0^1 u_x\,\widetilde{u}_x\,dx + \mu_0 \int_0^1 \varphi\,\widetilde{u}_x\,dx - \beta_0(\tau+1) \int_0^1 \theta\,\widetilde{u}_x\,dx$$
$$= \int_0^1 h_1\,\widetilde{u}\,dx \qquad \forall \widetilde{u} \in H_*^1(0,1) \tag{4.2.15}$$

which implies

$$-\mu_1 u_{xx} = -\rho_1 u + \mu_0 \varphi_x - \beta_0(\tau+1)\theta_x + h_1 \in L^2(0,1) \tag{4.2.16}$$

Consequently, by the regularity theory for the linear elliptic equation, it follows that

$$u \in H^2(0,1) \cap H_*^1(0,1)$$

Moreover, (4.2.15) is also true for any $\vartheta \in C^1([0,1]) \subset H_*^1(0,1)$. Hence, we have

$$\rho_1 \int_0^1 u\,\vartheta\,dx + \mu_1 \int_0^1 u_x\,\vartheta_x\,dx + \mu_0 \int_0^1 \varphi\,\vartheta_x\,dx - \beta_0(\tau+1) \int_0^1 \theta\,\vartheta_x\,dx - \int_0^1 h_1\,\vartheta\,dx = 0$$

for all $\vartheta \in C^1([0,1])$. Thus, using integration by parts and bearning in mind (4.2.16), we obtain

$$u_x(1)\,\vartheta(1) - u_x(0)\,\vartheta(0) = 0 \qquad \forall \vartheta \in C^1([0,1])$$

Therefore, $u_x(0) = u_x(1) = 0$. Consequently, we obtain

$$u \in H_*^2(0,1) \cap H_*^1(0,1)$$

Similarly, we obtain

$$-a_0\,\varphi_{xx} = -\mu_4\,\varphi - \mu_0\,u_x - \beta_1(\tau+1)\,\theta + h_2 \in L^2(0,1)$$

$$-\delta\,\theta_{xx} = -a(\tau+1)\,\theta - \beta_0\,u_x - \beta_1\,\varphi + h_3 \in L^2(0,1)$$

thus, we have

$$\varphi\,,\ \theta \in H^2(0,1) \cap H_0^1(0,1)$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $U \in D(A)$ such that (4.2.10) is satisfied. Consequently, A is a maximal operator. Hence, the result of Theorem 4.2.1 follows from Lummer-Phillips theorem (see [52, 76]) $\square$

4.3 Exponential stability

In this section, we use the energy method to prove that system (4.1.3)-(4.1.4) is exponentially stable. To achieve our goal, we need the following lemmas:

Lemma 4.3.1. *The energy functional, E, defined by*

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \left(\rho_1 u_t^2 + J\varphi_t^2 + a_0\varphi_x^2 + \delta\tau\theta_x^2 + a(\tau\theta_t+\theta)^2 + \mu_1\left(u_x + \frac{\mu_0}{\mu_1}\varphi\right)^2 + \left(\xi - \frac{\mu_0^2}{\mu_1}\right)\varphi^2\right)dx \\ &+ \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx \qquad t \geq 0. \end{aligned} \tag{4.3.17}$$

satisfies

$$E'(t) \leq -\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,ds\right)\int_0^1 \varphi_t^2 dx - \delta\int_0^1 \theta_x^2 dx. \tag{4.3.18}$$

Proof. Multiplying $(2,3)_1$ by u_t, $(2,3)_2$ by φ_t, $(2,3)_3$ by $(\tau\theta_t+\theta)$ and integrating over (0,1) and summing them up we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho_1 u_t^2 + \mu_1 u_x^2 + 2\mu_0\varphi\,u_x + J\varphi_t^2 + a_0\varphi_x^2 + \xi\varphi^2 + a(\tau\theta_t+\theta)^2 + \delta\tau\theta_x^2\right)dx \\ &\quad + \gamma_1\int_0^1 \varphi_t^2 dx + \delta\int_0^1 \theta_x^2 dx + \int_0^1 \varphi_t\int_{\tau_1}^{\tau_2} \gamma_2(s)\,z(x,1,s,t)\,ds\,dx = 0 \end{aligned} \tag{4.3.19}$$

Multiplying $(2,3)_4$ by $|\gamma_2(s)|\, z$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(s)|\;z^2(x,1,s,t)\,ds\,dx$$
$$-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(s)|\;z^2(x,0,s,t)\,ds\,dx=0 \tag{4.3.20}$$

by summing (4.3.19),(4.3.20) and using the fact that $z(x,0,s,t)=\varphi_t(x,t)$,we have

$$\frac{d}{dt}\left\{\frac{1}{2}\int_0^1\left(\rho_1\,u_t^2+\mu_1 u_x^2+2\mu_0\varphi\,u_x+J\,\varphi_t^2+a_0\,\varphi_x^2+\xi\varphi^2+a(\tau\,\theta_t+\theta)^2+\delta\tau\,\theta_x^2\right)dx\right.$$
$$\left.+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx\right\}$$
$$=-\gamma_1\int_0^1\varphi_t^2dx-\delta\int_0^1\theta_x^2dx-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\gamma_2(s)\;z(x,1,s,t)\,ds\,dx$$
$$-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(s)|\;z^2(x,1,s,t)\,ds\,dx+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\gamma_2(s)|\;ds\int_0^1\varphi_t^2dx$$

using the fact that

$$\mu_1 u_x^2+2\mu_0\varphi\,u_x+\xi\varphi^2=\mu_1\left(u_x+\frac{\mu_0}{\mu_1}\varphi\right)^2+\left(\xi-\frac{\mu_0^2}{\mu_1}\right)\varphi^2$$

we find

$$E(t)=\frac{1}{2}\int_0^1\left(\rho_1\,u_t^2+J\,\varphi_t^2+a_0\,\varphi_x^2+\delta\tau\,\theta_x^2+a(\tau\,\theta_t+\theta)^2+\mu_1\left(u_x+\frac{\mu_0}{\mu_1}\varphi\right)^2+\left(\xi-\frac{\mu_0^2}{\mu_1}\right)\varphi^2\right)dx$$
$$+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx$$

and

$$E'(t) = -\left(\gamma_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx - \delta\int_0^1 \theta_x^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \gamma_2(s)\, z(x,1,s,t)\, ds\, dx$$
$$-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, z^2(x,1,s,t)\, ds\, dx \tag{4.3.21}$$

using Young's inequality, we get

$$-\int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \gamma_2(s)\, z(x,1,s,t)\, ds\, dx \le \left(\frac{1}{2}\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, z^2(x,1,s,t)\, ds\, dx \tag{4.3.22}$$

inserting (4.3.22) in (4.3.21) , and using (4.1.5), we find

$$E'(t) \le -\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx - \delta\int_0^1 \theta_x^2 dx \le 0$$

Thus, the energy is decreasing and bounded above by $E(0)$, which concludes the proof. □

Lemma 4.3.2. *Let (u, φ, θ, z) be the solution of (4.1.3)-(4.1.4) then, the functional $I_1(t)$ defined by*

$$I_1(t) = \rho_1\int_0^1 u_t\, u\, dx - \beta_0\tau\int_0^1 \theta\, u_x\, dx \qquad t \ge 0, \tag{4.3.23}$$

satisfies

$$I_1'(t) \le -\frac{\mu_1}{2}\int_0^1 u_x^2\, dx + c_0\int_0^1 \left(\theta_x^2 + u_t^2 + \varphi^2\right) dx \qquad t \ge 0. \tag{4.3.24}$$

Proof. Multiplying $(2,3)_1$ by u and integrating over (0,1) we obtain

$$\rho_1\int_0^1 u_{tt}\, u\, dx = \mu_1\int_0^1 u_{xx}\, u\, dx + \mu_0\int_0^1 \varphi_x\, u\, dx - \beta_0\int_0^1 (\tau\,\theta_{tx} + \theta_x)\, u\, dx$$

and then integrating by parts together with the boundary conditions, we get

$$\begin{aligned}
\rho_1 \int_0^1 u_{tt}\, u\, dx + \rho_1 \int_0^1 u_t^2\, dx - \rho_1 \int_0^1 u_t^2\, dx &= -\mu_1 \int_0^1 u_x^2\, dx - \mu_0 \int_0^1 \varphi\, u_x\, dx + \beta_0 \int_0^1 (\tau\, \theta_t + \theta)\, u_x\, dx \\
&= -\mu_1 \int_0^1 u_x^2\, dx - \mu_0 \int_0^1 \varphi\, u_x\, dx + \beta_0\, \tau \int_0^1 \theta_t\, u_x dx + \beta_0 \int_0^1 \theta\, u_x dx \\
&+ \beta_0\, \tau \int_0^1 \theta\, u_{tx}\, dx - \beta_0\, \tau \int_0^1 \theta\, u_{tx}\, dx \\
&= -\mu_1 \int_0^1 u_x^2\, dx - \mu_0 \int_0^1 \varphi\, u_x\, dx + \beta_0\, \tau \int_0^1 \theta_t\, u_x dx + \beta_0 \int_0^1 \theta\, u_x dx \\
&+ \beta_0\, \tau \int_0^1 \theta\, u_{tx}\, dx + \beta_0\, \tau \int_0^1 \theta_x\, u_t\, dx
\end{aligned}$$

wich is equivalent to

$$\begin{aligned}
\frac{d}{dt}\left(\rho_1 \int_0^1 u_t\, u\, dx - \beta_0\, \tau \int_0^1 \theta\, u_x dx\right) &= \rho_1 \int_0^1 u_t^2\, dx - \mu_1 \int_0^1 u_x^2\, dx - \mu_0 \int_0^1 \varphi\, u_x\, dx \\
&+ \beta_0\, \tau \int_0^1 \theta_x\, u_t\, dx + \beta_0 \int_0^1 \theta\, u_x dx
\end{aligned}$$

then, we get

$$I_1\,(t) = \rho_1 \int_0^1 u_t\, u\, dx - \beta_0 \tau \int_0^1 \theta\, u_x\, dx \qquad t \geq 0 \tag{4.3.25}$$

and

$$I_1{}'\,(t) = \rho_1 \int_0^1 u_t^2\, dx - \mu_1 \int_0^1 u_x^2\, dx - \mu_0 \int_0^1 \varphi\, u_x\, dx + \beta_0\, \tau \int_0^1 \theta_x\, u_t\, dx + \beta_0 \int_0^1 \theta\, u_x dx \tag{4.3.26}$$

Using Young's and Poincar's inequalities and (0.0.14) we fined

$$-\mu_0\int_0^1 \varphi\, u_x\, dx \le \xi^2\int_0^1 \varphi^2\, dx + \frac{\mu_1}{4}\int_0^1 u_x^2\, dx \tag{4.3.27}$$

$$\beta_0\,\tau\int_0^1 \theta_x\, u_t\, dx \le \frac{\beta_0\,\tau}{2}\int_0^1 \theta_x^2\, dx + \frac{\beta_0\,\tau}{2}\int_0^1 u_t^2\, dx \tag{4.3.28}$$

$$\beta_0\int_0^1 \theta\, u_x dx \le \beta_0^2\int_0^1 \theta_x^2\, dx + \frac{\mu_1}{4}\int_0^1 u_x^2\, dx \tag{4.3.29}$$

inserting (4.3.27),(4.3.28),(4.3.29) in (4.3.26), we get

$$\begin{aligned} I_1{}'(t) &\le -\frac{\mu_1}{2}\int_0^1 u_x^2\, dx + \left(\rho_1 + \frac{\beta_0\,\tau}{2}\right)\int_0^1 u_t^2\, dx + \left(\beta_0^2 + \frac{\beta_0\,\tau}{2}\right)\int_0^1 \theta_x^2\, dx + \xi^2\int_0^1 \varphi^2\, dx \\ &\le -\frac{\mu_1}{2}\int_0^1 u_x^2\, dx + c_0\int_0^1 \left(u_t^2 + \theta_x^2 + \varphi^2\right) dx \end{aligned}$$

□

Lemma 4.3.3. *Let (u, φ, θ, z) be the solution of (4.1.3)-(4.1.4) then, the functional $I_2(t)$ defined by*

$$I_2(t) = J\int_0^1 \varphi_t\, \varphi\, dx + \frac{\gamma_1}{2}\int_0^1 \varphi^2 dx - \frac{\mu_0\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi(y)\, dy\right) dx \qquad t \ge 0, \tag{4.3.30}$$

satisfies for any $\varepsilon_1 > 0$ the estimate

$$\begin{aligned} I_2{}'(t) \le& -a_0\int_0^1 \varphi_x^2\, dx - \frac{\mu_3}{2}\int_0^1 \varphi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c_0\int_0^1 (\tau\,\theta_t + \theta)^2 dx \\ &+ c_0\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \varphi_t^2 dx + \frac{1}{\mu_3}\left(\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right)\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, z^2(x,1,s,t)\, ds\, dx \qquad t \ge 0, \end{aligned} \tag{4.3.31}$$

where $\mu_3 = \xi - \dfrac{\mu_0^2}{\mu_1}$

Proof. Multiplying $(2,3)_2$ by φ, integrating over (0,1) we obtain

$$J\int_0^1 \varphi_{tt}\,\varphi\,dx = a_0\int_0^1 \varphi_{xx}\,\varphi\,dx - \mu_0\int_0^1 u_x\,\varphi\,dx - \xi\int_0^1 \varphi^2\,dx + \beta_1\int_0^1 (\tau\,\theta_t+\theta)\,\varphi\,dx$$
$$-\gamma_1\int_0^1 \varphi_t\,\varphi\,dx - \int_0^1 \varphi\int_{\tau_1}^{\tau_2} \gamma_2(s)\;z(x,1,s,t)\,ds\,dx$$

using integration by parts together with the boundary conditions, we get

$$J\int_0^1 \varphi_{tt}\,\varphi\,dx + J\int_0^1 \varphi_t^2\,dx - J\int_0^1 \varphi_t^2\,dx = -a_0\int_0^1 \varphi_x^2\,dx - \mu_0\int_0^1 u_x\,\varphi\,dx - \xi\int_0^1 \varphi^2\,dx + \beta_1\int_0^1 (\tau\,\theta_t+\theta)\,\varphi\,dx$$
$$-\gamma_1\int_0^1 \varphi_t\,\varphi\,dx - \int_0^1 \varphi\int_{\tau_1}^{\tau_2} \gamma_2(s)\;z(x,1,s,t)\,ds\,dx$$

wich is equivalent to

$$\frac{d}{dt}\left(J\int_0^1 \varphi_t\,\varphi\,dx + \frac{\gamma_1}{2}\int_0^1 \varphi^2\,dx\right) = J\int_0^1 \varphi_t^2\,dx - a_0\int_0^1 \varphi_x^2\,dx - \mu_0\int_0^1 u_x\,\varphi\,dx - \xi\int_0^1 \varphi^2\,dx$$
$$+\beta_1\int_0^1 (\tau\,\theta_t+\theta)\,\varphi\,dx - \int_0^1 \varphi\int_{\tau_1}^{\tau_2} \gamma_2(s)\;z(x,1,s,t)\,ds\,dx \qquad (4.3.32)$$

Multiplying $(2,3)_1$ by $-\frac{\mu_0}{\mu_1}\int_0^x \varphi(y)\;dy$, integrating over (0,1) and then integrating by parts together with the boundary conditions, we get

$$-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_{tt}\left(\int_0^x \varphi(y)\;dy\right)\;dx = \mu_0\int_0^1 u_x\,\varphi\,dx + \frac{\mu_0^2}{\mu_1}\int_0^1 \varphi^2\,dx - \frac{\beta_0\,\mu_0}{\mu_1}\int_0^1 (\tau\,\theta_t+\theta)\,\varphi\,dx$$

witch is equivalent to

$$\frac{d}{dt}\left(-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi\left(y\right)\,dy\right)\,dx\right)=\mu_0\int_0^1 u_x\,\varphi\,dx+\frac{\mu_0^2}{\mu_1}\int_0^1 \varphi^2\,dx-\frac{\beta_0\,\mu_0}{\mu_1}\int_0^1 \left(\tau\,\theta_t+\theta\right)\varphi\,dx$$
$$-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi_t\left(y\right)\,dy\right)\,dx \tag{4.3.33}$$

we sum (4.3.32) and (4.3.33) we find

$$\frac{d}{dt}\left(J\int_0^1 \varphi_t\,\varphi\,dx+\frac{\gamma_1}{2}\int_0^1 \varphi^2\,dx-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi\left(y\right)\,dy\right)\,dx\right)$$
$$=J\int_0^1 \varphi_t^2\,dx-a_0\int_0^1 \varphi_x^2\,dx-\mu_3\int_0^1 \varphi^2\,dx+\left(\beta_1-\frac{\beta_0\,\mu_0}{\mu_1}\right)\int_0^1 \left(\tau\,\theta_t+\theta\right)\varphi\,dx$$
$$-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi_t\left(y\right)\,dy\right)\,dx-\int_0^1 \varphi\int_{\tau_1}^{\tau_2} \gamma_2\left(s\right)\,z\left(x,1,s,t\right)ds\,dx$$

where $\mu_3=\xi-\dfrac{\mu_0^2}{\mu_1}$ then, we get

$$I_2\left(t\right)=J\int_0^1 \varphi_t\,\varphi\,dx+\frac{\gamma_1}{2}\int_0^1 \varphi^2\,dx-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi\left(y\right)\,dy\right)\,dx\qquad t\geq 0 \tag{4.3.34}$$

and

$$I_2{}'\left(t\right)=J\int_0^1 \varphi_t^2\,dx-a_0\int_0^1 \varphi_x^2\,dx-\mu_3\int_0^1 \varphi^2\,dx-\int_0^1 \varphi\int_{\tau_1}^{\tau_2} \gamma_2\left(s\right)\,z\left(x,1,s,t\right)ds\,dx$$
$$+\left(\beta_1-\frac{\beta_0\,\mu_0}{\mu_1}\right)\int_0^1 \left(\tau\,\theta_t+\theta\right)\varphi\,dx-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi_t\left(y\right)\,dy\right)\,dx\qquad t\geq 0 \tag{4.3.35}$$

Using Young's and Cauchy Schwarz inequalities we obtain

$$-\frac{\mu_0\,\rho_1}{\mu_1}\int_0^1 u_t\left(\int_0^x \varphi_t\left(y\right)\,dy\right)\,dx\leq\varepsilon_1\int_0^1 u_t^2\,dx+\frac{1}{4\varepsilon_1}\left(\frac{\mu_0\,\rho_1}{\mu_1}\right)^2\int_0^1 \varphi_t^2\,dx \tag{4.3.36}$$

$$-\int_0^1 \varphi \int_{\tau_1}^{\tau_2} \gamma_2(s)\, z(x,1,s,t)\, ds\, dx \le \frac{1}{\mu_3}\left(\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right) \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, z^2(x,1,s,t)\, ds\, dx + \frac{\mu_3}{4}\int_0^1 \varphi^2\, dx$$

$$\left(\beta_1 - \frac{\beta_0\,\mu_0}{\mu_1}\right)\int_0^1 (\tau\,\theta_t + \theta)\,\varphi\, dx \le \frac{\mu_3}{4}\int_0^1 \varphi^2\, dx + \frac{1}{\mu_3}\left(\beta_1 - \frac{\beta_0\,\mu_0}{\mu_1}\right)^2 \int_0^1 (\tau\,\theta_t + \theta)^2\, dx \tag{4.3.37}$$

inserting (4.3.36) and (4.3.37) in (4.3.35) we get

$$I_2'(t) \le -a_0 \int_0^1 \varphi_x^2\, dx - \frac{\mu_3}{2}\int_0^1 \varphi^2\, dx + \varepsilon_1 \int_0^1 u_t^2\, dx + c_0 \int_0^1 (\tau\,\theta_t + \theta)^2\, dx + c_0\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \varphi_t^2\, dx$$
$$+ \frac{1}{\mu_3}\left(\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right)\int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\, z^2(x,1,s,t)\, ds\, dx \qquad t \ge 0$$

□

Lemma 4.3.4. *Let (u, φ, θ, z) be the solution of (4.1.3)-(4.1.4) then, the functional $I_3(t)$ defined by*

$$I_3(t) = -a\int_0^1 \tau^2 \theta_t\, \theta\, dx - \frac{a\tau}{2}\int_0^1 \theta^2\, dx \qquad t \ge 0, \tag{4.3.38}$$

satisfies for any $\varepsilon_2 > 0$ the estimate

$$I_3'(t) \le -\frac{a}{2}\int_0^1 (\tau\theta_t + \theta)^2\, dx + \varepsilon_2 \int_0^1 u_t^2\, dx + c_0 \int_0^1 \varphi_t^2\, dx + c_0\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \theta_x^2\, dx \qquad t \ge 0. \tag{4.3.39}$$

Proof. Multiplying $(2,3)_3$ by $-\tau\,\theta$, integrating over (0,1) we obtain

$$-a\,\tau^2 \int_0^1 \theta_{tt}\,\theta\, dx - a\,\tau \int_0^1 \theta_t\,\theta\, dx = \beta_0\,\tau \int_0^1 u_{tx}\,\theta\, dx + \beta_1\,\tau \int_0^1 \varphi_t\,\theta\, dx - \delta\tau \int_0^1 \theta_{xx}\,\theta\, dx$$

and then integrating by parts together with the boundary conditions, we get

$$-a\,\tau^2 \int_0^1 \theta_{tt}\,\theta\, dx - a\,\tau \int_0^1 \theta_t\,\theta\, dx = -\beta_0\,\tau \int_0^1 u_t\,\theta_x\, dx + \beta_1\,\tau \int_0^1 \varphi_t\,\theta\, dx + \delta\tau \int_0^1 \theta_x^2\, dx$$

wich is equivalent to

$$\frac{d}{dt}\left(-a\int_0^1 \tau^2\theta_t\,\theta\,dx-\frac{a\tau}{2}\int_0^1 \theta^2\,dx\right)=-a\tau^2\int_0^1 \theta_t^2 dx-\beta_0\,\tau\int_0^1 u_t\,\theta_x\,dx+\beta_1\,\tau\int_0^1 \varphi_t\,\theta\,dx+\delta\tau\int_0^1 \theta_x^2\,dx$$

then, we obtain

$$I_3\,(t)=-a\int_0^1 \tau^2\theta_t\,\theta\,dx-\frac{a\tau}{2}\int_0^1 \theta^2\,dx \qquad t\geq 0$$

and

$$I_3{}'\,(t)=-a\tau^2\int_0^1 \theta_t^2 dx-\beta_0\,\tau\int_0^1 u_t\,\theta_x\,dx+\beta_1\,\tau\int_0^1 \varphi_t\,\theta\,dx+\delta\tau\int_0^1 \theta_x^2\,dx \tag{4.3.40}$$

using Young's and Poincar's inequalities, we find

$$-\beta_0\,\tau\int_0^1 u_t\,\theta_x\,dx\leq \varepsilon_2\int_0^1 u_t^2\,dx+\frac{1}{4\varepsilon_2}(\beta_0\tau)^2\int_0^1 \theta_x^2\,dx \tag{4.3.41}$$

$$\beta_1\,\tau\int_0^1 \varphi_t\,\theta\,dx\leq \frac{1}{2}(\beta_1\tau)^2\int_0^1 \varphi_t^2\,dx+\frac{1}{2}\int_0^1 \theta_x^2\,dx \tag{4.3.42}$$

making use of the fact

$$-\int_0^1 (\tau\,\theta_t)^2 dx\leq -\frac{1}{2}\int_0^1 (\tau\theta_t+\theta)^2\,dx+\int_0^1 \theta_x^2\,dx$$

we get

$$-a\int_0^1 (\tau\,\theta_t)^2 dx\leq -\frac{a}{2}\int_0^1 (\tau\theta_t+\theta)^2\,dx+a\int_0^1 \theta_x^2\,dx \tag{4.3.43}$$

inserting (4.3.41), (4.3.42) and (4.3.43) in (4.3.40) we get

$$I_3{}'\,(t)\leq -\frac{a}{2}\int_0^1 (\tau\theta_t+\theta)^2\,dx+\varepsilon_2\int_0^1 u_t^2\,dx+c_0\int_0^1 \varphi_t^2\,dx+c_0\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \theta_x^2\,dx \qquad t\geq 0$$

□

Lemma 4.3.5. *Let (u,φ,θ,z) be the solution of (4.1.3)-(4.1.4) then, the functional $I_4(t)$ defined by*

$$I_4(t) = -\rho_1\, a \int_0^1 \left(\int_0^x u_t(y)\, dy \right) (\tau\,\theta_t + \theta)\, dx \qquad t \geq 0, \tag{4.3.44}$$

satisfies for any $\varepsilon_3 > 0$ the estimate

$$\begin{aligned} I_4'(t) \leq & -\frac{\beta_0\rho_1}{2} \int_0^1 u_t^2\, dx + \varepsilon_3 \int_0^1 u_x^2\, dx + c_0 \int_0^1 \varphi^2\, dx + c_0 \int_0^1 \varphi_t^2\, dx \\ & + c_0 \int_0^1 \theta_x^2\, dx + c_0 \left(1 + \frac{1}{\varepsilon_3}\right) \int_0^1 (\tau\,\theta_t + \theta)^2\, dx \qquad t \geq 0. \end{aligned} \tag{4.3.45}$$

Proof. Multiplying $(2,3)_3$ by $-\rho_1 \int_0^x u_t(y)\, dy$, integrating over (0,1) we obtain

$$\begin{aligned} -\rho_1\, a \int_0^1 \left(\int_0^x u_t(y)\, dy \right) (\tau\,\theta_t + \theta)_t\, dx = {} & \beta_0\,\rho_1 \int_0^1 u_{tx} \left(\int_0^x u_t(y)\, dy \right) dx + \beta_1\,\rho_1 \int_0^1 \varphi_t \left(\int_0^x u_t(y)\, dy \right) dx \\ & - \delta\,\rho_1 \int_0^1 \theta_{xx} \left(\int_0^x u_t(y)\, dy \right) dx \end{aligned}$$

using integration by parts together with the boundary conditions, we get

$$-\rho_1\, a \int_0^1 \left(\int_0^x u_t(y)\, dy \right) (\tau\,\theta_t + \theta)_t\, dx = -\beta_0\,\rho_1 \int_0^1 u_t^2\, dx + \delta\,\rho_1 \int_0^1 \theta_x\, u_t dx + \beta_1\,\rho_1 \int_0^1 \varphi_t \left(\int_0^x u_t(y)\, dy \right) dx$$

wich is equivalent to

$$\begin{aligned} \frac{d}{dt} \left(-a\rho_1 \int_0^1 (\tau\,\theta_t + \theta) \left(\int_0^x u_t(y)\, dy \right) dx \right) = {} & -a\rho_1 \int_0^1 (\tau\,\theta_t + \theta) \left(\int_0^x u_{tt}(y)\, dy \right) dx + \delta\,\rho_1 \int_0^1 \theta_x\, u_t dx \\ & - \beta_0\,\rho_1 \int_0^1 u_t^2\, dx + \beta_1\,\rho_1 \int_0^1 \varphi_t \left(\int_0^x u_t(y)\, dy \right) dx \end{aligned}$$

using $(2,3)_1$ we obtain

$$\frac{d}{dt}\left(-a\rho_1\int_0^1(\tau\theta_t+\theta)\left(\int_0^x u_t(y)\,dy\right)\,dx\right)=-a\mu_1\int_0^1(\tau\theta_t+\theta)\,u_x\,dx-a\mu_0\int_0^1(\tau\theta_t+\theta)\,\varphi\,dx$$
$$+a\,\beta_0\int_0^1(\tau\theta_t+\theta)^2dx-\beta_0\,\rho_1\int_0^1 u_t^2\,dx$$
$$+\beta_1\,\rho_1\int_0^1\varphi_t\left(\int_0^x u_t(y)\,dy\right)dx+\delta\,\rho_1\int_0^1\theta_x\,u_t dx$$

then, we have

$$I_4(t)=-a\,\rho_1\int_0^1(\tau\theta_t+\theta)\left(\int_0^x u_t(y)\,dy\right)\,dx\qquad t\geq 0$$

and

$$I_4{}'(t)=-a\mu_1\int_0^1(\tau\theta_t+\theta)\,u_x\,dx-a\mu_0\int_0^1(\tau\theta_t+\theta)\,\varphi\,dx+a\,\beta_0\int_0^1(\tau\theta_t+\theta)^2dx$$
$$-\beta_0\,\rho_1\int_0^1 u_t^2\,dx+\beta_1\,\rho_1\int_0^1\varphi_t\left(\int_0^x u_t(y)\,dy\right)dx+\delta\,\rho_1\int_0^1\theta_x\,u_t dx \tag{4.3.46}$$

Using Young's and Cauchy Schwarz inequalities we find

$$\delta\,\rho_1\int_0^1\theta_x\,u_t dx\leq\frac{\beta_0\,\rho_1}{4}\int_0^1 u_t^2\,dx+\frac{\delta^2\rho_1}{\beta_0}\int_0^1\theta_x^2\,dx \tag{4.3.47}$$

$$\beta_1\,\rho_1\int_0^1\varphi_t\left(\int_0^x u_t(y)\,dy\right)dx\leq\frac{\beta_0\,\rho_1}{4}\int_0^1 u_t^2\,dx+\frac{\beta_1\,\rho_1}{\beta_0}\int_0^1\varphi_t^2\,dx \tag{4.3.48}$$

$$-a\mu_0\int_0^1(\tau\theta_t+\theta)\,\varphi\,dx\leq\frac{a\,\mu_0}{2\varepsilon_3}\int_0^1(\tau\theta_t+\theta)^2\,dx+\frac{a\,\varepsilon_3\,\mu_0}{2}\int_0^1\varphi^2\,dx \tag{4.3.49}$$

$$-a\mu_1\int_0^1(\tau\theta_t+\theta)\,u_x\,dx\leq\varepsilon_3\int_0^1 u_x^2\,dx+\frac{1}{4\varepsilon_3}(a\,\mu_1)^2\int_0^1(\tau\theta_t+\theta)^2\,dx \tag{4.3.50}$$

inserting (4.3.47), (4.3.48), (4.3.49) and (4.3.50) in (4.3.46) we get

$$I_4{}'(t) \leq -\frac{\beta_0 \rho_1}{2}\int_0^1 u_t^2\,dx + \varepsilon_3 \int_0^1 u_x^2\,dx + c_0 \int_0^1 \varphi^2\,dx + c_0 \int_0^1 \varphi_t^2\,dx$$
$$+ c_0 \int_0^1 \theta_x^2\,dx + c_0\left(1+\frac{1}{\varepsilon_3}\right)\int_0^1 (\tau\,\theta_t+\theta)^2\,dx \qquad t \geq 0$$

□

Lemma 4.3.6. *Let (u, φ, θ, z) be the solution of (4.1.3)-(4.1.4) then, the functional $I_5(t)$ defined by*

$$I_5(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx \qquad t \geq 0, \tag{4.3.51}$$

satisfies the estimate

$$I_5{}'(t) \leq -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,z^2(x,1,s,t)\,ds\,dx - \eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx$$
$$+ \gamma_1 \int_0^1 \varphi_t^2\,dx \qquad t \geq 0. \tag{4.3.52}$$

Proof. Multiplying $(2,3)_4$ by $e^{-s\rho}\,|\gamma_2(s)|\,z$, integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$ we obtain

$$\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\,|\gamma_2(s)|\;z_t\,z\,ds\,d\rho\,dx = -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\gamma_2(s)|\;z_\rho\,z\,ds\,d\rho\,dx$$

wich is equivalent to

$$\frac{d}{dt}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx = -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\gamma_2(s)|\;\frac{\partial}{\partial\rho}z^2(x,\rho,s,t)\;ds\,d\rho\,dx$$

Integration by part and using the fact that $z(x,0,s,t) = \varphi_t(x,t)$ gives

$$\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\left|\gamma_2(s)\right| z^2(x,\rho,s,t)\,ds\,d\rho\,dx = -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\left|\gamma_2(s)\right| z^2(x,1,s,t)\,ds\,dx$$
$$-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\ \left|\gamma_2(s)\right| z^2(x,\rho,s,t)\,ds\,d\rho\,dx + \left(\int_{\tau_1}^{\tau_2}\left|\gamma_2(s)\right| ds\right)\int_0^1 \varphi_t^2\,dx$$

then, we have

$$I_5(t) = \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\left|\gamma_2(s)\right| z^2(x,\rho,s,t)\,ds\,d\rho\,dx \qquad t \geq$$

and

$$I_5{}'(t) = -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\left|\gamma_2(s)\right| z^2(x,1,s,t)\,ds\,dx + \left(\int_{\tau_1}^{\tau_2}\left|\gamma_2(s)\right| ds\right)\int_0^1 \varphi_t^2\,dx$$
$$-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,e^{-s\rho}\ \left|\gamma_2(s)\right| z^2(x,\rho,s,t)\,ds\,d\rho\,dx$$

using the fact that $e^{-s} \leq e^{-s\rho} \leq 1$ we get for all $\rho \in [0,1]$

$$I_5{}'(t) \leq -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\left|\gamma_2(s)\right| z^2(x,1,s,t)\,ds\,dx + \left(\int_{\tau_1}^{\tau_2}\left|\gamma_2(s)\right| ds\right)\int_0^1 \varphi_t^2\,dx$$
$$-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,e^{-s}\ \left|\gamma_2(s)\right| z^2(x,\rho,s,t)\,ds\,d\rho\,dx$$

Since $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$.
Finally , setting $\eta_1 = e^{-\tau_2}$ and recalling (4.1.5) we get ((4.3.52)) □

Now, we define the Lyapunov functional $L(t)$ by

$$L(t) = N\,E(t) + N_1\,I_1(t) + N_2\,I_2(t) + N_3\,I_3(t) + N_4\,I_4(t) + N_5\,I_5(t) \qquad \forall t \geq 0, \tag{4.3.53}$$

where N, N_1, N_2, N_3, N_4 and N_5 are positive real numbers to be chosen appropriately later

Lemma 4.3.7. *Let (u, φ, θ, z) be the solution of (4.1.3)-(4.1.4) then, there exists two positive constants b_1 and b_2 such that the Lyapunov functional $L(t)$ satisfies*

$$b_1 E(t) \leq L(t) \leq b_2 E(t) \qquad \forall t \geq 0, \tag{4.3.54}$$

and

$$L'(t) \leq -\varsigma E(t) \quad , \quad \varsigma > 0. \tag{4.3.55}$$

Proof. From (4.3.53), we can write

$$\begin{aligned}
|L(t) - N E(t)| &\leq N_1 |I_1(t)| + N_2 |I_2(t)| + N_3 |I_3(t)| + N_4 |I_4(t)| + N_5 |I_5(t)| \\
&\leq N_1 \rho_1 \int_0^1 |u_t| \, |u| \, dx + N_1 \beta_0 \tau \int_0^1 |u_x| \, |\theta| \, dx + N_2 J \int_0^1 |\varphi_t| \, |\varphi| \, dx + \frac{N_2 \gamma_1}{2} \int_0^1 \varphi^2 dx \\
&+ \frac{N_2 \mu_0 \rho_1}{\mu_1} \int_0^1 |u_t| \left(\int_0^x |\varphi(y)| \, dy \right) dx + N_3 a \tau^2 \int_0^1 |\theta_t| \, |\theta| \, dx + \frac{N_3 a \tau}{2} \int_0^1 \theta^2 dx \\
&+ N_4 \rho_1 a \int_0^1 |\tau \theta_t + \theta| \left(\int_0^x |u_t(y)| \, dy \right) dx + N_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s \, e^{-s\rho} \, |\gamma_2(s)| \, z^2 ds \, d\rho \, dx
\end{aligned}$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, (4.3.17) and the fact that $e^{-s\rho} \leq 1$ for all $\rho \in [0, 1]$ we obtain

$$\begin{aligned}
|L(t) - N E(t)| &\leq c \int_0^1 \left(u_t^2 + \varphi_t^2 + (\tau \theta_t + \theta)^2 + \varphi_x^2 + \varphi^2 + \theta_x^2 + \left(u_x + \frac{\mu_0}{\mu_1} \varphi \right)^2 \right) dx \\
&+ c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s \, |\gamma_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx \\
&\leq c E(t)
\end{aligned}$$

Consequently, we have

$$(N - c) E(t) \leq L(t) \leq (N + c) E(t)$$

Choosing N is sufficiently large and depends on N_i , $i = \overline{1,5}$, we obtain (4.3.55)

Now, By differentiating (4.3.53) and recalling (4.3.18) , (4.3.24) , (4.3.31) , (4.3.39) , (4.3.45) , (4.3.52) we get

$$\begin{aligned}
L'(t) \leq & -\left(\frac{N_4\,\beta_0\,\rho_1}{2} - N_3\,\varepsilon_2 - N_2\,\varepsilon_1 - N_1\,c_0\right)\int\limits_0^1 u_t^2 dx - N_2\,a_0 \int\limits_0^1 \varphi_x^2 dx \\
& - \left(N\left(\gamma_1 - \int\limits_{\tau_1}^{\tau_2} |\gamma_2(s)|\, ds\right) - N_2\,c_0\left(1+\frac{1}{\varepsilon_1}\right) - N_3\,c_0 - N_4\,c_0 - N_5\,\gamma_1\right)\int\limits_0^1 \varphi_t^2 dx \\
& - \left(\delta\,N - N_1\,c_0 - N_3\,c_0\left(1+\frac{1}{\varepsilon_2}\right) - N_4\,c_0\right)\int\limits_0^1 \theta_x^2 dx - \left(\frac{N_1\,\mu_1}{2} - N_4\,\varepsilon_3\right)\int\limits_0^1 u_x^2 dx \\
& - \left(\frac{N_2\,\mu_3}{2} - N_1\,c_0 - N_4\,c_0\right)\int\limits_0^1 \varphi^2 dx - \left(\frac{N_3\,a}{2} - N_2\,c_0 - N_4\,c_0\left(1+\frac{1}{\varepsilon_3}\right)\right)\int\limits_0^1 (\tau\,\theta_t + \theta)^2 dx \\
& - \left(N_5\,\eta_1 - \frac{N_2\,\gamma_1}{\mu_3}\right)\int\limits_0^1 \int\limits_{\tau_1}^{\tau_2} |\gamma_2(s)|\, z^2(x,1,s,t)\, ds\, dx \\
& - N_5\,\eta_1 \int\limits_0^1 \int\limits_0^1 \int\limits_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\, z^2(x,\rho,s,t)\, ds\, d\rho\, dx
\end{aligned}$$

At this point, we set $\varepsilon_1 = \dfrac{\beta_0\,\rho_1\,N_4}{8N_2}$, $\varepsilon_2 = \dfrac{\beta_0\,\rho_1\,N_4}{4N_2}$ and $\varepsilon_3 = \dfrac{\mu_1\,N_1}{4N_4}$, we end up with

$$\begin{aligned}
L'(t) \leq & -\left(\frac{N_4\,\beta_0\,\rho_1}{8} - N_1\,c_0\right)\int_0^1 u_t^2 dx - N_2\,a_0\int_0^1 \varphi_x^2 dx \\
& -\left(N\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,ds\right) - N_2\,c_0\left(1+\frac{8N_2}{N_4\,\beta_0\,\rho_1}\right) - N_3\,c_0 - N_4\,c_0 - N_5\,\gamma_1\right)\int_0^1 \varphi_t^2 dx \\
& -\left(\delta\,N - N_1\,c_0 - N_3\,c_0\left(1+\frac{4N_3}{N_4\,\beta_0\,\rho_1}\right) - N_4\,c_0\right)\int_0^1 \theta_x^2 dx - \frac{N_1\,\mu_1}{4}\int_0^1 u_x^2 dx \\
& -\left(\frac{N_2\,\mu_3}{2} - N_1\,c_0 - N_4\,c_0\right)\int_0^1 \varphi^2 dx - \left(\frac{N_3\,a}{2} - N_2\,c_0 - N_4\,c_0\left(1+\frac{4N_4}{N_1\,\mu_1}\right)\right)\int_0^1 (\tau\,\theta_t + \theta)^2 dx \\
& -\left(N_5\,\eta_1 - \frac{N_2\,\gamma_1}{\mu_3}\right)\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,z^2(x,1,s,t)\,ds\,dx \\
& - N_5\,\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\gamma_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx
\end{aligned}$$

Then, we fixed N_1 and we choose N_4 large enough such that

$$\kappa_1 = \frac{N_4\,\beta_0\,\rho_1}{8} - N_1\,c_0 > 0$$

once N_4 is fixed, we take N_2 large enough such that

$$\kappa_2 = \frac{N_2\,\mu_3}{2} - N_1\,c_0 - N_4\,c_0 > 0$$

after we fixed N_2, we choose N_3 and N_3 large enough such that

$$\kappa_3 = \frac{N_3\,a}{2} - N_2\,c_0 - N_4\,c_0\left(1+\frac{4N_4}{N_1\,\mu_1}\right) > 0 \qquad , \qquad \kappa_4 = N_5\,\eta_1 - \frac{N_2\,\gamma_1}{\mu_3} > 0$$

Finally, select N that is large enough (even larger such that (4.3.54) is still valid) to

$$\kappa_5 = N\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)|\,ds\right) - N_2\,c_0\left(1+\frac{8N_2}{N_4\,\beta_0\,\rho_1}\right) - N_3\,c_0 - N_4\,c_0 - N_5\,\gamma_1 > 0$$

and

$$\kappa_6 = \delta N - N_1 c_0 - N_3 c_0 \left(1 + \frac{4N_3}{N_4 \beta_0 \rho_1}\right) - N_4 c_0 > 0$$

Moreover, we set

$$\kappa_7 = \delta \frac{N_1 \mu_1}{4} \quad , \quad \kappa_8 = N_2 a_0 \quad , \quad \kappa_9 = N_5 \eta_1$$

we obtain (4.3.55) □

In what follows, we'll use the equivalence relationship (4.3.54) to estimate the system's energy (4.1.3)-(4.1.4) using the estimated (4.3.55) based on the assumption (4.1.5). The stability result can now be stated as follows.

Theorem 4.3.1. *Let (u, φ, θ, z) be the solution of (4.1.3)-(4.1.4) and we assume that (0.0.14) hold, Then, the solution (u, φ, θ, z) decays exponentially, i. e . there exist two positive constants λ_1 and λ_2 such that*

$$E(t) \leq \lambda_2 e^{-\lambda_1 t} \qquad \forall t \geq 0. \tag{4.3.56}$$

Proof. using the equivalence of $E(t)$ and $L(t)$ we deduce that

$$L'(t) \leq -\lambda_1 L(t) \qquad \forall t \geq 0 \tag{4.3.57}$$

where $\lambda_1 = \dfrac{\varsigma}{b_2} > 0$, Asimple integration of (4.3.57) gives

$$L(t) \leq L(0)\, e^{-\lambda_1 t} \qquad \forall t \geq 0$$

which yields the serial result (4.3.56) and by using the other side of (4.3.54) again. The proof is complete □

CONCLUSION

In this research, we conducted an in-depth qualitative study of hyperbolic-type problems under the influence of various forms of delay, some of which are characterized by the absence of damping effects. The analysis was developed within a rigorous mathematical framework, relying on the theory of semigroups to derive fundamental results regarding the existence and uniqueness of solution. The assumptions were carefully selected to reflect realistic modeling scenarios while ensuring mathematical feasibility.

To gain a deeper understanding of the dynamic properties of the solutions, particular attention was given to the asymptotic analysis of the behavior of the system. This was achieved by applying the energy method, in which a carefully constructed Lyapunov functional based on the multiplicative approach was used to obtain precise decay estimates. This methodology enabled us to demonstrate that the energy associated with the system decays exponentially over time, confirming the stabilizing effect of the integrated delay mechanisms under the proposed assumptions. For one of these systems, a numerical study was conducted to validate the analytically obtained results.

Building on this foundation, we plan to carry out a detailed numerical investigation of the remaining systems presented in this work. The benefits of this numerical study will extend beyond validating the analytical results; it will also provide visual and quantitative insights into the time evolution of the solutions. Furthermore, it will facilitate the exploration of parameter sensitivity and the effectiveness of delay in practical applications, thus paving the way for future research on control strategies and optimization in similar dynamical systems.

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