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Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: ***Dynamical systems***

Maximum number of periodic solutions of certain differential equations

Presented by:

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Dedication

In memory of my father, whose life was an inspiration for me and whose values of integrity and perseverance continue to guide me. To my mother, whose unconditional love, constant support, patience, and unwavering belief have been my guiding light. To my family, for their unwavering belief, continuous support, and loving care throughout my life.

Acknowledgements

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I wish to thank my colleagues and friends for their motivational words and the stimulating discussions we shared during this journey.

Finally, I extend my heartfelt appreciation to my family for their love, patience, and encouragement, which gave me the strength to complete this work.

Abstract

This thesis investigates the dynamics of periodic solutions and bifurcations in nonlinear dynamical systems. We use averaging theory to study the maximum number of isolated periodic solutions (i.e. limit cycles) in a second-order differential system. Also, using the same theory we examine zero-Hopf bifurcation for finding periodic solutions in a modified hyperchaotic Chen system and a three-dimensional Kolmogorov system. Through these studies, we demonstrate the emergence of limit cycles and establish conditions for their existence. We provide numerical examples to illustrate our results.

Keywords: Differential system, averaging theory, periodic solution, limit cycle, zero-Hopf bifurcation.

Resumé

Cette thèse étudie la dynamique des solutions périodiques et des bifurcations dans les systèmes dynamiques non linéaires. Nous utilisons la théorie de moyennisation pour analyser le nombre maximum de solutions périodiques isolées (i.e. cycles limites) dans un système différentiel d'ordre deux. Ainsi, en utilisant la même théorie nous examinons la bifurcation de zéro-Hopf pour chercher les solutions periodiques dans un système de Chen hyperchaotique modifié ainsi que dans un système de Kolmogorov tridimensionnel. À travers ces études, nous démontrons l'émergence de cycles limites et établissons des conditions pour leur existence. Nous fournissons des exemples numériques pour illustrer nos résultats.

Mots clés: Système différentiel, théorie de moyennisation, solution périodique, cycle limite, bifurcation de zéro-Hopf.

الملخص

تستكشف هذه الأطروحة ديناميات الحلول الدورية والتشعبات في الأنظمة الديناميكية غير الخطية. سنستخدم نظرية المتوسط لدراسة الحد الأقصى من الحلول الدورية المعزولة (الدورات الحدية) في الأنظمة التفاضلية من الرتبة الثانية. أيضا، باستعمال نفس النظرية سنفحص تشعبات صفر-هوف لإيجاد الحلول الدورية في كل من نظام تشين مفرط الفوضوية المعدَّل ونظام كولموغوروف ثلاثي الأبعاد. من خلال هذه الدراسات، نوضح ظهور الدورات الحدية ونحدد شروط وجودها. نقدم أمثلة عددية لتوضيح نتائجنا.

كلمات مفتاحية: نظام تفاضلي، حل دوري، دورة حدية، نظرية المتوسط، تشعب صفر-هوف.

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Introduction

Nature manifests in countless fascinating ways, often showing complex and mesmerizing patterns. From the rhythmic beating of a heart to the spiraling motion of galaxies, our world is filled with phenomena that reveal unexpected patterns within what seems to be chaos. These phenomena are governed by dynamical systems that offer a precise framework to study and understand the phenomenon's complex behaviors with its chaotic processes.

A dynamical system is a mathematical model used to describe the evolution of a system over time. This system is typically represented using differential equations, difference equations, or iterative maps, and it can model a wide range of phenomena from physical processes to economic systems.

The origins of dynamical system theory date back to the classical work of Isaac Newton in the 17th century, who put the framework of the laws of motion under the influence of forces and laid the foundation for the study of systems governed by differential equations. The modern theory of dynamical systems derives substantially from the work of Henri Poincare on the three-body problem of celestial mechanics in the 19th century. Birkhoff, followed by Kolmogorov, Arnold, and Moser, expanded upon Poincaré's geometric methods, leading to a significantly deeper comprehension of classical mechanics. On the theoretical front, the study of nonlinear differential equations, particularly nonlinear oscillators, spurred the development of new mathematical techniques by mathematicians like Van der Pol, Andronov, Littlewood, Cartwright, Levinson, and Smale. In 1963, Lorenz [28] made a groundbreaking discovery saying that the solutions to his equations did not settle into equilibrium or periodic states but instead oscillated in an irregular, aperiodic manner. Furthermore, even a slight difference in initial conditions led to vastly divergent outcomes, revealing the system's inherent unpredictability. This implied that small errors in measuring the current state of the atmosphere could rapidly amplify, resulting in highly inaccurate forecasts. In 1971, Ruelle and Takens [42] introduced a new theory for the onset of turbulence in fluids, based on the concept of strange attractors.

Among the various phenomena observed in dynamical systems, limit cycles play a particularly essential role. A limit cycle represents an isolated closed periodic orbit in phase space i.e. it is a closed orbit in the neighborhood where no other periodic orbit can exist. Understanding the conditions under which limit cycles exist, as well as their stability and number, is essential for the analysis of oscillatory processes in both natural and engineered systems.

The concept of limit cycles was first introduced by Poincaré in 1881 [36] and rigorously studied in the early 20th century. His work laid the foundation for the qualitative theory of differential equations. In 1901, extending Poincaré's work, the Swedish mathematician Ivar Bendixon published a paper proving rigorously what is known now as the Poincaré-Bendixson Theorem. Bendixon showed that, for a continuous dynamical system defined by a smooth vector field on a two-dimensional plane, if a trajectory of the system remains within a bounded, closed, and finite region of the plane for all future time, and if this region contains no equilibrium points, then the trajectory must either Approach a periodic orbit (i.e., a limit cycle) or be a periodic orbit itself. In simple words, if a trajectory does not escape a certain region and there is no equilibrium point in that region, the system's behavior must eventually become periodic. This rules out complex behaviors like chaos in such systems, which are possible in higher dimensions.

In recent decades, research has expanded to cover various forms of dynamical systems, including those with discontinuities [37, 38], time delays, and non-autonomous components. Despite these advancements, several questions remain unsolved, particularly regarding the maximum number and relative positions of the limit cycles of a polynomial differential system of degree n. This unsolved question is part of the 16th problem listed among 23 open problems established by David Hilbert in 1900 at the International Conference of Mathematicians in Paris. This problem is still open even in the simplest case where $n = 2$.

In nature and engineering, we often encounter dynamical systems described by differential equations that are difficult to solve directly, especially when these systems involve oscillatory components and nonlinear terms, or are subject to periodic perturbations. In many cases, such systems are analytically unsolvable and usually require approximation methods like perturbation methods and numerical simulations to understand their behavior. The averaging theory is one of the most important methods that are used to approximate the long-term behavior of these systems, especially nonlinear systems where the dynamic includes rapidly oscillating components or periodic forces, by focusing on the average effect of their oscillations rather than looking for the exact dynamic at every moment, see [3, 4, 14]. It transforms a difficult problem into a more manageable one by reducing the influence of fast, small-scale changes and focusing on the dominant, slower dynamics that govern the system's behavior over longer periods.

Consider a system of differential equations with a small parameter ε

$$\dot{x} = \varepsilon f(t, x), \tag{0.0.1}$$

where $f(t, x)$ is periodic in time. Since this system can exhibit complex oscillatory behavior, it may be difficult to analyze directly. The idea of averaging is to replace this oscillatory system with the averaged system

$$\dot{y} = \varepsilon \overline{f}(y),$$

where $\overline{f}(y)$ represents the time-averaged behavior of $f(t, x)$ over one period, often denoted by T.

Many important problems from celestial mechanics to electrical circuits can be transformed into the equation (0.0.1). There are many forms and theorems in averaging theory [43, 35, 25], and several papers have been published concerning the application of this method, see for instance [26, 18, 14, 3, 4].

The organization of this thesis consists of an introduction, five chapters, and a conclusion.

Chapter 1: Preliminaries

This chapter introduces key preliminaries of the qualitative theory of dynamical systems, which are essential for understanding the core topics of this thesis. It begins with a general introduction to dynamical systems, followed by an examination of continuous dynamical systems and a detailed discussion of equilibrium points and the linearization method. The classification of equilibria in $\mathbb{R}^2$ and their stability properties are explored, alongside an analysis of periodic solutions and limit cycles, including theorems concerning the existence of limit cycles. The chapter concludes with an introduction to bifurcation theory, focusing on the various types of bifurcations, with emphasis on Hopf and zero-Hopf bifurcations.

Chapter 2: The averaging theory

This chapter focuses on the averaging method, a crucial technique for detecting periodic solutions in dynamical systems. This method serves as the primary tool for proving the key results in this thesis. The chapter outlines the theoretical framework and highlights its importance in simplifying complex system behaviors for analysis.

Chapter 3: Maximum Number of limit cycles of a second order differential system

In this chapter, we investigate the maximum number of limit cycles in the following differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1 + \sin^n(\theta)\cos^m(\theta))H(x,y), \end{aligned}$$

where the dot denotes the derivative with respect to time t, $\varepsilon > 0$ is a small parameter, m and n are arbitrary non negative integers, $H(x,y)$ is a polynomial of degree $l \geq 1$ and $\theta = \arctan(y/x)$. Using first-order averaging theory, we establish an upper bound for the maximum number of limit cycles. Additionally, we present examples to confirm and illustrate our results.

This study was published in the journal **"Boletim da Sociedade Paranaense de Matemática"**, see [7].

Chapter 4: Zero-Hopf bifurcation in a modified hyperchaotic Chen system

This chapter is dedicated to examining the zero-Hopf bifurcation in a modified hyperchaotic Chen system. By applying averaging theory, we analytically prove the existence of two periodic orbits arising from the zero-Hopf equilibrium at the origin. An example is provided to illustrate our result.

This study is published in the journal **"Mathematics in Engineering, Science & Aerospace (MESA)"**, see [8].

Chapter 5: Three-dimensional zero-Hopf bifurcation in a cubic Kolmogorov system

In this chapter, we investigate the zero-Hopf bifurcation of a Kolmogorov system of degree 3 in $\mathbb{R}^3$. Utilizing second-order averaging theory, we demonstrate that the system can generate at most 5 periodic solutions bifurcating from the zero-Hopf equilibrium at the point $(1, 1, 1)$.

This study is currently being prepared for submission.

The final chapter concludes this thesis, summarizes the key findings, and suggests potential directions for future research.

It is noteworthy that all computations presented in this thesis were performed using Maple.

CHAPTER

1

Preliminaries

1.1 Introduction to dynamical systems

Dynamical system theory is a mathematical framework for analyzing systems that change over time. A dynamical system, which may be considered an object of any nature, is governed by a set of rules or equations that determine how the system's state evolves. Depending on the character of the state variables, dynamical systems can be classified as discrete or continuous.

Definition 1.1.1. *(General definition) A dynamical system can be generalized as a tuple* (X, Φ, T) *where*

i. X *is the state space (usually a subset of* $\mathbb{R}^n$ *or a more abstract space).*

ii. T *is the time set (could be continuous or discrete), which is an additive semi-group i.e.* $0 \in T$ *and for* $t_1, t_2 \in T$ *also* $t_1 + t_2 \in T$.

iii. $\Phi : T \times X \longrightarrow X$ *is a family of maps that describe the system's evolution, often referred to as the flow.*

The flow map Φ *satisfies the (semi-)group property:*

$$\textit{For any} \quad x \in X, \quad \Phi(0, x) = x,$$

$$\textit{and for any} \quad t, s \in T, \quad \Phi(t + s, x) = \Phi(t, \Phi(s, x)).$$

Dynamical systems are often categorized into two fundamental types:

1. **Continuous dynamical systems:** These are governed by differential equations, where the state evolves continuously over time. For example, a system defined by an ordinary differential equation $\frac{dx(t)}{dt} = f(x(t))$.
2. **Discrete dynamical system:** These are governed by difference equations, where the state changes in discrete time steps. For example, a system described by $x_{n+1} = f(x_n)$.

In the sequel, we shall consider only continuous dynamical systems.

1.2 Continous dynamical systems

Definition 1.2.1. *A continuous dynamical system is represented by a differential equation*

$$\frac{dx}{dt} = f(x,t); \quad \text{where } x \in E \subseteq \mathbb{R}^n\,,\ t \in \mathbb{R}. \tag{1.2.1}$$

having a unique solution $x(t,t_0) = x(t)$ satisfying the condition $x(t_0) = x_0$.

Definition 1.2.2. *An autonomous dynamical system is described by a system (1.2.1) where the right-hand side of the equation does not explicitly depend on the independent variable t i.e. $f(x,t) = f(x)$. If the system does depend on t, it is referred to as a non-autonomous system.*

Definition 1.2.3. *A linear dynamical system is characterized by the linearity of the function f of system (1.2.1), meaning that it satisfies the principle of superposition and can be described by equations of the form*

$$\frac{dx}{dt} = Ax,$$

where A is a matrix. The principle of superposition implies that if $x_1(t)$ and $x_2(t)$ are solutions to the system, then any linear combination of these solutions is also a solution.

In contrast, a nonlinear dynamical system is one in which the function f is nonlinear, meaning that the system's behavior cannot be described by a linear function of the state *variables. Nonlinear systems often exhibit complex behaviors such as bifurcations, chaos, and multiple equilibrium points, and are described by equations where f does not satisfy the principle of superposition.*

Remark 1.2.1. *If the matrix A of the linear system is constant, then the system is autonomous. If A depends on the independent variable t i.e. $A = A(t)$, then the system is non-autonomous.*

Example 1.2.1. *A simple example of a continuous dynamical system is the predator-prey model, also known as the Lotka-Volterra system*

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= \delta xy - \gamma y, \end{aligned} \tag{1.2.2}$$

where $x(t)$ is the population of prey at time t, $y(t)$ the population of predators at time t, α is the growth rate of the prey population in the absence of predators, β is predation rate, γ is the death rate of predators in the absence of prey, and δ is the rate at which predators increase their population by consuming prey.

The system (1.2.2) is an autonomous nonlinear dynamical system. In more detail, the nonlinear terms $-\beta xy$ and δxy in the equations make the system nonlinear, while the absence of explicit time dependence classifies it as autonomous.

1.2.1 The vector field

Definition 1.2.4. *Given a subset $\mathcal{U}$ of $\mathbb{R}^n$, a vector field F is defined as*

$$F : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

where for each point $x \in \mathcal{U}$, the vector field $F(x)$ assigns a vector in $\mathbb{R}^n$.

Remark 1.2.2. *If $x = (x_1, x_2, \dots, x_n)$ represents a point in $\mathbb{R}^n$, then $F(x)$ can be written as*

$$F(x) = (F_1(x), F_2(x), \dots, F_n(x))$$

where each $F_i(x)$, for $i = 1, \dots, n$, is a scalar function that depends on the coordinates $x_1, x_2, \dots, x_n$.

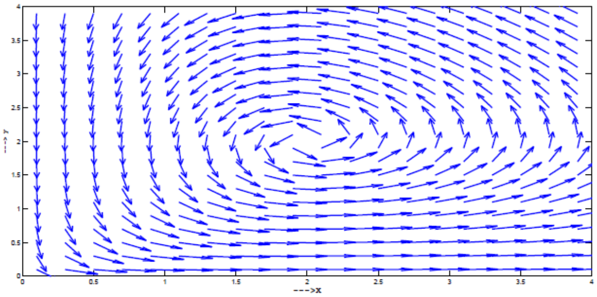

Figure 1.1: Vector field of the Lotka Volterra system with $\alpha = \gamma = 2$

Example 1.2.2. *Take the Lotka-Volterra system from the previous example*

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy,\\ \frac{dy}{dt} &= \delta xy - \gamma y,\end{aligned}$$

with $\alpha = \gamma = 2$ and $\beta = \delta = 1$. The vector field for this system is shown in Figure 1.1.

1.2.2 Flow

Definition 1.2.5. *A flow in a continous dynamical system (1.2.1) is a function $\phi : \mathbb{R} \times \mathcal{U} \longrightarrow \mathcal{U}$ such that, for all $x \in \mathcal{U} \subseteq \mathbb{R}^n$ and for all real numbers s and t,*

$$\begin{aligned}\phi(x, 0) &= x,\\ \phi(\phi(x, t), s) &= \phi(x, s + t).\end{aligned}$$

The flow that is determined by a vector field is called a vector flow.

1.2.3 Orbits and phase portraits

Consider the autonomous nonlinear differential system

$$\dot{x} = f(x)\,;\ \text{with } x \in \mathcal{U} \subseteq \mathbb{R}^n, \tag{1.2.3}$$

where $\dot{x} = \dfrac{dx}{dt}$ is the derivative of x with respect to time t. $\mathcal{U}$ is called the phase-space of the system (1.2.3).

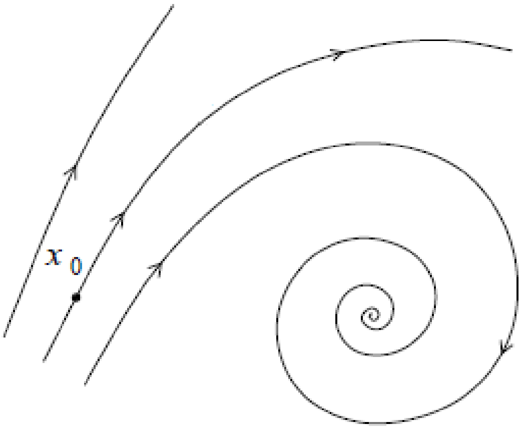

Figure 1.2: Orbits

Definition 1.2.6. *An orbit starting at* x_0 *is an ordered subset of the phase space* $\mathcal{U}$

$$\mathcal{O}(x_0) = \{x \in \mathcal{U} : x = \phi(t, x_0),\ \textit{for all}\, t \in \mathbb{R}\ \textit{such that}\, \phi(t, x_0)\ \textit{is defined}\},$$

where ϕ *is the flow of the system (1.2.3).*

Orbits of a system (1.2.3) are curves in the phase space $\mathcal{U}$ parametrized by time t and oriented by its direction of increase, see Figure 1.2. Orbits are also called trajectories.

Definition 1.2.7. *A phase portrait of a system (1.2.3) is a graphical representation of the orbits of this system in the phase space. It is represented by drawing some significant orbits indicating the orientation by arrows.*

Phase portraits are important for visualizing the qualitative behavior of dynamical systems, such as identifying stable and unstable equilibria, periodic orbits, and the overall flow structure in the phase space.

Example 1.2.3. *Consider the system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x. \end{aligned} \tag{1.2.4}$$

This system represents the simple harmonic oscillator where the solutions are sinusoidal functions, leading to circular orbits in the phase space. See Figure 1.3.

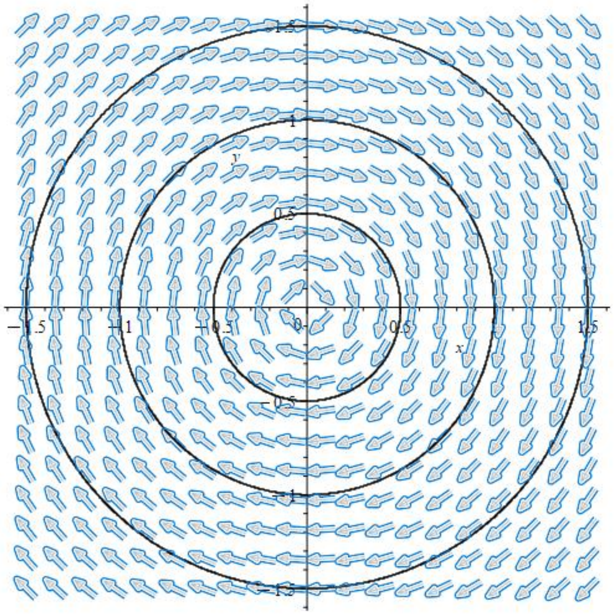

Figure 1.3: Phase portrait of a simple harmonic oscillator

1.3 Equilibrium points and linearization

Consider the nonlinear differential system

$$\dot{x} = f(x), \tag{1.3.5}$$

where $f : \mathcal{U} \longrightarrow \mathbb{R}^n$, a C^1 function, and $\mathcal{U}$ is an open subset of $\mathbb{R}^n$.

1.3.1 Equilibrium points

Definition 1.3.1. *A point* $x_0 \in \mathcal{U}$ *is called an equilibrium point of the equation (1.3.5) if* $f(x_0) = 0$.

Remark 1.3.1. *In many textbooks, an equilibrium point is sometimes called a singular or critical point.*

1.3.2 Linearization

In order to analyze equilibrium points one should begin by linearizing the differential equation in the neighborhood of the equilibrium point. Linearization is important as it makes analyzing the behavior of a nonlinear system near a given point possible by studying its linear part.

The linearization of a function is the first-order term of its Taylor expansion around the point of interest (i.e. equilibrium point). For a system defined by the equation (1.3.5) the linearized system can be written as

$$\dot{x} \approx f(x_0) + Df(x_0)(x - x_0),$$

where x_0 is the equilibrium point of (1.3.5) and $Df(x_0)$ is the Jacobian of $f(x)$ evaluated at x_0.

Since x_0 is an equilibrium point, we have $f(x_0) = 0$. Thus, the previous linearized system is now given by

$$\dot{x} \approx Df(x_0)(x - x_0). \tag{1.3.6}$$

The point x_0 is often translated to the origin of the phase space by doing the simple change of variable $y = x - x_0$. Also, we can simplify the notation of the Jacobian of f at x_0 by $Df(x_0) = A$, where A is a $n \times n$ matrix with constant coefficients. Consequently, the system (1.3.6) is simplified as follows

$$\dot{y} = A\,y. \tag{1.3.7}$$

We assume that $\det(A) \neq 0$ excluding the case where A is a singular matrix.

To analyze the equilibrium point of (1.3.6) we need to determine the eigenvalues of A which are characterized by the zeros of the equation

$$\det(A - \lambda I) = 0.$$

We note these eigenvalues by $\lambda_1, \ldots, \lambda_n$.

From the theorem of The Jordan canonical form, there exists an invertible matrix P such that $P^{-1}AP$ is in Jordan form. For detailed information, see [34].

We define the linear transformation of coordinates $z = P^{-1}y$ where P is the invertible matrix defined in the theorem. Thus, after simple computations the system (1.3.7) transforms into the following equation in the variable z

$$\dot{z} = P^{-1}APz. \tag{1.3.8}$$

Remarks 1.3.1.

- *If the eigenvalues* $\lambda_1, \dots, \lambda_n$ *are distinct then* $P^{-1}AP$ *is a diagonal matrix with the eigenvalues as diagonal elements.*
- *The linear transformation simplifies the equilibrium point analysis even if there are some equal eigenvalues.*

1.4 Classification of equilibria in $\mathbb{R}^2$

Consider the equation (1.3.8) with A a 2×2 matrix with two eigenvalues λ_1 and λ_2. The two eigenvalues can be both real or complex conjugate.

If $\lambda_1 \neq \lambda_2$ then the Jordan form $J = P^{-1}AP$ is of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

In this case, the general solution $z(t)$ of the equation (1.3.8) is given by

$$z(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}. \tag{1.4.9}$$

with c_1 and c_2 arbitrary constants. We put $z_1(t) = c_1 e^{\lambda_1 t}$ and $z_2(t) = c_2 e^{\lambda_2 t}$.

For different choices of λ_1 and λ_2, we distinguish the following cases.

(**i**). The equilibrium point (the origin of the equation (1.3.8)) is called a node if the eigenvalues are real and have the same sign.

If $\lambda_1 \neq \lambda_2$, then from the solutions $z_1(t)$ and $z_2(t)$ we have the relation $|z_1| = c|z_2|^{\lambda_1/\lambda_2}$ with c a constant. This yield in the phase-plane orbits in the form of parabolas, see Figure 1.4.

If $\lambda_1 = \lambda_2 = \lambda$, then J is in general

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and the solutions of (1.3.8) are

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} + c_2 t e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix},$$

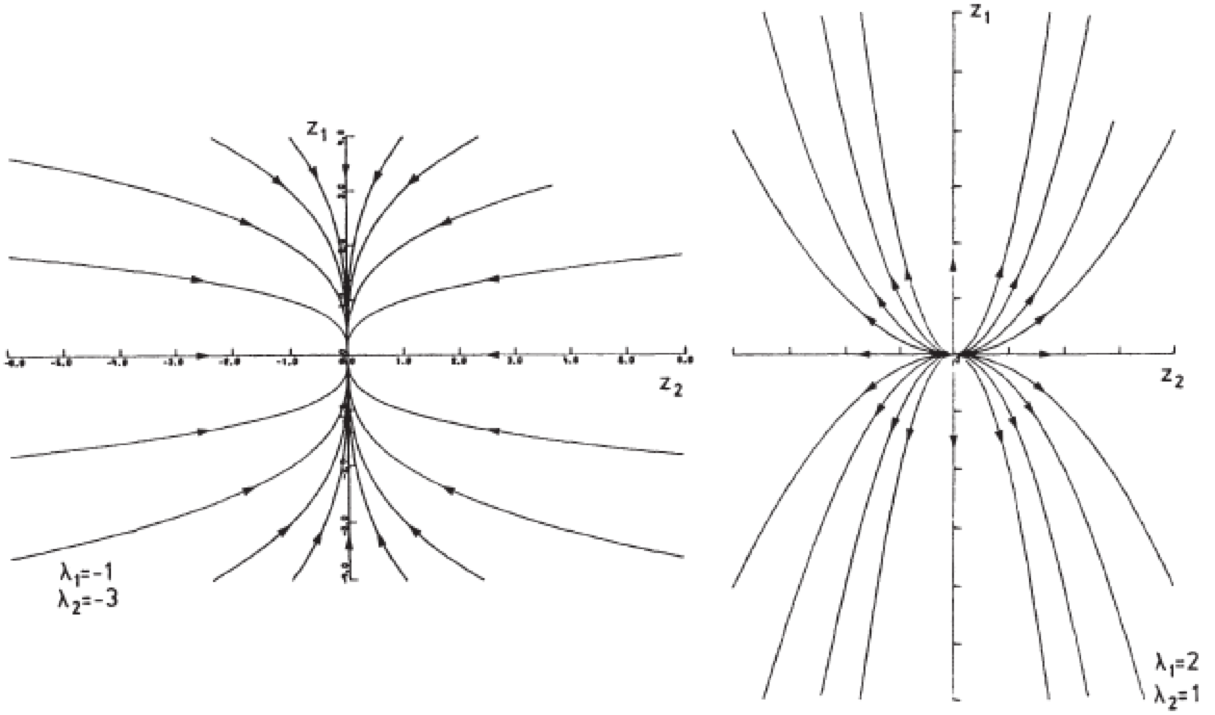

Figure 1.4: Phase portraits of a node

with c_1 and c_2 arbitrary constants. See Figure 1.5.

(**ii**). The equilibrium point is called a saddle point if the eigenvalues λ_1 and λ_2 are real and have different signs. The solutions of (1.3.8) are of the form (1.4.9). So, in the phase plane, the orbits are given by $|z_1| = c|z_2|^{\lambda_1/\lambda_2}$ and their behavior is hyperbolic. See Figure 1.6.

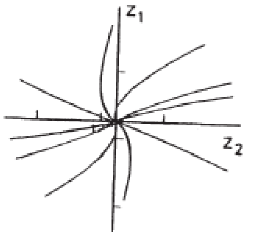

Figure 1.5: Phase portrait of a node in the case of repeated eigenvalues

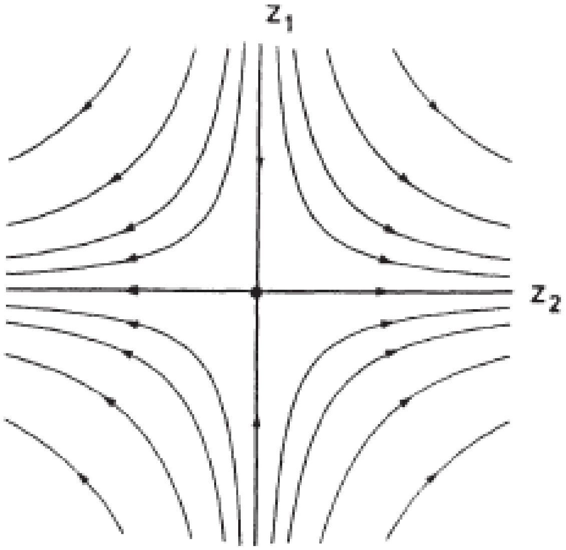

Figure 1.6: Phase portrait of a saddle

(**iii**). The equilibrium point is called a focus if the eigenvalues λ_1 and λ_2 are complex conjugate, i.e. $\lambda_{1,2} = \alpha \pm \beta i$ with $\alpha\beta \neq 0$. The solutions of the equation (1.3.8) are complex of the form $z_{1,2} = \exp((\alpha \pm \beta i)t)$ with α and β are real numbers, however, the linear combination of these solutions yields real independent solutions of the form $\exp(\alpha t)\cos(\beta t), \exp(\alpha t)\sin(\beta t)$. If $\alpha < 0$ then the orbits are spiraling in with respect to the origin of the plane, see Figure 1.7. If $\alpha > 0$ then the orbits are spiraling out with respect to the origin of the plane.

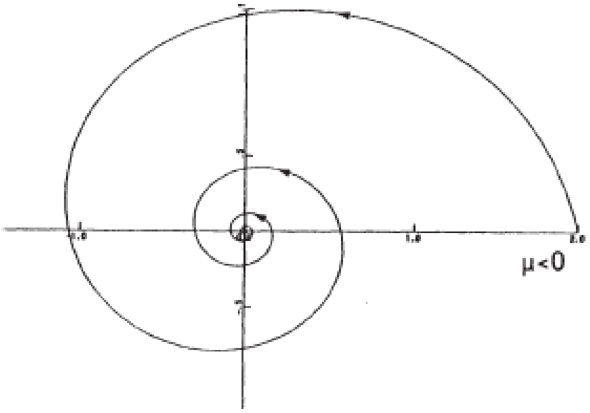

Figure 1.7: Phase portrait of a focus

(**iv**). The equilibrium point is called a center if the eigenvalues $\lambda_{1,2} = \pm\beta i$ with *beta* is real. In particular, this is a special case of the previous one with $\alpha = 0$. The solutions can be represented by a combination of $\cos(\beta t)$ and $\sin(\beta t)$, and the orbits are circles in the phase plane. See Figure 1.8.

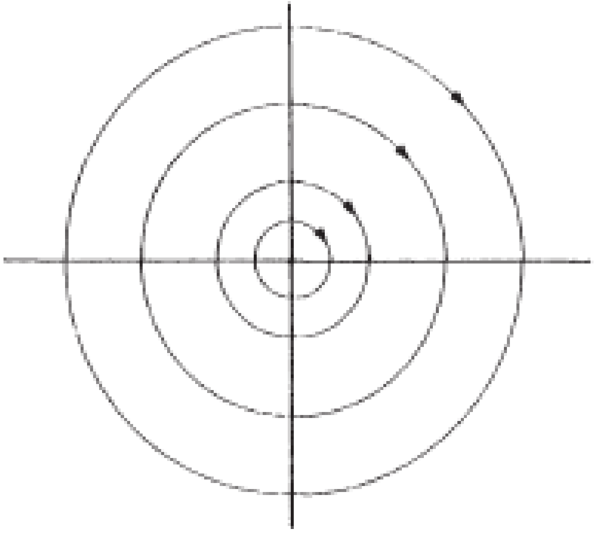

Figure 1.8: Phase portrait of a center

1.5 Stability of equilibrium points

Definition 1.5.1. *Consider the system (1.3.5) where f is a smooth function. Let x^* be an equilibrium point, i.e., $f(x^*) = 0$.*

i. An equilibrium point x^* is said to be Lyapunov stable if and only if

$$\forall\epsilon > 0,\ \exists\delta > 0 \ \textit{ such that if } \ \| x(0) - x^* \| < \delta\,,\ \textit{then} \ \ \| x(t) - x^* \| < \epsilon \ \textit{ for all } \ t \geq 0.$$

ii. An equilibrium point x^* is said to be asymptotically stable if it is Lyapunov stable, and additionally, there exists a $\delta > 0$ such that

$$\textit{if} \ \ \| x(0) - x^* \| < \delta\,, \ \ \textit{then} \ \lim_{t\to\infty} \| x(t) - x^* \| = 0.$$

iii. An equilibrium point x^* is unstable if it is not Lyapunov stable, meaning there exists some $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, there are initial conditions within δ-distance from x^* that lead the trajectories away from the equilibrium.

One powerful method for characterizing the stability of an equilibrium point x^* of the system (1.3.5) is through linearization. By approximating the system near x^*, we compute the Jacobian $A = Df(x^*)$, which represents the linearization of the system around the equilibrium.

Theorem 1.5.1. *Let x^* be the equilibrium point of the system (1.3.5).*

i. If all the eigenvalues of A have negative real parts, the equilibrium point x^* is asymptotically stable.

ii. If at least one eigenvalue has a positive real part, the equilibrium is unstable.

iii. If all eigenvalues have negative real parts, and at least one eigenvalue has a zero real part, then nothing can be said about the stability of the equilibrium x^*, and further analysis is needed to determine the system's behavior.

1.6 Periodic solutions and limit cycles

A periodic solution in a dynamical system is a solution where the system's state repeats itself after a constant period of time.

Definition 1.6.1. *A solution $x = \zeta(t)$ of the system (1.3.5) is called a periodic solution if there exists a positive number T such that $\zeta(t + T) = \zeta(t)$ for all $t \in \mathbb{R}$.*

Remark 1.6.1. *If the solution $\zeta(t)$ has a period T, then it also has periods $2T$, $3T$, etc. If T is the smallest period among all the periods of $\zeta(t)$, then this solution is called T-periodic.*

We mention in the following points some of the characteristics and applications of periodic solutions.

- In the phase-space, periodic solutions (orbits) are characterized by closed curves, they return to their starting point after a fixed interval of time.
- In this case of periodicity, the system's behavior is predictable as it will have the same sequence of states over each period.
- Periodic orbits can be stable or unstable. The stable periodic orbits attract nearby trajectories, while unstable ones repel them.

Periodic solutions are important in various fields. For instance:

- Understanding the motion of celestial bodies where they interact with each other by mutual gravitational force which may lead to periodic trajectories.
- In nonlinear dynamics, when analyzing systems that have complex behaviors, periodic solutions can help in understanding the structure of chaotic attractors.
- In mathematical physics, studying Hamiltonian systems where periodic solutions get involved as the energy levels and stability of the system.

Example 1.6.1. *Consider the equation of the pendulum governed by the equation*

$$\ddot{x} + \sin(x) = 0, \tag{1.6.10}$$

which can be transformed into the system

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -\sin(x_1), \end{aligned}$$

where x_1 represents the angular displacement and x_2 represents the angular velocity.
In the phase plane, we see a family of closed orbits corresponding with periodic solutions, see Figure 1.9.

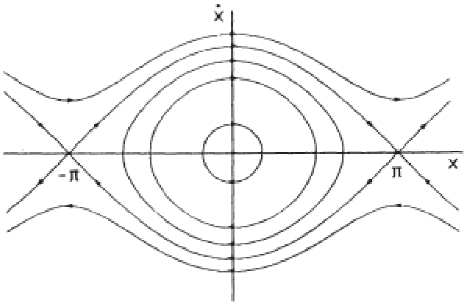

Figure 1.9: Periodic solutions of the pendulum equation (1.6.10)

Definition 1.6.2. *A limit cycle is a periodic orbit in the phase space which has the following properties.*

- *The limit cycle is isolated, meaning no other periodic orbits are arbitrarily close to it.*
- *All nearby trajectories either approach the limit cycle as time tends to infinity (stable limit cycle) or diverge away from it (unstable limit cycle).*

1.6.1 Stability of periodic orbits

Theorem 1.6.1. *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and suppose that $f \in C^1(\mathcal{U})$. Let $\gamma(t)$ be a T-periodic solution of (1.3.5). Then, $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla \cdot f(\gamma(t))dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \nabla \cdot f(\gamma(t))dt > 0.$$

It may be a stable, unstable, or semi-stable limit cycle if this quantity equals zero.

1.6.2 Bendixon's and Dulac's criteria

Consider the autonomous planar system

$$\dot{x} = F(x,y)\,, \quad \dot{y} = G(x,y), \tag{1.6.11}$$

with $(x, y) \in D \subset \mathbb{R}^2$.

Theorem 1.6.2. *(Criterion of Bendixon) Assume that the domain D is simply connected and the functions, F and G, are continuously differentiable in D. If the divergence $\nabla.(F, G)$ of the vector field (F, G) is not identically zero and does not change sign in D, then the system (1.6.11) has no periodic orbit contained entirely in D.*

Example 1.6.2. *Consider the nonlinear oscillator system with nonlinear damping*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -m(x) - n(x)y, \end{aligned}$$

where $m(x)$ and $n(x)$ are smooth functions, with $n(x) > 0$ for all $x \in \mathbb{R}$.
Let $(F(x, y), G(x, y)) = (y, -m(x) - n(x)y)$. The divergence of (F, G) is negative definite, i.e. $\nabla.(F, G) = -n(x)$. From Bendixon's criterion theorem, the previous system has no periodic solutions.

A more general result of the theorem of Bendixon is given by Dulac's Criterion.

Theorem 1.6.3. *(Criterion of Dulac) Assume that the domain D is simply connected and the function, F and G, are continuously differentiable in D. If there exists a continuously differentiable function H in D such that $\nabla.(H(F, G))$ is not identically zero and does not change sign in D, then the following statements hold*

i. *The system (1.6.11) has no periodic orbit contained entirely in D.*

ii. *If S is an annular region contained in D on which $\nabla.(H(F, G))$ does not change sign, then there exists at most one limit cycle of the system (1.6.11).*

1.6.3 The Poincaré-Bendixon theorem

Consider the equation (1.3.5) $\dot{x} = f(x)$ with $x \in \mathcal{U} \subseteq \mathbb{R}^n$. In subsection 1.2.3, we defined an orbit $\mathcal{O}$ (or trajectory) passing through a point x_0 in $\mathcal{U}$ as the set

$$\mathcal{O}(x_0) = \{x \in \mathcal{U} : x = \phi(t, x_0), \text{ for all } t \in \mathbb{R} \text{ such that } \phi(t, x_0) \text{ is defined}\},$$

where ϕ is the flow of the system (1.3.5).

In this section, we also define the positive half-trajectory through the point $x_0 \in \mathcal{U}$ by the curve

$$\mathcal{O}^+(x_0) = \{x \in \mathcal{U} : x = \phi(t, x_0),\ t \geq 0\}.$$

Likewise, we define $\mathcal{O}^-(x_0)$ and we obviously note that $\mathcal{O}(x_0) = \mathcal{O}^+(x_0) \cup \mathcal{O}^-(x_0)$.

Remark 1.6.2. *If the point x_0 is not specifically involved in the motion of the trajectory, we simply denote it by $\mathcal{O}$. Similarly, we note that $\mathcal{O} = \mathcal{O}^+ \cup \mathcal{O}^-$.*

Definition 1.6.3. *If there exists a sequence $t_n \longrightarrow \infty$ such that*

$$\lim_{n \to -\infty} \phi(t_n, x) = \ell_1, \quad \textit{where } \ell_1 \in \mathcal{U},$$

then the point ℓ_1 is called an ω-limit point of the trajectory $\mathcal{O}$ of the system (1.3.5). Similarly, If there exists a sequence $t_n \longrightarrow -\infty$ such that

$$\lim_{n \to -\infty} \phi(t_n, x) = \ell_2, \quad \textit{where } \ell_2 \in \mathcal{U},$$

then the point ℓ_2 is called an α-limit point of the trajectory $\mathcal{O}$ of the system (1.3.5).

Definition 1.6.4. *An ω-limit set of a trajectory $\mathcal{O}$ is the set of all ω-limit points of $\mathcal{O}$, and is denoted by $\omega(\mathcal{O})$. Similarly, An α-limit set of a trajectory $\mathcal{O}$ is the set of all α-limit points of $\mathcal{O}$, and is denoted by $\alpha(\mathcal{O})$. The limit set of $\mathcal{O}$, $\omega(\mathcal{O}) \cup \alpha(\mathcal{O})$, is the set of all limit points of $\mathcal{O}$.*

Theorem 1.6.4. *If ℓ is an ω-limit point of a trajectory $\mathcal{O}$ of (1.3.5), i.e. $\ell \in \omega(\mathcal{O})$, then $\mathcal{O}_\ell \subset \omega(\mathcal{O})$. Similarly, if $\ell \in \alpha(\mathcal{O})$ then $\mathcal{O}_\ell \subset \alpha(\mathcal{O})$.*

This theorem shows that given a point $\ell \in \omega(\mathcal{O})$ of a system (1.3.5), then all other points of the trajectory $\phi(., \ell)$ of (1.3.5) through the point ℓ are also ω-limit points of $\mathcal{O}$. Analogously, the same result applies in the case of α-limit points.

Remarks 1.6.1.

- *An equilibrium point x_0 of system (1.3.5) is its own α and ω-limit set.*

- *A stable node (or focus) is the ω-limit set of every trajectory in some neighborhood of the point.*

Definition 1.6.5. *If ℓ is any regular point in $\alpha(\mathcal{O})$ or $\omega(\mathcal{O})$ then the trajectory through ℓ is called a limit orbit of $\mathcal{O}$.*

Example 1.6.3. *We consider the following differential system*

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2),\\ \dot{y} &= x + y(1 - x^2 - y^2).\end{aligned}$$

When we change the system into polar coordinates we get

$$\begin{aligned}\dot{r} &= r(1 - r^2),\\ \dot{\theta} &= 1.\end{aligned}$$

The origin is an equilibrium point for the system and in the phase plane, the flow:

- *Spirals around the origin in a counter-clockwise direction.*
- *Spirals outward for $0 < r < 1$ because $\dot{r} > 0$.*
- *Spirals inward for $r > 1$ because $\dot{r} < 0$.*

Since $\dot{r} = 0$ on $r = 1$, the flow approaches the unit circle in the counter-clockwise direction describing a trajectory $\mathcal{O}_0$. This trajectory passes through the point $(\cos(\theta_0), \sin(\theta_0))$ on the unit circle at $t = 0$, see Figure 1.10. The trajectory $\mathcal{O}_0$ is called a stable limit cycle. It is the ω-limit set of every trajectory of this system except the equilibrium point at the origin. $\mathcal{O}_0$ consist of one limit orbit and it is its own α and ω-limit set.

Theorem 1.6.5. *(The Poincaré-Bendixon theorem) Consider the system (1.3.5) where $f \in C^1(\mathcal{U})$ with $\mathcal{U} \subset \mathbb{R}^2$. Assume that (1.3.5) has a trajectory $\mathcal{O}$ with $\mathcal{O}^+$ contained in a compact subset D of $\mathcal{U}$ and has a finite number of equilibrium points in D. Then, ω-limit set $\omega(\mathcal{O})$ is either an equilibrium point of (1.3.5), a periodic orbit of (1.3.5), or that $\omega(\mathcal{O})$ is composed of a finite number of equilibria $\ell_1, \ldots, \ell_m$ of (1.3.5) and a countable number of limit orbits of (1.3.5) whose α and ω limit sets belong to $\{\ell_1, \ldots, \ell_m\}$.*

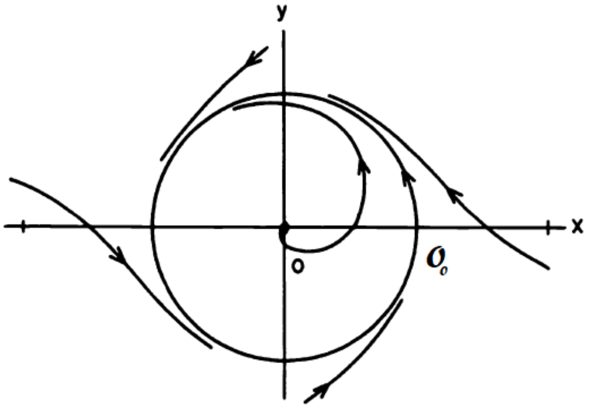

Figure 1.10: A stable limit cycle $\mathcal{O}_0$

The Poincaré-Bendixon theorem is proved in [34].

Example 1.6.4. *Consider the system*

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2 - 2x - 3), \\ \dot{y} &= x + y(x^2 + y^2 - 2x - 3). \end{aligned} \tag{1.6.12}$$

The system (1.6.12) has one equilibrium point located at the origin $(0, 0)$*. First, to determine the possibility of periodic orbits for the system we apply the criterion of Bendixon, so we compute the divergence of the vector function on the righthand side of system (1.6.12) we get*

$$4x^2 + 4y^2 - 6x - 6 = 4\left[(x - \frac{3}{4})^2 + y^2 - \frac{33}{16}\right].$$

We deduce that no closed orbits can be contained in the interior of the circle with center $(\frac{3}{4}, 0)$ *and radius* $\dfrac{\sqrt{33}}{4}$ *because inside that circle (Bendixon circle) the divergence is sign definite. Closed orbits are possible only if they enclose or intersect with this Bendixon circle.*

We write the system (1.6.12) in polar coordinates we find

$$\begin{aligned} \dot{r} &= r(r^2 - 2r\cos(\theta) - 3), \\ \dot{\theta} &= 1. \end{aligned} \tag{1.6.13}$$

If $r < 1$ *we have* $\dot{r} < 0$*, and if* $r > 3$ *we have* $\dot{r} > 0$*. According to the Poincaré-Bendixon theorem, the annulus* $1 < r < 3$ *must contain one or more limit cycles.*

1.7 Bifurcation theory

1.7.1 Overview

Bifurcation theory is a branch of mathematics that studies changes in the qualitative structure of a given family of curves, usually the solutions to a family of differential equations and dynamical systems. It explains how slight changes in a system's parameters can trigger abrupt and substantial shifts in its overall dynamics, leading to phenomena like the emergence or disappearance of equilibria, limit cycles, or other crucial behaviors.

The foundation of the theory was introduced by Poincaré in 1885 when he studied the branching of solutions (i.e. splitting or creation of new solutions) in the three-body problem in celestial mechanics.

1.7.2 Bifurcation types

Bifurcation theory branches into two main classes:

- **Local bifurcation**: this class focuses on the analysis of changes that occur near a specific equilibrium point or periodic orbit. Types of local bifurcations include saddle-node bifurcation, Hopf bifurcation, pitchfork bifurcation, transcritical bifurcation, and Neimark-Sacker bifurcation.

- **Global bifurcation**: global bifurcations often occur when larger invariant sets, intersect or interact with one another or with the system's equilibria. Unlike local bifurcations, which can be detected by analyzing the stability of equilibrium points, global bifurcations often necessitate a broader examination of the system's overall dynamics. Types of global bifurcations include homoclinic bifurcation, heteroclinic bifurcation, and infinite-period bifurcation.

In this section, we emphasize local bifurcations, particularly the Hopf bifurcation and zero-Hopf bifurcation. Some examples of local bifurcations in one-dimensional systems are introduced below.

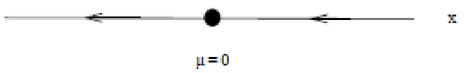

Figure 1.11: Phase portrait of Example 1.7.1

Example 1.7.1. *Consider the equation*

$$\dot{x} = f(x, \mu), \tag{1.7.14}$$

where $f(x, \mu) = \mu - x^2$*. We distinguish three cases:*

1. *If* $\mu > 0$*, the system has two equilibria* $p_{1,2} = \pm\sqrt{\mu}$*. With a simple computation, we get the following results*
   (a) $Df(\sqrt{\mu}, \mu) = -2\sqrt{\mu} < 0$ *which means that* $p_1 = \sqrt{\mu}$ *is stable.*
   (b) $Df(-\sqrt{\mu}, \mu) = 2\sqrt{\mu} > 0$ *which means that* $p_2 = -\sqrt{\mu}$ *is unstable.*
2. *If* $\mu < 0$*, there are no equilibria.*
3. *When* $\mu = 0$ *the system has only one equilibrium point, the origin* $x = 0$*. In this case, the equilibrium point is nonhyperbolic because* $Df(0, 0) = 0$*, hence we cannot analyze its stability by linearizing the system. Nevertheless, we can examine it by a phase portrait, see Figure 1.11.*

We conclude that the origin is an unstable saddle node. The system (1.7.14) has a saddle-node bifurcation at $\mu = 0$*, See Figure 1.12.*

Example 1.7.2. *Consider the equation*

$$\dot{x} = f(x, \mu), \tag{1.7.15}$$

where $f(x, \mu) = \mu x - x^2$*. The equation (1.7.15) has two equilibrium points* $p_1 = 0$ *and* $p_2 = \mu$*. Computing the derivative of* f *we get* $Df(x, \mu) = \mu - 2x$*, hence*

1. $Df(0, \mu) = \mu$ *which means that* p_1 *is stable if* $\mu < 0$ *and unstable if* $\mu > 0$*.*

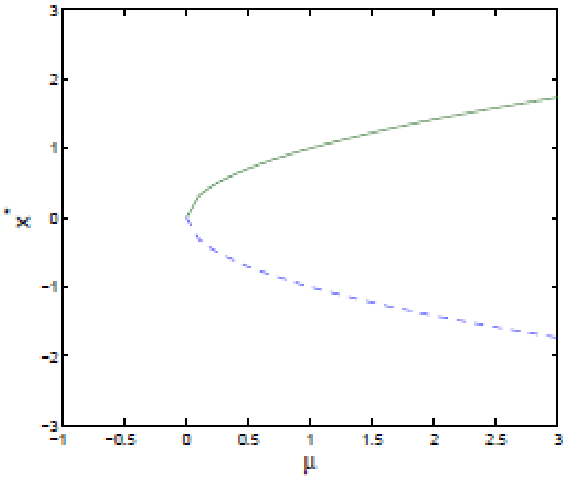

Figure 1.12: Bifurcation diagram for Example 1.7.1

2. *$Df(\mu,\mu) = -\mu$ which means that p_2 is stable if $\mu > 0$ and unstable if $\mu < 0$.*

If $\mu = 0$, then (1.7.15) has one equilibrium at the origin and $Df(0,0) = 0$ implying that this equilibrium is nonhyperbolic. In this case, we use the phase portrait to analyze the stability of the equilibrium point, which is the same phase portrait as in the previous example shown in Figure 1.11. Thus, the equilibrium at the origin is a saddle, hence unstable, when $\mu = 0$. The equilibria of the system (1.7.15) exchange stability as the parameter μ crosses $\mu = 0$ in what is called a transcritical bifurcation.

Example 1.7.3. *Consider the equation governed by*

$$\dot{x} = f(x,\mu), \tag{1.7.16}$$

where $f(x,\mu) = \mu x - x^3$. The derivative of f is $Df(x,\mu) = \mu - 3x^2$.

Again we distinguish three cases:

1. *If $\mu < 0$ the equation (1.7.16) has only one equilibrium point located at the origin which is stable since $Df(0,\mu) = \mu < 0$.*

2. *If $\mu > 0$ (1.7.16) has three equilibrium points $p_1 = 0$ and $p_{2,3} = \pm\sqrt{\mu}$.*

    (a) *The equilibrium point $p_1 = 0$ is unstable since $Df(0,\mu) = \mu > 0$.*

(b) *Both of the equilibrium points* $p_{2,3} = \pm\sqrt{\mu}$ *are stable because* $Df(\pm\sqrt{\mu}, \mu) = -2\mu < 0$.

3. *If* $\mu = 0$, *the only equilibrium is at the origin of the system (1.7.16) which is nonhyperbolic because* $Df(0, 0) = 0$. *So, we look at the phase portrait as shown in Figure 1.13. We see that the equilibrium point is stable. Hence, the system (1.7.16) encounters a pitchfork bifurcation at* $\mu = 0$.

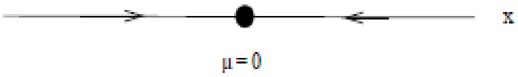

Figure 1.13: Phase portrait of Example 1.7.3

For more details about these types of bifurcations on higher dimensional systems, see [34].

1.7.3 Hopf bifurcation

Consider the dynamical system

$$\dot{x} = f(x, \mu) \ , \quad \mu \in \mathbb{R}. \tag{1.7.17}$$

Hopf bifurcation, also known as Poincaré-Andronov-Hopf bifurcation, is a local bifurcation in which an equilibrium point x_0 of a system (1.7.17) loses stability as a pair of complex conjugate eigenvalues of the matrix $Df(x_0)$, while all other eigenvalues have non-zero real parts, crosses the imaginary axis of the complex plane when a parameter μ crosses a critical value μ_0.

Under generic assumptions about the system (1.7.17), the equilibrium point x_0 changes stability as μ passes through the bifurcation value μ_0, leading to the emergence of a small-amplitude limit cycle.

Definition 1.7.1. *Consider the system (1.7.17) with an equilibrium point x_0 at $\mu = \mu_0$. Suppose that at $\mu = \mu_0$, the Jacobian matrix $Df(x_0)$ has a pair of purely imaginary eigenvalues $\pm i\omega$ and no other eigenvalues with zero real part, and the following condition is satisfied*

$$\frac{d}{d\mu}\Re(\lambda(\mu))\bigg|_{\mu=\mu_0},$$

where $\lambda(\mu)$ are the eigenvalues of the matrix $Df(x_0)$.

Then, the system encounters a Hopf bifurcation at $\mu = \mu_0$, and the following cases can occur.

- ***Case 1: Supercritical Hopf bifurcation***
  The equilibrium point loses stability and a stable limit cycle emerges as μ increases through μ_0. The system exhibits stable oscillations with small amplitude.

- ***Case 2: Subcritical Hopf bifurcation***
  As μ increases through μ_0, an unstable limit cycle emerges. The equilibrium point remains stable until the bifurcation occurs, at this point, the system can transition to large-amplitude oscillations, potentially causing unstable behavior.

The following Hopf bifurcation theorem gives sufficient conditions for the emergence of periodic solutions from a Hopf bifurcation in planar systems.

Theorem 1.7.1. *Consider the planar system*

$$\begin{aligned} \dot{x} &= F_\mu(x,y), \\ \dot{y} &= G_\mu(x,y), \end{aligned} \tag{1.7.18}$$

where μ is a real parameter. Let the point (x_0, y_0) be the equilibrium point of the system (1.7.18), which may depend on μ, and the eigenvalues of the linearized system near that equilibrium be given by $\lambda(\mu)$, $\overline{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$. Assume that for a value of μ, $\mu = \mu_0$, the following conditions are satisfied.

i. $\alpha(\mu_0) = 0$, $\beta(\mu_0) = \omega \neq 0$, *such that* $\operatorname{sgn}(\omega) = \operatorname{sgn}\left[\frac{\partial g_\mu}{\partial x}\bigg|_{\mu=\mu_0}(x_0, y_0)\right].$

ii. $\left.\dfrac{d\alpha(\mu)}{d\mu}\right|_{\mu=\mu_0} = d \neq 0.$

iii. $a \neq 0$, *such that*

$$a = \frac{1}{16}(f_{xxx}+f_{xyy}+g_{xxy}+g_{yyy})+\frac{1}{16\omega}(f_{xy}(f_{xx}+f_{yy})-g_{xy}(g_{xx}+g_{yy})-f_{xx}g_{xx}+f_{yy}g_{yy}),$$

with $f_{xy} = \left.\dfrac{\partial^2 f_\mu}{\partial x \partial y}\right|_{\mu=\mu_0}(x_0, y_0)$, *etc.*

Then a unique curve of periodic solutions bifurcates from the equilibrium point (x_0, y_0) *into the region* $\mu > \mu_0$ *if* $ad < 0$ *or* $\mu < \mu_0$ *if* $ad > 0$. *And,*

- *The equilibrium point is stable for* $\mu > \mu_0$ *(resp.* $\mu < \mu_0$*) and unstable for* $\mu < \mu_0$ *(resp.* $\mu > \mu_0$*) if* $d < 0$ *(resp.* $d > 0$*). At the same time, the periodic solutions are stable (resp. unstable) if the equilibrium point is unstable (resp. stable) on the side of* $\mu = \mu_0$ *where these solutions exist.*

- *The amplitude of these periodic orbits grows proportional to* $\sqrt{|\mu - \mu_0|}$ *whilst their periods tend to* $\dfrac{2\pi}{|\omega|}$ *as* μ *tends to* μ_0.

- *The bifurcation is called supercritical if the bifurcating periodic solutions are stable, and subcritical if they are unstable.*

Example 1.7.4. *Consider the Lienard system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (\mu - x^2)y, \end{aligned} \tag{1.7.19}$$

where μ *is a real parameter. The system has an equilibrium point located at the origin* $(0,0)$*, with the eigenvalues* $\lambda(\mu)$*, where* $\overline{\lambda}(\mu) = \frac{1}{2}\left(\mu \pm i\sqrt{4-\mu^2}\right)$. *When* $\mu = 0$*, the system undergoes a Hopf bifurcation. Since* $\omega = -1$*,* $d = \frac{1}{2}$*, and* $a = -\frac{1}{8}$*, the bifurcation is supercritical and there is a stable isolated periodic orbit (i.e. stable limit cycle) if* $\mu > 0$ *for* μ *sufficiently small, see Figure 1.14.*

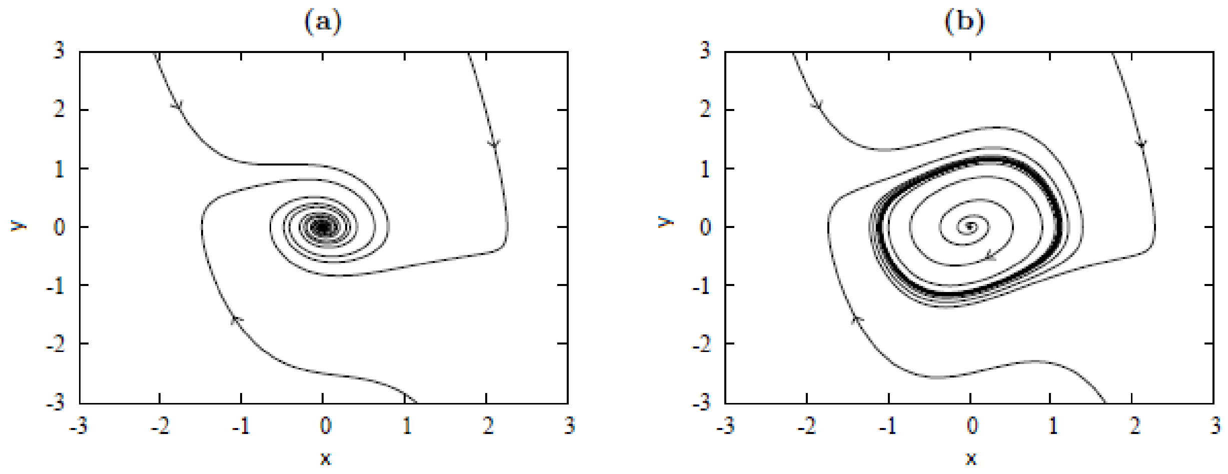

Figure 1.14: Phase portraits of Example 1.7.4 for (a) $\mu = -0.2$, (b) $\mu = 0.3$. At $\mu = 0$ there is a supercritical Hopf bifurcation.

1.7.4 Zero-Hopf bifurcation

The zero-Hopf bifurcation is a codimension-2 bifurcation, which occurs when an equilibrium point exhibits both zero and purely imaginary eigenvalues. This phenomenon is closely related to the standard Hopf bifurcation, where oscillatory behavior emerges from purely imaginary eigenvalues. However, the presence of a zero eigenvalue introduces additional complexity.

Definition 1.7.2. *The zero-Hopf bifurcation, also called fold-Hopf bifurcation, saddle-node bifurcation, or Gavrilov-Guckenheimer bifurcation, is a bifurcation of an equilibrium point in a two-parameter family of autonomous ordinary differential systems at which the critical equilibrium has at least one zero eigenvalue and a pair of purely imaginary eigenvalues. When, in addition to the pair of purely imaginary eigenvalues, all other eigenvalues are zero, this denotes a complete zero-Hopf bifurcation.*

In the following example, we present a zero-Hopf bifurcation analysis of a three-dimensional Rössler system authored by Jaume Llibre, see [23].

Example 1.7.5. *Consider the following Rössler system*

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= bx - cz + xz. \end{aligned} \tag{1.7.20}$$

This system was designed by Otto Rössler as one of the simplest mathematical models that exhibit chaos in three dimensions. Nevertheless, it is studied not only for its simplicity but also for its rich dynamical behaviors.

The system (1.7.20) has an equilibrium point at the origin of coordinates. We shall characterize when this point becomes a zero-Hopf equilibrium point. The characteristic polynomial of the linear part of the system (1.7.20) at the origin is given by

$$p(\lambda) = -\lambda^3 + (a - c)\lambda^2 + (ac - 1 - b)\lambda + ab - c.$$

To determine when the origin is a zero-Hopf equilibrium, we impose the condition that $p(\lambda) = -\lambda(\lambda^2 + \omega^2)$*, where* $\omega > 0$ *is a real number. Expanding this condition gives*

$$p(\lambda) = -\lambda^3 - \omega^2\lambda.$$

Equating coefficients of like powers of λ *in the two polynomials, we obtain the following system of equations*

$$a - c = 0,$$

$$ac - 1 - b = -\omega^2,$$

$$ab - c = 0.$$

Solving these equations, we find:

1. *If* $a = c = \pm\sqrt{2 - \omega^2}$ *and* $b = 1$*, then* $\omega \in (0, \sqrt{2})$.

2. *If* $a = c = 0$ *and* $b = \omega^2 - 1$*, then* $\omega \in (0, \infty)$.

Thus, there are two one-parameter families of Rössler systems for which the origin is a zero-Hopf equilibrium point:

$$\begin{aligned} &(i) \quad a = c \in (-\sqrt{2}, \sqrt{2}) \text{ and } b = 1, \\ &(ii) \quad a = c = 0 \text{ and } b \in (-1, \infty). \end{aligned}$$

Remarks 1.7.1.

- *The second one-parameter family does not provide periodic orbits bifurcating from the zero–Hopf equilibrium point localized at the origin of coordinates of the Rössler system (1.7.20). This statement is proved in [23].*

- *We are primarily interested in the first one-parameter family, which exhibits a zero-Hopf bifurcation of periodic orbits.*

If $(a,b,c) = (\bar{a}+\varepsilon\alpha, 1+\varepsilon\beta, \bar{a}+\varepsilon\gamma)$*, with* $\varepsilon > 0$ *a sufficiently small parameter, then the Rössler system (1.7.20) becomes*

$$\begin{aligned} \dot{x} &= -y-z, \\ \dot{y} &= x+(\bar{a}+\varepsilon\alpha)y, \\ \dot{z} &= (1+\varepsilon\beta)x-(\bar{a}+\varepsilon\gamma)z+xz. \end{aligned} \tag{1.7.21}$$

Rescaling the variables as $(x,y,z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$*, the system in the new variables* (X,Y,Z) *becomes*

$$\begin{aligned} \dot{X} &= -Y-Z, \\ \dot{Y} &= X+\bar{a}Y+\varepsilon\alpha Y, \\ \dot{Z} &= X-\bar{a}Z+\varepsilon(\beta X-\gamma Z+XZ). \end{aligned} \tag{1.7.22}$$

The linear part of the system at the origin, when $\varepsilon = 0$*, can be written in its real Jordan normal form, i.e. as*

$$\begin{pmatrix} 0 & -\sqrt{2-\bar{a}^2} & 0 \\ \sqrt{2-\bar{a}^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This transformation is achieved through the linear change of variables $(X,Y,Z) \to (u,v,w)$*, defined by*

$$\begin{aligned} X &= \frac{(\bar{a}^2-2)v-\bar{a}(\sqrt{2-\bar{a}^2}u+w)}{\bar{a}^2-2}, \\ Y &= \frac{\sqrt{2-\bar{a}^2}u+w}{\bar{a}^2-2}, \\ Z &= \frac{-\bar{a}(2-\bar{a}^2)v+\sqrt{2-\bar{a}^2}(\bar{a}^2-1)u+w}{\bar{a}^2-2}. \end{aligned} \tag{1.7.23}$$

In these new variables, the system becomes

$$\begin{aligned}
\dot{u} &= -\sqrt{2-\bar{a}^2}v + \varepsilon G_1(u,v,w),\\
\dot{v} &= \sqrt{2-\bar{a}^2}u + \varepsilon G_2(u,v,w),\\
\dot{w} &= \varepsilon G_3(u,v,w),
\end{aligned} \tag{1.7.24}$$

where G_1, G_2, and G_3, are explicit functions given by

$$\begin{aligned}
G_1(u,v,w) &= \frac{1}{(2-\bar{a}^2)^{3/2}}\Big[\alpha(1-\bar{a}^2)(\sqrt{2-\bar{a}^2}u+w) + \big((\bar{a}^2-2)v - \bar{a}(\sqrt{2-\bar{a}^2}u+w)\big)\\
&\quad \cdot\Big(\beta + \frac{(\bar{a}^2-1)u}{\sqrt{2-\bar{a}^2}} + \bar{a}v - \frac{w}{\bar{a}^2-2}\Big) + \gamma\Big(\bar{a}(2-\bar{a}^2)v + \sqrt{2-\bar{a}^2}(\bar{a}^2-1)u + w\Big)\Big],\\
G_2(u,v,w) &= \frac{\alpha\bar{a}}{\bar{a}^2-2}\big(\sqrt{2-\bar{a}^2}u+w\big),\\
G_3(u,v,w) &= \frac{1}{\bar{a}^2-2}\Big[\alpha\big(-\sqrt{2-\bar{a}^2}u-w\big) + \bar{a}\Big((\bar{a}^2-2)v - \big(\sqrt{2-\bar{a}^2}u+w\big)\Big)\\
&\quad \cdot\Big(\beta - \frac{w}{\bar{a}^2-2} + \frac{(\bar{a}^2-1)u}{\sqrt{2-\bar{a}^2}} + \bar{a}v\Big) + \gamma\Big(\bar{a}(2-\bar{a}^2)v + \sqrt{2-\bar{a}^2}(\bar{a}^2-1)u + w\Big)\Big].
\end{aligned}$$

Introducing cylindrical coordinates (r,θ,w) via $u = r\cos\theta$ and $v = r\sin\theta$, the system becomes

$$\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon F_1(\theta,r,w),\\
\frac{dw}{d\theta} &= \varepsilon F_2(\theta,r,w),
\end{aligned} \tag{1.7.25}$$

where F_1 and F_2 are periodic in θ with period 2π, and are given by

$$\begin{aligned}
F_1(\theta,r,w) &= \frac{1}{(2-\bar{a}^2)^{3/2}}r\cos\theta\Big[\alpha(1-\bar{a}^2)(\sqrt{2-\bar{a}^2}r\cos\theta+w) + \big(\beta - \frac{w}{\bar{a}^2-2}\big)\\
&\quad \cdot\big(-\bar{a}w - \bar{a}\sqrt{2-\bar{a}^2}r\cos\theta + (\bar{a}^2-2)r\sin\theta\big) + \gamma\big(w + \sqrt{2-\bar{a}^2}(\bar{a}^2-1)r\cos\theta\big)\Big],\\
F_2(\theta,r,w) &= \frac{1}{(2-\bar{a}^2)^{3/2}}\Big[-\alpha\big(\sqrt{2-\bar{a}^2}r\cos\theta+w\big) + \big(\beta - \frac{w}{\bar{a}^2-2}\big)\\
&\quad \cdot\big(-\bar{a}w - \bar{a}\sqrt{2-\bar{a}^2}r\cos\theta + (\bar{a}^2-2)r\sin\theta\big) + \gamma\big(w + \sqrt{2-\bar{a}^2}(\bar{a}^2-1)r\cos\theta\big)\Big].
\end{aligned}$$

We shall Apply the averaging method of first order to the system (1.7.25). We compute the *averaged equations, and we get*

$$\begin{aligned} f_1(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,w)d\theta \\ &= \frac{r}{2(2-\bar{a}^2)^{5/2}}\Big[2(\alpha-\gamma)+\bar{a}\big(\bar{a}(-3\alpha+\bar{a}(-w+\beta+\bar{a}(\alpha-\gamma))+3\gamma)-2\beta\big)\Big], \\ f_2(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,w)d\theta \\ &= \frac{1}{2(2-\bar{a}^2)^{5/2}}\Big[2w(\gamma-\alpha)\bar{a}^2+2\bar{a}\big(r^2+w(w+2\beta)\big)+4w(\alpha-\gamma)-\bar{a}^3\big(r^2+2w\beta\big)\Big]. \end{aligned}$$

Solving $f_1(r,w)=f_2(r,w)=0$, we find a unique solution (r^*,w^*) with $r^*>0$, provided that the following conditions hold

$$\bar{a}\neq 0, \text{ and } \big(-\alpha+\bar{a}(1-\bar{a}^2)\beta+\gamma\big)\,\big((\bar{a}^2-1)\alpha+\bar{a}\beta+(1-\bar{a}^2)\gamma\big)<0.$$

The determinant of the Jacobian matrix

$$J=\left.\frac{\partial(f_1,f_2)}{\partial(r,w)}\right|_{(r,w)=(r^*,w^*)}$$

takes the non-zero value

$$-\frac{(\alpha+\bar{a}\beta-\gamma)(-\alpha+\bar{a}(1-\bar{a}^2)\beta+\gamma)}{2(\bar{a}^2-2)^3}.$$

Moreover, the eigenvalues of J are given by

$$e_{1,2}=\frac{A\pm\sqrt{B}}{2\bar{a}^2(2-\bar{a}^2)^{3/2}},$$

where

$$\begin{aligned} A =&(2-\bar{a}^2)(\alpha-\bar{a}\beta-\gamma), \\ B =& (3\bar{a}^4-4)\alpha^2+2\bar{a}(2\bar{a}^6-3\bar{a}^4+4)\alpha\beta-2(3\bar{a}^4-4)\alpha\gamma \\ &+\bar{a}^2(3\bar{a}^4-4)\beta^2-2\bar{a}(2\bar{a}^6-3\bar{a}^4+4)\beta\gamma+(3\bar{a}^4-4)\gamma^2. \end{aligned}$$

Theorem 2.2.1 guarantees that for $\varepsilon>0$ sufficiently small, there exists a periodic solution $(r(\theta,\varepsilon),w(\theta,\varepsilon))$ of system (1.7.25) such that

$$(r(0,\varepsilon),w(0,\varepsilon))\to(r^*,w^*) \quad \text{as } \varepsilon\to 0.$$

Thus, system (1.7.24) has the periodic solution

$$\begin{pmatrix} u(\theta,\varepsilon) \\ v(\theta,\varepsilon) \\ w(\theta,\varepsilon) \end{pmatrix} = \begin{pmatrix} r(\theta,\varepsilon)\cos\theta \\ r(\theta,\varepsilon)\sin\theta \\ w(\theta,\varepsilon) \end{pmatrix}, \tag{1.7.26}$$

for $\varepsilon > 0$ *sufficiently small. Consequently, system (1.7.22) has the periodic solution* $(X(\theta), Y(\theta), Z(\theta))$ *obtained from (1.7.26) through the change of variables (1.7.23).*

Finally, for $\varepsilon > 0$ *sufficiently small, system (1.7.21) has a periodic solution*

$$(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta)),$$

which tends to the origin as $\varepsilon \to 0$*. Therefore, this is a periodic solution that starts at the zero-Hopf equilibrium point located at the origin when* $\varepsilon = 0$*. The stability of this periodic orbit is determined by the eigenvalues* e_1 *and* e_2*. If both eigenvalues are negative, the periodic orbit is stable; otherwise, it is unstable.*

CHAPTER 2

Averaging Theory

2.1 Introduction

The concept of averaging has its roots in celestial mechanics, particularly in the work of the mathematicians Clairaut, Lagrange, and Laplace in the 18th century. They used early forms of averaging to study the motion of planets, moons, and other celestial bodies. The challenge was to deal with small perturbations in their orbits over long timescales, leading to the need for techniques that could average out these small effects.

In the 19th century, the theory of averaging advanced significantly with the work of the mathematicians Jacobi and Poisson, who formalized aspects of averaging in the context of Hamiltonian systems. They laid the groundwork for the development of perturbation methods used in classical mechanics, which later influenced the formal theory of averaging. The first mathematical formalization was done in 1928 by Fatou [17], and was greatly refined in 1934 by the Kiev School of Mathematics led by Bogoliubov and Krylov [21]. Important practical and theoretical contributions were made in 1945 by Bogoliubov [2], and in 1961 by Bogoliubov and Mitropolsky [1]. They played a crucial role in rigorously developing averaging methods for nonlinear oscillatory systems, expanding the technique to handle broader classes of systems, such as those with fast and slow timescales.

From the latter half of the 20th century to the present, averaging methods continued to evolve with contributions from various fields, including nonlinear dynamics and applied mathematics. These methods became essential tools for analyzing a wide range of physical, biological, and engineering systems, especially those exhibiting limit cycles, bifurcations, and chaotic behavior. For a comprehensive modern exposition of averaging theory, refer to the work of Sanders, Verhulst, and Murdock [43].

The idea is to consider a dynamical system given by

$$\dot{x} = \varepsilon f(t, x, \varepsilon), \tag{2.1.1}$$

where $x \in \mathbb{R}^n$ is the state vector, ε is a small parameter $(0 < \varepsilon << 1)$ and $f(t, x, \varepsilon)$ is a periodic function in time t of period T.

The objective here is to approximate this system by focusing on its long-term dynamics. To achieve this, the theory involves computing the averaged function $\overline{f}(x)$ defined as the time average of $f(t, x, \varepsilon)$ over one period T of the oscillation

$$\overline{f}(x) = \frac{1}{T}\int_0^T f(t, x, 0)\, dt.$$

Note that $\overline{f}$ depends only on the state x simplifying the system.

The averaged system is then described by the equation

$$\dot{y} = \varepsilon \overline{f}(y),$$

where $y(t)$ represents the state vector. This averaged system provides a simpler way to analyze the existence, stability, and behavior of limit cycles and other long-term behaviors in the original system.

2.2 Averaging method of first order for periodic orbits

Consider the differential system

$$\dot{\mathbf{x}} = \varepsilon f(t, \mathbf{x}) + \varepsilon^2 g(t, \mathbf{x}, \varepsilon), \tag{2.2.2}$$

with $\mathbf{x} \in E \subset \mathbb{R}^n$, E a bounded domain, and $t \in \mathbb{R}^+$. The functions $f(t, \mathbf{x})$ and $g(t, \mathbf{x}, \varepsilon)$ are T-periodic in t.

We define the averaged system associated to the system (2.2.2) by

$$\dot{y} = \varepsilon \overline{f}(y), \tag{2.2.3}$$

where

$$\overline{f}(y) = \frac{1}{T}\int_0^T f(\theta, y)\, d\theta. \tag{2.2.4}$$

The following theorem states the conditions under which the critical points of the averaged system (2.2.3) give rise to T-periodic orbits in system (2.2.2).

Theorem 2.2.1. *Consider the system (2.2.2) and assume that the functions* f*,* g*,* $D_{\mathbf{x}}f$*,* $D_{\mathbf{x}}^2 f$ *and* $D_{\mathbf{x}}^2 g$ *are continuous and bounded by a constant* M*, independent of* ε*, in* $[0,\infty)\times E$ *with* $-\varepsilon_0 < \varepsilon < \varepsilon_0$*. Furthermore,* f *and* g *are* T*-periodic in* t*, with* T *independent of* ε*.*

i. If $a \in E$ *is a critical point of the averaged system (2.2.3) such that* $\det(D_{\mathbf{x}}\overline{f}(a)) \neq 0$ *then, for* $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system (2.2.2) such that* $\mathbf{x}(0,\varepsilon) \longrightarrow a$ *as* $\varepsilon \longrightarrow 0$*.*

ii. If the critical point $y = a$ *of the averaged system (2.2.3) has all its eigenvalues with negative real part then, for* $|\varepsilon| > 0$ *sufficiently small, the corresponding periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system (2.2.2) is asymptotically stable. If one eigenvalue has a positive real part, the solution* $\mathbf{x}(t,\varepsilon)$ *is unstable.*

For a proof of this theorem, see [48].

Example 2.2.1. *Consider the Liénard system*

$$\begin{aligned} \dot{x} &= y - \varepsilon(a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5), \\ \dot{y} &= -x, \end{aligned} \tag{2.2.5}$$

where $a_i \neq 0$ *for* $i = 1..5$*, and* ε *is sufficiently small.*

We transform the system (2.2.5) into polar coordinates (r,θ)*, where* $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$*, by doing the change of variables*

$$\begin{aligned} \dot{r} &= \frac{1}{r}(x\dot{x} + y\dot{y}), \\ \dot{\theta} &= \frac{1}{r^2}(x\dot{y} - y\dot{x}), \end{aligned}$$

and we obtain

$$\begin{aligned}
\dot{r} &= \big(-\cos(\theta)^6 a_5\, r^5 - \cos(\theta)^5 a_4\, r^4 - \cos(\theta)^4 a_3\, r^3 - \cos(\theta)^3 a_2\, r^2 - \cos(\theta)^2 a_1 r \big)\varepsilon, \\
\dot{\theta} &= -1 + \big(\cos(\theta)^5 \sin(\theta) a_5 r^4 + \cos(\theta)^4 \sin(\theta) a_4 r^3 + \cos(\theta)^3 \sin(\theta) a_3 r^2 + \cos(\theta)^2 \sin(\theta) a_2 r \\
&\quad + \cos(\theta)\sin(\theta) a_1 \big)\varepsilon.
\end{aligned} \tag{2.2.6}$$

Taking θ as the independent variable we write

$$\frac{dr}{d\theta} = \varepsilon\big(\cos(\theta)^6 a_5 r^5 + \cos(\theta)^5 a_4 r^4 + \cos(\theta)^4 a_3 r^3 + \cos(\theta)^3 a_2 r^2 + \cos(\theta)^2 a_1 r \big) + O(\varepsilon^2). \tag{2.2.7}$$

Again taking $\mathbf{x} = r$,$t = \theta$,$T = 2\pi$, and $f(t, \mathbf{x}) = \cos(\theta)^6 a_5 r^5 + \cos(\theta)^5 a_4 r^4 + \cos(\theta)^4 a_3 r^3 + \cos(\theta)^3 a_2 r^2 + \cos(\theta)^2 a_1 r$, the system (2.2.7) is in the normal form (2.1.1) for applying Theorem 2.2.1.

Now, we compute the averaged function $\overline{f}(r)$

$$\begin{aligned}
\overline{f}(r) &= \frac{1}{T}\int_0^T f(t, \mathbf{x})\, dt, \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\theta)^6 a_5 r^5 + \cos(\theta)^5 a_4 r^4 + \cos(\theta)^4 a_3 r^3 + \cos(\theta)^3 a_2 r^2 + \cos(\theta)^2 a_1 r\, d\theta, \\
&= \frac{5}{16} a_5\, r^5 + \frac{3}{8} a_3\, r^3 + \frac{1}{2} a_1 r.
\end{aligned} \tag{2.2.8}$$

Solving the equation $\overline{f}(r) = 0$ we find five solutions

$$r_1 = 0,\ r_2 = \frac{\sqrt{5}\sqrt{a_5\left(-3a_3 + \sqrt{-40a_5a_1 + 9a_3^2}\right)}}{5a_5},$$

$$r_3 = -\frac{\sqrt{5}\sqrt{a_5\left(-3a_3 + \sqrt{-40a_5a_1 + 9a_3^2}\right)}}{5a_5}, r_4 = \frac{\sqrt{5}\sqrt{-a_5\left(3a_3 + \sqrt{-40a_5a_1 + 9a_3^2}\right)}}{5a_5},$$

$$\text{and} \qquad r_5 = -\frac{\sqrt{5}\sqrt{-a_5\left(3a_3 + \sqrt{-40a_5a_1 + 9a_3^2}\right)}}{5a_5}.$$

Since $r > 0$, we take only r_2 and r_4 with $a_5 > 0$, $-40a_5a_1 + 9a_3^2 > 0$ and $-3a_3 + \sqrt{-40a_5a_1 + 9a_3^2} > 0$, or r_3 and r_5 with $a_5 < 0$, $-40a_5a_1 + 9a_3^2 > 0$ and $3a_3 + \sqrt{-40a_5a_1 + 9a_3^2} > 0$. Furthermore, we have

$$\frac{d\overline{f}(r_{2,3})}{dr} = \frac{3\sqrt{5}\sqrt{a_5\left(-3a_3 + \sqrt{-40a_5a_1 + 9a_3^2}\right)}\left(-87a_3 + 29\sqrt{-40a_5a_1 + 9a_3^2} - 36a_5\right)}{160a_5^2},$$

$$\frac{d\overline{f}(r_4)}{dr} = -\frac{3\sqrt{5}\sqrt{-a_5\left(3a_3+\sqrt{-40a_5a_1+9a_3^2}\right)}\left(87a_3+29\sqrt{-40a_5a_1+9a_3^2}+36a_5\right)}{160a_5^2},$$

$$\frac{d\overline{f}(r_5)}{dr} = \frac{3\sqrt{5}\sqrt{-a_5\left(3a_3+\sqrt{-40a_5a_1+9a_3^2}\right)}\left(87a_3+29\sqrt{-40a_5a_1+9a_3^2}+36a_5\right)}{160a_5^2}.$$

If $-87a_3+29\sqrt{-40a_5a_1+9a_3^2}-36a_5 \neq 0$, *then* $\dfrac{d\overline{f}(r_{2,3})}{dr} \neq 0$. *And, if* $87a_3+29\sqrt{-40a_5a_1+9a_3^2}+36a_5 \neq 0$, *then* $\dfrac{d\overline{f}(r_{4,5})}{dr} \neq 0$.

Thus, under the conditions above it follows from Theorem 2.2.1 that the system (2.2.5) has two limit cycles.

For instance, let $a_1 = \dfrac{1}{8}, a_3 = -1$, *and* $a_5 = 1$. *Hence, the averaged function is given by*

$$\overline{f}(r) = \frac{5}{16}r^5 - \frac{3}{8}r^3 + \frac{1}{16}r$$

has two positive real roots $r_1 = 1$ *and* $r_2 = \dfrac{\sqrt{5}}{5}$, *and their values by the derivative of* $\overline{f}$ *are* $\dfrac{d\overline{f}(r_1)}{dr} = \dfrac{327}{32} > 0$ *and* $\dfrac{d\overline{f}(r_2)}{dr} = -\dfrac{21\sqrt{5}}{160} < 0$ *respectively.*
Consequently, by Theorem 2.2.1 the system (2.2.5) with the coefficients $a_1 = \dfrac{1}{8}, a_3 = -1$, *and* $a_5 = 1$ *has two limit cycles, the one which corresponds to* $r_2 = \dfrac{\sqrt{5}}{5}$ *is stable while the other which corresponds to* $r_1 = 1$ *is unstable. See Figure 2.1.*

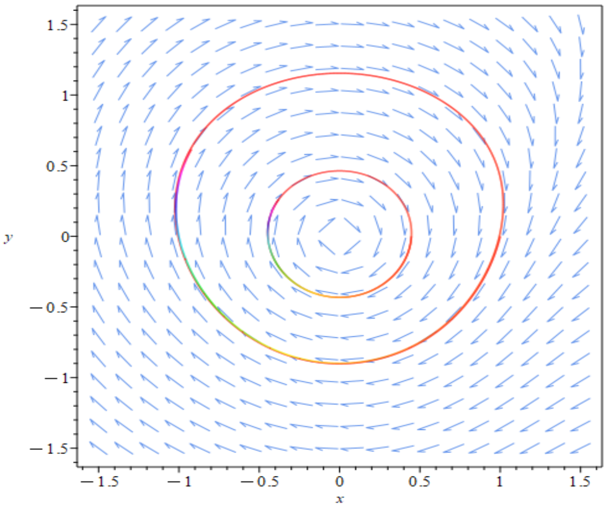

Figure 2.1: Two limit cycles in a Lienard system with $\varepsilon = 0.01$

2.3 Averaging methods for finding periodic orbits via Brouwer degree

In [10], the authors have developed the averaging method for finding the periodic solutions up to the third order using the Brouwer degree theory. This section presents the averaging method of first and second order.

2.3.1 First-order averaging method

Here, we state the theorem of the first-order averaging method using the Brouwer degree.

Theorem 2.3.1. *Consider the differential system*

$$\dot{\mathbf{x}} = \varepsilon f_1(t, \mathbf{x}) + \varepsilon^2 g(t, \mathbf{x}, \varepsilon), \tag{2.3.9}$$

where $f_1 : \mathbb{R}\times E \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}\times E\times(-\varepsilon_f, \varepsilon_f) \longrightarrow \mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable and* E *is an open subset of* $\mathbb{R}^n$*. We define* $\overline{f}_1 : E \longrightarrow \mathbb{R}^n$ *as follows*

$$\overline{f}_1(\mathbf{z}) = \frac{1}{T}\int_0^T f_1(s, \mathbf{z})\, ds, \tag{2.3.10}$$

and suppose that

(i) f_1 *and* g *are locally Lipschitz with respect to* $\mathbf{x}$.

(ii) For $a \in E$ *with* $\overline{f}_1(a) = 0$*, there exists a neighborhood* V *of* a *such that* $\overline{f}_1(\mathbf{z}) \neq 0$ *for all* $\mathbf{z} \in \overline{V}\backslash\{a\}$ *and* $d_B(\overline{f}_1, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $\phi(., \varepsilon)$ *of system (2.3.9) such that* $\phi(., \varepsilon) \longrightarrow a$ *as* $\varepsilon \longrightarrow 0$.

Theorem 2.3.1 has weaker hypotheses than that analogous result Theorem 2.2.1.

2.3.2 Second-order averaging method

The following theorem presents the second-order averaging method for finding periodic orbits via the Brouwer degree.

Theorem 2.3.2. *Consider the following differential system*

$$\dot{\mathbf{x}} = \varepsilon f_1(t, \mathbf{x}) + \varepsilon^2 f_2(t, \mathbf{x}) + \varepsilon^3 g(t, \mathbf{x}, \varepsilon), \tag{2.3.11}$$

where the functions $f_1,\ f_2 :\ \mathbb{R} \times E \longrightarrow \mathbb{R}^n,\ g : \mathbb{R} \times E \times (-\varepsilon_f, \varepsilon_f) \longrightarrow \mathbb{R}^n$ *are continuous and* T*-periodic in* t*.* E *is an open subset of* $\mathbb{R}^n$*. Assume that*

(i) $f_1(t, \cdot) \in C^1(E)$ *for all* $t \in \mathbb{R}$*,* $f_1,\ f_2,\ g$ *and* $D_{\mathbf{x}} f_1$ *are locally Lipschitz with respect to* $\mathbf{x}$ *and* g *is differentiable with respect to* ε*.*

The functions $\overline{f}_1,\ \overline{f}_2 :\ E \longrightarrow \mathbb{R}^n$ *are defined as follows*

$$\begin{aligned} \overline{f}_1(\mathbf{z}) &= \frac{1}{T}\int_0^T f_1(s, \mathbf{z})\, ds, \\ \overline{f}_2(\mathbf{z}) &= \frac{1}{T}\int_0^T \left[D_{\mathbf{z}} f_1(s, \mathbf{z}) \cdot \int_0^s f_1(t, \mathbf{z})\, dt + f_2(s, \mathbf{z}) \right] ds. \end{aligned}$$

Moreover, assume that

(ii) for $U \subset E$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$*, there exists* $a_\varepsilon \in U$ *such that* $\overline{f}_1(a_\varepsilon) + \varepsilon \overline{f}_2(a_\varepsilon) = 0$ *and* $d_B(\overline{f}_1 + \varepsilon \overline{f}_2, U, a_\varepsilon) \neq 0$*.*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $\phi(., \varepsilon)$ *of system (2.3.11).*

Theorems 2.3.1 and 2.3.2 are proved in [10].

The expression $d_B(\overline{f}_1 + \varepsilon \overline{f}_2, U, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $\overline{f}_1(a_\varepsilon) + \varepsilon \overline{f}_2(a_\varepsilon) :\ U \longrightarrow \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of $\overline{f}_1(a_\varepsilon) + \varepsilon \overline{f}_2(a_\varepsilon)$ at a_ε is not zero.

If $\overline{f}_1$ is identically zero and $\overline{f}_2$ is not identically zero, then the zeros of $\overline{f}_1 + \varepsilon \overline{f}_2$ are mainly the zeros of $\overline{f}_2$ for ε sufficiently small. In this case, the previous result provides the averaging method of second order.

Example 2.3.1. *(Smooth perturbed harmonic oscillator) consider the differential equation*

$$\ddot{x} + x + b_\varepsilon \dot{x} = g_\varepsilon\left(x, \dot{x}\right), \tag{2.3.12}$$

where

$$b_\varepsilon = \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 O(1) > 0, \ \textit{and } g_\varepsilon(x,y) = \varepsilon g_1(x,y) + \varepsilon^2 g_2(x,y) + \varepsilon^3 O(1)$$

is a sufficiently differentiable function.

By taking $y = \dot{x}$, the second order differential equation (2.3.12) writes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_\varepsilon y + g_\varepsilon(x,y). \end{aligned} \tag{2.3.13}$$

We write system (2.3.12) in the standard form (2.3.11) in order to apply the averaging theorem.

Writting system (2.3.13) in polar coordinates $(x,y) = (r\cos(\theta), r\sin(\theta))$, we get

$$\begin{aligned} \dot{r} &= \sin(\theta)\left(g_\varepsilon(r\cos(\theta), r\sin(\theta)) - b_\varepsilon r\sin(\theta)\right), \\ \dot{\theta} &= -1 + \frac{\cos(\theta) g_\varepsilon(r\cos(\theta), r\sin(\theta))}{r} - b_\varepsilon \cos(\theta)\sin(\theta). \end{aligned}$$

Thus, taking θ as the new independent variable we obtain

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r\sin(\theta)\left(g_\varepsilon(r\cos(\theta), r\sin(\theta)) - b_\varepsilon r\sin(\theta)\right)}{-r + \cos(\theta) g_\varepsilon(r\cos(\theta), r\sin(\theta)) + b_\varepsilon\cos(\theta)\sin(\theta)},$$

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon\sin(\theta)\left(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta)\right) \\ &+ \frac{\varepsilon^2}{r}\sin(\theta)\big(\cos(\theta)\left(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta))\right)^2 \\ &+ r\left(b_2 r\sin(\theta) - g_2(r\cos(\theta), r\sin(\theta))\right)\big) + \varepsilon^3 O(1), \\ &= \varepsilon f_1(\theta, r) + \varepsilon^2 f_2(\theta, r) + \varepsilon^3 g(\theta, r, \varepsilon), \end{aligned} \tag{2.3.14}$$

where

$$\begin{aligned} \overline{f}_1(\theta, r) =& \sin(\theta)\left(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta)\right), \\ \overline{f}_2(\theta, r) =& \frac{1}{r}\sin(\theta)\big(\cos(\theta)\left(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta))\right)^2 \\ & + r\left(b_2 r\sin(\theta) - g_2(r\cos(\theta), r\sin(\theta))\right)\big). \end{aligned}$$

Now, let the functions g_1 *and* g_2 *be given by*

$$g_1(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$g_2(x,y) = \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3.$$

We Compute the first-order averaged function and its zeros.

$$\overline{f}_1(r) = \pi b_1 r - \frac{\pi\left(3a_{03}+a_{21}\right)}{4}r^2.$$

The equation $\overline{f}_1(r) = 0$ *has 3 solutions*

$$r_1 = 0, r_2 = -2\sqrt{\frac{b_1}{3a_{03}+a_{21}}}, \text{ and } r_3 = 2\sqrt{\frac{b_1}{3a_{03}+a_{21}}}.$$

Assuming $b_1\left(3a_{03}+a_{21}\right) > 0, r^* = r_3$ *is the only positive solution. Hence, the first order averaging method provides the existence of a periodic solution* $r(\theta,\varepsilon)$ *of the differential equation (2.3.14) such that* $r(\cdot,\varepsilon) \to r^*$ *as* $\varepsilon \to 0$. *Accordingly, one gets the existence of a periodic solution* $(x(t,\varepsilon), x'(t,\varepsilon))$ *of the differential equation (4) satisfying* $|(x(\cdot,\varepsilon), x'(\cdot,\varepsilon))| \to r^*$ *as* $\varepsilon \to 0$. *See Figure 2.2.*

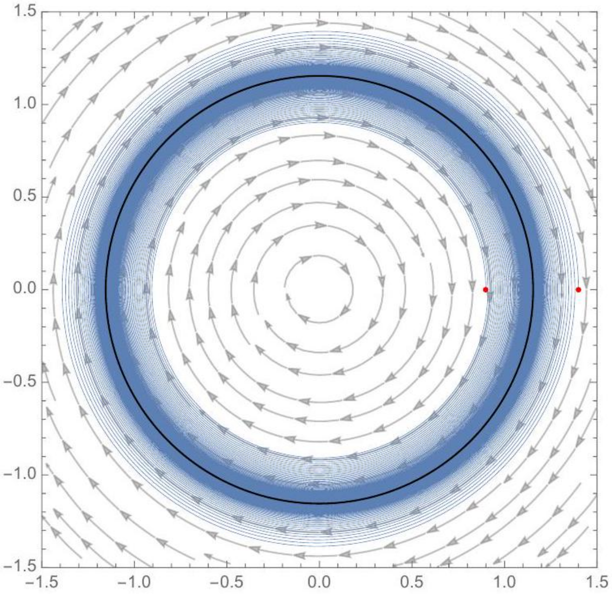

Figure 2.2: Mathematica simulation of two backwards trajectories of the differential system $(\dot{x},\dot{y}) = (y, -x-\varepsilon y+\varepsilon y^3)$ for $\varepsilon = 0.1$. The red dots indicate the initial conditions $(0.9, 0)$ and $(1.4, 0)$. The black closed curve indicates the limit cycle.

In order to ensure that $\overline{f}_1 = 0$ *and compute the second-order averaged function and its zeros, we assume that* $b_1 = 0$ *and* $a_{21} = -3a_{03}$. *Under this condition, we compute the averaged function* $\overline{f}_2$

$$\overline{f}_2(r) = \frac{\pi r}{8}\left(a_{03}\left(a_{12} + 3a_{30}\right) r^4 - 2\left(3\alpha_{03} + \alpha_{21}\right) r^2 + 8b_2\right).$$

Conditions can be assumed so that the equation $\overline{f}_2(r) = 0$ *has zero, one or two positive simple solutions. For instance, by taking* $g_1(x, y) = y^3 + 2xy^2 - 3x^2y$, $g_2(x, y) = 5x^2y + 2xy^2$, *and* $b_2 = 1$, *we get that*

$$\overline{f}_2(r) = \frac{\pi r}{4}\left(r^4 - 5r^2 + 4\right).$$

The equation $\overline{f}_2(r) = 0$ *has two positive solutions* $r_1^* = 1$ *and* $r_2^* = 2$. *Hence, the second order averaging method provides the existence of two periodic solutions* $r_1(\theta, \varepsilon)$ *and* $r_2(\theta, \varepsilon)$ *of the differential equation (2.3.14) such that* $r_1(\cdot, \varepsilon) \to r_1^*$ *and* $r_2(\cdot, \varepsilon) \to r_2^*$ *as* $\varepsilon \to 0$. *Accordingly, one gets the existence of periodic solutions* $(x_1(t, \varepsilon), \dot{x}_1(t, \varepsilon))$ *and* $(x_2(t, \varepsilon), \dot{x}_2(t, \varepsilon))$ *of the differential equation (2.3.12) satisfying* $|(x_1(\cdot, \varepsilon), \dot{x}_1(\cdot, \varepsilon))| \to r_1^*$ *and* $|(x_2(\cdot, \varepsilon), \dot{x}_2(\cdot, \varepsilon))| \to r_2^*$ *as* $\varepsilon \to 0$. *See Figure 2.3.*

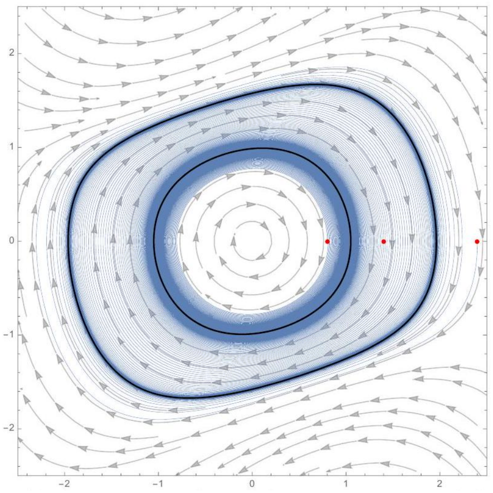

Figure 2.3: Mathematica simulation of some trajectories of the differential system $(\dot{x}, \dot{y}) = \left(y, -x + \varepsilon\left(y^3 + 2xy^2 - 3x^2y\right) + \varepsilon^2\left(-y + 5x^2y + 2xy^2\right)\right)$ for $\varepsilon = 0.1$. The red dots indicate the initial conditions $(0.8, 0)$, $(1.4, 0)$, and $(2.4, 0)$. The blue continuous lines indicate the backward trajectory for initial condition $(0.8, 0)$, the orbit for initial condition $(1.4, 0)$, and the forward trajectory for the initial condition $(2.4, 0)$. The black closed curves indicate limit cycles.

2.4 Other first-order averaging method for periodic orbits

Consider the bifurcation problem of T-periodic solutions of the differential system of the form

$$\dot{\mathbf{x}} = f_0(t, \mathbf{x}) + \varepsilon f_1(t, \mathbf{x}) + \varepsilon^2 g(t, \mathbf{x}, \varepsilon), \tag{2.4.15}$$

with ε small enough. The functions $f_0,\ f_1 :\ \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathbb{R}^n$ are C^2 functions, T-periodic in t and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = f_0(t, \mathbf{x}), \tag{2.4.16}$$

has a k-dimensional submanifold of periodic solutions.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the system (2.4.16) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system (2.4.16) along the periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{y} = D_{\mathbf{x}} f_0(t, \mathbf{x}(t, \mathbf{z}, 0))y, \tag{2.4.17}$$

where y is a $n \times n$ matrix.

In what follows, we denote by $M_z(t)$ some fundamental matrix of the linear differential system (2.4.17).

We assume that there exists an open set V with $Cl(V) \subset \Omega$ such that for each $\mathbf{z} \in Cl(V)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T-periodic, where $\mathbf{x}(t, \mathbf{z}, 0)$ denotes the solution of the unperturbed system (2.4.16) with $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$. The set $Cl(V)$ is isochronous for the system (2.4.15); i.e., it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the bifurcation problem of T-periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $Cl(V)$ is given in the following theorem.

Theorem 2.4.1. *(Perturbation of an isochronous set) We assume that there is an open and bounded set V with $Cl(V) \subset \Omega$ such that for each $\mathbf{z} \in Cl(V)$, the solution $\mathbf{x}(t, \mathbf{z}, 0)$ is T-periodic. Consider a function $\mathcal{F} : Cl(V) \longrightarrow \mathbb{R}^n$ defined by*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T}\int_0^T M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z}),0)dt. \tag{2.4.18}$$

If there is an $a \in V$ *with* $\mathcal{F}(a) = 0$ *and* $\det\left((\frac{\partial \mathcal{F}}{\partial \mathbf{z}})(a)\right) \neq 0$, *then there is a* T*-periodic solution* $\varphi(t,\varepsilon)$ *to system (2.4.15) such that* $\varphi(0,\varepsilon) \to a$ *as* $\varepsilon \to 0$.

Theorem 2.4.1 is a classical result due to Malkin [30] and Roseau [39]. For a simplified proof of this theorem see [9].

The following example examines the zero-Hopf bifurcation of a Michelson system, as studied by Llibre and Zang in [27].

Example 2.4.1. *Consider the Michelson system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= c^2 - y - \frac{x^2}{2}, \end{aligned} \tag{2.4.19}$$

with $(x,y,z) \in \mathbb{R}^3$ *and* $c \geq 0$.

The system (2.4.19) clearly has two critical points $p_1 = (-\sqrt{2}c,0,0)$ *and* $p_2 = (\sqrt{2}c,0,0)$ *for* $c \geq 0$. *If* $c = 0$, *the system has a critical point at the origin.*

For any $\varepsilon \neq 0$ *we do the re-scale of variables* $x = \varepsilon x_1, y = \varepsilon y_1, z = \varepsilon z_1$ *and* $c = \varepsilon d$, *then the system (2.4.19) becomes*

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = z_1, \quad \dot{z}_1 = -y_1 + \varepsilon d^2 - \varepsilon\frac{1}{2}x_1^2. \tag{2.4.20}$$

Now doing the change of variables $x_1 = X$, $y_1 = r\sin\theta$ *and* $z_1 = r\cos\theta$, *system (2.4.19) is given by*

$$\dot{X} = r\sin\theta, \quad \dot{r} = \frac{\varepsilon}{2}\left(2d^2 - X^2\right)\cos\theta, \quad \dot{\theta} = 1 - \frac{\varepsilon}{2r}\left(2d^2 - X^2\right)\sin\theta. \tag{2.4.21}$$

This system can be written as

$$\begin{aligned} \frac{dX}{d\theta} &= r\sin(\theta) + \frac{\varepsilon}{2}(2d^2 - X^2)\sin^2(\theta) + \varepsilon^2 g_1(\theta,r,\varepsilon), \\ \frac{dr}{d\theta} &= \frac{\varepsilon}{2}(2d^2 - X^2)\cos(\theta) + \varepsilon^2 g_2(\theta,r,\varepsilon), \end{aligned} \tag{2.4.22}$$

where g_1 and g_2 are analytic functions.

The unperturbed system

$$\begin{aligned} \frac{dX}{d\theta} &= r\sin(\theta), \\ \frac{dr}{d\theta} &= 0, \end{aligned} \tag{2.4.23}$$

has the 2π-periodic solution

$$X(\theta) = r_0 + X_0 - r_0\cos\theta, \quad r(\theta) = r_0, \tag{2.4.24}$$

for arbitrary $(X_0, r_0) \neq (0,0)$, such that $X(0) = X_0$ and $r(0) = r_0$.
It has the fundamental solution matrix

$$M = \begin{pmatrix} 1 & 1-\cos\theta \\ 0 & 1 \end{pmatrix}.$$

which is independent of the initial condition (X_0, r_0). Applying Theorem 2.4.1 to the system (2.4.22) we have ,along the solution (2.4.24), that

$$\mathcal{F}(X_0, r_0) = \frac{1}{2}\int_0^{2\pi} M^{-1}\begin{pmatrix} (2d^2 - X^2)\sin^2\theta \\ (2d^2 - X^2)\cos\theta \end{pmatrix} d\theta.$$

Then $\mathcal{F}(X_0, r_0) = (f_1(X_0, r_0), f_2(X_0, r_0))$ with

$$f_1(X_0, r_0) = \frac{1}{4}\left(4d^2 - 5r_0^2 - 6r_0X_0 - 2X_0^2\right), \quad f_2(X_0, r_0) = \frac{1}{2}r_0(X_0 + r_0).$$

The equation $\mathcal{F}(X_0, r_0) = 0$ has a unique non-trivial solution $X_0 = -2d$ and $r_0 = 2d$. In addition, we have $\det D\mathcal{F}(X_0, r_0)|_{X_0=-2d, r_0=2d} = d^2$. Consequently, it follows from Theorem 2.4.1 that for any given $d > 0$ and for $|\varepsilon| > 0$ sufficiently small system (2.4.22) has a periodic orbit $(X(\theta,\varepsilon), r(\theta,\varepsilon))$ of period 2π, such that $(X(0,\varepsilon), r(0,\varepsilon)) \to (-2d, 2d)$ as $\varepsilon \to 0$. See Figure 2.4.

Going back to system (2.4.19) we get that, for $c > 0$ small enough, the Michelson system has a periodic orbit of period 2π given by $x(t) = -2c\cos t + O(c), y(t) = 2c\sin t + O(c)$ and $z(t) = 2c\cos t + O(c)$.

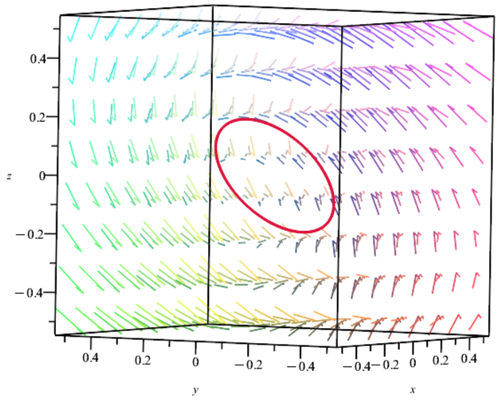

Figure 2.4: A periodic orbit of period 2π for the system 2.4.19 with $c = 0.1$

CHAPTER

3

Maximum Number of limit cycles of a second order differential system

3.1 Introduction and statement of the main results

In two-dimensional systems, we often encounter centers which are isolated equilibrium points surrounded by periodic orbits. These orbits follow circular or elliptical paths without spiraling toward or away from the equilibrium point. This behavior arises when the linearized system at the equilibrium point has purely imaginary eigenvalues.

One way to produce limit cycles is by perturbing a system that initially has a center. In the unperturbed system, the trajectories around the center are closed and periodic. However, by introducing a small perturbation, these periodic orbits may either shrink or expand, giving rise to a stable or unstable limit cycle.

The Mathieu differential equation [31]

$$\ddot{x} + b(1 + \cos(\theta))x = 0, \tag{3.1.1}$$

where b is a real constant, is the simplest mathematical model of an excited system depending on a parameter. This equation was first introduced and discussed in 1868 by Mathieu while studying the problem of vibrations of an elliptical drumhead. It has many applications in engineering and physics, see [41, 46, 5, 47].

The system form of the Mathieu differential equation is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -b(1+\cos(\theta))x. \end{aligned}$$

In [13], T. Chen and J. Llibre studied the limit cycles of the differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x-\varepsilon(1+\cos^m(\theta))Q(x,y), \end{aligned} \tag{3.1.2}$$

where $\varepsilon > 0$ sufficiently small, m is an arbitrary non negative integer, $Q(x, y)$ is a polynomial of degree $n \geq 1$ and $\theta = \arctan(y/x)$. They provided an upper bound of the maximum number of limit cycles that can bifurcate from the linear center $\dot{x} = y$, $\dot{y} = -x$ of the system (3.1.2).

In this chapter, using the averaging theory of first order, we study the maximum number of limit cycles of the following differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x-\varepsilon(1+\sin^n(\theta)\cos^m(\theta))H(x,y), \end{aligned} \tag{3.1.3}$$

where $\varepsilon > 0$ is a small parameter, m and n are arbitrary non negative integers, $H(x, y)$ is a polynomial of degree $l \geq 1$ and $\theta = \arctan(y/x)$.

The following theorem presents the main result.

Theorem 3.1.1. *Using the averaging theory of first order, the maximum number of limit cycles of the differential system (3.1.3) that bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ is as follows*

(a) If l is even

(a.1) *We have at most $l - 1$ limit cycles, if m and n have different parity.*

(a.2) *We have at most $(l - 2)/2$ limit cycles, if m and n have the same parity.*

(b) If l is odd

(b.1) *We have at most l limit cycles, if m is even and n is odd.*

(b.2) *We have at most $(l - 1)/2$ limit cycles, if m is odd and n is even, or if m and n have the same parity.*

To prove Theorem 3.1.1, we shall use the following auxiliary theorem.

Theorem 3.1.2. *(Descartes Theorem) Let*

$$p(r) = a_{i_1} r^{i_1} + a_{i_2} r^{i_2} + \ldots + a_{i_n} r^{i_n}$$

be a polynomial with real coefficients, with $0 \leq i_1 < i_2 < \ldots < i_n$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \ldots, n\}$*. When* $a_{i_j} a_{i_{j+1}} < 0$*, we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m*, then* $p(r)$ *has at most* m *positive real zeros. Furthermore, we can choose the coefficients of* $p(r)$ *in such a way that* $p(r)$ *has exactly* $n-1$ *positive real zeros.*

Descartes theorem provides a useful tool for estimating the number of positive real roots of a polynomial by analyzing the sign changes in its coefficients. For a proof of this theorem, see [12].

3.2 Proof of the main results

Assume that the polynomial $H(x, y) = \sum_{i+j=0}^{l} a_{ij} x^i y^j$. By applying change of the variables (x, y) into the polar coordinates (r, θ) defined by $x = r\cos(\theta)$, $y = r\sin(\theta)$, with $r > 0$, the system (3.1.3) becomes

$$\begin{array}{rcl} \dot{r} & = & -\varepsilon \sum_{i+j=0}^{l} R_{ij}(\theta) r^{i+j}, \\ \dot{\theta} & = & -1 - \varepsilon \sum_{i+j=0}^{l} \Theta_{ij}(\theta) r^{i+j-1}, \end{array}$$

where

$$\begin{array}{rcl} R_{ij}(\theta) & = & a_{ij}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta)), \\ \Theta_{ij}(\theta) & = & a_{ij}(\cos^{i+1}(\theta)\sin^j(\theta) + \cos^{i+m+1}(\theta)\sin^{j+n}(\theta)). \end{array}$$

From the previous differential system, we take θ as the new independent variable as follows

$$\frac{dr}{d\theta} = \varepsilon \sum_{i+j=0}^{n} R_{ij}(\theta) r^{i+j} + O(\varepsilon^2) = \varepsilon f(r,\theta) + O(\varepsilon^2). \tag{3.2.4}$$

Now, we compute the averaged function $\overline{f}$, associated to the equation (3.2.4), which is given by

$$\overline{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta)\, d\theta. \tag{3.2.5}$$

To compute the averaged function (3.2.5) , we shall use the following formulas

$$\int_0^{2\pi} \cos(\theta)^p sin(\theta)^{2q} d\theta = \frac{(2q-1)!!}{(2q+p)(2q+p-2)...(p+2)} \int_0^{2\pi} \cos(\theta)^p d\theta,$$

$$p \in \mathbb{R} \backslash \{-2,-4,...\}, q \in \mathbb{N}.$$

$$\int_0^{2\pi} \cos(\theta)^p \sin(\theta)^{2q+1} d\theta = 0,\ p \in \mathbb{R}\backslash\{-1,-3,...\}, \quad q \in \mathbb{N}.$$

$$\int_0^{2\pi} \cos(\theta)^{2l} d\theta = \frac{(2l-1)!!}{2^l l!} 2\pi, \quad l > 0.$$

$$\int_0^{2\pi} \cos(\theta)^{2l+1} d\theta = 0, \quad l \geq 0.$$

For more information on these integrals, see [50].

We distinguish two cases for the parity of l, each case has four subcases for the parity of m and n.

Case (1) Suppose that l is even, we have four subcases for studying $\overline{f}(r)$.

Subcase (1.1) If m is even and n is odd, we have

$$
\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=0}^{l} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=1}^{l+1} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=1}^{l+1} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{2q+2p=2}^{l} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=2}^{l} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{q+p=1}^{l/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{q+p=1}^{l/2} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta\\
&\quad+\frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p} r^{2q+2p}\int_0^{2\pi}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)d\theta\\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi\\
&\quad+\frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p} r^{2q+2p}\times\\
&\quad\frac{(2p+n)!!}{(2p+n+2q+m+1)(2p+n+2q+m-1)...(2q+m+2)}\frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!}2\pi\\
&= \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}\\
&\quad+\sum_{q+p=1}^{l/2} a_{2q,2p} r^{2q+2p}\times\\
&\quad\frac{(2p+n)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m+1)(2p+n+2q+m-1)...(2q+m+2)}\\
&= \sum_{k=1}^{l} A_k r^k.
\end{aligned}
$$

Subcase (1.2) If m and n are even, we have

$$
\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij} r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right.\\
&\quad\left.+\sum_{i+j=0}^{l} a_{ij} r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.
\end{aligned}
$$

$$+\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\Bigg]\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1}\cos^{i}(\theta)\sin^{2p}(\theta)$$

$$+\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\Bigg]\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^{2} q(\theta)\sin^{2p}(\theta)$$

$$+\sum_{2q+2p=1}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\Bigg]\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l} a_{2q,2p-1} r^{2q+2p-1}\cos^{2} q(\theta)\sin^{2p}(\theta)$$

$$+\sum_{2q+2p=2}^{l} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\Bigg]\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q}(\theta)\sin^{2p}(\theta)$$

$$+\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\Bigg]\,d\theta$$

$$=\frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\Bigg[\int_0^{2\pi}\cos^{2q}(\theta)\sin^{2p}(\theta)$$

$$+\int_0^{2\pi}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\Bigg]\,d\theta$$

$$=\frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\Bigg[\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^{q}q!}2\pi$$

$$+\frac{(2p+n-1)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!}2\pi\Bigg]$$

$$=\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\Bigg[\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}$$

$$+\frac{(2p+n-1)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\Bigg]$$

$$=\sum_{k=1}^{l} B_k r^{2k-1}.$$

Subcase (1.3) If m and n are odd, we have

$$\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left.+\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left.+\sum_{i+2p=0}^{l} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left.+\sum_{2q-1+2p=0}^{l} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left.+\sum_{2q+2p=2}^{l+1} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left.+\sum_{2q+2p=2}^{l} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{l/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left.+\sum_{q+p=1}^{l/2} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\right]d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} r^{2q+2p-1}\left[a_{2q,2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta\right.\\
&\left. a_{2q-1,p}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)d\theta\right]\\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi\right.\\
&+a_{2q-1,2p}\frac{(2p+n)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)}\\
&\left.\times\frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}2\pi\right]\\
&= \sum_{q+p=1}^{l/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}\right.\\
&+a_{2q-1,2p}\\
&\left.\times\frac{(2p+n)!!(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)}\right]\\
&= \sum_{k=1}^{l/2} C_k r^{2k-1}.
\end{aligned}$$

Subcase (1.4) If m is odd and n is even, we have

$$\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right.\\
&\quad\left.+\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.\\
&\quad\left.+\sum_{i+2p-1=0}^{l} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\right]d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&+ \sum_{i+2p=1}^{l+1} a_{i,2p} r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&+ \sum_{2q+2p=2}^{l} a_{2q-1,2p-1} r^{2q+2p-2}\cos^{2q-1+m}(\theta)\sin^{2p+n}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&+ \sum_{2q+2p=2}^{l} a_{2q-1,2p-1} r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&+ \sum_{q+p=1}^{l/2} a_{2q-1,2p-1} r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta) d\theta \\
&+ \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q-1,2p-1} r^{2q+2p-2}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta) d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi \\
&+ \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q-1,2p-1} r^{2q+2p-2} \\
&\times \frac{(2p+n-1)!!}{(2p+n+2q+m-1)(2p+n+2q+m-1-2)...(2q+m-1+2)} \\
&\times \frac{(2q+m-2)!!}{2^{q+m/2}(q+m/2)!}2\pi
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1} \frac{(2p-1)!!(2q-1)!!}{2^{p+q} q!(p+q)(p+q+1)...(q+1)} \\
&\quad + \sum_{q+p=1}^{l/2} a_{2q-1,2p-1} r^{2q+2p-2} \\
&\quad \times \frac{(2p+n-1)!!}{(2p+n+2q+m-1)(2p+n+2q+m-3)...(2q+m+1)} \\
&\quad \times \frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!} \\
&= \sum_{k=1}^{l} D_k r^k.
\end{aligned}
$$

Case (2) If l is odd, then we have four subcases for studying $\overline{f}(r)$.

Subcase (2.1) If m is even and n is odd, we have

$$
\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} a_{ij} r^{i+j} (\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta)) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^i(\theta)\sin^{j+1}(\theta) \right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^{i+m}(\theta)\sin^{j+n+1}(\theta) \right] d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta)\sin^{2p}(\theta) \right. \\
&\quad \left. + \sum_{i+2p=0}^{l} a_{i,2p} r^{i+2p} \cos^{i+m}(\theta)\sin^{2p+n+1}(\theta) \right] d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta)\sin^{2p}(\theta) \right. \\
&\quad \left. + \sum_{i+2p=1}^{l+1} a_{i,2p} r^{i+2p} \cos^{i+m}(\theta)\sin^{2p+n+1}(\theta) \right] d\theta
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=1}^{l+1} a_{2q,2p} r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=2}^{l+1} a_{2q,2p} r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{q+p=1}^{(l+1)/2} a_{2q,2p} r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta \\
&\quad + \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p} r^{2q+2p}\int_0^{2\pi}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi \\
&\quad + \frac{1}{2\pi}\sum_{q+p=1}^{(l+2)/2} a_{2q,2p} r^{2q+2p} \\
&\quad \times \frac{(2p+n)!!}{(2p+n+1+2q+m)(2p+n+1+2q+m-2)...(2q+m+2)} \\
&\quad \times \frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!}2\pi \\
&= \sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)} \\
&\quad + \sum_{q+p=1}^{(l+1)/2} a_{2q,2p}\, r^{2q+2p} \\
&\quad \times \frac{(2p+n)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m+1)(2p+n+2q+m-1)...(2q+m+2)} \\
&= \sum_{k=1}^{l+1} E_k r^k.
\end{aligned}$$

Subcase (2.2) If m and n are even, we have

$$\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=1}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}\, r^{2q+2p-1} \\
&\quad \times \int_0^{2\pi}\left[\cos^2 q(\theta)\sin^{2p}(\theta) + \cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\left[\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q\, q!}2\pi\right.\\
&\quad+\frac{(2p+n-1)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\\
&\quad\left.\times\frac{(2q+m-1)!!}{2^{q+m/2}\,(q+m/2)!}2\pi\right]\\
&= \sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}\, q!(p+q)(p+q+1)...(q+1)}\\
&\quad+\sum_{q+p=1}^{(l+1)/2} a_{2q,2p} r^{2q+2p}\\
&\quad\times\frac{(2p+n-1)!!(2q+m-1)!!}{2^{q+m/2}\,(q+m/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\\
&= \sum_{k=1}^{(l+1)/2} F_k r^{2k-1}.
\end{aligned}$$

Subcase (2.3) If m and n are odd, we have

$$\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right.\\
&\quad\left.+\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.\\
&\quad\left.+\sum_{2q-1+j=0}^{l} a_{2q-1,j}r^{2q-1+j}\cos^{2q-1+m}(\theta)\sin^{j+n+1}(\theta)\right]d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.\\
&\quad\left.+\sum_{2q+j=2}^{l+1} a_{2q-1,j}r^{2q+j-1}\cos^{2q+m-1}(\theta)\sin^{j+n+1}(\theta)\right]d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=2}^{l+1} a_{2q-1,2p} r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{q+p=1}^{(l+1)/2} a_{2q-1,2p} r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\Bigg] d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\Bigg[a_{2q,2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta \\
&\quad + a_{2q-1,p}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)d\theta\Bigg] \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\Bigg[a_{2q,2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi \\
&\quad + a_{2q-1,2p}\frac{(2p+n)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)} \\
&\quad \times \frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}2\pi\Bigg] \\
&= \sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\Bigg[a_{2q,2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)} + a_{2q-1,2p} \\
&\quad \times \frac{(2p+n)!!(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)}\Bigg] \\
&= \sum_{k=1}^{(l+1)/2} G_k r^{2k-1}.
\end{aligned}
$$

Subcase (2.4) If m is odd and n is even, we have

$$
\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \\
&\quad + \sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q-1+2p-1=0}^{l} a_{2q-1,2p-1}r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=2}^{l+1} a_{2q-1,2p-1}r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=2}^{l+1} a_{2q-1,2p-1}r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{q+p=1}^{(l+1)/2} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\Bigg]d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta\right.\\
&\left.+a_{2q-1,p-1}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)d\theta\right]\\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi\right.\\
&+a_{2q-1,2p-1}\frac{(2p+n-1)!!}{(2p+n+2q+m-1)(2p+n+2q+m-3)...(2q+m+1)}\\
&\left.\times\frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}2\pi\right]\\
&= \sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}+a_{2q-1,2p-1}\right.\\
&\times\frac{(2q+m-2)!!}{(2p+n+2q+m-1)(2p+n+2q+m-3)...(2q+m+1)}\\
&\left.\times\frac{(2p+n-1)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}\right]\\
&= \sum_{k=1}^{(l+1)/2} I_k r^{2k-1}.
\end{aligned}
$$

Having reached the result, we discuss the above cases. So, from the subcases (1.1), (1.4), and (2.1) we obtain that the averaged function $\overline{f}(r)$ is generated by a linear combination of a set $\zeta_1 = \{r, r^2, ..., r^p\}$ with $p \in \{l, l+1\}$. Using Descartes Theorem, it results that $\overline{f}(r)$ can have at most $l-1$ solutions if l is even while m and n have different parity. Also, it can have l solutions if l is odd while m is even and n is odd. Consequently, by Theorem 2.2.1, for ε small enough, the differential system (3.1.3) can have at most $l-1$ or l limit cycles. From the subcases (1.2), (1.3), (2.2), (2.3), and (2.4) we obtain that the averaged function $\overline{f}(r)$ is generated by a linear combination of a set $\zeta_2 = \{r, r^3, ..., r^p\}$ with $p \in \{l-1, l\}$. Using Descartes Theorem, it results that $\overline{f}(r)$ can have at most $(l-2)/2$ solutions if l is even while m and n have the same parity. Also, it can have $(l-1)/2$ solutions if l is odd while m is odd and n is even, or when m and n have the same parity. Similarly, by using the Theorem 2.2.1 and for ε small enough, the system (3.1.3) can have at most $(l-2)/2$ or $(l-1)/2$ limit cycles.

3.3 Applications

In this section, we present several numerical examples to illustrate the theoretical results discussed.

Example 3.3.1. *Consider the differential system (3.1.3)*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1 + \sin^n(\theta)\cos^m(\theta))H(x,y), \end{aligned} \tag{3.3.6}$$

where $n = 1,\ m = 2,$ *and*

$$H(x,y) = x^5y - x^3y^3 + \frac{16}{5}y^5 - \frac{5456}{175}y^4 + \frac{613}{420}y^2 + \frac{13}{560}y - \frac{1}{280}$$

a polynomial of degree $l = 6$. *According to Theorem 3.1.1, the system (3.3.6) can have at most* $l - 1 = 5$ *limit cycles.*

Transforming system (3.3.6) into polar coordinates ($x = r\cos(\theta)$ *and* $y = r\sin(\theta)$) *with*

$$\begin{aligned} \dot{r} &= \frac{1}{r}(x\dot{x} + y\dot{y}), \\ \dot{\theta} &= \frac{1}{r^2}(x\dot{y} - y\dot{x}), \end{aligned}$$

gives us

$$\begin{aligned} \dot{r} = -&\left(2\sin(\theta)\left(\cos(\theta)^9 - \frac{3}{2}\cos(\theta)^7 - (-\frac{1}{2} + \sin(\theta))cos(\theta)^5 + \frac{\sin(\theta)\cos(\theta)^3}{2}\right)r^6\right. \\ &+ 2\sin(\theta)\left(\frac{8}{5}\cos(\theta)^8 - \frac{24}{5}\cos(\theta)^6 - \frac{8}{5}(\sin(\theta) - 3)\cos(\theta)^4 + (\frac{16}{5}\sin(\theta) - \frac{8}{5})\cos(\theta)^2\right. \\ &\left.-\frac{8}{5}\sin(\theta)\right)r^5 + 2\sin(\theta)\left(\frac{2728}{175}\sin(\theta)\cos(\theta)^6 - \frac{8}{5}(\frac{682}{35}\sin(\theta) - \frac{341}{35})\cos(\theta)^4 + (\frac{2728}{175}\sin(\theta)\right. \\ &\left.-\frac{5456}{175})\cos(\theta)^2 + \frac{2728}{175}\right)r^4 + 2\sin(\theta)\left(-\frac{613}{840}\cos(\theta)^6 + \frac{613}{420}\cos(\theta)^4 + (\frac{613}{840}\sin(\theta)\right. \\ &\left.-\frac{613}{840})\cos(\theta)^2 - \frac{613}{840}\sin(\theta)\right)r^3 + 2\sin(\theta)\left(-\frac{37}{40}\sin(\theta)\cos(\theta)^4 + (\frac{37}{40}\sin(\theta) - \frac{37}{40})\cos(\theta)^2\right. \\ &\left.+\frac{37}{40}\right)r^2 + 2\sin(\theta)\left(\frac{13}{1120}\cos(\theta)^4 - \frac{13}{1120}\cos(\theta)^2 - \frac{13}{1120}\sin(\theta)\right)r \\ &\left.+2\sin(\theta)\left(\frac{\sin(\theta)\cos(\theta)^2}{560} + \frac{1}{560}\right)\right)\varepsilon, \end{aligned}$$

$$\begin{aligned}\dot{\theta} = -1 + \Bigg(&\left(2\cos(\theta)^{10} - 3\cos(\theta)^8 - 2(-\frac{1}{2} + \sin(\theta))\cos(\theta)^6 + \sin(\theta)\cos(\theta)^4\right) r^5 \\ &+ \left(\frac{16}{5}\cos(\theta)^9 - \frac{48}{5}\cos(\theta)^7 - \frac{16}{5}(\sin(\theta) - 3)\cos(\theta)^5 + \frac{32}{5}(-\frac{1}{2} + \sin(\theta))\cos(\theta)^3 \right. \\ &\left. - \frac{16}{5}\sin(\theta)\cos(\theta)\right) r^4 + \left(\frac{5456}{175}\sin(\theta)\cos(\theta)^7 - \frac{16}{5}(\frac{682}{35}\sin(\theta) - \frac{341}{35})\cos(\theta)^5 + \frac{32}{5}(\frac{341}{70}\sin(\theta) \right. \\ &\left. - \frac{341}{35})\cos(\theta)^3 + \frac{5456}{175}\cos(\theta)\right) r^3 + \left(-\frac{613}{420}\cos(\theta)^7 + \frac{613}{210}\cos(\theta)^5 + \frac{32}{5}(\frac{613}{2688}\sin(\theta) \right. \\ &\left. - \frac{613}{2688})\cos(\theta)^3 - \frac{613}{420}\sin(\theta)\cos(\theta)\right) r^2 + \left(-\frac{37}{20}\sin(\theta)\cos(\theta)^5 + \frac{32}{5}(\frac{37\sin(\theta)}{128} - \frac{37}{128})\cos(\theta)^3 \right. \\ &\left. + \frac{37}{20}\cos(\theta)\right) r + \frac{13}{560}\cos(\theta)^5 - \frac{13}{560}\cos(\theta)^3 - \frac{13}{560}\sin(\theta)\cos(\theta) + \left(\frac{\sin(\theta)\cos(\theta)^3}{280} + \frac{\cos(\theta)}{280}\right)\frac{1}{r}\Bigg)\varepsilon.\end{aligned}$$

Taking θ as an independent time variable,

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta},$$

we obtain an equation of the form

$$\frac{dr}{d\theta} = \epsilon F(r,\theta) + O(\epsilon^2),$$

where

$$\begin{aligned}F(r,\theta) = -2\sin(\theta)\Bigg(&(-\frac{24}{5}r^5 - \frac{613}{840}r^3 + \frac{2728}{175}\sin(\theta)r^4)\cos(\theta)^6 + \cos(\theta)^9 r^6 \\ &- \frac{8}{5}((r^4 + \frac{682}{35}r^3 + \frac{37}{64}r)\sin(\theta) - 3r^4 - \frac{341}{35}r^3 - \frac{613}{672}r^2 - \frac{13}{1792})\cos(\theta)^4 r \\ &+ \frac{1}{2}\sin(\theta)\cos(\theta)^3 r^6 + \frac{8}{5}\cos(\theta)^8 r^5 - \frac{3}{2}\cos(\theta)^7 r^6 + (-\frac{8}{5}r^5 - \frac{613}{840}r^3 - \frac{13}{1120}r)\sin(\theta) \\ &- (-\frac{1}{2} + \sin(\theta))r^6\cos(\theta)^5 + ((\frac{16}{5}r^5 + \frac{2728}{175}r^4 + \frac{613}{840}r^3 + \frac{37}{40}r^2 + \frac{1}{560})\sin(\theta) \\ &- \frac{8}{5}r(r^4 + \frac{682}{35}r^3 + \frac{613}{1344}r^2 + \frac{37}{64}r + \frac{13}{1792}))\cos(\theta)^2 + \frac{2728}{175}r^4 + \frac{37}{40}r^2 + \frac{1}{560}\Bigg).\end{aligned}$$

Now, we calculate the averaged function of $\overline{f}(r,\theta)$

$$\begin{aligned}\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta, \\ &= r^5 - \frac{341}{280}r^4 + \frac{613}{1120}r^3 - \frac{37}{320}r^2 + \frac{13}{1120}r - \frac{1}{2240}.\end{aligned}$$

The equation $\overline{f}(r) = 0$ has five positive real zeros

$$r_1 = \frac{1}{2},\ r_2 = \frac{1}{4},\ r_3 = \frac{1}{5},\ r_4 = \frac{1}{7}\ \textit{and}\ r_5 = \frac{1}{8}.$$

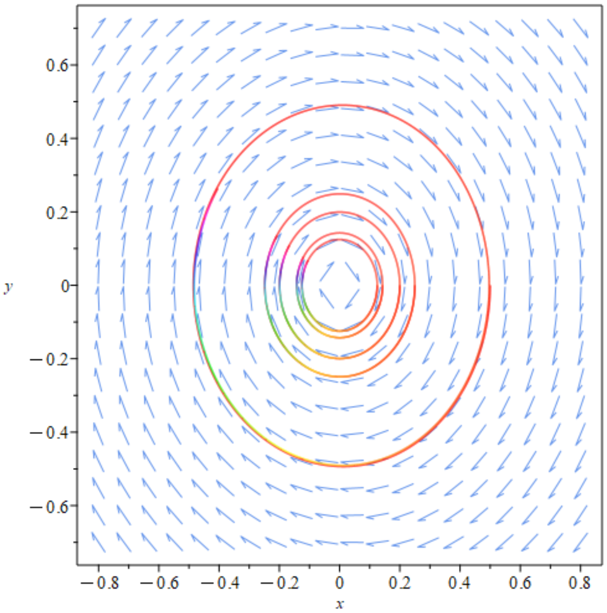

Figure 3.1: Limit cycles for system (3.3.6) with $\varepsilon = 0.005$

Furthermore, We have

$$\frac{d\overline{f}(r)}{dr} = 5r^4 - \frac{341}{70}r^3 + \frac{1839}{1120}r^2 - \frac{37}{160}r + \frac{13}{1120},$$

which yields

$$\frac{d\overline{f}(r_1)}{dr} = \frac{9}{896}, \quad \frac{d\overline{f}(r_2)}{dr} = -\frac{3}{17920}, \quad \frac{d\overline{f}(r_3)}{dr} = \frac{9}{14000}, \quad \frac{d\overline{f}(r_4)}{dr} = -\frac{3}{76832},$$

$$\text{and} \quad \frac{d\overline{f}(r_5)}{dr} = \frac{9}{143360}.$$

The derivatives of the averaged function $\overline{f}$ at the zeros r_1, r_2, r_3, r_4, and r_5 are all nonzero. Thus, by Theorem 2.2.1, the system (3.3.6) has exactly five limit cycles. In addition, two limit cycles which correspond to r_2 and r_4 are stable and the remaining ones are unstable. See Figures 3.1 and 3.2.

We now consider a similar system to (3.3.6) in which the parity of two parameters n and m is interchanged while the parity of the polynomial $H(x,y)$ of degree l remains the same. This system is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1 + \sin^2(\theta)\cos(\theta))H(x,y), \end{aligned} \tag{3.3.7}$$

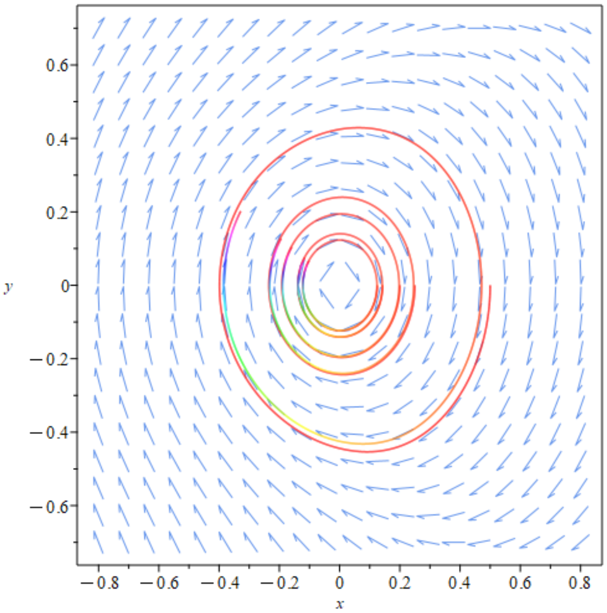

Figure 3.2: Limit cycles for system (3.3.6) with $\varepsilon = 0.05$

where

$$H(x,y) = x^6 + \frac{256}{3}x^5y - \frac{127}{15}y^5 + \frac{974}{15}xy^3 - \frac{409}{144}y^3 + \frac{35}{12}xy - \frac{1}{64}y + 1.$$

Since $n = 2$, $m = 1$ *and* $l = 6$, *then by Theorem 3.1.1, the system (3.3.7) can have at most* $l-1 = 5$ *limit cycles. Following the methodology of the previous computations, we shall skip the tedious computations for this system and straightforwardly present the averaged function, which is given by*

$$\overline{f}(r) = r^6 - \frac{127}{48}r^5 + \frac{487}{192}r^4 - \frac{409}{384}r^3 + \frac{35}{192}r^2 - \frac{1}{128}r,$$

The equation $\overline{f}(r) = 0$ *has five positive zeros*

$$r_1 = 1,\ r_2 = \frac{3}{4},\ r_3 = \frac{1}{2},\ r_4 = \frac{1}{3} \text{ and } r_5 = \frac{1}{16}.$$

Moreover, the derivatives of $\overline{f}$ *at the five zeros are all nonzero as shown below.*

$$\frac{d\overline{f}(r_1)}{dr} = \frac{5}{64},\ \frac{d\overline{f}(r_2)}{dr} = -\frac{55}{4096},\ \frac{d\overline{f}(r_3)}{dr} = \frac{7}{1536},\ \frac{d\overline{f}(r_4)}{dr} = -\frac{65}{15552},$$

$$\text{and}\quad \frac{d\overline{f}(r_5)}{dr} = \frac{5005}{1048576}.$$

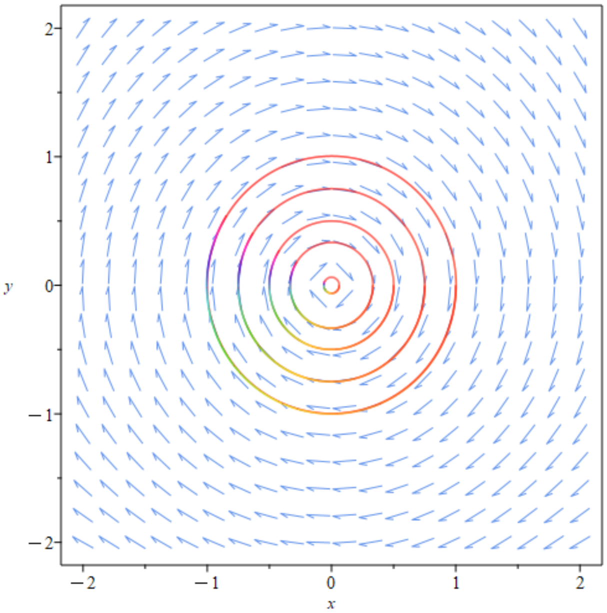

Figure 3.3: Limit cycles for system (3.3.7) with $\varepsilon = 0.0001$

Hence, by Theorem 2.2.1, the system (3.3.7) has exactly five limit cycles. Moreover, the limit cycles that correspond to r_2 *and* r_4 *are stable while the remaining ones are unstable. See Figures 3.3 and 3.4.*

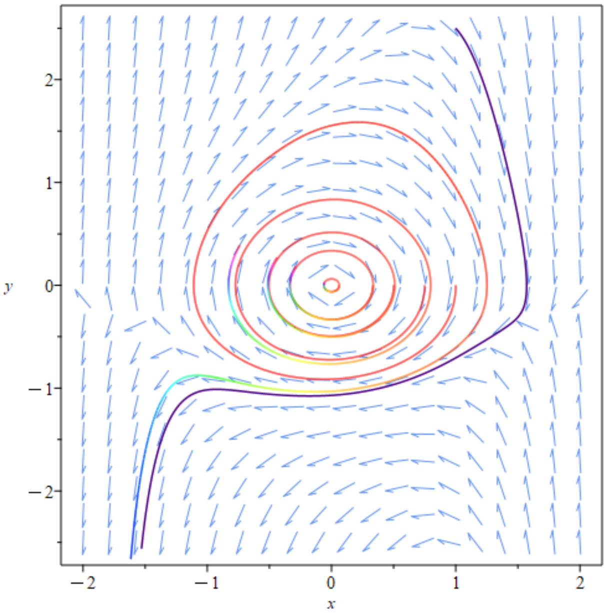

Figure 3.4: Limit cycles for system (3.3.7) with $\varepsilon = 0.004$

Example 3.3.2. *Consider the differential system (3.1.3)*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x-\varepsilon(1+\sin^n(\theta)\cos^m(\theta))H(x,y). \end{aligned} \tag{3.3.8}$$

where $n=2,\ m=2,$ *and*

$$H(x,y)=x^6+2y^5-x^2y-y^3+\frac{1}{16}y+1$$

a polynomial of degree $l=6$.

According to Theorem 3.1.1, the system (3.3.8) can have at most $(l-2)/2=2$ *limit cycles. Transforming system (3.3.8) into polar coordinates* ($x=r\cos(\theta)$ *and* $y=r\sin(\theta)$) *gives us*

$$\begin{aligned} \dot{r} = &-\Big((\cos(\theta)^4-\cos(\theta)^2-1)\cos(\theta)^6\sin(\theta)r^6+(\cos(\theta)^4-\cos(\theta)^2-1)(2\cos(\theta)^4-4\cos(\theta)^2 \\ &+2)\sin(\theta)^2r^5-(\cos(\theta)^4-\cos(\theta)^2-1)\sin(\theta)^2r^3+\frac{1}{16}((\cos(\theta)^4-\cos(\theta)^2-1)\sin(\theta)^2r) \\ &+(\cos(\theta)^4-\cos(\theta)^2-1)\sin(\theta)\Big)\,\varepsilon, \\ \dot{\theta} = &-1+((\cos(\theta)^{11}-\cos(\theta)^9-\cos(\theta)^7)r^5+(2\sin(\theta)\cos(\theta)^9-6\sin(\theta)\cos(\theta)^7+4\sin(\theta)\cos(\theta)^5 \\ &+2\sin(\theta)\cos(\theta)^3-2\sin(\theta)\cos(\theta))r^4+(-\sin(\theta)\cos(\theta)^5+\sin(\theta)\cos(\theta)^3+\sin(\theta)\cos(\theta))r^2 \\ &+\frac{1}{16}(\sin(\theta)\cos(\theta)^5-\sin(\theta)\cos(\theta)^3-sin(\theta)\cos(\theta))+(\cos(\theta)^5-\cos(\theta)^3-\cos(\theta))\frac{1}{r})\varepsilon. \end{aligned}$$

Taking θ *as an independent time variable, we get*

$$\frac{dr}{d\theta}=\epsilon F(r,\theta)+O(\epsilon^2),$$

where

$$\begin{aligned} F(r,\theta)= &-\sin(\theta)\Big((2\cos(\theta)^4r^5-4\cos(\theta)^2r^5+2r^5-r^3+\frac{1}{16}r)\sin(\theta) \\ &+r^6\cos(\theta)^6+1\Big)(\cos(\theta)^4-\cos(\theta)^2-1). \end{aligned}$$

Now, we calculate the averaged function of $\overline{f}(r,\theta)$

$$\begin{aligned} \overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi}F(r,\theta)d\theta, \\ &= \frac{87}{128}r^5-\frac{9}{16}r^3+\frac{9}{256}r. \end{aligned}$$

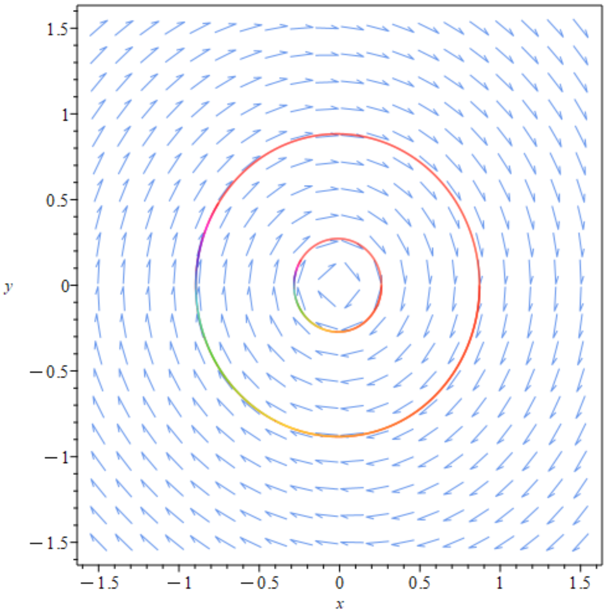

Figure 3.5: Limit cycles for system (3.3.8) with $\varepsilon = 0.01$

The equation $\overline{f}(r) = 0$ *has five zeros*

$$r_1 = 0,\ r_2 = \frac{\sqrt{1392 - 58\sqrt{402}}}{58},\ r_3 = -\frac{\sqrt{1392 - 58\sqrt{402}}}{58},\ r_4 = \frac{\sqrt{1392 + 58\sqrt{402}}}{58}$$

$$\text{and}\ \ r_5 = -\frac{\sqrt{1392 + 58\sqrt{402}}}{58}.$$

Since $r > 0$*, the equation* $\overline{f}(r) = 0$ *has only two positive real zeros* r_2 *and* r_4*.*

Computing the derivative of $\overline{f}$ *at* r_2 *and* r_4 *gives us*

$$\frac{d\overline{f}(r_2)}{dr} = \frac{603}{1856} - \frac{9\sqrt{402}}{464} \neq 0 \quad \text{and} \quad \frac{d\overline{f}(r_4)}{dr} = \frac{603}{1856} - \frac{9\sqrt{402}}{464} \neq 0.$$

Since $\frac{d\overline{f}(r_{2,3})}{dr} \neq 0$*, it follows from Theorem 2.2.1 that the system (3.3.8) have exactly two limit cycles. Furthermore, the limit cycle that corresponds to* r_2 *is stable while the one corresponding to* r_4 *is unstable. See Figures 3.5 and 3.6.*

We now consider a similar system to (3.3.8) in which the parity of the two parameters n *and* m *is even whereas the parity of the polynomial* $H(x, y)$ *of degree* l *remains odd. This system is given by*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1 + \sin(\theta)\cos(\theta))H(x, y), \end{aligned} \tag{3.3.9}$$

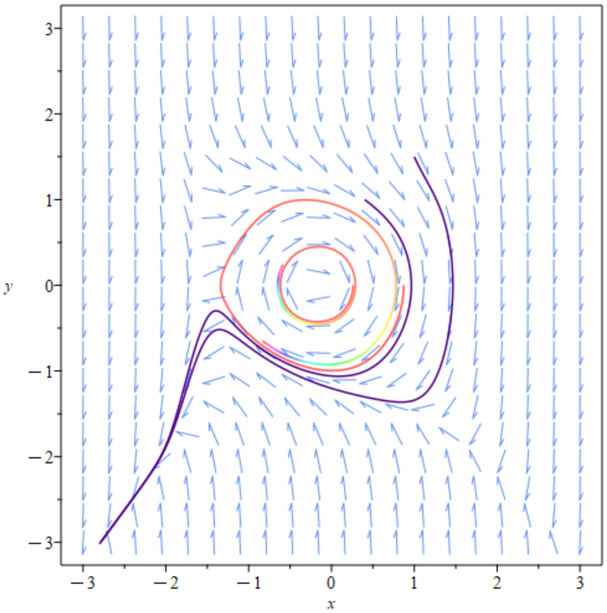

Figure 3.6: Limit cycles for system (3.3.8) with $\varepsilon = 0.15$

where

$$H(x,y) = x^6 + y^5 - y^3 + \frac{1}{8}y - 1.$$

Since $n = 1$, $m = 1$ *and* $l = 6$, *then by Theorem 3.1.1, the system (3.3.9) can have at most* $(l-2)/2 = 2$ *limit cycles.*

Doing the same computational steps as earlier, we straightforwardly present the computed averaged function

$$\overline{f}(r) = \frac{5}{16}r^5 - \frac{3}{8}r^3 + \frac{1}{16}r,$$

The equation $\overline{f}(r) = 0$ *has five real zeros*

$$r_1 = 0,\ r_2 = 1,\ r_3 = -1,\ r_4 = \frac{\sqrt{5}}{5} \text{ and } r_5 = -\frac{\sqrt{5}}{5}.$$

Given the requirement of positivity for r, *we shall only consider the two positive zeros* r_2 *and* r_4. *Moreover, the derivatives of* $\overline{f}$ *at* r_2 *and* r_4 *are given by* $\dfrac{d\overline{f}(r_2)}{dr} = \dfrac{1}{2} \neq 0$ *and* $\dfrac{d\overline{f}(r_4)}{dr} = -\dfrac{1}{10} \neq 0$ *respectively. Hence, by Theorem 2.2.1, the system (3.3.9) has exactly two limit cycles. In addition, the limit cycle that corresponds to* r_4 *is stable while the remaining one is unstable. See Figures 3.7 and 3.8.*

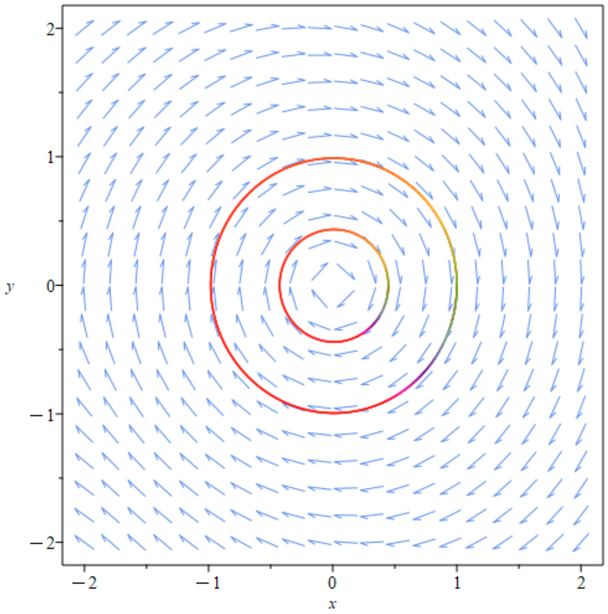

Figure 3.7: Limit cycles for system (3.3.9) with $\varepsilon = 0.01$

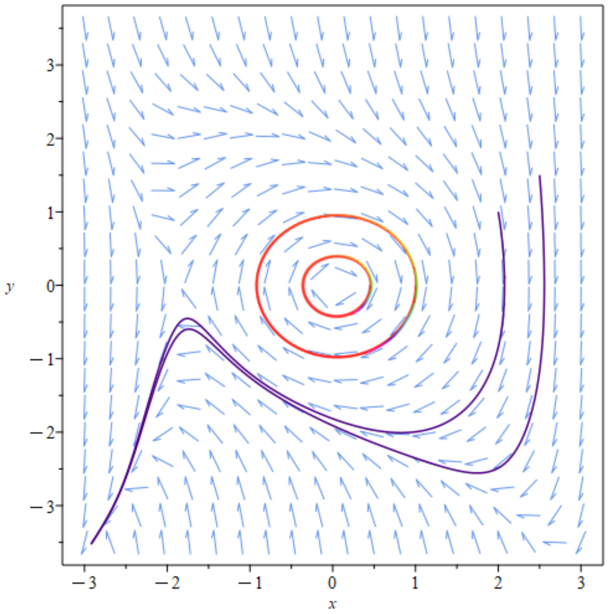

Figure 3.8: Limit cycles for system (3.3.9) with $\varepsilon = 0.05$

Example 3.3.3. *Consider the differential system (3.1.3)*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x-\varepsilon(1+\sin^n(\theta)\cos^m(\theta))H(x,y). \end{aligned} \tag{3.3.10}$$

where $n=3,\ m=2,$ *and*

$$H(x,y)=\frac{16}{5}y^5-\frac{5248}{15}x^4+\frac{4387}{270}y^3+y^2-\frac{4601}{27}x^2+\frac{499}{240}y-1$$

a polynomial of degree $l=5$.

According to Theorem 3.1.1, the system (3.3.10) can have at most $l=5$ *limit cycles.*

Transforming system (3.3.10) into polar coordinates ($x=r\cos(\theta)$ *and* $y=r\sin(\theta)$) *gives us*

$$\begin{aligned} \dot{r} = \Bigg(&\frac{16}{5}\sin(\theta)(\cos(\theta)^{10}-4\cos(\theta)^8+6\cos(\theta)^6+(\sin(\theta)-4)\cos(\theta)^4+\sin(\theta) \\ &+(-2\sin(\theta)+1)\cos(\theta)^2)r^5+\frac{5248}{5}\sin(\theta)(\sin(\theta)\cos(\theta)^8-\sin(\theta)\cos(\theta)^6-\cos(\theta)^4)r^4 \\ &+\frac{4387}{270}\sin(\theta)(-\cos(\theta)^8+3\cos(\theta)^6-3\cos(\theta)^4+(-\sin(\theta)+1)\cos(\theta)^2+\sin(\theta))r^3 \\ &+\sin(\theta)(\frac{4628}{27}\sin(\theta)\cos(\theta)^6-\frac{4655}{27}\sin(\theta)\cos(\theta)^4+((\sin(\theta))-\frac{4628}{27})\cos(\theta)^2+1)r^2 \\ &+\frac{499}{240}\sin(\theta)(\cos(\theta)^6-\cos(\theta)^4+\cos(\theta)^2+\sin(\theta))r \\ &+\sin(\theta)(\sin(\theta)\cos(\theta)^4-\sin(\theta)\cos(\theta)^2-1)\Bigg)\varepsilon, \end{aligned}$$

$$\begin{aligned} \dot{\theta} = -1+\Bigg(&(-\frac{16}{5}\cos(\theta)^{11}+4\cos(\theta)^9-6\cos(\theta)^7-(\sin(\theta)-4)\cos(\theta)^5-\sin(\theta)\cos(\theta) \\ &+2(\sin(\theta)-\frac{1}{2})\cos(\theta)^3)r^4+\frac{5248}{15}\left(-cos(\theta)^9\sin(\theta)+\cos(\theta)^7\sin(\theta)+\cos(\theta)^5\right)r^3 \\ &+\frac{4387}{270}\left(\cos(\theta)^9-3\cos(\theta)^7+3\cos(\theta)^5+(\sin(\theta)-1)\cos(\theta)^3-\sin(\theta)\cos(\theta)\right)r^2 \\ &+(-\frac{4628}{27}\cos(\theta)^7\sin(\theta)+\frac{4655}{27}\cos(\theta)^5\sin(\theta)+(-\sin(\theta)+\frac{4628}{27})\cos(\theta)^3-\cos(\theta))r \\ &+\frac{499}{240}(-\cos(\theta)^7+2\cos(\theta)^5-\cos(\theta)^3-\sin(\theta)\cos(\theta)) \\ &+(-\cos(\theta)^5\sin(\theta)+\sin(\theta)\cos(\theta)^3+\cos(\theta))\frac{1}{r}\Bigg)\varepsilon. \end{aligned}$$

Taking θ as an independent time variable, we get

$$\frac{dr}{d\theta} = \epsilon F(r,\theta) + O(\epsilon^2),$$

where

$$\begin{aligned}
F(r,\theta) = \frac{16}{5}\sin(\theta)\Bigg(& \cos(\theta)^{10} r^5 + \Big(\frac{328}{3}\sin(\theta) r^4 - 4r^5 - \frac{4387}{864} r^3\Big)\cos(\theta)^8 + \Big((-\frac{328}{3} r^4 + \frac{5785}{108} r^2)\sin(\theta) \\
& + 6r^5 + \frac{4387}{288} r^3 + \frac{499}{768} r\Big)\cos(\theta)^6 + \Big(-4r^5 - \frac{4387}{288} r^3 - \frac{499}{384} r - \frac{328}{3} r^4\Big)\cos(\theta)^4 \\
& + \Big((-2r^5 - \frac{4387}{864} r^3 + \frac{5}{16} r^2 - \frac{5}{16})\sin(\theta) + r(r^4 + \frac{4387}{864} r^2 + \frac{499}{768} - \frac{5785}{108} r)\Big)\cos(\theta)^2 \\
& + \Big(r^5 - \frac{23275}{432} r^2 + \frac{5}{16}\Big)\sin(\theta)\cos(\theta)^4 + \Big(r^5 + \frac{4387}{864} r^3 + \frac{499}{768} r\Big)\sin(\theta) + \frac{5}{16} r^2 - \frac{5}{16}\Bigg).
\end{aligned}$$

Now, we calculate the averaged function of $\overline{f}(r,\theta)$

$$\begin{aligned}
\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta, \\
&= r^5 - \frac{41}{10} r^4 + \frac{4387}{720} r^3 - \frac{1139}{288} r^2 + \frac{499}{480} r - \frac{1}{16}.
\end{aligned}$$

The equation $\overline{f}(r) = 0$ has five positive real zeros

$$r_1 = \frac{3}{2},\ r_2 = \frac{5}{4},\ r_3 = \frac{2}{3},\ r_4 = \frac{3}{5} \text{ and } r_5 = \frac{1}{12}.$$

Evaluating the derivative of $\overline{f}$ at the five zeros yields

$$\frac{d\overline{f}(r_1)}{dr} = \frac{17}{64},\quad \frac{d\overline{f}(r_2)}{dr} = -\frac{637}{5760},\quad \frac{d\overline{f}(r_3)}{dr} = \frac{49}{2592},\quad \frac{d\overline{f}(r_4)}{dr} = -\frac{403}{20000},$$

$$\text{and}\quad \frac{d\overline{f}(r_5)}{dr} = \frac{25823}{51840}.$$

The derivative of $\overline{f}$ at the five zeros are all nonzero. Thus, by Theorem 2.2.1, the system (3.3.10) has exactly five limit cycles. Moreover, the limit cycles corresponding to r_2 and r_4 are stable, while the remaining ones are unstable. See Figures 3.9 and 3.10.

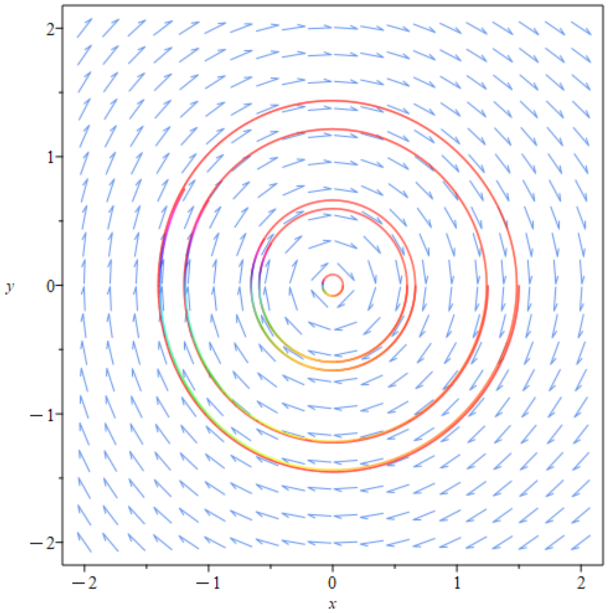

Figure 3.9: Limit cycles of system (3.3.10) with $\varepsilon = 0.0001$

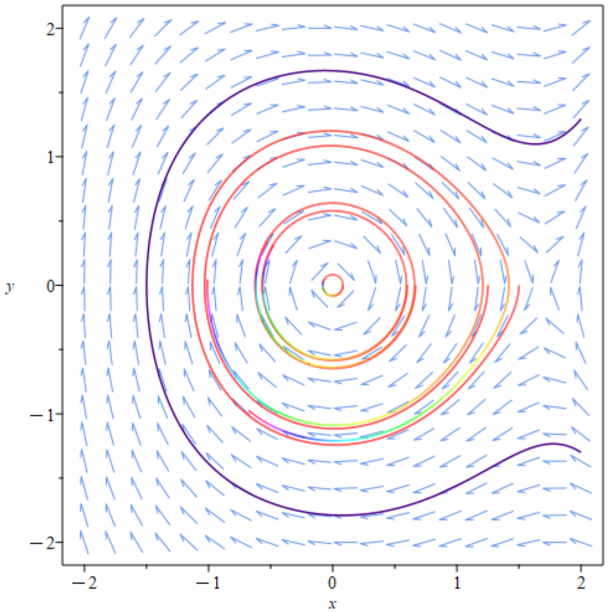

Figure 3.10: Limit cycles of system (3.3.10) with $\varepsilon = 0.0005$

We now consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1+\sin^2(\theta)\cos(\theta))H(x,y), \end{aligned} \tag{3.3.11}$$

where

$$H(x,y) = a_{0,7}y^7 + a_{1,5}xy^5 + a_{0,5}y^5 + a_{1,3}xy^3 + a_{2,1}x^2y + a_{1,1}xy + a_{0,1}y + a_{0,0}.$$

Since, $n = 2$, $m = 1$ and $l = 7$, then by Theorem 3.1.1, the system (3.3.11) can have at most $l = 7$ limit cycles.

Doing the same computational steps as earlier, we straightforwardly present the computed averaged function

$$\overline{f}(r) = \frac{35}{128}a_{0,7}r^7 + \frac{7}{256}a_{1,5}r^6 + \frac{5}{16}a_{0,5}r^5 + \frac{5}{128}a_{1,3}r^4 + \frac{1}{8}a_{2,1}r^3 + \frac{1}{16}r^2a_{1,1} + \frac{1}{2}a_{0,1}r.$$

Using Descartes Theorem 3.1.2 the equation $\overline{f}(r) = 0$ can have at most 6 positive real zeros, note that $r = 0$ is also a zero for the equation.

For the following choice of coefficients

$$\left\{a_{0,1} = \frac{1}{945},\ a_{0,5} = \frac{33602}{4725}, a_{0,7} = \frac{128}{35},\ a_{1,1} = \frac{-82}{315},\ a_{1,3} = \frac{-491}{21},\ a_{1,5} = \frac{-198848}{2205},\ a_{2,1} = \frac{275}{189}\right\}. \tag{3.3.12}$$

The equation $\overline{f}(r) = 0$ has exactly six positive real zeros

$$r_1 = 1,\ r_2 = \frac{1}{2},\ r_3 = \frac{4}{7},\ r_4 = \frac{1}{5},\ r_5 = \frac{1}{9},\ \textit{and}\ r_5 = \frac{1}{12}.$$

Furthermore, the derivative of $\overline{f}$ at the six zeros are given by

$$\begin{aligned} &\frac{d\overline{f}(r_1)}{dr} = \frac{44}{315},\quad \frac{d\overline{f}(r_2)}{dr} = \frac{1}{1152},\quad \frac{d\overline{f}(r_3)}{dr} = -\frac{15457}{10588410},\quad \frac{d\overline{f}(r_4)}{dr} = -\frac{26}{140625}, \\ &\frac{d\overline{f}(r_5)}{dr} = \frac{116}{2657205},\quad \textit{and}\quad \frac{d\overline{f}(r_6)}{dr} = -\frac{451}{8957952}. \end{aligned}$$

Hence, the system (3.3.11) with the choice of coefficients (3.3.12) has six limit cycles, three of which correspond to r_3, r_4, and r_6 are stable, whereas the remaining ones are unstable. See Figures 3.11 and 3.12.

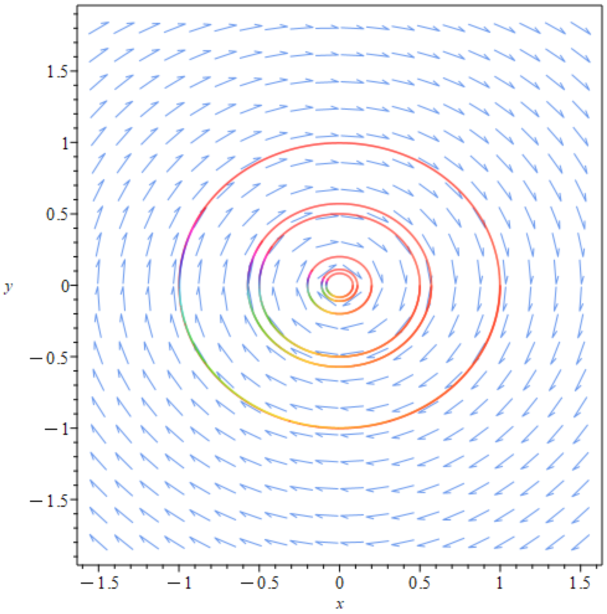

Figure 3.11: Limit cycles for system (3.3.11) with $\varepsilon = 0.0001$

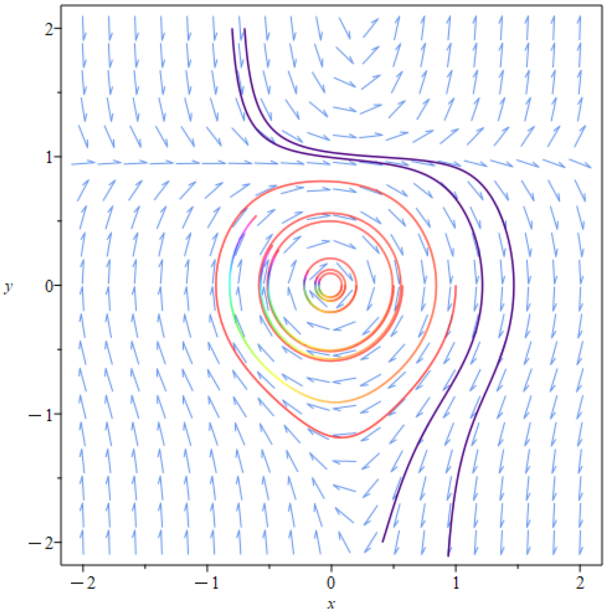

Figure 3.12: Limit cycles for system (3.3.11) with $\varepsilon = 0.01$

Example 3.3.4. *Consider the differential system (3.1.3)*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1 + \sin^n(\theta)\cos^m(\theta))H(x,y). \end{aligned} \tag{3.3.13}$$

where $n = 1,\ m = 1,$ *and*

$$H(x,y) = a_{0,3}y^3 + a_{1,2}xy^2 + a_{2,1}x^2y + a_{3,0}x^3 + a_{0,2}y^2 + a_{1,1}xy + a_{2,0}x^2 + a_{0,1}y + a_{1,0}x + a_{0,0}$$

a general polynomial of degree $l = 3$*. According to Theorem 3.1.1, the system (3.3.13) can have at most* $(l-1)/2 = 1$ *limit cycle.*
Transforming system (3.3.13) into polar coordinates ($x = r\cos(\theta)$ *and* $y = r\sin(\theta)$) *gives us*

$$\begin{aligned} \dot{r} = -\Bigg(&\Big((a_{2,1} - a_{0,3})\cos(\theta)^5 - (-a_{1,2} + a_{3,0})\sin(\theta)\cos(\theta)^4 - (a_{2,1} + a_{3,0} - 2a_{0,3} - a_{1,2})\cos(\theta)^3 \\ &- (a_{2,1} - a_{0,3} + a_{1,2})\sin(\theta)\cos(\theta)^2 + (-a_{0,3} - a_{1,2})\cos(\theta) - a_{0,3}\sin(\theta)\Big)\sin(\theta)r^3 \\ &+ \Big(a_{1,1}\cos(\theta)^4 + (a_{0,2} - a_{2,0})\sin(\theta)\cos(\theta)^3 - (-a_{0,2} + a_{1,1} + a_{2,0})\cos(\theta)^2 \\ &+ (-a_{0,2} - a_{1,1})\sin(\theta)\cos(\theta) - a_{0,2}\Big)\sin(\theta)r^2 + \Big(a_{0,1}\cos(\theta)^3 - a_{1,0}\sin(\theta)\cos(\theta)^2 \\ &+ (-a_{0,1} - a_{1,0})\cos(\theta) - a_{0,1}\sin(\theta)\Big)\sin(\theta)r + \Big(-a_{0,0}\sin(\theta)\cos(\theta) - a_{0,0}\Big)\sin(\theta)\Bigg)\varepsilon, \end{aligned}$$

$$\begin{aligned} \dot{\theta} = -1 + \Bigg(&\Big((a_{2,1} - a_{0,3})\cos(\theta)^6 - (a_{3,0} - a_{1,2})\sin(\theta)\cos(\theta)^5 - (a_{2,1} + a_{3,0} - 2a_{0,3} - a_{1,2})\cos(\theta)^4 \\ &+ -(a_{2,1} - a_{0,3} + a_{1,2})\sin(\theta)\cos(\theta)^3 - (a_{0,3} + a_{1,2})\cos(\theta)^2 - a_{0,3}\sin(\theta)\cos(\theta)\Big)r^2 \\ &+ \Big(a_{1,1}\cos(\theta)^5 + (a_{0,2} - a_{2,0})\sin(\theta)\cos(\theta)^4 - (-a_{0,2} + a_{1,1} + a_{2,0})\cos(\theta)^3 \\ &- (a_{0,2} + a_{1,1})\sin(\theta)\cos(\theta)^2 - a_{0,2}\cos(\theta)\Big)r + a_{0,1}\cos(\theta)^4 - a_{1,0}\sin(\theta)\cos(\theta)^3 \\ &- (a_{0,1} + a_{1,0})\cos(\theta)^2 - a_{0,1}\sin(\theta)\cos(\theta) + \Big(-a_{0,0}\sin(\theta)\cos(\theta)^2 - a_{0,0}\cos(\theta)\Big)\frac{1}{r}\Bigg)\varepsilon. \end{aligned}$$

Taking θ *as an independent time variable, we get*

$$\frac{dr}{d\theta} = \epsilon F(r,\theta) + O(\epsilon^2),$$

where

$$\begin{aligned}F(r,\theta) = &-\sin(\theta)\Bigg(r^3(a_{2,1}-a_{0,3})\cos(\theta)^5-(r(a_{3,0}-a_{1,2})\sin(\theta)-a_{1,1})r^2\cos(\theta)^4-a_{0,3}r^3\sin(\theta)\\ &+(((-a_{0,2}-a_{1,1})r^2-a_{0,0})\sin(\theta)-((a_{0,3}+a_{1,2})r^2+a_{0,1}+a_{1,0})r)\cos(\theta)\\ &-(-r(a_{0,2}-a_{2,0})\sin(\theta)+(a_{2,1}+a_{3,0}-2a_{0,3}-a_{1,2})r^2-a_{0,1})r\cos(\theta)^3-a_{0,1}r\sin(\theta)\\ &-(((a_{2,1}-a_{0,3}+a_{1,2})r^2+a_{1,0})\sin(\theta)-r(a_{0,2}-a_{1,1}-a_{2,0}))r\cos(\theta)^2-a_{0,2}r^2-a_{0,0})\Bigg)\end{aligned}$$

Now, we calculate the averaged function of $\overline{f}(r,\theta)$

$$\begin{aligned}\overline{f}(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta,\\ &= \frac{1}{16}(6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0})r^3+\frac{1}{8}(4a_{0,1}+a_{1,0})r.\end{aligned}$$

The equation $\overline{f}(r)=0$ *has three zeros*

$$r_1=0,\ r_2=\frac{\sqrt{-2(6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0})(4a_{0,1}+a_{1,0})}}{6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0}},$$

$$\text{and}\ \ r_3=-\frac{\sqrt{-2(6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0})(4a_{0,1}+a_{1,0})}}{6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0}}.$$

If $6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0}>0$ *and* $4a_{0,1}+a_{1,0}<0$*, then* r_2 *is the only positive real zero for* $\overline{f}(r)=0$*. Conversely, if* $6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0}<0$ *and* $4a_{0,1}+a_{1,0}>0$ *then* r_3 *is the only positive real zero for* $\overline{f}(r)=0$*. Furthermore, evaluating the derivative of* $\overline{f}$ *at* r_2 *or* r_3 *gives us*

$$\frac{d\overline{f}(r_2)}{dr}=-(a_{0,1}+\frac{a_{1,0}}{4}),$$

Since $-(a_{0,1}+\frac{a_{1,0}}{4})=-\frac{1}{4}(4a_{0,1}+a_{1,0})\neq 0$*, it follows that* $\frac{d\overline{f}(r_2)}{dr}\neq 0$.

By Theorem 2.2.1, if $6a_{0,3}+a_{1,2}+2a_{2,1}+a_{3,0}$ *and* $4a_{0,1}+a_{1,0}$ *are nonzero real values having opposite signs, then the system (3.3.13) has one limit cycle.*

Furthermore, it is stable if $4a_{0,1}+a_{1,0}>0$ *and unstable if* $4a_{0,1}+a_{1,0}<0$*. That is, if* r_3 *is a zero for* $\overline{f}(r)=0$*, the corresponding limit cycle is stable, whereas if* r_2 *is the zero, the corresponding limit cycle is unstable.*

For the following choice of coefficients

$$a_{0,1}=1,\ a_{0,3}=-1,\ a_{2,1}=-6,\ a_{1,0}=0\ a_{1,2}=0,\ \ \text{and}\ \ a_{3,0}=0, \tag{3.3.14}$$

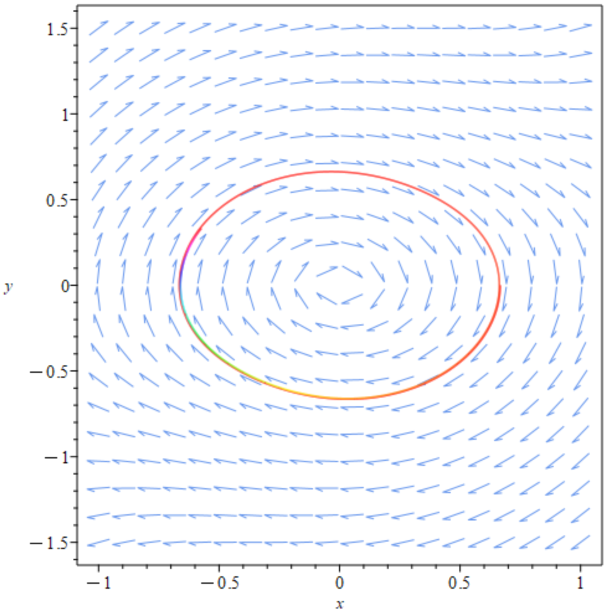

Figure 3.13: A stable Limit cycle of system (3.3.13) with coefficients (3.3.14) and $\varepsilon = 0.1$

the equation $\overline{f}(r) = 0$ *has only one positive zero* $r_3 = \dfrac{2}{3}$. *Also, we have* $\dfrac{d\overline{f}(r_3)}{dr} = -1$. *Hence, the system (3.3.13) has one stable limit cycle, see Figure 3.13.*

We now consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1+\sin^2(\theta)\cos^2(\theta))H(x,y), \end{aligned} \tag{3.3.15}$$

where

$$H(x,y) = a_{0,3}y^3 + a_{1,2}xy^2 + a_{2,1}x^2y + a_{3,0}x^3 + a_{0,2}y^2 + a_{1,1}xy + a_{2,0}x^2 + a_{0,1}y + a_{1,0}x + a_{0,0}.$$

Since $n = 2$, $m = 2$ *and* $l = 3$, *then by Theorem 3.1.1, the system (3.3.15) can have at most* $(l-1)/2 = 1$ *limit cycles.*

Doing the same computational steps as earlier, we straightforwardly present the computed averaged function

$$\overline{f}(r) = \frac{53a_{0,3} + 19a_{2,1}}{128}r^3 + \frac{9a_{0,1}}{16}r.$$

The equation $\overline{f}(r) = 0$ *has three zeros*

$$r_1 = 0,\ r_2 = \frac{6\sqrt{-2(53a_{0,3}+19a_{2,1})a_{0,1}}}{53a_{0,3}+19a_{2,1}} \ \text{and}\ r_3 = -\frac{6\sqrt{-2(53a_{0,3}+19a_{2,1})a_{0,1}}}{53a_{0,3}+19a_{2,1}}.$$

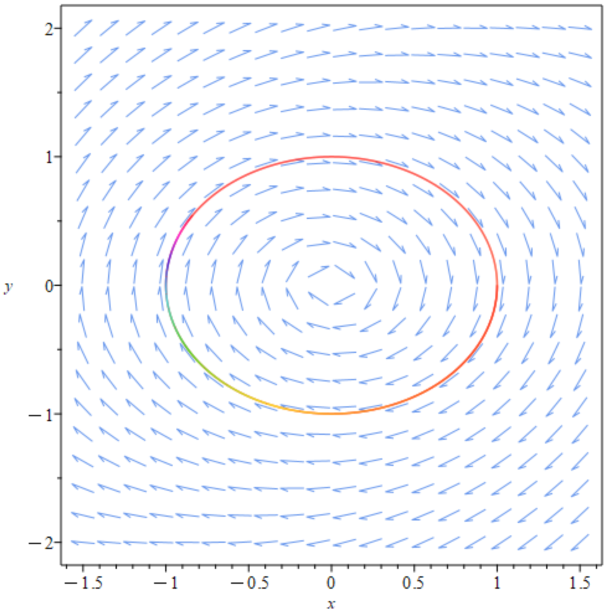

Figure 3.14: A stable limit cycle for system (3.3.15) with coefficients (3.3.16) and $\varepsilon = 0.1$

If $53a_{0,3} + 19a_{2,1} > 0$ *and* $a_{0,1} < 0$*, then* r_2 *is the only positive real zero for* $\overline{f}(r) = 0$*. Conversely, if* $53a_{0,3} + 19a_{2,1} < 0$ *and* $a_{0,1} > 0$ *then* r_3 *is the only positive real zero for* $\overline{f}(r) = 0$*. Furthermore, evaluating the derivative of* $\overline{f}$ *at* r_2 *or* r_3 *gives us*

$$\frac{d\overline{f}(r_{2,3})}{dr} = -\frac{9a_{0,1}}{8},$$

Since $a_{0,1} \neq 0$*, then we have* $\dfrac{d\overline{f}(r_{2,3})}{dr} \neq 0$*.*

By theorem 2.2.1, if $53a_{0,3} + 19a_{2,1}$ *and* $a_{0,1}$ *are nonzero real numbers having opposite signs then the system (3.3.15) has one limit cycle. Furthermore, it is stable if* $a_{0,1} > 0$ *and unstable if* $a_{0,1} < 0$*.*

For the following choice of coefficients

$$a_{0,1} = 1, \ a_{0,3} = -1, \ \ \textit{and} \ \ a_{2,1} = -1, \tag{3.3.16}$$

the equation $\overline{f}(r) = 0$ *has only one positive zero* $r^* = 1$*. Also, we have* $\dfrac{d\overline{f}(r^*)}{dr} = -\dfrac{9}{8} < 0$*. Hence, the system (3.3.15) has one stable limit cycle, see Figure 3.14.*

CHAPTER

4

Zero-Hopf bifurcation in a modified hyperchaotic Chen system

4.1 Introduction and statement of the main results

Chaos has been extensively studied in mathematics and physics since the discovery of the first chaotic attractor in a third-order autonomous system by Lorenz in 1963 [28]. This research led to the emergence of hyperchaos as a key area of interest, particularly after Otto Rössler [40] introduced one of the first hyperchaotic attractors in 1979.

A hyperchaotic system is typically defined as a chaotic system with at least two positive Lyapunov exponents, indicating that its chaotic dynamics unfold in multiple directions simultaneously. These systems are more complex than traditional chaotic systems. Due to their complexity, hyperchaos has found applications in enhancing security for chaotic communication systems, encryption, fluid mixing, nonlinear circuits, and other fields.

It is important to note that the minimal dimension for a hyperchaotic system is four. However, several well-known systems, such as the hyperchaotic Lorenz-Haken system, the hyperchaotic Rössler attractor, the hyperchaotic Lü system, and Chua's circuit, have been constructed by extending third-order chaotic systems, which is a common method for generating hyperchaotic systems.

In [22], the authors introduced the hyperchaotic Chen system

$$\begin{cases} \dot{x} &= a(y-x)+w, \\ \dot{y} &= dx+cy-xz, \\ \dot{z} &= xy-bz, \\ \dot{w} &= yz+rw, \end{cases} \tag{4.1.1}$$

where a,d,c,b and r are real arbitrary parameters. The dot denotes the derivative with respect to an independent variable t. Their study has led to significant advancements in various areas, including secure communications and power system protection, see [44, 33, 15]. The zero-Hopf bifurcation of this system was characterized in [32]. Other works studied zero-Hopf bifurcation of 3-dimensional systems, for instance [6, 16], but there is very little work done on n-dimensional ($n > 3$) zero-Hopf bifurcation due to the complexity related to higher dimensional problems.

In this chapter, we shall use the averaging method for studying the periodic orbits that bifurcate from the zero-Hopf of a modified 4-dimensional hyperchaotic Chen system. More precisely, we study the zero-Hopf bifurcation of the following 4-dimensional hyperchaotic Chen system

$$\begin{cases} \dot{x} &= \alpha(a_1y-a_2x+w), \\ \dot{y} &= dx+cy-xz, \\ \dot{z} &= xy-bz, \\ \dot{w} &= yz+rw, \end{cases} \tag{4.1.2}$$

where α,a_1,a_2,d,c,b and r are real arbitrary parameters.

Recall that a 4-dimensional zero-Hopf equilibrium refers to an equilibrium point where the eigenvalues consist of two zeros and a pair of purely imaginary conjugate numbers.

In what follows, we present the main results.

Proposition 4.1.1. *The hyperchaotic Chen system 4.1.2 has a zero-Hopf bifurcation if and only if $r=b=0$, and $\alpha(a_2^2\alpha+a_1d)<0$. In this case, the zero-Hopf equilibrium is the only singular point of the system localized at the origin.*

This proposition shows that, under certain conditions on the parameters, the modified hyperchaotic Chen system (4.1.2) has a zero-Hopf equilibrium point at the origin of the system.

The following theorem characterizes the conditions under which periodic orbits bifurcate from this zero-Hopf equilibrium point.

Theorem 4.1.1. *Let* $c = \alpha a_2$ *with* $\alpha > 0$, *and* $(r, b) \longrightarrow (\varepsilon r, \varepsilon b)$, *where* ϵ *is sufficiently small. If* $a_2 r b > 0$, *and* $\alpha(a_2^2\alpha + a_1 d) < 0$, *the modified* 4*-dimensional hyperchaotic system (4.1.2) exhibit two periodic orbits bifurcating from the zero-Hopf equilibrium as parameters vary.*

4.2 Proof of the main results

4.2.1 Proof of Proposition 4.1.1

First, we compute the equilibrium points of the modified hyperchaotic Chen system (4.1.2). It is straightforward to verify that system (4.1.2) has a unique equilibrium point at $(0, 0, 0, 0)$ for any choice of the parameters.

Linearizing the system at the origin yields the following characteristic polynomial

$$p(\lambda) = (b + \lambda)(r - \lambda)(-\lambda^2 + (c - \alpha a_2 c + \alpha a_1 d)). \tag{4.2.3}$$

The eigenvalues associated with the equilibrium point are

$$\lambda_1 = -b\,,\ \lambda_2 = r\,,\ \lambda_{3,4} = \frac{1}{2}\left(c - \alpha a_2 \pm \sqrt{\alpha^2 a_2^2 + c^2 + 2\alpha(a_2 c + 2a_1 d)}\right). \tag{4.2.4}$$

We will determine the values of the parameters for which $p(\lambda)$ has two eigenvalues equal to zero and a pair of purely imaginary roots, $\pm i\beta$, with $\beta > 0$. To achieve this, we impose

$$p(\lambda) = \lambda^2(\lambda^2 + \beta^2),$$

and we obtain that $r = 0 = b,\ c = \alpha a_2$ and $-\beta = \alpha(a_2^2\alpha + a_1 d) < 0$. Thus, the origin is the only zero-Hopf equilibrium of the system (4.1.2).

4.2.2 Proof of Theorem 4.1.1

Under the assumptions $c = \alpha a_2$, $\alpha(a_2^2\alpha + a_1 d) < 0$, and $(r, b) \longrightarrow (\varepsilon r, \varepsilon b)$ with $\varepsilon > 0$ being a sufficiently small parameter, the system (4.1.2) takes the form

$$\begin{cases} \dot{x} &= \alpha(a_1 y - a_2 x + w), \\ \dot{y} &= dx + \alpha a_2 y - xz, \\ \dot{z} &= xy - \varepsilon bz, \\ \dot{w} &= yz + \varepsilon rw. \end{cases} \tag{4.2.5}$$

Doing the re-scale of variables $(x, y, z, w) \longrightarrow (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$, the system (4.2.5) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \alpha(a_1 y - a_2 x + w) \\ dx + \alpha a_2 y \\ 0 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ -xz \\ xy - bz \\ yz + rw \end{pmatrix}. \tag{4.2.6}$$

System (4.2.6) is the normal form of system (2.4.15), as described in Section 2.4 of Chapter 2. First, we need to solve the unperturbed system of (4.2.6)

$$\begin{cases} \dot{x} &= \alpha(a_1 y - a_2 x + w), \\ \dot{y} &= dx + \alpha a_2 y, \\ \dot{z} &= 0, \\ \dot{w} &= 0, \end{cases} \tag{4.2.7}$$

with the initial condition $\mathbf{z} = (x(0), y(0), z(0), w(0)) = (x_0, y_0, z_0, w_0) \in \mathbb{R}^4$. After performing some computations, we find that the solution $\mathbf{x}(t, \mathbf{z}) = (x(t), y(t), z(t), w(t))$, which satisfies the initial condition $\mathbf{z} = (x_0, y_0, z_0, w_0)$, is given by

$$\begin{cases} x(t) &= (x_0 + \frac{\alpha^2 a_2}{\Omega^2} w_0) \cos(\Omega t) + (\frac{\alpha a_1}{\Omega} y_0 - \frac{\alpha a_2}{\Omega} x_0 + \frac{\alpha}{\Omega} w_0) \sin(\Omega t) - \frac{\alpha^2 a_2}{\Omega^2} w_0, \\ y(t) &= (y_0 - \frac{d\alpha}{\Omega^2} w_0) \cos(\Omega t) + (\frac{d}{\Omega} x_0 + \frac{\alpha a_2}{\Omega}) \sin(\Omega t) + \frac{d\alpha}{\Omega^2} w_0, \\ z(t) &= z_0, \\ w(t) &= w_0, \end{cases} \tag{4.2.8}$$

where $\Omega = \sqrt{-\alpha(a_2^2\alpha + a_1 d)}$ with $\alpha(a_2^2\alpha + a_1 d) < 0$. We note that all the solutions (4.2.8) of (4.2.7) are periodic of period $T = \dfrac{2\pi}{\Omega}$, which confirms that the system (4.2.6) is a perturbation of an isochronous system when $\alpha(a_2^2\alpha + a_1 d) < 0$. Thus, we can apply Theorem 2.4.1.

The fundamental matrix M of solution $\mathbf{x}(t, \mathbf{z})$ is

$$\begin{pmatrix} \cos(\Omega t) - \dfrac{\alpha a_2}{\Omega}\sin(\Omega t) & \dfrac{\alpha a_1}{\Omega}\sin(\Omega t) & 0 & \dfrac{1}{\Omega^2}(\alpha^2 a_2\cos(\Omega t) + \Omega\alpha\sin(\Omega t) - \alpha^2 a_2) \\ \dfrac{d}{\Omega}\sin(\Omega t) & \cos(\Omega t) + \dfrac{\alpha a_2}{\Omega}\sin(\Omega t) & 0 & \dfrac{d\alpha}{\Omega^2}(1 - \cos(\Omega t)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A simple method for calculating the inverse of a 4-dimensional matrix is to use it in block matrix form, i.e.

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are 2-dimensional matrices (note that C is a 2×2 zero matrix and D is the 2×2 identity matrix). So M can be inverted block-wise as follows

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix},$$

with $C = 0_{2,2}$ and $D = I_2$. Note that A is invertible and $D - CA^{-1}B = D$ which is also an invertible matrix.

Simplifying the previous formula gives us a simple matrix

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_2 \end{pmatrix}.$$

Hence, the matrix M^{-1} becomes

$$\begin{pmatrix} \cos(\Omega t) + \dfrac{\alpha a_2}{\Omega}\sin(\Omega t) & -\dfrac{\alpha a_1}{\Omega}\sin(\Omega t) & 0 & -\dfrac{\alpha^2 a_2}{\Omega^2}\cos(\Omega t) + \dfrac{\alpha}{\Omega}\sin(\Omega t) + \dfrac{\alpha^2 a_2}{\Omega^2} \\ -\dfrac{d}{\Omega}\sin(\Omega t) & \cos(\Omega t) - \dfrac{\alpha a_2}{\Omega}\sin(\Omega t) & 0 & \dfrac{d\alpha}{\Omega^2}(\cos(\Omega t) - 1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, we compute the bifurcation function (2.4.18), we obtain

$$\mathcal{F}(\mathbf{z}) = \frac{\Omega}{2\pi}\int_0^{\frac{2\pi}{\Omega}} M_{\mathbf{z}}^{-1}(t) f_1(t, x_{\mathbf{z}})\, dt = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4),$$

where

$$\begin{aligned}
\mathcal{F}_1(\mathbf{z}) &= \frac{1}{2}\left(\frac{d\alpha - \alpha^2 a_1 a_2}{\Omega^2} x_0 z_0 + \frac{\alpha^2 a_1^2}{\Omega^2} y_0 z_0 + \frac{\alpha^2 a_1}{\Omega^2} z_0 w_0 + \frac{3 d\alpha^3 a_2}{\Omega^4} z_0 w_0 + \frac{2\alpha^2 a_2}{\Omega^2} r w_0\right), \\
\mathcal{F}_2(\mathbf{z}) &= \frac{1}{2}\left(\frac{d\alpha a_1}{\Omega^2} x_0 z_0 + \frac{\alpha^2 a_1 a_2 + d\alpha}{\Omega^2} y_0 z_0 - 3\frac{d^2\alpha^2}{\Omega^4} z_0 w_0 - 2\frac{d\alpha r}{\Omega^2} w_0\right), \\
\mathcal{F}_3(\mathbf{z}) &= \frac{1}{2}\left(-2\frac{\alpha^2 a_2^2}{\Omega^2} x_0 y_0 + 2\frac{\alpha^2 a_2}{\Omega^2} y_0 w_0 - \frac{d\alpha a_2}{\Omega^2} x_0^2 + \frac{\alpha^2 a_1 a_2}{\Omega^2} y_0^2 - 3\frac{d\alpha^3 a_2}{\Omega^4} w_0^2 - 2b z_0\right), \\
\mathcal{F}_4(\mathbf{z}) &= \left(\frac{d\alpha}{\Omega^2} z_0 + r\right) w_0.
\end{aligned} \tag{4.2.9}$$

Solving the nonlinear system $\mathcal{F}_1(\mathbf{z}) = \mathcal{F}_2(\mathbf{z}) = \mathcal{F}_3(\mathbf{z}) = \mathcal{F}_4(\mathbf{z}) = 0$ we find two solutions

$$\zeta_{1,2} = \left(\mp\Omega\frac{\sqrt{a_2 r b}}{d}, \pm\Omega\frac{\sqrt{a_2 r b}}{\alpha a_2}, \frac{r(\alpha a_2^2 + d a_1)}{d}, \mp\frac{(\alpha a_2^2 + d a_1)\sqrt{\alpha a_2 r b}}{d\alpha a_2}\right),$$

where $\Omega = \sqrt{-\alpha(a_2^2\alpha + a_1 d)}$. Since $\alpha > 0$, the solutions $\zeta_{1,2}$ exists if $a_2 r b > 0$. Note that $\alpha(a_2^2\alpha + a_1 d) < 0$ implies that $d \neq 0$.

The determinant of the Jacobian matrix of $\mathcal{F}$ at the points $\zeta_{1,2}$ is

$$\det(D\mathcal{F}(\zeta_{1,2})) = \frac{b r^3 (d^2 - d\alpha a_1^3 - \alpha^2 a_1^2 a_2^2)}{2d^2}.$$

We have $d^2 - d\alpha a_1^3 - \alpha^2 a_1^2 a_2^2 = d^2 - a_1^2\alpha(a_2^2\alpha + a_1 d) > 0$. Furthermore, from $a_2 r b > 0$ we have that $br \neq 0$. Thus, we get that $\det(D\mathcal{F}(\zeta_{1,2})) \neq 0$ which means that $\zeta_{1,2}$ are simple zeros of $\mathcal{F}$. Consequently, by Theorem 2.4.1 the system (4.2.5) has two T-periodic solutions $\phi_{1,2}(t,\varepsilon)$ with period $\frac{2\pi}{\Omega}$ such that $\phi_{1,2}(0,\varepsilon) \longrightarrow \zeta_{1,2}$ as $\varepsilon \longrightarrow 0$.

Since we have applied the re-scaling $(x, y, z, w) \longrightarrow (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$ for obtaining system (4.2.5) from system (4.1.2), the solutions of system (4.2.5) yield periodic orbits $\varepsilon\phi_{1,2}(t,\varepsilon)$ for system (4.1.2), which tend to the zero-Hopf equilibrium as $\varepsilon \longrightarrow 0$.

Below, we present an example of a system (4.1.2) with two periodic solutions.

4.3 Example

Consider the system (4.2.5) with the following parameters

$$\alpha = 2 (\Rightarrow c = 2), a_1 = -2, a_2 = 2, d = 2, r = b = -1.$$

So, system (4.2.5) becomes

$$\begin{cases} \dot{x} &= 2(-2y - x + w), \\ \dot{y} &= 2x + 2y - xz, \\ \dot{z} &= xy + \epsilon z, \\ \dot{w} &= yz - \epsilon w. \end{cases} \tag{4.3.10}$$

Straightforward computations show that the bifurcation function $\mathcal{F}(x_0, y_0, z_0, w_0)$ is equal to

$$\begin{aligned} \mathcal{F}_1(\mathbf{z}) &= \frac{1}{2}(3x_0z_0 + 4y_0z_0 + z_0w_0 - 2w_0), \\ \mathcal{F}_2(\mathbf{z}) &= \frac{1}{2}(-2x_0z_0 - y_0z_0 - 3z_0w_0 + 2w_0), \\ \mathcal{F}_3(\mathbf{z}) &= \frac{1}{2}(-2x_0y_0 + 2y_0w_0 - x_0^2 - 2y_0^2 - 3w_0^2 + 2z_0), \\ \mathcal{F}_4(\mathbf{z}) &= (z_0 - 1)w_0. \end{aligned} \tag{4.3.11}$$

The system $\mathcal{F}(x_0, y_0, z_0, w_0) = 0$ has two solutions given by

$$\zeta_{1,2} = (\mp 1, \pm 1, 1, \pm 1).$$

Since the determinant of the Jacobian matrix of $\mathcal{F}$ at the points $\zeta_{1,2}$ is nonzero, Theorem 2.4.1 guarantees the existence of two T-periodic orbits $\phi_{1,2}(t, \varepsilon)$ of the system (4.3.10), such that $\phi_{1,2}(0, \varepsilon) \longrightarrow \zeta_{1,2}$ as $\varepsilon \longrightarrow 0$.

Since we have applied the re-scaling $(x, y, z, w) \longrightarrow (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$, returning back to the system (4.3.10) yields the periodic orbits $\varepsilon\phi_{1,2}(t, \varepsilon)$ for system (4.1.2), which tend to the zero-Hopf equilibrium as $\varepsilon \longrightarrow 0$.

CHAPTER 5

Three-dimensional zero-Hopf bifurcation in a cubic Kolmogorov system

5.1 Introduction

Kolmogorov systems, named after the mathematician Andrey Kolmogorov, are a class of differential equations used to model the dynamics of populations in biological systems or other systems with interacting components. In 1926, Alfred J. Lotka [29] and Vito Volterra [49] proposed these systems of degree 2 in the plane for studying the interactions between two species, and in 1936 Kolmogorov [20] extended them to any dimensions and degrees.

These systems can take various forms, but they generally involve a set of first-order ordinary differential equations that describe the evolution of multiple interacting variables over time.

The general form of a Kolmogorov system is:

$$\frac{dx_i}{dt} = x_i f_i(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n.$$

Where x_i represents the state variables (these variables are usually non-negative e.g., population sizes of different species), and f_i are functions that describe the interactions between the variables.

In ecological and biological contexts, Kolmogorov systems have been used to model various types of interactions, such as competition, predation, mutualism, and parasitism, where one species benefits at the expense of another. Also, they have been used to model other phenomena including hydrodynamics [11], chemical reactions [19], plasma dynamics, population economics [45], market dynamics, etc. It is worth noting that the region of ecological interest in 3-dimensional Kolmogorov systems is the first octant.

One of the main goals of the qualitative theory of differential systems is to analyze and characterize the behavior of systems through their periodic solutions. Periodic solutions, a key concept in this analysis, are closed trajectories representing stable or unstable periodic orbits in phase space. Determining the number of periodic solutions of a differential system is a critical and challenging process. It can provide profound insights into the system's oscillatory dynamics and responses to perturbations.

In [24], Llibre et al. considered the following Kolmogorov system of degree three in the space, that is,

$$\begin{aligned} \dot{x} &= -x(a_1(x-1)+a_2(y-1)+a_3(z-1)+a_4(x-1)^2+a_5(x-1)(y-1) \\ &\quad +a_6(x-1)(z-1)+a_7(y-1)^2+a_8(y-1)(z-1)+a_9(z-1)^2), \\ \dot{y} &= -y(b_1(x-1)+b_2(y-1)+b_3(z-1)+b_4(x-1)^2+b_5(x-1)(y-1) \\ &\quad +b_6(x-1)(z-1)+b_7(y-1)^2+b_8(y-1)(z-1)+b_9(z-1)^2), \\ \dot{z} &= -z(c_1(x-1)+c_2(y-1)+c_3(z-1)+c_4(x-1)^2+c_5(x-1)(y-1) \\ &\quad +c_6(x-1)(z-1)+c_7(y-1)^2+c_8(y-1)(z-1)+c_9(z-1)^2), \end{aligned} \tag{5.1.1}$$

where a_i,b_i, and c_i for $i = 1, .., 9$ are real parameters. Using the averaging method of first-order they have proved the existence of periodic solutions for the system (5.1.1) bifurcating from a zero-Hopf equilibrium point.

In their work, they provided sufficient conditions where the system showed the emergence of two periodic solutions bifurcating from the zero-Hopf equilibrium point $(1, 1, 1)$.

We recall that a zero-Hopf equilibrium point in a 3-dimensional system is an isolated equilibrium point having a pair of pure complex eigenvalues and the remaining eigenvalue is zero.

In this paper, we shall extend the analysis of the Kolmogorov system (5.1.1) to the second-order averaging method. The main goal is to determine the maximum number of periodic orbits that can bifurcate from the equilibrium point $(1, 1, 1)$ of the system (5.1.1) using the averaging theory of second order provided with some illustrative examples.

5.2 Main results

The following theorem is the main result of this paper on the zero-Hopf bifurcation of the system (5.1.1). For the sake of clarity, we set the following notations

$$\begin{aligned}
\alpha_1 &= a_1^2a_3b_1 - a_1^3b_3 + a_1(a_3b_1b_2 - b_3(2a_2b_1+\omega^2)) + b_1(a_2(a_3b_1-b_2b_3)+a_3(b_2^2+\omega^2)),\\
\alpha_2 &= -a_1^2a_2b_3 - a_2^2b_1b_3 - a_1a_2(-a_3b_1+b_2b_3) + a_3b_2(b_2^2+\omega^2) - a_2(-2a_3b_1b_2+b_3(b_2^2+\omega^2)),\\
\alpha_3 &= -a_3^2b_1 + a_3(a_1-b_2)b_3 + a_2b_3^2,\\
\alpha_4 &= -\omega(a_3^2b_2^2(b_2^2+\omega^2) + 2a_2a_3b_2(a_3b_1b_2 - b_3(a_1b_2+b_2^2+\omega^2)) + a_2^2(a_3^2b_1^2 - 2a_3b_1(a_1+b_2)b_3\\
&\quad + b_3^2(a_1^2+2a_1b_2+b_2^2+\omega^2))).
\end{aligned}$$

Theorem 5.2.1. *The following statements hold*

1. *By applying the averaging theory of first order, the system (5.1.1) has at most 2 periodic solutions bifurcating from the zero-Hopf equilibrium point* $(1,1,1)$ *when* $\varepsilon = 0$.

2. *By applying the averaging theory of second order, the system (5.1.1) has at most 5 periodic solutions bifurcating from the zero-Hopf equilibrium point* $(1,1,1)$ *when* $\varepsilon = 0$.

5.3 Proof of Theorem 5.2.1

Consider the Kolmogorov system (5.1.1). Obviously, the point $(1,1,1)$ is an equilibrium point for the system in the interior of the first octant in space.

By linearizing the system (5.1.1) at $(1,1,1)$ and calculating the characteristic polynomial P, we obtain

$$\begin{aligned}
P(\lambda) = {} & \lambda^3 - (-a_1-b_2-c_3)\lambda^2 - (-a_1b_2 - a_1c_3 + a_2b_1 + a_3c_1 - b_2c_3 + b_3c_2)\lambda \\
& + a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.
\end{aligned} \tag{5.3.2}$$

Imposing that $P(\lambda) = -\lambda(\lambda^2+\omega^2)$, we obtain

$$c_1 = \frac{\alpha_1}{\alpha_3}, \quad c_2 = \frac{\alpha_2}{\alpha_3}, \quad c_3 = -a_1 - b_2.$$

Under these conditions, the Jacobian matrix of (5.1.1) evaluated at $(1,1,1)$ has the eigenvalues 0 and $\pm i\omega$, with $\omega > 0$. It follows that $(1,1,1)$ is a zero-Hopf equilibrium point for the system (5.1.1).

Now, we assume that

$$c_1 = \frac{\alpha_1}{\alpha_3} + \varepsilon c_{11} + \varepsilon^2 c_{12}\ , \quad c_2 = \frac{\alpha_2}{\alpha_3} + \varepsilon c_{21} + \varepsilon^2 c_{22}\ , \quad c_3 = -a_1 - b_2 + \varepsilon c_{31} + \varepsilon^2 c_{32}, \quad (5.3.3)$$

where ε is a sufficiently small parameter.

Translating the equilibrium point $(1,1,1)$ to the origin by doing the change of variables $(x,y,z) \to (X,Y,Z)$, where $(X,Y,Z) = (x-1, y-1, z-1)$, the system (5.1.1) becomes

$$\begin{aligned}
\dot{x} &= -(X+1)(a_1X + a_2Y + a_3Z + a_4X^2 + a_5XY + a_6XZ + a_7Y^2 + a_8YZ + a_9Z^2,) \\
\dot{y} &= -(Y+1)(b_1X + b_2Y + b_3Z + b_4X^2 + b_5XY + b_6XZ + b_7Y^2 + b_8YZ + b_9Z^2), \\
\dot{z} &= -(Z+1)(c_1X + c_2Y + c_3Z + c_4X^2 + c_5XY + c_6XZ + c_7Y^2 + c_8YZ + c_9Z^2).
\end{aligned} \quad (5.3.4)$$

Now, we shall transform the linear part of the system (5.3.4) into its Jordan normal form J using the auxiliary matrix

$$P = \begin{pmatrix} \dfrac{\beta_1}{\gamma} & \dfrac{\beta_2}{\gamma} & \dfrac{a_2b_3 - a_3b_2}{a_1b_2 - a_2b_1} \\ \dfrac{\beta_3}{\gamma} & \dfrac{\beta_4}{\gamma} & -\dfrac{b_3a_1 - b_1a_3}{a_1b_2 - a_2b_1} \\ 1 & 0 & 1 \end{pmatrix}, \quad \text{such that } \ J = PAP^{-1},$$

where

$$\begin{aligned}
\beta_1 &= a_1a_2b_2b_3 - a_1a_3b_2^2 - a_1a_3\omega^2 - a_2^2b_1b_3 + a_2a_3b_1b_2 - a_2b_3\omega^2, \\
\beta_2 &= -a_1a_2b_3\omega + a_2a_3b_1\omega - a_2b_2b_3\omega + a_3b_2^2\omega + a_3\omega^3, \\
\beta_3 &= -a_1^2b_2b_3 + a_1a_2b_1b_3 + a_1a_3b_1b_2 - a_2a_3b_1^2 - a_3b_1\omega^2 - b_2b_3\omega^2, \\
\beta_4 &= a_1^2b_3\omega - a_1a_3b_1\omega + a_2b_1b_3\omega - a_3b_1b_2\omega + b_3\omega^3, \\
\gamma &= \omega^2a_1^2 + 2a_2b_1\omega^2 + \omega^2b_2^2 + \omega^4 + (a_1b_2 - a_2b_1)^2,
\end{aligned}$$

and A the Jacobian matrix of the system (5.3.4) with the eigenvalues $0, \pm i\omega$ is given by

$$A = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \\ -\dfrac{\alpha_1}{\alpha_3} & -\dfrac{\alpha_2}{\alpha_3} & a_1 + b_2 \end{pmatrix}.$$

Through the transformation $(x_1, y_1, z_1) = P(X, Y, Z)$, we obtain the new system in the variables (x_1, y_1, z_1) such that the zero-Hopf equilibrium point is located at the origin and its linear part is in Jordan normal form. We are unable to present this system due to its large expressions, which would require several pages to fully write out.

To prove the statements of Theorem 5.2.1, we want to write the new system into the normal form for applying the second-order averaging method. Therefore, we should take the following steps:

- Firstly, we transform the system into cylindrical coordinates $(x_1, y_1, z_1) \to (r\sin(\theta), r\cos(\theta), z_1)$ through the change of variables

  $$\dot{r} = \frac{x_1\dot{x_1} + y_1\dot{y_1}}{r}, \quad \dot{\theta} = \frac{x_1\dot{y_1} - y_1\dot{x_1}}{r^2}, \quad \dot{z_1} = \dot{z_1}.$$

  The transformed system will be in the variables (r, θ, z_1).

- Second, we do the rescaling of the variables $(r, z_1) \to (\varepsilon R, \varepsilon Z_1)$. Consequently, we obtain a system in the form

  $$\begin{aligned} \dot{R} &= H_{11}(R,\theta,Z_1)\varepsilon + H_{21}(R,\theta,Z_1)\varepsilon^2 + O(\varepsilon^3), \\ \dot{\theta} &= \omega + H_{12}(R,\theta,Z_1)\varepsilon + H_{22}(R,\theta,Z_1)\varepsilon^2 + O(\varepsilon^3), \\ \dot{Z_1} &= H_{13}(R,\theta,Z_1)\varepsilon + H_{23}(R,\theta,Z_1)\varepsilon^2 + O(\varepsilon^3), \end{aligned} \tag{5.3.5}$$

  where H_{ij} are 2π-periodic in the variable θ.

- Finally, we take θ as the new independent variable. As a result, we can write the system (5.3.5) in the variables (R, Z_1) as follows

  $$\begin{aligned} \frac{dR}{d\theta} &= \varepsilon f_{11}(R,\theta,Z_1) + \varepsilon^2 f_{21}(R,\theta,Z_1) + O(\varepsilon^3), \\ \frac{dZ_1}{d\theta} &= \varepsilon f_{12}(R,\theta,Z_1) + \varepsilon^2 f_{22}(R,\theta,Z_1) + O(\varepsilon^3). \end{aligned} \tag{5.3.6}$$

We take $x = (R, Z_1)$, $t = \theta$ and

$$f_1(t,x) = (f_{11}(R,\theta,Z_1), f_{12}(\theta,R,Z_1)),$$
$$f_2(t,x) = (f_{21}(R,\theta,Z_1), f_{22}(\theta,R,Z_1)).$$

Performing these steps in the system (5.3.4), we obtain a system of the form (5.3.6) where $f_1(t,x)$ and $f_2(t,x)$ are periodic functions of class C^1 with period $T = 2\pi$. It is important to note that we cannot provide the functions f_1 and f_2, as they are too extensive to present.

Now, applying the first-order averaging method, we obtain the first-order averaged function $\overline{f}_1 = (\overline{f}_{11}, \overline{f}_{12})$ that has the form

$$\overline{f}_{1i} = \frac{1}{2\pi}\int_0^{2\pi} f_{1i}(\theta,R,Z_1)d\theta,$$

where $i = 1, 2$. By doing these integrations, we get

$$\overline{f}_{11}(R,Z_1) = R(C + DZ_1),$$

and

$$\overline{f}_{12}(R,Z_1) = ER^2 - (2C + \frac{c_{31}}{\omega})Z_1 + FZ_1^2,$$

with the expressions C, D, E and F are in Appendix A.

The system of equations $\overline{f}_{11}(R,Z_1) = 0$, $\overline{f}_{12}(R,Z_1) = 0$ has the solutions

- $S_1 = (0, (2C + \frac{c_{31}}{\omega})/F)$ with $(2C + \frac{c_{31}}{\omega})F \neq 0$,
- $S_2 = (\sqrt{-\Big(C(D(2C + \frac{c_{31}}{\omega}) + CF)/E\Big)}/D, -C/D)$ with $C(D(2C + \frac{c_{31}}{\omega}) + CF)/E < 0$, $C \neq 0$ and $D > 0$.

Computing the Jacobian matrix of f, we obtain

$$\begin{pmatrix} C + DZ_1 & DR \\ 2ER & -(2C + \frac{c_{31}}{\omega}) + 2FZ_1 \end{pmatrix}.$$

Furthermore, computing the Jacobian determinant at solution S_1 gives us $(2C+\frac{c_{31}}{\omega})(D(2C+\frac{c_{31}}{\omega}) + CF)/F$, and at solution S_2 we get $2C(D(2C + \frac{c_{31}}{\omega}) + CF)/D$.

According to Theorem 2.2.1, if $(D(2C+\frac{c_{31}}{\omega})+CF)\neq 0$ then for $\varepsilon>0$ sufficiently small, the system (5.3.6) has two 2π-periodic solutions (limit cycles) $(R_1(\theta,\varepsilon),Z_{11}(\theta,\varepsilon))$ and $(R_2(\theta,\varepsilon),Z_{12}(\theta,\varepsilon))$ such that $(R_i(\theta,\varepsilon),Z_{1i}(\theta,\varepsilon))\to(R_i,Z_{1i})=S_i$ for $i=1,2$ when $\varepsilon\to 0$.

Moreover, the eigenvalues of the Jacobian matrix of f at S_1 are $\lambda_1=(2C+\frac{c_{31}}{\omega})$ and $\lambda_2=(D(2C+\frac{c_{31}}{\omega})+CF)/F$ and the eigenvalues of the Jacobian matrix of f at S_2 are

$$\lambda_{3,4}=\frac{-2C\left(D+F\right)\omega-Dc_{31}\pm\sqrt{\left(-12D^2\omega^2+4F^2\omega^2\right)C^2-4D\omega c_{31}\left(D-F\right)C+D^2c_{31}^2}}{2D\omega}.$$

If $\lambda_1<0$ and $\lambda_2<0$, then the periodic solution that corresponds to S_1 is stable. Otherwise, it is unstable.

Similarly, if $\lambda_3<0$ and $\lambda_4<0$ then the periodic solution that corresponds to S_2 is stable. Otherwise, it is unstable.

Going back through the changes of variables, we obtain two periodic solutions $(x_j(t,\varepsilon),y_j(t,\varepsilon),z_j(t,\varepsilon))$, for $j=1,2$, bifurcating from $(1,1,1)$ with a period tending to 2π when $\varepsilon\to 0$. Furthermore, $(x_j(t,\varepsilon),y_j(t,\varepsilon),z_j(t,\varepsilon))=(1,1,1)+O(\varepsilon)$ for $j=1,2$. This completes the proof of the statement **(a)** of Theorem 5.2.1. This case was studied in [24].

For proving statement **(b)** of Theorem 5.2.1, we use the second-order averaging method. So, we must annul the first-order averaged function $(f_{11}(R,Z_1),f_{12}(R,Z_1))$. For that reason, we take the following conditions

$$\begin{aligned}
a_5&=-\frac{a_1a_2-a_2a_6-a_2b_8-2a_2c_9+2a_3b_7+a_3c_8}{a_3},\\
c_4&=-\frac{1}{a_3^3b_1}(a_1^3a_9b_1+a_1^2a_2b_1b_9-a_1^2a_3a_6b_1+a_1^2a_3b_1c_9+a_1^2b_9\omega^2-a_1a_2a_3b_1b_6+2a_1a_2a_9b_1^2\\
&\quad+a_1a_3^2a_4b_1-a_1a_3^2b_1c_6-a_1a_3a_8b_1^2-a_1a_3b_6\omega^2+a_1a_9b_1\omega^2+2a_2^2b_1^2b_9+a_2a_3^2b_1b_4\\
&\quad-a_2a_3b_1^2b_8+2a_2a_3b_1^2c_9+3a_2b_1b_9\omega^2-a_3^2b_1^2c_8+a_3^2b_4\omega^2-a_3b_1b_8\omega^2+a_3b_1c_9\omega^2+b_9\omega^4),\\
c_7&=-\frac{1}{a_3^3b_1}(a_1a_2^2a_9b_1-a_1a_2a_3a_8b_1+a_1a_3^2a_7b_1+a_2^3b_1b_9-a_2^2a_3b_1b_8+a_2^2a_3b_1c_9+a_2^2b_9\omega^2\\
&\quad+a_2a_3^2b_1b_7-a_2a_3^2b_1c_8-a_2a_3b_8\omega^2+a_3^2b_7\omega^2),\\
b_2&=0,\quad b_3=0,\quad c_{21}=0,\quad c_{31}=0.
\end{aligned}$$

From the second-order averaging method, we have that $\overline{f}_2 = (\overline{f}_{21}, \overline{f}_{22})$ where

$$\overline{f}_2(R, Z_1) = \frac{1}{2\pi}\int_0^{2\pi}\left[D_{(R,Z_1)}f_1(\theta, R, Z_1)\cdot\int_0^{\theta} f_1(t, R, Z_1)\,dt + f_2(\theta, R, Z_1)\right]d\theta.$$

Computing these integrals, we obtain

$$\begin{aligned}\overline{f}_{21}(R, Z_1) &= G_1R^3 + G_2RZ^2 + G_3RZ + G_4R,\\ \overline{f}_{22}(R, Z_1) &= K_1R^3 + K_2R^2Z + K_3R^2 - G3Z^2 + K_4Z,\end{aligned}$$

where G_i and K_i, for $i = 1\cdots 4$ are given in Appendix A.

Assuming that $G_1 = \dfrac{1}{4G_2}, G_4 = \dfrac{1}{2G_2}, G_3 = -1$, and $K_3 = 0$, we solve the system $\overline{f}_{21}(R, Z_1) = 0$, $\overline{f}_{22}(R, Z_1) = 0$ and we obtain nine roots, namely,

$$\begin{aligned}&(R_1, Z_{11}) = (0, 0),\ (R_2, Z_{12}) = (\sqrt{2}, 0),\ (R_3, Z_{13}) = (-\sqrt{2}, 0),\\ &(R_4, Z_{14}) = (0, \frac{1+\sqrt{-4K_1K_4+1}}{2K_1}),\ (R_5, Z_{15}) = (0, \frac{1-\sqrt{-4K_1K_4+1}}{2K_1}),\\ &(R_6, Z_{16}) = \left(\sqrt{2}\frac{1}{|4G_2^2K_2+K_1|}\sqrt{-(A+BC^*)}, \frac{1}{8G_2^2K_2+2K_1}(-4G_2K_2+1+C^*)\right),\\ &(R_7, Z_{17}) = \left(-\sqrt{2}\frac{1}{|4G_2^2K_2+K_1|}\sqrt{-(A+BC^*)}, \frac{1}{8G_2^2K_2+2K_1}(-4G_2K_2+1+C^*)\right),\\ &(R_8, Z_{18}) = \left(\sqrt{2}\frac{1}{|4G_2^2K_2+K_1|}\sqrt{-(A-BC^*)}, \frac{1}{8G_2^2K_2+2K_1}(-4G_2K_2+1-C^*)\right),\\ &(R_9, Z_{19}) = \left(-\sqrt{2}\frac{1}{|4G_2^2K_2+K_1|}\sqrt{-(A-BC^*)}, \frac{1}{8G_2^2K_2+2K_1}(-4G_2K_2+1-C^*)\right),\end{aligned}$$

where

$$\begin{aligned}A &= 8G_2^4K_2K_4 + 4G_2^3K_2 + 2G_2^2K_1K_4 - G_2^2 - G_2K_1 - K_1^2,\\ B &= G_2K_1 + G_2^2,\\ C^* &= \sqrt{-16K_2(K_2+K_4)G_2^2 - 8G_2K_2 - (8K_2+4K_4)K_1 + 1},\end{aligned}$$

with $4G_2^2K_2 + K_1 \neq 0$ and $K_1 \neq 0$.

We take the non-negative roots $(R_2, Z_{12}), (R_4, Z_{14}), (R_5, Z_{15}), (R_6, Z_{16})$, and (R_8, Z_{18}), with the following conditions

- $-16K_2\left(K_2+K_4\right)G_2^2-8G_2K_2-\left(8K_2+4K_4\right)K_1+1>0$.
- $1-4K_1K_4>0$.
- $A+BC^*<0$ and $A-BC^*<0$.

The inequalities $A+BC^*<0$ and $A-BC^*<0$ hold simultaneously if and only if $\dfrac{A}{C^*}<B<-\dfrac{A}{C^*}$ with $A<0$.

Now, let $S=(R^*, Z_1^*)$ be a solution of the polynomial system $\overline{f}_{21}(R, Z_1)=0, \overline{f}_{22}(R, Z_1)=0$. According to the averaging method, to verify that the solution S is a periodic solution, we must have

$$det\begin{pmatrix}\dfrac{\partial f_{21}}{\partial R} & \dfrac{\partial f_{21}}{\partial Z_1}\\ \dfrac{\partial f_{22}}{\partial R} & \dfrac{\partial f_{22}}{\partial Z_1}\end{pmatrix}\Bigg|_{(R,Z_1)=(R^*,Z_1^*)} \neq 0.$$

Thus, by Theorem 2.3.2 the system 5.1.1 has at most five 2π-periodic solutions (limit cycles)
$(R_2(\theta,\varepsilon), Z_{12}(\theta,\varepsilon))$, $(R_4(\theta,\varepsilon), Z_{14}(\theta,\varepsilon))$, $(R_5(\theta,\varepsilon), Z_{15}(\theta,\varepsilon))$, $(R_6(\theta,\varepsilon), Z_{16}(\theta,\varepsilon))$, and
$(R_8(\theta,\varepsilon), Z_{18}(\theta,\varepsilon))$ such that $(R_k(\theta,\varepsilon), Z_{1k}(\theta,\varepsilon)) \to (R_k, Z_{1k})$, for $k=2,4,5,6,8$,
when $\varepsilon \to 0$.

Finally, going back through the changes of variables, we obtain the five periodic solutions $(x_j(t,\varepsilon), y_j(t,\varepsilon), z_j(t,\varepsilon))$, for $j=1,\ldots,5$, bifurcating from $(1,1,1)$ with a period tending to 2π when $\varepsilon \to 0$. Furthermore, $(x_j(t,\varepsilon), y_j(t,\varepsilon), z_j(t,\varepsilon)) = (1,1,1)+O(\varepsilon)$ for $j=1,\ldots,5$. This completes the proof of the statement **(b)** of Theorem 5.2.1.

5.4 Applications

Here, we provide two examples illustrating the results of Theorem 5.2.1. The first example corresponds to part **(a)** of the theorem, while the second corresponds to part **(b)**.

Example 5.4.1. *Consider the Kolmogorov system*

$$\begin{aligned}
\dot{x} \quad &= (x+1)\left(x - \frac{7}{4}y - \frac{1}{4}z - 2x^2 + 2xy + xz + y^2 - yz + z^2\right), \\
\dot{y} \quad &= -(y+1)(x^2 - xy + xz - \frac{1}{2}y^2 + 6yz - z^2 + x - y), \\
\dot{z} \quad &= -(z+1)((\varepsilon+1)x + (-\varepsilon-1)y + \varepsilon z + 3x^2 - 2xy + xz - y^2 + yz + 2z^2),
\end{aligned} \tag{5.4.7}$$

with ε *is sufficiently small. The singular point* $(1,1,1)$ *of the system (5.4.7) is a zero-Hopf equilibrium point meaning that the eigenvalues of the Jacobian matrix of (5.4.7) at* $(1,1,1)$ *are* $0, \pm i$.

Doing the same steps detailed in the proof of the statement ***(a)*** *of Theorem 5.2.1, that is, we begin by translating the point* $(1,1,1)$ *to the origin, and writing the Jacobian matrix at* $(1,1,1)$ *in Jordan normal form using the auxiliary matrix*

$$P = \begin{pmatrix} 1 & 1 & -\frac{1}{3} \\ 1 & 0 & -\frac{1}{3} \\ 1 & 0 & 1 \end{pmatrix},$$

such that $J = PAP^{-1}$, *where* A *is the Jacobian matrix of the system (5.4.7).*
Hence, we write the new system in the variables (x_1, y_1, z_1) *as follows*

$$\begin{aligned}
\dot{x_1} \quad &= \left(-\frac{1}{4}x_1^2 - \frac{1}{4}x_1y_1 - \frac{1}{2}z_1x_1 - \frac{1}{4}z_1y_1 - \frac{1}{4}z_1^2 - \frac{1}{4}x_1 - \frac{1}{4}y_1 - \frac{1}{4}z_1\right)\epsilon - \frac{41x_1^3}{8} - \frac{41x_1^2}{8} \\
&\quad -\frac{77z_1x_1^2}{24} - \frac{11y_1x_1^2}{4} - \frac{3x_1y_1^2}{2} - \frac{15x_1y_1}{4} - \frac{7z_1x_1y_1}{6} + \frac{13z_1^2x_1}{8} - \frac{43z_1x_1}{12} - \frac{3y_1^2}{2} \\
&\quad -\frac{z_1y_1^2}{2} - y_1 + \frac{z_1^2y_1}{4} - \frac{5z_1y_1}{12} - \frac{85z_1^3}{72} + \frac{53z_1^2}{24}, \\
\dot{y_1} \quad &= \frac{3}{2}z_1^3 + 2y_1^3 + 2y_1^2 + \frac{7}{2}x_1^3 + \frac{9}{2}x_1^2 - \frac{10}{3}z_1x_1y_1 + \frac{1}{2}z_1x_1^2 + y_1x_1^2 + 4x_1y_1^2 + 4x_1y_1 \\
&\quad -\frac{91}{18}z_1^2x_1 + \frac{4}{3}z_1x_1 - \frac{8}{3}z_1y_1^2 - \frac{7}{9}z_1^2y_1 - z_1y_1 - \frac{9}{2}z_1^2 + x_1, \\
\dot{z_1} \quad &= \left(-\frac{3}{4}x_1^2 - \frac{3}{4}x_1y_1 - \frac{3}{2}z_1x_1 - \frac{3}{4}z_1y_1 - \tfrac{3}{4}z_1^2 - \frac{3}{4}x_1 - \frac{3}{4}y_1 - \frac{3}{4}z_1\right)\epsilon + \frac{9x_1^3}{8} + \frac{9x_1^2}{8} \\
&\quad -\frac{49z_1x_1^2}{8} - \frac{9y_1x_1^2}{4} - \frac{3x_1y_1^2}{2} - \frac{9x_1y_1}{4} - \frac{7z_1x_1y_1}{2} - \frac{199z_1^2x_1}{24} - \frac{7z_1x_1}{4} - \frac{3y_1^2}{2} - \frac{5z_1y_1^2}{2} \\
&\quad +\frac{z_1^2y_1}{12} - \frac{z_1y_1}{4} - \frac{11z_1^3}{72} - \frac{85z_1^2}{24}.
\end{aligned} \tag{5.4.8}$$

Next, we transform the system (5.4.8) into cylindrical coordinates $(x_1, y_1, z_1) = (r\sin(\theta), r\cos(\theta), z_1)$, *we get*

$$
\begin{aligned}
\dot{r} = {} & \Bigg(\Bigg(-\frac{\cos(\theta)^3}{4} - \frac{\sin(\theta)\cos(\theta)^2}{4}\Bigg) r^2 + \Bigg(-\frac{\cos(\theta)^2}{2} - \frac{\cos(\theta)\sin(\theta)}{4}\Bigg) z_1 r + \Bigg(-\frac{\cos(\theta)^2}{4} \\
& -\frac{\cos(\theta)\sin(\theta)}{4}\Bigg) r - \frac{z_1^2\cos(\theta)}{4} - \frac{z_1\cos(\theta)}{4}\Bigg)\epsilon + \Bigg(-\frac{21\cos(\theta)^4}{8} - \frac{13\sin(\theta)\cos(\theta)^3}{4} - \frac{9\cos(\theta)^2}{2} \\
& +4\cos(\theta)\sin(\theta) + 2)\, r^3 + \Bigg(\frac{5\cos(\theta)^3}{8} + 2\sin(\theta)\cos(\theta)^2 - \frac{23\cos(\theta)}{6} - \frac{8\sin(\theta)}{3}\Bigg) z_1\, r^2 \\
& + \Bigg(-\frac{61\cos(\theta)^3}{8} - \frac{5\sin(\theta)\cos(\theta)^2}{4} + \frac{5\cos(\theta)}{2} + 2\sin(\theta)\Bigg) r^2 + \Bigg(\frac{173\cos(\theta)^2}{72} \\
& -\frac{173\cos(\theta)\sin(\theta)}{36} - \frac{7}{9}\Bigg) z_1^2 r + \Bigg(-\frac{31\cos(\theta)^2}{12} + \frac{11\cos(\theta)\sin(\theta)}{12} - 1\Bigg) z_1 r \\
& + \Bigg(-\frac{85\cos(\theta)}{72} + \frac{3\sin(\theta)}{2}\Bigg) z_1^3 + \Bigg(\frac{53\cos(\theta)}{24} - \frac{9\sin(\theta)}{2}\Bigg) z_1^2, \\
\dot{\theta} = {} & 1 + \Bigg(\Bigg(-\frac{\cos(\theta)^3}{4} + \frac{\sin(\theta)\cos(\theta)^2}{4} + \frac{\cos(\theta)}{4}\Bigg) r + \Bigg(-\frac{\cos(\theta)^2}{4} + \frac{\cos(\theta)\sin(\theta)}{2} + \frac{1}{4}\Bigg) z_1 \\
& -\frac{\cos(\theta)^2}{4} + \frac{\cos(\theta)\sin(\theta)}{4} + \frac{1}{4} + \frac{\sin(\theta)z_1^2}{4r} + \frac{\sin(\theta)z_1}{4r}\Bigg)\epsilon + \Bigg(-\frac{13\cos(\theta)^4}{4} + \frac{21\sin(\theta)\cos(\theta)^3}{8} \\
& +\frac{27\cos(\theta)^2}{4} + \frac{7\cos(\theta)\sin(\theta)}{2}\Bigg) r^2 + \Bigg(2\cos(\theta)^3 - \frac{5\sin(\theta)\cos(\theta)^2}{8} - \frac{3\cos(\theta)}{2} + \frac{\sin(\theta)}{2}\Bigg) z_1 r \\
& + \Bigg(-\frac{5\cos(\theta)^3}{4} + \frac{61\sin(\theta)\cos(\theta)^2}{8} + \frac{23\cos(\theta)}{4} + \frac{3\sin(\theta)}{2}\Bigg) r + \Bigg(-\frac{173\cos(\theta)^2}{36} \\
& -\frac{173\cos(\theta)\sin(\theta)}{72} - \frac{1}{4}\Bigg) z_1^2 + \Bigg(\frac{11\cos(\theta)^2}{12} + \frac{31\cos(\theta)\sin(\theta)}{12} + \frac{5}{12}\Bigg) z_1 \\
& + \frac{\left(\frac{3\cos(\theta)}{2} + \frac{85\sin(\theta)}{72}\right) z_1^3}{r} + \frac{\left(-\frac{9\cos(\theta)}{2} - \frac{53\sin(\theta)}{24}\right) z_1^2}{r}, \\
\dot{z_1} = {} & \Bigg(\Bigg(-\frac{3\cos(\theta)^2}{4} - \frac{3\cos(\theta)\sin(\theta)}{4}\Bigg) r^2 + \Bigg(-\frac{3\cos(\theta)}{2} - \frac{3\sin(\theta)}{4}\Bigg) z_1 r + \Bigg(-\frac{3\cos(\theta)}{4} \\
& -\frac{3\sin(\theta)}{4}\Bigg) r - \frac{3z_1^2}{4} - \frac{3z_1}{4}\Bigg)\epsilon + \Bigg(\frac{21\cos(\theta)^3}{8} - \frac{9\sin(\theta)\cos(\theta)^2}{4} - \frac{3\cos(\theta)}{2}\Bigg) r^3 + \Bigg(-\frac{29\cos(\theta)^2}{8} \\
& -\frac{7\cos(\theta)\sin(\theta)}{2} - \frac{5}{2}\Bigg) z_1\, r^2 + \Bigg(\frac{21\cos(\theta)^2}{8} - \frac{9\cos(\theta)\sin(\theta)}{4} - \frac{3}{2}\Bigg) r^2 + \Bigg(-\frac{199\cos(\theta)}{24} \\
& +\frac{\sin(\theta)}{12}\Bigg) z_1^2 r + \Bigg(-\frac{7\cos(\theta)}{4} - \frac{\sin(\theta)}{4}\Bigg) z_1 r - \frac{11z_1^3}{72} - \frac{85z_1^2}{24}.
\end{aligned}
$$

Following this, we do the re-scale of variables by $(r, z_1) \to (\varepsilon R, \varepsilon Z_1)$*, hence we obtain a system of the form*

$$\begin{aligned} \dot{R} &= H_{11}(R, \theta, Z_1)\varepsilon + O(\varepsilon^2), \\ \dot{\theta} &= 1 + O(\varepsilon), \\ \dot{Z}_1 &= H_{12}(R, \theta, Z_1)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{5.4.9}$$

where H_{11} *and* H_{12} *are smooth functions and* 2π*-periodic in the variable* θ*, given by*

$$\begin{aligned} H_{11}(R, \theta, Z_1) = &-\frac{21}{8}\left(\frac{61\cos(\theta)^3}{21} + \frac{10\sin(\theta)\cos(\theta)^2}{21} - \frac{20\cos(\theta)}{21} - \frac{16\sin(\theta)}{21}\right)R^2 \\ &-\frac{21}{8}\left(\frac{62\cos(\theta)^2}{63} - \frac{22}{63}\cos(\theta)\sin(\theta) + \frac{8}{21}\right)ZR - \frac{21}{8}\left(\frac{2\cos(\theta)^2}{21}\right. \\ &\left.+\frac{2\cos(\theta)\sin(\theta)}{21}\right)R - \frac{21}{8}\left(-\frac{53\cos(\theta)}{63} + \frac{12\sin(\theta)}{7}\right)Z^2 - \frac{Z\cos(\theta)}{4}, \\ H_{12}(R, \theta, Z_1) = &\frac{21}{8}\left(\cos(\theta)^2 - \frac{6\cos(\theta)\sin(\theta)}{7} - \frac{4}{7}\right)R^2 + \frac{21}{8}\left(-\frac{2\cos(\theta)}{3} - \frac{2\sin(\theta)}{21}\right)ZR \\ &+\frac{21}{8}\left(-\frac{2\cos(\theta)}{7} - \frac{2\sin(\theta)}{7}\right)R - \frac{85Z^2}{24} - \frac{3Z}{4}. \end{aligned}$$

Taking θ *as the new independent variable, we write*

$$\begin{aligned} \frac{dR}{d\theta} &= f_{11}(R, \theta, Z_1)\varepsilon + O(\varepsilon^2), \\ \frac{dZ_1}{d\theta} &= f_{12}(R, \theta, Z_1)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{5.4.10}$$

where f_{11} *and* f_{12} *are smooth functions and* 2π*-periodic in* θ*, given by*

$$\begin{aligned} f_{11}(R, \theta, Z_1) = &\left(-\frac{61\cos(\theta)^3}{8} - \frac{5\sin(\theta)\cos(\theta)^2}{4} + \frac{5\cos(\theta)}{2} + 2\sin(\theta)\right)R^2 - \left(\frac{31\cos(\theta)^2}{12}\right. \\ &\left.-\frac{11\cos(\theta)\sin(\theta)}{12} + 1\right)ZR + \left(-\frac{\cos(\theta)^2}{4} - \frac{\cos(\theta)\sin(\theta)}{4}\right)R \\ &+\left(\frac{53\cos(\theta)}{24} - \frac{9\sin(\theta)}{2}\right)Z^2 - \frac{Z\cos(\theta)}{4}, \end{aligned}$$

$$\begin{aligned} f_{12}(R, \theta, Z_1) = &\left(\frac{21\cos(\theta)^2}{8} - \frac{9\cos(\theta)\sin(\theta)}{4} - \frac{3}{2}\right)R^2 + \left(-\frac{7\cos(\theta)}{4} - \frac{\sin(\theta)}{4}\right)ZR \\ &+\left(-\frac{3\cos(\theta)}{4} - \frac{3\sin(\theta)}{4}\right)R - \frac{85Z^2}{24} - \frac{3Z}{4}. \end{aligned}$$

The system (5.4.10) is in normal form (2.4.15) with $T = 2\pi$, $\mathbf{x} = (R, Z_1)$, $t = \theta$, *and* $\varepsilon^2 g(t, \mathbf{x}, \varepsilon) = O(\varepsilon^2)$.

Applying the first-order averaging method, we obtain the first-order averaged function as follows

$$\overline{f}_{11}(R, Z_1) = \frac{1}{T}\int_0^T f_1(R, \theta, Z_1)\, d\theta = -\frac{55}{24} ZR - \frac{1}{8} R,$$
$$\overline{f}_{12}(R, Z_1) = \frac{1}{T}\int_0^T f_1 R, \theta, Z_1)\, d\theta = \frac{3}{16} R^2 - \frac{85}{24} Z^2 - \frac{3}{4} Z.$$

The system of equations $\overline{f}_{11}(R, Z_1) = 0$, $\overline{f}_{12}(R, Z_1) = 0$ *has four roots, namely,*

$$(R_1, Z_{11}) = (\frac{7\sqrt{10}}{55}, -\frac{3}{55}), \;\; (R_2, Z_{12}) = (-\frac{7\sqrt{10}}{55}, -\frac{3}{55}), \;\; (R_3, Z_{13}) = (0, -\frac{18}{85}),$$

and $(R_4, Z_{14}) = (0, 0)$.

We consider only the two solutions (R_1, Z_{11}) *and* (R_3, Z_{13}). *Furthermore, we compute the determinant of the Jacobian matrix of* $(\overline{f}_{11}, \overline{f}_{12})$ *at these two solutions, and we get*

$$det \begin{pmatrix} -\frac{55Z}{24} - \frac{1}{8} & -\frac{55}{24}R \\ -\frac{3R}{8} & -\frac{85Z}{12} - \frac{3}{4} \end{pmatrix} \Bigg|_{(R_1, Z_{11}) = (\frac{7\sqrt{10}}{55}, -\frac{3}{55})} = \frac{49}{352} \neq 0,$$

and

$$det \begin{pmatrix} -\frac{55Z}{24} - \frac{1}{8} & -\frac{55}{24}R \\ -\frac{3R}{8} & -\frac{85Z}{12} - \frac{3}{4} \end{pmatrix} \Bigg|_{(R_3, Z_{13}) = (0, -\frac{18}{85})} = \frac{147}{544} \neq 0.$$

Consequently, it follows from Theorem 2.3.1 that for $|\varepsilon|$ *sufficiently small, system (5.4.10) has two* 2π*-periodic solutions (limit cycles).*

Moreover, the eigenvalues associated to (R_1, Z_{11}) *are* $\lambda_{1,2} = -\frac{2}{11} \pm \frac{\sqrt{1334}}{8}$ *meaning that its associated periodic solution is unstable, while the eigenvalues associated to* (R_3, Z_{13}) *are* $\lambda_1 = \frac{3}{4}$ *and* $\lambda_2 = \frac{49}{136}$ *meaning that its associated periodic solution is also unstable.*

Example 5.4.2. *Consider the Kolmogorov system*

$$\begin{aligned}
\dot{x} &= (x+1)\left(2y+z-x^2-11xy-5xz-2y^2+\frac{275}{2}yz+\frac{83}{2}z^2\right),\\
\dot{y} &= (y+1)\left(-x-2x^2-102xy+xz+8y^2+5yz+z^2\right),\\
\dot{z} &= (z+1)\left((1+\epsilon+\epsilon^2)\,x-\frac{\epsilon^2}{2}y-\epsilon^2 z+4x^2+322xy-xz-8y^2-yz+z^2\right),
\end{aligned} \tag{5.4.11}$$

with ε is sufficiently small. The singular point $(1,1,1)$ of the system (5.4.7) is a zero-Hopf equilibrium point since the eigenvalues of the Jacobian matrix of (5.4.11) at $(1,1,1)$ are $0, \pm i$.

*Doing the same steps detailed in the proof of statement **(b)** of Theorem 5.2.1, we translate the zero-Hopf equilibrium point located at $(1,1,1)$ to the origin, write the Jacobian matrix of system (5.4.11) at the origin in Jordan normal form, and obtain a new system in the variables (x_1,y_1,z_1). Next, we introduce cylindrical coordinates $(x_1,y_1,z_1)=(r\sin(\theta), r\cos(\theta), z_1)$, re-scale the variables (r,z_1) to $(\varepsilon R, \varepsilon Z_1)$, and take θ as the independent variable. This yields a system of the form (5.3.6), where the functions $f_1(R,\theta,Z_1)=(f_{11}(R,\theta,Z_1), f_{12}(R,\theta,Z_1))$ and $f_2(R,\theta,Z_1)=(f_{21}(R,\theta,Z_1), f_{22}(R,\theta,Z_1))$ are 2π-periodic and of class C^1.*

Consequently, we compute the second-order averaged function $\overline{f}_2=(\overline{f}_{21},\overline{f}_{22})$, where

$$\overline{f}_2(R,Z_1)=\frac{1}{2\pi}\int_0^{2\pi}\left[D_{(R,Z_1)}f_1(\theta,R,Z_1)\cdot\int_0^{\theta}f_1(t,R,Z_1)\,dt+f_2(\theta,R,Z_1)\right]d\theta,$$

and we obtain

$$\begin{aligned}
\overline{f}_{21}(R,Z_1) &= \frac{1}{8}R^3-2RZ^2+RZ-\frac{1}{4}R,\\
\overline{f}_{22}(R,Z_1) &= Z^3-Z^2-\frac{1}{2}R^2Z+\frac{3}{2}Z.
\end{aligned}$$

Remark 5.4.1. *Note that the averaged function f_1 is identically zero for the system (5.4.11). For that reason, we are able to apply the averaging method of second order.*

The system of equations $\overline{f}_{21}(R,Z_1)=0$, $\overline{f}_{22}(R,Z_1)=0$ has nine roots, five of which are non-negative, and are given by

$$S_1=(0,-\frac{1}{2}-\frac{\sqrt{7}}{2}),\ \ S_2=(0,-\frac{1}{2}+\frac{\sqrt{7}}{2}),\ \ S_3=(\sqrt{2},0),$$
$$S_4=(\frac{\sqrt{2}\sqrt{11-2\sqrt{3}}}{3},\frac{1}{6}+\frac{\sqrt{3}}{6}),\quad \textit{and}\quad S_5=(\frac{\sqrt{2}\sqrt{11+2\sqrt{3}}}{3},\frac{1}{6}-\frac{\sqrt{3}}{6}).$$

We compute the Jacobian of $\overline{f}_1$ *we get*

$$D\overline{f}_1=\begin{pmatrix}\frac{3}{8}R^2-2Z^2+Z-\frac{1}{4} & -4RZ+R\\ -RZ & -3Z^2-2Z-\frac{1}{2}R^2+\frac{3}{2}\end{pmatrix}.$$

Evaluating the determinant of the Jacobian $A=D\overline{f}_1$ *at the points* S_j*, for* $j=1,\cdots,5$*, gives us*

$$\det(A_{(R,Z_1)=S_1})=\frac{175}{8}+\frac{61\sqrt{7}}{8}\neq 0,\ \ \det(A_{(R,Z_1)=S_2})=\frac{175}{8}-\frac{61\sqrt{7}}{8}\neq 0,$$
$$\det(A_{(R,Z_1)=S_3})=\frac{1}{4}\neq 0,\ \ \det(A_{(R,Z_1)=S_4})=-\frac{3}{4}-\frac{5\sqrt{3}}{36},\ \ \textit{and}\ \ \det(A_{(R,Z_1)=S_5})=-\frac{3}{4}+\frac{5\sqrt{3}}{36}.$$

Thus, it follows from Theorem 2.3.2 that system (5.4.11) has five 2π*-periodic solutions (limit cycles). Moreover, the eigenvalues of the Jacobian matrix at the points* S_1*,* S_2*,* S_3*,* S_4*, and* S_5 *are*

$$\left(\lambda_{11}=-\frac{7}{2}-\frac{\sqrt{7}}{2},\lambda_{12}=-\frac{19}{4}-\frac{3\sqrt{7}}{2}\right),$$
$$\left(\lambda_{21}=-\frac{19}{4}+\frac{3\sqrt{7}}{2},\lambda_{22}=-\frac{7}{2}+\frac{\sqrt{7}}{2}\right),\ \left(\lambda_{31}=\frac{1}{2},\lambda_{32}=\frac{1}{2}\right),$$
$$\left(\lambda_{41}=-\frac{7\sqrt{3}}{36}+\frac{1}{9}+\frac{\sqrt{1135+124\sqrt{3}}}{36},\lambda_{42}=-\frac{7\sqrt{3}}{36}+\frac{1}{9}-\frac{\sqrt{1135+124\sqrt{3}}}{36}\right),$$
$$\textit{and}\ \left(\lambda_{51}=\frac{7\sqrt{3}}{36}+\frac{1}{9}+\frac{\sqrt{1135-124\sqrt{3}}}{36},\lambda_{52}=\frac{7\sqrt{3}}{36}+\frac{1}{9}-\frac{\sqrt{1135-124\sqrt{3}}}{36}\right)$$

respectively. These values indicate that the periodic solutions associated to S_1 *and* S_2 *are stable, whereas the remaining ones which are associated to* S_3*,* S_4*, and* S_5 *are unstable. See Figures 5.1 and 5.2.*

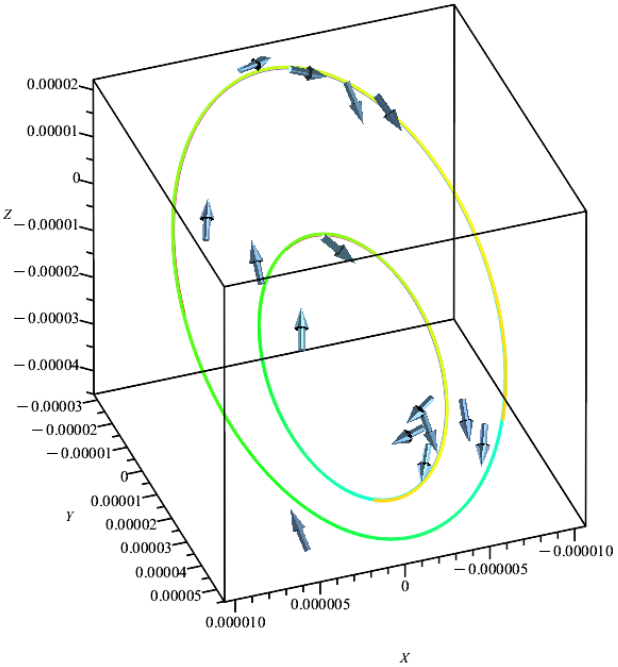

Figure 5.1: Two stable limit cycles of (5.4.11) with $\varepsilon = 10^{-5}$

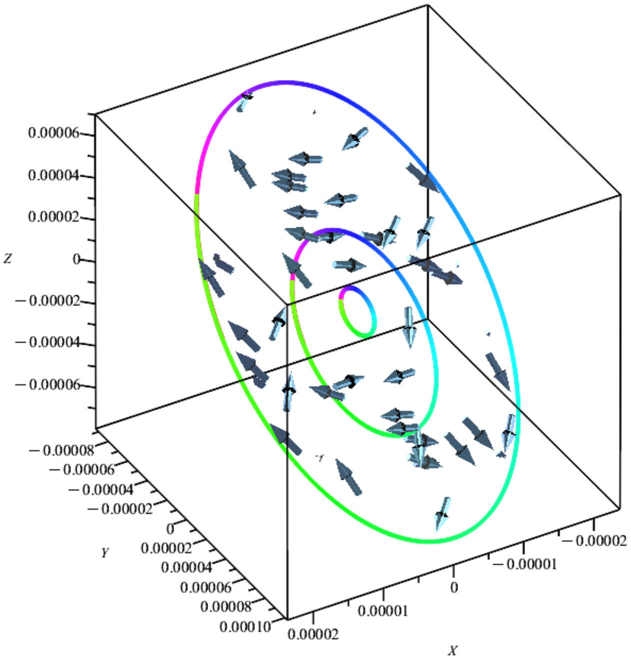

Figure 5.2: Three unstable limit cycles of (5.4.11) with $\varepsilon = 10^{-5}$

Conclusion and future direction

In this thesis, we have explored the qualitative aspects of certain dynamical systems, particularly focusing on the existence of limit cycles and bifurcations. Our investigation was guided by the application of averaging theory, which proved to be a crucial tool for analyzing the periodic behaviors of the systems under study.

Overall, the results of this thesis offered insights into the maximum number of limit cycles and bifurcations in certain nonlinear dynamical systems. The application of first and second-order averaging methods has helped to shed light on some aspects of periodic behavior in these systems, providing useful examples and laying the groundwork for further exploration.

While this thesis offers some interesting insights, there are plenty of exciting directions for future research to explore. Further investigations could explore higher-order averaging techniques to detect more intricate periodic behaviors in dynamical systems. Moreover, applying these methods to other classes of systems, such as higher-dimensional or more complex nonlinear models, would expand the scope of this work.

APPENDIX A

Appendix A

Here, we provide the expressions introduced in the proof of Theorem 5.2.1.

$$
\begin{aligned}
C &= \frac{1}{2\omega^3}(a_3b_2c_{11} - a_2b_3c_{11} - a_3b_1c_{21} + a_1b_3c_{21} + a_2b_1c_{31} - a_1b_2c_{31} + c_{31}\omega^2),\\
D &= (2a_1^4b_3(a_9b_2^2 + b_3(-a_8b_2 + a_7b_3)) - 2a_3^4b_1(b_2^2c_4 - b_1b_2c_5 + b_1^2c_7)\\
&\quad + a_2^2b_3(2a_2(a_9b_1^3 - a_6b_1^2b_3 + a_4b_1b_3^2 + b_2b_3^2b_4 - b_1b_2b_3b_6 + b_1^2b_2b_9 + b_3^3c_4 - b_1b_3^2c_6 + b_1^2b_3c_9)\\
&\quad + (2a_9b_1^2 - b_3(a_6b_1 + b_3(b_5 + c_6) + b_1(b_2 - b_8 - 2c_9)))\omega^2) + a_3^3(-2a_7b_1^3b_2 - 2a_4b_1b_2^3\\
&\quad - 2a_2b_1b_2^2b_4 - 2b_2^4b_4 + 2a_2b_1^2b_2b_5 + 2b_1b_2^3b_5 - 2a_2b_1^3b_7 - 2b_1^2b_2^2b_7 + 4a_2b_1b_2b_3c_4 - 2b_2^3b_3c_4\\
&\quad - 2a_2b_1^2b_3c_5 + 2b_1b_2^2b_3c_5 - 2a_2b_1^2b_2c_6 - 2b_1^2b_2b_3c_7 + 2a_2b_1^3c_8 - 2a_4b_1b_2\omega^2 + b_1^2b_2\omega^2 - 2b_2^2b_4\omega^2\\
&\quad + b_1b_2b_5\omega^2 - b_1b_2c_6\omega^2 + b_1^2c_8\omega^2 + a_5b_1^2(2b_2^2 + \omega^2)) - a_2a_3(2a_2^2b_1(b_3^2b_4 - b_1b_3b_6 + b_1^2b_9) + b_3(2a_8b_1^2\\
&\quad - a_6b_1b_2 - b_1b_2^2 - a_5b_1b_3 + b_1b_2b_3 - 2b_2b_3b_5 + 2b_1b_3b_7 + b_1b_2b_8 - 2b_2b_3c_6 + b_1b_3c_8 + 2b_1b_2c_9)\omega^2\\
&\quad + 2a_2(a_9b_1^3b_2 + a_8b_1^3b_3 - 2a_6b_1^2b_2b_3 - a_5b_1^2b_3^2 + 3a_4b_1b_2b_3^2 + 3b_2^2b_3^2b_4 - b_1b_2b_3^2b_5 - 2b_1b_2^2b_3b_6 + b_1^2b_2^2b_9\\
&\quad + b_1^2b_2b_3b_8 + 3b_2b_3^3c_4 - b_1b_3^3c_5 - 2b_1b_2b_3^2c_6 + b_1^2b_3^2c_8 + b_1^2b_2b_3c_9 + b_3^2b_4\omega^2 - b_1b_3b_6\omega^2 + b_1^2b_9\omega^2))\\
&\quad + a_3^2(2a_2^2b_1(b_2(2b_3b_4 - b_1b_6) - b_3^2c_4 + b_1b_3(-b_5 + c_6) + b_1^2(b_8 - c_9)) + b_3(2a_7b_1^2 + b_2(-a_5b_1\\
&\quad - b_2(b_5 + c_6) + b_1(b_2 + 2b_7 + c_8)))\omega^2 + a_2(2a_8b_1^3b_2 + 2a_7b_1^3b_3 - 4a_5b_1^2b_2b_3 + 6a_4b_1b_2^2b_3 + 6b_2^3b_3b_4\\
&\quad - 4b_1b_2^2b_3b_5 - 2b_1b_2^3b_6 + 2b_1^2b_2b_3b_7 + 2b_1^2b_2^2b_8 + 6b_2^2b_3^2c_4 - 4b_1b_2b_3^2c_5 - 2b_1b_2^2b_3c_6 + 2b_1^2b_3^2c_7 + b_1^2b_2\omega^2\\
&\quad + 2b_1^2b_2b_3c_8 + 2a_4b_1b_3\omega^2 + 4b_2b_3b_4\omega^2 - b_1b_3b_5\omega^2 - 2b_1b_2b_6\omega^2 + b_1^2b_8\omega^2 + b_1b_3c_6\omega^2 - 2b_1^2c_9\omega^2\\
&\quad - a_6b_1^2(2b_2^2 + \omega^2))) + a_1^3(2a_2b_3(-2a_9b_1b_2 + a_8b_1b_3 + a_6b_2b_3 - a_5b_3^2 + b_3^2b_7 - b_2b_3b_8 + b_2^2b_9)\\
&\quad + a_3(-2a_9b_1b_2^2 + b_3(4a_8b_1b_2 - 2a_6b_2^2 - 6a_7b_1b_3 + 2a_5b_2b_3 + 2b_3^2c_7 - 2b_2b_3c_8 + 2b_2^2c_9 + b_2\omega^2)))\\
&\quad + a_1^2(a_3^2(-2a_8b_1^2b_2 + 2a_6b_1b_2^2 + 6a_7b_1^2b_3 - 4a_5b_1b_2b_3 + 2a_4b_2^2b_3 + 2b_2b_3^2c_5 - 2b_2^2b_3c_6 - 6b_1b_3^2c_7\\
&\quad + 4b_1b_2b_3c_8 - 2b_1b_2^2c_9 - b_1b_2\omega^2 + b_2b_3\omega^2) + b_3(2a_2^2(a_9b_1^2 - a_6b_1b_3 + a_4b_3^2 - b_3^2b_5 + b_2b_3b_6 + b_1b_3b_8
\end{aligned}
$$

$$
\begin{aligned}
&- 2b_1b_2b_9) + 2(a_9b_2^2 - a_8b_2b_3 + a_7b_3^2)\omega^2 + a_2(2a_9b_1b_2^2 - 2a_8b_1b_2b_3 + 2a_7b_1b_3^2 + 2b_2b_3^2b_7 - 2b_2^2b_3b_8 \\
&+ 2b_2^3b_9 + 2b_3^3c_7 - 2b_2b_3^2c_8 + 2b_2^2b_3c_9 + b_2b_3\omega^2)) + a_3(-2a_9b_1b_2^3 + 2a_8b_1b_2^2b_3 - 2a_7b_1b_2b_3^2 - 2b_2^2b_3^2b_7 \\
&+ 2b_2^3b_3b_8 - 2b_2^4b_9 - 2b_2b_3^3c_7 + 2b_2^2b_3^2c_8 - 2b_2^3b_3c_9 - a_6b_2b_3\omega^2 + a_5b_3^2\omega^2 + b_2b_3^2\omega^2 + b_2b_3b_8\omega^2 \\
&- 2b_2^2b_9\omega^2 + b_3^2c_8\omega^2 - 2b_2b_3c_9\omega^2 + a_2(4a_9b_1^2b_2 - 4a_8b_1^2b_3 + 4a_5b_1b_3^2 - 4a_4b_2b_3^2 + 2b_2b_3^2b_5 - 2b_2^2b_3b_6 \\
&- 6b_1b_3^2b_7 + 4b_1b_2b_3b_8 - 2b_1b_2^2b_9 - 2b_3^3c_5 + 2b_2b_3^2c_6 + 2b_1b_3^2c_8 - 4b_1b_2b_3c_9 - b_1b_3\omega^2 - b_3^2\omega^2))) \\
&- a_1(a_3^3(2a_7b_1^3 - 2a_5b_1^2b_2 + 2a_4b_1b_2^2 - 2b_2^2b_3c_4 + 4b_1b_2b_3c_5 - 2b_1b_2^2c_6 - 6b_1^2b_3c_7 + 2b_1^2b_2c_8 + b_1b_2\omega^2) \\
&+ a_3^2(2a_8b_1^2b_2^2 - 2a_6b_1b_2^3 - 4a_7b_1^2b_2b_3 + 2a_5b_1b_2^2b_3 + 2b_2^3b_3b_5 - 2b_2^4b_6 - 4b_1b_2^2b_3b_7 + 2b_1b_2^3b_8 + 2b_2^2b_3^2c_5 \\
&- 2b_2^3b_3c_6 - 4b_1b_2b_3^2c_7 + 2b_1b_2^2b_3c_8 - a_6b_1b_2\omega^2 + b_1b_2^2\omega^2 + 2a_5b_1b_3\omega^2 - 2a_4b_2b_3\omega^2 + 2b_1b_2b_3\omega^2 \\
&+ b_2^2b_3\omega^2 + b_2b_3b_5\omega^2 - 2b_2^2b_6\omega^2 + b_1b_2b_8\omega^2 - b_2b_3c_6\omega^2 + 2b_1b_3c_8\omega^2 - 2b_1b_2c_9\omega^2 - a_2(2a_8b_1^3 \\
&- 2a_6b_1^2b_2 - 2a_5b_1^2b_3 + 4a_4b_1b_2b_3 + 2b_2^2b_3b_4 - 4b_1b_2b_3b_5 + 2b_1b_2^2b_6 + 6b_1^2b_3b_7 - 2b_1^2b_2b_8 - 4b_2b_3^2c_4 \\
&+ 4b_1b_3^2c_5 - 4b_1^2b_3c_8 + 4b_1^2b_2c_9 + b_1^2\omega^2 + b_1b_3\omega^2)) + a_2b_3(-2a_2^2(b_3^2b_4 - b_1b_3b_6 + b_1^2b_9) \\
&+ (4a_9b_1b_2 - b_3(2a_8b_1 + a_6b_2 + b_2^2 - a_5b_3 + b_2b_3 + 2b_3b_7 - b_2b_8 + b_3c_8 - 2b_2c_9))\omega^2 + a_2(4a_9b_1^2b_2 \\
&- 2a_8b_1^2b_3 - 2a_6b_1b_2b_3 + 2a_5b_1b_3^2 + 2b_2b_3^2b_5 - 2b_2^2b_3b_6 - 2b_1b_2b_3b_8 + 4b_1b_2^2b_9 + 2b_3^3c_5 - 2b_2b_3^2c_6 \\
&- 2b_1b_3^2c_8 + 4b_1b_2b_3c_9 + b_1b_3\omega^2 + b_3^2\omega^2)) + a_3(2a_2^2(a_9b_1^3 - a_6b_1^2b_3 + a_4b_1b_3^2 + 2b_2b_3^2b_4 - 2b_1b_3^2b_5 \\
&+ 2b_1^2b_3b_8 - 2b_1^2b_2b_9 - b_3^3c_4 + b_1b_3^2c_6 - b_1^2b_3c_9) + b_3(-2a_8b_1b_2 + a_6b_2^2 + b_2^3 + 4a_7b_1b_3 - a_5b_2b_3 + b_2^2b_3 \\
&+ 2b_2b_3b_7 - b_2^2b_8 + b_2b_3c_8 - 2b_2^2c_9)\omega^2 + a_2(-4a_9b_1^2b_2^2 + 4a_7b_1^2b_3^2 - 4a_5b_1b_2b_3^2 - 4b_2^2b_3^2b_5 + 4b_2^3b_3b_6 \\
&+ 4b_1b_2b_3^2b_7 - 4b_1b_2^3b_9 - 4b_2b_3^3c_5 + 4b_2^2b_3^2c_6 + 4b_1b_3^3c_7 - 4b_1b_2^2b_3c_9 + 2a_4b_3^2\omega^2 - 2b_2b_3^2\omega^2 - b_3^2b_5\omega^2 \\
&+ 2b_2b_3b_6\omega^2 + b_1b_3b_8\omega^2 - 4b_1b_2b_9\omega^2 + b_3^2c_6\omega^2 - 2b_1b_3c_9\omega^2 + a_6b_1b_3(4b_2^2 - \omega^2))))))/(2(a_3^2b_1 \\
&+ a_3(-a_1 + b_2)b_3 - a_2b_3^2)\omega^5),
\end{aligned}
$$

$$
\begin{aligned}
E = {}& \frac{1}{2((b_2^2 + \omega^2)a_1^2 - 2a_1a_2b_1b_2 + \omega^4 + 2a_2b_1\omega^2 + b_2^2\omega^2 + b_1^2a_2^2)(a_1a_3b_3 + b_3^2a_2 - a_3^2b_1 - a_3b_2b_3)\omega^3} \\
&\times ((-a_1b_2 + a_2b_1)(-(c_7b_1^2 - b_2c_5b_1 + c_4(b_2^2 + \omega^2))b_1a_3^4 + (((-b_1^2c_5 + 2b_1b_2c_4)a_2 - c_7(b_2 - 3a_1)b_1^2 \\
&- c_5(2a_1b_2 - b_2^2 + \omega^2)b_1 - c_4(b_2^2 + \omega^2)(b_2 - a_1))b_3 - ((b_7 - c_8)b_1^2 - b_2(b_5 - c_6)b_1 + b_4(b_2^2 + \omega^2))b_1a_2 \\
&- a_7(b_2 + a_1)b_1^3 + (-b_2(c_8 - a_5)a_1 + (-b_7 + a_5)b_2^2 - \omega^2(b_7 - c_8))b_1^2 + ((c_6 - a_4)a_1 \\
&+ b_2(b_5 - a_4))(b_2^2 + \omega^2)b_1 - b_4(b_2^2 + \omega^2)^2)a_3^3 + ((-b_1c_4a_2^2 + (c_7b_1^2 - 2c_5(b_2 - a_1)b_1 \\
&+ c_4(-2a_1b_2 + 3b_2^2 + \omega^2))a_2 - c_7(3a_1^2 - 2a_1b_2 + \omega^2)b_1 - c_5(b_2 - a_1)(a_1b_2 + \omega^2))b_3^2
\end{aligned}
$$

$$+(-((b_5-c_6)b_1-2b_2b_4)b_1a_2^2+(b_1^3a_7+((3b_7-2c_8-a_5)a_1+b_2(b_7+c_8-2a_5))b_1^2+(-2b_2(b_5-a_4)a_1$$

$$+(-2b_5-c_6+3a_4)b_2^2-2\omega^2(-\frac{a_4}{2}+b_5-\frac{c_6}{2}))b_1+3b_4(b_2+\frac{a_1}{3})(b_2^2+\omega^2))a_2+a_7(3a_1^2+2a_1b_2+\omega^2)b_1^2$$

$$+(2b_2(c_8-a_5)a_1^2+2(b_7-\frac{c_8}{2}-\frac{a_5}{2})(b_2^2+\omega^2)a_1+2\omega^2b_2(c_8-a_5))b_1-((c_6-a_4)a_1^2+b_2(b_5-c_6)a_1$$

$$+\omega^2(b_5-a_4))(b_2^2+\omega^2))b_3+b_1^2((b_8-c_9)b_1-b_2b_6)a_2^2+2(\frac{a_8(b_2+a_1)b_1^2}{2}+(-\frac{b_2(b_8-2c_9+a_6)a_1}{2}$$

$$+(\frac{b_8}{2}-\frac{a_6}{2})b_2^2+\omega^2(b_8-c_9))b_1-\frac{b_6(b_2^2+\omega^2)(b_2-a_1)}{2})b_1a_2+a_8(b_2+a_1)(-a_1b_2+\omega^2)b_1^2$$

$$+(b_2^2+\omega^2)((-c_9+a_6)a_1^2-b_2(b_8-a_6)a_1+\omega^2(b_8-c_9))b_1+b_6a_1(b_2^2+\omega^2)^2)a_3^2$$

$$+(((b_1c_5-3c_4(b_2-\frac{a_1}{3}))a_2^2+(-2a_1c_7b_1+c_5(-a_1^2+2a_1b_2+\omega^2))a_2-c_7(a_1^2+\omega^2)(b_2-a_1))b_3^3$$

$$+(-b_1b_4a_2^3+((-c_8+a_5)b_1^2+((2b_5-c_6-a_4)a_1+b_2(b_5+2c_6-3a_4))b_1-b_4(2a_1b_2+3b_2^2+\omega^2))a_2^2$$

$$+(-2a_7a_1b_1^2+((-3b_7+c_8+2a_5)a_1^2-2b_2(b_7-a_5)a_1-\omega^2(b_7+c_8-2a_5))b_1+b_2(b_5+c_6-2a_4)a_1^2$$

$$+2(b_2^2+\omega^2)(b_5-c_6)a_1+\omega^2b_2(b_5+c_6-2a_4))a_2-(a_1^2+\omega^2)(a_7(b_2+3a_1)b_1+b_2(c_8-a_5)a_1$$

$$+(b_7-c_8)b_2^2+\omega^2(b_7-a_5)))b_3^2+(b_1^2b_6a_2^3+2(-\frac{b_1^2a_8}{2}+((-b_8+\frac{c_9}{2}+\frac{a_6}{2})a_1-\frac{b_2(b_8+c_9-2a_6)}{2})b_1$$

$$+b_6(b_2^2+\omega^2))b_1a_2^2+(-2a_8(a_1^2+\omega^2)b_1^2-2(-a_1b_2+\omega^2)((b_8-c_9)a_1+b_2(c_9-a_6))b_1$$

$$+b_6(b_2^2+\omega^2)(-a_1^2-2a_1b_2+\omega^2))a_2-(a_1^2+\omega^2)(a_8(-2a_1b_2-b_2^2+\omega^2)b_1-((c_9-a_6)a_1$$

$$+b_2(b_8-c_9))(b_2^2+\omega^2)))b_3-(b_1^2a_2^2+2(-a_1b_2+\omega^2)b_1a_2+(a_1^2+\omega^2)(b_2^2+\omega^2))(b_1b_9a_2$$

$$+a_9(b_2+a_1)b_1+b_9(b_2^2+\omega^2)))a_3+b_3(a_2(c_4a_2^2-a_1c_5a_2+c_7(a_1^2+\omega^2))b_3^3+(((-c_6+a_4)b_1$$

$$+b_4(b_2+a_1))a_2^3+(a_1(c_8-a_5)b_1+(a_4-b_5)a_1^2-b_2(b_5-c_6)a_1-\omega^2(c_6-a_4))a_2^2$$

$$+(a_1^2+\omega^2)(b_1a_7+(b_7-a_5)a_1+b_2(b_7-c_8))a_2+a_7(a_1^2+\omega^2)^2)b_3^2+(((c_9-a_6)b_1$$

$$-b_6(b_2+a_1))b_1a_2^3+(b_1^2a_1a_8+((b_8-a_6)a_1^2+b_2(b_8-2c_9+a_6)a_1+2(c_9-a_6)\omega^2)b_1$$

$$-b_6(b_2+a_1)(-a_1b_2+\omega^2))a_2^2+(a_1^2+\omega^2)(-a_8(b_2-a_1)b_1-b_2(b_8-a_6)a_1+(-b_8+c_9)b_2^2$$

$$+(c_9-a_6)\omega^2)a_2-b_2a_8(a_1^2+\omega^2)^2)b_3+(b_1^2a_2^2+2(-a_1b_2+\omega^2)b_1a_2+(a_1^2+\omega^2)(b_2^2+\omega^2))((a_9b_1$$

$$+b_9(b_2+a_1))a_2+a_9(a_1^2+\omega^2))))),$$

$$
\begin{aligned}
F = {} & \frac{1}{2(a_1b_2 - a_2b_1)(-a_3^2b_1 + b_3(a_1 - b_2)a_3 + b_3^2a_2)\omega^3}(-2((-b_1^3c_7 + b_1^2b_2c_5 - b_1b_2^2c_4)a_3^4 + (((-b_1^2c_5 \\
& + 2b_1b_2c_4)a_2 - c_7(b_2 - 3a_1)b_1^2 + b_2c_5(b_2 - 2a_1)b_1 - b_2^2c_4(b_2 - a_1))b_3 - ((b_7 - c_8)b_1^2 \\
& - b_2(b_5 - c_6)b_1 + b_2^2b_4)b_1a_2 - a_7(b_2 + a_1)b_1^3 + (-b_2(c_8 - a_5)a_1 + (-b_7 + a_5)b_2^2 - \omega^2b_7)b_1^2 \\
& + b_2(b_2(c_6 - a_4)a_1 + (b_5 - a_4)b_2^2 + \omega^2b_5)b_1 - b_2^2b_4(b_2^2 + \omega^2))a_3^3 + ((-b_1c_4a_2^2 + (c_7b_1^2 - 2c_5(b_2 - a_1)b_1 \\
& + 3c_4b_2(b_2 - \frac{2a_1}{3}))a_2 + 2a_1(c_7(b_2 - \frac{3a_1}{2})b_1 - \frac{b_2c_5(b_2 - a_1)}{2}))b_3^2 + (-((b_5 - c_6)b_1 - 2b_2b_4)b_1a_2^2 \\
& + (b_1^3a_7 + ((3b_7 - 2c_8 - a_5)a_1 + b_2(b_7 + c_8 - 2a_5))b_1^2 + (-2b_2(b_5 - a_4)a_1 + (-2b_5 - c_6 + 3a_4)b_2^2 \\
& - \omega^2b_5)b_1 + 2b_2(\omega^2 + \frac{3}{2}b_2^2 + \frac{1}{2}a_1b_2)b_4)a_2 + a_7(3a_1^2 + 2a_1b_2 + \omega^2)b_1^2 + (2b_2(c_8 - a_5)a_1^2 \\
& + ((2b_7 - c_8 - a_5)b_2^2 + 2\omega^2b_7)a_1 - \omega^2b_2a_5)b_1 + b_2(-b_2(c_6 - a_4)a_1^2 + ((-b_5 + c_6)b_2^2 - \omega^2b_5)a_1 \\
& + \omega^2b_2a_4))b_3 + (b_1((b_8 - c_9)b_1 - b_2b_6)a_2 + a_8(b_2 + a_1)b_1^2 + (b_2(c_9 - a_6)a_1 + (b_8 - a_6)b_2^2 + b_8\omega^2)b_1 \\
& - b_2b_6(b_2^2 + \omega^2))(-a_1b_2 + a_2b_1))a_3^2 + (((b_1c_5 - 3c_4(b_2 - \frac{a_1}{3}))a_2^2 - 2(b_1c_7 - c_5(b_2 - \frac{a_1}{2}))a_1a_2 \\
& - a_1^2c_7(b_2 - a_1))b_3^3 + (-b_1b_4a_2^3 + ((-c_8 + a_5)b_1^2 + ((2b_5 - c_6 - a_4)a_1 + b_2(b_5 + 2c_6 - 3a_4))b_1 \\
& - b_4(2a_1b_2 + 3b_2^2 + \omega^2))a_2^2 + (-2a_7a_1b_1^2 + ((-3b_7 + c_8 + 2a_5)a_1^2 - 2b_2(b_7 - a_5)a_1 + \omega^2a_5)b_1 \\
& + b_2(b_5 + c_6 - 2a_4)a_1^2 + ((2b_5 - 2c_6)b_2^2 + \omega^2b_5)a_1 - 2\omega^2b_2a_4)a_2 - 2a_1(a_7(\omega^2 + \frac{1}{2}a_1b_2 + \frac{3}{2}a_1^2)b_1 \\
& + \frac{b_2(c_8 - a_5)a_1^2}{2} + ((\frac{b_7}{2} - \frac{c_8}{2})b_2^2 + \frac{\omega^2b_7}{2})a_1 - \frac{\omega^2b_2a_5}{2}))b_3^2 - (-a_1b_2 + a_2b_1)(-b_1b_6a_2^2 + (b_1^2a_8 \\
& + ((2b_8 - c_9 - a_6)a_1 + b_2(b_8 + c_9 - 2a_6))b_1 - b_6(a_1b_2 + 2b_2^2 + \omega^2))a_2 + a_8(2a_1^2 + a_1b_2 + \omega^2)b_1 \\
& + b_2(c_9 - a_6)a_1^2 + ((b_8 - c_9)b_2^2 + b_8\omega^2)a_1 - \omega^2b_2a_6)b_3 - (-a_1b_2 + a_2b_1)^2(b_1b_9a_2 + a_9(b_2 + a_1)b_1 \\
& + b_9(b_2^2 + \omega^2)))a_3 + (a_2(a_1^2c_7 - a_1c_5a_2 + c_4a_2^2)b_3^3 + (((-c_6 + a_4)b_1 + b_4(b_2 + a_1))a_2^3 + (a_1(c_8 - a_5)b_1 \\
& + (a_4 - b_5)a_1^2 - b_2(b_5 - c_6)a_1 + \omega^2a_4)a_2^2 - (-a_1a_7b_1 + (-b_7 + a_5)a_1^2 - b_2(b_7 - c_8)a_1 + \omega^2a_5)a_1a_2 \\
& + a_1^2a_7(a_1^2 + \omega^2))b_3^2 + (-a_1b_2 + a_2b_1)(((c_9 - a_6)b_1 - b_6(b_2 + a_1))a_2^2 + (a_1a_8b_1 + (b_8 - a_6)a_1^2 \\
& + b_2(b_8 - c_9)a_1 - \omega^2a_6)a_2 + a_1a_8(a_1^2 + \omega^2))b_3 + (-a_1b_2 + a_2b_1)^2((a_9b_1 + b_9(b_2 + a_1))a_2 \\
& + a_9(a_1^2 + \omega^2)))b_3)),
\end{aligned}
$$

$$
\begin{aligned}
G_1 = {} & \frac{1}{8\omega^3(a_1^2\omega^2 + a_2^2b_1^2 + 2a_2b_1\omega^2 + \omega^4)b_1a_3^2}(-c_5b_1(3b_7b_1^2 + b_4\omega^2)a_3^5 + (-3c_8a_7b_1^4 + ((-2b_7 - c_8)\omega^2 \\
& + 6b_7a_1(b_7 + c_8) - 3a_2((b_5 - c_6)b_7 - c_5b_8))b_1^3 - 2\omega^2(-\frac{a_1b_7}{2} - c_5b_8 \\
& + (b_5 - \frac{3c_6}{2} - a_4)b_7 + \frac{c_9c_5}{2} + \frac{c_8(b_5 - c_6 + a_4)}{2})b_1^2 - 2\omega^2(a_4\omega^2 + (-\frac{c_5b_6}{2} - b_4(b_7 + c_8))a_1 \\
& + \frac{b_4a_2(b_5 - c_6)}{2})b_1 + b_4\omega^4(c_6 + a_1 + 2a_4))a_3^4 + ((-3a_7a_8a_1 - 3a_2(a_7b_8 - 2a_7c_9 - a_8c_8))b_1^4 \\
& + (((-a_7 - a_8)a_1 + (c_8 + c_9)a_2 - 2a_7b_8 + 2a_8c_8 + 4a_7c_9 + a_7a_6)\omega^2 + 3b_7(a_2 + a_8)a_1^2 \\
& - 6a_2((b_7 + c_8)b_8 + 2b_7c_9)a_1 + 3a_2^2(-b_9c_5 + b_7b_6 + (b_5 - c_6)b_8))b_1^3 \\
& + \omega^2(c_8\omega^2 + a_1^2b_7 + ((-b_8 + c_6)a_2 - a_9c_5 + (-2b_7 - 3c_8)b_8 - 4b_7c_9 + (-b_5 + c_6 - a_4)a_8 \\
& + a_6(b_7 + 2c_8))a_1 + 3a_2(-\frac{5b_9c_5}{3} + (b_5 - \frac{4c_6}{3} - a_4)b_8 + (\frac{2a_4}{3} + \frac{b_5}{3} - \frac{c_6}{3})c_9 \\
& + \frac{5(b_7 + \frac{2}{5}c_8)b_6}{3}))b_1^2 + 2\omega^2(((-\frac{b_8}{2} + \frac{c_6}{2} - \frac{a_4}{2} + \frac{a_6}{2})a_1 - b_9c_5 + (-\frac{c_6}{2} - a_4)b_8 + c_9a_4 \\
& + b_6(b_7 + \frac{c_8}{2}) + \frac{a_4a_6}{2})\omega^2 + (\frac{b_4a_2}{2} - \frac{b_9c_5}{2} + (-b_7 - c_8)b_6 + \frac{a_8b_4}{2})a_1^2 - 2(c_9b_4 - \frac{b_6(b_5 - c_6)}{4})a_2a_1 \\
& + \frac{b_4b_6a_2^2}{2})b_1 + \omega^4((b_4 - b_6)a_1^2 + (-2c_9b_4 + (-c_6 - 2a_4)b_6 - b_4a_6)a_1 + b_4b_6a_2))a_3^3 \\
& + (3a_2((2a_9a_7 + a_8^2)a_1 + a_2(2a_7b_9 - a_9c_8 + a_8(b_8 - 2c_9)))b_1^4 + ((((a_9 + a_8)a_2 + 2a_9a_7 + 2a_8^2)a_1 \\
& + 4a_2((\frac{b_9}{4} - \frac{3c_9}{4})a_2 + 2a_7b_9 - a_9c_8 + a_8(b_8 - 2c_9 - \frac{a_6}{4})))\omega^2 - 3a_2((a_2b_8 + b_8a_8 + 2a_9b_7)a_1^2 \\
& + (-4b_8c_9 - 2c_8b_9)a_2a_1 + a_2^2((b_5 - c_6)b_9 + b_6b_8)))b_1^3 + \omega^2((a_1a_8 + (b_9 - 5c_9)a_2 + 2a_7b_9 - a_9c_8 \\
& + a_8(b_8 - 3c_9 - a_6))\omega^2 + ((-3b_8 - c_9 + a_6)a_2 + (-2b_7 - c_8)a_9 - a_8(b_8 + c_9 - 2a_6))a_1^2 + a_2((b_6 + b_9)a_2 \\
& + 6c_8b_9 + (b_5 - c_6 + 2a_4)a_9 - b_8^2 + (10c_9 + a_6)b_8 + 2a_8b_6 - 4c_9a_6)a_1 - 5a_2^2((b_5 - \frac{6c_6}{5} - \frac{4a_4}{5})b_9 \\
& + b_6(b_8 + \frac{3c_9}{5})))b_1^2 - 2\omega^2(c_9\omega^4 + ((\frac{b_8}{2} + c_9 - \frac{a_6}{2})a_1^2 + ((-\frac{b_6}{2} - b_9)a_2 - c_8b_9 + (-c_9 - \frac{a_6}{2})b_8 + c_9a_6 \\
& + \frac{a_6^2}{2} - \frac{a_8b_6}{2})a_1 + a_2((b_5 - 2c_6 - 3a_4)b_9 + b_6(b_8 + c_9)))\omega^2 + ((\frac{b_6a_2}{2} + (-b_7 - c_8)b_9 + \frac{a_8b_6}{2})a_1^2 \\
& + a_2((\frac{b_5}{2} - \frac{c_6}{2})b_9 + a_9b_4 - 2c_9b_6)a_1 + a_2^2(b_9b_4 + \frac{b_6^2}{2}))a_1)b_1 + (b_9(c_6 + a_1 + 2a_4)\omega^2 - 2((\frac{b_6}{2} - \frac{b_9}{2})a_1^2 \\
& + ((-\frac{c_6}{2} - a_4)b_9 - b_6(c_9 + \frac{a_6}{2}))a_1 + a_2(b_9b_4 + \frac{b_6^2}{2}))a_1)\omega^4)a_3^2 + (-3a_2^2(3a_8a_9a_1 + (2a_8b_9 \\
& + a_9(b_8 - 2c_9))a_2)b_1^4 - 6((\frac{a_9(a_2 + 5a_8)a_1}{3} + (\frac{b_9a_2}{6} + 2a_8b_9 + (b_8 - 2c_9 - \frac{a_6}{6})a_9)a_2)\omega^2 \\
& - \frac{1}{2}(a_2((b_9a_2 + a_8b_9 + 2a_9b_8)a_1^2 + 2b_9a_2(b_8 - 2c_9)a_1 + b_6b_9a_2^2)))a_2b_1^3 - 3((a_9(a_2 + \frac{2a_8}{3})a_1 \\
& + a_2(\frac{2b_9a_2}{3} + \frac{7a_8b_9}{3} + a_9(b_8 - \frac{8c_9}{3} - \frac{2a_6}{3})))\omega^2 - a_2(\frac{4b_9a_2}{3} + \frac{a_8b_9}{3} + a_9(b_8 + \frac{4c_9}{3} - \frac{4a_6}{3}))a_1^2 \\
& + \frac{a_9(a_2 + 2a_8)a_1^3}{3} + a_2^2((-\frac{10b_8}{3} + \frac{16c_9}{3} + a_6)b_9 + a_9b_6)a_1 - 2b_6b_9a_2^3)\omega^2b_1^2
\end{aligned}
$$

$$
\begin{aligned}
&-\omega^2((a_1a_9+b_9a_2+a_8b_9-2(c_9+\frac{a_6}{2})a_9)\omega^4+(a_1^3a_9+(-2b_9a_2-2(c_9+\frac{a_6}{2})a_9)a_1^2\\
&+2a_2((-2b_8+4c_9+2a_6)b_9+a_9b_6)a_1-4b_6b_9a_2^2)\omega^2-(b_9(a_2+a_8)a_1^2+2a_2(a_9b_6-2b_9c_9)a_1+3b_6b_9a_2^2)a_1^2)b_1\\
&+((a_1^2+(-2c_9-a_6)a_1+b_6a_2)\omega^2+a_1^4+(-2c_9-a_6)a_1^3+3b_6a_1^2a_2)\omega^4b_9)a_3+4((a_1a_9+b_9a_2)b_1+b_9\omega^2)\\
&\times(\frac{3a_2^2a_9b_1^3}{2}+(\frac{5}{2}\omega^2a_2a_9-\frac{3}{2}b_9a_1a_2^2)b_1^2+\omega^2(\omega^2a_9-\frac{3}{2}b_9a_1a_2+a_1^2a_9)b_1-\frac{\omega^2b_9a_1(a_1^2+\omega^2)}{2})a_2),
\end{aligned}
$$

$$
\begin{aligned}
G_2=&\frac{1}{2a_3^2b_1a_2^2\omega^3}(-3a_3^5b_1b_7c_5+(-3((b_5-c_6)b_7-c_5b_8)b_1a_2-3b_1^2c_8a_7-2(\omega^2-3a_1(b_7+c_8))b_7b_1\\
&-2\omega^2b_7(b_5-\frac{c_6}{2}-\frac{a_1}{2}-a_4))a_3^4+(3(-b_9c_5+b_7b_6+(b_5-c_6)b_8)b_1a_2^2+((-3a_7b_8+6a_7c_9+3a_8c_8)b_1^2\\
&+((b_8+c_8)\omega^2-6a_1(-\frac{a_1b_7}{2}+(b_7+c_8)b_8+2b_7c_9))b_1+2\omega^2(-\frac{b_8a_1}{2}+(b_5-\frac{c_6}{2}-a_4)b_8+\frac{3b_7b_6}{2}))a_2\\
&-2(\frac{3a_8b_1a_1}{2}+\omega^2(b_8-c_9+\frac{a_1}{2}-\frac{a_6}{2}))(-a_1b_7+a_7b_1))a_3^3+2(-\frac{3((b_5-c_6)b_9+b_6b_8)b_1a_2^2}{2}\\
&+((\frac{3}{2}b_8a_8+3a_7b_9-\frac{3}{2}a_9c_8-3a_8c_9)b_1^2+(-\omega^2c_9+6a_1(b_8c_9-\frac{1}{4}b_8a_1+\frac{1}{2}c_8b_9))b_1\\
&-(-\frac{b_9a_1}{2}+\frac{3b_6b_8}{2}+b_9(b_5-\frac{c_6}{2}-a_4))\omega^2)a_2+3a_1(a_9a_7+\frac{a_8^2}{2})b_1^2+((\frac{a_1a_8}{2}+b_8a_8+3a_7b_9\\
&-a_8(c_9+\frac{a_6}{2}))\omega^2-3(a_9b_7+\frac{b_8a_8}{2})a_1^2)b_1-3(\frac{b_8a_1}{6}+\frac{b_8^2}{3}+(-\frac{c_9}{3}-\frac{a_6}{6})b_8+b_7b_9)\omega^2a_1)a_2a_3^2\\
&-2a_2^2(-\frac{3b_1b_6b_9a_2^2}{2}+((\frac{3}{2}a_9b_8+3a_8b_9-3a_9c_9)b_1^2-3(b_8-2c_9+\frac{a_1}{2})a_1b_9b_1-\frac{3b_6b_9\omega^2}{2})a_2\\
&+\frac{9a_8a_9b_1^2a_1}{2}+((\frac{a_1a_9}{2}+a_9b_8+3a_8b_9-(c_9+\frac{a_6}{2})a_9)\omega^2-3a_1^2(a_9b_8+\frac{a_8b_9}{2}))b_1\\
&-4\omega^2(b_8-\frac{c_9}{4}+\frac{a_1}{8}-\frac{a_6}{8})b_9a_1)a_3+6a_2^3(-b_9a_1+a_9b_1)(a_1a_9b_1+a_2b_1b_9+b_9\omega^2)),
\end{aligned}
$$

$$
G_3=\frac{(c_2^2c_9-c_2c_3c_8+c_3^2c_7)c_{11}}{c_2\omega^3},\qquad G_4=\frac{-a_2b_1c_{32}+a_3b_1c_{22}-c_{32}\omega^2}{2\omega^3},
$$

$$
\begin{aligned}
K_1=&\frac{1}{b_3^2b_1b_2^2\omega^3}(b_3^5b_1b_7b_5+(-b_1(b_5b_8+b_7(-b_5+b_6))b_2+b_1^2b_8b_7-2b_7b_1(b_7+b_8)b_1+\omega^2b_7b_5)b_3^4\\
&+(-(-b_9b_5+b_7b_6+(b_5-b_6)b_8)b_1b_2^2+((b_7b_8-2b_7b_9-b_8b_8)b_1^2+2b_1(-\frac{b_1b_7}{2}+(b_7+b_8)b_8+2b_7b_9)b_1\\
&-\omega^2(b_5b_8+b_7b_6))b_2-(b_8b_1b_1+b_8\omega^2)(b_1b_7-b_7b_1))b_3^3+b_2((b_6b_8-(-b_5+b_6)b_9)b_1b_2^2\\
&+((-2b_7b_9-b_8b_8+2b_8b_9+b_9b_8)b_1^2+(b_8b_1^2+(-4b_8b_9-2b_8b_9)b_1)b_1+\omega^2(b_5b_9+b_6b_8))b_2\\
&+(-2b_9b_7-b_8^2)b_1b_1^2+(b_1^2(b_8b_8+2b_9b_7)-(2b_7b_9+b_8b_8)\omega^2)b_1+2b_1(b_7b_9+\frac{b_8^2}{2})\omega^2)b_3^2-b_2^2(b_1b_6b_9b_2^2\\
&+((-2b_8b_9-b_9b_8+2b_9b_9)b_1^2+2(b_8-2b_9+\frac{b_1}{2})b_1b_9b_1+b_6b_9\omega^2)b_2-3b_8b_9b_1^2b_1\\
&+(b_1^2(b_8b_9+2b_9b_8)-\omega^2(2b_8b_9+b_9b_8))b_1+3b_1b_8b_9\omega^2)b_3+2b_2^3(b_9b_1-b_9b_1)(b_1b_9b_1+b_2b_1b_9+b_9\omega^2))
\end{aligned}
$$

$$
\begin{aligned}
K_2 = {} & \frac{1}{2\omega^3(a_1^2\omega^2 + a_2^2b_1^2 + 2a_2b_1\omega^2 + \omega^4)b_1a_3^2}(3(a_2^2a_9 - a_2a_3a_8 + a_3^2a_7)(c_8a_3^2 + ((b_8 - 2c_9)a_2 + a_1a_8)a_3 \\
& - 2a_2^2b_9 - 2a_1a_9a_2)b_1^4 + (3b_7c_5a_3^5 + (c_8\omega^2 + (-3c_5b_8 + 3(b_5 - c_6)b_7)a_2 - 6b_7a_1(b_7 + c_8))a_3^4 \\
& + (((b_8 - c_9)a_2 + 3a_7b_8 - 2a_8c_8 + a_1a_8 - 2a_7c_9)\omega^2 + (3b_9c_5 + (-3b_5 + 3c_6)b_8 - 3b_7b_6)a_2^2 \\
& + 6a_1(-\frac{a_1b_7}{2} + (b_7 + c_8)b_8 + 2b_7c_9)a_2 - 3a_8b_7a_1^2)a_3^3 + (((-b_9 + c_9)a_2^2 + (-a_1a_9 - 8a_7b_9 + 4a_9c_8 \\
& - 5(b_8 - \frac{6c_9}{5})a_8)a_2 - 2a_1(a_9a_7 + a_8^2))\omega^2 + 3a_2(a_2^2((b_5 - c_6)b_9 + b_6b_8) + a_1(b_8a_1 - 4b_8c_9 - 2c_8b_9)a_2 \\
& + a_1^2(b_8a_8 + 2a_9b_7)))a_3^2 + 7a_2((\frac{a_2^2b_9}{7} + (\frac{a_1a_9}{7} + \frac{12a_8b_9}{7} + (-\frac{10c_9}{7} + b_8)a_9)a_2 + \frac{10a_8a_9a_1}{7})\omega^2 \\
& - \frac{3a_2(b_6b_9a_2^2 + 2b_9(b_8 - 2c_9 + \frac{a_1}{2})a_1a_2 + a_1^2(a_8b_9 + 2a_9b_8))}{7})a_3 + (-10a_1a_2^2a_9^2 - 16a_2^3a_9b_9)\omega^2 \\
& + 6b_9^2a_1a_2^4 + 6b_9a_1^2a_2^3a_9)b_1^3 + 5((-\frac{2c_5b_8}{5} + \frac{c_9c_5}{5} + (\frac{b_5}{5} - \frac{c_6}{5} + \frac{a_4}{5})c_8 + \frac{3b_7(b_5 - \frac{2c_6}{3})}{5})a_3^4 \\
& + (\frac{b_8\omega^2}{5} + (-\frac{a_1c_6}{5} + b_9c_5 + (-\frac{4b_5}{5} + \frac{3c_6}{5} + \frac{a_4}{5})b_8 + (-b_7 - \frac{2c_8}{5})b_6 - \frac{c_9(b_5 - c_6 + 2a_4)}{5})a_2 \\
& + \frac{(a_9c_5 + (b_7 + 3c_8)b_8 + (b_5 - c_6 + a_4)a_8 + 2b_7c_9 - 2a_6(b_7 + c_8))a_1}{5})a_3^3 + (((\frac{c_9}{5} - \frac{b_9}{5})a_2 \\
& - \frac{2a_7b_9}{5} + \frac{a_9c_8}{5} - \frac{2(b_8 - \frac{c_9}{2})a_8}{5})\omega^2 + (-\frac{b_6a_1}{5} + (-c_6 - \frac{2a_4}{5} + \frac{6b_5}{5})b_9 + b_6(b_8 + \frac{3c_9}{5}))a_2^2 \\
& - \frac{((-2b_8 - c_9 + a_6)a_1 + 6c_8b_9 + (b_5 - c_6 + 2a_4)a_9 + 2a_8b_6 - 2b_8^2 + 8b_8c_9 - 4c_9a_6)a_1a_2}{5} \\
& + \frac{2a_1^2((b_7 + \frac{c_8}{2})a_9 + \frac{a_8(b_8+c_9-2a_6)}{2})}{5})a_3^2 + ((\frac{2a_2^2b_9}{5} + (\frac{a_1a_9}{5} + \frac{7a_8b_9}{5} + (b_8 - \frac{4c_9}{5})a_9)a_2 \\
& + \frac{2a_8a_9a_1}{5})\omega^2 - \frac{6b_6b_9a_2^3}{5} + \frac{3(-b_9a_1 + (-\frac{11b_8}{3} + \frac{14c_9}{3} + \frac{2a_6}{3})b_9 + a_9b_6)a_1a_2^2}{5} \\
& - \frac{3a_1^2(-\frac{a_1a_9}{3} + \frac{a_8b_9}{3} + a_9(b_8 + \frac{4c_9}{3} - \frac{4a_6}{3}))a_2}{5} + \frac{2a_1^3a_9a_8}{5})a_3 \\
& - \frac{14a_2(a_9(b_9a_2 + \frac{2a_1a_9}{7})\omega^2 - \frac{6b_9^2a_1a_2^2}{7} - \frac{b_9a_1^2a_2a_9}{7} + \frac{2a_1^3a_9^2}{7})}{5})\omega^2b_1^2 + (b_4c_5a_3^5 + (b_4a_2(b_5 - c_6) \\
& - 2a_1(\frac{c_5b_6}{2} + b_4(b_7 + c_8)))a_3^4 + ((2b_9c_5 + (-b_5 + a_4)b_8 - 2b_6(b_7 + \frac{c_8}{2}))\omega^2 - b_4b_6a_2^2 \\
& - a_1(b_4a_1 + b_6(b_5 - c_6) - 4c_9b_4)a_2 - (-b_9c_5 + (-2b_7 - 2c_8)b_6 + a_8b_4)a_1^2)a_3^3 \\
& + (((-b_6a_1 + (4b_5 - 2c_6 - 2a_4)b_9 + 2b_6(b_8 + c_9))a_2 - a_1(b_8a_6 + a_8b_6 - b_8^2 + 2c_8b_9))\omega^2 \\
& + 2(a_2^2(b_9b_4 + \frac{b_6^2}{2}) + (\frac{b_6a_1}{2} + (\frac{b_5}{2} - \frac{c_6}{2})b_9 + a_9b_4 - 2c_9b_6)a_1a_2 + \frac{a_1^2((-2b_7 - 2c_8)b_9 + a_8b_6)}{2})a_1)a_3^2 \\
& + ((b_9a_2 + a_8b_9 + a_9b_8)\omega^4 + (-4b_6b_9a_2^2 + 2((-3b_8 + 2c_9 + a_6)b_9 + a_9b_6)a_1a_2 + a_1^2a_9b_8)\omega^2 \\
& - 3(b_6b_9a_2^2 + \frac{2a_1(a_9b_6 - 2b_9c_9 + \frac{1}{2}b_9a_1)a_2}{3} + \frac{a_1^2a_8b_9}{3})a_1^2)a_3 - 4a_2b_9(\omega^4a_9 + (-2b_9a_1a_2 + \frac{1}{2}a_1^2a_9)\omega^2 \\
& - \frac{a_1^3(a_1a_9 + b_9a_2)}{2}))\omega^2b_1 + \omega^4(b_5a_3^2 + (-b_8a_1 - b_6a_2)a_3 + 2b_9a_1a_2)(b_4a_3^2 - b_6a_1a_3 + b_9(a_1^2 + \omega^2))),
\end{aligned}
$$

$$
\begin{aligned}
K_3 &= -\frac{1}{\omega^3(\omega^4 + (a_1^2 + 2a_2a_1)\omega^2 + a_2^2a_1^2)}(a_2a_{11}(a_9\omega^4 + ((\frac{3a_9a_2}{2} + (-\frac{a_8}{2} + \frac{a_9}{2})a_3 \\
&\quad + \frac{a_1a_9}{2})a_1 + \frac{a_1^2a_9}{2} - \frac{a_6a_1a_3}{2} + \frac{a_4a_3^2}{2})\omega^2 - \frac{a_1^2(-a_2^2a_9 + a_2a_3a_8 - a_3^2a_7)}{2})), \\
K_4 &= \frac{(a_2c_{32} - a_3c_{22})b_1}{\omega^3}.
\end{aligned}
$$

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