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Thesis

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Contribution to the study of certain functional differential equations

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Abstract in English

Contribution to the Study of Certain Functional Differential Equations

In this work, three nonlinear functional differential equations with delays and iterative terms are investigated. By virtue of a hybrid approach that combines the Banach, Schauder, and Krasnoselskii fixed point theorems with the Green's functions method, we establish some sufficient conditions that guarantee the existence, uniqueness and stability of positive periodic solutions. The key idea consists to define a Banach space and a subset of it in order to facilitate the study on the one hand, and on the other hand to ensure some desired requirements before converting the proposed problem into an equivalent integral equation whose kernel is a Green's function. Then the application of certain fixed point theorems with the aid of some useful properties of the obtained Green's kernel and some functional analysis tools help us to prove the sought results.

Keywords: Continuous dependence, existence, fixed point theorem, functional differential equation, iterative differential equation, Green's function, uniqueness.

Abstract in French

Contribution à l'étude de certaines équations différentielles fonctionnelles

Dans ce travail, trois classes d'équations différentielles fonctionnelles non linéaires avec retards et termes itératifs sont étudiées. Grâce à une approche hybride combinant les théorèmes du point fixe de Banach, Schauder et Krasnoselskii avec la méthode des fonctions de Green, nous établissons des conditions suffisantes qui garantissent l'existence, l'unicité et la stabilité des solutions positives et périodiques. L'idée principale consiste à définir un espace de Banach et un sous-ensemble d'une part, pour faciliter l'étude et, d'autre part, pour garantir certaines exigences souhaitées avant de convertir le problème proposé en une équation intégrale équivalente dont le noyau est une fonction de Green. Ensuite, l'application de certains théorèmes de point fixe à l'aide de certaines propriétés utiles du noyau de Green obtenu et de quelques outils d'analyse fonctionnelle nous aident à prouver les résultats souhaités.

Mots-clés: Dépendance continue, existence, théorème de point fixe, équation différentielle fonctionnelle, équation différentielle itérative, fonction de Green, unicité.

Abstract in Arabic

مساهمة في دراسة بعض المعادلات التفاضلية الدالية

في هذا العمل، تم دراسة ثلاث معادلات تفاضلية دالية غير خطية ذات تأخيرات و بحدود تكرارية. بفضل منهجية هجينة تمزج بين نظريات النقطة الثابتة لباناخ، شاودر و كراسنوسيلسكي مع طريقة دوال غرين قمنا بإنشاء بعض الشروط الكافية التي تضمن وجود، وحدانية و استقرار الحلول الموجبة الدورية. تتمثل الفكرة الرئيسية في تحديد فضاء باناخي ومجموعة جزئية منه وذلك لتسهيل الدراسة من ناحية ولاستيفاء بعض المتطلبات المرغوبة من ناحية أخرى قبل تحويل المسألة المقترحة إلى معادلة تكاملية مكافئة نواتها هي دالة غرين ومن ثم فإن تطبيق بعض نظريات النقطة الثابتة بمساعدة بعض الخصائص المفيدة لنواة غرين المتحصل عليها وبعض أدوات التحليل الدالي يساعدنا في إثبات النتائج المرجوة.

كلمات مفتاحية: الارتباط المستمر، الوجود، نظرية النقطة الثابتة، المعادلة التفاضلية الدالية، المعادلة التفاضلية التكرارية، دالة غرين، الوحدانية.

Dedication

To my parents

To my sisters and my brother

To my friends and family

To myself

Acknowledgements

With all that is in my heart and with everything that I have been through. First of all, I thank God Almighty for everything I am today. I thank God for granting me success and for giving me sufficient strength to complete this way and the long path. I thank God for everything. If it were not for God, none of this would have happened!

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Finally, I thank my family and everyone who supported me in completing this study.

Acronyms

Abbreviation	**Meaning**
FDE	Functional differential equation
ODE	Ordinary differential equation
DDE	Delayed differential equation
ADE	Advanced differential equation
NDE	Neutral differential equation
SDDE	state-dependent delay differential equation
IDE	Iterative differential equation

List of symbols

Sets and numbers

$\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).

$\mathbb{R}^*$: the set of all non-zero real numbers.

$\mathbb{R}^n$: n-dimensional real Euclidean space.

$[a, b]$: the interval of numbers between a and b, including a and b.

(a, b) : an open interval.

$[a, +\infty)$: left-closed and right–unbounded interval.

$\mathcal{C}^1\left(\mathbb{R}^{n+1}, \mathbb{R}\right)$: space of continuously differentiable functions from $\mathbb{R}^{n+1}$ into $\mathbb{R}$.

$B\left(0, R_1\right)$: circle with center 0 and radius R_1.

T : a period.

$p = \sup_{t\in[0,T]} p\left(t\right)$.

P_T : the *Banach* space of all continuous and periodic functions with period T.

$\Omega = \left\{x \in P_T : 0 < R_0 \leq x\left(t\right) \leq R_1,\ \left|x\left(t_2\right) - x\left(t_1\right)\right| \leq L\left|t_2 - t_1\right|,\ \forall t_1, t_2 \in [0, T]\right\}$.

Functions

$|\cdot|$: absolute value.

$\|.\|_E$: a norm on E.

$\|f\|_\infty$: the uniform norm defined by $\sup|f(x)|$.

$x^{[n]}(t)$: the composition of the function $x(t)$ with itself n times or the n^{th} iterate of the function $x(t)$.

$\sum\limits_{m=1}^{n}$: the summation from index $m = 1$ to $m = n$.

$\lim\limits_{x \longrightarrow x_0}$: limit as x approaches x_0.

$\approx$: approximately equal to.

$x'(t) := \dfrac{dx(t)}{dt}$: the first derivative of the function $x(t)$ with respect to t.

sup : the supremum.

max : the maximum.

min : the minimum.

$\exp x$: the exponential function of x.

$G(t, x)$: the *Green's* function.

When the other notations appear, they are explained.

CHAPTER 1

General Introduction

1.1 Introduction

Functional differential equations (FDEs) are a type of differential equations that depend not only on their current values, but also on their past or future values. They find relevance in almost every sphere of life and play a significant role in modelling many phenomena in various fields, such as life sciences, medical science, physical sciences, economic sciences, and engineering.

Starting in the early 20th century when functional differential equations knew there hour of glory , many scholars gained a more profound understanding of the prominent role of the deviating arguments in modelling real-world phenomena with memory effects. Consequently, they made much progress in handling with these equations which in turn spurred the accelerated development of the theory of functional differential equations whether delay differential equations (DDEs) or advanced differential equations (ADEs). Indeed, many scholars came to the conclusion that a growing number of phenomena encountered in our life, in physics, in biology, in chemistry, etc., require delay periods for their occurrence, where there are time lags between the action on the system and the system's response to this action and hence they found that delay differential equations are the best means to model and to deal with such phenomena. For more than half a century, the majority of research have been devoted to deal with state-dependent delay differential equations (SDDEs) which appeared for the first time in the early nineteenth century as an article published by the French mathematician Poisson [72] on functional differential equations with an example of a state-dependent delay. There are also many references dedicated to study these equations such as [3] , [35], and [36].

Recently, the mathematical community has got interested in a special type of functional differential equations which is called "iterative differential equations". These equations which are relationships between unknown functions, their derivatives, and their iterates, find their applications in various fields such as biology, ecology, physics, chemistry, electrodynamics, and engineering (see [71] and [34]). Due to their peculiarities, these equations are challenging to study, particularly when applying the existence and uniqueness theories. Numerous methods have been employed to handle these kinds of equations, including the Picard successive approximation method, the nonexpansive operator method, and the fixed point theory (see to [8, 34, 88, 89, 90] and references therein). The last method has shown to be incredibly successful and is still a very useful, strong, and adaptable tool to study nonlinear functional differential equations.

The primary objective of this maniscript is to investigate three classes of first-order differential equations with retarded and iterative terms by using this theory as well as the Green's functions approach. Since converting the problem at hand to a fixed point problem is one of the most elegant approaches to achieve our goals, we started the study of each problem with transforming it into an integral equation with a Green's kernel before proving the existence of fixed points of an integral operator. To be more precise, we used the fixed point theorems of Banach, Schauder and Krasnosleskii combined with the Green's functions method to establish the existence, uniqueness and stability of positive periodic solutions.

1.2 Brief Historical Background

Population dynamics is the dominant branch of mathematical biology, that studies the fluctuations of the size or the density of one or more populations for certain species over time and space. It is also interested in the environmental influences on the increase and the decrease of the populations.

The simple exponential growth model

The modern foundations of population dynamics date from the end of the 18 th century with the model of the British clergyman and economist Thomas Robert Malthus which is known today as "Malthusian growth model" or " exponential growth model". In 1798, Malthus published his pioneering and famous essay "An Essay on the Principle of Population" in which he argued that, "Population, when unchecked, increases in a geometrical ratio. Subsistence only increases in an arithmetical ratio."

He considered an ideal homogeneous animal population by neglecting variations in age, size and possible periodicity for birth or mortality where individuals lived alone in an invariable environment or coexisted with other species without any direct or indirect influence. The variation in the number of individuals in a population $N(t)$ over time is given by

$$\underbrace{\text{Variation}}_{N'(t)} = \underbrace{\text{number of births}}_{N_n} - \underbrace{\text{number of deaths}}_{N_m}.$$

This leads to the following differential equation:

$$N'(t) = N_n(t) - N_m(t),$$

where N_n is the number of births and N_m is the number of deaths.

Malthus predicted that population would increase exponentially unless if it is constrained,. So the Malthusian model has the following form:

$$N(t) = N(t_0)\ e^{a(t-t_0)},$$

where $N(t_0)$ is the initial population size, $a = \lambda - \mu$ stands for the population growth rate (also called the Malthusian parameter of the population growth or the intrinsic rate of increase), λ is the natality rate, and μ is the mortality rate. This model can also be written as a differential equation

$$\frac{dN(t)}{dt} = a\ N(t),$$

where $\frac{dN(t)}{dt}$ represents the instantaneous rate of change of the population as a function of time.

The logistic equation

One of the major problems with Malthusian model is the fact that the exponential growth is somewhat rare. It generally leads to an extinction or an explosion of the population, especially at the beginning of the invasion of the environment in which resources are overabundant.

In 1838, the Belgian biologist Pierre François Verhulst proposed a more realistic model of population evolution known as a logistic growth model which is a kind of generalization of the Malthusian model but with a maximum number of individuals of the species that the environment can support indefinitely. He introduced a corrective term when the population grows which is called carrying capacity, biotic capacity or even carrying capacity of the environment. He then proposed the following differential equation:

$$\frac{dN\left(t\right)}{dt}=rN\left(t\right)\left(1-\frac{N\left(t\right)}{K}\right)=f\left(N\left(t\right)\right),$$

where r represents the growth rate and K denotes the carrying capacity.

The delay logistic equation (the Hutchinson equation)

One disadvantage of the logistic model is that the growth rate of a population $N'\left(t\right)$ at time t depends on the relative number of individuals at that time. In reality, the process of reproduction is not instantaneous but there is often significant delays. The delay is generally a positive constant depending on time or state or even distributed and can represent for example, but not limited to, the resource regeneration time, the gestation time, the larval phase, the duration of the cell cycle, the transit time or the duration of a cellular transformation in the dynamics of cell populations , the development period, the juvenile phase, the maturation period or the feeding time in the population dynamics of certain human, animal and plant species, etc.

In order to take into account the effect of these delays on the population dynamics, G.E. Hutchinson (1948) incorporated a delay in the logistic equation as follows:

$$N'\left(t\right)=rN\left(t\right)\left[1-\frac{N\left(t-\tau\right)}{K}\right],$$

where r and K have the same meaning as in the logistic equation, and τ represents a time delay.

This equation, which is known as the Hutchinson's equation (also known as delayed logistic equation or Wright's equation), exhibit much more complicated dynamics than ordinary differential equations. It has been studied by several researchers like Wright in 1945 and Markus in 1958, Jones in 1962, Kaplan and Yorke in 1975 and others.

The emergence of iterative differential equations which can be considered as a special type of functional differential equations with deviating arguments depending on state and time, dates back to the 19th century. As far as we know, Babbage [6] can be considered as one of the first researchers interested in this kind of differential equations, it was his first published document in this topic in 1815 under the title "An Essay Towards the Calculus of Function", in which he investigated the problem of determining a function that equals its n^{th} iterate $x^{[n]}(t)$. In 1968, Andrzej [5] studied the first order iterative differential equation $x'(t) = f(t, x(t), x(x(t)))$ with the initial value $x(0) = c$ $(c > 0)$ by using Picard's successive approximation, with 0 serving as the domain's left end point. In 1984, Eder [34] applied the contraction mapping principle for proving the existence of the unique monotone solution of the equation $x'(t) = x(x(t))$ with the condition $x(t_0) = t_0$, $t_0 \in [-1, 1]$. Later, in 1990, Wang [84] used Schauder's fixed point theorem for studying the existence results of the equation $x'(t) = f(x(x(t)))$ associated with $x(a) = a$, where a is the end point of the interval. Consequently, in 1993, Fečkan [37] studied the local solution of the same equation of Wang with the initial value $x(0) = 0$ by the aid of the contraction mapping principle. The previous equation was the focus of much of attention, for example, in 1995 Buica [18] conducted a study of the existence, uniqueness and continuous dependence of its solutions with initial value problems by applying fixed point theorems and in 1997 Ge and Mo [39] also investigated it with the initial value problem $x(t_0) = x_0$ and in 2010 Berinde [8] extended the findings of Buica [18] by introducing the method of nonexpansive operators for studying it.

Recently, there have been many researchers who have published papers about this type of first order differential equations and their applications in many fields (see [12, 54, 57, 85, 88, 89]), in ecology such as population dynamics models with iterative terms (see [63, 64, 65]) and iterative Nicholson's blowflies models (see [16, 49], and in hematology such as iterative hematopoiesis models (see [10, 47, 48]).

1.3 Problem Statement

Studying functional differential equations with delays and iterative terms that have certain distinguishing features is more difficult than it may appear at first glance. Their developing theory, which is still in its early stages of development, their iterative terms, which create numerous insurmountable obstacles during investigation, and the fact that most conventional methods are either rarely applicable or frequently lead to a dead end are some of the reasons for this.

The present dissertation is motivated by the desire to answer the following fundamental research questions as possible, and also to illustrate these answers with examples, whenever feasible.

(i) How to control the iterative terms?

(ii) Does the problem have solutions?

(iii) What are the conditions for the existence of a unique positive periodic solution?

(iv) How does the system respond to parameter changes?

1.4 Research Aims and Objectives

This study seeks to establish the existence, uniqueness and stability results for three classes of first-order functional differential equations with retarded and iterative terms by using the fixed point theory as well as some properties of an obtained Green's function. More precisely, it gives criteria that ensure some qualitative and quantitative properties of the solutions such as the existence, uniqueness, positivity, boundedness, periodicity and continuous dependence on parameters of solutions for three problems arising in biology and population dynamics. The first problem is a first-order iterative differential equation; the second one is a first-order iterative differential equation with an iterative and delayed harvesting term, and the last one is a first-order neutral differential equations with an iterative harvesting term.

1.5 Motivation

Usually, the study of iterative-delay differential equations was restricted to some simple types of them, by applying an approach among a limited number of usual methods. In case of more complex iterative-delay differential equations, their distinctive characteristics often hamper the application of these methods.

The current study was motivated by the interest in contributing to the development of the emergent theory of these equations and also by an endeavor to explore how our hybrid technique could be used to get rid of hitches and challenges that are often encountered in the investigation of such kind of equations.

1.6 Contributions

The following succinctly describes the fascinating aspects of our contributions:

- According to numerous studies, the majority of real phenomena involve delays related to time and state, which can lead to the appearance of iterative terms. This kind of delay is included in the studied equations that model phenomena in life sciences.

- The technique used here is effective and successful and can be applied to many different delay models in many different fields. It is especially important to handle many iterative models that are common in life sciences, like population models, disease transmission models, blood cell production models, two-body problems of classical electrodynamics models and so on.

1.7 Brief Research Methodology

This work is concerned with the existence, uniqueness, and stability of positive periodic solutions for three functional differential equations with delayed and iterative terms. Because of the difficulties surrounding the study of this kind of equations, there is no specific methodology applicable to deal with them in a fruitful manner. Herein, we have two functional differential equations that include second iterates where one of them has, in addition, a time varying delay whereas the third equations involves terms with a time varying delay and other terms with iterations of the state at least up to its n-th iterates. At first sight, one may be tempted to cave in and to consider every effort for studying them useless. Despite all these constraints, we tried to pave the way for better application of our hybrid technique that demonstrates the efficiency and efficacy of combining the fixed point theory with other methods, such as the Green's functions method and with the use of some functional analysis tools. This technique boils down to turning the problem at hand into a fixed point one. Constructing the cornerstones of the approach such as an appropriate Banach space and a suitable subset of it, is one way to lay the foundations for applying the chosen fixed point theorems, controlling the iterative terms, and achieving the desired results which in turn help us to overcome any unforeseen challenges that may arise during our study. The next step is to transform our nonlinear iterative problems into equivalent integral equations with the same Green's kernel. Thereafter, by virtue of the aforemensionned integral equations, the Arzelà-Ascoli theorem and several helpful properties of the obtained Green's kernel, we define integral operators that fulfill the requirements of the used fixed point theorems. To conclude, the proof of the existence of solutions can be strengthened by using the Schauder or Krasnosleskii fixed point theorems or by introducing some additional conditions under which the Banach fixed point theorem can be used to establish the existence of a unique solution that depends continuously on parameters.

1.8 Thesis Overview

This manuscript has been divided into five chapters. Its structure is as follows: the first two chapters are introductories whereas the final three chapters contain new results that are published in reputable international journals. The first chapter serves as a general introduction that gives a succinct synopsis of the topic, a brief history, the research goals and methodology, the motivation behind the work, and its contributions. It also presents the outlines of the thesis which is divided into five chapters and concludes with a general conclusion, some suggestions for future research, a list of our published papers and a bibliography. In the second chapter, some basic concepts and functional analysis definitions are presented. We also mention the basic tools used to establish the existence, uniqueness, and stability results such as the fixed point theorems of Schauder, Banach and Krasnosleskii which will be referred to throughout the rest of the work.

In the third chapter, we disclose the outcomes published in [65]. It focuses on the existence, uniqueness and stability results for a Musca Domestica model that is governed by the following first-order nonlinear differential equation with iterative terms:

$$x'(t)+p\,(t)\,x(t)=b\beta x^{[2]}\,(t)-b^{2}\alpha\left(x^{[2]}\,(t)\right)^{2},$$

where $x^{[2]}\,(t)=x\,(x\,(t))$, $p\in\mathcal{C}\,(\mathbb{R},]0,\infty[)$ is a $T-$periodic function and the remainder parameters are positive. For achieving our goals, we first determine whether or not our problem is equivalent to a certain integral equation. Afterward, we use the well-known Schauder's fixed point theorem to demonstrate that there is at least one positive periodic solution. Finally, we add an extra condition that makes it easier to apply the Banach fixed point theorem and hence ensures the existence of a unique positive bounded periodic solution that continuously depends on the adult mortality rate p. In addition, we give two concrete examples illustrating the main theorems.

In the fourth chapter, we present results published in [63] which concentrates on the existence, uniqueness and continuous dependence on parameters of positive periodic solutions for the following first-order delay differential equation with iterative terms:

$$x'\,(t)+p\,(t)\,x\,(t)=ax^{[2]}\,(t)-d\left(x^{[2]}\,(t)\right)^{2}-qx^{[2]}\,(t)\,E\,(t,x\,(t)\,,x\,(t-\tau\,(t)))\,,$$

where $a,d,q>0$, and $p\in\mathcal{C}\,(\mathbb{R},(0,+\infty))\,,\tau\in\mathcal{C}\,(\mathbb{R},(0,+\infty))$ and $E\in\mathcal{C}\,(\mathbb{R}^{3},(0,+\infty))$are T-periodic functions with respect to the time variable. In order to study this iterative-delay differential equation, we use the same technique as in the previous chapter. It is based on the Banach and Schauder fixed point theorems together with the Green's functions method as key tools in obtaining the required outcomes. Furthermore, we provide two examples to show that our main theoretical findings are valid.

In the fifth chapter, we showcase the findings reported in [64], which deals with the existence, uniqueness and stability of positive periodic solutions for the following first-order neutral delay differential equation with an iterative harvesting term:

$$\frac{d}{dt}\left[x\,(t)-cx\,(t-\tau\,(t))\right]=-p\,(t)\,x\,(t)+f\,(t,x\,(t-\tau\,(t)))-H\left(t,x\,(t)\,,x^{[2]}\,(t)\,,...,x^{[n]}\,(t)\right),$$

where the iterate $x^{[n]}(t)$ stands for x composed with itself n times, i.e. $x^{[2]}(t) = x(x(t)), ..., x^{[n]}(t) = x^{[n-1]}(x(t))$, $c \in (0,1)$, $p, \tau \in C(\mathbb{R}, (0,\infty))$, $f \in \mathcal{C}([0,T] \times \mathbb{R}, (0,\infty))$ and $H \in \mathcal{C}([0,T] \times \mathbb{R}^n, (0,\infty))$ are $T-$periodic functions with respect to the time variable. To investigate this neutral delay differential equation with iterative terms, we mainly use the same steps of the previously used approach, but instead of using the Schauder fixed-point theorem, we use the Krasnosleskii fixed-point theorem combined with the Banach fixed-point theorem and the Green's functions method. Finally, we give an example to illustrate our main findings.

CHAPTER 2

Preliminaries and Reminders

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In this introductory chapter, we introduce basic notions and some necessary background materials for ready reference including necessary functional analysis tools, certain fixed point theorems as well as the concepts of *Green*'s function and iterative functional differential equations.

2.1 Preliminaries and Background Material

This section starts with a reminder of basic definitions of normed and Banach spaces. Furthermore, we introduce some definitions and preliminary notions.

2.1.1 Normed and Banach spaces

Definition 2.1 [56] A normed space $\mathbb{X}$ is a vector space with a norm defined on it. A Banach space is a complete normed space (complete in the metric defined by the norm).

Example 2.1 The space $\mathcal{C}\left([a,b],\mathbb{R}\right)$ of continuous functions from $[a,b]$ into $\mathbb{R}$ provided with the norm

$$\|f\| = \sup_{t\in[a,b]} |f(t)|,$$

is a Banach space.

Example 2.2 The space

$$P_T = \left\{x \in \mathcal{C}\left([a,b],\mathbb{R}\right) : x(t+T) = x(t)\right\},$$

of all continuous and periodic functions with period T equipped with the supermum norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|,$$

is a Banach space.

2.1.2 Definitions and Basic Properties of a Normed Space

Definition 2.2 [27] Let $(\mathbb{X}, \|.\|)$ be a normed space.

- A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$ is said to converge to $a \in \mathbb{X}$ if

$$\forall\varepsilon > 0, \exists N_0 \in \mathbb{N} : \forall n \in \mathbb{N},\ \ (n \geq N_0) \Longrightarrow \left(\|x_n - a\| < \varepsilon\right).$$

In this case one writes $\lim_{n\to\infty} x_n = a$ or $x_n \to a$ as $n \to \infty$.

- A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$ is called a Cauchy sequence if

$$\forall\varepsilon > 0, \exists N_0 \in \mathbb{N} : \forall n, m \in \mathbb{N},\ \ (n \geq N_0 \text{ and } m \geq N_0) \Longrightarrow \left(\|x_n - x_m\| < \varepsilon\right).$$

Definition 2.3 Let $\mathbb{X}$ be a vector space over $\mathbb{F}$. A convex subset of $\mathbb{X}$ is a subset $\mathbb{M} \subseteq \mathbb{X}$ such that for every pair of points $x, y \in \mathbb{M}$ and for every $\alpha \in [0,1]$ we have that

$$\alpha x + (1-\alpha)\, y \in \mathbb{M}.$$

2.1.3 Bounded, Closed and Compact Subset in a Normed Vector Space

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$ and $\mathbb{M}$ be a set of it.

Definition 2.4 $\mathbb{M}$ is called bounded if and only if there is a positive number $r \geq 0$ such that $\|x\|_{\mathbb{X}} \leq r$ for all $x \in \mathbb{M}$.

Definition 2.5 $\mathbb{M}$ is called closed if and only if each sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{M}$ which converges to an element $x \in \mathbb{X}$, then $x \in \mathbb{M}$.

Definition 2.6 [74] The closure of a set $\mathbb{M}$ (denoted by $\overline{\mathbb{M}}$) is the smallest closed set that contains $\mathbb{M}$.

Definition 2.7 [74] A set $\mathbb{M}$ is called relatively compact if its closure $\overline{\mathbb{M}}$ is compact.

Definition 2.8 $\mathbb{M}$ is called relatively compact if and only if each sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{M}$ has a convergent subsequence $x_{n_k} \to x \in \mathbb{X}$ as $k \to \infty$.

Definition 2.9 $\mathbb{M}$ is called compact if and only if each sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{M}$ has a convergent subsequence $x_{n_k} \to x \in \mathbb{M}$ as $k \to \infty$.

2.1.4 Continuous, Lipschitz Continuous and Compact Operators

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two Banach vector spaces over the same field $\mathbb{F}$.

Definition 2.10 [70] Let $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ be an operator and let $x_0 \in \mathbb{X}$. Then,

1. $\mathcal{S}$ is said to be continuous at $x_0 \in \mathbb{X}$ if for any sequence (x_n) in $\mathbb{X}$ which converges to x_0, the sequence $(\mathcal{S}x_n)$ converges to $\mathcal{S}x_0$ in $\mathbb{Y}$. In orther words,$\mathcal{S}$ is said to be continuous at x_0 if

   $$(x_n \to x_0) \Longrightarrow (\mathcal{S}x_n \to \mathcal{S}x_0)\,.$$

   i.e.,

   $$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, x_0) > 0, \forall x \in \mathbb{X}, \ \ (\|x - x_0\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x_n - \mathcal{S}x_0\|_{\mathbb{Y}} < \varepsilon)\,.$$

   If $\mathcal{S}$ is continuous at every point of $\mathbb{X}$, then $\mathcal{S}$ is said to be continuous on $\mathbb{X}$, i.e.,

   $$\forall \varepsilon > 0, \forall x \in \mathbb{X}, \exists \delta > 0, \forall y \in \mathbb{X}, (\|x - y\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} < \varepsilon)\,.$$

2. $\mathcal{S}$ is said to be uniformly continuous on $\mathbb{X}$ if

   $$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall x, y \in \mathbb{X}, (\|x - y\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} < \varepsilon)\,.$$

Definition 2.11 [79] A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is called *Lipschitz continuous* if there is a positive constant C such that

$$\forall x, y \in \mathbb{X} : \|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} \leq C\,\|x - y\|_{\mathbb{X}}\,.$$

If $C \in [0, 1[$, $\mathcal{S}$ is called a contraction mapping.

Remark 2.1 If $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ then

$$(\mathcal{S} \text{ is a contraction}) \Longrightarrow (\mathcal{S} \text{ is Lipschitz continuous}) \Longrightarrow (\mathcal{S} \text{ is continuous on } \mathbb{X})\,.$$

Theorem 2.1 *[74] A continuous mapping on a closed bounded interval is bounded and attains its bounds.*

Lemma 2.1 *[3] All continuous mappings on compact sets are uniformly continuous.*

Remark 2.2 Theorem 2.1 and Lemma 2.1 are hidden in the proof of many theorems and lemmas in the rest of this thesis where we integrate a continuous function over a compact interval.

Definition 2.12 [70] An operator $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be compact if it maps every bounded subset of $\mathbb{X}$ into a relatively compact subset of $\mathbb{Y}$. i.e.,

$$[\mathcal{S} \text{ compact}] \Longleftrightarrow \left[\forall M \subset E, \ \ (M \text{ bounded}) \Longrightarrow \left(\overline{\mathcal{S}(M)} \text{ compact}\right)\right].$$

Equivalently, $\mathcal{S}$ is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, the sequence $(\mathcal{S}x_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}$.

Example 2.3 Let $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ be a bounded operator with finite dimensional range. Then, $\mathcal{S}$ is compact.

Definition 2.13 [70] An operator $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be completely continuous if for any sequence $(x_n)_{n\in\mathbb{N}}$ converging weakly to x, the sequence $(\mathcal{S}x_n)_{n\in\mathbb{N}}$ converges to $\mathcal{S}x$. That is,

$$x_n \rightharpoonup x \Longrightarrow \mathcal{S}x_n \to \mathcal{S}x.$$

Theorem 2.2 *Let $\mathcal{S}$ be an operator from a reflexive Banach space $\mathbb{X}$ into a Banach space $\mathbb{Y}$. If $\mathcal{S}$ is completely continuous then $\mathcal{S}$ is continuous and compact.*

Proof. Let $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ be completely continuous. We need to show the compactness of $\mathcal{S}$. Let M be a bounded subset of $\mathbb{X}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M. Reflexivity of $\mathbb{X}$ implies that $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(x_{n_k})_{n\in\mathbb{N}}$. Assume that $x_{n_k} \rightharpoonup x$. Because $\mathcal{S}$ is completely continuous, $\mathcal{S}x_{n_k} \to \mathcal{S}x$. This proves that $\mathcal{S}(M)$ is relatively compact. ∎

Definition 2.14 [82] If $f(x,t)$ is a continuous function and its derivative $f_x(x,t)$ is x piecewise continuous, then we get

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(x,t)\,dt = f(x, g_2(x))\,g_2'(x) - f(x, g_1(x))\,g_1'(x) + \int_{g_1(x)}^{g_2(x)} f_x(x,t)\,dt,$$

where $g_1(x)$ and $g_2(x)$ are differentiable.

2.1.5 Arzelà-Ascoli Theorem

Let $\mathbb{X}$ be a compact subset of a normed vector space over $\mathbb{F}$ and let $\mathcal{C}(\mathbb{X})$ be the normed vector space of real valued continuous functions on $\mathbb{X}$ with the $sup-$norm

$$\|f\|_\infty = \sup_{x\in\mathbb{X}} |f(x)|.$$

Let $\mathcal{F}$ be a collection of functions in $\mathcal{C}(\mathbb{X})$.

Definition 2.15 [70] (Uniformly bounded sequence) A sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}$ is called uniformly bounded if there is a number $M \geq 0$ such that

$$\|f_n\|_\infty = \sup_{x\in\mathbb{X}} |f_n(x)| \leq M, \ \forall f_n \in \mathcal{F}.$$

Definition 2.16 [70] (equicontinuous sequence) A sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}$ is called equicontinuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon,$$

whenever $\|x - y\|_{\mathbb{X}} < \delta$ for all functions f_n in the sequence.

Theorem 2.3 *[70] (The Ascoli–Arzelà theorem) If a sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}$ is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges uniformly, that means $\mathcal{F}$ is relatively compact in $\mathcal{C}(\mathbb{X})$.*

Corollary 2.1 *A collection $\mathcal{F} \subset \mathcal{C}(\mathbb{X})$ be compact if and only if it be closed, uniformly bounded and equicontinuous.*

2.2 Some Fixed Point Theorems

The fixed point theory is a versatile and fundamental theory that is broadly applicable in dealing with various types of equations including functional differential equations. Indeed, it is one of the very helpful and powerful approaches for establishing the existence and uniqueness of solutions to different types of equations.

Definition 2.17 [4] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$. A fixed point of a mapping $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ of $\mathbb{X}$ into itself is an $x \in \mathbb{X}$ which is mapped onto itself, that is

$$\mathcal{S}(x) = x.$$

This is equivalent to saying that the equation $S(x) - x = 0$ has a solution.

2.2.1 Schauder's Fixed Point Theorem

The Schauder fixed point theorem relies on the compactness of the operator in a Banach space.

Theorem 2.4 *[79] Let* $\mathbb{M}$ *be a non-empty bounded closed convex subset of a Banach space* $(\mathbb{X}, \|.\|)$ *and let* $\mathcal{S} : \mathbb{M} \longrightarrow \mathbb{M}$ *be a compact and continuous mapping. Then* $\mathcal{S}$ *has a fixed point in* $\mathbb{M}$.

An alternative version of the *Schauder* fixed point theorem can be stated as follows:

Theorem 2.5 *[79] Let* $\mathbb{M}$ *be a non-empty compact convex subset of a Banach space* $(\mathbb{X}, \|.\|)$ *and let* $\mathcal{S} : \mathbb{M} \longrightarrow \mathbb{M}$ *be a continuous mapping. Then* $\mathcal{S}$ *has a fixed point in* $\mathbb{M}$.

2.2.2 Banach's Fixed Point Theorem

The Banach fixed point theorem (known also as the Banach contraction principle) which was published by the mathematician Stefan Banach in 1922, is one of the useful tools that is extensively applicable in establishing the existence and uniqueness of solutions.

Theorem 2.6 *[69] Let* S *be a contraction on a Banach space* $\mathbb{X}$. *Then* S *has a unique fixed point* $\overline{x} \in X$.

Theorem 2.7 *[43] If* $\mathbb{M}$ *is a closed subset of a Banach space* $\mathbb{X}$ *and* $\mathcal{S} : \mathbb{M} \to \mathbb{M}$ *is a contraction, then* $\mathcal{S}$ *has a unique fixed point in* $\mathbb{M}$.

2.2.3 Krasnoselskii's Fixed Point Theorem

The Krasnoselskii fixed point theorem which is a combination of the Banach contraction mapping theorem and the Schauder fixed point theorem, is a theorem for a sum of two operators; one of them is continuous and compact and the other is a contraction. In fact, many problems can be written as fixed point equation having the form

$$x = Ax + Bx \;\; (\forall x \in \mathbb{M}),$$

where A and B are two operators and $\mathbb{M}$ is a closed convex nonempty subset of a functional Banach space.

Theorem 2.8 *(Krasnoselskii's fixed point theorem) [73] Let* $\mathbb{M}$ *be a closed convex nonempty subset of a Banach space* $(\mathbb{X}, \|.\|)$. *Suppose that* A *and* B *map* $\mathbb{M}$ *into* $\mathbb{X}$ *such that*

(i) $Ax + By \in \mathbb{M}\,(\forall x, y \in \mathbb{M})$,

(ii) *A is continuous and compact,*

(iii) *B is a contraction with constant* $\alpha < 1$.

Then there is a $z \in \mathbb{M}$ *with* $Az + Bz = z$.

Remark 2.3 The theorem becomes Banach's fixed point theorem if $A = 0$. In the event if $B = 0$, the theorem is Schauder's fixed point theorem.

2.3 Green's Function

The first appearance of the concept of the Green's function which is introduced by the British mathematician George Green [40], was in 1828. It is one of the potent means for solving nonhomogeneous linear differential equations associated with initial conditions or boundary conditions of the form

$$Ly(t) = f(t),$$

where L is a linear operator and f is the source term, or transforming them into integral equations whose kernels are a Green's functions.

Definition 2.18 (Green's function) [20] We will consider two – point n^{th}– order linear boundary value problems of the form

$$\begin{cases} L_n y(t) = \sigma(t), \ t \in I \equiv [c, d], \\ U_i(y) = \xi_i, \ i = \overline{1, m}, \end{cases} \tag{P_1}$$

where

$$L_n y(t) = a_0(t) y^{(n)}(t) + a_1(t) y^{(n-1)}(t) + ... + a_{n-1}(t) y'(t) + a_n(t) y(t),$$

and

$$U_i(y) = \sum_{j=0}^{n-1} \left(\alpha_j^i y^{(j)}(c) + \beta_j^i y^{(j)}(d)\right), \; i = \overline{1,m}, \; m \leq n,$$

being α_j^i, β_j^i and ξ_i real constants for all $i = \overline{1,m}$ and $j = \overline{0,n-1}$, σ and a_k continuous real functions for all $k = \overline{0,n}$, and $a_0(t) \neq 0$ for all $t \in I$.

We say that G is a Green's function for problem (P_1) if it satisfies the following properties:

(**G1**) G is defined on the square $I \times I$.

(**G2**) For $k = \overline{0,n-2}$, the partial derivatives $\dfrac{\partial^k G}{\partial t^k}$ exist and they are continuous on $I \times I$.

(**G3**) $\dfrac{\partial^{k-1} G}{\partial t^{k-1}}$ and $\dfrac{\partial^k G}{\partial t^k}$ exist and are continuous on the triangles $c \leq s < t \leq d$ and $c \leq t < s \leq d$.

(**G4**) For each $t \in (c,d)$ there exist the lateral limits

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^+\right),$$

and

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^-\right),$$

(i.e., the limits when $(t,s) \to (t,t)$ with $s > t$ or with $s < t$) and, moreover

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^+\right) - \frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t, t^-\right) = -\frac{1}{a_0(t)}.$$

(**G5**) For each $s \in (c,d)$, the function $t \to G(t,s)$ is a solution of the differential equation $L_n y = 0$ on $t \in [c,s)$ and $t \in (s,d]$. That is,

$$a_0(t) \frac{\partial^n G}{\partial t^n}(t,s) + a_1(t) \frac{\partial^{n-1} G}{\partial t^{n-1}}(t,s) + ... + a_{n-1}(t) \frac{\partial G}{\partial t}(t,s) + a_n(t) G(t,s) = 0,$$

on both intervals.

(**G6**) For each $s \in (c,d)$, the function $t \to G(t,s)$ satisfies the boundary conditions

$$\sum_{j=0}^{n-1} \left(\alpha_j^i \frac{\partial^j G}{\partial t^j}(c,s) + \beta_j^i \frac{\partial^j G}{\partial t^j}(d,s)\right) = 0, \; i = \overline{1,m}.$$

Theorem 2.9 *[20] Let us suppose that the homogeneous problem*

$$\begin{cases} L_n y(t) = 0, \ t \in I \equiv [c,d], \\ U_i(y) = 0, \ i = \overline{1,n}, \end{cases} \tag{P_2}$$

has only the trivial solution. Then there exists a unique Green's function, $G(t,s)$, related to (P_2). Moreover, for each continuous function σ, the unique solution of the problem

$$\begin{cases} L_n y(t) = \sigma(t), \ t \in I \equiv [c,d], \\ U_i(y) = 0, \ i = \overline{1,n}, \end{cases}$$

is given by the expression

$$y((t)) = \int_c^d G(t,s)\,\sigma(s)\,ds.$$

Example 2.4 We solve the following first-order periodic problem by using Green's function method:

$$\begin{cases} y'(t) + \eta y(t) = \sigma(t), t \in [a,b], \\ \qquad y(a) = y(b). \end{cases}$$

Multiply the both sides of the equation by $e^{\eta t}$ and we get

$$y'(t)\,e^{\eta t} + \eta y(t)\,e^{\eta t} = \sigma(t)\,e^{\eta t},$$

we have

$$\left(y(t)\,e^{\eta t}\right)' = y'(t)\,e^{\eta t} + \eta y(t)\,e^{\eta t},$$

so,

$$\left(y(t)\,e^{\eta t}\right)' = \sigma(t)\,e^{\eta t},$$

by integration

$$y(t)\,e^{\eta t} - y(a)\,e^{\eta a} = \int_a^t \sigma(s)\,e^{\eta s} ds, t \in [a,b],$$

the solution given by

$$y(t) = y(a)\,e^{-\eta(t-a)} + \int_a^t \sigma(s)\,e^{-\eta(t-s)} ds.$$

We have $y(a) = y(b)$, so

$$y(a) = y(a)\,e^{-\eta(b-a)} + \int_a^b \sigma(s)\,e^{-\eta(b-s)} ds,$$

hence,

$$y(a) = \frac{1}{1-e^{-\eta(b-a)}} \int_a^b \sigma(s) e^{-\eta(b-s)} ds,$$

after substituting in the solution we find

$$\begin{aligned} y(t) &= \frac{e^{-\eta(t-a)}}{1-e^{-\eta(b-a)}} \int_a^b \sigma(s) e^{-\eta(b-s)} ds + \int_a^t \sigma(s) e^{-\eta(t-s)} ds \\ &= \frac{e^{-\eta(t-a)}}{1-e^{-\eta(b-a)}} \int_a^b \sigma(s) e^{-\eta(b-s)} ds + \int_a^b \chi_{(a,t)} \sigma(s) e^{-\eta(t-s)} ds \\ &= \int_a^b G(t,s) \sigma(s) ds. \end{aligned}$$

Where $\chi_{(a,t)}$ is the indicator function of the interval (a,t) and

$$G(t,s) = \frac{1}{e^{b\eta} - e^{a\eta}} \begin{cases} e^{\eta(b+s-t)}, a \le s < t \le b, \\ e^{\eta(a+s-t)}, a < t < s \le b. \end{cases}.$$

Lemma 2.2 *The Green's function $G(t,s)$ defined by:*

$$G(t,s) = \frac{\exp\left(\int_t^s p(u)\, du\right)}{\exp\left(\int_0^T p(u)\, du\right) - 1},$$

satisfies these the following properties:

(1) $G(t+T, s+T) = G(t,s), \forall t, s \in [0,T]$.

(2) *There exist two positive constants*

$$\gamma_0 = \frac{\exp\left(-\int_0^T p(u)\, du\right)}{\exp\left(\int_0^T p(u)\, du\right) - 1}, \ \gamma_1 = \frac{\exp\left(\int_0^T p(u)\, du\right)}{\exp\left(\int_0^T p(u)\, du\right) - 1},$$

such that

$$0 < \gamma_0 \le G(t,s) \le \gamma_1, \forall s,t \in [0,T], \tag{2.1}$$

(3) *For all $t_1, t_2 \in [0,T]$ and $t_1 < t_2$ we have*

$$\begin{aligned} \int_{t_1}^{t_1+T} &\left| \exp\left(\int_{t_2}^s p(u)\, du \right) - \exp\left(\int_{t_1}^s p(u)\, du \right) \right| ds \\ &\le Tp \exp\left(\int_0^T p(u)\, du \right) |t_2 - t_1|. \end{aligned} \tag{2.2}$$

Proof. It is easy to prove that both properties (1) and (2) are true. So, we will move on to proving the last property For all $t_1, t_2 \in [0,T]$ and $t_1 < t_2$ we have

$$\begin{aligned} & \int_{t_1}^{t_1+T} \left| \exp\left(\int_{t_2}^{s} p(u)\,du \right) - \exp\left(\int_{t_1}^{s} p(u)\,du \right) \right| ds \\ = & \int_{t_1}^{t_1+T} \left| \exp\left(\int_{t_2}^{s} p(u)\,du \right) \right| \left| 1 - \exp\left(\int_{t_1}^{t_2} p(u)\,du \right) \right| ds. \end{aligned}$$

Using the mean value theorem on the function

$$g(t) = \exp\left(\int_{t}^{t_2} p(u)\,du \right), \text{ for all } t \in [t_1, t_2],$$

it is easy to show that the function g is defined and continuous in the interval $[t_1, t_2]$ and differentiable on the interval $]t_1, t_2[$, so

$$\begin{aligned} g(t_2) - g(t_1) &= 1 - \exp\left(\int_{t_1}^{t_2} p(u)\,du \right) \\ &= -p(\vartheta) \exp\left(\int_{\vartheta}^{t_2} p(u)\,du \right) (t_2 - t_1), \ \forall \vartheta \in]t_1, t_2[, \end{aligned}$$

then

$$\begin{aligned} & \int_{t_1}^{t_1+T} \left| \exp\left(\int_{t_2}^{s} p(u)\,du \right) - \exp\left(\int_{t_1}^{s} p(u)\,du \right) \right| ds \\ = & \int_{t_1}^{t_1+T} \left| \exp\left(\int_{t_2}^{s} p(u)\,du \right) \right| \left| p(\vartheta) \exp\left(\int_{\vartheta}^{t_2} p(u)\,du \right) (t_2 - t_1) \right| ds \\ \leq & \int_{t_1}^{t_1+T} |p(\vartheta)| \, |t_2 - t_1| \exp\left(\int_{\vartheta}^{s} p(u)\,du \right) ds \\ \leq & \ p \, |t_2 - t_1| \int_{t_1}^{t_1+T} \exp\left(\int_{0}^{T} p(u)\,du \right) ds \\ \leq & \ Tp \exp\left(\int_{0}^{T} p(u)\,du \right) |t_2 - t_1|, \end{aligned}$$

therefore

$$\int_{t_1}^{t_1+T} \left| \exp\left(\int_{t_2}^{s} p(u)\,du \right) - \exp\left(\int_{t_1}^{s} p(u)\,du \right) \right| ds \leq Tp \exp\left(\int_{0}^{T} p(u)\,du \right) |t_2 - t_1|.$$

The proof is finished ■

2.4 Iterative Functional Differential Equations

2.4.1 Iterations

Definition 2.19 The composition $x \circ y$ of the function x with the function y is

$$(x \circ y)(t) = x(y(t)),$$

the domain of $x \circ y$ is the set of all t in the domain of y such that $y(t)$ is in the domain of x.

Definition 2.20 For $x : E \longrightarrow E$, the n^{th} iterate of function x, denoted by $x^{[n]}$ for some nonnegative integer n, is defined recursively by

$$x^{[0]} = Id_E,$$

and

$$x^{[n+1]} = x \circ x^{[n]},$$

where Id_E is the identity mapping on E.

Lemma 2.3 *In the space P_T defined in the Example (2.2), we consider the subset*

$$\Omega = \{x \in P_T : 0 < R_0 \leq x(t) \leq R_1, \ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, \ \forall t_1, t_2 \in [0, T]\},$$

then,

(1) *[89] For all $x, y \in \Omega$*

$$\left\|x^{[k]} - y^{[k]}\right\| \leq \sum_{j=0}^{k-1} L^j \|x - y\|, \ k = 1, 2, \ldots$$

where $x^{[k]} = x \circ x \circ \ldots \circ x$, k times.

(2) *Ω is closed convex and bounded.*

(3) *Ω is compact.*

Proof.

(1) We will prove this inequality by induction. So, the proof will now proceed in two steps:

The basis step: For $k = 1$, we have

$$\|x - y\| \leq \|x - y\| .$$

then, the inequality holds for $k = 1$

The inductive step: Now, we assume that the inequality holds for a given $k = m$ and we want to show that it also holds for $k = m + 1$. Suppose that

$$\left\|x^{[m]} - y^{[m]}\right\| \leq \sum_{j=0}^{m-1} L^j \|x - y\| ,$$

then

$$\begin{aligned} \left|x^{[m+1]}(t) - y^{[m+1]}(t)\right| &\leq \left|x\left(x^{[m]}(t)\right) - x\left(y^{[m]}(t)\right)\right| + \left|x\left(y^{[m]}(t)\right) - y\left(y^{[m]}(t)\right)\right| \\ &\leq L\left|x^{[m]}(t) - y^{[m]}(t)\right| + \left|x\left(y^{[m]}(t)\right) - y\left(y^{[m]}(t)\right)\right| , \end{aligned}$$

so

$$\begin{aligned} \left\|x^{[m+1]} - y^{[m+1]}\right\| &\leq L\left\|x^{[m]} - y^{[m]}\right\| + \|x - y\| \\ &\leq L\sum_{j=0}^{m-1} L^j \|x - y\| + \|x - y\| \\ &\leq \left(\sum_{j=0}^{m-1} L^{j+1} + 1\right) \|x - y\| \\ &\leq \sum_{j=0}^{m} L^j \|x - y\| . \end{aligned}$$

By induction we deduce that

$$\left\|\varphi^{[m]} - \psi^{[m]}\right\| \leq \sum_{j=0}^{m-1} L^j \|\varphi - \psi\| \qquad \forall m \in \mathbb{N} ,$$

which finishes the proof.

(2) **<u>Ω is closed:</u>** Let $(x_n)_{n\in\mathbb{N}} \subset \Omega$ be a converge sequence to $x \in P_T$,

then

$$0 < R_0 \leq \lim_{n\to\infty} x_n \leq R_1.$$

Which is

$$0 < R_0 \leq x \leq R_1.$$

On the other hand

$$\begin{aligned} \left|x\left(t_2\right)-x\left(t_1\right)\right| &= \left|x\left(t_2\right)-x_n\left(t_2\right)+x_n\left(t_2\right)-x_n\left(t_1\right)+x_n\left(t_1\right)-x\left(t_1\right)\right| \\ &\leq \left|x\left(t_2\right)-x_n\left(t_2\right)\right|+\left|x_n\left(t_2\right)-x_n\left(t_1\right)\right|+\left|x_n\left(t_1\right)-x\left(t_1\right)\right| \\ &\leq \left|x_n\left(t_2\right)-x_n\left(t_1\right)\right| \\ &\leq L\left|t_2-t_1\right|. \end{aligned}$$

So $x \in \Omega$ and Ω is closed.

Ω is convex: Let $x_1, x_2 \in \Omega$ and $\alpha \in [0,1]$.

we have

$$0 < \alpha R_0 \leq \alpha x_1(t) \leq \alpha R_1,$$

and

$$0 < (1-\alpha) R_0 \leq (1-\alpha) x_2(t) \leq (1-\alpha) R_1.$$

So

$$0 < \alpha R_0 + (1-\alpha) R_0 \leq \alpha x_1(t) + (1-\alpha) x_2(t) \leq \alpha R_1 (1-\alpha) R_1.$$

Then

$$0 < R_0 \leq \alpha x_1(t) + (1-\alpha) x_2(t) \leq R_1.$$

Furthermore

$$\begin{aligned} &\left|\left(\alpha x_1+(1-\alpha) x_2\right)\left(t_2\right)-\left(\alpha x_1+(1-\alpha) x_2\right)\left(t_1\right)\right| \\ &= \left|\alpha\left(x_1\left(t_2\right)-x_1\left(t_1\right)\right)+(1-\alpha)\left(x_2\left(t_2\right)-x_2\left(t_1\right)\right)\right| \\ &\leq \alpha\left|x_1\left(t_2\right)-x_1\left(t_1\right)\right|+(1-\alpha)\left|x_2\left(t_2\right)-x_2\left(t_1\right)\right| \\ &\leq \alpha L\left|t_2-t_1\right|+(1-\alpha) L\left|t_2-t_1\right| \\ &\leq L\left|t_2-t_1\right|. \end{aligned}$$

Which proves the convexity of Ω.

Ω is bounded: If $x \in \Omega$, we have

$$0 < R_0 \leq x(t) \leq R_1.$$

But $0 < R_0 \leq R_1$, then

$$|x(t)| \leq R_1.$$

So $\Omega \subset B(0, R_1)$ and Ω is bounded.

(3) **Ω is Compact:** According to the two conditions $0 < R_0 \leq x(t) \leq R_1$ and $|x(t_2) - x(t_1)| \leq L|t_2 - t_1|$ in the definition of the set Ω, we deduce that this set is uniformly bounded and equicontinuous, therefore the Ascoli–Arzelà theorem ensures that it is relatively compact and as it is closed then it is compact.

■

CHAPTER 3

Existence, Uniqueness and Continuous Dependence on Parameters of Solutions for an Iterative Houseflies Model

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In this chapter, we derived some suitable criteria under which the existence, uniqueness, and stability results for an iterative Musca Domestica model were established.

3.1 Introduction

Musca domestica, the common housefly, is a global pest that harms both humans and animals. It is well known for its ability to harbor and spread a wide range of illnesses, such as cholera, yaws, amoebic dysentery, poliomyelitis, typhoid fever, diarrhea, and so forth. It can also contaminate food.

Among the first models with delays that have been used to describe the growth of single-species insect populations, we can cite the work of Maynard Smith [62] who suggested the following first order differential equation with a constant delay for modelling a single-species population with two stages; larva and adult:

$$M'(t) = -pM(t) + bM(t-\tau)\left[1 - M(t-\tau)\right],$$

with $M(t)$ is denoting the size of the population at time t, $p > 0$ is the adult mortality rate, b is the number of eggs laid per adult and the time delay τ stands for the time elapsing between oviposition and adult eclosion.

In 1976, Taylor and Sokal [81] developed the following delay differential equation, which was inspired by the previous equation and used to explain the oscillations in the dynamics of housefly populations in lab settings:

$$M'(t) = -pM(t) + bM(t-\tau)\left[\beta - b\alpha M(t-\tau)\right],$$

where β is the maximum egg-adult survival rate and α is the reduction in survival produced by each additional egg.

A growing number of experiments that have clearly manifested that the delays in many hereditary phenomena are generally depending on both time and state, have fascinated us and attracted our interests to consider a housefly model with iterative terms resulting from a time and state dependent delay. Indeed, several studies have revealed that due many factors such as the competition for food during the three larval stages, the duration of the fly life cycle depends in fact on the time and the population size. Simply said, adult females lay clusters of eggs in several batches by stacking them on top of each other which affects the duration of their life cycles and hence, the life cycles of the maggots at the bottom (near the moist and nutrient-rich place) are faster than those superimposed above them, which in turn are faster than those at the top which means that the duration of the life cycles τ varies depending not only on the time but also on the number of adult flies that lay eggs. By taking into account this information, we can revisit Taylor and Sokal model to the following one:

$$M'(t) = -p(t) M(t) + bM(t - \tau(t, M(t))) [\beta - b\alpha M(t - \tau(t, M(t)))],$$

and by assuming that $\tau(t, M(t)) = t - M(t)$, we arrive at the following first order iterative differential equation that involves implicitly the above time and state dependent delay:

$$x'(t) + p(t) x(t) = b\beta x^{[2]}(t) - b^2\alpha \left(x^{[2]}(t)\right)^2, \tag{3.1}$$

where $x^{[2]}(t) = x(x(t))$, $p \in \mathcal{C}(\mathbb{R},]0, \infty[)$ is a $T-$periodic function and the remainder parameters are positive.

Equation (3.1) is an iterative differential equation and equations of this kind have tremendous applications in an extremely wide range of areas, including biology, medicine, classical electrodynamics, physics, epidemiology, hematology, population dynamics and many other branches of science and technology. They have been of vital importance in modelling various natural phenomena over the last three centuries- and have been even more so in the last ten years- thanks to some papers (see [9]-[54], [64], [89] and references therein). Here, we would like to mention some recent works on iterative problems that arise in life sciences.

In [16], the authors have used Schauder's fixed point theorem to study the following Nicholson's blowflies equation with an iterative harvesting effort:

$$N'(t) = -p(t) N(t) + a(t) N(t-\tau) e^{-\gamma(t)N(t-\tau)} - qN(t-\tau) E\left(t, N(t), N^{[2]}(t), ..., N^{[n]}(t)\right),$$

where $N(t)$ denotes the population density of the sheep blowfly, Lucilia Cuprina.

In [10], by virtue of the same aforementioned fixed point theorem, Bouakkaz has investigated the following class of first-order iterative differential equations with application to three iterative hematopoiesis models for humans and animals.

$$x'(t) = -p(t) x(t) + a(t) x^m(t - \tau(t)) f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right),$$

where $x(t)$ is the density of mature cells in blood circulation at time t, $p(t)$ and $a(t)$ are, respectively the death and the production rates of blood cells.

In [64], Mezghiche et al. have applied Banach and Krasnoselskii's fixed point theorems together with the Green's functions method to establish the existence, uniqueness and stability results for the following class of first order neutral delay differential equations with an iterative harvesting term:

$$\frac{d}{dt}\left[x(t)-cx(t-\tau(t))\right]=-p(t)x(t)+f(t,x(t-\tau(t)))-H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right).$$

Here $x(t)$ can represent for example, a size of a population, a number of individuals or a blood cell density and H is the harvesting function.

In [48], Khemis et al. have utilized the same approach adopted in [64] to discuss the existence, uniqueness and stability of positive periodic solutions for the following Lasota–Wazewska model with an iterative production term and a delayed harvesting one:

$$x'(t)=-p(t)x(t)+a(t)e^{-\gamma x^{[2]}(t)}-H(t,x(t-\tau)),$$

where $x(t)$ is the density of mature erythrocytes in an animal at time t.

So, here we draw our motivation first and foremost from these works and as we said before, from the fact that delays in life sciences depend generally on the time and the state and also from our contribution in enriching and complementing some earlier publications whether on insect population dynamics or iterative problems. It is worth noting here that these latter are obviously burning topics and hence in many cases they are difficult to deal with. For this, the theory of such kind of equations which can be considered as a special type of the so-called functional differential equations with delays depending upon both the time and the state variables; has not yet been well established (see [9]-[54], [64], [89]). The chief problem lies in the iterative terms that generally impede the application of usual methods and could make the study somewhat difficult.

The main purpose of this chapter is to establish some sufficient criteria for ensuring the existence, uniqueness and stability of positive periodic solutions of the iterative differential equation (3.1). For achieving our goals, we use an attractive technique based on converting the problem at hand into an equivalent integral equation before constructing an integral operator with a Green's function type kernel. Next, through the fixed point theory, some functional analysis tools together with some properties of the obtained Green's function, we success in establishing some new existence, uniqueness and stability results for our problem.

The plan of the chapter is as follows. In Section 2, we introduce our notations, assumptions and some preliminaries. In Section 3, we state and prove our main results concerning the existence, uniqueness and continuous dependence on parameters of positive periodic solutions for the proposed model, whilst Section 4 is dedicated to give an example to corroborate the effectiveness of the obtained findings.

3.2 Preliminaries

At the beginning of this section, we provide some definitions and hypotheses that we will need in the rest of the chapter.

Let P_T the *Banach* space of all continuous and periodic functions with the period T given in the Example 2.2, and let Ω the subset that is given in Lemma 2.3.

In our study, we will be assumed that

($\mathbf{H}_1$) For all $x \in \Omega$ and $s \in [0, T]$, we suppose that

$$\min_{s\in[0,T]} \left\{ b\beta x^{[2]}(s) - b^2\alpha \left(x^{[2]}(s)\right)^2 \right\} \geq \frac{R_0}{\gamma_0 T}. \tag{3.2}$$

($\mathbf{H}_2$) The following estimates are satisfied:

$$\gamma_1 T b\beta \leq 1, \tag{3.3}$$

$$\left(\gamma_1 pT + 2\gamma_1\right)\left(b\beta R_1 + b^2\alpha R_1^2\right) \leq L. \tag{3.4}$$

Here, we prove an equivalence between equation (3.1) with the periodic properties and a certain integral equation.

Lemma 3.1 *$x \in \Omega \cap \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is a solution of equation (3.1) if and only if $x \in \Omega$ is a solution of the following integral equation:*

$$x(t) = \int_t^{t+T} G(t,s)\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right] ds, \tag{3.5}$$

where G is a Green's function given in Lemma 2.2.

Proof. Let $x \in \Omega \cap \mathcal{C}^1(\mathbb{R},\mathbb{R})$ be a solution of equation (3.1). Multiplying both sides of this equation by $\exp\left(\int_0^t p(u)\,du\right)$ we get

$$\frac{d}{ds}\left[x(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds = \left[b\beta x^{[2]}(t) - b^2\alpha\left(x^{[2]}(t)\right)^2\right]\exp\left(\int_0^t p(u)\,du\right).$$

The integration from t to $t+T$ gives

$$\begin{aligned}\int_t^{t+T}\frac{d}{ds}&\left[x(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds\\ &= \int_t^{t+T}\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right]\exp\left(\int_0^s p(u)\,du\right)ds.\end{aligned}$$

The fact that $x(t) = x(t+T)$ implies that

$$\begin{aligned}\int_t^{t+T}\frac{d}{ds}&\left[x(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds\\ &= x(t)\left[\exp\left(\int_0^{t+T} p(u)\,du\right) - \exp\left(\int_0^t p(u)\,du\right)\right]\\ &= x(t)\left[\exp\left(\int_0^t p(u)\,du\right)\left(\exp\left(\int_t^{t+T} p(u)\,du\right) - 1\right)\right].\end{aligned}$$

Therefore

$$\begin{aligned}x(t) &= \int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_t^{t+T} p(u)\,du\right) - 1}\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds\\ &= \int_t^{t+T} G(t,s)\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds,\end{aligned}$$

which completes the first step of the proof.

Conversely, if we assume that x satisfies (3.5) and by the derivation of this integral equation, we infer that x satisfies equation (3.1). ■

Remark 3.1 By using the inequality of Zhao 2.3 for $k = 2$. We have

For all $x, y \in \Omega$,

$$\begin{aligned}\left|x^{[2]}(t) - y^{[2]}(t)\right| &\le |x(x(t)) - x(y(t))| + |x(y(t)) - y(y(t))|\\ &\le L\,|x(t) - y(t)| + \|x - y\|\\ &\le L\,\|x - y\| + \|x - y\|.\end{aligned}$$

So

$$\left\|x^{[2]} - y^{[2]}\right\| = \sup_{t\in[0,T]}\left|x^{[2]}(t) - y^{[2]}(t)\right| \le (1 + L)\,\|x - y\|. \tag{3.6}$$

3.3 Existence of Positive Periodic Solutions

In the first part of this section, we will use the Schauder fixed point theorem with some properties of the obtained Green's function to prove the existence of at least one positive periodic solution for equation (3.1). For this end and by virtue of Lemma 3.1, we define an operator $\mathcal{K} : \Omega \to P_T$ as follows:

$$(\mathcal{K}x)(t) = \int_t^{t+T} G(t,s) \left[b\beta x^{[2]}(s) - b^2\alpha \left(x^{[2]}(s) \right)^2 \right] ds. \tag{3.7}$$

So, fixed points of $\mathcal{K}$ are solutions of equation (3.1) and vice versa. This means that our main task is clearly to show that the operator $\mathcal{K}$ has fixed points which are solutions to equation (3.1).

Let us begin by pointing out that by virtue of the periodic properties, the operator $\mathcal{K}$ is well-defined. Next, we state and prove the following lemma which establishes the continuity and compactness of $\mathcal{K}$.

Lemma 3.2 *The operator $\mathcal{K}$: $\Omega \to P_T$ given by (3.7) is continuous and compact.*

Proof. Thanks to Lemma 2.3, Ω is a compact subset of P_T. So, to show that $\mathcal{K}$ is a compact operator it suffices to show that it is continuous. For $x, y \in \Omega$, we have

$$\begin{aligned} |(\mathcal{K}x)(t) - (\mathcal{K}y)(t)| &\le b\beta \int_t^{t+T} G(t,s) \left| x^{[2]}(s) - y^{[2]}(s) \right| ds \\ &\quad + b^2\alpha \int_t^{t+T} G(t,s) \left| x^{[2]}(s) + y^{[2]}(s) \right| \left| x^{[2]}(s) - y^{[2]}(s) \right| ds. \end{aligned}$$

Taking into account the second property of Green's function (2.1) and (3.6), we obtain

$$\begin{aligned} |(\mathcal{K}x)(t) - (\mathcal{K}y)(t)| &\le \left(\gamma_1 Tb\beta (1+L) + 2\gamma_1 Tb^2\alpha R_1 (1+L) \right) \|x-y\| \\ &= \mu \|x-y\|, \end{aligned}$$

where,

$$\mu = \gamma_1 Tb (\beta + 2b\alpha R_1)(1+L),$$

which shows that the operator $\mathcal{K}$ is Lipschitz continuous and hence continuous. Therefore, $\mathcal{K}$ is compact. ■

Lemma 3.3 *If conditions (3.2) and (3.3) hold, then*

$$R_0 \leq (\mathcal{K}x)(t) \leq R_1,$$

for all $x \in \Omega$.

Proof. Let $x \in \Omega$. In view of (3.2) we have

$$\begin{aligned}(\mathcal{K}x)(t) &= \int_t^{t+T} G(t,s)\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right] ds\\ &\geq \gamma_0 T \min_{s\in[0,T]}\left\{b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right\}\\ &\geq \gamma_0 T \frac{R_0}{\gamma_0 T} = R_0,\end{aligned}$$

and by using (3.3) we find

$$\begin{aligned}(\mathcal{K}x)(t) &= \int_t^{t+T} G(t,s)\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right] ds\\ &\leq b\beta \int_t^{t+T} G(t,s)\, x^{[2]}(s)\, ds\\ &\leq \gamma_1 T b\beta R_1\\ &\leq R_1.\end{aligned}$$

Consequently, $R_0 \leq (\mathcal{K}x)(t) \leq R_1$. ■

Lemma 3.4 *If condition (3.4) holds, then*

$$\left|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)\right| \leq L\left|t_2 - t_1\right|,$$

for all $t_1, t_2 \in \mathbb{R}$.

Proof. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. For $x \in \Omega$, we have

$$\begin{aligned}\left|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)\right| = &\left|\int_{t_2}^{t_2+T} G(t_2,s)\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right] ds\right.\\ &\left.- \int_{t_1}^{t_1+T} G(t_1,s)\left[b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right] ds\right|.\end{aligned}$$

Thereby

$$\begin{aligned}\left|(\mathcal{K}x)(t_2)-(\mathcal{K}x)(t_1)\right| = \Bigg| & \int_{t_2}^{t_1} G(t_2,s)\left[b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds \\ & + \int_{t_1}^{t_1+T} G(t_2,s)\left[b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds \\ & + \int_{t_1+T}^{t_2+T} G(t_2,s)\left[b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds \\ & - \int_{t_1}^{t_1+T} G(t_1,s)\left[b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds\Bigg|,\end{aligned}$$

from which we infer that

$$\begin{aligned}\left|(\mathcal{K}x)(t_2)-(\mathcal{K}x)(t_1)\right| \le & \int_{t_2}^{t_1} G(t_2,s)\left[b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds \\ & + \int_{t_1+T}^{t_2+T} G(t_2,s)\left[b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right]ds \\ & + \int_{t_1}^{t_1+T} \left|G(t_2,s)-G(t_1,s)\right|\left|b\beta x^{[2]}(s)-b^2\alpha\left(x^{[2]}(s)\right)^2\right|ds.\end{aligned}$$

It follows from the two properties of Green's function (2.1), (2.2) and (3.4) that

$$\begin{aligned}\left|(\mathcal{K}x)(t_2)-(\mathcal{K}x)(t_1)\right| & \le 2\gamma_1\left(b\beta R_1+b^2\alpha R_1^2\right)|t_2-t_1| \\ & + \gamma_1 pT\left(b\beta R_1+b^2\alpha R_1^2\right)|t_2-t_1| \\ & = \left(\gamma_1 pT+2\gamma_1\right)\left(b\beta R_1+b^2\alpha R_1^2\right)|t_2-t_1| \\ & \le L\,|t_2-t_1|\,.\end{aligned}$$

So, $\left|(\mathcal{K}x)(t_2)-(\mathcal{K}x)(t_1)\right| \le L\,|t_2-t_1|$ for all $t_1,t_2\in\mathbb{R}$. ■

Our first main result is the following theorem:

Theorem 3.1 *Suppose that conditions (3.2) − (3.4) hold, then equation (3.1) has at least one positive periodic solution $x\in\Omega$.*

Proof. As a result of Lemmas 3.3 and 3.4, $\mathcal{K}$ maps Ω into itself, i.e. $\mathcal{K}(\Omega)\subset\Omega$ and from Lemma 3.2, $\mathcal{K}$ is a compact and continuous operator, so all requirements of the Schauder's fixed point theorem are satisfied. This shows that $\mathcal{K}$ has at least one fixed point in Ω, which means that equation (3.1) admits at least one positive periodic solution. ■

3.4 Uniqueness of Positive Periodic Solutions

The second part of this section will be devoted to establishing the uniqueness of solutions by using the Banach contraction principle.

Theorem 3.2 *Besides of the assumptions (3.2) − (3.4) if we further assume that $\mu < 1$, then equation (3.1) has one and only one solution $x \in \Omega$.*

Proof. From Lemma 3.2, for all $x, y \in \Omega$ we arrived at

$$\left|(\mathcal{K}x)(t) - (\mathcal{K}y)(t)\right| \leq \mu \left\|x - y\right\|.$$

Thanks to condition $\mu < 1$ and the Banach fixed point theorem, operator $\mathcal{K}$ has a unique fixed point in Ω. From Lemma 3.1, this unique fixed point is the unique positive periodic solution of equation (3.1). ■

3.5 Continuous Dependence on Parameters

Now, we establish the continuous dependence of the solution upon the adult mortality rate.

Theorem 3.3 *The solution obtained in Theorem 3.2 depends continuously on the function p.*

Proof. Let

$$x_1(t) = \int_t^{t+T} G_1(t,s)\left[b\beta x_1^{[2]}(s) - b^2\alpha\left(x_1^{[2]}(s)\right)^2\right] ds,$$

be the unique solution of equation (3.1) and let

$$x_2(t) = \int_t^{t+T} G_2(t,s)\left[b\beta x_2^{[2]}(s) - b^2\alpha\left(x_2^{[2]}(s)\right)^2\right] ds,$$

be a solution of the perturbed equation with a small perturbation in the adult mortality rate $p_1(t)$ where

$$G_1(t,s) = \frac{\exp\left(\int_t^s p_1(u)\,du\right)}{\exp\left(\int_0^T p_1(u)\,du\right) - 1}, \quad G_2(t,s) = \frac{\exp\left(\int_t^s p_2(u)\,du\right)}{\exp\left(\int_0^T p_2(u)\,du\right) - 1}.$$

Estimating the difference between $x_1(t)$ and $x_2(t)$, we obtain

$$\begin{aligned}|x_1(t)-x_2(t)| \leq & \int_t^{t+T} |G_1(t,s)-G_2(t,s)| \left| b\beta x_1^{[2]}(s) - b^2\alpha \left(x_1^{[2]}(s)\right)^2 \right| ds \\ & + \int_t^{t+T} G_2(t,s) \left[\left| b\beta x_1^{[2]}(s) - b\beta x_2^{[2]}(s) \right| \right. \\ & \left. + \left| b^2\alpha \left(x_1^{[2]}(s)\right)^2 - b^2\alpha \left(x_2^{[2]}(s)\right)^2 \right| \right] ds.\end{aligned}$$

Thanks to the mean value theorem, we get

$$\int_t^{t+T} |G_1(t,s)-G_2(t,s)|\, ds \leq \eta \|p_1-p_2\|, \tag{3.8}$$

where

$$\eta = \frac{T^2 e^{T(\|p_2\|+\max(\|p_1\|,\|p_2\|))}}{\left(\exp\left(\int_0^T p_1(u)\,du\right)-1\right)\left(\exp\left(\int_0^T p_2(u)\,du\right)-1\right)} + \frac{T^2 e^{T\max(\|p_1\|,\|p_2\|)}}{\exp\left(\int_0^T p_1(u)\,du\right)-1}.$$

It follows from (2.1), (3.8) and (3.6) that

$$\begin{aligned}|x_1(t)-x_2(t)| \leq & \, TbR_1(\beta-b\alpha R_1)\,\eta\,\|p_1-p_2\| \\ & + \gamma_1 Tb(1+L)(\beta+2b\alpha R_1)|x_1(t)-x_2(t)|.\end{aligned}$$

So

$$\|x_1-x_2\| \leq \frac{TbR_1\eta(\beta-b\alpha R_1)}{1-\mu}\|p_1-p_2\|.$$

This completes the proof. ■

3.6 Example

Here is a concrete example illustrating Theorems 3.1, 3.2 and 3.3.

Example 3.1 Consider the following iterative houseflies model:

$$x'(t)+\left(0.025+0.024\sin^2\frac{2\pi t}{35}\right)x(t) = (0.05)(0.4)\,x^{[2]}(t)-(0.05)^2(0.000226)\left(x^{[2]}(t)\right)^2, \tag{3.9}$$

where

$$p(t) = 0.025+0.024\sin^2\frac{2\pi t}{35},\ b=0.05,\ \beta=0.4 \text{ and } \alpha=0.000226.$$

Let

$$\Omega_1 = \{x \in P_T : 0 < R_0 \leq x(t) \leq R_1,\ \ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\},$$

where $T = 35$, $R_0 = 0.00721$, $R_1 = 0.8$ and $L = 0.2$.

We define an integral operator $\mathcal{K}_1 : \Omega_1 \to P_T$ as follows:

$$(\mathcal{K}_1 x)(t) = \int_t^{t+35} G(t,s)\left[(0.02)\, x^{[2]}(s) - \left(565 \times 10^{-9}\right)\left(x^{[2]}(s)\right)^2\right] ds,$$

where its kernel is the following Green's function:

$$G(t,s) = \frac{\exp\left(\int_t^{t+35}\left(0.025 + 0.024 \sin^2 \frac{2\pi u}{35}\right) du\right)}{\exp\left(\int_t^{t+35}\left(0.025 + 0.024 \sin^2 \frac{2\pi u}{35}\right) du\right) - 1}.$$

Thanks to the periodic properties, the operator $\mathcal{K}_1$ is well-defined.

We have

$$p = 0.049,\ \ \gamma_0 \approx 0.103\,32,\ \ \gamma_1 \approx 1.3772 \text{ and } \mu = \gamma_1 T b(\beta + 2b\alpha R_1)(1 + L) \approx 1.1569 > 1.$$

Moreover, we find

$$\min_{s\in[0,35]}\left\{b\beta x^{[2]}(s) - b^2\alpha\left(x^{[2]}(s)\right)^2\right\} \approx 1.9996 \times 10^{-3} \geq \frac{R_0}{\gamma_0 T} \approx 1.993\,8 \times 10^{-3},$$

which means that condition (3.2) is satisfied. And

$$\gamma_1 T b\beta = 0.96404 < 1,$$

which implies that condition (3.3) is fulfilled. We have also

$$(\gamma_1 p T + 2\gamma_1)\left(b\beta R_1 + b^2\alpha R_1^2\right) \approx 8.1863 \times 10^{-2} \leq L = 0.2.$$

Then condition (3.2) is also satisfied.

Finally, we conclude that all conditions of Theorem 3.1 hold and hence equation (3.9) has at least one positive periodic solution in Ω_1. Indeed since conditions (3.2) – (3.4) are satisfied, then Lemmas 3.3 and 3.4 show that $\mathcal{K}_1$ maps Ω_1 into itself. Furthermore, we get

$$|(\mathcal{K}_1 x)(t) - (\mathcal{K}_1 y)(t)| \leq 1.1569\ \ \|x - y\|,$$

and therefore the continuity of the operator $\mathcal{K}_1$ results immediately afterwards. In addition, Arzelà-Ascoli theorem ensures the compactness of the departure set Ω_1 which, in turn, proves the compactness of the continuous operator $\mathcal{K}_1$. Therefore, we conclude by the Schauder's fixed point theorem that the operator $\mathcal{K}_1$ has at least one fixed point in Ω_1 which is a positive periodic solution of equation (3.9).

But

$$\mu = \gamma_1 T b \left(\beta + 2b\alpha R_1\right)\left(1 + L\right) \approx 1.1569 > 1.$$

Therefore, Theorems 3.2 and 3.3 cannot be applied here. Indeed, the additional criterion (3.8) is not fulfilled and hence the solution of equation (3.9) is not necessarily unique and we cannot get any information about the stability of solutions.

CHAPTER 4

Existence, Uniqueness and Continuous Dependence on Parameters of Solutions for a Recruitment Model with Iterative Terms and a Nonlinear Harvesting

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In this chapter, we are interested in studying the existence, uniqueness and continuous dependence of positive periodic solutions of a first order delay differential equation with iterative terms by means of the Schauder and Banach fixed point fixed point theorems along with the Green's function method.

4.1 Introduction

Our foremost concern in this chapter is to establish some sufficient criteria that assure the existence, uniqueness and stability of positive periodic and bounded solutions to the following first-order differential equation with a time-varying delay and iterative terms:

$$x'(t) + p(t)x(t) = ax^{[2]}(t) - d\left(x^{[2]}(t)\right)^2 - qx^{[2]}(t)E(t, x(t), x(t-\tau(t))), \tag{4.1}$$

where $x^{[2]}(t) = x(x(t))$ is the second iterate of x, $a, d, q > 0$, p, $\tau \in \mathcal{C}(\mathbb{R}, (0, +\infty))$ are two T-periodic functions and $E \in \mathcal{C}(\mathbb{R}^3, (0, +\infty))$ is a T-periodic function with respect to the first variable and satisfies the following Lipschitz condition:

$$|E(t, x_1, x_2) - E(t, y_1, y_2)| \leq \ell_1 |x_1 - y_1| + \ell_2 |x_2 - y_2|. \tag{4.2}$$

equation (4.1) is usually relevant to the dynamics of single-species population growth with harvesting strategy where $x(t)$ represents the population density at time t, $p(t)$ is per capita daily adult mortality rate, $ax^{[2]}(t) - d\left(x^{[2]}(t)\right)^2$ is the recruitment term and $qx^{[2]}(t)E(t, x(t), x(t-\tau(t)))$ is the harvesting term which may be due to live capture, fishing, hunting or trapping individuals where q is the catchability coefficient, $\tau(t)$ is referred to as the time delay required for harvesting mature individuals and E is the harvesting or fishery effort that depends on both the current and the past densities. So, our new results shed lights on an important question which is about the effect of the harvesting strategy on the population dynamics. Indeed, such external factor which reduces the population affects the mathematical model as it plays a key role in the dynamics of the population and, in some cases, can even lead to the eventual extinction.

Despite the long history of iterative differential equations, there were very little works available in the literature that dealt with these equations as realistic models. But although the authors generally face some difficulties in studying them, such equations have recently attracted considerable attention that led to several recent contributions including (see [9]-[89]).

The purpose of this work is twofold: first, it aims to contribute to the emerging literature on this topic, and, secondly, to highlight the impact of the harvesting strategy on the population dynamics as our new findings highlights on this effect where the harvesting term involves two delays, the first lag depends on time while the second one which gives the second iterate $x^{[2]}(t)$, depends not only on the time but also depends on the population density.

The plan of this manuscript is organized as follows. In Section 2, we present some definitions and materials needed to establish our main results. In Section 3, we give certain conditions for which the Schauder's fixed point theorem could be applied and hence could guarantee the existence of at least one positive periodic and bounded solution of equation (4.1). Further, by means of the Banach contraction principle, we are also able to derive the existence and uniqueness result and also establish the continuous dependence of the unique solution on parameters.

4.2 Preliminaries

In this section, we shall recall some relevant preliminaries, which are crucial in our arguments.

Let the Banach space P_T of all continuous and periodic functions with the period T defined in the Example 2.2, and the subset Ω given in Lemma 2.3.

Remark 4.1 It follows from Lemma 2.3 that

$$\left\| \left(x^{[2]}\right)^2 - \left(y^{[2]}\right)^2 \right\| \leq 2R_1(1+L)\|x-y\|,$$

for all $x, y \in \Omega$.

We first begin by giving an equivalence between our equation (4.1) and an integral one.

Lemma 4.1 *$x \in \Omega \cap C^1(\mathbb{R}, \mathbb{R})$ is a solution of equation (4.1) if and only if $x \in \Omega$ is a solution of the following integral equation:*

$$x(t) = \int_t^{t+T} G(t,s) \left\{ ax^{[2]}(s) - d\left(x^{[2]}(s)\right)^2 - qx^{[2]}(s) E(s, x(s), x(s-\tau(s))) \right\} ds, \tag{4.3}$$

where G is a Green's function given in Lemma 2.2.

Proof. Let $x \in \Omega \cap C^1(\mathbb{R}, \mathbb{R})$ be a solution of equation (4.1). Multiplying both sides of this equation by $\exp\left(\int_0^t p(u)\,du\right)$ we arrive at

$$\begin{aligned}&\frac{d}{ds}\left[x(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds\\&=\left[ax^{[2]}(t)-d\left(x^{[2]}(t)\right)^2-qx^{[2]}(s)E(t,x(t),x(t-\tau(t)))\right]\exp\left(\int_0^t p(u)\,du\right).\end{aligned}$$

Integrating from t to $t+T$ we get

$$\begin{aligned}&\int_t^{t+T}\frac{d}{ds}\left[x(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds\\&=\int_t^{t+T}\left[ax^{[2]}(s)-d\left(x^{[2]}(s)\right)^2-qx^{[2]}(s)E(s,x(s),x(s-\tau(s)))\right]\exp\left(\int_0^s p(u)\,du\right)ds.\end{aligned}$$

By the periodic properties we find that

$$\begin{aligned}&\int_t^{t+T}\frac{d}{ds}\left[x(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds\\&=x(t)\left[\exp\left(\int_0^{t+T}p(u)\,du\right)-\exp\left(\int_0^t p(u)\,du\right)\right]\\&=x(t)\left[\exp\left(\int_0^t p(u)\,du\right)\left(\exp\left(\int_t^{t+T}p(u)\,du\right)-1\right)\right].\end{aligned}$$

Thus

$$x(t)=\int_t^{t+T}G(t,s)\left\{ax^{[2]}(s)-d\left(x^{[2]}(s)\right)^2-qx^{[2]}(s)E(s,x(s),x(s-\tau(s)))\right\}ds.$$

Now, let us recover equation (4.1).

Conversely, assume that x satisfies the integral equation (4.3), by differentiation one can easily verify that x is a solution of equation (4.1). ■

From the last lemma, we define an operator $\mathcal{F}:\Omega\to P_T$ as follows:

$$(\mathcal{F}x)(t)=\int_t^{t+T}G(t,s)\left\{ax^{[2]}(s)-d\left(x^{[2]}(s)\right)^2-qx^{[2]}(s)E(s,x(s),x(s-\tau(s)))\right\}ds. \tag{4.4}$$

So equation (4.1) associated with the periodic properties can be converted into a fixed point problem. In other words, if x is a fixed point of the operator $\mathcal{F}$ then x is a solution of equation (4.1) and vice versa.

4.3 Existence of Periodic Solutions

Now, we intend to state and prove our first main result. For this purpose, we will use Schauder's fixed point theorem to prove that operator $\mathcal{F}$ has at least one fixed point in Ω which means that equation (4.1) has at least one positive periodic and bounded solution.

Lemma 4.2 *Operator $\mathcal{F}$ is continuous.*

Proof. It is not difficult to show that $\mathcal{F}(t+T) = \mathcal{F}(t)$. For $x, y \in \Omega$, we have

$$\begin{aligned}
\left|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\right| \leq{}& a\int_t^{t+T} G(t,s)\left|x^{[2]}(s) - y^{[2]}(s)\right| ds \\
&+ d\int_t^{t+T} G(t,s)\left|\left(x^{[2]}(s)\right)^2 - \left(y^{[2]}(s)\right)^2\right| ds \\
&+ q\int_t^{t+T} G(t,s)\left|x^{[2]}(s)E(s,x(s),x(s-\tau(s)))\right. ds \\
&\left.-y^{[2]}(s)E(s,y(s),y(s-\tau(s)))\right| ds.
\end{aligned}$$

But

$$\begin{aligned}
&\left|x^{[2]}(s)E(s,x(s),x(s-\tau(s))) - y^{[2]}(s)E(s,y(s),y(s-\tau(s)))\right| \\
&\leq \left|E(s,x(s),x(s-\tau(s)))\right|\left|x^{[2]}(s) - y^{[2]}(s)\right| \\
&+ y^{[2]}(s)\left|E(s,x(s),x(s-\tau(s))) - E(s,y(s),y(s-\tau(s)))\right|.
\end{aligned}$$

By using (4.2) and Remark 4.1 we obtain

$$\left|E(s,x(s),x(s-\tau(s)))\right| \leq E_0 + (\ell_1+\ell_2)R_1, \tag{4.5}$$

with

$$E_0 = \max_{t\in[0,T]} E(t,0,0),$$

and

$$\begin{aligned}
&\left|x^{[2]}(s)E(s,x(s),x(s-\tau(s))) - x^{[2]}(s)E(s,y(s),y(s-\tau(s)))\right| \\
&\leq (E_0 + (\ell_1+\ell_2)R_1)(L+1)\|x-y\| + R_1(\ell_1+\ell_2)\|x-y\| \\
&\leq (E_0(L+1) + R_1(L+2)(\ell_1+\ell_2))\|x-y\|. \qquad (4.6)
\end{aligned}$$

It follows from (2.1), (4.6), Lemma 2.3 and Remark 4.1 that

$$\left\|\mathcal{F}x-\mathcal{F}y\right\|\leq\lambda\left\|x-y\right\|,$$

where

$$\lambda=\gamma_1 T\left(\left(L+1\right)\left(a+2dR_1\right)+q\left(E_0\left(L+1\right)+R_1\left(L+2\right)\left(\ell_1+\ell_2\right)\right)\right),$$

from which we infer that $\mathcal{F}$ is a Lipschitz continuous operator and hence is a continuous operator. ■

Lemma 4.3 *If* $aT\gamma_1\leq 1$ *and*

$$ax^{[2]}\left(s\right)-d\left(x^{[2]}\left(s\right)\right)^2-qx^{[2]}\left(s\right)E\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)\geq\frac{R_0}{T\gamma_0}. \tag{4.7}$$

Then

$$R_0\leq\left(\mathcal{F}x\right)\left(t\right)\leq R_1,$$

for all $x\in\Omega$ *and* $t\in\mathbb{R}$.

Proof. Let $x\in\Omega$. Since $aT\gamma_1\leq 1$, it follows from (2.1) that

$$\begin{aligned}\left(\mathcal{F}x\right)\left(t\right)&\leq a\int_t^{t+T}G\left(t,s\right)x^{[2]}\left(s\right)ds\\&\leq aT\gamma_1 R_1\\&\leq R_1,\end{aligned}$$

and by taking into account (2.1), (4.5) and (4.7) we obtain

$$\begin{aligned}\left(\mathcal{F}x\right)\left(t\right)&=\int_t^{t+T}G\left(t,s\right)\left\{ax^{[2]}\left(s\right)-d\left(x^{[2]}\left(s\right)\right)^2-qx^{[2]}\left(s\right)E\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)\right\}\\&\geq T\gamma_0\min_{s\in[0,T]}\left\{ax^{[2]}\left(s\right)-d\left(x^{[2]}\left(s\right)\right)^2-qx^{[2]}\left(s\right)E\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)\right\}\\&\geq T\gamma_0\frac{R_0}{T\gamma_0}=R_0.\end{aligned}$$

Consequently, $R_0\leq\left(\mathcal{F}x\right)\left(t\right)\leq R_1$ for all $x\in\Omega$ and $t\in\mathbb{R}$. ■

Lemma 4.4 *If*

$$\gamma_1 R_1 (pT+2)(qE_0 + a + dR_1 + q\ell_1 R_1 + q\ell_2 R_1) \leq L, \tag{4.8}$$

then

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq L|t_2 - t_1|,$$

for all $t_1,t_2 \in \mathbb{R}$ *and* $x \in \Omega$.

Proof. Let $t_1,t_2 \in \mathbb{R}$ and $x \in \Omega$. We have

$$\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq{}& a\int_{t_2}^{t_1} x^{[2]}(s)\,G(t_2,s)\,ds + a\int_{t_1+T}^{t_2+T} x^{[2]}(s)\,G(t_2,s)\,ds \\
&+ a\int_{t_1}^{t_1+T} x^{[2]}(s)\,|G(t_2,s) - G(t_1,s)|\,ds \\
&+ d\int_{t_2}^{t_1} \left(x^{[2]}(s)\right)^2 G(t_2,s)\,ds + d\int_{t_1+T}^{t_2+T} \left(x^{[2]}(s)\right)^2 G(t_2,s)\,ds \\
&+ d\int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\left(x^{[2]}(s)\right)^2 ds \\
&+ q\int_{t_2}^{t_1} G(t_2,s)\,x^{[2]}(s)\,E(s,x(s),x(s-\tau(s)))\,ds \\
&+ q\int_{t_1+T}^{t_2+T} G(t_2,s)\,x^{[2]}(s)\,E(s,x(s),x(s-\tau(s)))\,ds \\
&+ q\int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\,x^{[2]}(s)\,E(s,x(s),x(s-\tau(s)))\,ds.
\end{aligned}$$

From (2.1), (4.5) and (4.5) we achieve

$$\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq{}& 2aR_1\gamma_1|t_2 - t_1| + R_1 aTp\gamma_1|t_2 - t_1| \\
&+ 2dR_1\gamma_1|t_2 - t_1| + dR_1Tp\gamma_1|t_2 - t_1| \\
&+ 2q\gamma_1 R_1(E_0 + (\ell_1 + \ell_2)R_1)|t_2 - t_1| \\
&+ qR_1(E_0 + (\ell_1 + \ell_2)R_1)Tp\gamma_1|t_2 - t_1| \\
\leq{}& \gamma_1 R_1(pT+2)(qE_0 + a + dR_1 + q\ell_1 R_1 + q\ell_2 R_1)|t_2 - t_1|.
\end{aligned}$$

It follows from (4.8) and Lemma 2.3 that

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq L|t_2 - t_1|,$$

for all $t_1,t_2 \in \mathbb{R}$ and $x \in \Omega$. ■

Theorem 4.1 *Suppose that conditions (4.7), (4.8) and $aT\gamma_1 \leq 1$ hold, then equation (4.1) has at least one positive periodic and bounded solution in Ω.*

Proof. From Lemmas 4.3 and 4.4 we conclude that operator $\mathcal{F}$ maps the compact subset Ω into itself and since Lemma 4.2 guarantees the continuity of the operator $\mathcal{F}$, then all conditions of Schauder's fixed point theorem are satisfied. Accordingly, $\mathcal{F}$ has at least one fixed point $x \in \Omega$ such that $\mathcal{F}x = x$. Thanks to Lemma 4.1, equation (4.1) has at least one positive periodic and bounded solution. ■

4.4 Existence and Uniqueness of Periodic Solutions

Theorem 4.2 *Suppose that conditions (4.7), (4.8) and $aT\gamma_1 \leq 1$ are fulfilled. If $\lambda < 1$, then equation (4.1) has a unique positive periodic and bounded solution $x \in \Omega$.*

Proof. Let $x, y \in \Omega$. From the proof of Lemma 4.2 we have

$$\left\| \mathcal{F}x - \mathcal{F}y \right\| \leq \lambda \left\| x - y \right\|.$$

Since $\lambda < 1$, then $\mathcal{F}$ is a contraction. So, by the Banach fixed point theorem, $\mathcal{F}$ has a unique fixed point which is the unique positive periodic and bounded solution of equation (4.1). ■

4.5 Stability

Theorem 4.3 *The unique solution obtained in Theorem 4.2 depends continuously on the death rate p and the harvesting effort E.*

Proof. Let

$$x_1(t) = \int_t^{t+T} G_1(t,s) \left\{ a x_1^{[2]}(s) - d\left(x_1^{[2]}(s) \right)^2 - q x_1^{[2]}(s) E_1(s, x_1(s), x_1(s-\tau(s))) \right\} ds,$$

and

$$x_2(t) = \int_t^{t+T} G_2(t,s) \left\{ a x_2^{[2]}(s) - d\left(x_2^{[2]}(s) \right)^2 - q x_2^{[2]}(s) E_2(s, x_2(s), x_2(s-\tau(s))) \right\} ds,$$

where

$$G_1(t,s)=\frac{\exp\left(\int_t^s p_1(u)\,du\right)}{\exp\left(\int_0^T p_1(u)\,du\right)-1}\text{ and }G_2(t,s)=\frac{\exp\left(\int_t^s p_2(u)\,du\right)}{\exp\left(\int_0^T p_2(u)\,du\right)-1},$$

are two different solutions of equation (4.1). We have

$$\begin{aligned}|x_1(t)-x_2(t)|\le{}& a\int_t^{t+T}\left|x_1^{[2]}(s)\,G_1(t,s)-x_2^{[2]}(s)\,G_2(t,s)\right|ds\\&+d\int_t^{t+T}\left|\left(x_1^{[2]}(s)\right)^2G_1(t,s)-x_2^{[2]}(s)\,G_2(t,s)\right|ds\\&+q\int_t^{t+T}\Big|x_1^{[2]}(s)\,E_1(s,x_1(s),x_1(s-\tau(s)))\,G_1(t,s)\\&-x_2^{[2]}(s)\,E_2(s,x_2(s),x_2(s-\tau(s)))\,G_2(t,s)\Big|\,ds.\end{aligned}$$

Thus

$$\begin{aligned}&|x_1(t)-x_2(t)|\\&\le a\int_t^{t+T}G_1(t,s)\left|x_1^{[2]}(s)-x_2^{[2]}(s)\right|ds+a\int_t^{t+T}x_2^{[2]}(s)\,|G_1(t,s)-G_2(t,s)|\,ds\\&+d\int_t^{t+T}G_1(t,s)\left|\left(x_1^{[2]}(s)\right)^2-\left(x_2^{[2]}(s)\right)^2\right|ds+d\int_t^{t+T}\left(x_2^{[2]}(s)\right)^2|G_1(t,s)-G_2(t,s)|\,ds\\&+q\int_t^{t+T}G_1(t,s)\,|E_1(s,x_1(s),x_1(s-\tau(s)))-E_2(s,x_1(s),x_1(s-\tau(s)))|\,ds\\&+q\int_t^{t+T}x_1^{[2]}(s)\,E_2(s,x_1(s),x_1(s-\tau(s)))\,|G_1(t,s)-G_2(t,s)|\,ds\\&+q\int_t^{t+T}x_1^{[2]}(s)\,G_2(t,s)\,|E_2(s,x_1(s),x_1(s-\tau(s)))-E_2(s,x_2(s),x_2(s-\tau(s)))|\,ds\\&+q\int_t^{t+T}\left|x_1^{[2]}(s)-x_2^{[2]}(s)\right|G_2(t,s)\,E_2(s,x_2(s),x_2(s-\tau(s)))\,ds.\end{aligned}$$

The mean value theorem leads to

$$\int_t^{t+T}|G_1(t,s)-G_2(t,s)|\,ds\le\sigma\,\|p_1-p_2\|\,,\tag{4.9}$$

where

$$\sigma=\frac{T^2e^{T(\|p_2\|+\max(\|p_1\|,\|p_2\|))}}{\left(\exp\left(\int_0^T p_1(u)\,du\right)-1\right)\left(\exp\left(\int_0^T p_2(u)\,du\right)-1\right)}+\frac{T^2e^{T\max(\|p_1\|,\|p_2\|)}}{\exp\left(\int_0^T p_1(u)\,du\right)-1}.$$

It follows from (4.2), (2.1), (4.5), (4.9), Lemma 2.3 and Remark 4.1 that

$$\begin{aligned}\|x_1-x_2\| \leq{}& aT\gamma_1(1+L)\|x_1-x_2\|+aR_1\sigma\|p_1-p_2\| \\ &+2dT\gamma_1R_1(1+L)\|x_1-x_2\|+dR_1\sigma\|p_1-p_2\| \\ &+qR_1T\gamma_1\|E_1-E_2\|+q(E_0+(\ell_1+\ell_2)R_1)\sigma\|p_1-p_2\| \\ &+qR_1T\gamma_1(\ell_1+\ell_2)\|x_1-x_2\| \\ &+qT\gamma_1(E_0+(\ell_1+\ell_2)R_1)(1+L)\|x_1-x_2\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_1-x_2\| \leq{}& \frac{1}{1-\lambda}\left[R_1\sigma(a+dR_1+q(E_0+(\ell_1+\ell_2)R_1))\|p_1-p_2\|\right. \\ &\left.+qR_1T\gamma_1\|E_1-E_2\|\right].\end{aligned}$$

This completes the proof. ■

CHAPTER 5

Existence, Uniqueness and Continuous Dependence on Parameters of Solutions for a first order Neutral Differential Equation with Iterative Terms

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In this chapter, we are concerned with the study of the existence, uniqueness and stability of positive periodic solutions for a first order neutral iterative differential equations by using the Krasnoselskii and Banach fixed point fixed point theorems as well as the Green's function method.

5.1 Introduction

Recently, a great attention has been devoted to investigate neutral differential equations where the highest order derivatives occur with delays. These equations arise in various applied sciences such as population dynamic, biology, hematology, physics, economics, chemistry, and so forth. For any further and detailed information, we refer the interested reader to [11, 21, 51, 59, 75] and the references cited therein. Among the papers that dealt with the existence of periodic solutions for first order neutral differential equations, we cite some of them which are relevant to what we are discussing in this work.

In 1991, Serra [75] used the Mawhin coincidence degree theory to investigate the existence of periodic solutions for the following neutral differential equation:

$$\frac{d}{dt}\left[x\left(t\right)-cx\left(t-\tau\right)\right]=f\left(t,x\left(t\right)\right),$$

with $t\in\mathbb{R}$, $c\in\mathbb{R}$, $\tau\in\left]0,2\pi\right[$, $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is Caratheodory function and $2\pi-$periodic with respect to the time variable.

In 2008, by means of the Krasnoselskii fixed point theorem, Luo et al. [59] gave the sufficient conditions that guarantee the existence of positive periodic solutions of the following neutral differential equation:

$$\frac{d}{dt}\left[x\left(t\right)-cx\left(t-\tau\left(t\right)\right)\right]=-a\left(t\right)x\left(t\right)+f\left(t,x\left(t-\tau\left(t\right)\right)\right),$$

where $a\in C\left(\mathbb{R},\left(0,\infty\right)\right)$, $\tau\in C\left(\mathbb{R},\mathbb{R}\right)$ and $f\in C\left(\mathbb{R}\times\mathbb{R},\mathbb{R}\right)$ are common periodic functions with respect to the time variable.

In 2016, Candan [21] employed the same aforementioned fixed point theorem to discuss the positive periodic solutions for the following neutral differential equation:

$$\frac{d}{dt}\left[x\left(t\right)-P\left(t\right)x\left(t-\tau\right)\right]=-a\left(t\right)x\left(t\right)+f\left(t,x\left(t-\tau\right)\right),$$

here $\tau>0$, $a\in C\left(\mathbb{R},\left(0,\infty\right)\right)$, $P\in C^1\left(\mathbb{R},\mathbb{R}\right)$ and $f\in C\left(\mathbb{R}\times\mathbb{R},\mathbb{R}\right)$ are common periodic functions with respect to the time variable.

In the present work, we give new sufficient criteria for the existence, uniqueness and continuous dependence of positive periodic solutions for the following first order neutral differential equation with iterative terms:

$$\frac{d}{dt}\left[x(t)-cx(t-\tau(t))\right]=-p(t)x(t)+f(t,x(t-\tau(t)))-H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right), \tag{5.1}$$

where the iterate $x^{[n]}(t)$ stands for x composed with itself n times, i.e. $x^{[2]}(t) = x(x(t)),...,x^{[n]}(t)=x^{[n-1]}(x(t))$, $c\in(0,1)$, $p,\tau\in C(\mathbb{R},(0,\infty))$, $f\in\mathcal{C}([0,T]\times\mathbb{R},(0,\infty))$ and $H\in\mathcal{C}([0,T]\times\mathbb{R}^n,(0,\infty))$ are $T-$periodic functions with respect to the time variable. Furtheremore, the functions $f(t,x)$ and $H(t,x_1,x_2,...,x_n)$ are supposed globally Lipschitz in x and $x_1,...,x_n$ respectively, i.e. there exist a positive constant k and n positive constants $l_1,l_2,...,l_n$ such that

$$|f(t,x)-f(t,z)|\leq k\|x-z\|, \tag{5.2}$$

and

$$|H(t,x_1,x_2,...,x_n)-H(t,z_1,z_2,...,z_n)|\leq\sum_{i=1}^{n}l_i\|x_i-z_i\|. \tag{5.3}$$

It is noteworthy that equation (5.1) is a first order iterative differential equation and such equations as we said before are used to model a variety of phenomena observed in an extremely wide range of areas, including life sciences (see, e.g., [8, 16, 85], and references cited therein). For instance, it can model many biological and ecological equations such as: Neutral Mackey-Glass models with harvesting, Neutral Wazewska-Lasota model with harvesting, Neutral Nicholson's blowflies model with harvesting and Neutral houseflies model with harvesting.

So, to put forward a more meaningful and realistic model that can describe a biological phenomenon and contain the minimal basic biological information about it, we assume that the production f (flux or recruitment) term incorporates a time-varying delay and also we take into account a harvesting strategy H with time and state dependent delays that lead to the appearance of the iterates $x^{[i]}(t)$.

The strong interest in this work is motivated by the fact that the harvesting of individuals provides a good description of the population dynamics and plays a prominent role in getting a better understanding of its effects on the management of biological resources. Moreover, in many biological and vital phenomena, the delays are generally depending on both the time and the state variable that can give the iterations in the model such as in infectious diseases spread, blood cell production and insect population growth.

We now outline some key features of our work by the following items:

- It should be pointed out that equation (5.1) is more general than those investigated in [16, 21, 59, 75]. Furthermore, to the best of our knowledge, there are no published papers that address this problem with a time varying delay and an iterative harvesting term that involves implicitly $(n-1)$ time and state dependent delays.

- Despite that recent years have gradually witnessed an unprecedented interest towards such kind of equations (we can mention, for instance, [8], [13]-[16], [22]-[50], [54], [60], [85]-[89]), the investigation in this direction remains scarce and their theory has not yet been developed enough. So, it is our belief that our work is of significance because it contributes in the literature of this emergent theory.

- There is no doubt that the harvesting strategy plays a crucial role in the population dynamics and it is also quite normal that it involves many delays. So, on account of these facts, we attempt to understand the effect of the harvesting strategy on the population dynamics by adding an iterative harvesting term involving implicitly many time and state dependent delays.

- We are interested in the positivity, boundedness and periodicity of solutions which makes our results even more powerful and biologically meaningful. This is due to the fact that the state $x(t)$ in biological phenomena could, for example, stand for an amount of cells, a density, a number of individuals or a size of the population which should be positive and bounded quantities and generally periodic.

The rest of the chapter is furnished as follows: In Section 2, we start with some preliminaries which will be needed in what follows. In Section 3, by virtue of the Krasnoselskii fixed point theorem and some Green's function properties, we construct some new results about the existence of positive periodic solutions for equation (5.1). In Section 4, the existence, uniqueness and continuous dependence on parameters of the solutions are established by using the Banach fixed point theorem. In Section 5, we exhibit an example to which our key outcomes can be applied.

5.2 Preliminaries

In this Section, we present some notations and preliminary material needed to establish our main findings and we also state and prove some preliminary results.

Let the Banach space P_T of all continuous and periodic functions with the period T defined in the Example 2.2, and the subset Ω given in Lemma 2.3.

For convenience, throughout this work, we introduce the following notations:

$$f_1 = \max_{t\in[0,T]} \left|f\left(t,0\right)\right|, \ H_1 = \max_{t\in[0,T]} \left|H\left(t,0,0,...,0\right)\right|, \Lambda = \sum_{i=1}^{n} l_i \sum_{j=0}^{i-1} L^j.$$

In the sequel we will assume that the following hypotheses are satisfied: There exists a positive constant $f_0 > 0$ such that

$$f\left(t,x\right) \geq f_0, \ \forall t \in \mathbb{R}, \ \forall x \in \left(0,\infty\right). \tag{5.4}$$

The following estimates are satisfied:

$$cR_1 + \gamma_1 T\left(kR_1 + f_1\right) \leq R_1, \tag{5.5}$$

$$\gamma_0 T f_0 - \gamma_1 T\left(R_1\Lambda + H_1\right) - cT\gamma_1 pR_1 + cR_0 \geq R_0, \tag{5.6}$$

and

$$\gamma_1\left(2+pT\right)\left(H_1 + f_1 + R_1\left(k+\Lambda+cp\right)\right) + L\left(1+L\right)c \leq L. \tag{5.7}$$

Now, we state and prove the following lemma, which we intend to use later.

Lemma 5.1 *$x \in \Omega \cap \mathcal{C}^1(\mathbb{R},\mathbb{R})$ is a solution of (5.1) if and only if $x \in \Omega$ satisfies the following integral equation:*

$$x\left(t\right) = \int_t^{t+T} G\left(t,s\right)\left[f\left(s,x\left(s-\tau\left(s\right)\right)\right) - H\left(s,x\left(s\right),x^{[2]}\left(s\right),...,x^{[n]}\left(s\right)\right)\right.$$
$$\left. - cp\left(s\right)x\left(s-\tau\left(s\right)\right)\right]ds + cx\left(t-\tau\left(t\right)\right), \tag{5.8}$$

where

$$G\left(t,s\right) = \frac{\exp\left(\int_t^s p\left(u\right)du\right)}{\exp\left(\int_0^T p\left(u\right)du\right)-1}. \tag{5.9}$$

Proof. Let $x \in \Omega \cap \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ be a solution of (5.1). Multiplying both sides of (5.1) by $\exp\left(\int_0^t p(u)\,du\right)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\left(x(t) - cx(t-\tau(t))\right)\exp\left(\int_0^t p(u)\,du\right)\right] \\
&= \left[f\left(t, x(t-\tau(t))\right) - H\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)\right]\exp\left(\int_0^t p(u)\,du\right) \\
&- cp(t)\,x(t-\tau(t))\exp\left(\int_0^t p(u)\,du\right).
\end{aligned}$$

It follows from the periodic properties and the integration from t to $t+T$ that

$$\begin{aligned}
&\left(x(t) - cx(t-\tau(t))\right)\left(\exp\left(\int_0^{t+T} p(u)\,du\right) - \exp\left(\int_0^t p(u)\,du\right)\right) \\
&= \int_t^{t+T}\left\{f\left(s, x(s-\tau(s))\right) - H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right. \\
&\left.-c(s)\,p(s)\,x(s-\tau(s))\right\}\exp\left(\int_0^s p(u)\,du\right) ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
x(t) = &\int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right) - 1}\left\{f\left(s, x(s-\tau(s))\right) - H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right. \\
&\left.-cp(s)\,x(s-\tau(s))\right\} ds + cx(t-\tau(t)).
\end{aligned}$$

This completes the proof of the lemma. ∎

For all $t, s, t_1, t_2 \in \mathbb{R}$ and in view of (5.2), (5.3) and Lemma 2.3 we get

$$\left|f\left(s, x(s-\tau(s))\right)\right| \leq kR_1 + f_1, \tag{5.10}$$

and

$$\left|H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| \leq R_1\Lambda + H_1, \tag{5.11}$$

for all $x \in \Omega$.

5.3 Existence of Positive Periodic Solutions

In this section, we will use the Krasnoselskii fixed point theorem to prove the existence of positive periodic solutions of equation (5.1). For this and, by virtue of Lemma 5.1, we define an operator $\mathcal{S} : \Omega \to P_T$ as follows:

$$(\mathcal{S}x)(t) = (F_1x)(t) + (F_2x)(t),$$

where $F_1, F_2 : \Omega \to P_T$ are defined as follows:

$$\left(F_1 x\right)(t) = \int_t^{t+T} G(t,s)\left[f\left(s, x\left(s-\tau(s)\right)\right) - H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right.$$
$$\left. -cp(s)\, x\left(s-\tau(s)\right)\right] ds,$$

and

$$\left(F_2 x\right)(t) = cx\left(t-\tau(t)\right).$$

So, the existence of solutions for equation (5.1) is equivalent whether or not the operator $\mathcal{S}$ has a fixed point.

Since $c \in (0,1)$, then F_2 is a contraction. So, to apply Krasnoselskii's fixed point theorem, it suffices to prove that F_1 is continuous and compact and that $F_1 x + F_2 y \in \Omega$ for all $x, y \in \Omega$.

We start by proving the compactness and the continuity of F_1.

Lemma 5.2 *Operator $F_1 : \Omega \to P_T$ is continuous and compact.*

Proof. Since Ω is a compact subset of P_T, then the compactness of F_1 follows immediately from its continuity.

Let us prove that F_1 is continuous. Indeed, for all $x, y \in \Omega$, we have

$$\begin{aligned}\left|\left(F_1 x\right)(t) - \left(F_1 y\right)(t)\right| \leq & \int_t^{t+T} \left|f\left(s, x\left(s-\tau(s)\right)\right) - f\left(s, y\left(s-\tau(s)\right)\right)\right| G(t,s)\, ds \\ & + \int_t^{t+T} G(t,s)\left|H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds\right. \\ & \left. -H\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right)\right| ds \\ & + \int_t^{t+T} cp(s)\left|x\left(s-\tau(s)\right) - y\left(s-\tau(s)\right)\right| G(t,s)\, ds.\end{aligned}$$

It follows from (5.2), (5.3) and (2.1) that

$$\left\|F_1 x - F_1 y\right\| \leq \gamma_1 T k \left\|x-y\right\| + \gamma_1 T \sum_{i=1}^{n} l_i \left\|x^{[i]} - y^{[i]}\right\| + \gamma_1 T c p \left\|x-y\right\|.$$

By using Lemma 2.3, we get

$$\left\|F_1 x - F_1 y\right\| \leq \gamma_1 T\left(k + cp + \Lambda\right)\left\|x-y\right\|, \tag{5.12}$$

which establishes that the operator F_1 is Lipschitz continuous and hence continuous. Therefore, F_1 is compact. ■

Lemma 5.3 *Let $\tau \in \Omega$. If conditions (5.4)-(5.7) hold, then*

$$F_1x + F_2y \in \Omega,$$

for all $x, y \in \Omega$.

Proof. Let $x, y \in \Omega$. From (5.5), (2.1) and (5.10) we obtain

$$\begin{aligned}(F_1x)(t) + (F_2y)(t) &\leq cy(t-\tau(t)) + \int_t^{t+T} G(t,s) f(s, x(s-\tau(s)))\, ds \\ &\leq cR_1 + \gamma_1 T(kR_1 + f_1) \\ &\leq R_1,\end{aligned}$$

and from (5.4), (5.6), (2.1) and (5.11), we arrive at

$$\begin{aligned}(F_1x)(t) + (F_2y)(t) &\geq \gamma_0 T f_0 - \gamma_1 T(R_1\Lambda + H_1) - cT\gamma_1 pR_1 + cR_0 \\ &\geq R_0.\end{aligned}$$

Consequently,

$$R_0 \leq (px)(t) + (By)(t) \leq R_1, \tag{5.13}$$

for all $x, y \in \Omega$.

Let $\tau \in \Omega$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. For all $x, y \in \Omega$, we have

$$\begin{aligned}&\left|((F_1x) + (F_2y))(t_2) - ((F_1x) + (F_2y))(t_1)\right| \\ &\leq \left|\int_{t_2}^{t_2+T} G(t_2,s) f(s, x(s-\tau(s)))\, ds - \int_{t_1}^{t_1+T} G(t_1,s) f(s, x(s-\tau(s)))\, ds\right| \\ &+ \left|\int_{t_2}^{t_2+T} G(t_2,s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds\right. \\ &- \left.\int_{t_1}^{t_1+T} G(t_1,s) H\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds\right| \\ &+ \left|\int_{t_2}^{t_2+T} G(t_2,s)\, cp(s)\, x(s-\tau(s))\, ds - \int_{t_1}^{t_1+T} G(t_1,s)\, cp(s)\, x(s-\tau(s))\, ds\right| \\ &+ \left|cy(t_2-\tau(t_2)) - cy(t_1-\tau(t_1))\right|.\end{aligned}$$

By using (2.1), (2.2) and (5.10), we get

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} G(t_2,s) f(s,x(s-\tau(s)))\,ds - \int_{t_1}^{t_1+T} G(t_1,s) f(s,x(s-\tau(s)))\,ds\right| \\
&\leq \left|\int_{t_2}^{t_1} G(t_2,s) f(s,x(s-\tau(s)))\,ds\right| + \left|\int_{t_1+T}^{t_2+T} G(t_2,s) f(s,x(s-\tau(s)))\,ds\right| \\
&+ \int_{t_1}^{t_1+T} \left|G(t_2,s)-G(t_1,s)\right| f(s,x(s-\tau(s)))\,ds. \\
&\leq (2+pT)\,\gamma_1\,(kR_1+f_1)\,|t_2-t_1|\,. \qquad (5.14)
\end{aligned}$$

From (2.1), (2.2) and (5.11), we obtain

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} G(t_2,s) H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right) ds\right. \\
&\left.-\int_{t_1}^{t_1+T} G(t_1,s) H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right) ds\right| \\
&\leq \left|\int_{t_2}^{t_1} G(t_2,s) H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right) ds\right. \\
&+\left|\int_{t_1+T}^{t_2+T} G(t_2,s) H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right) ds\right| \\
&+\int_{t_1}^{t_1+T} \left|G(t_2,s)-G(t_1,s)\right| H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right) ds\Big| \\
&\leq (2+pT)\,(R_1\Lambda+H_1)\,\gamma_1\,|t_2-t_1|\,. \qquad (5.15)
\end{aligned}$$

In view of (2.1) we get

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} G(t_2,s)\,cp(s)\,x(s-\tau(s))\,ds - \int_{t_1}^{t_1+T} G(t_1,s)\,cp(s)\,x(s-\tau(s))\,ds\right| \\
&\leq \left|\int_{t_2}^{t_1} G(t_2,s)\,cp(s)\,x(s-\tau(s))\,ds\right| + \left|\int_{t_1+T}^{t_2+T} G(t_2,s)\,cp(s)\,x(s-\tau(s))\,ds\right| \\
&+ \int_{t_1}^{t_1+T} \left|G(t_2,s)-G(t_1,s)\right| cp(s)\,x(s-\tau(s))\,ds \\
&\leq (2+pT)\,\gamma_1 c R_1 p\,|t_2-t_1|\,. \qquad (5.16)
\end{aligned}$$

Since $\tau \in \Omega$, then

$$\begin{aligned}
\left|cy(t_2-\tau(t_2)) - cy(t_1-\tau(t_1))\right| &\leq c\left|y(t_2-\tau(t_2)) - y(t_1-\tau(t_1))\right| \\
&\leq Lc\left|t_2-t_1+\tau(t_2)-\tau(t_1)\right| \\
&\leq Lc\left(|t_2-t_1| + L|t_2-t_1|\right) \\
&\leq L(1+L)\,c\,|t_2-t_1|\,. \qquad (5.17)
\end{aligned}$$

Thus, it follows from (5.7) and (5.14)-(5.17) that

$$\left|\left(\left(F_1x\right)+\left(F_2y\right)\right)\left(t_2\right)-\left(\left(F_1x\right)+\left(F_2y\right)\right)\left(t_1\right)\right| \leq L\left|t_2-t_1\right|, \tag{5.18}$$

for all $t_1, t_2 \in \mathbb{R}$ and $x, y \in \Omega$.

According to (5.13) and (5.18) we conclude the desired result. ■

Now we are ready to present our first existence theorem.

Theorem 5.1 *Let $\tau \in \Omega$. If conditions (5.4)-(5.7) hold, then equation (5.1) has at least one positive periodic solution in Ω.*

Proof. Based on Lemmas 5.2 and 5.3, the fact that Ω is a compact subset of P_T and that F_2 is a contraction, we conclude by the Krasnoselskii fixed point theorem that there exists at least $x \in \Omega$ satisfies $\mathcal{S}\left(x\left(t\right)\right)=x\left(t\right).$ Thanks to Lemma 5.1, x is a solution of equation (5.1). ■

5.4 Existence and Uniqueness of Positive Periodic Solutions

Theorem 5.2 *Let $\tau \in \Omega$. If conditions (5.4)-(5.7) and the following estimate:*

$$T\gamma_1\left(k+\Lambda+cp\right)+c<1, \tag{5.19}$$

are fulfilled, then equation (5.1) has a unique positive periodic solution $x \in \Omega$.

Proof. Let $x, y \in \Omega$. By repeating the same steps as those in the proof of Lemma 5.3, we infer that $\mathcal{S}\left(\Omega\right) \subset \Omega$ and similarly as in the proof of Lemma 5.2, we get

$$\left\|\mathcal{S}x-\mathcal{S}y\right\| \leq \left(T\gamma_1\left(k+\Lambda+cp\right)+c\right)\left\|x-y\right\|.$$

According to (5.19) and the Banach fixed point theorem, $\mathcal{S}$ is a contraction and hence $\mathcal{S}$ has a unique fixed point which is the unique solution of (5.1). ■

5.5 Stability

Remark 5.1 If

$$G_1(t,s)=\frac{\exp\left(\int_t^s p_1(u)\,du\right)}{\exp\left(\int_0^T p_1(u)\,du\right)-1},\quad G_2(t,s)=\frac{\exp\left(\int_t^s p_2(u)\,du\right)}{\exp\left(\int_0^T p_2(u)\,du\right)-1},$$

then

$$\int_t^{t+T}\left|G_1(t,s)-G_2(t,s)\right|ds\leq\mu\left\|p_1-p_2\right\|, \tag{5.20}$$

where

$$\mu=\frac{T^2\exp\left(T\left(\|p_2\|+\max\left(\|p_1\|,\|p_2\|\right)\right)\right)}{\left(\exp\left(\int_0^T p_1(u)\,du\right)-1\right)\left(\exp\left(\int_0^T p_2(u)\,du\right)-1\right)}+\frac{T^2\exp\left(T\max\left(\|p_1\|,\|p_2\|\right)\right)}{\exp\left(\int_0^T p_1(u)\,du\right)-1}.$$

Theorem 5.3 *The unique solution obtained in Theorem 5.2 depends continuously on functions p, f and H.*

Proof. If G_1 and G_2 are given as in Remark 5.1, let

$$\begin{aligned}x_1(t)=\int_t^{t+T}&G_1(t,s)\left[f_1(s,x_1(s-\tau(s)))-H_1\left(s,x_1(s),x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\right.\\&\left.-cp_1(s)\,x_1(s-\tau(s))\right]ds+cx_1(t-\tau(t)),\end{aligned}$$

and

$$\begin{aligned}x_2(t)=\int_t^{t+T}&G_2(t,s)\left[f_2(s,x_2(s-\tau(s)))-H_2\left(s,x_2(s),x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)\right.\\&\left.-cp_2(s)\,x_2(s-\tau(s))\right]ds+cx_2(t-\tau(t)),\end{aligned}$$

be two different solutions of equation (5.1). We have

$$\begin{aligned}\left|x_1(t)-x_2(t)\right|\leq&\int_t^{t+T}\left|G_1(t,s)\,f_1(s,x_1(s-\tau(s)))-G_2(t,s)\,f_2(s,x_2(s-\tau(s)))\right|ds\\&+\int_t^{t+T}\left|G_1(t,s)\,H_1\left(s,x_1(s),x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\right.\\&\left.-G_2(t,s)\,H_2\left(s,x_2(s),x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)\right|ds\\&+\int_t^{t+T}\left|G_1(t,s)\,cp_1(s)\,x_1(s-\tau(s))-G_2(t,s)\,cp_2(s)\,x_2(s-\tau(s))\right|ds\\&+\left|cx_1(t-\tau(t))-cx_2(t-\tau(t))\right|.\end{aligned}$$

Using (5.2), (5.3), (2.1), (5.10), (5.11) and (5.20), we get

$$\begin{aligned}
&\int_t^{t+T} \left|G_1(t,s) f_1(s, x_1(s-\tau(s))) - G_2(t,s) f_2(s, x_2(s-\tau(s)))\right| ds \\
&\leq \int_t^{t+T} G_1(t,s) \left|f_1(s, x_1(s-\tau(s))) - f_2(s, x_1(s-\tau(s)))\right| ds \\
&+ \int_t^{t+T} f_2(s, x_1(s-\tau(s))) \left|G_1(t,s) - G_2(t,s)\right| ds \\
&+ \int_t^{t+T} G_2(t,s) \left|f_2(s, x_1(s-\tau(s))) - f_2(s, x_2(s-\tau(s)))\right| ds \\
&\leq T\gamma_1 \|f_1 - f_2\| + \mu(kR_1 + f_1)\|p_1 - p_2\| + T\gamma_1 k \|x_1 - x_2\|,
\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}
&\int_t^{t+T} \left|G_1(t,s) H_1\left(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\right) - G_2(t,s) H_2\left(s, x_2(s), x_2^{[2]}(s), ..., x_2^{[n]}(s)\right)\right| ds \\
&\leq \int_t^{t+T} G_1(t,s) \left|H_1\left(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\right) - H_2\left(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\right)\right| ds \\
&+ \int_t^{t+T} H_2\left(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\right) \left|G_1(t,s) - G_2(t,s)\right| ds \\
&+ \int_t^{t+T} G_2(t,s) \left|H_2\left(s, x_1(s), x_1^{[2]}(s), ..., x_1^{[n]}(s)\right) - H_2\left(s, x_2(s), x_2^{[2]}(s), ..., x_2^{[n]}(s)\right)\right| ds \\
&\leq T\gamma_1 \|H_1 - H_2\| + \mu(\Lambda R_1 + H_1)\|p_1 - p_2\| + T\gamma_1 \Lambda \|x_1 - x_2\|.
\end{aligned} \tag{5.22}$$

On the other hand, we have

$$\begin{aligned}
&\int_t^{t+T} \left|G_1(t,s) cp_1(s) x_1(s-\tau(s)) - G_2(t,s) cp_2(s) x_2(s-\tau(s))\right| ds \\
&\leq \int_t^{t+T} cp_1(s) G_1(t,s) \left|x_1(s-\tau(s)) - x_2(s-\tau(s))\right| ds \\
&+ \int_t^{t+T} cp_1(s) x_2(s-\tau(s)) \left|G_1(t,s) - G_2(t,s)\right| ds \\
&+ \int_t^{t+T} cx_2(s-\tau(s)) G_2(t,s) \left|p_1(s) - p_2(s)\right| ds \\
&\leq T\gamma_1 c \|p_1\| \|x_1 - x_2\| + c\|p_1\| R_1 \mu \|p_1 - p_2\| + T\gamma_1 c R_1 \|p_1 - p_2\|,
\end{aligned} \tag{5.23}$$

and

$$\left|cx_1(t-\tau(t)) - cx_2(t-\tau(t))\right| \leq c\|x_1 - x_2\|. \tag{5.24}$$

Thanks to (5.21)-(5.24), we get

$$\begin{aligned}\|x_1-x_2\| &\leq T\gamma_1\|f_1-f_2\|+\mu(kR_1+f_1)\|p_1-p_2\|+T\gamma_1k\|x_1-x_2\|\\ &+T\gamma_1\|H_1-H_2\|+\mu(\Lambda R_1+H_1)\|p_1-p_2\|+T\gamma_1\Lambda\|x_1-x_2\|\\ &+T\gamma_1c\|p_1\|\|x_1-x_2\|+c\|p_1\|R_1\mu\|p_1-p_2\|+T\gamma_1cR_1\|p_1-p_2\|\\ &+c\|x_1-x_2\|.\end{aligned}$$

Taking into account (5.19) we obtain the following estimate:

$$\begin{aligned}\|x_1-x_2\| \leq \frac{1}{1-T\gamma_1(k+\Lambda+cp_1)}&(T\gamma_1\|f_1-f_2\|+T\gamma_1\|H_1-H_2\|\\ &(\mu(kR_1+f_1)+\mu(\Lambda R_1+H_1)+\mu c\|p_1\|R_1+T\gamma_1cR_1)\|p_1-p_2\|),\end{aligned}$$

which completes the proof. ■

5.6 Example

In this section, we are going to perform an illustrating application of our obtained findings.

Example 5.1 Consider the following neutral differential equation with an iterative harvesting term:

$$\begin{aligned}\frac{d}{dt}[x(t)-0.001x(t-\tau(t))] &= -\left(\frac{1}{70}+\frac{1}{70}\sin^4\left(\frac{2\pi}{35}t\right)\right)x(t)\\ &+\left(\left(\frac{1}{9\pi^3}+\frac{1}{36\pi^3}\sin^2\left(\frac{2\pi}{35}t\right)\right)+\frac{1}{7\pi^3}\sin^2\left(\frac{2\pi}{35}t\right)x(t-\tau(t))\right)\\ &-\left(\frac{1}{9\pi^8}\sin^2\left(\frac{2\pi}{35}t\right)+\frac{1}{20\pi^8}\sin^2\left(\frac{2\pi}{35}t\right)x(t)+\frac{1}{30\pi^8}\sin^2\left(\frac{2\pi}{35}t\right)x^{[2]}(t)\right),\end{aligned} \tag{5.25}$$

where $R_0=0.05$, $R_1=1.5$, $L=\pi$ and $c=0.001$.

We have

$$\begin{aligned}&l_1=\frac{1}{20\pi^8},\ l_2=\frac{1}{30\pi^8},\ H_1=\frac{1}{9\pi^8},\ \Lambda\simeq 1.9819\times 10^{-5},\\ &f_0=\frac{1}{9\pi^3},\ f_1=\frac{5}{36\pi^3},\ k=\frac{1}{7\pi^3},\ p=\frac{1}{35},\ \gamma_0\simeq 0.50856,\ \gamma_1\simeq 2.0114.\end{aligned}$$

So

$$\begin{aligned}
&cR_1 + \gamma_1 T\left(kR_1 + f_1\right) = 0.80337 < R_1 = 1.5, \\
&\gamma_0 T f_0 - \gamma_1 T\left(R_1\Lambda + H_1\right) - cT\gamma_1 pR_1 + cR_0 = 0.05.7901 > R_0 = 0.05, \\
&\gamma_1\left(2 + pT\right)\left(\left(kR_1 + f_1\right) + \left(R_1\Lambda + H_1\right) + cR_1p\right) + L\left(1 + L\right)c \simeq 0.082252 \leq L = \pi,
\end{aligned}$$

and

$$\gamma_1 Tk + \gamma_1 T\Lambda + \gamma_1 Tcp + c \simeq 0.32876 < 1.$$

Then all conditions of Theorems 5.1 and 5.2 are satisfied. Thereby equation (5.25) has one and only one positive periodic solution in Ω that depends continuously on p, f and H.

General Conclusion and Perspectives

Functional differential equations with delays or iterative terms have captivated the attention of the majority of the scientific community and now occupy a more central place in all fields of science due to the increasing number of their applications.

This work is concerned with the study of three first order iterative-delay differential equations most often arising either from biological phenomena or from ones in population dynamics. The first chapter's goal was to provide a quick introduction to the subject and a synopsis of the dissertation's content. In the second chapter, we have presented some concepts, tools and sufficient preliminary results to allow us to develop a better understanding of the remaining chapters in which we have investigated these equations by means of a compelling technique that combines the fixed point theory along with the Green's functions method. Thus, by using this hybrid approach, we have been able to prove some existence, uniqueness and stability results which have been published in reputable journals.

Now, we will outline the steps we took to get the intended results. Our first and key step was to choose a suitable Banach space and a subset of it. They have satisfied all our requirements, whether biological or mathematical ones, such as the control of the iterative terms as well as the periodicity, the positivity and the boundedness of the sought solutions. The second step is to reformulate each problem from a differential equation with the periodic conditions to an integral one with a Green's kernel whose solutions were solutions of the suggested equations and vice versa. The third step is to apply the Schauder or the Krasnosleskii fixed point theorems for proving the existence of positive bounded and periodic solutions. Finally, under certain additional hypotheses, the Banach contraction principle has guaranteed the existence and continuous dependence on the parameters of the unique solution.

More work can be done on this topic, especially since the theory of iterative differential equations is not very well developed yet. So, there are a lot of prominent research prospects of this direction including:

-The possibility of applying the used technique to deal with many iterative-delay differential equations resulting from several phenomena such Nicholson's blowflies model, neural network models, fisheries management models, hematopoiesis models, etc.

- It appears to be significant to study the existence of almost-periodic, pseudo-almost-periodic or anti-periodic solutions for iterative-delay differential equations.

- We can expand the scope of the study by paying attention to fractional iterative differential equations and iterative-delay differential equations of higher order.

- It would also seem valuable to employ numerical methods or software to get approximate solutions or even to provide numerical simulations to show the outcomes.

To sum, we think that the outcomes in this dissertation, which extend and enhance a lot of previous findings in the literature, can contribute even a little to the development of the theory of iterative differential problems.

List of Publications

1. Mezghiche, L., Khemis, R., Bouakkaz, A.: Positive periodic solutions for a neutral differential equation with iterative terms arising in biology and population dynamics. International Journal of Nonlinear Analysis and Applications. **13**(2), 1041–1051 (2022). Indexed by Scopus

2. Mezghiche, L., Khemis, R., Bouakkaz, A.: Some existence and uniqueness results of a houseflies model with a delay depending on time and state. International Journal of Nonlinear Analysis and Applications. **14**(1), 865–876 (2023). Indexed by Scopus

3. Mezghiche, L., Khemis, R.: On Periodic Solutions of a Recruitment Model with Iterative Terms and a Nonlinear Harvesting. Bol. Soc. Paran. Mat. **41**, 1–9 (2023). Indexed by Scopus

BIBLIOGRAPHY

[1] Abel, N.H.: Oeuvres compétes. Christiana I I. 36–39 (1881)

[2] Ahmad, B., Ntouyas, S. K., Alsaedi, A.: Existence results for boundary value problems of differential inclusions with three-point integral boundary conditions. International Scholarly Research Network. 1–13 (2011)

[3] Agarwal, R.P., Bohner, M., Wan-Tong, L.: Nonoscillation and Oscillation: Theory for Functional Differential Equations. Marcel Dekker. Inc. New York. (2004)

[4] Almezel, S., Ansari, Q. H., & Khamsi, M. A. (Eds.). Topics in fixed point theory. Switzerland: Springer. (2014)

[5] Andrzej, P.: On some iterative-differential equation I. Zeszyty Naukowe UJ. Prace Mat. **12**, 53–56 (1968)

[6] Babbage, C.: An essay towards the calculus of functions. Philos. Trans. R. Soc. Lond. **105**, 389–432 (1815)

[7] Benchohra, M., Nieto, J. J., Ouahab, A.: Second-order boundary value problem with integral boundary conditions. Boundary Value Problems. **2011**(1), 260-309 (2010)

[8] Berinde, V.: Existence and approximation of solutions of some first order iterative differential equations. Miskolc Math. Notes. **11**(1), 13–26 (2010)

[9] Bouakkaz, A.: Bounded solutions to a three-point fourth-order iterative boundary value problem. Rocky Mt. J. Math. **52**(3), 793–803 (2022)

[10] Bouakkaz, A.: Positive periodic solutions for a class of first-order iterative differential equations with an application to a hematopoiesis model. Carpathian J. Math. **38**(2), 347–355 (2022)

[11] Bouakkaz, A., Ardjouni, A., Djoudi, A.: Existence of positive periodic solutions for a second-order nonlinear neutral differential equation by the Krasnoselskii's fixed point theorem. Nonlinear Dyn. Syst. Theory. **17**(3), 230–238 (2017)

[12] Bouakkaz, A., Ardjouni, A., Djoudi, A.: Periodic solutions for a nonlinear iterative functional differential equation. Electron. J. Math. Anal. Appl. **7**(1), 156–166 (2019)

[13] Bouakkaz, A., Ardjouni, A., Djoudi, A.: Periodic solutions for a second order nonlinear functional differential equation with iterative terms by Schauder fixed point theorem. Acta Math. Univ. Comen. **87**(2), 223–235 (2018)

[14] Bouakkaz, A., Ardjouni, A., Khemis, R., Djoudi, A.: Periodic solutions of a class of third-order functional differential equations with iterative source terms. Bol. Soc. Mat. Mex. **26**, 443–458 (2020)

[15] Bouakkaz, A., Khemis, R: Positive periodic solutions for a class of second-order differential equations with state-dependent delays. Turkish J. Math. **44**(4), 1412–1426 (2020)

[16] Bouakkaz, A., Khemis, R.: Positive periodic solutions forrevisited Nicholson's blowflies equation with iterative harvesting term. J. Math. Anal. Appl. **494**(2), 124663 (2021)

[17] Boucherif, A.: Second-order boundary value problems with integral boundary conditions. Nonlinear Anal. **70(1)**, 364–371 (2009)

[18] Buica, A.: Existence and continuous dependence of solutions of some functional differential equations. Seminar of Fixed Point Theory. **3**, 1-14 (1995)

[19] Brown, R. F.: A Topological Introduction to Nonlinear Analysis. Third Edition. Springer International Publishing Birkhauser. (2014)

[20] Cabada, A., Cid, J.A., Villamarin, B.M.: Compulation of Green's functions for boundary value problems with Mathematica, Appl. Math. Comput. **219(**4**)**, 1919–1936 (2012)

[21] Candan, T.: Existence of positive periodic solutions of first order neutral differential equations with variable coefficients. Appl. Math. Lett. **52**, 142–148 (2016)

[22] Cheraiet, S., Bouakkaz, A., Khemis, R.: Bounded positive solutions of an iterative three-point boundary-value problem with integral boundary conditions. J. Appl. Math. Comput. **65**, 597–610 (2021)

[23] Cheraiet, S., Bouakkaz, A., Khemis, R.: Some new findings on bounded solution of a third order iterative boundary-value problem. J. Interdiscip. Math. **25**(4), 1153–1162 (2022)

[24] Chouaf, S., Bouakkaz, A., Khemis, R.: On bounded solutions of a second-order iterative boundary value problem. Turkish J. Math. **46**(2), 453–464 (2022)

[25] Chouaf, S., Khemis, R., Bouakkaz, A.: Some existence results on positive solutions for an iterative second-order boundary-value problem with integral boundary conditions. Bol. Soc. Parana. Mat. **40**, 1–10 (2022)

[26] Clarke, F.: Functional Analysis. Calculus of Variations and Optimal Control. Graduate Texts in Mathematics. Springer. London. **264**, (2013)

[27] Course of normed-spaces. MATH 310, Xi'an Jiaotong-Liverpool University. China. Https://www.coursehero.com/file/196628281/normed-spaces2014-11-12pdf/.

[28] Danchin, R.: Cours de topologie et d'analyse fonctionnelle Master première année. Année 2009-2010

[29] Driver, R.D.: A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics. Proc. Inter Symposium on Nonli Diff Eqs and Nonlinear Mechanics. Academic Press. New York. 474–484 (1963)

[30] Driver, R.D.: A two-body problem of classical electrodynamics: the one dimensional case. Ann. Phys. **21**(1), 122–142 (1963)

[31] Driver, R.D.: Delay-differential equations with an application to a two-body problem of classical electrodynamics. Tech. Report. University of Minnesota. Minneapolis. (1960)

[32] Driver, R.D.: Existence theory for a delay differential system. Contrib. Differential Eq. **1**(3), 317–336 (1963)

[33] Driver, R.D., Norris, M.: Note on uniqueness for a one-dimensional two-body problem of classical electrodynamics. Annals of Physics. **42**(2), 347-351 (1967)

[34] Eder, E.: The Functional Differential Equation $x'(t) = x(x(t))$. Journal of Differential Equations. **54**(3), 390–400 (1984)

[35] Erbe, L.H, Zhang, B.G, Kong, Q.: Oscillation Theory for Functional Differential Equations. CRC Press. (1995)

[36] Faria, T., Freitas, P.: Topics in Functional Differential and Difference Equations. American Mathematical Society Providence, RI. (2001)

[37] Fečkan, M., Wang, J., Zhao, H.Y.: Maximal and minimal nondecreasing bounded solutions of iterative functional differential equations. Appl. Math. Lett. **113**, 106886 (2021)

[38] Galvis, J., Rojas, E.M., Sinitsyn, A.V.: Existence of positive solutions of a nonlinear second-order boundary-value problem with integral boundary conditions. Electronic Journal of Differential Equations. **2015**(236), 1–7 (2015)

[39] Ge, W., Mo, Y.: Existence of solutions to differential-iterative equation. Journal of Beijing Institute of Technology. **6**(3), 192–200, (1997)

[40] Green, G.: An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism. Nottingham. (1828)

[41] Guerfi, A., Ardjouni, A.: Periodic solutions for second order totally nonlinear iterative differential equations. The journal of Analysis. **30**(1), 353–367 (2021)

[42] Gurney, W.S.C., Blathe, S.P., Nishbt, R.M.: Nicholson's blow‡ies revisited. Nature **287**, 17–21, (1980)

[43] Hale, J.K., Lunel, S.M.V.: Introduction to functional differential equations. Springer Science and Business Media. **99**, (2013)

[44] Hutchinson, G.E.: Circular causal systems in ecology. Annals of the New York Academy of Sciences. vol. **50**(4), 221–246 (1948)

[45] Infante, G.**:** Positive solutions of nonlocal boundary value problems with singularities. Discrete and Continuous Dynamical Systems. **2009(**No.special**),** 377–384 (2009)

[46] Kaufmann, E.R.: Existence and uniqueness of solutions for a second-order iterative boundary-value problem functional differential equation. Electron. J. Differ. Equ. **2018**(150), 1–6 (2018)

[47] Khemis, M., Bouakkaz, A.: Existence and uniqueness results for a neutral erythropoiesis model with iterative production and harvesting terms. Bol. Soc. Paran. Mat. **41**, 1–9 (2023)

[48] Khemis, M., Bouakkaz, A., Khemis, R.: Existence, uniqueness and stability results of an iterative survival model of red blood cells with a delayed nonlinear harvesting term. J. Math. Model. **10**(3), 515–528 (2022)

[49] Khemis, R.: Existence, uniqueness and stability of positive periodic solutions for an iterative Nicholson's blowflies equation. J. Appl. Math. Comput. **69**, 1903–1916 (2023)

[50] Khemis, R., Ardjouni, A., Bouakkaz, A., Djoudi, A.: Periodic solutions of a class of third–order differential equations with two delays depending on time and state. Comment. Math. Univ. Carol. **60**(3), 379–399 (2019)

[51] Khemis, R., Ardjouni, A., Djoudi A.: Existence of periodic solutions for a second-order nonlinear neutral differential equation by the Krasnoselskii's fixed point technique. Matematiche. **72**(1), 145–156 (2017)

[52] Khemis, R., Bouakkaz, A.: New results on periodic solutions for a nonlinear fourth-order iterative differential equation. J. Prime Res. Math. **18**(2), 88–99 (2022)

[53] Khemis, R., Bouakkaz, A., Chouaf, S.: On the existence of periodic solutions of a second order iterative differential equation. Acta Math. Univ. Comen. **92**(1), 9-22 (2023)

[54] Khuddush, M., Prasad, K.R.: Nonlinear two-point iterative functional boundary value problems on time scales. J. Appl. Math. Comput. **68**, 4241–4251 (2022)

[55] Krasnov, M.L., Kisselev, A.I., Makarenko, G.I.: Equations intégrales: Problèmes et exercices. Moscou. (1977)

[56] Kreyszig, E.: Introductory functional analysis with applications. University of windsor. Canada. (1978)

[57] Li, Y., Kuang, Y.: Periodic solutions in periodic state-dependent delay equations and population models. Proc. Amer. Math. Soc. **130**(5), 1345–1353 (2002)

[58] Liu, Y., Ge, W.: Positive periodic solutions of nonlinear Duffing equations with delay and variable coefficients. Tamsui Oxf. J. Math. Sci. **20(**2**)**, 235–255 (2004)

[59] Luo, Y., Wang, W., Shen, J.: Existence of positive periodic solutions for two kinds of neutral functional differential equations. Appl. Math. Lett. **21**(6), 581–587 (2008)

[60] Mansouri, B., Ardjouni, A., Djoudi, A.: Periodicity and continuous dependence in iterative differential equations. Rend. Circ. Mat. Palermo, II. Ser **69**, 561–576 (2020)

[61] Martin, A.: Equations aux dérivées partielles. Exercices résolus. Dunod. (1991)

[62] Maynard, S.J.: Mathematical ideas in biology. Cambridge University Press, London, (1968)

[63] Mezghiche, L., Khemis, R.: On Periodic Solutions of a Recruitment Model with Iterative Terms and a Nonlinear Harvesting. Bol. Soc. Paran. Mat. **41**, 1–9 (2023)

[64] Mezghiche, L., Khemis, R., Bouakkaz, A.: Positive periodic solutions for a neutral differential equation with iterative terms arising in biology and population dynamics. International Journal of Nonlinear Analysis and Applications. **13**(2), 1041–1051 (2022)

[65] Mezghiche, L., Khemis, R., Bouakkaz, A.: Some existence and uniqueness results of a houseflies model with a delay depending on time and state. International Journal of Nonlinear Analysis and Applications. **14**(1), 865–876 (2023)

[66] Nicholson, A.J.: An outline of the dynamics of animal populations. Austral. J. Zoo. **2**, 9–65 (1954)

[67] Nicholson, A.J.: Compensatory Reactions of Populations to Stresses, and Their Evolutionary Significance. Australian Journal of Zoology. **2,** 1–8 (1954)

[68] Nicholson, A.J.: The self adjustment of population to change. Cols. Spring Harb's Syrup. Quant. Bzol. **22**, 153–173 (1957)

[69] Pata, V.: Fixed Point Theorems and Applications. Springer International Publishing, (2019)

[70] Pathak, H.K.: An Introduction to Nonlinear Analysis and Fixed Point Theory. Springer. (2018)

[71] Petuhov, V.R.: On a boundary value problem, (Russian. English summary). Trudy Sem.Teor. Differencial. Uravneniis Otklon. Argumentom Univ. Druzby Narodov Patrisa Limumby. **3**, 252-255 (1965)

[72] Poisson, S.D.: Sur les équations aux différences melées. Journal de l'École Polytechnique. **6**, 126–147 (1806)

[73] Rus, I.A., Petrusel, A., Petrusel, G., Fixed point theory 1950-2000 : Romanian contributions, House of the Book of Science, Cluj-Napoca. (2002)

[74] Rynne, B.P., Youngson, M.: Linear Functional Analysis. Springer Undergraduate Mathematics Series. (2008)

[75] Serra, E.: Periodic solutions for some nonlinear differential equations of neutral type. Nonlinear Anal. Theory Methods Appl. **17**(2), 139–151 (1991)

[76] Si, J.G., Li, W.R., Cheng, S.S.: Analytic solutions of an iterative functional differential equation. Comput. Math. Appl. **33(6)**, 47–51 (1997)

[77] Si, J.G., Wang, X.P.: Smooth solutions of a nonhomogeneous iterative functional differential equation with variable coefficients. J. Math. Anal. Appl. **226**(2), 377–392 (1998)

[78] Schröder, E.: Über iterate funktionen. Math. Ann. **3**, 295–322 (1871)

[79] Smart, D.R.: Fixed point theorems. Cambridge Tracts in Mathematics. no. 66. Cambridge University Press. London-New York. (1974)

[80] Staněk, S.: On global properties of solutions of functional differential equation $x'(t) = x(t) + x(x(t))$. Dynam. Systems Appl. **4**, 263–278 (1995)

[81] Taylor, C.E., Sokal, R.R.: Oscillations in Housefly Population Sizes Due to Time Lags. Ecology. **57,** 1060-1067 (1976)

[82] Teeravarapaug, P.: Introduction to Green's functions. College of Education Bangkok, Thailand, (1977)

[83] Turab, A., Sintunavarat, W.: A unique solution of the iterative boundary value problem for a second-order differential equation approached by fixed point results. Alexandria Engineering Journal. **60**(6), 5797–5802 (2021)

[84] Wang, Y., Lian, H., Ge, W.: Periodic solutions for a second order nonlinear functional differential equation. Appl. Math. Lett. **20(1)**, 110-115 (2007)

[85] Yang, D., Zhang, W.: Solutions of equivariance for iterative differential equations. Appl. Math. Lett. **17**(7), 759-765 (2004)

[86] Zhang, P.: Analytic solutions for iterative functional differential equations. Electron. J. Differential Equations. **180**, 1-7 (2012)

[87] Zhang, P., Gong, X.: Existence of solutions for iterative differential equations. Electron. J. Differential Equations. **7**, 1-10 (2014)

[88] Zhao, H.Y., Fečkan, M.: Periodic solutions for a class of differential equations with delays depending on state. Math. Commun. **23(1)**, 29-42 (2018)

[89] Zhao, H.Y., Liu, J.: Periodic solutions of an iterative functional differential equation with variable coefficients. Math. Methods Appl. Sci. **1(**40**)**, 286-292 (2017)

[90] Zhao, H.Y., Liu, J.: Periodic Solutions of a Second-Order Functional Differential Equation with State-Dependent Argument. Mediterr. J. Math. **15**(214), (2018)