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Memory
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Homogenization of the Parabolic Problem in perforated domains with small holes

Option : NAPDE$_s$

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All praise is due to Allah, through whose grace efforts are completed and endeavors are sealed. No servant overcomes obstacles and difficulties except by His guidance and assistance. So, praise be to Allah at the beginning and at the end.

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DEDICATION

I dedicate the fruit of my humble efforts.

To my parents, who gave me life, hope, and nurtured in me a passion for learning and knowledge. May God protect them.

To my sisters, Manar and Nesrin, who God has blessed me with their presence in my life. To her and to the years of life I have been fortunate to enjoy, witnessing her presence around me in every moment... I long for her and for the days with her... to my grandmother Fatima, may God have mercy on her.

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Abstract

In this memory we study a class of parabolic problems in periodically perforated domains with a homogeneous Newmann condition on the boundary of the holes. We focus on the homogenezation of these equations. The proof is based on th periodic unfolding method in perforated domains.

Key words : hyperbolic problem, periodic unfolding, perforated domain, homognisation.

Résumé

Nous considérons dans cette mémoire une class de Problème parabolique dans un domaine perforé avec des petits trous périodiquement et de condition de Newmann hmogène. Nous nous concentrons sur l'homogenisation de ces équation. Lépreuve est basée sur la méthod d'eclatement periodique dans les domaines perforés.

Mots clés : probléme hyperbolique, éclatement périodique, domaine perforé, homogénéisation.

ملخص

في هذا العمل في هذا العمل قمنا بدراسة قسم من المسائل القطع المكافئ في المجالات المسامية مع وضع شروط نيومان المتجانسة على حدود الثقوب نهتم بدراسة تجانس هذه المعادلات و ذلك بالاعتماد على طريقة الانفجار الدوري في المجالات المسامية.

كلمات مفتاحية: قطع المكافئ، الانفجار دوري، مجال مسامي، تجانس.

CONTENTS

INTRODUCTION

Various physical phenomena can be described in terms of conduction thermique. In these studies, researchers aim to find averaged characteristics of the conduction such as energy transfer.

To drive some quantities, mathematicians used several mathematical approaches collectively known as Homogenization Theory.

Energing in the late 1960's and early 1970's, this theory became a significant part of mathematics concerning partial differential equations and their numerical approximations. It originated from works in physics and machanics, where equivalent macroscopic stuctures were obtained for microscopic heterogeneous madia, as seen in works by Bruggman [4](1935), Hashin and Shtrikman [9](1963), among others. Applied mathmaticians and numerical analysts became increasingly interested in the late 1960's, with seminal works by De Giorgi and Spagnolo [8](1973), Babuska [1](1975), Bensoussan, Lions, and Papanicolaou [3](1978), and Sanchez-Palencia [12](1980), among others.

Homogenization theory enables the replacement of problems strongly oscillating coefficients with approximate problems having constant coefficients. This implies that the solution of a boundary value problem, dependent on a small parameter, converges to the solution of a limit boundary value problem explicity describes, making it much easier to process numerically.

In our paper, we utilize the periodic unfolding method due to its simplicity in handing perforated domains without the need for extension operators. This metod also allows us to adress second-order operators with highly oscillating coefficients, which may not be feasible with order approaches.

The periodic unfolding method was first introduced in Cioranescu, Damlamian and Griso [6] for the case of fixed domains (see [7] for more details)

We extend the unfolding operator introduced in pervious works [6] to time-dependent functions and thoroughly examine its properties. Then we apply the periodic unfolding method to homogenization results for the heat equation with oscillating coefficients in domains containing small holes.

The paper is structured as follows: chapters 1,2 provide background on the geometric framework of perforated domains and define properties of unfolding operators for fixed and perforated domains with small holes. Additionally, we present the extension of the local average operator to time dependent functions in this chapter. Chapter 4 presents the main homogenization results for the heat equation and the main results for the problem.

CHAPTER

1

THE PERIODIC UNFOLDING METHOD IN PERFORATED DOMAINS

In this chapter we recall the definition and some properties of the periodic unfolding operators in perforated domains $\mathcal{T}_\varepsilon$ for the classical homogenization.

1.1 The periodic unfolding operator $\mathcal{T}_\varepsilon$

In this section, we introduce the periodic unfolding operator in the case of perforated domains introduced by Cioranscu et all [5] and [6].

Definition 1.1 *[5]. Let* $\varphi \in L^p(\Omega_\varepsilon)$, $p \in [1 + \infty]$. *We define the function*

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = \tilde{\varphi}(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y) \tag{1.1}$$

for every $x \in \mathbb{R}^N$ *and* $y \in Y$.

Remark 1.1 *Notice that the oscillations due one see immediately the interest of the unfolding operator. Indeed, when trying to pass to the limit in a sequence defined on* Ω_ε *one needs first, while using standard methods, to extend it to a fixed domain. With* $\mathcal{T}_\varepsilon$*, such extensions are more necessary.The main properties given in for fixed domains can easily be adapted for the perforated ones without any major difficulty in the proofs. These properties are listed in the proposition below. To do so, let us first define the following domain:*

$$\widetilde{\Omega_\varepsilon}{=}int\left(\bigcup_{\xi\in\Lambda_\varepsilon}(\xi+Y)\right),$$

where

$$\Lambda_\varepsilon\{\xi\in\mathbb{Z}^N,\varepsilon(\xi+\bar{Y})\cap\Omega\neq 0\}.$$

The set Ω_ε *is the smallest finite union of* εY *cells containing* Ω*.*

Proposition 1.1 *[5]:The unfolding operator* $\mathcal{T}$ *has the following properties :*

1. $\mathcal{T}_\varepsilon$ *is a linear operator.*

2. $\mathcal{T}_\varepsilon(\varphi)\Big(x,\{\frac{x}{\varepsilon}\}_Y\Big)=\varphi(x)\quad\forall\varphi\in L^p(\Omega_\varepsilon)$ *and* $x\in\mathbb{R}^N.$

3. $\mathcal{T}_\varepsilon(\varphi\psi)=\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi),\quad\forall\varphi,\psi\in L^p(\Omega_\varepsilon).$

4. *Let* φ *in* $L^p(Y)$ *or* $L^p(Y)$ *be a periodic function. Set* $\varphi_\varepsilon(x)=\varphi\Big(\frac{x}{\varepsilon}\Big)$*. Then,* $\mathcal{T}_\varepsilon(\varphi_\varepsilon)(x,y)=\varphi(y),$ *a.e. in* $\widetilde{\Omega}_\varepsilon.$

5. *One has the integration formula*

$$\int_{\Omega_\varepsilon}\varphi\ dx=\frac{1}{|Y|}\int_{\widetilde{\Omega}_\varepsilon\times Y}\mathcal{T}_\varepsilon(\varphi)\ dx\ dy\quad,\forall\varphi\in L^1(\Omega_\varepsilon).$$

6. *For every* $\varphi \in L^2(\Omega_\varepsilon), \mathcal{T}_\varepsilon(\varphi)$ *belongs to* $L^2(\mathbb{R}^N \times Y)$ *it also belongs to* $L^2(\tilde{\Omega}_\varepsilon \times Y)$.

7. *For every* $\varphi \in L^2(\Omega_\varepsilon)$

$$||\mathcal{T}_\varepsilon(\varphi)||_{L^2(R^N \times Y)} = \sqrt{|Y|}||\varphi||_{L^2(\Omega_\varepsilon)}$$

8. $\nabla_y \mathcal{T}_\varepsilon(\varphi)(x, y) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(x, y)$ *for every* $(x, y) \in \Omega \times Y$.

9. *If* $\varphi \in H^1(\tilde{\Omega}_\varepsilon)$*, then* $\mathcal{T}_\varepsilon(\varphi)$ *is in* $L^2\left(\mathbb{R}^N, H^1(Y)\right)$.

10. *One has the estimate*

$$||\nabla_y \mathcal{T}_\varepsilon(\varphi)||_{(L^2(\Omega \times Y))^N} = \varepsilon\sqrt{|Y|}||\nabla_x \varphi||_{(L^2(\Omega_\varepsilon))^N}$$

1.2 proof

1. Let $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned}
\mathcal{T}_\varepsilon(\alpha\varphi + \beta\psi) &= \varphi(\alpha(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y)) + \psi(\beta(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y)) \\
&= \alpha\varphi(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y) + \beta\psi(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y) \\
&= \alpha\mathcal{T}_\varepsilon(\varphi) + \beta\mathcal{T}_\varepsilon(\psi).
\end{aligned}$$

Then $\mathcal{T}_\varepsilon$ is a linear operator.

2.

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi)(x,\{\frac{x}{\varepsilon}\}) &= \varphi(\varepsilon[\frac{x}{\varepsilon}]+\varepsilon\{\frac{x}{\varepsilon}\})\\ &= \varphi(\varepsilon[\frac{x}{\varepsilon}]+x-\varepsilon[\frac{x}{\varepsilon}])\\ &= \varphi(x).\end{aligned}$$

3. Let $\varphi, \psi \in L^2(\Omega_\varepsilon)$, we have by definition 1.1

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi\psi) &= \varphi\psi\left(\varepsilon[\frac{x}{\varepsilon}]_y+\varepsilon y\right)\\ &= \varphi\left(\varepsilon[\frac{x}{\varepsilon}]_y+\varepsilon y\right)\psi\left(\varepsilon[\frac{x}{\varepsilon}]_y+\varepsilon y\right)\\ &= \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi).\end{aligned}$$

4. and by definition for all $\varphi_\varepsilon = \varphi(\frac{x}{\varepsilon})$, $\varphi \in L^2(Y)$, we have

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi_\varepsilon)(x,y) &= \varphi\left(\varepsilon\left[\frac{\{\frac{x}{\varepsilon}\}}{\varepsilon}\right]+\varepsilon\left\{\frac{\{\frac{x}{\varepsilon}\}}{\varepsilon}\right\}\right)\\ &= \varphi\{\frac{x}{\varepsilon}\} = \varphi(y).\end{aligned}$$

5. According the definition 1.1, we have

$$\begin{aligned}\frac{1}{|\bar{Y}|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)\;dx\;dy &= \frac{1}{|Y|}\int_{\widetilde{\Omega_\varepsilon}\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)\;dx\;dy\\ &= \frac{1}{|Y|}\sum_{\xi\in\Xi_\varepsilon}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)\;dx\;dy\end{aligned}$$

on each set $(\varepsilon\xi+\varepsilon Y)\times Y$ with $\xi \in \Xi_\varepsilon$, the function $\mathcal{T}_\varepsilon(\varphi)(x,y)\;dx\;dy = \varphi(\varepsilon\xi+\varepsilon y)$ is constant in x, it is consequence of (1.1). So, for each integral in the sum of the right member, we have :

$$\begin{aligned}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)\;dx\;dy &= |\varepsilon\xi+\varepsilon Y|\int_Y\varphi(\varepsilon\xi+\varepsilon y)dy\\ &= \varepsilon^N|Y|\int_Y\varphi(\varepsilon\xi+\varepsilon y)dy = |Y|\int_{(\varepsilon\xi+\varepsilon Y)}\varphi(x)dx\end{aligned} \tag{1.2}$$

we suppose $\boldsymbol{x} = \boldsymbol{\varepsilon\xi} + \boldsymbol{\varepsilon y}$, this implies $\boldsymbol{dx} = \boldsymbol{\varepsilon}^{\boldsymbol{N}}\boldsymbol{dy} \Longrightarrow \boldsymbol{dy} = \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^{\boldsymbol{N}}}$ by summing in Ξ, the right member be $\int_{\widetilde{\boldsymbol{\Omega}}_{\varepsilon}} \boldsymbol{\varphi(x)}\ \boldsymbol{dx}.$

so we get:

$$\sum_{\xi\in\Xi_\varepsilon} \boldsymbol{\varepsilon}^{\boldsymbol{N}} |\boldsymbol{Y}| \int_{(\varepsilon\xi+\varepsilon Y)} \boldsymbol{\varphi(x)} \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^{\boldsymbol{N}}} \boldsymbol{dx} = \int_{\widetilde{\boldsymbol{\Omega}}} \boldsymbol{\varphi(x)}\ \boldsymbol{dx}.$$

Now, we remplace this expression in (1.2), we get the result.

7. For all $\varphi \in \boldsymbol{L}^2(\boldsymbol{\Omega}_\varepsilon)$

$$\begin{aligned}
\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N\times Y)} &= \left(\int |\mathcal{T}_\varepsilon(\varphi)|^2\ \boldsymbol{dx}\ \boldsymbol{dy}\right)^{\frac{1}{2}}\\
&= \left(\mathbf{1}\int_{\Omega\times Y} |\varphi|^2\ \boldsymbol{dx}\ \boldsymbol{dy}\right)^{\frac{1}{2}} = \left(\int \mathbf{1}\ \boldsymbol{dy}\right)^{\frac{1}{2}} \left(\int |\varphi|^2 \boldsymbol{dx}\right)^{\frac{1}{2}}\\
&= \sqrt{|\boldsymbol{Y}|}||\varphi||_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

8. For all $\varphi \in \boldsymbol{L}^2(\boldsymbol{\Omega}_\varepsilon)$

$$\begin{aligned}
\nabla_y \mathcal{T}_\varepsilon(\varphi)(\boldsymbol{x},\boldsymbol{y}) &= \nabla_y \varphi\left(\varepsilon\left[\frac{\boldsymbol{x}}{\varepsilon}\right]_{\boldsymbol{y}} + \varepsilon \boldsymbol{y}\right)\\
&= \varepsilon \nabla_x \varphi\left(\varepsilon\left[\frac{\boldsymbol{x}}{\varepsilon}\right]_{\boldsymbol{y}} + \varepsilon \boldsymbol{y}\right)\\
&= \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(\boldsymbol{x},\boldsymbol{y}).
\end{aligned}$$

10. We apply proposition 1.1, we obtain

$$\begin{aligned}
\|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega\times Y))^N} &= \|\varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)\|_{(L^2(\Omega\times Y))^N}\\
&= \varepsilon\sqrt{|\boldsymbol{Y}|}||\nabla_x \varphi||_{(L^2(\Omega_\varepsilon))^N}.
\end{aligned}$$

CHAPTER

2

THE TIME-DEPENDENT UNFOLDING OPERATOR IN PERFORATED DOMAINS

In this chapter, we adapt the unfolding operator $\boldsymbol{\mathcal{T}_\varepsilon^*}$ in to time-dependent functions We present the unfolding operator $\boldsymbol{\mathcal{T}_\varepsilon^*}$ which maps functions defined on the oscillating domaine $\boldsymbol{\Omega_\varepsilon^* \times (0,T)}$ into functions defined on the fixed domaine $\boldsymbol{\Omega \times Y^* \times (0,T)}$. This avoids the use of any extension operator. We also give some properties by extending the previous ones for more details and for the classical unfolding theory) For any $\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{N}}$, we use $[\boldsymbol{z}]_{\boldsymbol{y}}$ to denote the T, unique integer combination $\sum_{\boldsymbol{j=1}}^{\boldsymbol{n}} \boldsymbol{k_J b_J}$ of the period such that $\boldsymbol{z} - [\boldsymbol{z}]_{\boldsymbol{Y}} \in \boldsymbol{Y}$ Set:

$$\{\boldsymbol{z}\}_{\boldsymbol{Y}} = \boldsymbol{z} - [\boldsymbol{z}]_{\boldsymbol{Y}} \in \boldsymbol{Y} \quad \boldsymbol{a.efor} \quad \boldsymbol{z} \in \mathbb{R}^{\boldsymbol{N}} \tag{2.1}$$

Then for each $\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{N}}$, we have:

$$\boldsymbol{x} = \boldsymbol{\varepsilon} \left(\left\{ \frac{\boldsymbol{x}}{\boldsymbol{\varepsilon}} \right\}_{\boldsymbol{Y}} + \left[\frac{\boldsymbol{x}}{\boldsymbol{\varepsilon}} \right]_{\boldsymbol{Y}} \right). \tag{2.2}$$

a.e for $x \in \mathbb{R}^N$. Let us first recall the unfolding operator $\mathcal{T}_\varepsilon$ for the fixed domain $\Omega \times (0, T)$ introduced in [17], where the properties of $\mathcal{T}_\varepsilon$, are shown without proofs.

Definition 2.1 *for* $p \in [1, +\infty)$ *and* $q \in [1, \infty]$, *let* ϕ *be in* $L^q(0, T; L^p(\Omega))$. *The unfolding operator* $\mathcal{T}_\varepsilon$:$L^q(0, T; L^p(\Omega)) \mapsto L^q(0, T; L^p(\Omega \times Y))$ *is defined as follows:*

$$\mathcal{T}_\varepsilon(\phi)(x, y, t) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right) & a.e. \quad for \ (x, y, t) \in \hat{\Omega}_\varepsilon \times Y \times (0, T), \\ 0 & a.e. \quad for \ (x, y, t) \in \Lambda_\varepsilon \times Y \times (0, T). \end{cases}$$

In a similar way, we extend the unfolding operator for the perforated domain Ω^*_ε *to the following unfolding operator* $\mathcal{T}^*_\varepsilon$ *for the perforated domain* $\Omega^*_\varepsilon \times (0, T)$.

Definition 2.2 *For* $p \in [1, +\infty)$ *and* $q \in [1, \infty]$, *let* ϕ *be in* $\text{Ł}^q(0; T; L^p(\Omega^*_\varepsilon))$. *The unfolding operator* $\mathcal{T}^*_\varepsilon$:$L^q(0, T; L^p(\Omega^*_\varepsilon)) \mapsto L^q(0, T; L^p\Omega \times Y^*)$ *is defined as follows:*

$$\mathcal{T}^*_\varepsilon(\phi) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right) & a.e. \quad for \ (x, y, t) \in \hat{\Omega}_\varepsilon \times Y^* \times (0, T), \\ 0 & a.e. \quad for \ (x, y, t) \in \Lambda_\varepsilon \times Y^* \times (0, T). \end{cases}$$

From this definition, the following properties are immediate:

$$\begin{aligned} &\mathcal{T}^*_\varepsilon(vw) = \mathcal{T}^*_\varepsilon(v)\mathcal{T}^*_\varepsilon(w), \quad \forall w, v \in L^q(0, T; L^p(\Omega^*_\varepsilon)), \\ &\mathcal{T}^*_\varepsilon(\psi\varphi) = \varphi\mathcal{T}^*_\varepsilon(\psi), \quad \forall \psi \in L^p(\Omega^*_\varepsilon) \quad and \quad \varphi \in L^q(0, T), \\ &\nabla_y(\mathcal{T}^*_\varepsilon(\phi)) = \varepsilon\mathcal{T}^*_\varepsilon(\nabla\phi), \quad \forall \phi \in L^q(0, T; W^{1,P}(\Omega^*_\varepsilon)). \end{aligned} \tag{2.3}$$

Remark 2.1 *Concerning* $\mathcal{T}_\varepsilon$ *and* $\mathcal{T}^*_\varepsilon$, *we have the following :*

$$\mathcal{T}^*_\varepsilon(\omega|_{\Omega^*_\varepsilon \times (0,T)}) = \mathcal{T}_\varepsilon(\omega)|_{\Omega \times Y^* \times (0,T)},$$

$$\mathcal{T}^*_\varepsilon(\psi) = \mathcal{T}_\varepsilon(\tilde{\psi})|_{\Omega \times Y^* \times (0,T)},$$

where ω *and* ψ *are defined on* $\Omega \times (0, T)$ *and* $\Omega^*_\varepsilon \times (0, T)$, *respectively.*
In Definitions 2.1 and 2.2, if ϕ *is independent of* t, *then* $\mathcal{T}_\varepsilon$ *and* $\mathcal{T}^*_\varepsilon$ *are the classical unfoliding operators defined in [6] and [1], respectively.*
For simplicity, we always write $\mathcal{T}^*_\varepsilon(\phi)$ *instead of* $\mathcal{T}^*_\varepsilon(\phi|_{\Omega^*_\varepsilon \times (0,T)})$ *for any function* ϕ *defined in* $\Omega \times (0, T)$.

Next we list some properties of the unfolding operator $\mathcal{T}_\varepsilon^*$ used in this paper. The proofs are essentially the same as those in [4] and [7].

Proposition 2.1 *for* $p \in [1, +\infty)$ *and* $q \in [1, \infty]$ *the operator* $\mathcal{T}_\varepsilon^*$ *linear and continuous from* $L^q(0,T;L^p(\Omega_\varepsilon^*))$ *to* $L^q(0,T;L^p(\Omega \times Y^*))$. *Let* $\phi \in L^q(0,T;L^1(\Omega_\varepsilon^*))$ *and* $\omega, v \in L^q(0,T;L^p(\Omega_\varepsilon^*))$.
For a.e. $t \in (o, T)$, *we have :*

(i) $\displaystyle\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(\phi)(x,y,t)\,dx\,dy = \int_{\widehat{\Omega}_\varepsilon^*}\phi(x,t)dx = \int_{\Omega_\varepsilon^*}\phi(x,t)dx - \int_{\Lambda_\varepsilon^*}\phi(x,t)dx,$

(ii) $||\mathcal{T}_\varepsilon^*(\omega)||_{L^p(\Omega\times Y^*)} = |Y|^{\frac{1}{p}}||\omega||_{L^p(\Omega_\varepsilon^*)} \leqslant |Y|^{\frac{1}{p}}||\omega||_{L^P(\widehat{\Omega}_\varepsilon^*)}.$

Proposition 2.2 *for* $q \in [1, +\infty]$, *let* ϕ_ε *be in* $L^q(0,T;L^1(\Omega_\varepsilon^*))$ *and satisfy*

$$\int_o^T\int_{\Lambda_\varepsilon^*}|\phi_\varepsilon|dx\ dt \to 0,$$

then

$$\int_0^T\int_{\Omega_\varepsilon^*}\phi_\varepsilon dx\ dt - \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(\phi_\varepsilon)dx\ dy\ dt \to 0.$$

As usual, this convergence is denoted by

$$\int_0^T\int_{\Omega_\varepsilon^*}\phi_\varepsilon dx\ dt \backsimeq \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(\phi_\varepsilon)dx\ dy\ dt \to 0.$$

Moreover,we have the following convergences :

Proposition 2.3 *(i) For* $p, q \in (1, +\infty]$, *let* $\phi_\varepsilon \in L^q(0,T;L^p(\Omega_\varepsilon^*))$ *and* $\psi \in L^{q'}(0,T;L^{p'}(\Omega_\varepsilon^*))$
$(1/p + 1/p' = 1, 1/q + 1/q' = 1)$ *such that*

$$||\phi_\varepsilon||_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leqslant C \quad and \quad ||\psi_\varepsilon||_{L^{q'}(0,T;L^{p'}(\Omega_\varepsilon^*))} \leqslant C\ , \tag{2.4}$$

then

$$\int_0^T\int_{\Omega_\varepsilon^*}\phi_\varepsilon\psi dx\ dt \backsimeq \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(\phi_\varepsilon)\mathcal{T}_\varepsilon^*(\psi)\ dx\ dy\ dt.$$

(ii) *For* $\boldsymbol{p, q} \in (1, +\infty]$*, let* $\phi_\varepsilon \in L^q(0,T;L^p(\Omega_\varepsilon^*))$ *and* $\psi_\varepsilon \in L^{q'}(0,T;L^{po}(\Omega_\varepsilon^*))$ $(1/p+1/po<1, 1/q+1/q'=1)$ *such that*

$$||\phi_\varepsilon||_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leqslant C \quad and \quad ||\psi_\varepsilon||_{L^{q'}(0,T;L^{po}(\Omega_\varepsilon^*))} \leqslant C, \tag{2.5}$$

Then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi_\varepsilon dx\ dt \backsimeq \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y^*} \tau^*(\phi_\varepsilon)\mathcal{T}^*(\psi_\varepsilon) dx\ dy\ dt.$$

Proposition 2.4 *(some convergence properties)*

(i) *For* $p, q \in [1, \infty)$*, let* $\omega \in L^q(0,T;L^p(\Omega))$*. Then*

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \to \omega \quad strongly\ in\ L^q(0,T;L^p(\Omega\times Y^*)).$$

(ii) *For* $p, q \in [1, \infty)$*, let* $\{\omega_\varepsilon\}$ *be a sequence in* $L^q(0,T;L^p(\Omega))$ *such that*

$$\omega_\varepsilon \to \omega \quad strongly\ in\ L^q(0,T;L^p(\Omega)).$$

then

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \to \omega \quad strongly\ in\ L^q(0,T;L^p(\Omega\times Y^*)).$$

(iii) *For* $p \in (1, \infty)$ *and* $q \in (1, \infty]$*, let* $\{\omega_\varepsilon\}$ *be a sequence in* $L^q(0,T;L^p(\Omega_\varepsilon^*))$ *such that*

$$||\omega_\varepsilon||_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq C.$$

If

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \rightharpoonup \omega\ weakly\ in\ L^q(0,T;L^P(\Omega\times Y^*)),$$

then we have

$$\tilde{\omega}_\varepsilon \rightharpoonup \theta\mathcal{M}_{Y^*}\ weakly\ in\ L^q(0,T;L^p(\Omega)).$$

For $q = \infty$ *the weak convergences above are replaced by the weak* convergences, respectively*

proof:

(i) Using proposition 2.4 and (2.3), we have:

$$||\mathcal{T}_\varepsilon^*(\omega)-\omega||_{L^q(0,T;L^p(\Omega\times Y^*))} = ||\mathcal{T}_\varepsilon^*(\omega-\phi\varphi)+\mathcal{T}_\varepsilon^*(\phi\varphi)-\phi\varphi-\omega||_{L^q(0,T;L^p(\Omega\times Y^*))},$$

$$\leqslant 2|Y|^{\frac{1}{p}}||\phi\varphi-\omega||_{L^q(0,T;L^p(\Omega))}+||(\mathcal{T}_\varepsilon^*(\phi)-\phi)\varphi||_{L^q(0,TL^p(\Omega\times Y^*))}$$

for any $\phi \in \mathcal{D}(\Omega)$ *and* $\phi \in \mathcal{D}(0,T)$. *Since*

$$\mathcal{T}_\varepsilon^*(\phi) \rightarrow \phi \textit{ strongly in } L^p(\Omega \times Y^*),$$

then

$$||(\mathcal{T}_\varepsilon^*(\phi)-\phi)\varphi||_{L^q(0.T;L^p(\Omega\times Y^*))} \rightarrow 0.$$

Consequently, we have

$$\limsup_{\varepsilon\longrightarrow 0} ||\mathcal{T}_\varepsilon^*(\omega)-\omega||_{L^q(0,T;L^p(\Omega\times Y^*))} \leqslant 2|Y|^{\frac{1}{p}}||\phi\varphi-\omega||_{L^q(0,T;L^p(\Omega))},$$

which implies statement (i) due to the density of $D(0,T)\otimes D(\Omega)$ *in* $L^q(0,T;L^p(\Omega))$.

(ii) From proposition 2.1 follows from statment (i).

(iii) Let $\psi \in L^{q'}(0,T;L^{p'}(\Omega))$ $(1/p+1/p'=1, 1/q+1/q'=1)$ *from proposition 2.3(i)*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \tilde{\omega}_\varepsilon\psi\,dx\,dt = \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*}\omega_\varepsilon\psi dx\,dt = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(\omega_\varepsilon)\mathcal{T}_\varepsilon^*(\psi)\,dx\,dy\,dt.$$

Since statement (i) gives

$$\mathcal{T}_\varepsilon^*(\psi) \rightarrow \psi \textit{ strongly in } L^{q'}(0,T;(L^{p'}\Omega\times Y^*)),$$

then we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \tilde{\omega}_\varepsilon\psi dx\,dt = \int_0^T\int_\Omega\left\{\frac{1}{|Y|}\int_{Y^*}\hat{\omega}dy\right\}\psi dx\,dt = \theta\int_0^T\int_\Omega \mathcal{M}_{Y^*}(\hat{\omega})\psi dx\,dt.$$

which implies the desired result.

Now we complete this subsection with a convergence result related to the space $L^q(0,T;W^{1,p}(\Omega_\varepsilon^*))$ *this proof can be directly obtained from that of Theorem 2.1 in [4].*

Proposition 2.5 *([15], [17]) let* $p \in]1, +\infty[$ *and* $\{\varphi_\varepsilon\}$ *be a sequence in the space* $L^\infty(0, T; W_0^{1,p}(\Omega))$ *such that*

$$||\nabla \varphi_\varepsilon||_{L^\infty(0,T;L^p(\Omega))} \leqslant C.$$

Then there exist $\varphi \in L^\infty(0, T; W_0^{1,p}(\Omega))$ *and* $\hat{\varphi} \in L^\infty(0, T; W_{per}^{1,p}(Y))$ *such that up to a subsequence,*

$$\mathcal{T}_\varepsilon(\nabla \varphi_\varepsilon) \rightharpoonup \varphi \quad \textit{weakly}^* \textit{ in } L^\infty(0, T; L^p(\Omega; W^{1,p}(Y))),$$

$$\mathcal{T}_\varepsilon(\nabla \varphi_\varepsilon) \rightharpoonup \nabla_x \varphi + \nabla_y \hat{\varphi} \quad \textit{weakly}^* \textit{ in } L^\infty(0, T; L^p(\Omega \times Y)).$$

We end this section by recalling the definition of the mean value operator $\mathcal{M}_Y$ *and that of the local average* $\mathcal{M}_Y^\varepsilon$ *and give some of their properties that will be useful in the sequel.*

Theorem 2.1 *For* $p \in (1, +\infty)$, *let* $\{\omega_\varepsilon\}$ *be a sequence in* $L^\infty(0, T; W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega))$ *such that*

$$||\nabla \omega_\varepsilon||_{L^\infty(0,T;L^p(\Omega_\varepsilon^*))} \leq C \quad \textit{and} \quad \left|\left|\frac{\partial \omega_\varepsilon}{\partial t}\right|\right|_{L^\infty(0,T;L^p(\Omega_\varepsilon^*))} \leq C. \tag{2.6}$$

Then there exist $\omega \in L^\infty(0, T; W_0^{1,p}(\Omega))$ *with* $\dfrac{\partial \omega}{\partial t} \in L^\infty(0, T; L^p(\Omega))$ *and* $\hat{\omega} \in L^\infty(0, T; W_{per}^{1,p}(Y))$ *with* $\mathcal{M}_Y^* \equiv 0$, *such that, up to a subsequence,*

$$\begin{aligned}
&\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \rightharpoonup \omega \quad \textit{weakly} * \ \textit{in } \L^\infty(0, T; (\Omega; W^{1,p}(Y^*))),\\
&\mathcal{T}_\varepsilon^*(\nabla \omega_\varepsilon) \rightharpoonup \nabla \omega + \nabla_y \hat{\omega} \quad \textit{weakly} * \ \textit{in } L^\infty(0, T; L^p(\Omega \times Y^*)),\\
&\mathcal{T}_\varepsilon^*\left(\frac{\partial \omega_\varepsilon}{\partial t}\right) \rightharpoonup \frac{\partial \omega}{\partial t} \quad \textit{weakly} * \ \textit{in } L^\infty(0, T; L^p(\Omega \times Y^*)),\\
&\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \longrightarrow \omega \quad \textit{strongly in } L^q(0, T; L^p(\Omega; W^{1,p}(Y^*))),\\
&||\omega_\varepsilon - \omega||_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \longrightarrow 0,
\end{aligned} \tag{2.7}$$

where q is any number in $(1, +\infty)$

Definition 2.3 *let* $p \in [1, +\infty[$ ***and*** $q \in [1, +\infty]$. *The mean value operator*

$\mathcal{M}_Y$:$L^q(0,T;L^p(\Omega\times Y))\longmapsto L^q(0,T;L^p(\Omega))$ *is defined by*

$$\mathcal{M}_Y(u)(x,t)=\frac{1}{|Y|}\int_Y u(x,y,t)\ dy,$$

for every $u\in L^q(0,T;L^p(\Omega\times Y))$.

Definition 2.4 *Let* $p\in[1,+\infty[$ *and* $q\in[1,+\infty]$. *The local average operator* $\mathcal{M}^\varepsilon_Y$:$L^q(0,T;L^p(\Omega))\longmapsto L^q(0,T;L^p(\Omega))$ *is defined by*

$$\mathcal{M}^\varepsilon_Y(\varphi)(x,t)=\frac{1}{|Y|}\int_Y \mathcal{T}_\varepsilon(\varphi)(x,y,t)\ dy,$$

for every $\varphi\in L^q(0,T;L^p(\Omega))$.

Remark 2.2 *In connection, some of the properties of* $\mathcal{T}_\varepsilon$ *(in the case of dependence on time) can be derived directly for those of the unfolding operator for fixed domains from [4] with the time* t *as a mere parameter.*
As a consequence, we have the following result.

Proposition 2.6 *Let* $p\in[1,\infty[$ *and* $q\in[1,\infty]$.

(1) For $\varphi\in L^q(0,T;L^p(\Omega))$, *one has*

$$\mathcal{T}_\varepsilon(\mathcal{M}^\varepsilon_Y(\varphi))(x,y,t)=\mathcal{M}_Y(\mathcal{T}_\varepsilon(\varphi))(x,t)=M^\varepsilon_Y(\varphi)(x,t)\quad \text{in}\quad \Omega\times[0,t].$$

(2) Let $\{\omega_\varepsilon\}$ *be a sequence in* $L^q(0,T,L^P(\Omega))$ *such that*

$$\omega_\varepsilon\to\omega\quad \text{strongly in } L^q(0,T,L^p(\Omega)).$$

Then

$$\mathcal{M}^\varepsilon_Y(\omega_\varepsilon)\to\mathcal{M}_Y(\omega)=\omega\quad \text{strongly in } L^q(0,T,L^p(\Omega)).$$

(3) For $\varphi\in L^q(0,T;L^p(\Omega))$,

$$||\mathcal{M}_Y(\varphi)||_{L^q(0,T;L^p(\Omega))}\leqslant|Y|^{\frac{1-p}{p}}||\varphi||_{L^q(0,T;L^p(\Omega))}.$$

CHAPTER

3

UNFOLDING OPERATOR IN DOMAINS DEPENDING ON TWO PARAMETERS

In this chapter we recall the definition and some of its properties of the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending on two mall parameters ε and δ, as introduced in [6].

Definition 3.1 *[6] Let* $p \in [1, +\infty[$. *For* $\phi \in L^p(\Omega)$, *the unfolding operator* $\mathcal{T}_{\varepsilon,\delta}$ *is the function* $\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ *defined by*

$$\mathcal{T}_{\varepsilon,\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x,\delta z) & \textit{if } (x,z) \in \hat{\Omega}_\varepsilon \times \dfrac{1}{\delta} Y \\ 0 & \textit{otherwise}, \end{cases}$$

where $\mathcal{T}_\varepsilon$ *is the operator for fixed domains as introduced in [4].*
To go further, let us introduce what is called a perforated domain with small holes, denoted here $\Omega^*_{\varepsilon,\delta}$. *Let* $B \subset\subset Y$ *and denote* $Y^*_\delta = Y/\delta\bar{B}$. *Then* $\Omega^*_{\varepsilon,\delta}$ *is defined as*

$$\Omega^*_{\varepsilon,\delta} = \{x \in \Omega \quad \textit{such that} \quad \{\frac{x}{\varepsilon}\}_Y \in Y^*_\delta\},$$

where $\delta \to 0$ *with* ε. *This definition means that* $\Omega^*_{\varepsilon,\delta}$, *is a domain* ε*-periodically perforated by holes* $\varepsilon\delta B$, *see Figure3.*

Remark 3.1 *As shown in [6], it turns out that the operator* $\mathcal{T}_{\varepsilon,\delta}$ *is well-adapted for domains with small holes when dealing with functions which vanish on the boundary of* $\Omega^*_{\varepsilon,\delta}$. *It is precisely the case we treat in this work. We will deal with functions belonging in particular, to* $H^1_0(\Omega)$. *The extensions of these functions by zero to the whole of* Ω, *belong to* $H^1_0(\Omega)$. *Consequently in the sequel, we will not distinguish the elements of* $H^1_{\Omega^*_{\varepsilon,\delta}}$ *and their extensions from* $H^1_0(\Omega)$.

Proposition 3.1 *[6]*

(1) for any $v, w \in L^p(\Omega), \mathcal{T}_{\varepsilon,\delta}(v,w) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon\delta}(w)$

(2) for any $u \in L^2(\Omega)$,

$$\delta^N \int_{\Omega\times\mathbb{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)| dx\ dz \leqslant \int_\Omega |u| dx.$$

(3) for any $u \in L^2(\Omega)$,

$$||\mathcal{T}_{\varepsilon,\delta}(u)||^2_{L^2(\Omega\times R^N)} \leqslant \frac{1}{\delta^N}||u||^2_{L^2(\Omega)}.$$

(4) for any $u \in L^1(\Omega)$,

$$|\int_\Omega u\ dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u) dx\ dz| \leqslant \int_{\Lambda_\varepsilon} |u| dx.$$

(5) Let $u \in H^1(\Omega)$. *Then*

$$\mathcal{T}_{\varepsilon\delta}(\nabla_\varepsilon u) = \frac{1}{\varepsilon\delta}\nabla_\varepsilon(\mathcal{T}_{\varepsilon,\delta}(u)), \quad in\Omega \times \frac{1}{\delta}Y.$$

(6) Suppose $N \geq 3$ *and let* $\omega \subset \mathbb{R}^N$ *be open and bounded. The following estimates hald:*

$$||\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))||^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leqslant \frac{\varepsilon^2}{\delta^{N-2}}||\nabla u||^2_{L^2(\Omega)},$$

$$||(\mathcal{T}_{\varepsilon,\delta}(u - M_Y^\varepsilon(u))||^2_{L^2(\Omega;L^{2*}(\mathbb{R}^N))} \leqslant \frac{C\varepsilon^2}{\delta^{N-2}}||\nabla u||^2_{L^2(\Omega)},$$

$$||\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))||^2_{L^2(\Omega\times w)} \leqslant \frac{2C\varepsilon^2}{\delta^{N-2}}|w|^{2/N}||\nabla u||^2_{L^2(\Omega)} + 2|\omega|||u||^2_{L^2(\Omega)},$$

Definition 3.2 *A sequence $\{v_{\varepsilon,\delta}\}$ in $L^1(\Omega)$ satisfies the unfolding criterion for integrals(u.c.i) if*

$$\int_\Omega v_{\varepsilon,\delta}\ dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})dxdz \to 0,$$

for every sequence $(\varepsilon,\delta) \to (0^+,0^+)$. This property is denoted

$$\int_\Omega v_{\varepsilon,\delta}\ dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})dx\ dz.$$

Proposition 3.2 *[6] (u.c.i) if $\{v_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |u_\varepsilon| dx \to 0,$$

then it satisfies u.c.i..

Corollaire 3.1 *[6] Let $\{u_\varepsilon\}$ be bounded in $L^2(\Omega)$ and $\{v_\varepsilon\}$ be bounded in $L^p(\Omega)$ with $p > 2$. Then $\{u_\varepsilon v_\varepsilon\}$ satisfies u.c.i*

Remark 3.2 *As observed in [6], for any $\psi \in D(\Omega)$, one has*

$$||\mathcal{T}_{\varepsilon,\delta}(\psi) - \psi||_{L^\infty(\hat{\Omega}\times\frac{1}{\delta}Y)} \longrightarrow 0,$$

3.1 Time-dependent Unfolding Operator In Domains With Two Parameters

In this subsection, we extend the operator $\mathcal{T}_{\varepsilon,\delta}$, defined in the previous section to time-dependent functions by adapting what is done in [15]. We start by defining the unfolding operator for time-dependent functions in the domain $\Omega^*_{\varepsilon,\delta}\times]0,T[$, depending on ε and δ. In what follows, we have $(\varepsilon,\delta) \to (0,0)$ through any sequence and subsequence.

Definition 3.3 *Let* $p \in [1,+\infty[$ *and* $q \in L^q(0,T;L^p(\Omega))$ *The unfolding operator* $\mathcal{T}_{\varepsilon,\delta} : L^q(0,T;L^p(\Omega)) \to L^q(0,T;L^p(\Omega))$ *is definded as*

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(x,\delta z,t) & if \quad (x,z,t) \in \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y\times]0,T[\\ 0 & otherwise, \end{cases}$$

that is,

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\delta z,t) & if \quad (x,z,t) \in \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y\times]0,T[\\ 0 & otherwise, \end{cases}$$

As mentioned above, for $\delta = 1$ *we are in presence of the unfolding operator for fixed domains introduced in [4].*

Remark 3.3 *From now on, if a function does not depend on t, by* $\mathcal{T}_{\varepsilon,\delta}(\varphi)$ *we simply mean the operator introduced in Definition 3.1.*
Being defined by means of the operator $\mathcal{T}_\varepsilon$*, the unfolding operator* $\mathcal{T}_{\varepsilon,\delta}$*, inherits most of the general properties of it. In particular, the following proposition is straightforward:*

Proposition 3.3 *Let* $p \in [1,+\infty[$ *and* $q \in [1,+\infty[$.

1. $\mathcal{T}_{\varepsilon,\delta}$ *is linear and continuous from* $L^q(0,T;L^p(\Omega))$ *to* $L^q(0,T;L^p(\Omega\times\mathbb{R}^N))$.
2. $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon\delta}(v)\mathcal{T}_{\varepsilon\delta}(\omega)$ *for every* $v,w \in L^q(0,T;L^p(\Omega))$.
3. $\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi)) = \varepsilon\delta\mathcal{T}_{\varepsilon,\delta}(\nabla\varphi)$ *in* $\Omega \times \frac{1}{\delta}Y\times]0,T[$ *for all* $\varphi \in L^q(0,T;H^1_0(\Omega))$.

Theorem 3.1 *Let* $p \in [1,+\infty[$ *and* $q \in [1,+\infty[$.

(1) Let $\varphi \in L^q(0,T,L^p(\Omega))$.

$$\frac{\delta^N}{|Y|}\int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)dxdz = \int_{\hat{\Omega}_\varepsilon} \varphi(x,t)dx$$

$$= \int_\Omega \varphi(x,t)dx - \int_{\Lambda_\varepsilon} \varphi(x,t)dx$$

for a.e. $t \in]o,T[$

The continuity of the operator $\mathcal{T}_{\varepsilon,\delta}$, *from Proposition 3.3 reads as follows:*

$$||\mathcal{T}_{\varepsilon,\delta}(\varphi)||_{L^q(0,T;L^p(\Omega))} \leqslant \left(\frac{|Y|}{\delta^N}\right)^{\frac{1}{p}} ||\varphi||_{L^q(0,T;L^p(\Omega))}.$$

(2) Let $\varphi \in L^q(0,T;H^1(\Omega))$ *and* $N \geqslant 3$. *Then for a.e.* $t \in]0,T[$,

$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi))\|_{L^p(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon|Y|^{\frac{1}{p}}}{\delta^{\frac{N}{p}-1}}\|\nabla\varphi\|_{L^p(\Omega)}.$$

Proposition 3.4 *Let* $q \in [1,+\infty]$ *and* $\varphi_\varepsilon \in L^q(0,T;L^1(\Omega))$ *satisfying*

$$\int_0^T\int_{\Lambda_\varepsilon}\varphi_\varepsilon dxdt \to 0 \quad ,$$

Then

$$\int_0^T\int_\Omega \varphi_\varepsilon dxdt \cong \frac{\delta^N}{|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon)dx\ dz\ dt.$$

The proof of the following proposition is essentially the same as that of [15], Proposition 2.6].

Proposition 3.5 *Let* $p,q \in]1,+\infty]$. *Let* $\{\varphi_\varepsilon\}$ *be a sequence in* $L^q(0,T;L^p(\Omega))$ *and* $\{\psi_\varepsilon\}$ *be a sequence in* $L^{q'}(0,t;L^{p_0}(\Omega))$, *such that*

$$\|\varphi_\varepsilon\|_{L^q(0,T;L^p(\Omega))} \leq C \quad and \quad \|\psi_\varepsilon\|_{L^{q'}(0,T;L^{p_0}(\Omega))} \leq C \quad ,$$

where $\dfrac{1}{p}+\dfrac{1}{p_0} \leq 1$ *and* $\dfrac{1}{q}+\dfrac{1}{q'} = 1$. *Then,*

$$\int_0^T\int_{\widehat{\Omega}_\varepsilon}\varphi_\varepsilon\psi_\varepsilon dxdt \cong \frac{\delta^N}{|Y|}\int_0^T\int_{\Omega\times\frac{1}{\delta}Y}\mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon\psi_\varepsilon)dxdzdt \quad .$$

The next two propositions extend to time-dependent functions some properties given in [6], Theorem 2.11].

Proposition 3.6 *Let* $u \in L^q(0,T;H^1(\Omega))$. *For* $q \in [1,+\infty[$, *one has the estimates*

$$\|\mathcal{T}_{\varepsilon,\delta}(u-\mathcal{M}^\varepsilon_Y(u))\|_{L^q(0,T;L^p(\Omega;L^{p*}(\mathbb{R}^N))} \leq \frac{C\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}}\|\nabla u\|_{L^q(0,T;L^p(\Omega))} \quad ,$$

and for ω *an open and bounded subset of* $\mathbb{R}^N$,

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|_{L^q(0,T;L^p(\Omega\times\omega))}$$

$$\leq \frac{2C\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}}\|\nabla u\|_{L^q(0,T;L^p(\Omega))} + 2|\omega||Y|^{\frac{1-p}{p}}\|u\|_{L^q(0,T;L^p(\Omega))}$$

where C is the Sobolev-Poincar'e-Wirtinger constant for $H^1(Y)$.

Theorem 3.2 *Let* $p \in [1,+\infty[$, $q \in [1,+\infty]$, $N \geq 3$, $\{w_{\varepsilon,\delta}\}$ *be a sequence in* $L^q(0,T;H^1(\Omega))$ *which is uniformly bounded with respect to* ε *and* δ *as* $(\varepsilon,\delta) \longrightarrow (0,0)$. *Then up to a subsequence, there exists* W *in* $L^q(0,T;^P(\Omega;L^{p^*}(\mathbb{R}^N)))$ *with* $\nabla_z W$ *in* $L^q(0,T;^P(\Omega;L^p(\Omega\times\mathbb{R}^N)))$ *such that:*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - \mathcal{M}^{\varepsilon}_{Y}(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \rightharpoonup W \text{ weakly in } L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N))), \text{ and}$$

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}))1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \text{ weakly in } L^q(0,T;L^p(\Omega\times\mathbb{R}^N)) \quad .$$

Furthermore, if :

$$k^* = \limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon} < +\infty \ ,$$

then one can choose the subsequence above and some $U \in L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N)))$ *with :*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \text{ weakly in } L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N))) \ .$$

CHAPTER

4

HOMOGENIZATION OF A CLASS OF PARABOLIC PROBLEMS

In this chapter, we use the adapted unfolding method presented in Section 2 to study the asymptotic behavior of a class of parbolic problems in perforated domains. To introduce the coefficient matrix, we define, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$ with $\mathbf{0} < \boldsymbol{\alpha} < \boldsymbol{\beta}$, the set $\boldsymbol{M}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mathcal{O}})$ of the $\boldsymbol{n} \times \boldsymbol{n}$ matrix- valued functions in $\boldsymbol{L}^{\infty}(\boldsymbol{\mathcal{O}})$ such that:

$$\boldsymbol{A}(\boldsymbol{x})\boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq \boldsymbol{\alpha}|\boldsymbol{\lambda}|^{\mathbf{2}} \quad , \quad |\boldsymbol{A}(\boldsymbol{x})\boldsymbol{\lambda}| \leq \boldsymbol{\beta}|\boldsymbol{\lambda}|$$

for any $\boldsymbol{\lambda} \in \mathbb{R}^{\boldsymbol{N}}$ and a.e. on $\boldsymbol{\mathcal{O}}$.

For any $\boldsymbol{\varepsilon}$, we suppose that:

$$\begin{cases} \boldsymbol{A}^{\varepsilon} \in \boldsymbol{M}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Omega}) \quad , \\ \boldsymbol{A}^{\varepsilon} \quad \textit{\textbf{symmetric.}} \end{cases} \tag{4.1}$$

Consider the following parabolic problem with homogeneous Dirichlet-Neumann boundary :

$$\begin{cases} u_\varepsilon' - div(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & in\ \Omega_\varepsilon^* \times (0,T), \\ u_\varepsilon = 0 & on\ \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0 & on\ \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u_\varepsilon^0 & in\ \Omega_\varepsilon^* \end{cases} \tag{4.2}$$

where n_ε is the outward unit normal vector fietd defined on ∂S_ε.

We suppose that :

$$\begin{cases} u_\varepsilon^0 \in V^\varepsilon, \\ f_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon^*)). \end{cases} \tag{4.3}$$

Set

$$\mathcal{W}_\varepsilon = \left\{ v_\varepsilon | v_\varepsilon \in L^2(0,T;V^\varepsilon) \quad , v_\varepsilon' \in L^2((0,T;L^2(\Omega_\varepsilon^*)) \right\} \tag{4.4}$$

with the norm defined by

$$\|v_\varepsilon\|_{\mathcal{W}_\varepsilon} = \|v_\varepsilon\|_{L^2(0,T;V^\varepsilon)} + \|v_\varepsilon'\|_{L^2(0,T;L^2(\Omega_\varepsilon^*))} \quad .$$

4.1 Homogenization of parabolic problem in the perforated domain Ω_ε

4.1.1 variational formulation

We multiply (4.2) by a test function v, where $v \in V^\varepsilon$ we get:

$$\int_{\Omega_\varepsilon^*} u_\varepsilon' v\ dx \ - \int_{\Omega_\varepsilon^*} (div A^\varepsilon \nabla u_\varepsilon) v\ dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v\ dx \tag{4.5}$$

we integrate on Ω_ε^* and by Green's formula, we get :

$$-\int_{\Omega_\varepsilon^*} (div A^\varepsilon \nabla u_\varepsilon) v\ dx = \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v\ dx - \int_{\partial S_\varepsilon \cap \partial\Omega} A^\varepsilon \nabla u_\varepsilon v n_\varepsilon\ d\Gamma \tag{4.6}$$

this implay that

$$\begin{aligned} -\int_{\Omega_\varepsilon^*}(divA^\varepsilon\nabla u_\varepsilon)v\ dx = \int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v\ dx - \int_{\partial S_\varepsilon} A^\varepsilon\nabla u_\varepsilon v n_\varepsilon\ d\Gamma \\ + \int_{\partial\Omega} A^\varepsilon\nabla u_\varepsilon v n_\varepsilon\ d\Gamma \end{aligned} \tag{4.7}$$

Then

$$-\int_{\Omega_\varepsilon^*} div(A^\varepsilon\nabla u_\varepsilon)v\ dx \ = \int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v\ dx \tag{4.8}$$

we replace (4.7) in (4.5), we get:

$$\int_{\Omega_\varepsilon^*} u_\varepsilon^{'} v\ dx \ - \int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v\ dx \ = \int_{\Omega_\varepsilon^*} f_\varepsilon v\ dx \tag{4.9}$$

Now, the variational formulation of problem (4.2) is to find $u_\varepsilon \in \mathcal{W}_\varepsilon$ such that

$$\begin{cases} \langle u_\varepsilon^{'}, v\rangle_{(V^\varepsilon)',V^\varepsilon} \ + \int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v\ dx \ = \int_{\Omega_\varepsilon^*} f_\varepsilon v\ dx \\ \quad in\ \mathcal{D}^{'}(0,T)\ for\ all\ v \in V^\varepsilon\ , \\ \quad u_\varepsilon(0,x) = u_\varepsilon^0\ in\ \Omega_\varepsilon^*\ . \end{cases} \tag{4.10}$$

For every fixed ε, classical results provide that problem (4.9) has a unique solution u_ε such that

$$u_\varepsilon \in \mathcal{C}^0\left([0,T];V^\varepsilon\right) \cap \mathcal{C}^1\left([0,T];L^2(\Omega_\varepsilon^*)\right) \quad .$$

In order to study the homogenization of problem (4.2), we suppose that there exists a matrix $A = (a_{ij})_{1\leq i,j\leq n}$ such that

$$\mathcal{T}_\varepsilon^*(A^\varepsilon) \longrightarrow A\ strongly\ in\ (L^1(\Omega\times Y^*))^{n\times n} \quad , \tag{4.11}$$

which implies $A \in \mathcal{M}(\alpha,\beta,\Omega\times Y^*)$ (see also [4] and [7]). Concerning the initial data, we assume that

$$\begin{cases} \quad \|u_\varepsilon^0\|_{V^\varepsilon} \leq\ C, \\ \tilde{u}_\varepsilon^0 \rightharpoonup \theta u^0\ weakly\ in\ L^2(\Omega) \\ \tilde{u}_\varepsilon^1 \rightharpoonup \theta u^1\ weakly\ in\ L^2(\Omega) \\ \tilde{f}_\varepsilon \rightharpoonup \theta f\ weakly\ in\ L^2(0,T;L^2(\Omega))\ , \end{cases} \tag{4.12}$$

where C is a constant independent of ε. under these assumptions, classical results show that problem (4.9) has a unique solution u_ε with the following uniform estimate:

$$\|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega^*_\varepsilon))} + \|u'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega^*_\varepsilon))} \leq C \tag{4.13}$$

where the constant C does not depend on ε.

Now we state the main theorem of this section.

Theorem 4.1 *Let A^ε satisfy (4.1) and (4.10). suppose that u_ε is the solution of problem (4.2) with 4.3 and 4.11. Then there exist $u \in L^\infty(0,T;H^1_0(\Omega))$ with $u' \in L^\infty(0,T;L^2(\Omega))$ and $\hat{u} \in L^\infty(0,T;L^2(\Omega, H^1_{per}(Y^*)))$ with $\mathcal{M}_{Y^*}(\hat{u}) = 0$, such that*

$$\begin{cases} \mathcal{T}^*_\varepsilon(u_\varepsilon) \rightharpoonup u \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega,H^1(Y^*))) , \\ \mathcal{T}^*_\varepsilon(u'_\varepsilon) \rightharpoonup u' \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega \times (Y^*)) , \\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \hat{u} \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega \times (Y^*)) , \\ \mathcal{T}^*_\varepsilon(u_\varepsilon) \longrightarrow u \text{ strongly in } L^q(0,T;L^2(\Omega,H^1(Y^*))) , \\ \|u_\varepsilon - u\|_{L^q(0,T;L^2(\Omega^*\varepsilon))} \longrightarrow 0 , \end{cases} \tag{4.14}$$

where q is any number in $(1,+\infty)$. the pair $(u,\hat{u})$ with $\mathcal{M}_{Y^*}(\hat{u}) = 0$ is the unique solution of the following problem :

$$\begin{cases} \theta \int_0^T \int_\Omega u\Psi\varphi' \, dxdt + \dfrac{1}{|Y|}\int_0^T \int_{\Omega\times Y^*} A(\nabla u \nabla_y \hat{u})(\nabla\Psi\nabla_y\Phi)\varphi \, dx \, dy \, dt \\ \qquad = \theta \int_0^T \int_\Omega f\Psi\varphi \, dx \, dt \\ \text{for any } \Psi \in H^1_0(\Omega), \ \Phi \in L^2(\Omega; H^1_{per}(Y^*)) \text{ and } \varphi \in \mathcal{D}(0,T) , \\ \qquad u = 0 \text{ on } \Omega \times (0,T) \\ \qquad u(x,0) = u^0 \ \text{ in } \Omega \end{cases} \tag{4.15}$$

We also have

$$\hat{u} = \Sigma_{j=1}^n \frac{\partial u}{\partial x_j} \mathcal{X}_j \tag{4.16}$$

with $\mathcal{X}_j \in L^\infty(\Omega; H^1_{per}(Y^*))$ $(j = 1, \dots, n)$ *being the solution of the cell problem*

$$\begin{cases} -div_y(A\nabla_y(\mathcal{X}_j + y_j)) = 0 \ in \ Y^* \ , \\ A\nabla_y(\mathcal{X}_j + y_j) \cdot n_1 = 0 \ on \ \partial S \ , \\ \mathcal{M}_{Y^*}(\mathcal{X}_j)(x, .) = 0, \ \mathcal{X}_j(x, .)Y - periodic. \end{cases} \tag{4.17}$$

Remark 4.1 *:Observe that the homogenized matrix* A^0 *here, as defined in [4], differs from the classical one (see for instance [9]) by a factor of* θ^{-1}*. We also notice that here* A^0 *depends up on* x *while the one in [9] does not.*

proof of theorem 4.1: In view of (4.12) and theorem 2.1, we get that there exist $u \in L^\infty(0, T; L^2(\Omega))$ with $u' \in L^\infty(0, T; L^2(\Omega))$ and $\hat{u} \in L^\infty(0, T; L^2(\Omega, H^1_{per}(Y^*)))$ with $\mathcal{M}_{Y^*} = 0$,such that, up to a subsequence (still denoted by ε), (4.13) holds. From proposition 2.5, we further get that

$$\tilde{u}_\varepsilon \rightharpoonup \theta\mathcal{M}_{Y^*}(u) \ weakly^* \ in \ L^\infty(0, T; L^2(\Omega))$$

$$A^\varepsilon\widetilde{\nabla u_\varepsilon} \rightharpoonup \theta\mathcal{M}_{Y^*}[A(\nabla u + \nabla_y\hat{u})] \ weakly^* \ in \ L^\infty(0, T; L^2(\Omega))) \tag{4.18}$$

Since u is independent of y, then convergence holds true for the above subsequence.
Let $\Psi, \phi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{per}(Y^*)$. Set

$$v_\varepsilon(x) = \Psi(x) + \varepsilon\phi(x)\psi^\varepsilon(x) \qquad with \ \psi^\varepsilon(x) = \psi(\frac{x}{\varepsilon}) \tag{4.19}$$

Then

$$\begin{aligned} \nabla_x v_\varepsilon &= \nabla_x \left(\Psi(x) + \varepsilon\phi(x)\psi^\varepsilon(x)\right) \\ &= \nabla_x\Psi(x) + \varepsilon\nabla_x \left[\phi(x)\psi(\frac{x}{\varepsilon})\right] \\ &= \nabla_x\Psi(x) + \varepsilon\nabla_x\phi(x)\psi(\frac{x}{\varepsilon}) + \varepsilon\frac{1}{\varepsilon}\phi(x)\nabla_y\psi(x_\varepsilon) \\ &= \nabla_x\Psi(x) + \varepsilon\psi^\varepsilon\nabla_x\phi + \phi(\nabla_y\psi(\frac{\cdot}{\varepsilon})). \end{aligned} \tag{4.20}$$

By Proposition, we have

$$\begin{cases} \mathcal{T}^*_\varepsilon(v_\varepsilon) \longrightarrow \Psi \textit{ strongly in } L^2(\Omega \times Y^*) \\ \mathcal{T}^*_\varepsilon(\phi\psi^\varepsilon) \longrightarrow \Phi \textit{ strongly in } L^2(\Omega \times Y^*) \textit{ with } \Phi = \phi(x)\psi(y), \\ \mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \longrightarrow \nabla\Psi + \nabla_y\Phi \textit{ strongly in } L^2(\Omega \times Y^*) \end{cases} \tag{4.21}$$

Let $\varphi \in \mathcal{D}(0,T)$. From (4.13) and (4.19) and proposition 2.6, we deduce

$$\int_0^T \int_{\Omega^*_\varepsilon} u^{'}_\varepsilon v_\varepsilon \varphi \; dx \; dt \longrightarrow \theta \int_0^T \int_\Omega u\Psi\varphi^{'} \; dxdt. \tag{4.22}$$

$$\int_0^T \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi \; dxdt \longrightarrow \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla\Psi\nabla_y\Phi)\varphi \; dx \; dy \; dt. \tag{4.23}$$

By virtue of 4.11, we have

$$\int_0^T \int_{\Omega^*_\varepsilon} f_\varepsilon v_\varepsilon \varphi \; dxdt \longrightarrow \theta \int_0^T \int_\Omega f\Psi\varphi \; dxdt. \tag{4.24}$$

Now, choosing $v_\varepsilon\varphi$ as test function in the variational formulation (4.9), we get

$$\int_0^T \int_{\Omega^*_\varepsilon} u^{'}_\varepsilon v_\varepsilon \varphi \; dx \; dt - \int_0^T \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx \; dt = \int_0^T \int_{\Omega^*_\varepsilon} f_\varepsilon v_\varepsilon \varphi dx \; dt$$

Now, we pass at the limit in this expression term by term, we start by the first term

$$\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_{\Omega^*_\varepsilon} u^{'}_\varepsilon v_\varepsilon \varphi \; dxdt = \lim_{\varepsilon \longrightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi^{'}) dxdydt$$

By theorem 4.1, (4.13) proposition 2.6, (4.22) :

$$\lim_{\varepsilon \longrightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi^{'}) dxdydt = \lim_{\varepsilon \longrightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} u\Psi\varphi^{'} \; dxdydt$$

By Fubini's theorem:

$$\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_{\Omega^*_\varepsilon} u^{'}_\varepsilon v_\varepsilon \varphi \; dxdt = \lim_{\varepsilon \longrightarrow 0} \frac{1}{|Y|} \int_0^T \int_\Omega u\Psi\varphi^{'} \; dxdt \int_{Y^*} \; dy$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon^{'} v_\varepsilon\varphi\; dxdt = \frac{-|Y^*|}{|Y|}\int_0^T\int_\Omega u\Psi\varphi^{'}\; dxdt$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon^{'} v_\varepsilon\varphi\; dxdt = -\theta\int_0^T\int_\Omega u\Psi\varphi^{'}\; dxdt$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon^{'} v_\varepsilon\varphi\; dxdt \longrightarrow \theta\int_0^T\int_\Omega u\Psi\varphi^{'}\; dx\; dt \tag{4.25}$$

For the second term we use here the operator $\mathcal{T}_\varepsilon$ we get

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi\; dx\; dt = \lim_{\varepsilon\longrightarrow 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)\; dx\; dt$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi\; dx\; dt = \lim_{\varepsilon\longrightarrow 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\; dx\; dy\; dt$$

Now the limit of the second term is

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi\; dx\; dt = \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\; dx\; dy\; dt \tag{4.26}$$

Finally for the last term we use also the operator $\mathcal{T}_\varepsilon$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\; dx\; dt\; = \lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon(f_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)\; dx\; dy\; dt$$

we have:

$\mathcal{T}_\varepsilon(v_\varepsilon)\longrightarrow\Psi$; $\mathcal{T}_\varepsilon(\varphi)\longrightarrow\varphi$; $\mathcal{T}_\varepsilon(f_\varepsilon)\longrightarrow f$ this implies that

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\; dx\; dt\; = \lim_{\varepsilon\longrightarrow 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} f\Psi\varphi\; dx\; dy\; dt$$

applying Fubbinni's Theorem, we obtain:

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\; dx\; dt\; = \frac{1}{|Y|}\int_T^0 f\Psi\varphi\; dx\; dt\int_{Y^*}\; dy$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\; dx\; dt\; = \frac{|Y^*|}{|Y|}\int_T^0\int_\Omega f\Psi\varphi\; dx\; dt$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\; dx\; dt\; = \theta\int_0^T\int_\Omega f\Psi\varphi\; dx\; dt$$

$$\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx \, dt \longrightarrow \theta \int_0^T \int_\Omega f \Psi \varphi \, dx \, dt \tag{4.27}$$

Choosing $v_\varepsilon \varphi$ as test function in the variational formulation (4.10), passing to the limit and making use of (4.22)-(4.24), we obtain:

$$\theta \int_0^T \int_\Omega u \Psi \varphi' \, dx \, dt + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt$$

$$= \theta \int_0^T \int_\Omega f \Psi \varphi \, dx \, dt$$

This gives the equation in (4.15), due to the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ and the density of $\mathcal{D}(\Omega) \otimes H_{per}^1(Y^*)$ in $L^2(\Omega, H_{per}^1(Y^*))$.
Setting $\Psi = 0$ in (4.15), we get

$$div_y A(\nabla u + \nabla_y \hat{u}) = 0$$

Since u is independent of y and $\mathcal{M}_{Y^*} = 0$, we obtain (4.16). Then by standard computation, convergence (4.1.1). Moreover, we have the following identity:

$$\int_{Y^*} A(\nabla u + \nabla_y \hat{u}) \nabla \Psi \, dy = |Y^*| A^0 \nabla u \nabla \Psi, \tag{4.28}$$

Substituting (4.16) and (4.28) into (4.15), we get that

$$u' - div(A^0 \nabla u) = f \qquad in \ \Omega \times (0, T)$$

which gives the equation in (4.18) and $u' \in L^2(0, T; H^{-1}(\Omega))$. Hence from classical results, we have:

$$u \in \mathcal{C}^0([0, T]; L^2(\Omega)) \ and \ u' \in \mathcal{C}^0([0, T]; H^{-1}(\Omega))$$

Now, in order to check the initial condition, let v_ε be given by (4.19) and $\varphi \in C^\infty([0, T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Choosing $v_\varepsilon \varphi$ as test function in the variational formulation (4.10) and integrating by parts, we have:

$$- \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi \, dx \, dt + \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx \, dt$$

$$= \int_0^T \langle u_\varepsilon', v_\varepsilon\rangle_{(V^\varepsilon)', V^\varepsilon} \varphi \, dt \tag{4.29}$$

$$= \int_0^T \int_{\Omega_\varepsilon^*} u' v_\varepsilon \varphi \, dx \, dt$$

$$= \int_{\Omega_\varepsilon^*} (u_\varepsilon \varphi)|_0^T v_\varepsilon \, dx - \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi' \, dx \, dt$$

$$= \int_{\Omega_\varepsilon^*} v_\varepsilon \varphi(T) u_\varepsilon(T) \, dx - \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon(0) \varphi(0) \, dx - \int_0^T \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon \varphi' \, dx \, dt$$

$$= - \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon(0) \varphi(0) \, dx - \int_0^T \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon \varphi' \, dx \, dt$$

$$= - \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon(0) - \int_0^T \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon \varphi' \, dx \, dt$$

Passing to the limit, we have:

$$= -\theta \int_\Omega u^0 \Psi \, dx - \theta \int_0^T \int_\Omega \Psi u \varphi' \, dx \, dt, \tag{4.30}$$

also we integrating by parts the term $\int_0^T \int_\Omega \Psi u \varphi' \, dx \, dt$, we get

$$-\theta \int_0^T \int_\Omega \Psi u \varphi' \, dx \, dt = -\theta \int_\Omega u\varphi|_0^T \Psi \, dx + \theta \int_0^T \int_\Omega \varphi u' \, dx \, dt$$

$$-\theta \int_0^T \int_\Omega \Psi u \varphi' \, dx \, dt = \int_\Omega u(T)\varphi(T)\Psi \, dx - \int_\Omega u(0)\varphi(0)\Psi \, dx - \int_0^T \int_\Omega \varphi u' \Psi \, dx \, dt$$

$$- \theta \int_0^T \int_\Omega \Psi u \varphi' \, dx \, dt = -\theta \int_\Omega u(0,x)\Psi \, dx - \int_0^T \int_\Omega \varphi u' \Psi \, dx \, dt \tag{4.31}$$

By (4.30) and some calculus, we obtain:

$$- \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon(0) - \int_0^T \int_{\Omega_\varepsilon^*} v_\varepsilon u_\varepsilon \varphi' \, dx \, dt = -\theta \int_\Omega u^0 \Psi \, dx - \theta \int_\Omega u(0,x)\Psi \, dx - \int_0^T \int_\Omega \varphi u' \Psi \, dx \, dt$$

Passing to the limit, we get:

$$-\frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi)\varphi \, dx \, dy \, dt + \theta \int_0^T \int_\Omega f \Psi \varphi \, dx \, dt$$

$$= -\theta \int_\Omega u^0 \Psi \, dx - \theta \int_0^T \int_\Omega \Psi u \varphi' \, dx \, dt \tag{4.32}$$

$$= -\theta \int_\Omega u^0 \Psi \, dx - \theta \int_\Omega u(0,x)\Psi \, dx - \int_0^T \int_\Omega \varphi u' \Psi \, dx \, dt$$

Combining this with (4.15), we have:

$$u(0,x) = u^0$$

4.2 Homogenization of parabolic problem in the perforated domain $\Omega_{\varepsilon,\delta}$

We want to study now the asumptotic behavior of our problem as $\varepsilon \longrightarrow 0$.

$$\begin{cases} u'_{\varepsilon,\delta}(x,t) - div(A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t) = f_{\varepsilon,\delta}(x,t) & in\ \Omega^*_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,t) = 0 & on\ \partial\Omega^*_{\varepsilon,\delta}\times]0,T[, \\ A^\varepsilon \nabla u_{\varepsilon,\delta}\cdot n_{\varepsilon,\delta} = 0 & on\ \partial S_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta} & in\ \Omega^*_{\varepsilon,\delta} \end{cases} \tag{4.33}$$

We suppose that the data satisfy the following assumptions:

$$\begin{cases} A^\varepsilon \in M(\alpha,\beta,\Omega) \quad , A^\varepsilon symmetric. \\ f_{\varepsilon,\delta} \in L^2(0,T;L^2(\Omega)) \\ u^0_{\varepsilon,\delta} \in L^2(\Omega) \end{cases} \tag{4.34}$$

Moreover, we assume that :

$$\begin{cases} u^0_{\varepsilon,\delta} \rightharpoonup u^0 \ weakly\ in\ L^2(\Omega), \\ f_{\varepsilon,\delta} \rightharpoonup f \ weakly\ in\ L^2(0,T;L^2(\Omega)), \end{cases} \tag{4.35}$$

Set

$$W_{\varepsilon,\delta} = \{v_{\varepsilon,\delta} \in L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta})); v'_{\varepsilon,\delta} \in L^2(0,T;H^{-1}(\Omega^*_{\varepsilon,\delta}))\}$$

equipped with the norm:

$$\|v_{\varepsilon,\delta}\|_{W_{\varepsilon,\delta}} = \|v_{\varepsilon,\delta}\|_{L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta}))} + \|v'_{\varepsilon,\delta}\|_{L^2(0,T;H^{-1}(\Omega^*_{\varepsilon,\delta}))} \quad .$$

4.2.1 Statement of the main homogenization result

In this section, we suppose that $N \geq 3$ and that ε and $\delta = \delta(\varepsilon)$ holds, that is, there exists the following limit and is finite:

$$k^* \doteq \lim_{\varepsilon \longrightarrow 0} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty \quad .$$

We also denote by $M(\alpha, \beta, \Omega)$ the set of $N \times N$ matrices $A = (a_{ij})_{1 \leq i,j \leq N}$ in $(L^\infty(\Omega))^{N\times N}$ such that

(i) $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$,

(ii) $|A(x)\lambda| \leq \beta|\lambda|$,

for any $\lambda \in \mathbb{R}^N$ and almost everywhere on Ω, where $\alpha, \beta \in \mathbb{R}^N$ such that $0 < \alpha < \beta$.

4.2.2 Variational formulation

We multiply 4.33 by a test function v, where $v \in V^\varepsilon$,and we integrate on the domain $\Omega_{\varepsilon,\delta}$, we get

$$\int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta} v \; dx \; - \int_{\Omega^*_{\varepsilon,\delta}} (div A^\varepsilon \nabla u_{\varepsilon,\delta}) v \; dx = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v \; dx, \tag{4.36}$$

and by Green's formula, we get:

$$\begin{aligned} - \int_{\Omega^*_{\varepsilon,\delta}} (div A^\varepsilon \nabla u_{\varepsilon,\delta}) v \; dx &= \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v \; dx - \int_{\partial S_{\varepsilon,\delta} \cap \partial\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} v n_{\varepsilon,\delta} \; d\Gamma \\ - \int_{\Omega^*_{\varepsilon,\delta}} (div A^\varepsilon \nabla u_{\varepsilon,\delta}) v \; dx &= \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v \; dx - \int_{\partial S_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} v n_{\varepsilon,\delta} \; d\Gamma \\ &\quad + \int_{\partial\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} v n_{\varepsilon,\delta} \; d\Gamma \end{aligned} \tag{4.37}$$

Then

$$- \int_{\Omega^*_{\varepsilon,\delta}} div(A^\varepsilon \nabla u_{\varepsilon,\delta}) v \; dx \; = \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v \; dx \tag{4.38}$$

we replace (4.38) in (4.36), we get:

$$\int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta} v \; dx \; - \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v \; dx \; = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v \; dx \tag{4.39}$$

The variational formulation of problem (4.33) is to find $u_{\varepsilon,\delta} \in W_{\varepsilon,\delta}$ such that, for all $v \in H_0^1(\Omega^*_{\varepsilon,\delta})$

$$\begin{cases} \langle u'_{\varepsilon,\delta}, v\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))', H_0^1(\Omega^*_{\varepsilon,\delta})} + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v \, dx = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v \, dx \\ \qquad in\ \mathcal{D}'(0,T)\ for\ all\ v \in V^\varepsilon, \\ \qquad u_{\varepsilon,\delta}(0,x) = u^0_{\varepsilon,\delta}\ in\ \Omega^*_{\varepsilon,\delta}. \end{cases} \tag{4.40}$$

For this problem, classical results [8, 19] provide for every fixed ε and δ the existance and uniqueness of a solution of problem (4.40), such that:

$$u_{\varepsilon,\delta} \in L^2(0,T; H_0^1(\Omega^*_{\varepsilon,\delta})) \cap \mathcal{C}^0([0,T]; L^2(\Omega^*_{\varepsilon,\delta}))$$

and satisfies the estimate

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;H_0^1(\Omega^*_{\varepsilon,\delta}))} + \|u'_{\varepsilon,\delta}\|_{(0,T,L^2(\Omega^*_{\varepsilon,\delta}))} \le C \tag{4.41}$$

Where C is independent of ε and δ.

Remark 4.2 *In the following, we identify functions in $H_0^1(\Omega^*_{\varepsilon,\delta})$ with their zero extension to $H_0^1(\Omega)$ so that we can write (4.41) as*

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u'_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega))} \le C \tag{4.42}$$

where C is independent of ε and δ.

We adapt here for the evolution problem some arguments introduced in [6]. Let us introduce the functional space

$$K_B = \{\Phi \in L^2(0,T; L^{2^*}(\mathbb{R}^N)) : \nabla\Phi \in L^2(0,T; L^2(\mathbb{R}^N)),\ \Phi\ is\ constant\ on\ B\} \tag{4.43}$$

We also need the following lemmas from [6] in order to pass to the limit in equation (4.40)

Lemma 4.1 *([6]). Let $N \ge 3$. Then, for every $\delta_0 > 0$, the set*

$$\cup_{0<\delta<\delta_0}\{\phi \in H^1_{per}(Y) : \phi = 0\ on\ \partial B\},$$

is dense in $H^1_{per}(Y)$.

Lemma 4.2 *:([6]). Let* $v \in \mathcal{D}(\mathbb{R}^N) \cap K_B$ *(i.e.,* $v = v(B)$ ***is constant on*** B*) and set*

$$w_{\varepsilon,\delta}(x) = v(B) - v\left(\frac{1}{\delta}\{\frac{x}{\varepsilon}\}_Y\right) \ \ \textit{for } x \in \ \mathbb{R}^N.$$

Then

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \textit{ weakly in } H^1(\Omega) \tag{4.44}$$

Remark 4.3 *(1) From the definition of* $w_{\varepsilon,\delta}$ *above, one has*

$$\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})(x,z) = v(B) - v(z) \qquad \textit{in } \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y,$$

and consequently (see [6]),

$$\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = \frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})) = -\frac{1}{\varepsilon\delta}\nabla_z v \qquad \textit{in } \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y \tag{4.45}$$

(2) Let $\{w_{\varepsilon,\delta}\}$ *be a sequence satisfying 4.44. We have,*

$$\mathcal{T}_\varepsilon(w_{\varepsilon,\delta}) \rightharpoonup v(B) \textit{ strongly in } L^2(\Omega \times Y) \tag{4.46}$$

Indeed, it was shown in [6] that $\{w_{\varepsilon,\delta}\}$ *is bounded in* $H^1(\Omega)$ *so that together with 4.44 and Rellich compactness theorem, one has* $w_{\varepsilon,\delta} \rightharpoonup v(B)$ ***strongly in*** $L^2(\Omega)$*; that is,*

$$\|w_{\varepsilon,\delta} - v(B)\|_{L^2(\Omega)} \longrightarrow 0.$$

(see [6]) This, together with Proposition 4.46(2) gives 4.46.

We state now a homogenization theorem for system 4.33:

Theorem 4.2 *Under assumptions (4.34) and (4.35), suppose that as* $\varepsilon \longrightarrow 0$*, there is a matrix field* A *such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,y) \longrightarrow A(x,y) \textit{ a.e. in } \Omega \times Y, \tag{4.47}$$

and as both $\varepsilon, \delta \longrightarrow 0$*, there exists* a *matrix field* A^0 *such that*

$$\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(x,z) \longrightarrow A^0(x,z) \text{ a.e. in } \Omega \times (\mathbb{R}^N \backslash B).$$

Let $u_{\varepsilon,\delta}$ *be the solution of* (4.40)*. Then there exists* u *in* $L^\infty(0,T;H_0^1(\Omega))$ *and* $\hat{u}$ *in* $L^\infty(0,T;L^2(\Omega;H^1_{per}Y)))$*, such that*

$$\begin{cases} u_{\varepsilon,\delta} \rightharpoonup u \text{ weakly}^* \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ \mathcal{T}_\varepsilon(u_{\varepsilon,\delta}) \rightharpoonup u \text{ weakly}^* \text{ in } \ L^\infty(0,T;L^2(\Omega;H^1Y))), \\ \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y \hat{u} \text{weakly}^* \text{ in } \ L^\infty(0,T;L^2(\Omega \times Y)). \end{cases} \tag{4.48}$$

Moreover, there exists $U \in L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N)))$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \text{ weakly in } L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N))) \tag{4.49}$$

with U *vanishing on* $\Omega \times B \times]0,T[$ *and* $U - k^* u \in L^2(0,T;L^2(\Omega;K_B))$ *(*K_B *being defined by 4.60).*

The couple $(u,\hat{u})$ *satisfies the limit equation*

$$\int_Y A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla_y \phi(y)\ dy = 0, \tag{4.50}$$

for a.e. $x \in \Omega$*, a.e.* $t \in]0,T[$ *and for* $\phi \in H^1_{per}(Y)$*. While the function* U *obeys*

$$\int_{\mathbb{R}^N \backslash B} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\ dz = 0, \tag{4.51}$$

for a.e. $x \in \Omega$*, a.e.* $t \in]0,T[$ *and for all* $v \in K_B$*, with* $v_B = 0$*. The ordered triplet* $(u,\hat{u},U)$ *satisfies the limit equation*

$$\begin{aligned} \langle u'(.,t),\psi\rangle_{(H_0^1(\Omega))',H_0^1(\Omega)} - k^* \int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t) v_B \psi(x)\ dx\ d\sigma_z \\ + \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\ dx\ dy \\ = \int_\Omega f(x,t)\psi(x)\ dx, \quad \text{for a.e. } t \in]0,T[\text{ and for all } \psi \in H_0^1(\Omega), \end{aligned} \tag{4.52}$$

$$u(x,0)=u^0 \qquad in\ \Omega.$$

where v_B is the inward normal to ∂B and $d\sigma_z$ its surface measure.

In what follows, we will use the notation $m_Y(.)$ for the average over Y defined as

$$m_Y(v)=\frac{1}{|Y|}\int_Y v(y)\ dy, \qquad \forall v\in L^1(Y)$$

The result below describes now the homogenized problem in the variable (x, t) in $\Omega\times]0,T[$. To this aim, let us consider the correctors $\hat{\mathcal{X}}_j, j=1,\cdots,N$ solutions of the cell problem; they are the same for domains without holes (see [2, 8]).

$$\hat{\mathcal{X}}_j\in L^\infty(\Omega;H^1_{per}(Y)),$$

$$\int_Y A\nabla(\hat{\mathcal{X}}_j-y_j)\nabla\varphi\ dy=0\ a.e.\ x\in\Omega,\forall\varphi\in H^1_{per}(Y) \tag{4.53}$$

$$m_Y(\hat{\mathcal{X}}_j)=0,$$

where A is given by (4.47).

We consider also the cell problem corresponding to the holes B defining the corrector θ for small holes, introduced in [6]

$$\theta\in L^\infty(\Omega;K_B), \qquad \theta(x,B)\equiv 1,$$

$$\int_{\mathbb{R}^N\setminus B} A^0(x,z)\nabla_z\theta(x,z)\nabla_z\Psi(z)\ dz=0 \tag{4.54}$$

$$a.e.\ for\ x\in\ \Omega,\ \forall\Psi\in K_B\ with\ \Psi(B)=0.$$

4.3 Proof of main results

Let us now present the proofs of the homogenization results stated in the previous section. We adapt here some ideas in [6, 15].

4.3.1 Proof of Theorem 4.2

We prove the results in several steps

Step 1. The existence of $u \in L^\infty(0,T;H_0^1(\Omega))$ such that up to subsequences, convergences (4.48) hold, follows from estimate (4.41) while the existence of $\hat{u}$ in $L^\infty(0,T;L^2(\Omega;H^1_{per}Y)))$ and such that convergence (4.48) hold, (see also Remark 4.3).

On the other hand, from (4.42) there exists a function
$W \in L^2(0,T;L^2(\Omega;L^{2*}(\mathbb{R}^N)))$ with $\nabla_z W \in L^2(0,T;L^2(\Omega\times\mathbb{R}^N))$ such that (up to a subsequence)

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})-\mathcal{M}^\varepsilon_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}) \rightharpoonup W \textit{ weakly in } L^2(0,T;L^2(\Omega;L^{2*}(\mathbb{R}^N))) \quad (4.55)$$

Moreover, there exists U such that (up to a subsequence) (4.49) holds.

Step 2. Let us check the properties of the function U. From (4.48) we have by compactness,

$$u_{\varepsilon,\delta} \rightharpoonup u \quad \textit{strongly in } L^2(0,T;L^2(\Omega)), \quad (4.56)$$

so that

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}^\varepsilon_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \rightharpoonup k^*u \quad \textit{strongly in } L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N))). \quad (4.57)$$

Thus, from (4.49), (4.55) and (4.57) we conclude that

$$U = W + k^*u \quad \textit{and} \quad \nabla_z U = \nabla_z W.$$

Moreover, we have

$$\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) = \frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad w- \ L^2(0,T;L^2(\Omega\times\mathbb{R}^N)) \quad (4.58)$$

Also, from Definition 3.3,

$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \quad \textit{in } \Omega\times B\times]0,T[,$$

and thus from (4.49), (4.57),

$$U = u = 0 \quad in\ \Omega \times B \times]0, T[. \tag{4.59}$$

This means that

$$W = U - k^* u \ \in L^2(0, T; L^2(\Omega; K_B)).$$

Step 3. Let us prove the first limit equation. Let$\psi \in \mathcal{D}(\Omega)$ and $\phi \in C^1_{per}(Y)$ vanishing in a neighborhood of $y = 0$, and set $v_\varepsilon(x) = \varepsilon \psi(x) \phi^\varepsilon(x)$ with $\phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. By Proposition 3.3,

$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \longrightarrow \psi \nabla_y \phi \quad strongly\ in\ L^2(\Omega \times Y). \tag{4.60}$$

Taking v_ε as a test function in (4.40), multiplying by $\varphi \in \mathcal{D}(0, T)$, and integrating over]0, T[, we obtain

$$\begin{aligned} &\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) v_\varepsilon(x) \varphi'(t)\ dx\ dt \\ &+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t) \nabla v_\varepsilon(x) \varphi(t)\ dx\ dt \\ &= \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) v_\varepsilon(x) \varphi(t)\ dx\ dt. \end{aligned} \tag{4.61}$$

Note that this equation can be written as

$$\begin{aligned} &\varepsilon \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) \psi(x) \phi^\varepsilon(x) \varphi'(t)\ dx\ dt \\ &+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t) \nabla v_\varepsilon(x) \varphi(t)\ dx\ dt \\ &= \varepsilon \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) \psi(x) \phi^\varepsilon(x) \varphi(t)\ dx\ dt. \end{aligned} \tag{4.62}$$

The aim is to pass at the limit in this expression.

We first use the unfolding operator $\mathcal{T}_\varepsilon$ to pass to the limit in the second term of the left-hand side of this equation. Using some propositions together with (4.47) and (4.60), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\ dx\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)\ dx\ dy\ dt\\
=&\frac{1}{Y}\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\psi(x)\nabla_y\phi(y)\varphi(t)\ dx\ dy\ dt.
\end{aligned}\tag{4.63}$$

On the other hand, the first term on the left-hand side of (4.70) as well as the term on the right-hand side goes to zero as $\varepsilon\longrightarrow 0$, which implies

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\ dx\ dt=0,$$

So that

$$\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\psi(x)\nabla_y\phi(y)\varphi(t)\ dx\ dy\ dt=0.$$

By Lemma 4.1, we obtain (4.50) which describes the asymptotic behavior of the problem based on the oscillations in the coefficients of (4.40).

Now, to take into account the effect of the perforations, let us use $w_{\varepsilon,\delta}\psi$ as a test function in (4.40), where $w_{\varepsilon,\delta}$ is the function defined in Lemma 4.2 and for $\psi\in\mathcal{D}(\Omega)$. Thus, we have

$$\begin{aligned}
&\langle u'_{\varepsilon,\delta}(x,t), w_{\varepsilon,\delta}(x)\psi(x)\rangle_{(H^1_0(\Omega^*_{\varepsilon,\delta}))',H^1_0(\Omega^*_{\varepsilon,\delta})}\\
&+\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\ dx\\
&+\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\nabla\psi(x)\ dx\\
&=\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\ dx.
\end{aligned}\tag{4.64}$$

Let $\varphi \in \mathcal{D}(0, T)$ and multiply the integrands in this equation and integrate over]0, T[,

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x) \psi(x) \varphi'(t)\ dx\ dt \\
&+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t) \nabla w_{\varepsilon,\delta}(x) \psi(x) \varphi(t)\ dx\ dt \\
&+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x) \nabla \psi(x) \varphi(t)\ dx\ dt \\
&= \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x) \psi(x) \varphi(t)\ dx\ dt.
\end{aligned} \tag{4.65}$$

Remark 4.4 *We take* $|Y| = 1$

For the first term on the left-hand side of this equation, we apply the operator $\mathcal{T}_\varepsilon$. Thus, Definition 3.3 together with Remark 4.3 and (4.48), we obtain,

$$\begin{aligned}
&\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x) \psi(x) \varphi'(t)\ dx\ dt \\
&= \lim_{\varepsilon \longrightarrow 0} \int_0^T \int_{\Omega \times Y} \mathcal{T}_\varepsilon(u_{\varepsilon,\delta}) \mathcal{T}_\varepsilon(w_{\varepsilon,\delta}) \mathcal{T}_\varepsilon(\psi) \mathcal{T}_\varepsilon(\varphi')\ dx\ dy\ dt \\
&= v(B) \int_0^T \int_{\Omega \times Y} u(x,t) \psi(x) \varphi'(t)\ dx\ dy\ dt.
\end{aligned} \tag{4.66}$$

So that :

$$\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x) \psi(x) \varphi'(t)\ dx\ dt \longrightarrow v(B) \int_0^T \int_{\Omega \times Y} u(x,t) \psi(x) \varphi'(t)\ dx\ dy\ dt \tag{4.67}$$

For the second term on the left-hand side of equation (4.65), we use the operator $\mathcal{T}_{\varepsilon,\delta}$ (4.47), (4.58), (4.59), and Remark 4.3, yield

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\ dx\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}\delta^N\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\psi(x))\mathcal{T}_{\varepsilon,\delta}(\varphi(t))\ dx\ dz\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}\delta^N\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\frac{1}{\varepsilon\delta}\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi(x))\mathcal{T}_{\varepsilon,\delta}(\varphi(t))\ dx\ dz\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}\delta^{\frac{N}{2}}\cdot\delta^{\frac{N}{2}}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\frac{1}{\varepsilon\delta}\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi(x))\mathcal{T}_{\varepsilon,\delta}(\varphi(t))\ dx\ dz\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}-\frac{\delta^{\frac{N}{2}}}{\varepsilon\delta}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\delta^{\frac{N}{2}}(\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}))(-\frac{1}{\varepsilon\delta}\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi(x))\mathcal{T}_{\varepsilon,\delta}(\varphi(t))\ dx\ dz\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}\left(-\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\frac{1}{\varepsilon\delta}\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi(x))\mathcal{T}_{\varepsilon,\delta}(\varphi(t))\ dx\ dz\ dt\right)\\
=&-k^*\int_0^T\int_{\Omega\times\mathbb{R}^N}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\ dx\ dz\ dt\\
=&-k^*\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\ dx\ dz\ dt,
\end{aligned}$$

So that :

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\ dx\ dt\longrightarrow\\
&-k^*\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\ dx\ dz\ dt,
\end{aligned}\tag{4.68}$$

For the third term on the left-hand side of (4.65), we use $\mathcal{T}_\varepsilon$. From Proposition 3.3(2)(4), Proposition 3.4(1), Definition 3.3 together with Remark 4.3,(4.46), Proposition 3.5(ii), passing to the limit gives

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)\ dx\ dt\\
=&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\mathcal{T}_\varepsilon(\varphi)\ dx\ dy\ dt\\
=&v(B)\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\ dx\ dy\ dt.
\end{aligned}$$

So that :

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)\;dx\;dt \longrightarrow\\ &v(B)\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\;dx\;dy\;dt\end{aligned} \tag{4.69}$$

For the term on the right-hand side of equation (4.65), we also apply $\mathcal{T}_\varepsilon$, Definition 3.3, Remark 4.3 and (4.35), passing to the limit, yield

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\;dx\;dt\\ =&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega\times Y} \mathcal{T}_\varepsilon(f_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\psi)\mathcal{T}_\varepsilon(\varphi)\;dx\;dy\;dt\\ =&v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\;dx\;dy\;dt\end{aligned}$$

So that :

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\;dx\;dt \longrightarrow\\ &v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\;dx\;dy\;dt.\end{aligned} \tag{4.70}$$

Thus, combining (4.66)-(4.70), the limit equation of (4.65) is

$$\begin{aligned}&v(B)\int_0^T\int_{\Omega\times Y} u(x,t)\psi(x)\varphi'(t)\;dx\;dy\;dt\\ &-k^*\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\;dx\;dz\;dt\\ &+v(B)\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\;dx\;dy\;dt\\ &=v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\;dx\;dy\;dt,\end{aligned}$$

which is true for all $\varphi\in\mathcal{D}(0,T)$,$\psi\in H^1_0(\Omega)$ and $v\in K(B)$. So, we obtain (4.51) for $v\in K(B)$ such that $v(B)=0$.

If $v(B)\neq 0$, by applying Stoke's formula and (4.51), we have

$$\begin{aligned}&\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\;dx\;dz\;dt\\ &=v(B)\int_0^T\int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)v_B\psi(x)\varphi(t)\;dx\;d\sigma_z\;dt,\end{aligned} \tag{4.71}$$

which used in (4.3.1) gives the first equation of problem (4.52).

Step 4. It remains now to check the limit initial condition. Let $v_\varepsilon = w_{\varepsilon,\delta}\psi$ where $w_{\varepsilon,\delta}$ is given by Lemma 4.2 and $\psi \in \mathcal{D}(\Omega)$. Let $\varphi \in C^\infty([0,T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Take $v_\varepsilon\varphi$ as a test function in (4.40).

Using the initial conditions in (4.40) and by integration by parts, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\;dx\;dt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\;dx\;dt\\
=&\int_0^T \langle u'_{\varepsilon,\delta}(x,t), v_\varepsilon(x)\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))',H_0^1(\Omega^*_{\varepsilon,\delta})}\varphi(t)\;dt\\
&= \int_{\Omega^*_{\varepsilon,\delta}} (u_{\varepsilon,\delta}(x,t)\varphi(t))|_0^T v_\varepsilon(x)\;dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\;dx\;dt\\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(T)\varphi(T)v_\varepsilon\;dx - \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(0)\varphi(0)v_\varepsilon\;dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\;dx\;dt\\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u^0_{\varepsilon,\delta}v_\varepsilon(x)\;dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\;dx\;dt
\end{aligned}$$

Passing to the limit, we get :

$$-\int_{\Omega^*_{\varepsilon,\delta}} u^0_{\varepsilon,\delta}v_\varepsilon(x)\;dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\;dx\;dt = \int_{\Omega\times\mathbb{R}^N} u^0\psi\;dx - \int_0^T \int_\Omega \psi u\varphi'\;dx\;dt$$

In view of (4.68)-(4.70) and (4.35), passing to the limit in this equation yields

$$\begin{aligned}
&v(B)\int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\;dx\;dy\;dt\\
&+ k^*\int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\;dx\;dz\;dt\\
&- v(B)\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\;dx\;dy\;dt\\
&= v(B)\int_\Omega u^0\psi(x)\;dx - v(B)\int_0^T \int_\Omega u(x,t)\psi(x)\varphi'(t)\;dx\;dt\\
&= -v(B)\int_\Omega u^0(x)\psi(x)\;dx + v(B)u(x,0)\psi(x)\;dx\\
&+ \langle u(x,t),\psi(x)\rangle_{(H_0^{-1}(\Omega),H_0^1(\Omega)}\varphi(t)\;dt.
\end{aligned} \tag{4.72}$$

Combining this equation with (4.3.1) yields

$$-\int_\Omega u^0(x)\psi(x)\;dx + \int_\Omega u(x,0)\psi(x)\;dx = 0, \quad \forall\psi \in \mathcal{D}(\Omega), \tag{4.73}$$

which implies $\boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{u}^{\mathbf{0}}(\boldsymbol{x})$.

This concludes the proof.

CONCLUSION

In this work we study the homogenization of the heat equation in the perforated domain Ω_ε and $\Omega_{\varepsilon,\delta}$, we get

For the domain Ω_ε, we have the following homogenized problem

$$\begin{cases} u' - div(A^{hom}\nabla u) = f & in\ \Omega \times (0,T), \\ u = 0 & on\ \partial\Omega \times (0,T), \\ u(x,0) = u^0 & in\ \Omega \end{cases}$$

were the homogenized matrix $A^{hom} = (a^0_{i,j})_{1\leq i,j\leq n}$ is defined by

$$a^0_{i,j}(x) = \mathcal{M}_{Y^*}\left(a_{i,j} + \Sigma_{k=1}^{n} a_{ik}\frac{\partial \mathcal{X}_j}{\partial y_k}\right)$$

And for the domain $\Omega_{\varepsilon,\delta}$, we have the following homogenized problem

$$\begin{cases} u' - div(\mathcal{A}^{hom}\nabla u) + (k^*)^2\Theta u = f\ in\ \Omega\times]0,T[\\ u = 0 \quad in\ \partial\Omega\times]0,T[, \\ u(x,0) = u^0, \qquad in\ \Omega \end{cases}$$

where the homogenized matrix field is

$$\boldsymbol{\mathcal{A}^{hom} = m_Y \left(a_{ij} + \sum_{k=1}^{N} a_{ik} \frac{\partial \hat{\mathcal{X}}_j}{\partial y_k} \right)},$$

and

$$\boldsymbol{\Theta = \int_{\partial B} A^0 \nabla_z \theta \nu_B \, d\sigma_z.}$$

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