

**UNIVERSITY 20 AUGUST 1955 OF SKIKDA**

**Faculty of Sciences**

**Department of Matter Sciences**

# **PHYSICS 1**

## **Mechanics of the Material Point**

**Courses and exercises for students of:**

- \* Matter Sciences (LMD SM)
- \* Sciences and Technologies (LMD ST)

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## Foreword

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This course booklet for Physics I - **Mechanics of Particle** - is intended for students in the first year in the field of Matter Sciences. The main objective of this subject is to acquire the basic formalisms in the mechanics of a particle and the mathematical representations of physical phenomena related to particle mechanics.

The program of the Mechanics of Material point in the first Semester consists of four chapters:

- **The first chapter** provides a review of vector algebra and dimensional analysis. Both cover the fundamental quantities that are used to express physical laws and principles. In addition to necessary reviews, this section aims to introduce clear definitions and appropriate notations.
- **The second chapter** is dedicated to Kinematics. The purpose is to describe the movements of objects without focusing on the causes that produce them. It deals exclusively with the motions of material points.
- **The third chapter** describes the dynamics of a particle within the framework of Newtonian mechanics, including its three laws or principles: the law of inertia, the fundamental law of dynamics, and the law of reciprocal actions. We consider general laws, referred to as force laws, established for several interactions, along with applications aimed at predicting the motion of bodies. The concepts of angular momentum particle with respect to the origin and pseudo-forces are also addressed in this chapter.
- **The fourth chapter** revolves around the third method of analysis, which is that of work and energy. This approach eliminates the need to calculate acceleration by directly relating force, mass, velocity, and displacement. We first consider the work done by a force and the kinetic energy of a particle. Subsequently, we delve into the concepts of potential and total energies, and we apply the principle of conservation of energy in various practical situations.

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# Chapter I

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## Dimensional Analysis and Vectors

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## I.1 Dimensional analysis

### I. 1. 1 Units and dimensions

#### I. 1. 1. 1. Physical quantities- Fundamental and Derived Quantities

All quantities in terms of which laws of physics can be expressed and which can be measured are called Physical Quantities. For example: *Distance, Speed, Mass, Force, etc.*

- The quantities that are independent of other quantities are called **fundamental quantities**. The units that are used to measure these fundamental quantities are called **fundamental units**. There are four systems of units, namely CGS, MKS, FPS, and SI.
- The quantities that are derived using the fundamental quantities are called **derived quantities**. The units that are used to measure these derived quantities are called **derived units**.

**a. Fundamental Quantity:** The quantity which is independent of other physical quantities. In mechanics, **mass**, **length** and **time** are called fundamental quantities. Units of these fundamental physical quantities are called **Fundamental units**.

Fundamental Physical Quantity	Fundamental unit
Mass	Kg, Gram, Pound
Length	Meter, Centimeter, Foot
Time	Second

**b. Derived Quantity:** The quantity which is derived from the fundamental quantities *e.g.* area is a derived quantity.

**Example:**

$$\text{Area} = \text{Length} \times \text{Breadth}$$

$$= \text{Length} \times \text{Length}$$

$$= (\text{Length})^2$$

$$\text{Speed} = \text{Distance} / \text{Time}$$

$$= \text{Length} / \text{Time}$$

The units for derived quantities are called **Derived Units**.

**Measurement:** In our daily life, we need to express and compare the magnitude of different quantities; this can be done only by measuring them.

Measurement is the comparison of an unknown physical quantity with a known fixed physical quantity.

**Unit:** The known fixed physical quantity is called unit.

**OR** *The quantity used as standard for measurement is called unit.*

For example, when we say that length of the class room is 10 meter. We compare the length of class room with standard quantity of length called meter.

Length of class room = 10 meter

Physical Quantity (Q) = Numerical value (n)  $\times$  unit (u), (Q = nu)

Q = Physical Quantity

n = Numerical value

u = Standard unit

**Example:**

Mass of table = 15 kg

Mass = Physical quantity

15 = Numerical value

Kg = Standard unit

### I. 1. 1. 2. Systems of Units: CGS, FPS, MKS, SI

For measurement of physical quantities, the following systems are commonly used (Table I. 1):

- ✓ **C.G.S system:** In this system, the unit of length is centimeter, the unit of mass is gram and the unit of time is second.
- ✓ **F.P.S system:** In this system, the unit of length is foot, the unit of mass is pound and the unit of time is second.
- ✓ **M.K.S:** In this system, the unit of length is meter, unit of mass is kg and the unit of time is second.
- ✓ **S.I System:** This system is an improved and extended version of M.K.S system of units. It is called international system of unit.

**Table I.1.** Systems of units

Fundamental Quantity	System of Units		
	CGS	MKS	FPS
Length	centimeter	meter	foot
Mass	gram	kilogram	pound
Time	second	second	second

With the development of science & technology, the three fundamental quantities like mass, length & time were not sufficient as many other quantities like electric current, heat etc. were introduced (Table I. 2).

Therefore, more fundamental units in addition to the units of mass, length and time are required. Thus, MKS system was modified with addition of four other fundamental quantities and two supplementary quantities (Table I. 3).

**Table I.2.** Fundamental physical quantities

Physical Quantity	Symbol	Unit	Symbol
Length	l	meter	m
Mass	M	kilogram	kg
Time	T	second	s
Electric current	I	ampere	A
Thermodynamic temperature	$\theta$	kelvin	K
Intensity of light	J	candela	cd
Quantity of substance	N	mole	mol

**Table I. 3.** Supplementary Quantities

Plane angle	Radian	rad
Solid angle	Steradian	sr

### I. 1. 1. 3. Definition of Fundamental (Basic) and Supplementary Unit of S.I.

#### a. Fundamental units

- **Meter (m):** The meter is the length of the path travelled by light in vacuum during a time interval of  $1/299\,792\,458$  of a second.
- **Kilogram (Kg) :** The kilogram is the mass of the platinum-iridium prototype which was approved by the “*Conférence Générale des Poids et Mesures*”, held in Paris in 1889, and kept by the “*Bureau International des Poids et Mesures*”.
- **Second (s):** The second is the duration of 9192631770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of Cesium-133 atom.
- **Ampere (A) :** The ampere is the intensity of a constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to  $2 \cdot 10^{-7}$  Newton per meter of length.
- **Kelvin (K):** Kelvin is the fraction  $1/273.16$  of the thermodynamic temperature of the triple point of water.
- **Candela (Cd):** The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency  $540 \times 10^{12}$  hertz and that has a radiant intensity in that direction of  $1/683$  watt per steradian.

- **Mole (mol):** The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of Carbon-12.

**b. Supplementary units:**

- **Radian (rad):** It is supplementary unit of plane angle. It is the plane angle subtended at the centre of a circle by an arc of the circle equal to the radius of the circle. It is denoted by  $\theta$ .

$\theta = l / r$ ;  $l$  is length of the arc and  $r$  is radius of the circle.

- **Steradian (Sr):** It is supplementary unit of solid angle. It is the angle subtended at the centre of a sphere by a surface area of the sphere having magnitude equal to the square of the radius of the sphere. It is denoted by  $\Omega$ .

$$\Omega = \Delta s / r^2$$

**Notes:**

1. Some important abbreviations, the following table resumes some important abbreviations

**Table I. 4.** Some important abbreviations

MACRO Prefixes	MICRO Prefixes
Kilo (K) $10^3$	milli (m) $10^{-3}$
Mega (M) $10^6$	micro ( $\mu$ ) $10^{-6}$
Giga (G) $10^9$	nano (n) $10^{-9}$
Tera (T) $10^{12}$	pico (p) $10^{-12}$
Peta (P) $10^{15}$	femto (f) $10^{-15}$
Exa (E) $10^{18}$	atto (a) $10^{-18}$
Zetta (Z) $10^{21}$	zepto (z) $10^{-21}$
Yotta (y) $10^{24}$	yocto (y) $10^{-24}$

2. The following are not used in the SI system.

- deca  $10^1$  deci  $10^{-1}$
- hecta  $10^2$  centi  $10^{-2}$

## 3. How to Write Units of Physical Quantities?

- ✓ Full names of the units, even when they are named after a scientist, should not be written with a capital letter. For example, newton, watt, ampere, meter
- ✓ 2. The unit should be written either in full or in agreed symbols only
- ✓ 3. Units do not take the plural form. For example, 10 kg but not 10 kgs, 20 w but not 20 ws
- ✓ 4. No full stop or punctuation mark should be used within or at the end of symbols for units. For example, 10 W but not 10 W.

## 4. Some Important Units of Length:

- 1 micron =  $10^{-6}$  m =  $10^{-4}$  cm
- 1 angstrom =  $1\text{\AA} = 10^{-10}$  m =  $10^{-8}$  cm
- 1 fermi = 1 fm =  $10^{-15}$  m
- 1 Light year = 1 ly =  $9.46 \times 10^{15}$  m
- 1 Parsec = 1 pc = 3.26 light year

## 5. Some conversion factor of mass:

- 1 Kilogram = 2.2046 pound
- 1 Pound = 453.6 gram
- 1 kilogram = 1000 gram
- 1 milligram =  $1/1000$  gram =  $10^{-3}$  gram
- 1 centigram =  $1/100$  gram =  $10^{-2}$  gram
- 1 decigram =  $1/10$  gram
- 1 quintal = 100 kg
- 1 metric ton = 1000 kilogram

**I. 1. 2. Dimensional Formula and SI Units of Physical Quantities****1. Definition of dimensions:**

Dimensions of a physical quantity are the powers to which the fundamental units are raised to obtain one unit of that quantity.

**Examples:**

1) Area = Length x Breadth

$$= [L^1] \times [L^1] = [L^2] = [M^0 L^2 T^0]$$

Power (0, 2, 0) of fundamental units are called *dimensions* of area in mass, length and time respectively.

2) Density = mass/volume

$$= [M]/[L^3]$$

$$= [M^1 L^{-3} T^0]$$

## 2. Dimensional analysis:

Dimensional analysis is the practice of checking relations between physical quantities by identifying the dimensions of the physical quantities. These dimensions are independent of the numerical multiples and constants, and all the quantities in the world can be expressed as a function of the fundamental dimensions.

## 3. Dimensional Formula and equation:

The expression showing the powers to which the fundamental units are to be raised to obtain one unit of a derived quantity is called the *dimensional formula* of that quantity. An equation obtained by equating the physical quantity with its dimensional formula is called **dimensional equation**.

The dimension of physical quantity Q is represented by [Q]. The dimensional equation can be written with the seven fundamental quantities as:

$$[Q] = L^a \cdot M^b \cdot T^c \cdot I^d \cdot \theta^e \cdot J^f \cdot N^g \quad (\text{I. 1})$$

Where: L (Length), M (Mass), T (Time), I (Electric Current),  $\theta$  (Temperature), J (Luminous Intensity) and N (Quantity of Matter). The exponents a, b, c, d, e, f and g are called dimensions.

**Example:** The dimensional equation of area, density & velocity are given as under:

$$\text{Area} = [M^0 L^2 T^0]$$

$$\text{Density} = [M^1 L^{-3} T^0]$$

$$\text{Velocity} = [M^0 L^1 T^{-1}]$$

## 4. Dimensional formula, SI unit of derived Physical Quantities

The following table shows dimensional formula and SI unit of some Physical Quantities:

**Table I. 5.** Table shows dimensional formula and SI unit of some Physical Quantities

Physical Quantity	Symbol	Formula	Dimensions	Unit	Name of unit	Symbol
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Speed or velocity	V	Distance/time	$L^1T^{-1}$	$m.s^{-1}$	–	–
acceleration	a	Velocity/time	$L^1T^{-2}$	$m.s^{-2}$	–	–
Force	F	Mass $\times$ acceleration	$M^1 L^1 T^{-2}$	$Kg.m.s^{-2}$	Newton	N
Work-Energy	W- E	Force $\times$ distance	$M^1 L^2 T^{-2}$	$Kg.m^2.s^{-2}$ or N.m	Joule	J
Power	P	Energy/time	$M^1 L^2 T^{-3}$	$Kg.m^2.s^{-3}$ or $J.s^{-1}$	Watt	W
Pressure	p	Force/area	$M^1 L^{-1} T^{-2}$	$Kg.m^{-1}.s^{-2}$ or $N.m^{-2}$	Pascal	Pa
Frequency	f	1/Time period	$T^{-1}$	$s^{-1}$	Hertz	Hz
Electric charge	q	current $\times$ time	$I^1 T^1$	A.s	Coulomb	C
Electric potential	V	work/charge	$M^1 L^2 T^{-3} I^{-1}$	$Kg.m^2.s^{-3}A^{-1}$	Volt	V
Electric resistance	R	Potential/current	$M^1 L^2 T^{-3} I^{-2}$	$Kg.m^2.s^{-3}A^{-2}$	ohm	$\Omega$
Electric field	E	force/charge	$M^1 L^1 T^{-3} I^{-1}$	$NC^{-1}, Vm^{-1}$	–	–
Area	S	Length $\times$ Breadth	$L^2$	$m^2$	–	–
Volume	V	length $\times$ breadth $\times$ height	$L^3$	$m^3$	–	–
Density	D	mass/volume	$M L^{-3}$	$Kg.m^{-3}$	–	–

### 5. Classification of Physical Quantity:

Physical quantity has been classified into following four categories on the basis of dimensional analysis.

- ✓ **Dimensional Constant:** These are the physical quantities which possess dimensions and have constant (fixed) value.

e.g. Planck's constant, gas constant, universal gravitational constant etc.

- ✓ **Dimensional Variable:** These are the physical quantities which possess dimensions and do not have fixed value.

e.g. velocity, acceleration, force etc.

- ✓ **Dimensionless Constant:** These are the physical quantities which do not possess dimensions but have constant (fixed) value.

e.g.  $e, \pi$ , numbers like 1,2,3,4,5 etc.

- ✓ **Dimensionless Variable:** These are the physical quantities which do not possess dimensions and have variable value.

e.g. angle, strain, specific gravity etc.

### I. 1. 3. Principle of Homogeneity of dimensions

It states that *the dimensions of all the terms on both sides of an equation must be the same.*

According to the principle of homogeneity, the comparison, addition & subtraction of all physical quantities is possible only if they are of the same nature i.e., they have the same dimensions. If the power of M, L and T on two sides of the given equation are same, then the physical equation is correct otherwise not. Therefore, this principle is very helpful to check the correctness of a physical equation.

#### **Example:** Period of a simple pendulum

A physical relation must be dimensionally homogeneous, i.e., all the terms on both sides of the equation must have the same dimensions.

We can think that the period  $T$  of a simple pendulum (figure I. 1) could depend on the length  $l$  of the string, the mass  $m$  of the body and the acceleration due to gravity  $g$ .

Let us establish the relation which describes this dependence.

Expression of  $T$  as a function of the previous quantities:

$$T = Km^\alpha l^\beta g^\gamma \quad (\text{I. 2})$$

$K$  is a dimensionless constant and  $\alpha$ ,  $\beta$  and  $\gamma$  are exponents to be determined.

Now, dimensions of various quantities in the equation are:

$$[T] = T, [m^\alpha] = M^\alpha, [l^\beta] = L^\beta, [g^\gamma] = (L \cdot T^{-2})^\gamma = L^\gamma \cdot T^{-2\gamma}$$

The dimensional equation of (I.2) is then:

$$T = M^\alpha \cdot L^\beta \cdot L^\gamma \cdot T^{-2\gamma}$$

Here the equation must be dimensionally homogeneous; the dimensions of all the terms on both sides of the equation must be the same. Therefore the following relations result:

$$\alpha = 0$$

$$\beta + \gamma = 0 \Leftrightarrow \beta = -\gamma$$

$$-2\gamma = 1 \Leftrightarrow \gamma = -\frac{1}{2} \Rightarrow \beta = \frac{1}{2}$$

The relation (I.2) then becomes:

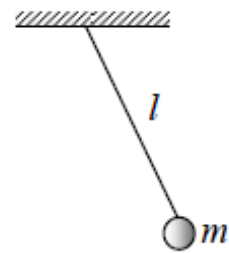


Figure I. 1

$$T = K \sqrt{\frac{l}{g}}$$

This analysis shows that the period of the pendulum does not depend on the mass  $m$ .

## I.2. Error and uncertainty

There is always a degree of uncertainty when measurements are taken; the uncertainty can be thought of as the difference between the **actual** reading taken (caused by the equipment or techniques used) and the **true value**. Uncertainties are not the same as errors:

- Errors can be thought of as issues with equipment or methodology that cause a reading to be different from the true value.
- The uncertainty is a range of values around a measurement within which the true value is expected to lie, and is an **estimate**.

**Example:** if the true value of the mass of a box is 950 g, but a systematic error with a balance gives an actual reading of 952 g, the uncertainty is  $\pm 2$  g.

### 1.2.1. Direct and Indirect Measurement

Two different measurement methods have to be distinguished, direct and indirect measurement.

#### a. Uncertainty in Direct Measurements

The direct measurement is the operative realization of the quantitative method, and is possible only for additive quantities.

**Example:** Measurement of the length of an object by a ruler and measurement of a mass by an equal-arm balance.

#### ➡ How to calculate uncertainty in direct measurement?

- ✓ **Take several measurements.** Let's say you want to calculate how long it takes a ball to drop to the floor from the height of a table. To get the best results, you'll have to measure the ball falling off the table top at least a few times, let's say five. Then, you'll have to find the average of the five measured times and then add or subtract the standard deviation from that number to get the best results.

Let's say you measured the five following times:  $M_1 = 0.43$  s,  $M_2 = 0.52$  s,  $M_3 = 0.35$  s,  $M_4 = 0.29$  s, and  $M_5 = 0.49$  s.

- ✓ **Find the *average* of the measurements.** Now, find the average by adding up the five different measurements and dividing the result by 5, the amount of measurements.  $0.43$

$s + 0.52 s + 0.35 s + 0.29 s + 0.49 s = 2.08 s$ . Now, divide 2.08 by 5.  $M_{average} = 2.08/5 = 0.42 s$ . The average time is 0.42 s.

In general, the **average** value of  $n$  several measurements  $M_1, M_2, \dots, M_n$  gives by:

$$M_{Average} = \frac{\text{sum of measured values}}{\text{number of measured}} = \frac{M_1 + M_2 + \dots + M_n}{n} = \frac{\sum_{i=1}^n M_i}{n} \quad (\text{I. 2})$$

- ✓ **Find the variance of these measurements.** To do this, first, find the difference between each of the five measurements and the average. To do this, just subtract the measurement from 0.42s. Here are the five differences:

$$\Delta M_1 = |0.43 - 0.42| = 0.01 \text{ s}$$

$$\Delta M_2 = |0.52 - 0.42| = 0.1 \text{ s}$$

$$\Delta M_3 = |0.35 - 0.42| = 0.07 \text{ s}$$

$$\Delta M_4 = |0.29 - 0.42| = 0.13 \text{ s}$$

$$\Delta M_5 = |0.49 - 0.42| = 0.07 \text{ s}$$

The parameter  $\Delta M$  is commonly called **absolute uncertainty**.

In general:

$$\Delta M_i = |M_i - M_{average}| \quad (\text{I. 3})$$

Find the average (or mean) of these values by dividing the adding result by 5.

$$\begin{aligned} \Delta M_{average} &= (\Delta M_1 + \Delta M_2 + \Delta M_3 + \Delta M_4 + \Delta M_5)/5 = (0.01 + 0.1 + 0.07 + 0.13 + 0.07)/5 \\ &= 0.38/5 = 0.076 \text{ s.} \end{aligned}$$

In general, the **average (or mean)** value of the **absolute uncertainty** can give by:

$$\Delta M_{Average} = \frac{\Delta M_1 + \Delta M_2 + \dots + \Delta M_n}{n} = \frac{\sum_{i=1}^n \Delta M_i}{n} \quad (\text{I. 4})$$

- ✓ **State the final measurement.** To do this, simply state the average of the measurements along with the added and subtracted standard deviation. Since the average of the

measurements is 0.42 s and the average of the absolute uncertainty is 0.076 s, the final measurement is:  $M = 0.42 \text{ s} \pm 0.076 \text{ s}$ .

The scientist would write  $M = M_{\text{average}} \pm \Delta M$ . The format is “*average value*” plus (+) or minus (-) “*uncertainty*”.

we can express the informations on the measure  $M$  in the following way:

$$M = (M_{\text{Average}} \pm \Delta M)u \quad (\text{I. 5})$$

where  $u$  represents the unit of measure and depends on the system of units we have adopted.

### Notes:

- The *true* value of a measurement can never be known.
- The value of  $\Delta M$  defines the number of significant digits with which it makes sense to express  $M$  that is **the last significant digit of  $M$  must have the same order of magnitude as  $\Delta M$ .**

For example, does not make sense (and therefore should be avoided) say that a measure of length is:  $l = (1.23456 \pm 0.001) \text{ m}$

because the absolute uncertainty blurs the evaluation of the millimeters so the digits relating to tenths and hundredths of a millimeter are meaningless. For this reason it is good practice to round the absolute uncertainty to one significant digit and to express as the last significant digit of the mean value the digit affected by the absolute uncertainty. The measure of the example must be properly written:

$$l = (1.235 \pm 0.001) \text{ m}$$

To measure directly the values of  $m$  and  $\Delta m$  we usually compare  $M$  with predetermined values provided by a tool consisting essentially of a graduated rule. For example, to measure a length we use a ruler or a vernier caliper; to measure duration of time we use a stopwatch; to measure a current we use an ammeter.

In this comparison we may be faced with two different occurrences:

- When repeating the measurement, we always get the same result (this happens if we use instruments with low precision), we assign to  $M$  the reading on the instrument and to  $\Delta M$  the value of the smallest division readable on the instrument.

- When, repeating the measurement we get many values randomly different (this happens if we use instruments with high precision) we assign to  $M$  the mean value and to  $\Delta M$  three times the sample standard deviation ( $\sigma$ ).

The ratio  $\frac{\Delta M}{M}$  is said **relative uncertainty**.

The relative uncertainty expresses the goodness of the measure. Where uncertainty is given as a percentage of the measurement:

$$\text{percentage uncertainty} = \frac{\text{uncertainty}}{\text{measured value}} \times 100\% = \frac{\Delta M}{M} \times 100\% \quad (1.6)$$

A small relative uncertainty means that the measure is very accurate; a large relative uncertainty implies that the measure is unreliable or even unusable.

### b. Uncertainty in indirect Measurements

Indirect measurement is the procedure by which the measure  $X(G)$  of a quantity  $G$  is obtained through analytical relations from the measures  $Y(A)$ ;  $Z(B)$ ;...of other quantities  $A$ ;  $B$ ;... directly measured. Quantities that could in principle be directly measured are often indirectly measured, for convenience or because of the practical difficulty of direct measurements.

**Example:** Velocity is indirectly measured; it is a ratio between a length and a time interval.

### ➡ Methods for Calculating Uncertainty through Functional Relationships

In other scientific applications, many more types of calculation may be required to calculate uncertainty. For example, calculations based on formulae which include trigonometric functions and/or more complicated logarithmic relationships would be common in the physical sciences. However, uncertainty associated with any measurement and its propagation through a defined functional relationship can be evaluated by two methods:

- **Differentiation (partial differentiation) method:** is a method used to derivative of a function  $f$ . In mathematics, the partial derivative of any function having several variables is its derivative with respect to one of those variables where the others are held constant. The partial derivative of a function  $f$  with respect to the differently  $x$  is variously denoted by  $\partial f/\partial x$ . Here  $\partial$  is the symbol of the partial derivative.

- **Logarithmic differentiation, (Derivative of Natural Logarithm (ln)) method:** differentiation by taking logarithms; is a method used to differentiate functions by employing the derivative logarithm of a function  $f$ .

The following example can explain how to use the both methods to calculate uncertainty through Functional Relationships.

**Example:** Suppose the functional relationship:  $f(x, y, z) = x^2y^3z^{-1}$

We use the partial differentiation method to calculate the relative and then the absolute uncertainties.

$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad (1.7)$$

$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = (2xy^3z^{-1})dx + (x^23y^2z^{-1})dy + (-x^2y^3z^{-2})dz$$

The partial derivative  $\frac{\partial f}{\partial x}$  means differentiating  $f$  with respect to  $x$  holding the other variables ( $y, z$ ) fixed.

Now the both sides of the equation divided by the functional  $f$ :

$$\frac{df}{f} = \frac{2xy^3z^{-1}}{x^2y^3z^{-1}} dx + \frac{x^23y^2z^{-1}}{x^2y^3z^{-1}} dy + \frac{-x^2y^3z^{-2}}{x^2y^3z^{-1}} dz = 2 \frac{dx}{x} + 3 \frac{dy}{y} - \frac{dz}{z}$$

The relative uncertainty of the functional  $f$  is given by:

$$\frac{\Delta f}{f} = 2 \frac{\Delta x}{x} + 3 \frac{\Delta y}{y} + \frac{\Delta z}{z}$$

The absolute uncertainty:

$$\Delta f = f \times \left( 2 \frac{\Delta x}{x} + 3 \frac{\Delta y}{y} + \frac{\Delta z}{z} \right)$$

Continuing with the same example to describe the second method (logarithmic differentiation or differentiation by taking logarithms functional), we can use the following steps:

- 1- Take the logarithm of the function.
- 2- Derived the logarithm of the function
- 3- Calculate relative and then absolute uncertainties.

$$f(x, y, z) = x^2 y^3 z^{-1}$$

$$\log f = 2 \log x + 3 \log y - \log z$$

Thus, taking the derivation of logarithm of f, we get:

$$\frac{df}{f} = 2 \frac{dx}{x} + 3 \frac{dy}{y} - \frac{dz}{z}$$

the relative uncertainty of functional f:

$$\frac{\Delta f}{f} = \left| 2 \frac{\Delta x}{x} + 3 \frac{\Delta y}{y} - \frac{\Delta z}{z} \right| = 2 \frac{\Delta x}{x} + 3 \frac{\Delta y}{y} + \frac{\Delta z}{z}$$

### Example:

Use the two previous methods to calculate uncertainty on the period T of a simple pendulum.

$$T = 2\pi \sqrt{\frac{l}{g}} = 2\pi \cdot l^{1/2} \cdot g^{-1/2}$$

#### 1. partial differentiation method

$$dT = \frac{\partial T}{\partial l} dl + \frac{\partial T}{\partial g} dg = 2\pi \left( \frac{1}{2} l^{-1/2} \cdot g^{-1/2} dl - \frac{1}{2} l^{1/2} \cdot g^{-3/2} dg \right)$$

$$\frac{dT}{T} = \frac{1}{2} \frac{l^{-1/2} \cdot g^{-1/2}}{l^{1/2} \cdot g^{-1/2}} dl - \frac{1}{2} \frac{l^{1/2} \cdot g^{-3/2}}{l^{1/2} \cdot g^{-1/2}} dg = \frac{1}{2} \frac{dl}{l} - \frac{1}{2} \frac{dg}{g}$$

$$\frac{\Delta T}{T} = \frac{1}{2} \frac{\Delta l}{l} + \frac{1}{2} \frac{\Delta g}{g}$$

#### 2. logarithmic differentiation method

$$\log T = \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\frac{dT}{T} = \frac{1}{2} \frac{dl}{l} - \frac{1}{2} \frac{dg}{g}$$

$$\frac{\Delta T}{T} = \frac{1}{2} \frac{\Delta l}{l} + \frac{1}{2} \frac{\Delta g}{g}$$

### I.3. Scalars and Vectors

The physical quantities that are used to describe the motion of objects can be divided into two categories. The quantity is either a vector or a scalar. These two categories can be distinguished from one another by their distinct definitions:

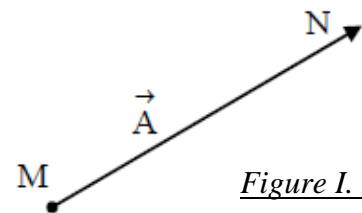
- **Scalars:** are quantities that are fully described by a magnitude (or numerical value) alone. The word scalar means representable by position on a line; having only magnitude. (Example: *distance, area, volume, speed, mass, time, energy, density, temperature, and work*).
- **Vectors:** are quantities that are fully described by both a magnitude and a direction. (Example: *displacement, velocity, force and acceleration*)

If you say that a car is traveling at 100 km/hr, you are using a scalar quantity, namely the number 100 with no direction attached, to describe the speed of the car. On the other hand, if you say that the car is traveling due north at 100 km/hr, your description of the car's velocity is a vector quantity since it includes both magnitude and direction.

#### I.3.1. Geometrical representation of vectors

##### 1. Definition

Because vectors are determined by both a magnitude and a direction, they are represented geometrically in 2 or 3 dimensional space as directed line segments or arrows. The length of the arrow corresponds to the *magnitude* of the vector while the direction of the arrow corresponds to the *direction* of the vector. The **tail** of the arrow is called the initial point of the vector while the **tip** of the arrow is called the terminal point of the vector. If the vector  $\overrightarrow{MN}$  has the point M as its initial point and the point N as its terminal point we will write,  $\vec{A} = \overrightarrow{MN}$ .



*Figure I. 2*

A vector  $\overrightarrow{MN}$  (figure I.1) is an oriented segment which has:

- Initial M and terminal point N.
- Magnitude or modulus  $|\overrightarrow{MN}|$ : the length of the segment MN.
- Direction: from M to N.

##### 2. Properties

Two related vectors  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  with different origins are:

- **Equal vectors:** which have the same length and same direction (figure I.3a), are said to be equal vectors even though they have different initial points and different terminal points.

$$\vec{AB} = \vec{CD}$$

- **Opposite vectors:** if they have the same modulus but opposite directions (figure I.3b); they are said to be "directly opposite" if they have the same support ( $\Delta$ ) (figure I.3c).  $\vec{AB} = -\vec{CD}$

- **Unit Vector:** A unit vector has a length (or magnitude) equal to one, which is basically used to show the direction of any vector. A *unit vector* is equal to the ratio of a vector and its magnitude. If  $\vec{A}$  is vector of arbitrary length and its magnitude is  $|\vec{A}|$ ,  $\vec{A} = |\vec{A}|\vec{u}$ , then the unit vector is given by:  $\vec{u} = \frac{\vec{A}}{|\vec{A}|}$ .

- **Zero Vector:** A vector with zero magnitudes is called a zero vector. The coordinates of zero vector are given by (0,0,0) and it is usually represented by 0 with an arrow ( $\rightarrow$ ) at the top or just 0. The sum of any vector with zero vector is equal to the vector itself,  $\vec{A} + \vec{0} = \vec{0} + \vec{A} = \vec{A}$ .

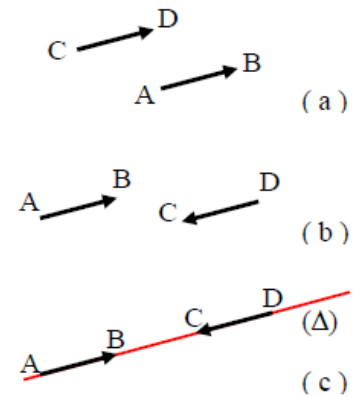


Figure I. 3

### 3. Magnitude (or modulus) of a Vector

The magnitude of a vector represents the length of the vector. Mathematically, the magnitude of a vector  $\vec{AB}$  is denoted by  $|\vec{AB}|$ , the length AB.

**Example:** suppose a vector  $\vec{V} = |\vec{V}|\vec{u} = 3\vec{u}$ ,  $|\vec{V}| = 3$  represents the modulus of vector  $\vec{V}$  and  $\vec{u}$  is unit vector (figure I.4).

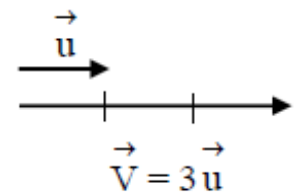


Figure I. 4

### 4. Operations on Vectors

#### 4.1. Addition of Vectors

The sum of two vectors  $\vec{U}$  and  $\vec{V}$ , denoted  $\vec{U} + \vec{V} = \vec{W}$  is a third vector  $\vec{W}$  obtained by the "parallelogram rule" (figure I.5). When the number of vectors to be added is

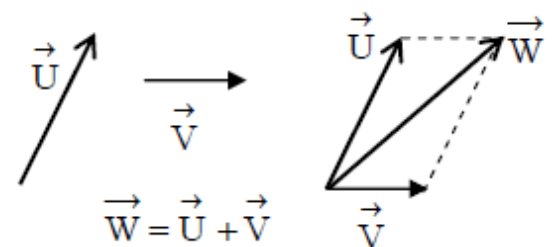


Figure I. 5

greater than two, the geometric method is applied which consists placing them end to end as shown in Figure I.6. Then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order.

**Properties:**

- ✓ Vector addition is commutative:  $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- ✓ Vector addition is associative:  $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- ✓ Distributive vector additions:  $(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}$

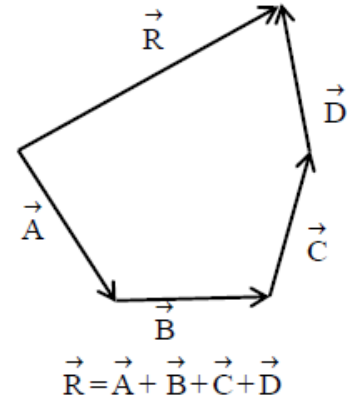


Figure I. 6

**4.2. Subtraction of Vectors**

Before going to the operation it is necessary to know about the reverse vector  $(-\vec{A})$ . A reverse vector  $-\vec{A}$  which is opposite of  $\vec{A}$  has a similar magnitude as  $\vec{A}$  but pointed in the opposite direction. First, we find the reverse vector. Then add them as the usual addition. Suppose two vectors  $\vec{U}$  and  $\vec{V}$ , the difference  $\vec{W} = (\vec{V} - \vec{U})$  can be written as:  $\vec{W} = (\vec{V}) + (-\vec{U})$ . We can then apply the parallelogram rule (figure I.7).

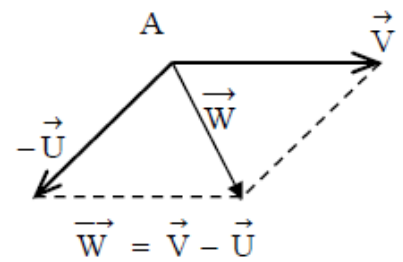


Figure I. 7

➤ **Chasles's relationship**

We suppose three points A, B and C (figure I. 8),  $\vec{AB} = \vec{AC} + \vec{CB}$

Particular case: If the three points A, B and C are aligned on the same axis, then we obtain the Chasles relation for algebraic measures:  $\vec{AB} = \vec{AC} + \vec{CB}$ .

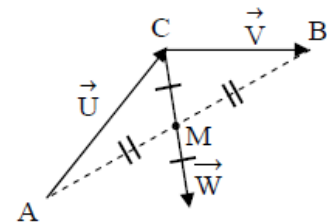


Figure I. 8

**4.3. Scalar Multiplication of Vectors**

Multiplication of a vector by a scalar quantity is called “Scaling.” In this type of multiplication, only the magnitude of a vector is changed not the direction.

The product of a vector  $\vec{V}$  by a scalar  $\alpha$  is a vector, denoted  $\alpha\vec{V}$  (figure I.9), such that:

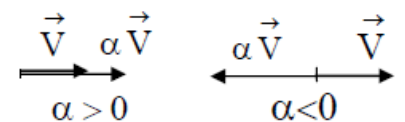


Figure I. 9

- ✓ The direction is that of  $\vec{V}$ ; if  $\alpha > 0$ , and that of  $-\vec{V}$  if  $\alpha < 0$ ;

- ✓ The modulus is equal to the product of that of  $\vec{V}$  by the absolute value of  $\alpha$ :  $|\alpha\vec{V}| = |\alpha| \cdot |\vec{V}|$

### Properties:

The multiplication of vector by a scalar verifies the following properties:

- ✓ Distributivity to the addition of vectors:  $\alpha(\vec{U} + \vec{V}) = \alpha\vec{U} + \alpha\vec{V}$
- ✓ Distributivity to the addition of scalars:  $(\alpha + \beta)\vec{U} = \alpha\vec{U} + \beta\vec{U}$
- ✓ Associativity:  $\alpha(\beta\vec{U}) = (\alpha\beta)\vec{U}$ .

## I.3.2. Cartesian coordinate systems

### 1. Two-Dimensional cartesian coordinates

A Cartesian coordinate system on a two-dimensional plane is defined by two perpendicular axes.

In other words, two lines drawn at right angles to each other on a flat surface (for example a flat sheet of paper, a thin pane of glass or the surface of a football field) provide a reference grid for every point on that surface. You might also see this type of system referred to as a *rectangular* or *orthogonal* coordinate system, because the reference axes are perpendicular.

A typical Cartesian coordinate system is defined by Ox and Oy axes. Each axis has a unit of length or distance (such as meters or miles). Any point within the coordinate system is described by a distance relative to both the x and y axes, (x,y). The axes cross at the point where the value of both x and y is zero; this is called *the origin (0,0)*. Note that the *direction* from the origin is also important, as this will determine whether the values of x and y are positive or negative.

The x-coordinate of a point is its perpendicular distance from the y-axis measured along the x-axis, and it is known as *Abcissa*.

The y-coordinate of a point is its perpendicular distance from the x-axis measured along the y-axis, and it is known as *Ordinate*.

The vectors  $\vec{i}$  and  $\vec{j}$  are determined the unit vectors along the x and y directions.

The position of a point M in (Oxy) plane is represented by the vector  $\overrightarrow{OM}$ . Let  $M_x$  and  $M_y$ , geometrically, the projections of M on the axes Ox and Oy, respectively. Notice that:

$$\overrightarrow{OM} = \overrightarrow{OM_x} + \overrightarrow{OM_y}$$

If:  $\overrightarrow{OM_x} = x \vec{i}$

$\overrightarrow{OM_y} = y \vec{j}$  then:  $\overrightarrow{OM} = x \vec{i} + y \vec{j}$

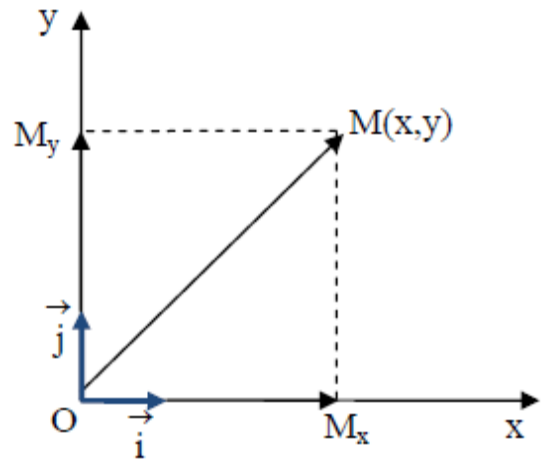


Figure I. 10

- The algebraic quantities x and y are the Cartesian coordinates of the point M in the system (O, x, y).
- The unit vectors  $\vec{i}$  and  $\vec{j}$  form an orthonormal basis (their modulus is equal to 1 and they are perpendicular each to other).

**Example:** Locate the following points on the Cartesian coordinate system. (2, 3), (-3, 1), (-1.5, -2.5), (0, 0). (Solution is shown in figure I. 11).

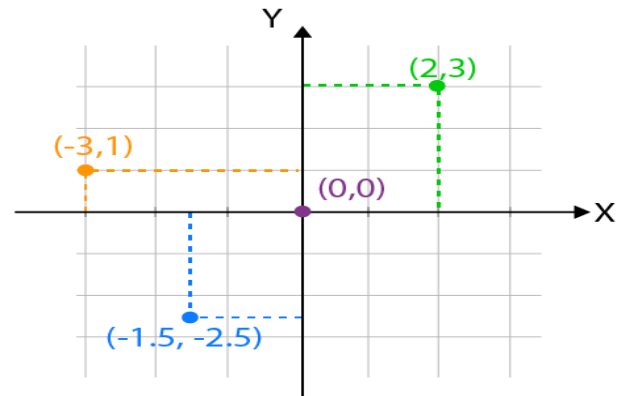


Figure I. 11

**2. Three-dimensional Cartesian system**

This system is used to identify point M in space (figure I.12). It is composed of three axes, Ox, Oy and Oz,

provided with the unit vectors  $\vec{i}$ ,  $\vec{j}$  and  $\vec{k}$  oriented positively. The position of a point M in space is given by the vector  $\overrightarrow{OM}$ . Let  $M_x$ ,  $M_y$  and  $M_z$  be the projections of M on the axes Ox, Oy and Oz, respectively.  $M'$  is the projection of M on the plane (O, x, y), note that:

$$\overrightarrow{OM} = \overrightarrow{OM_z} + \overrightarrow{OM'}$$

Note also that:

$$\overrightarrow{OM'} = \overrightarrow{OM_x} + \overrightarrow{OM_y}$$

Let:  $\overrightarrow{OM} = \overrightarrow{OM_x} + \overrightarrow{OM_y} + \overrightarrow{OM_z}$

If: 
$$\begin{cases} \overrightarrow{OM_x} = x\vec{i} \\ \overrightarrow{OM_y} = y\vec{j} \\ \overrightarrow{OM_z} = z\vec{k} \end{cases}$$

then: 
$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

- The algebraic quantities  $x$ ,  $y$  and  $z$  are the Cartesian coordinates of the point  $M$  in the system  $(O, x, y, z)$ .
- The unit vectors  $(\vec{i}, \vec{j}, \vec{k})$  form an orthonormal basis (*orthonormal basis is a basis consisting of unit-length*).
- Any point in space is represented with the coordinates  $(x, y, z)$ . The  $x$  value of the point  $(x, y, z)$  is referred to as the **abscissa**, the  $y$  value of the point is referred to as the **ordinate** and the  $z$  value is referred to as **applicate**.

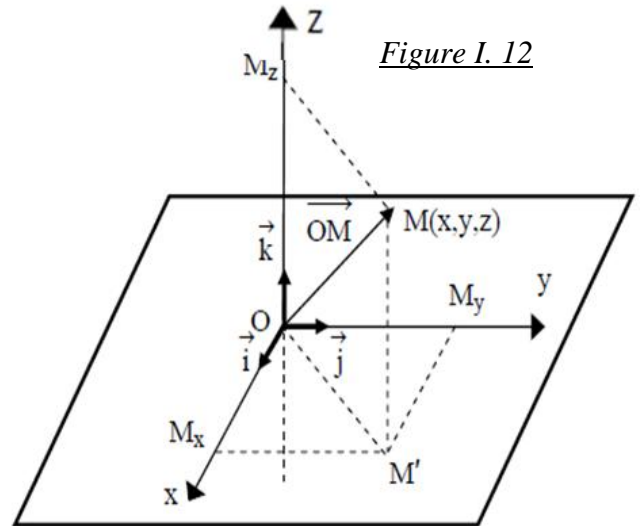


Figure I. 12

### 3. Vector Components

#### a- Project a vector onto Oxy plane (2D Cartesian coordinate)

Consider a vector lying in the  $(Oxy)$  plane and making an arbitrary angle  $\theta$  with the positive  $x$  axis as shown in Figure I. 13a. This vector can be expressed as the sum of two other component vectors,  $\vec{A}_x$  which is parallel to the  $x$  axis, and  $\vec{A}_y$ , which is parallel to the  $y$  axis. From Figure I.13b, we see that the two vectors form a right triangle and that:  $\vec{A} = \vec{A}_x + \vec{A}_y$ . We shall often refer to the **components of a vector**, written  $A_x$  and  $A_y$ . The component  $A_x$  represents the projection of along the  $x$  axis, and the component  $A_y$  represents the projection of along the  $y$  axis. These components can be positive or negative. The component  $A_x$  is positive if the component vector points in the positive  $x$  direction and is negative if points in the negative  $x$  direction. The same is true for the component  $A_y$ . From Figure I.13 and the definition of **sine** and **cosine**, we see that:

$$\cos \theta = \frac{A_x}{A}$$

$$\sin \theta = \frac{A_y}{A}$$

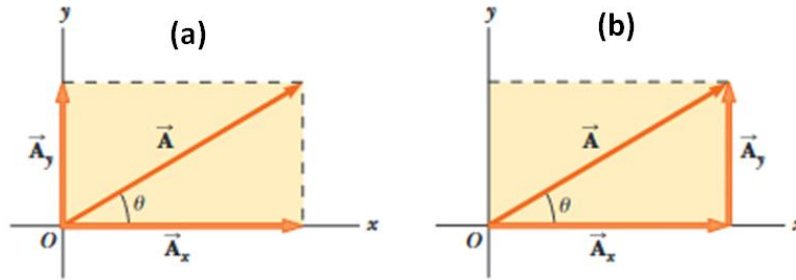


Figure I.13

Hence, the components of  $\vec{A}$  are:

$$A_x = A \cos \theta$$

$$A_y = A \sin \theta$$

The magnitudes of these components are the lengths of the two sides of a right triangle with a hypotenuse of length A. Therefore, the magnitude and direction of  $\vec{A}$  are related to its components through the expressions:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1} \left( \frac{A_y}{A_x} \right)$$

**b- Project a vector onto Oxyz space (3D Cartesian coordinate space)**

Let  $\vec{A}$  be an arbitrary vector,  $\vec{A}_x, \vec{A}_y, \vec{A}_z$  are the projections of  $\vec{A}$  on the axes of a Cartesian coordinate system (O, x, y, z), (figure I.14a). Let  $\vec{i}, \vec{j}$  and  $\vec{k}$  be unit vectors parallel to the x, y and z axis, respectively. The vector  $\vec{A}$  can be expanded in terms of these basis vectors (figure I. 14b):

$$\vec{A} = \vec{A}' + \vec{A}_z \quad \text{and} \quad \vec{A}' = \vec{A}_x + \vec{A}_y$$

Then : 
$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z \quad (\text{I. 8})$$

If in addition:

$$\vec{A}_x = x\vec{i}$$

$$\vec{A}_y = y\vec{j}$$

$$\vec{A}_z = z\vec{k}$$

Then: 
$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} \quad (\text{I.9})$$

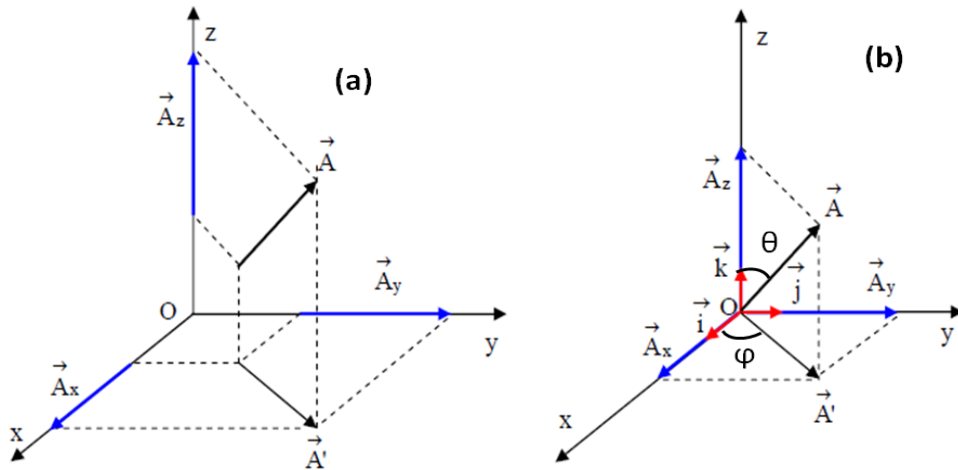


Figure I. 14

➤  $A_x, A_y$  and  $A_z$  are called **component** of the vector  $\vec{A}$ ; geometrically, they are the algebraic projections of  $\vec{A}$  along the three coordinate axes.

➤ Components can written as:  $\vec{A}(A_x, A_y, A_z)$  or  $\vec{A} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

$$A_x = A \cos \varphi \sin \theta$$

$$A_y = A \sin \varphi \sin \theta$$

$$A_z = A \cos \theta$$

➤ The modulus of  $\vec{A}$  is given by:  $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$  (I. 10)

**Notes:**

1. Let M ( $x_M, y_M, z_M$ ) and N ( $x_N, y_N, z_N$ ) be points in (O, x, y, z) system:

✓ The components of the vector  $\vec{MN}$  are obtained by writing:

$$\vec{MN} = \vec{MO} + \vec{ON} = \vec{ON} - \vec{OM} = (x_N - x_M)\vec{i} + (y_N - y_M)\vec{j} + (z_N - z_M)\vec{k} \quad (\text{I. 11})$$

Either:  $\vec{MN}((x_N - x_M), (y_N - y_M), (z_N - z_M))$

✓ The magnitude (modulus) of  $\vec{MN}$  is defined by:

$$|\vec{MN}| = \sqrt{(x_N - x_M)^2 + (y_N - y_M)^2 + (z_N - z_M)^2} \quad (\text{I.12})$$

✓ the midpoint of MN has the coordinates:

$$\left( \frac{x_M + x_N}{2}, \frac{y_M + y_N}{2}, \frac{z_M + z_N}{2} \right)$$

2. We suppose the vectors :  $\vec{A}(A_x, A_y, A_z)$  and  $\vec{B}(B_x, B_y, B_z)$  :

✓ If  $\vec{C} = \vec{A} + \vec{B}$ , to add vectors, add like components, the components of  $\vec{C}$  are

$$\text{given so: } \vec{C} \begin{pmatrix} C_x = A_x + B_x \\ C_y = A_y + B_y \\ C_z = A_z + B_z \end{pmatrix}$$

✓ If  $\vec{D} = \vec{A} - \vec{B}$ , the components of  $\vec{D}$  are given by:  $\vec{D} \begin{pmatrix} D_x = A_x - B_x \\ D_y = A_y - B_y \\ D_z = A_z - B_z \end{pmatrix}$

✓ To multiply by a scalar, multiply each component:

$$\alpha \vec{A} = \alpha A_x \vec{i} + \alpha A_y \vec{j} + \alpha A_z \vec{k}$$

**Example:** Vector  $\vec{V}(2, 2, 1)$ . Calculate the magnitude of  $\vec{V}$ . Find the components of the unit vector  $\vec{u}$ .

$$|\vec{V}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\vec{V} = |\vec{V}| \vec{u} \Rightarrow \vec{u} = \frac{\vec{V}}{|\vec{V}|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

unit vector in the direction of  $\vec{V}$  is  $\vec{u}(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

### I.3.3. Scalar product (dot product)

#### 1. Definition

The scalar product is a method for multiplying two vectors. Because the product of the multiplication is a scalar, the dot product is sometimes referred to as the scalar product. The dot product will be used to find an angle between two vectors and will have applications in finding distances between points and lines, points and planes, etc.

The scalar product of two vectors  $\vec{U}$  and  $\vec{V}$ , denoted  $\vec{U} \cdot \vec{V}$  is a scalar defined by:

$$\vec{U} \cdot \vec{V} = |\vec{U}| \cdot |\vec{V}| \cdot \cos \theta \quad (\text{I.13})$$

Where  $\theta$  is the angle between  $\vec{U}$  and  $\vec{V}$ .

**Note:** The scalar product is therefore positive for  $\theta$  acute, negative for  $\theta$  obtuse (figure I. 15)

- $\vec{U} \cdot \vec{V} > 0 \Rightarrow \cos \theta > 0$  if  $\theta$  is an **acute** angle
- $\vec{U} \cdot \vec{V} = 0 \Rightarrow \cos \theta = 0$  if  $\theta$  is a **right** angle
- $\vec{U} \cdot \vec{V} < 0 \Rightarrow \cos \theta < 0$  if  $\theta$  is an **obtuse** angle

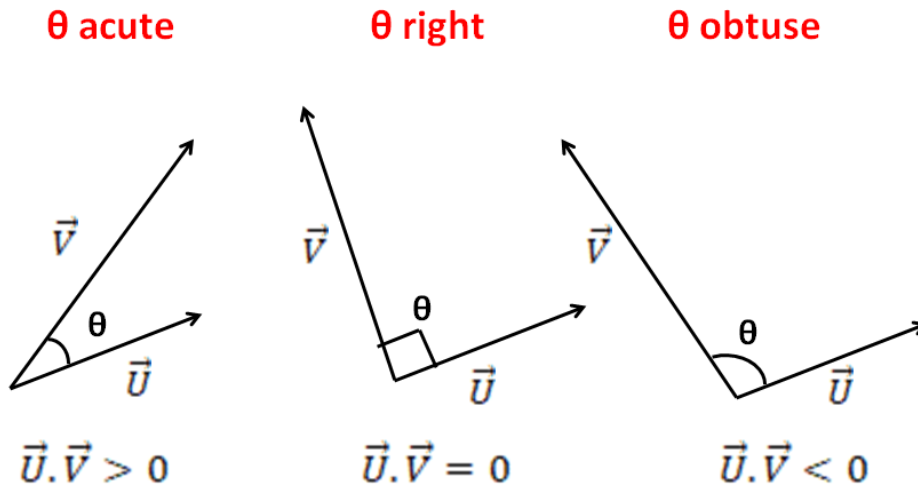


Figure I. 15

**2. Geometric form**

By definition of the scalar product  $\vec{U} \cdot \vec{V}$  :

$$\vec{U} \cdot \vec{V} = |\vec{U}| \cdot |\vec{V}| \cdot \cos \theta = |\vec{U}| \cdot |\vec{V}_U|$$

$\vec{V}_U$  is the algebraic projection of  $\vec{V}$  on  $\vec{U}$  (figure I.16).

The dot product of two vectors is equal to the product of the modulus of one vector by the measure algebraic of the projection of the other on its support line.

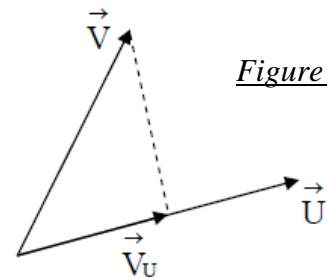


Figure I. 16

**3. Analytical form**

Let  $U_x, U_y, U_z$  and  $V_x, V_y, V_z$  be the components of  $\vec{U}$  and  $\vec{V}$  vectors respectively, in the orthonormal basis  $(\vec{i}, \vec{j}, \vec{k})$ . The scalar product of these two vectors is defined by the relation:

$$\begin{aligned} \vec{U} \cdot \vec{V} &= (U_x \vec{i} + U_y \vec{j} + U_z \vec{k}) \cdot (V_x \vec{i} + V_y \vec{j} + V_z \vec{k}) \\ \vec{U} \cdot \vec{V} &= U_x V_x + U_y V_y + U_z V_z \end{aligned} \tag{I.14}$$

Because  $\vec{i}, \vec{j}, \vec{k}$  are mutually perpendicular unit vectors:

$$\begin{aligned} \vec{i} \cdot \vec{i} &= \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \\ \vec{i} \cdot \vec{j} &= \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \end{aligned}$$

**Notes:**

- ✓ To find the dot product of two vectors, multiply the corresponding components together and add them up.
- ✓ Two vectors are perpendicular if their dot product is equal to zero. This means that they meet at right angles.

**Properties:**

- Scalar product is commutative:  $\vec{U} \cdot \vec{V} = \vec{V} \cdot \vec{U}$
- Distributive to the addition of vectors:  $\vec{W} \cdot (\vec{U} + \vec{V}) = \vec{W} \cdot \vec{U} + \vec{W} \cdot \vec{V}$
- $(\alpha\vec{U}) \cdot (\beta\vec{V}) = (\alpha\beta)(\vec{U} \cdot \vec{V})$ , ( $\alpha$  and  $\beta$  are scalars).
- $\vec{U} \cdot \vec{0} = 0$

**4. Orthogonal vectors**

$\vec{U}$  and  $\vec{V}$  are said to be orthogonal or perpendicular to each other if they meet at right angles.

$$\vec{U} \perp \vec{V} \Rightarrow \cos(\vec{U}, \vec{V}) = 0 \Rightarrow \vec{U} \cdot \vec{V} = 0$$

$$\vec{U} \perp \vec{V} \Leftrightarrow U_x V_x + U_y V_y + U_z V_z = 0$$

**5. Applications of the Dot Product in Geometry**

- Determination of the cosine of the angle between two vectors:  $\vec{U}(U_x, U_y, U_z)$  and  $\vec{V}(V_x, V_y, V_z)$ . By applying dot product:

$$\cos(\vec{U}, \vec{V}) = \frac{\vec{U} \cdot \vec{V}}{|\vec{U}| \cdot |\vec{V}|} = \frac{U_x V_x + U_y V_y + U_z V_z}{\sqrt{U_x^2 + U_y^2 + U_z^2} \cdot \sqrt{V_x^2 + V_y^2 + V_z^2}} \quad (I.15)$$

- Metric relations in a triangle:

$$\vec{BC} = \vec{BA} + \vec{AC}$$

Where:  $\vec{BC}^2 = (\vec{BA} + \vec{AC})^2$

$$\vec{BC}^2 = \vec{BA}^2 + \vec{AC}^2 + 2\vec{BA} \cdot \vec{AC}$$

Either:  $c^2 = a^2 + b^2 - 2a \cdot b \cdot \cos \theta$  (I.16)

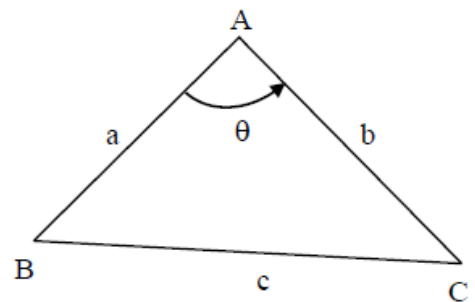


Figure I. 17

**Note:** in the case of a right triangle at A, we find the

Pythagorean Theorem:  $c^2 = a^2 + b^2$ , the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.

**I.3.4. Vector product (or cross product)**

**1. Definition**

The vector product of two vectors  $\vec{U}$  and  $\vec{V}$ , is a vector  $\vec{W}$ , denoted:

$$\vec{W} = \vec{U} \wedge \vec{V}$$

- The direction of  $\vec{W}$  is perpendicular to the plane containing  $\vec{U}$  and  $\vec{V}$  with its orientation given by the right-hand rule (figure I.20).
- of modulus:  $|\vec{W}| = |\vec{U}| \cdot |\vec{V}| \cdot \sin(\vec{U}, \vec{V})$  (I.17)

$|\vec{W}|$  measures the area of the parallelogram OABC (figure I. 18) from the adjoining diagram  $\vec{OA}$  and  $\vec{OB}$  of the vectors  $\vec{U}$  and  $\vec{V}$ . The area of a parallelogram is length of base  $\times$  height. Indeed H being the projection of A on OB, we have:

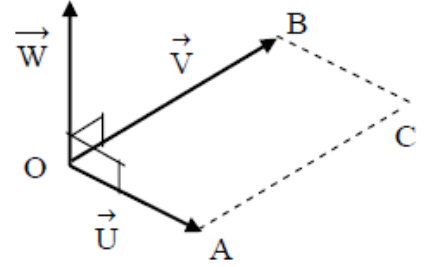


Figure I. 18

$$AH = OA \cdot \sin(\vec{U}, \vec{V}) = |\vec{U}| \cdot \sin(\vec{U}, \vec{V}) \quad (I.18)$$

Therefore the area is given by:

$$OB \cdot AH = |\vec{U}| \cdot |\vec{V}| \cdot \sin(\vec{U}, \vec{V}) = |\vec{W}| \quad (I.19)$$

**Note:**

$$\vec{i} \wedge \vec{j} = \vec{k}, \quad \vec{j} \wedge \vec{k} = \vec{i}, \quad \vec{k} \wedge \vec{i} = \vec{j} \quad ; \quad \vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = 0$$

**2. Analytical form**

We suppose  $U_x, U_y, U_z$  and  $V_x, V_y, V_z$  components of  $\vec{U}$  and  $\vec{V}$  in the orthonormal basis  $\vec{i}, \vec{j}, \vec{k}$ . The vector product of  $\vec{U}$  and  $\vec{V}$  can be obtained from the components of these vectors by:

$$\begin{aligned} \vec{W} = \vec{U} \wedge \vec{V} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ U_x & U_y & U_z \\ V_x & V_y & V_z \end{vmatrix} \\ &= \left( \underbrace{U_y V_z - U_z V_y}_{W_x} \right) \vec{i} + \left( \underbrace{U_z V_x - U_x V_z}_{W_y} \right) \vec{j} + \left( \underbrace{U_x V_y - U_y V_x}_{W_z} \right) \vec{k} \quad (I.20) \end{aligned}$$

To obtain the components of the vector product, a classical difference of the “cross products” of the components is carried out for  $W_x, W_y$  and  $W_z$  are successively deduced by performing a circular permutation on the indices:  $x \rightarrow y \rightarrow z \rightarrow x$ .

**Properties:**

- The vector product is anti-commutative:  $\vec{U} \wedge \vec{V} \neq \vec{V} \wedge \vec{U}, \vec{U} \wedge \vec{V} = -\vec{V} \wedge \vec{U}$ .
- Distributive to the addition vectors:  $\vec{W} \wedge (\vec{U} + \vec{V}) = \vec{W} \wedge \vec{U} + \vec{W} \wedge \vec{V}$ .
- $(\alpha \vec{U}) \wedge (\beta \vec{V}) = (\alpha \beta)(\vec{U} \wedge \vec{V})$ , ( $\alpha$  and  $\beta$  are scalars).
- $\vec{U} \parallel \vec{V} \Rightarrow |\vec{U}| \cdot |\vec{V}| \cdot \sin(\vec{U}, \vec{V}) = 0$  (I.21)

where :  $\vec{U} \wedge \vec{V} = 0$  (I.22)

### 3. Applications of the cross product in geometry

- We know that the area of the parallelogram (ABCD) is given by:  $|\vec{AB} \wedge \vec{AD}|$ ; the area of the triangle (ABC) is therefore equal to:  $\frac{1}{2} |\vec{AB} \wedge \vec{AD}|$ ; the area of the triangle is one half the area of the parallelogram.
- Cartesian equation of a straight line (D) passing through two points A and B of xOy plane: If a point  $M \in (D)$  Then  $\vec{AM} \wedge \vec{AB} = 0$

### 4. Applications of the cross product in physics

The moment of a vector about point O (figure I.19) is defined by the relationship:

$$\mathcal{M}_{(\vec{AB}/O)} = \vec{OA} \wedge \vec{AB} \quad (I.23)$$

Its modulus is:  $|\mathcal{M}_{(\vec{AB}/O)}| = |\vec{OA}| \cdot |\vec{AB}| \cdot \sin(\vec{OA}, \vec{AB})$

$$|\mathcal{M}_{(\vec{AB}/O)}| = \underbrace{|\vec{OA}| \cdot \sin(\vec{OA}, \vec{AB})}_{OH} \cdot |\vec{AB}|$$

Either:  $|\mathcal{M}_{(\vec{AB}/O)}| = OH \cdot |\vec{AB}| \quad (I.24)$

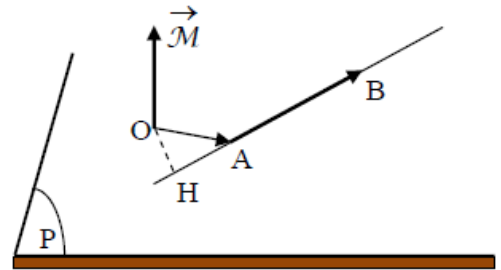


Figure I. 19

### 5. Orientation “right-hand rule”

The orientation of the cross product is given by the right-hand rule, “Rule of the three fingers”. (Figure I.17). We associate the basic vectors  $(\vec{i}, \vec{j}, \vec{k})$  with the axes of a rectangular trihedron formed by the three fingers of the right hand. To form a direct trihedron, we orient:

- ➡  $\vec{i}$  in the direction of the thumb.
- ➡  $\vec{j}$  in the direction of the index finger.
- ➡  $\vec{k}$  in the middle finger direction.

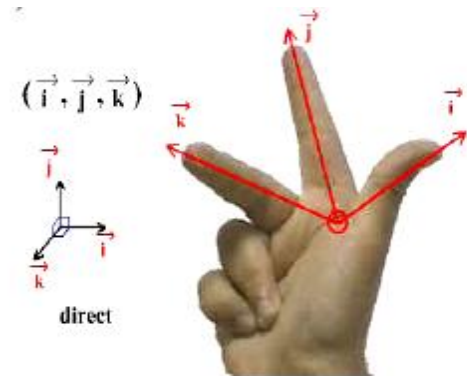


Figure I. 20

#### I.3.5. Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a triple product.

##### 1. Scalar triple product:

Let  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  be three vectors: The scalar triple product of these three vectors is defined as:

$$a = \vec{A} \cdot (\vec{B} \wedge \vec{C})$$

This is the dot product of one vector with the cross product of the other two vectors. The order of the vectors is important.

Supposed  $A_x, A_y, A_z, B_x, B_y, B_z$  and  $C_x, C_y, C_z$  as the respective components of  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  vectors in the orthonormal basis  $(\vec{i}, \vec{j}, \vec{k})$ , the scalar triple product of these three vectors is the scalar defined by the relation:

$$\begin{aligned}
 a = \vec{A} \cdot (\vec{B} \wedge \vec{C}) &= (A_x\vec{i} + A_y\vec{j} + A_z\vec{k}) \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
 &= A_x \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} + A_y \begin{vmatrix} B_z & B_x \\ C_z & C_x \end{vmatrix} + A_z \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix} \quad (I.25)
 \end{aligned}$$

➤ **Geometric interpretation**

Let's consider the vector  $\vec{V} = (\vec{B} \wedge \vec{C})$  whose magnitude is equal to the area of the parallelogram constructed on the representatives of  $\vec{B}$  and  $\vec{C}$ .

Notice that  $\vec{A} \cdot \vec{V} = OH \cdot |\vec{V}|$ , where OH is the projection of  $\vec{A}$  onto the support of  $\vec{V}$ . As a result, the absolute value of the mixed product  $a = \vec{A} \cdot (\vec{B} \wedge \vec{C})$  measures **the volume** of the parallelepiped (Figure I.21).

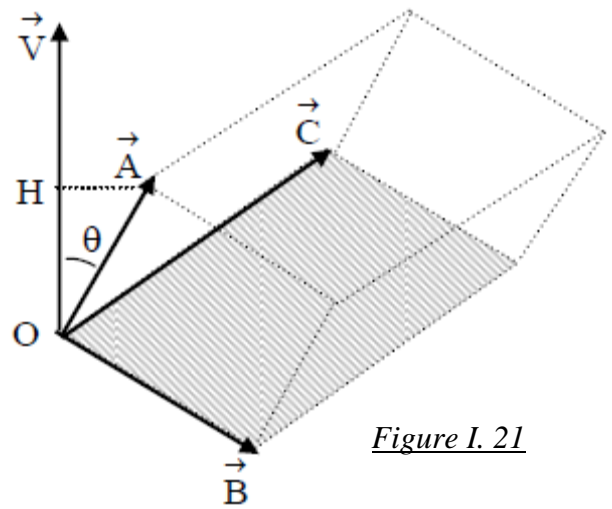


Figure I. 21

**Note:**

A circular permutation of vectors does not change the value of the scalar triple product:

$$a = \vec{A} \cdot (\vec{B} \wedge \vec{C}) = \vec{C} \cdot (\vec{A} \wedge \vec{B}) = \vec{B} \cdot (\vec{C} \wedge \vec{A}) \quad (I. 26)$$

➤ **Application of the Scalar triple product in physics**

The moment of a force vector  $\vec{F}$ , applied at point M in space, with respect to an axis (Oz) defined by the unit vector  $\vec{k}$  (Figure I.22), is given by the following relationship:

$$\mathcal{M}_{(\vec{F}/OZ)} = \vec{k} \cdot (\overrightarrow{OM} \wedge \vec{F})$$

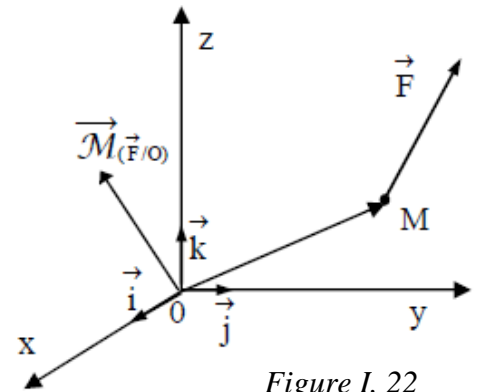


Figure I. 22

**2. Vector triple product**

The vector triple product involves taking the cross product of one vector with the cross product of two other vectors. It can be simplified by the so-called **BAC-CAB** rule:

$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

- ✓  $\vec{A} \cdot \vec{B}$  represents the dot product of vectors A and B.
- ✓  $\vec{A} \cdot \vec{C}$  represents the dot product of vectors A and C.

- ✓ The result of  $\vec{A} \wedge (\vec{B} \wedge \vec{C})$  is a vector.

### I.3.6. Derivative of vectors

The derivative of vectors or vector-valued functions can be defined similarly to the way we define the derivative of real-valued functions. Let's say we have the vector-valued function,

$$\vec{V}(t) = V_x(t)\vec{i} + V_y(t)\vec{j} + V_z(t)\vec{k}$$

we can define its derivative by the expression shown below:

$$\frac{d\vec{V}(t)}{dt} = \vec{V}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{V}(t+\Delta t) - \vec{V}(t)}{t+\Delta t - t} = V_x'(t)\vec{i} + V_y'(t)\vec{j} + V_z'(t)\vec{k} \quad (\text{I.27})$$

#### Notes:

Here are some other terms that you might encounter when dealing with the derivatives of vectors:

- The tangent line to the curve, that passes through a point, can be defined by the tangent vector,  $\vec{V}'(t) = \frac{d\vec{V}(t)}{dt}$ .
- We can calculate the **unit tangent vector** by dividing the derivative of  $\vec{V}'(t)$  by the absolute value of  $|\vec{V}'(t)|$ .

$$\vec{T}(t) = \frac{\vec{V}'(t)}{|\vec{V}'(t)|}$$

- Write down the derivative of vector functions by writing down the derivative of the components in form of:

$$\begin{aligned} \vec{V}'(t) &= V_x'(t)\vec{i} + V_y'(t)\vec{j} + V_z'(t)\vec{k} = \\ &= \frac{dV_x(t)}{dt}\vec{i} + \frac{dV_y(t)}{dt}\vec{j} + \frac{dV_z(t)}{dt}\vec{k} \end{aligned}$$

Now that we know the formal and practical definition of vector derivatives, let's break down the process of calculating the derivatives of different vector-valued functions.

Suppose that  $\vec{U}(t)$  and  $\vec{V}(t)$  are differentiable vector-valued functions,  $k$  is a scalar value, and  $f(t)$  is a real-valued function, then we have the following derivative rules:

#### ➤ **Scalar Multiple :**

$$\frac{d}{dt}(k\vec{U}(t)) = k \frac{d\vec{U}(t)}{dt} = k\vec{U}'(t)$$

➤ **Sum and Difference Vectors :**

$$\frac{d}{dt}(\vec{U}(t) + \vec{V}(t)) = \frac{d\vec{U}(t)}{dt} + \frac{d\vec{V}(t)}{dt} = \vec{U}'(t) + \vec{V}'(t)$$

$$\frac{d}{dt}(\vec{U}(t) - \vec{V}(t)) = \frac{d\vec{U}(t)}{dt} - \frac{d\vec{V}(t)}{dt} = \vec{U}'(t) - \vec{V}'(t)$$

➤ **Function Scalar and vector Product**

$$\frac{d}{dt}(f(t)\vec{U}(t)) = \frac{df(t)}{dt}\vec{U}(t) + f(t)\frac{d\vec{U}(t)}{dt} = f'(t)\vec{U}(t) + f(t)\vec{U}'(t)$$

➤ **Dot Product**

$$\frac{d}{dt}(\vec{U}(t) \cdot \vec{V}(t)) = \frac{d\vec{U}(t)}{dt} \cdot \vec{V}(t) + \vec{U}(t) \cdot \frac{d\vec{V}(t)}{dt} = \vec{U}'(t) \cdot \vec{V}(t) + \vec{U}(t) \cdot \vec{V}'(t)$$

➤ **Cross Product**

$$\frac{d}{dt}(\vec{U}(t) \wedge \vec{V}(t)) = \frac{d\vec{U}(t)}{dt} \wedge \vec{V}(t) + \vec{U}(t) \wedge \frac{d\vec{V}(t)}{dt} = \vec{U}'(t) \wedge \vec{V}(t) + \vec{U}(t) \wedge \vec{V}'(t)$$

➤ **Chain Rule (vector as function)**

$$\frac{d}{dt}(\vec{U}(f(t))) = \frac{d\vec{U}(f(t))}{df} \cdot \frac{df(t)}{dt} = \vec{U}'(f(t)) \cdot f'(t)$$

**Example:**

Find the derivative of vector:  $\vec{U}(t) = 2 \cos \theta \vec{i} + \sin \theta \vec{j}$ ,  $\theta(t) = t^2$

$$\frac{d\vec{U}(t)}{dt} = \frac{d\vec{U}(t)}{d\theta} \cdot \frac{d\theta}{dt} = (-2 \sin \theta \vec{i} + \cos \theta \vec{j}) \cdot 2t = -4t \sin \theta \vec{i} + 2t \cos \theta \vec{j}$$

**Note:** When  $\vec{U}(t) \cdot \vec{U}(t) = k$ , then we have  $\vec{U}(t) \cdot \vec{U}'(t) = 0$

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**EXERCISES**


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**A- Dimensional Analysis-Incertainties****Exercise 1:**

- Find the dimensions of the following physical quantities: charge  $Q$ , surface density  $\sigma$ , electric field  $E$ , electric potential  $V$ , electrical resistance  $R$ , resistivity  $\rho$ , capacitance  $C$ . Verify that  $RC$  is homogeneous with the time dimension.

- Find also dimensions of: force  $F$ , acceleration  $a$ , energy  $W$ , pressure  $P$ , and the constant  $k$  of the spring, where  $F=kx$ .

**Exercise 2:**

The resistance  $R$  for a wire of length  $L$  and cross-sectional area  $A$  is given by the formula  $R=\rho L/A$ . Find the dimensional equation for the physical quantity  $\rho$ .

**Exercise 3:**

The gravitational force between two objects of masses  $M$  and  $m$  separated by a distance  $r$  is known as the inverse square law and is expressed by:  $F = G \frac{mM}{r^2}$ , where  $G$  is the proportionality constant. If the force  $F$  has the SI units of  $\text{Kg} \cdot \text{m/s}^2$ , what are the units of  $G$ . ( $G$  is called the Universal Constant of Gravity.)

**Exercise 4:**

It was shown by experiment that the force acting on a ball immersed in a fluid is related to the viscosity coefficient  $\eta$ , the radius of the ball  $r$ , and the ball velocity  $v$ . Find the expression for this force, it is given by:  $F = K \eta^x r^y v^z$ , where  $K$  is a dimensionless constant and  $[\eta] = \text{ML}^{-1}\text{T}^{-1}$

**Exercise 5:**

The period of a simple pendulum is given by the formula:  $T = 2\pi \sqrt{\frac{l}{g}}$ , where  $l$  is the length of the pendulum and  $g$  is the free-fall acceleration. Find the expression of the relative uncertainty using the differentiation and differentiation logarithmic methods. Deduce the absolute uncertainty  $\Delta T$ .

**Exercise 6:**

The resistances  $R_1 = 2200 \Omega$  and  $R_2 = 120 \Omega$  were associated in parallel with an uncertainty of 10%. Calculate the equivalent resistance and deduce its relative uncertainty.

**Exercise 7:**

- A student is trying to remember some formulas from geometry. In what follows, assume  $A$  is area,  $V$  is volume, and all other variables are lengths. Determine which formulas are dimensionally consistent. (a)  $V = \pi r^2 h$ ; (b)  $A = 2\pi r^2 + 2\pi r h$ ; (c)  $V = 0.5bh$ ; (d)  $V = \pi d^2$ ; (e)  $V = \pi d^3/6$ .

- Consider the physical quantities  $s$ ,  $v$ ,  $a$ , and  $t$  with dimensions  $[s] = L$ ,  $[v] = LT^{-1}$ ,  $[a] = LT^{-2}$ , and  $[t] = T$ . Determine whether each of the following equations is dimensionally consistent. (a)  $v^2 = 2as$ ; (b)  $s = vt^2 + 0.5at^2$ ; (c)  $v = s/t$ ; (d)  $a = v/t$ .

- Consider the physical quantities  $m$ ,  $s$ ,  $v$ ,  $a$ , and  $t$  with dimensions  $[m]=M$ ,  $[s]=L$ ,  $[v] = LT^{-1}$ ,  $[a] = LT^{-2}$ , and  $[t] = T$ . Assuming each of the following equations is dimensionally consistent, find the dimension of the quantity on the left-hand side of the equation: (a)  $F = ma$ ; (b)  $K = 0.5mv^2$ ; (c)  $p = mv$ ; (d)  $W = mas$ ; (e)  $L = mvr$ .

**Exercise 8:**

Find the dimensions of the constants  $a$ ,  $b$  and  $c$  in the equation,  $x = a + bt^2 + ct^3$  where  $x$  is the position of a particle (in meters) and  $t$  is the time (in seconds).

- The volume of an object as a function of time is calculated by  $V = At^3 + B/t$ , where  $t$  is time measured in seconds and  $V$  is in cubic meters. Determine the dimension of the constants  $A$  and  $B$ .

- The velocity  $v$  of a particle depends upon the time 't' according to the equation:

$v = \sqrt{ab} + bt + \frac{c}{d+t}$ , Determine the units of  $a$ ,  $b$ ,  $c$  and  $d$ . What physical quantities they represent. All have SI units.

- Find the exponents  $n$  and  $m$  in the formula,  $a \propto r^n \cdot v^m$  where  $a$  is the acceleration of a particle moving in a circle of radius  $r$  with a constant speed  $v$ .

**Exercise 9:**

In quantum mechanics, the fundamental constant called Planck's constant,  $h$ , has dimensions of  $[ML^2T^{-1}]$ . Construct a quantity with the dimensions of length from  $h$ , a mass  $m$ , and  $c$ , the speed of light.

**Exercise 10:**

Percentage error in the measurement of height and radius of cylinder are and respectively. Find the percentage error in the measurement of volume. Which of the two measurements height or radius needs more attention?

**SOLUTION****Exercise 1:**

Dimension of physical quantities :

- Electric charge:  $Q = \text{current} \times \text{time} = i.t \rightarrow [Q] = [i].[t] = I.T$

- Surface density  $\sigma$ :  $\sigma = \text{charge/surface} = Q/S \rightarrow [\sigma] = [Q]/[S] = I.T/L^2 = I.T.L^{-2}$

- Electric field  $E$ :  $E = \text{force/charge} = F/Q \rightarrow [E] = [F]/[Q]$ ,

$F = m.a \rightarrow [F] = [m].[a]$ ,  $[a] = [v]/[t] = L.T^{-2} \rightarrow [F] = M.L.T^{-2} \rightarrow [E] = [F]/[Q] = (M.L.T^{-2})/I.T$   
 $[E] = M^1 L^1 T^{-3} I^{-1}$

- Electric potential:  $V = \text{Electric field} \times \text{distance} = E.l$  or  $V = \text{work/charge} = W/Q$

$[V] = [E].[l] = M^1.L^1 T^{-3} I^{-1}.L = M^1 L^2 T^{-3} I^{-1}$

- Electric Resistance:  $R = \text{Potential/current} = V/I \rightarrow [R] = [V]/[i] = (M^1 L^2 T^{-3} I^{-1})/I$

$[R] = M^1 L^2 T^{-3} I^{-2}$

- Resistivity  $\rho$ :  $\rho = (\text{Resistance} \times \text{surface})/\text{length} = R.S/l \rightarrow [\rho] = [R].[S]/[l] = (M^1 L^2 T^{-3} I^{-2}).L^2/L$

$[\rho] = M^1 L^3 T^{-3} I^{-2}$

- Capacitance  $C$ :  $C = \text{Charge/Potential} = Q/V \rightarrow [C] = [Q]/[V] = I.T/M^1 L^2 T^{-3} I^{-1}$

$[C] = M^{-1} L^{-2} T^4 I^2$

- Verify that  $RC$  is homogeneous with the time dimension:  $[R.C] = [R].[C] = T ?$

$[R].[C] = (M^1 L^2 T^{-3} I^{-2}) . (M^{-1} L^{-2} T^4 I^2) = T$

Therefore,  $RC$  is homogeneous with the time dimension.

▪ **dimensions of:**

- Acceleration  $a$ :  $[a] = [v]/[t] = L.T^{-2}$

- Force  $F$ :  $[F] = [m].[a] = [F] = M.L.T^{-2}$

- Energy  $W$ :  $W = \text{Force} \times \text{distance} = F.l$ ,  $[W] = [F].[l] = M.L.T^{-2}.L = M^1 L^2 T^{-2}$

- Pressure  $P$ :  $P = \text{Force/area} = F/S \rightarrow [P] = [F]/[S] = M.L.T^{-2}/L^2 = M^1 L^{-1} T^{-2}$

- The constant  $k$  of the spring, where:  $F = kx \rightarrow [k] = [F]/[x] = M.L.T^{-2}/L = M^1 T^{-2}$

**Exercise 2:**

Dimensional equation for the physical quantity  $\rho$  (Resistivity  $\rho$ )

$\rho = (\text{Resistance} \times \text{surface})/\text{length} = R.A/l$

$[\rho] = [R].[A]/[l] = (M^1 L^2 T^{-3} I^{-2}).L^2/L$

$[\rho] = M^1 L^3 T^{-3} I^{-2}$

**Exercise 3:**

$$F = G \frac{mM}{r^2}$$

if we denote the dimensions of  $F$  by  $[F]$  (and the same for the other quantities) then dimensions of the quantities in Newton's Law are:

$$[M] = M \text{ (mass)} \quad [m] = M, \quad [r] = L, \quad [F] = ML.T^{-2}$$

What we don't know (yet) is  $[G]$ , the dimensions of  $G$ . Putting the known dimensions into Newton's Law, we must have:

$$ML/T^2 = [G] M.M/L^2$$

since the dimensions must be the same on both sides. Doing some algebra with the dimensions, this gives:  $[G] = (ML/T^2)/(L^2M^2) = L^3/MT^2$

so the dimensions of  $G$  are  $L^3/(MT^2)$ .  $[G]=L^3/MT^2$  (in terms of the standards  $L$ ,  $M$  and  $T$ )

In the SI system,  $G$  has units of  $m^3/kg \cdot s^2$

#### Exercise 4:

$$F = k \eta^x r^y v^z$$

$$[\eta] = ML^{-1}T^{-1}, \quad [r] = L, \quad [v] = LT^{-1}$$

$$\text{Dimensions of force: } [F] = ML.T^{-2} \quad (1)$$

$$[F] = [K] [\eta]^x [r]^y [v]^z = (ML^{-1}T^{-1})^x \cdot L^y \cdot (LT^{-1})^z \quad (2)$$

$$(1)=(2) \rightarrow F = M^x \cdot L^{-x+y+z} \cdot T^{-x-z} = M \cdot L \cdot T^{-2}$$

By identification:

$$\begin{cases} x = 1 \\ -x + y + z = 1 \Rightarrow y = 1 \\ -x - z = -2 \Rightarrow z = 1 \end{cases}$$

So we deduce the expression for this force as:

$$F = k \eta \cdot r \cdot v \rightarrow F = 6\pi \cdot \eta \cdot r \cdot v \rightarrow \text{called Stoks Force}$$

#### Exercise 5:

Use the two methods differentiation and differentiation logarithmic methods to calculate **relative uncertainty** on the period  $T$  of a simple pendulum.

$$T = 2\pi \sqrt{\frac{l}{g}} = 2\pi \cdot l^{1/2} \cdot g^{-1/2}$$

#### 1. partial differentiation method

$$dT = \frac{\partial T}{\partial l} dl + \frac{\partial T}{\partial g} dg = 2\pi \left( \frac{1}{2} l^{-\frac{1}{2}} \cdot g^{-\frac{1}{2}} dl - \frac{1}{2} l^{\frac{1}{2}} \cdot g^{-\frac{3}{2}} dg \right)$$

$$\frac{dT}{T} = \frac{1}{2} \frac{l^{-\frac{1}{2}} \cdot g^{-\frac{1}{2}}}{l^{1/2} \cdot g^{-1/2}} dl - \frac{1}{2} \frac{l^{\frac{1}{2}} \cdot g^{-\frac{3}{2}}}{l^{1/2} \cdot g^{-1/2}} dg = \frac{1}{2} \frac{dl}{l} - \frac{1}{2} \frac{dg}{g}$$

$$\frac{\Delta T}{T} = \frac{1}{2} \frac{\Delta l}{l} + \frac{1}{2} \frac{\Delta g}{g}$$

## 2. logarithmic differentiation method

$$\log T = \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\frac{dT}{T} = \frac{1}{2} \frac{dl}{l} - \frac{1}{2} \frac{dg}{g}$$

$$\frac{\Delta T}{T} = \frac{1}{2} \frac{\Delta l}{l} + \frac{1}{2} \frac{\Delta g}{g}$$

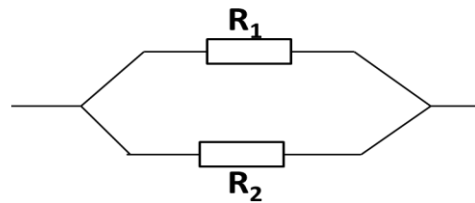
Deduce the absolute uncertainty  $\Delta T$ :  $\Delta T = T \left( \frac{1}{2} \frac{\Delta l}{l} + \frac{1}{2} \frac{\Delta g}{g} \right)$

### Exercise 6:

$R_1 = 2200\Omega$ ,  $R_2 = 120\Omega$

- Calculate the equivalent resistance:

$$\begin{aligned} \frac{1}{R_{eq}} &= \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 \cdot R_2} \Rightarrow R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2} \\ &= \frac{2200 \cdot 120}{2200 + 120} = 113.79\Omega \end{aligned}$$



- Calculate the relative uncertainty:

$$R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2} \Rightarrow \ln R_{eq} = \ln \left( \frac{R_1 \cdot R_2}{R_1 + R_2} \right)$$

$$\ln R_{eq} = \ln R_1 + \ln R_2 - \ln(R_1 + R_2)$$

$$\frac{dR_{eq}}{R_{eq}} = \frac{dR_1}{R_1} + \frac{dR_2}{R_2} - \frac{dR_1}{R_1 + R_2} - \frac{dR_2}{R_1 + R_2}$$

$$\frac{dR_{eq}}{R_{eq}} = dR_1 \left( \frac{1}{R_1} - \frac{1}{R_1 + R_2} \right) + dR_2 \left( \frac{1}{R_2} - \frac{1}{R_1 + R_2} \right)$$

$$\frac{dR_{eq}}{R_{eq}} = dR_1 \frac{R_2}{R_1(R_1 + R_2)} + dR_2 \frac{R_1}{R_2(R_1 + R_2)}$$

$$\frac{\Delta R_{eq}}{R_{eq}} = \frac{\Delta R_1}{R_1} \cdot \frac{R_2}{(R_1 + R_2)} + \frac{\Delta R_2}{R_2} \cdot \frac{R_1}{(R_1 + R_2)}$$

$$\frac{\Delta R_1}{R_1} = 10\% \quad , \quad \frac{\Delta R_2}{R_2} = 10\%$$

$$\frac{\Delta R_{eq}}{R_{eq}} = 10\% \cdot \frac{R_2}{(R_1 + R_2)} + 10\% \cdot \frac{R_1}{(R_1 + R_2)}$$

$$\frac{\Delta R_{eq}}{R_{eq}} = 10\% \frac{R_1 + R_2}{R_1 + R_2} = 10\% = 0.1$$

## B- Vectors

### Exercise 1:

Consider three vectors:  $\vec{a} = \vec{i} - 2\vec{j} + 2\vec{k}$ ,  $\vec{b} = 4\vec{i} + 4\vec{j} - 2\vec{k}$  and  $\vec{c} = -3\vec{i} + 4\vec{k}$  in an orthonormal basis  $(O, \vec{i}, \vec{j}, \vec{k})$ .

- Plot these vectors in the orthonormal basis and calculate their magnitude.
- Give the expression of the vector:  $\vec{d} = \vec{a} + \vec{b} - \vec{c}$  and find its modulus.
- Find the unit vector in the direction and in the direction opposite of the previous vector  $\vec{d}$ .
- Calculate the scalar and cross product:  $\vec{a} \cdot \vec{b}$ ,  $\vec{a} \wedge \vec{b}$

### Exercise 2:

Consider:  $\vec{u}(2, 1, 3)$ ,  $\vec{v}(3, 1, -2)$  and  $\vec{w}(4, -1, 1)$ . Find the following vectors:

$$\vec{u} + \vec{v}, \quad \vec{u} - \vec{v}, \quad 2\vec{w}, \quad 2\vec{u} - 3\vec{v}, \quad \vec{u} + 2\vec{v} - 3\vec{w}.$$

- Find the following lengths:

$$|\vec{u}|, \quad |\vec{v}|, \quad 2|\vec{w}|, \quad |\vec{u} + \vec{v}|, \quad |\vec{u} - \vec{v}|, \quad |\vec{v} - \vec{w}|$$

### Exercise 3:

Consider the displacement vectors:  $\vec{a} - \vec{b} = -\vec{i} - 4\vec{j} + 3\vec{k}$ ,  $\vec{a} + \vec{b} = 3\vec{i} + 2\vec{j} + \vec{k}$ .

- Find the components of the vectors:  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{a} - \vec{b}$ ,  $\vec{a} + \vec{b}$  and then their magnitude.
- Determine the angle between the following vectors:  $(\vec{a}, \vec{a} + \vec{b})$  and  $(\vec{b}, \vec{a} - \vec{b})$ .
- Calculate the area formed by the two vectors  $\vec{a}$  and  $\vec{a} + \vec{b}$ .
- Calculate the scalar triple product:  $\vec{a} \cdot [\vec{b} \wedge (\vec{a} + \vec{b})]$ .

### Exercise 4:

Find the x, y components of the vectors:  $\vec{A} = x\vec{i} + 3\vec{j}$  and  $\vec{B} = 2\vec{i} + y\vec{j}$  where:

- Each vector is orthogonal to the vector  $\vec{D} = 5\vec{i} + 6\vec{j}$

- Each vector is parallel to the same vector  $\vec{D}$ .

**Exercise 5:**

Consider two vectors:  $\vec{a}$  and  $\vec{b}$  in an orthonormal basis  $(O, \vec{i}, \vec{j}, \vec{k})$ . Show that when  $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$ , then  $\vec{a}$  is perpendicular to  $\vec{b}$ .

**Exercise 6:**

Find the volume of the parallelepiped having the following three vectors as edges.

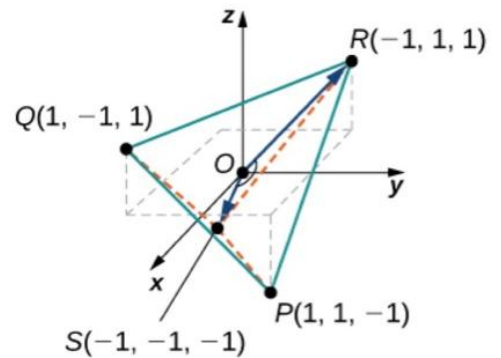
$\vec{U}(2, 3, 1)$ ,  $\vec{V}(3, 4, 3)$  and  $\vec{W}(4, 5, 6)$ .

**Exercise 7:**

A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points

$P(1, 1, -1), Q(1, -1, 1), R(-1, 1, 1)$ , and  $S(-1, -1, -1)$  (see figure).

- Find the distance between the hydrogen atoms located at  $P$  and  $R$ .
- Find the angle between vectors  $\vec{OS}$  and  $\vec{OR}$  that connect the carbon atom with the hydrogen atoms located at  $S$  and  $R$ , which is also called the *bond angle*.



**Exercise 8:**

Consider the following vectors:  $\vec{V}_1 = \sin t \vec{i} + \cos t \vec{j} + 3t \vec{k}$ ,  $\vec{V}_2 = e^t \vec{i} + 2 \cos 3t \vec{j} + 3 \sin 2t \vec{k}$ . Find the derivatives:  $\frac{d\vec{V}_1}{dt}, \frac{d\vec{V}_2}{dt}$

**Exercise 9:**

Let the vectors be the following:

$\vec{V}_2 = \sin t \vec{i} - \cos t \vec{j} + 3t \vec{k}$ ,  $\vec{V}_1 = 5t^3 \vec{i} + 3t \vec{j} - 2t^4 \vec{k}$

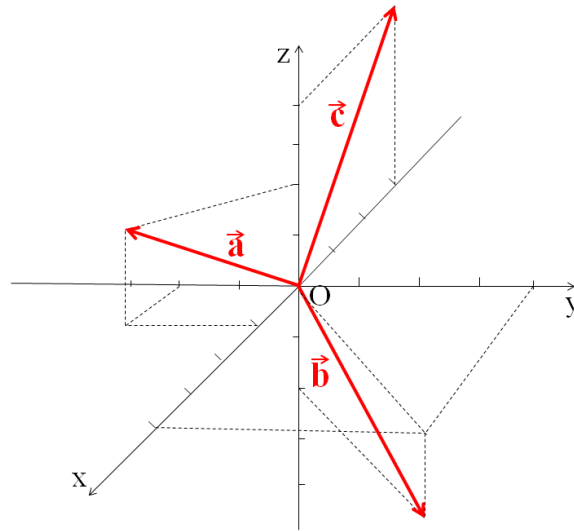
Find the following derivative expression in two ways:  $\frac{d}{dt}(\vec{V}_1 \cdot \vec{V}_2), \frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2)$ .

**SOLUTION**

**Exercise 1:**

$$\vec{a} = \vec{i} - 2\vec{j} + 2\vec{k}, \vec{b} = 4\vec{i} + 4\vec{j} - 2\vec{k}, \vec{c} = -3\vec{i} + 4\vec{k}$$

$$\vec{a} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \vec{b} \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}, \vec{c} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$



$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

$$|\vec{b}| = \sqrt{4^2 + 4^2 + (-2)^2} = 6$$

$$|\vec{c}| = \sqrt{(-3)^2 + 4^2} = 5$$

-The expression of the vector:  $\vec{d} = \vec{a} + \vec{b} - \vec{c}$

$$\vec{d} = \vec{a} + \vec{b} - \vec{c} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+4+3 \\ -2+4 \\ 2-2-4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$$

$$\vec{d} = 8\vec{i} + 2\vec{j} - 4\vec{k}$$

-The modulus:

$$|\vec{d}| = \sqrt{8^2 + 2^2 + (-4)^2} = \sqrt{84}$$

-The unit vector in the direction of the vector  $\vec{d}$ :

$$\vec{d} = |\vec{d}| \cdot \vec{u} \Rightarrow \vec{u} = \frac{\vec{d}}{|\vec{d}|} = \frac{8}{\sqrt{84}}\vec{i} + \frac{2}{\sqrt{84}}\vec{j} - \frac{4}{\sqrt{84}}\vec{k}$$

and in the direction opposite of the vector  $\vec{d}$ :

$$\Rightarrow \vec{u} = -\frac{\vec{d}}{|\vec{d}|} = -\frac{8}{\sqrt{84}}\vec{i} - \frac{2}{\sqrt{84}}\vec{j} + \frac{4}{\sqrt{84}}\vec{k}$$

-Calculate the scalar and cross product:  $\vec{a} \cdot \vec{b}$ ,  $\vec{a} \wedge \vec{b}$

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = 4 - 8 - 4 = -8$$

$$\vec{a} \wedge \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = \begin{vmatrix} 1 & -2 & 2 \\ 4 & 4 & -2 \end{vmatrix} = \begin{pmatrix} -4 \\ 10 \\ 12 \end{pmatrix}$$

**Exercise 4:**

Find the x, y components of the vectors:  $\vec{A} = x\vec{i} + 3\vec{j}$  and  $\vec{B} = 2\vec{i} + y\vec{j}$

1)- Each vector is orthogonal to the vector  $\vec{D} = 5\vec{i} + 6\vec{j}$

$$\vec{A} \perp \vec{D} \Rightarrow \vec{A} \cdot \vec{D} = 0 \Rightarrow 5x + 18 = 0 \Rightarrow x = -\frac{18}{5}$$

$$\vec{B} \perp \vec{D} \Rightarrow \vec{B} \cdot \vec{D} = 0 \Rightarrow 10 + 6y = 0 \Rightarrow y = -\frac{10}{6} = -\frac{5}{3}$$

2)- Each vector is parallel to the same vector  $\vec{D} = 5\vec{i} + 6\vec{j}$

$$\vec{A} \parallel \vec{D} \Rightarrow \vec{A} \wedge \vec{D} = \vec{0} \Rightarrow (x\vec{i} + 3\vec{j}) \wedge (5\vec{i} + 6\vec{j}) = (6x - 15)\vec{k} = \vec{0}$$

$$\vec{B} \parallel \vec{D} \Rightarrow \vec{B} \wedge \vec{D} = \vec{0} \Rightarrow (2\vec{i} + y\vec{j}) \wedge (5\vec{i} + 6\vec{j}) = (12 - 5y)\vec{k} = \vec{0}$$

$$\begin{cases} 6x - 15 = 0 \Rightarrow x = \frac{15}{6} = \frac{5}{2} \\ 12 - 5y = 0 \Rightarrow y = \frac{12}{5} \end{cases}$$

**Exercise 6:**

The volume of the parallelepiped having the following three vectors as edges:

$\vec{U}(2, 3, 1)$ ,  $\vec{V}(3, 4, 3)$  and  $\vec{W}(4, 5, 6)$ :

The absolute value of the mixed product  $a = \vec{U} \cdot (\vec{V} \wedge \vec{W})$  measures **the volume** of the parallelepiped so:

Firstly we calculate the cross product  $\vec{V} \wedge \vec{W}$ :

$$\vec{V} \wedge \vec{W} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{vmatrix} 3 & 4 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{pmatrix} 9 \\ -6 \\ -1 \end{pmatrix}$$

then we calculate the scalar product:  $a = \vec{U} \cdot (\vec{V} \wedge \vec{W})$ :

$$a = \vec{U} \cdot (\vec{V} \wedge \vec{W}) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -6 \\ -1 \end{pmatrix} = 18 - 18 - 1 = -1 \Rightarrow V = |a| = |-1| = 1 \text{ uv}$$

**Exercise 8:**

$\vec{V}_1 = \sin t \vec{i} + \cos t \vec{j} + 3t\vec{k}$  and  $\vec{V}_2 = e^t \vec{i} + 2 \cos 3t \vec{j} + 3 \sin 2t \vec{k}$ .

Find the derivatives:  $\frac{d\vec{V}_1}{dt}, \frac{d\vec{V}_2}{dt}$  ?

$$\vec{V}_1 = \sin t \vec{i} + \cos t \vec{j} + 3t\vec{k} \Rightarrow \frac{d\vec{V}_1}{dt} = \cos t \vec{i} - \sin t \vec{j} + 3\vec{k}$$

$$\vec{V}_2 = e^t \vec{i} + 2 \cos 3t \vec{j} + 3 \sin 2t \vec{k} \Rightarrow \frac{d\vec{V}_2}{dt} = e^t \vec{i} - 6 \sin 3t \vec{j} + 6 \cos 2t \vec{k}$$

**Exercise 9:**

Let the vectors be the following:

$$\vec{V}_1 = 5t^3\vec{i} + 3t\vec{j} - 2t^4\vec{k}$$

$$\vec{V}_2 = \sin t\vec{i} - \cos t\vec{j} + 3t\vec{k}$$

Find the following derivative expression in two ways:  $\frac{d}{dt}(\vec{V}_1 \cdot \vec{V}_2)$ ,  $\frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2)$ .

First method:

$$\begin{aligned}\frac{d}{dt}(\vec{V}_1 \cdot \vec{V}_2) &= \frac{d\vec{V}_1}{dt} \cdot \vec{V}_2 + \vec{V}_1 \cdot \frac{d\vec{V}_2}{dt} \\ \frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2) &= \frac{d\vec{V}_1}{dt} \wedge \vec{V}_2 + \vec{V}_1 \wedge \frac{d\vec{V}_2}{dt}\end{aligned}$$

Second method:

We calculate the scalar product (or cross product) and then the derivative of product.

# Chapter II

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## Kinematics of Material Point

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*Our study of kinematics is an important starting point for understanding the moving and interacting world around us. Because mathematics is the language of physics, we'll need a set of mathematical tools to describe and analyze all sorts of physical phenomena in our universe. Let's dive into some basic concepts of kinematics next: the key variables of kinematic motion and the kinematics equations behind these.*

## II.1 Introduction

**Kinematics** is the branch of mechanics that describes the motion of an object. It is the study of the motion of objects without reference to the forces that caused the motion. Most bodies studied by physicists are in motion. Motion occurs at all scales of the universe, from particles like electrons, protons, and neutrons that constitute atoms, to galaxies. It is crucial to define motion accurately in order to understand many phenomena that we observe around us. An object can have motion in various forms:

- Translation: the motion of a car on a road.
- Rotation: like the Earth's rotation about its axis.
- Vibration: small oscillations of a mass-spring system.
- A combination of several of these types of motion.

### II.1.1 Reference System

Rest and motion are two relative concepts. Indeed, an immobile observer  $A$  sees a tree in a fixed position, while the driver  $B$  of a nearby moving car sees it moving backward. This example demonstrates that describing motion requires specifying the nature of the observer. In physics, the study of motion is conducted by replacing the observer with a *coordinate system*, also referred to as a frame of reference or reference system. A reference system can be either fixed or mobile: the system associated with  $A$  is fixed, and the one associated with  $B$  is mobile. To express the concepts of rest and motion with respect to a reference frame, let's consider an orthonormal coordinate system  $\mathfrak{R}(O, x, y, z)$  in which the position of a body is located at  $M(x, y, z)$ . The body is at rest with respect to this reference frame if its coordinates remain constant over time. However, if at least one of them varies, the body is in motion with respect to  $\mathfrak{R}$ .

To describe mechanical motion, it is necessary to indicate the body relative to which the motion is considered. An arbitrarily chosen stationary body, in relation to which the movement of this body is considered, is called *reference body*.

### II.1.2 Material point in physics

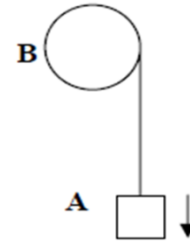
A material point in physics denotes a body, in cases where its size and shape can be neglected. It is considered that the material point has the mass of the original body, located at its center of gravity. **Material point** is called a body whose dimensions can be neglected under these conditions.

The word "material" here emphasizes the difference between this point and the geometric one. The geometric point has no physical properties. A material point can have mass, electric charge, and other physical characteristics.

**Example:**

1) Mass-string-pulley system from figure II.1:

- A can be reduced to a point mass;
- B and the string cannot be reduced to a point.



*Figure II. 1*

2) When studying the motion of planets around the Sun, they can be described by material points, and with the daily rotation of planets around their axis, such a model is inapplicable.

**Note:**

It is important to understand that material points do not exist in nature. A material point is an abstraction, a model for describing movement.

### II.1.3 Trajectory

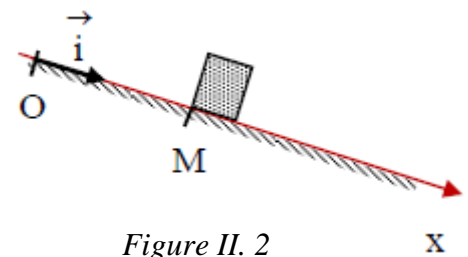
It is the locus of the successive positions occupied by the material point over time and with respect to the chosen reference system. For example a tennis ball during the first service of top tennis players mostly circumscribe linear trajectory, while during the second service its trajectory is parabolically curved. The trajectory equation can be fined by solving the parametric equations together and excluding the time  $t$  from them.

### II.2 Rectilinear motion

This movement is motivated by its simplicity: its description is easy and it is described by simple equations.

#### 1. Definition

If a particle is restricted to move along a straight line, its motion is called **rectilinear** (or linear) motion. Such a motion can be described using one coordinate only. Displacement of the particle and its derivatives i.e. velocity and acceleration all are one-dimensional vectors. Free-fall under the Earth's gravitational field, a car moving along a straight path can be approximated as rectilinear motions.



*Figure II. 2*

To qualitatively study a rectilinear motion, a one-dimensional reference frame consisting of an axis (X-axis) and an origin at  $O$  ( $x = 0$ ) is considered (figure II. 2).

The position M of the mobile is identified by the position vector:

$$\overrightarrow{OM} = x \vec{i} \quad (\text{II.1})$$

### 2. Space Diagram

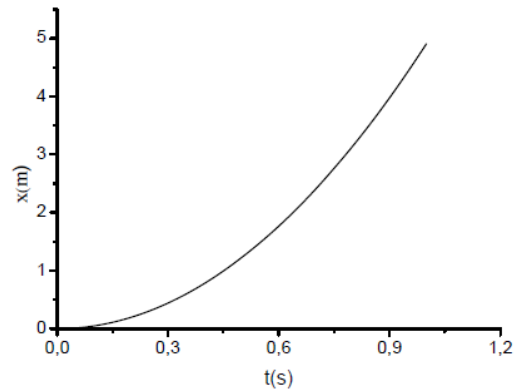
The position M of a mobile depends on time. Therefore, at each moment t, it can be identified by the vector:

$$\overrightarrow{OM}(t) = x(t)\vec{i} \quad (\text{II.2})$$

The relation  $x = f(t)$  is the **time equation (parametric equation)** of the motion.

**Example:**

$x = \frac{1}{2}gt^2$  for a free fall of an object released from the origin O of a vertically oriented axis directed downward. The graph of  $x(t)$  represents the position-time graph, also known as the displacement-time graph. Position-time graph for free fall, (figure II. 3)



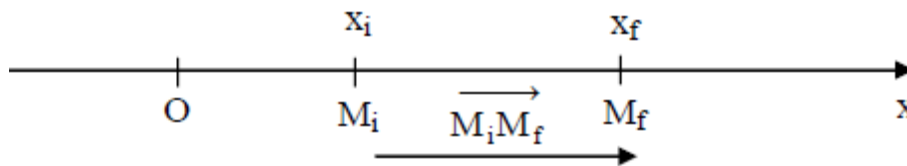
*Figure II. 3*

**Note:** The position-time graph is not necessarily a straight line, even in the case of rectilinear motion. It should not be confused with the trajectory.

### 3. Displacement Vector

Let  $M_i$  and  $M_f$  be two positions of a mobile on the (Ox) axis at instants  $t_i$  and  $t_f$ , respectively.

The vector  $\overrightarrow{M_iM_f}$  is called the displacement vector between  $t_i$  and  $t_f$ .



*Figure II. 4*

According to the Chasles' relation:

$$\overrightarrow{M_iM_f} = \overrightarrow{M_iO} + \overrightarrow{OM_f} \quad (\text{II.3})$$

The relationship between this displacement vector and the position vectors is then:

$$\overrightarrow{M_iM_f} = \Delta\overrightarrow{OM} = \overrightarrow{OM_f} - \overrightarrow{OM_i} \quad (\text{II.4})$$

As a result, its component along the (Ox) axis is:

$$\Delta x = x_f - x_i \quad (\text{II.5})$$

**Note:** One should not confuse its magnitude with the distance traveled. Indeed, if we consider the displacement  $M_i \rightarrow O \rightarrow M_f$  of a moving object on an Ox axis,  $|\overline{M_i M_f}| = |x_f - x_i|$  and the distance traveled is given by:

$$d = |\overline{M_i O}| + |\overline{OM_f}| = |x_i| + |x_f|$$

➔ **Distance** is a measurement of the total movement an object has traveled without reference to the direction of motion. In other words, we sum up the absolute value of the length of each segment along a path to find the total distance covered. Both displacement and distance are also measured in units of length. (see figure II. 5)

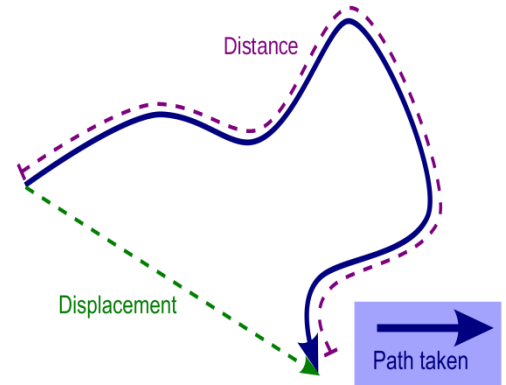


Figure II. 5

#### 4. Time

An important and deceptively simple variable that we rely on for both day-to-day structure and many physics problems is time, particularly elapsed time.

➔ **Elapsed time** is a measurement of how long an event takes, or the amount of time taken for observable changes to happen. We measure a time interval  $\Delta t$  as the difference between the final timestamp and initial timestamp, or:

$$\Delta t = t_f - t_i \quad (\text{II.6})$$

We record time typically in units of seconds, denoted by the symbol “s” in physics problems.

#### 5. Velocity

The **velocity vector**  $\vec{V}$  of a particle represents the rate of change of its position vector (displacement)  $\overline{OM}$  over time.

- ✓ This change can involve the direction of  $\overline{OM}$ , its magnitude, or both.
- ✓ The unit of velocity in the International System (SI) is meters per second (m/s).

In other words, the velocity variable describes how much an object changes its position for each unit of time that passes.

➔ **Speed** is a similar variable, but instead calculated using the total distance covered during some period of elapsed time.

**Note:** Speed is a scalar but velocity is a vector having direction same as displacement.

**a. Average Speed**

The average speed of a mobile object between two moments  $t_i$  and  $t_f$  corresponding to the positions  $M_i$  and  $M_f$  is defined by the ratio:

$$(\overline{V_m})_{t_i}^{t_f} = \frac{\Delta \overline{OM}}{\Delta t} = \frac{\overline{OM_f} - \overline{OM_i}}{t_f - t_i} \quad (\text{II. 7})$$

In algebraic terms, and in the case of linear motion along an axis (Ox):

$$(V_{mx})_{t_i}^{t_f} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} \quad (\text{II. 8})$$

**Example:**

Let's consider the free fall motion of a ball described by:  $x_1=45\text{m}$  at  $t_1=3\text{s}$  and  $x_2=125\text{m}$  at  $t_2=5\text{s}$

$$(V_{mx})_3^5 = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{125 - 45}{5 - 3} = 40(\text{m/s})$$

**Notes:**

1) In the diagram of the spaces in Figure II.6:

$$(V_{mx})_{t_A}^{t_B} = \frac{\Delta x}{\Delta t} = \text{slope of the secant (the line segment) AB}$$

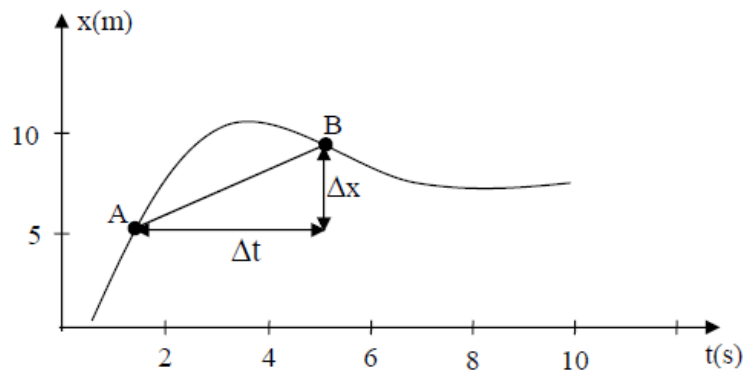


Figure II. 6

2) The scalar average velocity is given by the ratio:

$$V = \frac{\text{distance traveled}}{\text{time taken}} = \frac{d}{\Delta t} \quad (\text{II. 9})$$

3) An average velocity  $(\overline{V_m})_{t_i}^{t_f}$  characterizes the time interval  $[t_i, t_f]$  in which it is determined.

### b. Instantaneous Velocity

Sometimes we are interested in the velocity of a particle at a specific moment  $t$  corresponding to a given position. Let's consider the example of the space diagram in Figure II.7. The average velocity  $(\vec{V}_m)_{t_A}^{t_B}$  characterizes the interval  $[t_A, t_B]$ . To obtain a velocity that refers to the instant  $t_A$ , intuitively, it is appropriate to reduce the interval  $[t_A, t_B]$  to  $[t_A, t_A]$ . This is equivalent to making  $t_B$  approach  $t_A$ , which can be expressed as:

$$V_x(t_A) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \left. \frac{dx}{dt} \right|_{t=t_A} \quad (\text{II. 10})$$

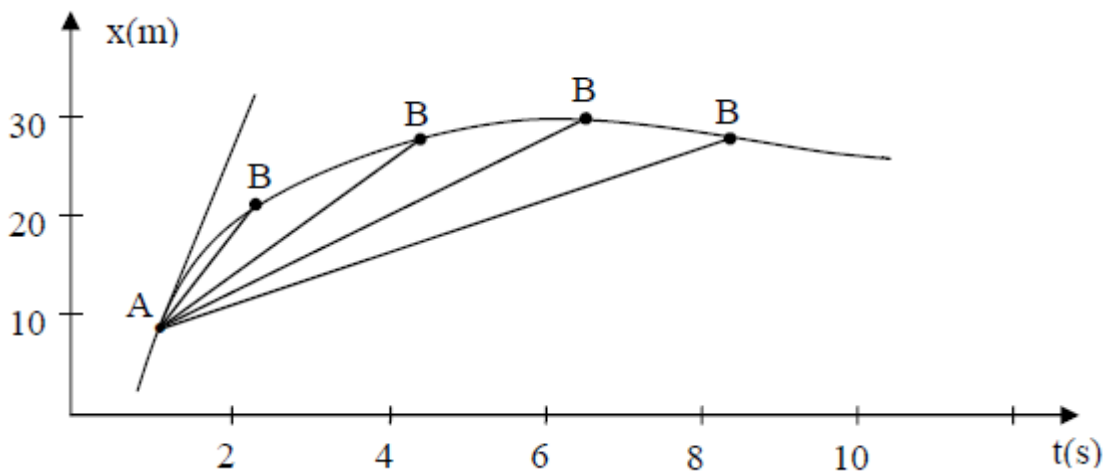


Figure II. 7

Thus, the slope of segment AB will approach that of the tangent to the graph at point A (see Figure II.7). As  $x$  is the algebraic projection of the position vector onto the axis, this suggests the following definition:

The **instantaneous velocity** vector of a mobile object at time  $t$  is given by the relation:

$$\vec{V}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{OM}}{\Delta t} = \frac{d\vec{OM}}{dt} \quad (\text{II. 11})$$

Algebraically, in the case of rectilinear motion along the (Ox) axis:

$$V_x(t) = \frac{dx}{dt}$$

= *slope of the tangent to the space diagram at the point corresponding to instant "t"*

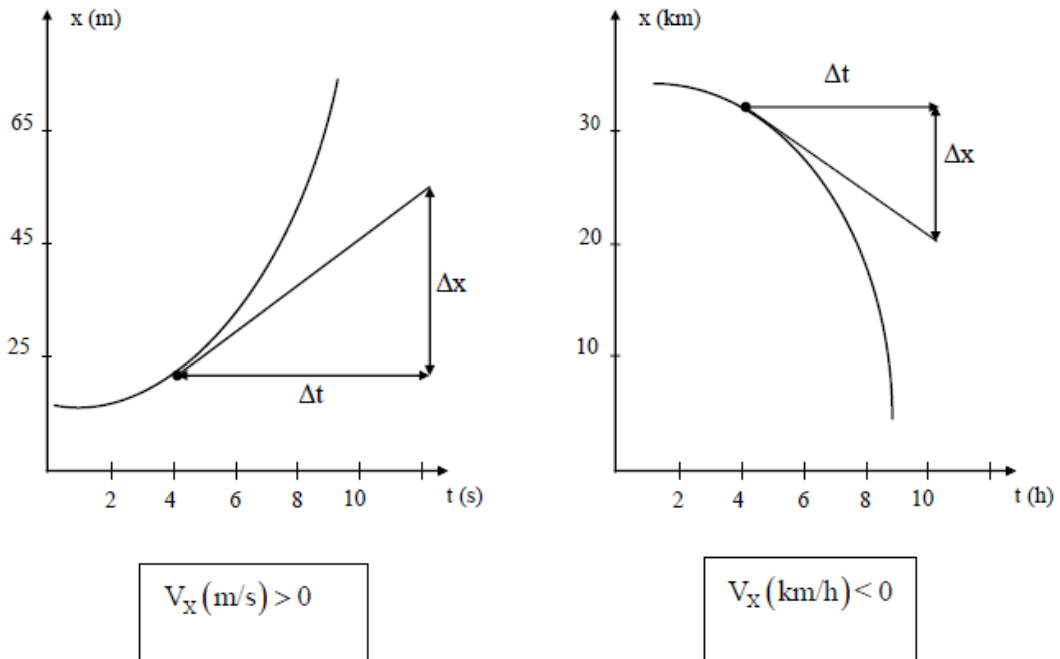
The graph of  $V_x(t)$  is called the **velocity diagram**.

➤ Measurement of Velocity

- ✓ **First Method:** If we have the space diagram, we can obtain the algebraic values,  $V_x$ , by measuring the slopes of the tangents to the graph at the points in question. It is important to pay attention to units and signs.

**Example:**

Figure II. 8



- ✓ **Second Method:** When we have a sufficient number of pairs of values  $(t, V_x(t))$ , we can plot the graph of  $V_x(t)$  and obtain unknown values through extrapolation or interpolation, as illustrated in Figure II.9.

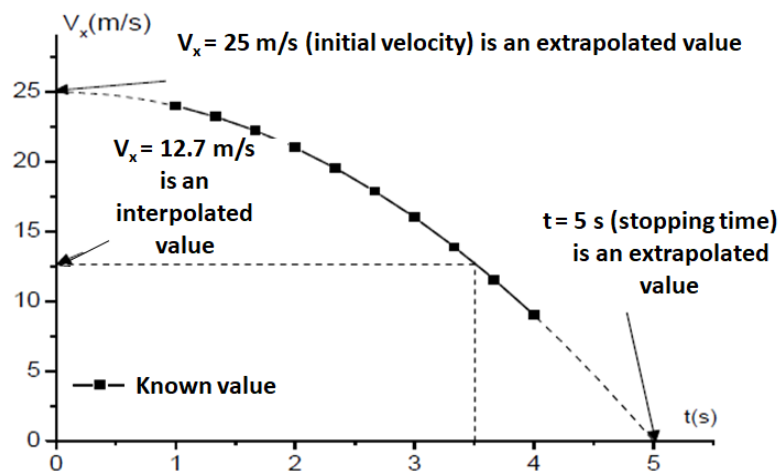


Figure II. 9

✓ **Third method**: one can confuse the average velocity with the instantaneous velocity in two cases:

**1st case**: When the velocity is constant (uniform motion) and we have:

$$\vec{V}(t) = (\overline{V_m})_{t_i}^{t_f} \quad \forall t, t_i, \text{ and } t_f \quad (\text{II. 12})$$

**2nd case**: When the time interval  $\Delta t = t_f - t_i$  is sufficiently small, we can confuse the average velocity  $(\overline{V_m})_{t_i}^{t_f}$  with the instantaneous velocity  $\vec{V}(t)$  in the middle of the interval:  $[t_i, t_f]$ :

$$\Delta t = t_f - t_i < \varepsilon \text{ (petit)} \Rightarrow \vec{V}(t) = (\overline{V_m})_{t_i}^{t_f} \quad \text{with } t = \frac{t_i + t_f}{2} \quad (\text{II. 13})$$

**Notes:**

- $\Delta t = t_f - t_i$  is sufficiently small when the difference between  $\vec{V}(t)$  and  $(\overline{V_m})_{t_i}^{t_f}$  is not significant compared to measurement errors.
- The midpoint of the interval  $[t_i, t_f]$  does not necessarily correspond to the midpoint of the segment  $M_i M_f$ .

**6. Acceleration**

The **acceleration** of an object is its change in velocity over an increment of time. Since velocity is a vector  $\vec{V}$  (indicating magnitude and direction), acceleration  $\vec{a}$  is also a vector. If the **magnitude** and/or **direction** of the velocity changes, so too does that of the acceleration. If the velocity is slowing down, the object is said to decelerate or having negative acceleration. The unit of acceleration in the International System is meters per second squared (m/s<sup>2</sup>). Considerations about acceleration include calculating average acceleration and instantaneous acceleration, as well as looking at uniform acceleration in a straight line and circular motion. By following a similar approach to the introduction of velocity, we can define the following quantities:

**a. Average acceleration**

Average acceleration is the change of velocity over a period of time. The average acceleration of a mobile between two moments'  $t_i$  and  $t_f$  is given by the ratio:

$$(\overline{a_m})_{t_i}^{t_f} = \frac{\Delta \vec{V}}{\Delta t} = \frac{\vec{V}_f - \vec{V}_i}{t_f - t_i} \quad (\text{II. 14})$$

In algebraic terms, and in the case of rectilinear motion along (Ox),

$$(a_{mx})_{t_i}^{t_f} = \frac{\Delta V_x}{\Delta t} = \frac{V_{xf} - V_{xi}}{t_f - t_i} \quad (\text{II. 15})$$

### b. Instantaneous Acceleration:

Instantaneous acceleration is the change of velocity over an instance of time. Instantaneous acceleration is defined as the limit of the average acceleration when the interval of time considered approaches 0. It is also defined in a similar manner as the derivative of velocity with respect to time. It is the rate at which velocity changes at a specific instant in time. The equation for instantaneous acceleration is:

$$\vec{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{V}}{\Delta t} = \frac{d\vec{V}}{dt} \quad (\text{II. 16})$$

Algebraically, in the case of rectilinear motion along (Ox):

$$a_x(t) = \frac{dV_x}{dt} = \text{slope of the tangent to the velocity diagram} \quad (\text{II. 17})$$

**Notes:** Note that:

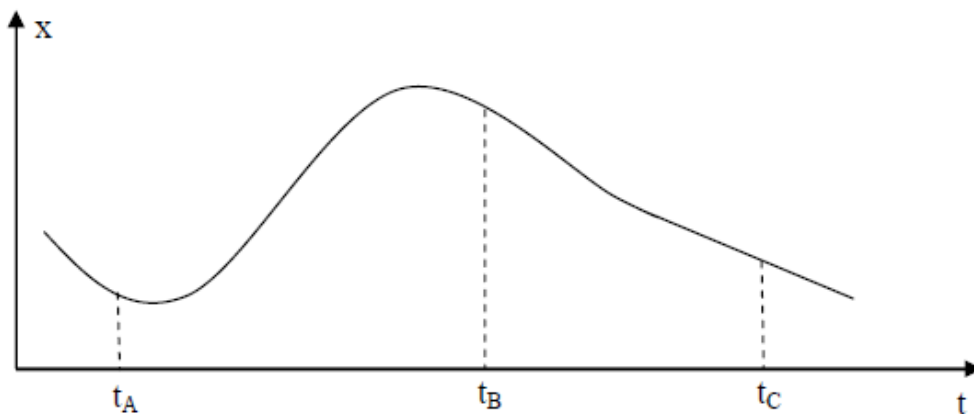
$$a_x(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta V}{\Delta t} = \frac{dV_x}{dt} = \frac{d^2x}{dt^2} \quad (\text{II. 18})$$

Thus, the orientation of the curvature of the position diagram indicates the sign of  $a_x$ :

- $a_x > 0$  curvature towards the positive direction of the axis (Ox),
- $a_x < 0$  curvature towards the negative direction of the axis (Ox).

**Example:** For the motion described by the position diagram in Figure II.10:

$$a_x(t_A) > 0 \quad a_x(t_B) < 0 \quad a_x(t_C) = 0$$



*Figure II. 10*

➤ **Acceleration Measurement:**

Similar to velocity, acceleration can be determined by:

- Measuring the slopes of tangents to velocity diagrams.
- Using interpolation or extrapolation on acceleration graphs.
- Confusing the average acceleration  $(\overline{a_m})_{t_i}^{t_f}$  with the instantaneous acceleration  $\vec{a}(t)$  in the middle of the time interval  $[t_i, t_f]$  if it is sufficiently small:

$$\Delta t = t_f - t_i < \varepsilon \text{ (petit)} \Rightarrow \vec{a}(t) = (\overline{a_m})_{t_i}^{t_f} \text{ Avec } t = \frac{t_i + t_f}{2}$$

## 7. Integral Relationships

In the previous discussions, it is evident that we can pass from position to velocity and from velocity to acceleration through differentiation. In this section, we will demonstrate that the reverse transition can be achieved through integration.

### a. Transition from velocity to position

Relation between position and velocity is:

$$V_x(t) = \frac{dx}{dt}$$

Let:

$$dx = V_x(t)dt$$

And by integrating between two moments'  $t_i$  and  $t_f$ , we obtain:

$$x(t_f) - x(t_i) = \int_{t_i}^{t_f} V_x(t)dt \quad (\text{II. 19})$$

**Consequence:** If we know the position  $x_0$  of a mobile at a specific moment  $t_0$ , and the expression of its velocity as a function of time, it is then possible to determine its position  $x(t)$  at any instant  $t$  by writing:

$$x(t) = x_0 + \int_{t_0}^t V_x(t)dt \quad (\text{II. 20})$$

### Example:

A mobile is in a motion along an axis (Ox) with a velocity  $V_x(t) = 2t + 1$  m/s, passes through a point with an abscissa of  $x_0 = 2$  m at time  $t_0 = 0$  s. Determine the equation describing the position as a function of time.

### Answer:

$$x(t) = x_0 + \int_{t_0}^t V_x(t)dt = 2 + \int_0^t (2t + 1)dt = t^2 + t + 2 \text{ (m)}$$

**Graphically**, the equivalence between the value of the integral of a function and the area under the curve of its graph and above the x-axis (figure II. 11) allows us to write:

$$x(t_f) - x(t_i) = \int_{t_i}^{t_f} V_x(t) dt$$

= Algebraic area bounded by the curve of  $V_x(t)$ ,  
the time axis, and the lines  $t = t_i$  and  $t = t_f$

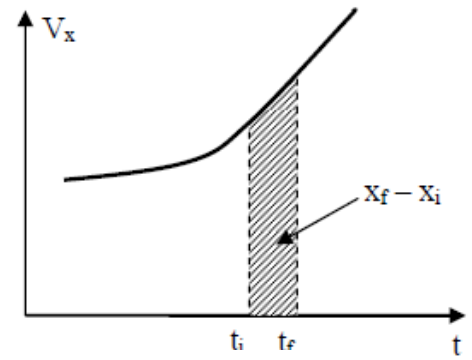


Figure II. 11

**Example:** Let's consider the velocity diagram in Figure II.12. Given  $x(0s) = 1$  m, let's determine  $x(2s)$  and  $x(5s)$ .

$$x(2s) = x(0s) + \int_0^2 V_x(t) dt = 1 + \text{air } A_1$$

$$= 1 + \frac{2(-1)}{2} = 0 \text{ m}$$

$$x(5s) = x(2s) + \int_2^5 V_x(t) dt = 0 + \text{air } A_2$$

$$= \frac{[(5-3)+(5-2)] \times 1}{2} = 2.5 \text{ m}$$

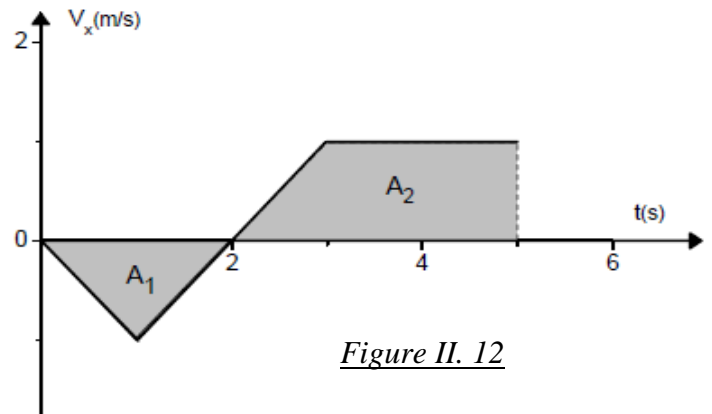


Figure II. 12

**Notes:**

- ✓ Since the areas are algebraic, any portion below the (Ot) axis is negative, and any portion above it is positive. In the previous example:

$$A_1 = \int_0^2 V_x(t) dt = -1m < 0$$

$$A_2 = \int_2^5 V_x(t) dt = 2.5m > 0$$

- ✓ To calculate the distance traveled between 0s and 5s, we sum the absolute values of the considered areas:

$$d = |A_1| + |A_2| = 1 + 2.5 = 3.5 \text{ m}$$

**b. Transition from acceleration to velocity**

Relation between Acceleration and velocity is:

$$a_x(t) = \frac{dV_x}{dt}$$

Let:

$$dV_x = a_x(t) dt \quad (\text{II.21})$$

And by integrating between two moments'  $t_i$  and  $t_f$ , we obtain:

$$V_x(t_f) - V_x(t_i) = \int_{t_i}^{t_f} a_x(t)dt \quad (\text{II. 22})$$

**Consequence:** If we know the velocity  $V_{x0}$  of a mobile at a specific moment  $t_0$ , and the expression of its acceleration  $a_x(t)$  as a function of time, it is possible to determine its velocity  $V_x(t)$  at any instant  $t$  by writing:

$$V_x(t) = V_{x0} + \int_{t_0}^t a_x(t)dt \quad (\text{II. 23})$$

Graphically,

$$V_x(t_f) - V_x(t_i) = \int_{t_i}^{t_f} a_x(t)dt \quad (\text{II. 24})$$

This expression represents the algebraic area bounded by the curve of  $a_x(t)$ , the time axis, and the lines  $t = t_i$  and  $t = t_f$ .

**Example:**

From the acceleration diagram in Figure II.13, let's determine  $V_x(2s)$  and  $V_x(6s)$ , given that  $V_x(0s) = 1m/s$ .

$$V_x(2s) = V_x(0s) + \int_0^2 a_x(t)dt = 1 + \text{air } A_1 = 1 + 2(-1) = -1 \text{ m/s}$$

$$V_x(6s) = V_x(2s) + \int_2^6 a_x(t)dt = -1 + \text{air } A_2 = -1 + 4 \times 2 = 7 \text{ m/s}$$

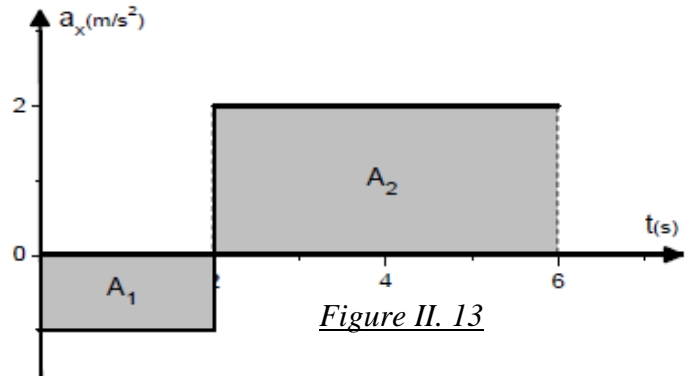


Figure II. 13

**8. Kinematic Study of Specific Rectilinear Motions**

The Kinematics of rectilinear motion is the study of the motion of particles or objects along a straight line path. It is a branch of mechanics that deals with the motion of objects in a straight line. Kinematics of rectilinear motion is a mathematical study of the relationships between the position, velocity, and acceleration of a particle or object moving in a straight line.

**a. Uniform Rectilinear Motion**

When an object travels at a constant speed with zero acceleration it is known as uniform rectilinear motion. This is a motion without acceleration according to the relation:

$$a_x(t) = \frac{dV_x}{dt} = 0$$

From the integral formula, for  $x_0 = x(0)$ , we obtain:

$$x(t) = x_0 + \int_0^t V_x(t)dt = V_x t + x_0 \quad (\text{II. 25})$$

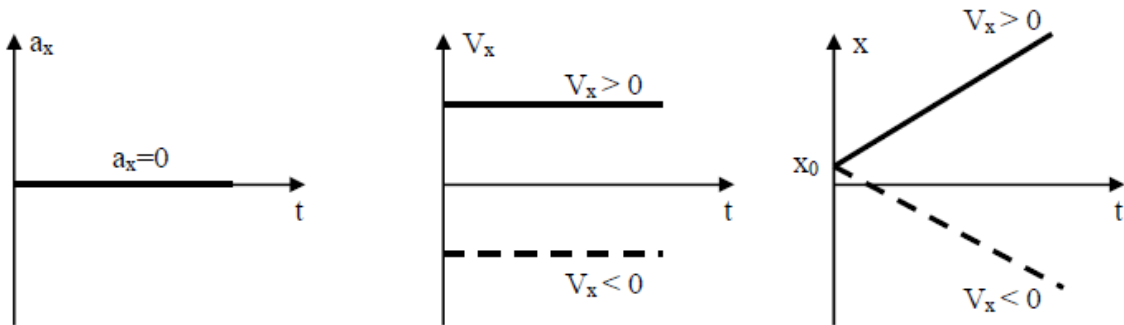


Figure II. 14

**b. Uniformly accelerated rectilinear motion:**

When an object travels with constant acceleration it is known as uniformly accelerated rectilinear motion. Using integral calculus, we obtain:

- For the velocity:

$$V_x(t) = V_{x0} + \int_0^t a_x(t)dt = a_x t + V_{x0} \quad \text{with} \quad V_{x0} = V_x(0)$$

- For the position:

$$x(t) = x_0 + \int_0^t V_x(t)dt = x_0 + \int_0^t (a_x t + V_{x0})dt = \frac{1}{2} a_x t^2 + V_{x0} t + x_0$$

**Examples of diagrams:**

- **Case where  $a_x > 0$**

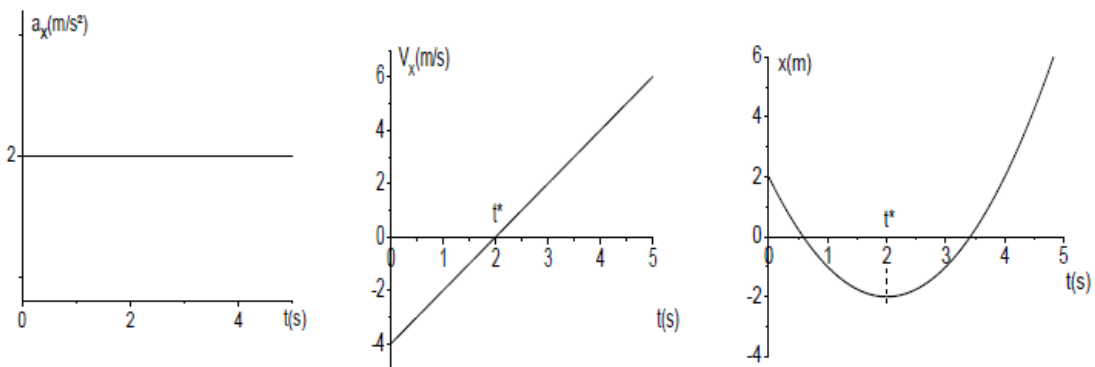
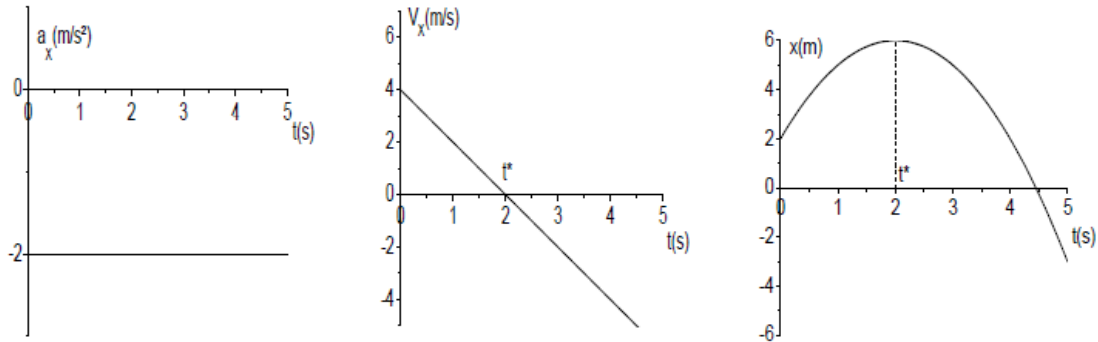


Figure II. 15

➤ **Case where  $a_x < 0$**



*Figure II. 16*

**c. Time-Independent Formula**

This formula is valid only for uniformly accelerated motion ( $a_x = \text{constant}$ ). By definition:

$$a_x = \frac{dV_x}{dt} \Rightarrow dV_x = a_x dt$$

By multiplying both sides by  $V_x$ , we obtain:

$$V_x dV_x = a_x V_x dt$$

As:

$$V_x = \frac{dx}{dt} \Rightarrow V_x dt = dx$$

it follows:

$$V_x dV_x = a_x dx$$

Hence:

$$\int_{V_{x1}}^{V_{x2}} V_x dV_x = \int_{x_1}^{x_2} a_x dx$$

Since  $a_x$  is constant, we obtain:  $V_{x2}^2 - V_{x1}^2 = 2a_x(x_2 - x_1)$  (II.26)

**d. Types of Rectilinear Motion**

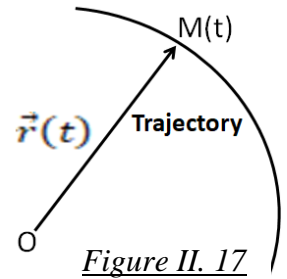
- $V_x = \text{constant}$ : the motion is uniform rectilinear.
- $a_x = \text{constant}$ : the motion is uniformly varied rectilinear.
- $a_x \cdot V_x > 0$ : the motion is accelerated rectilinear (uniformly if  $a_x = \text{constant}$ ).
- $a_x \cdot V_x < 0$ : the motion is decelerated or retarded rectilinear (uniformly if  $a_x = \text{constant}$ ).

**II.3. Motion in Space**

The motion in the trajectory is not necessarily a straight line.

**1. Position Tracking**

To describe the motion of an object, we consider O as a reference point (origin). The position M(t) is located at each instant t by the position vector  $\vec{r}(t) = \overrightarrow{OM}(t)$  (Figure II.17). To analyze the motion, a coordinate system must be defined. The choice of this system depends on the specific properties of the problem at hand.



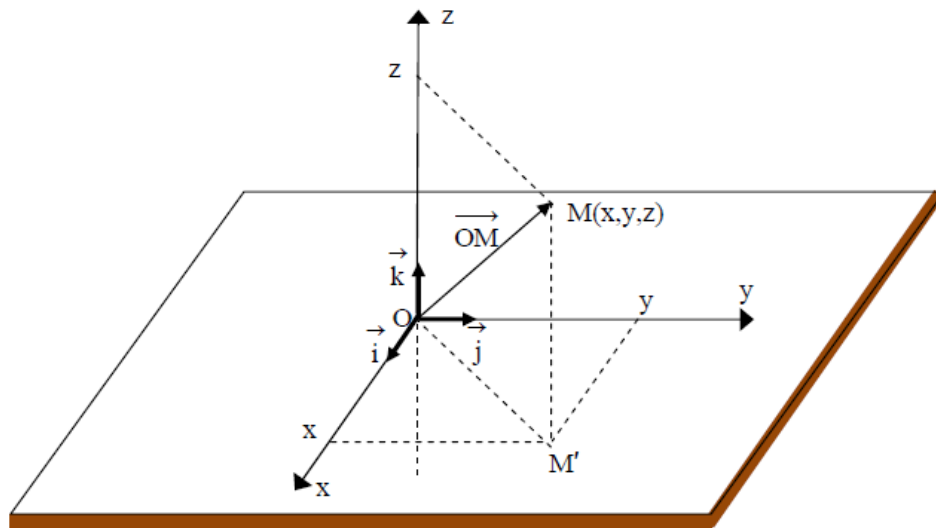
To simplify the study of motion in space, we will initially opt to use the Cartesian coordinate system. Other coordinate systems will be discussed in this chapter.

The Cartesian coordinate system is composed of three axes (Ox, Oy, Oz), with the unit vectors  $(\vec{i}, \vec{j}, \vec{k})$  (Figure II.18). The unit vectors  $(\vec{i}, \vec{j}, \vec{k})$  form an orthonormal basis (they are perpendicular to each other and have magnitudes equal to 1).

In this reference frame, the position vector of an object M is expressed as follows:

$$\vec{r} = \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} \quad (\text{II.27})$$

Here, x, y, and z are the coordinates of point M in our reference frame. As position varies with time, these coordinates are functions of the variable t.



*Figure II. 18*

The relationships:  $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$  form the *parametric equations* of the motion.

**Example:**

Figure II.19 shows the trajectory of a helical motion. At each instant, the position M of the object is located by its Cartesian coordinates

given by the parametric equations:  $\begin{cases} x = x_0 \cos(\omega t) \\ y = x_0 \sin(\omega t) \\ z = V_z t \end{cases}$

where  $x_0$  and  $V_z$  are constants.

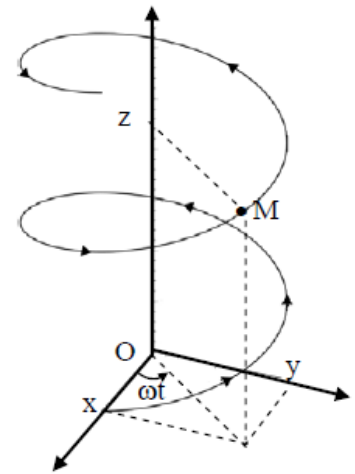


Figure II. 19

**Note:**

- In the general case, we locate a position using the three coordinates in a three-axis system (three-dimensional). When the motion occurs in a plane (planar motion), the coordinate system can be reduced to a two-dimensional system, for example, with axes (Ox, Oy) lying within the plane of motion.
- The equation of the trajectory is obtained by eliminating the variable t between the parametric equations. For example, for the motion of a projectile launched from the origin O with an initial horizontal velocity,  $\vec{V}_0$ , the parametric equations are:

$$\begin{cases} x = V_0 t \\ y = -\frac{1}{2} g t^2 \end{cases}$$

for an ascending (Oy) axis. We can then write:

$$t = \frac{x}{V_0}$$

By substituting this expression into the equation for y, we obtain the trajectory equation:

$$y = -\frac{1}{2} \frac{g}{V_0^2} x^2$$

When the parametric equations involve trigonometric functions of time, it is important to try to use relationships that characterize them. For example, if:

$$\begin{cases} x(t) = \cos t^2 \\ y(t) = \sin t^2 \end{cases}$$

Then:  $x^2 + y^2 = (\cos t^2)^2 + (\sin t^2)^2 = 1$

### 2. Displacement Vector

If an object is located at  $M_1$  at time  $t_1$ , such that:

$$\overrightarrow{OM_1}(t_1) = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \quad (\text{II.28})$$

and at time  $t_2$ , it is located at  $M_2$  such that:

$$\overrightarrow{OM_2}(t_2) = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} \quad (\text{II.29})$$

then the displacement vector is the vector  $\overrightarrow{M_1M_2}$  (figure II.20). Its relationship with position vectors is then:

$$\overrightarrow{M_1M_2} = \Delta\overrightarrow{OM} = \Delta\vec{r} = \overrightarrow{OM_2} - \overrightarrow{OM_1} \quad (\text{II.30})$$

In the Cartesian coordinate system:

$$\Delta\vec{r} = \Delta\overrightarrow{OM} = \Delta x\vec{i} + \Delta y\vec{j} + \Delta z\vec{k} \quad \text{where:} \quad \begin{cases} \Delta x = x_2 - x_1 \\ \Delta y = y_2 - y_1 \\ \Delta z = z_2 - z_1 \end{cases} \quad (\text{II.31})$$

### 3. Velocity

#### a. Average Velocity: $(\overrightarrow{V_m})_{t_1}^{t_2}$

Let  $M_1$  be the position of the object at time  $t_1$ , and  $M_2$  be its position at time  $t_2$  (figure II. 22). In this case as well, we define the average velocity vector between these two moments as:

$$(\overrightarrow{V_m})_{t_1}^{t_2} = \frac{\overrightarrow{M_1M_2}}{\Delta t} = \frac{\Delta\overrightarrow{OM}}{\Delta t} \quad (\text{II.32})$$

#### Characteristics:

- Its magnitude is:  $|\overrightarrow{V_m}|_{t_1}^{t_2} = \frac{|\overrightarrow{M_1M_2}|}{\Delta t} = \frac{|\Delta\overrightarrow{OM}|}{\Delta t} \quad (\text{II.33})$
- It has the same direction as  $\overrightarrow{M_1M_2}$ .
- It is a sliding vector; its point of application is a point on the segment  $[M_1M_2]$ .

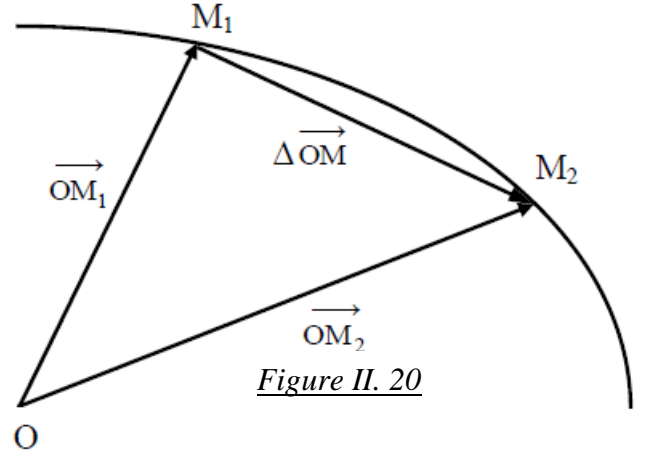
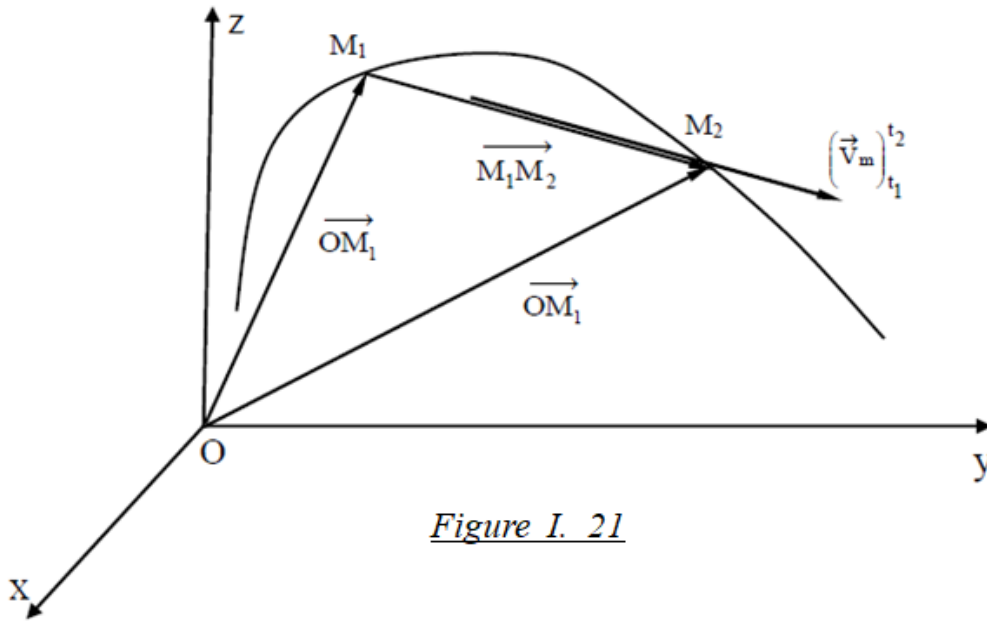


Figure II. 20



Its analytical expression in the Cartesian coordinate system is:

$$(\overline{V}_m)_{t_1}^{t_2} = \frac{\Delta x}{\Delta t} \vec{i} + \frac{\Delta y}{\Delta t} \vec{j} + \frac{\Delta z}{\Delta t} \vec{k} = V_{mx} \vec{i} + V_{my} \vec{j} + V_{mz} \vec{k} \quad (\text{II. 34})$$

### b. Instantaneous Velocity: $\vec{V}(t)$

Similar to rectilinear motion, the instantaneous velocity, in general, provides more precise information than the average velocity vector: it defines the object's velocity at each instant. The instantaneous velocity is also obtained from the average velocity by reducing the time interval  $\Delta t$  to zero. Thus, the instantaneous velocity  $\vec{V}_1(t_1)$  is obtained by considering the limit of  $(\overline{V}_m)_{t_1}^{t_2}$  as  $M_2$  approaches  $M_1$ . Graphically, the direction of the displacement vector approaches that of the tangent at  $M_1$ . Mathematically, this is expressed as:

$$\vec{V}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \overline{OM}}{\Delta t} = \frac{d\overline{OM}}{dt} \quad (\text{II. 35})$$

### Characteristics:

- The instantaneous velocity vector is *tangential to the trajectory at each instant*.
- Its direction is that of the motion.

➤ **Components in the Cartesian Coordinate System:** In the  $(O, \vec{i}, \vec{j}, \vec{k})$ , basis,  $\vec{V}$  is expressed as:

$$\vec{V}(t) = \frac{d}{dt}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k} \quad (\text{II.36})$$

Where :

$$\vec{V}(t) \begin{cases} V_x = \frac{dx}{dt} = \text{slope of the tangent to the graph of } x(t) \\ V_y = \frac{dy}{dt} = \text{slope of the tangent to the graph of } y(t) \\ V_z = \frac{dz}{dt} = \text{slope of the tangent to the graph of } z(t) \end{cases} \quad (\text{II.37})$$

And:

$$|\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2} \quad (\text{II.38})$$

➤ **Planar Motion:**

When the basis is reduced to  $(O, \vec{i}, \vec{j})$  the expression of  $\vec{V}$  is:

$$\vec{V}(t) = \frac{d}{dt}(x\vec{i} + y\vec{j}) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} = V_x\vec{i} + V_y\vec{j}$$

$$\vec{V}(t) \begin{cases} V_x = \frac{dx}{dt} \\ V_y = \frac{dy}{dt} \end{cases} \quad \text{and} \quad |\vec{V}| = \sqrt{V_x^2 + V_y^2}$$

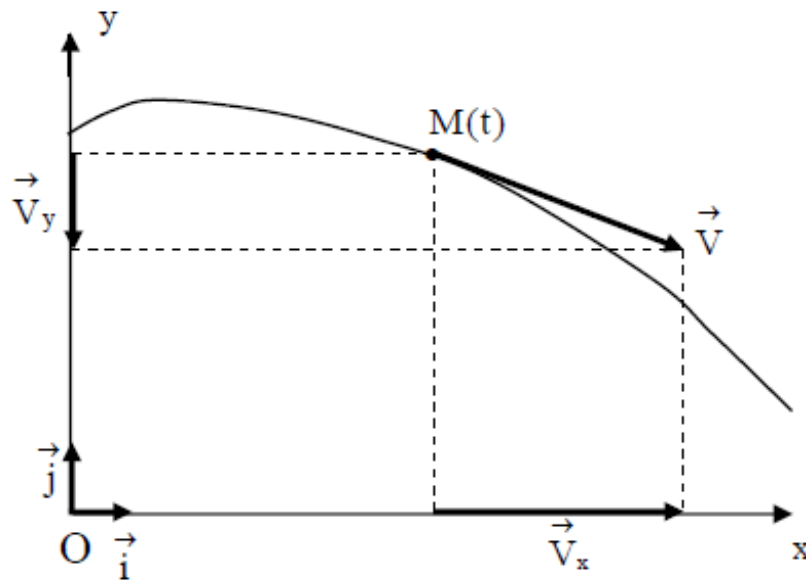


Figure II. 22

#### 4. Acceleration

a. **Average acceleration :**  $(\vec{a}_m)_{t_1}^{t_2}$

The relative change in velocity during the time interval  $\Delta t = t_2 - t_1$  is given by the average acceleration vector:

$$(\vec{a}_m)_{t_1}^{t_2} = \frac{\vec{V}_2 - \vec{V}_1}{\Delta t} = \frac{\Delta \vec{V}}{\Delta t} \quad (\text{II. 39})$$

- Its magnitude is:

$$|\vec{a}_m|_{t_1}^{t_2} = \frac{|\Delta \vec{V}|}{\Delta t} \quad (\text{II. 40})$$

- It has the same direction as  $\Delta \vec{V}$ .
- Generally,  $(\vec{a}_m)_{t_1}^{t_2}$  is applied at point M where the object is located at time t, the midpoint of the interval (Figure II.23), where:

$$t = \frac{t_1 + t_2}{2}$$

In the Cartesian coordinate system we can write that:

$$(\vec{a}_m)_{t_1}^{t_2} = \frac{\Delta V_x}{\Delta t} \vec{i} + \frac{\Delta V_y}{\Delta t} \vec{j} + \frac{\Delta V_z}{\Delta t} \vec{k} = a_{mx} \vec{i} + a_{my} \vec{j} + a_{mz} \vec{k} \quad (\text{II. 41})$$

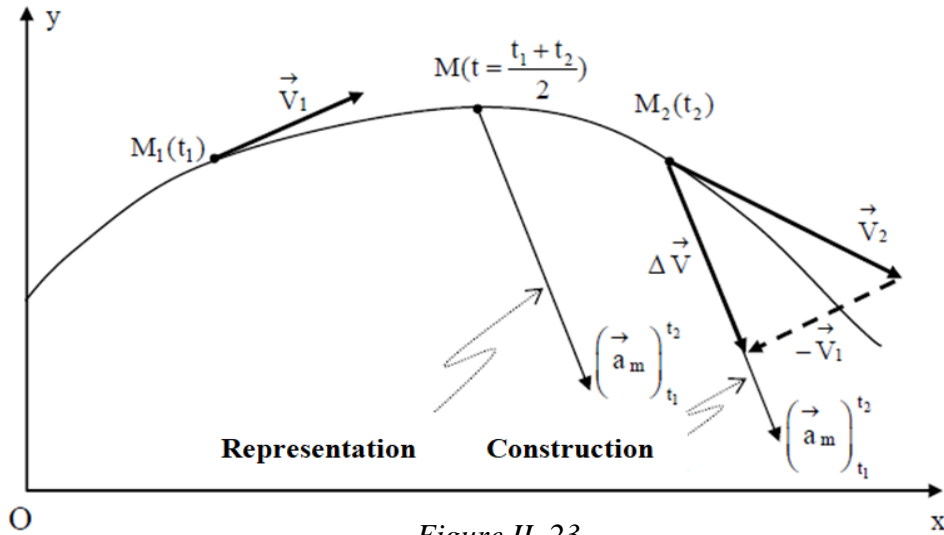


Figure II. 23

**b. Instantaneous acceleration :  $\vec{a}(t)$**

As before, we will take the limit  $\Delta t \rightarrow 0$  to obtain the instantaneous acceleration:

$$\vec{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{V}}{\Delta t} = \frac{d\vec{V}}{dt}$$

Its magnitude and direction can only be specified by introducing its components in a reference system. However,  $\vec{a}(t)$  always points *toward the concave side of the trajectory*.

In the Cartesian coordinate system,  $\vec{a}$  is expressed as:

$$\vec{a}(t) = \frac{d}{dt} (V_x \vec{i} + V_y \vec{j} + V_z \vec{k}) = \frac{dV_x}{dt} \vec{i} + \frac{dV_y}{dt} \vec{j} + \frac{dV_z}{dt} \vec{k} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \quad (\text{II. 42})$$

$$\text{Where: } \vec{a}(t) \begin{cases} a_x = \frac{dV_x}{dt} = \text{slope of the tangent to the graph of } V_x(t) \\ a_y = \frac{dV_y}{dt} = \text{slope of the tangent to the graph of } V_y(t) \\ a_z = \frac{dV_z}{dt} = \text{slope of the tangent to the graph of } V_z(t) \end{cases} \quad (\text{II.43})$$

$$\text{and: } |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad (\text{II.44})$$

### 5. Transition from acceleration to velocity and position

It is known that  $\vec{V}(t)$  and  $\vec{a}(t)$  are obtained by differentiation of  $\vec{r}(t)$ :

$$\vec{V}(t) = \frac{d\vec{r}}{dt} \quad \text{and} \quad \vec{a}(t) = \frac{d\vec{V}}{dt}$$

There are instances where the acceleration is known. Thus, we need to move from acceleration to velocity and then to position through integration. To achieve this, we can write the two previous relations in the following form:

$$d\vec{r}(t) = \vec{V}(t)dt \quad \text{and} \quad d\vec{V}(t) = \vec{a}(t)dt$$

Algebraically, these relations yield:

$$\begin{aligned} dx(t) &= V_x(t)dt ; & dy(t) &= V_y(t)dt ; & dz(t) &= V_z(t)dt \\ dV_x(t) &= a_x(t)dt ; & dV_y(t) &= a_y(t)dt ; & dV_z(t) &= a_z(t)dt \end{aligned}$$

The integration of these equations leads to:

- If we know the position vector  $\overline{OM}_0(x_0, y_0, z_0)$  of a mobile at a particular instant  $t_0$ , and the time-based expression of its velocity  $\vec{V}(t)(V_x(t), V_y(t), V_z(t))$ , we can determine its position at any instant  $t$  by writing:

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t V_x(t)dt \\ y(t) &= y_0 + \int_{t_0}^t V_y(t)dt \\ z(t) &= z_0 + \int_{t_0}^t V_z(t)dt \end{aligned} \quad (\text{II.45})$$

- If we know the velocity vector  $\vec{V}_0(V_{x0}, V_{y0}, V_{z0})$  of a mobile at a particular instant  $t_0$ , and the time-based expression of its acceleration  $\vec{a}(t)(a_x(t), a_y(t), a_z(t))$ , we can determine its position at any instant  $t$  by writing:

$$\begin{aligned}
 V_x(t) &= V_{x0} + \int_{t_0}^t a_x(t)dt \\
 V_y(t) &= V_{y0} + \int_{t_0}^t a_y(t)dt \\
 V_z(t) &= V_{z0} + \int_{t_0}^t a_z(t)dt
 \end{aligned}
 \tag{II.46}$$

**6. Approximation of instantaneous quantities using average quantities**

Similar to rectilinear motion, if  $\Delta t = t_f - t_i$  is sufficiently small, we can approximate:

- ✓ the average velocity  $(\vec{V}_m)_{t_i}^{t_f}$  with the instantaneous velocity  $\vec{V}(t)$  at the midpoint of the time interval  $[t_i, t_f]$ ;
- ✓ The average acceleration  $(\vec{a}_m)_{t_i}^{t_f}$  can be approximated by the instantaneous acceleration  $\vec{a}(t)$  at the midpoint of the time interval  $[t_i, t_f]$ , given:
- ✓  $\Delta t = t_f - t_i < \varepsilon$  (small)  $\Rightarrow \begin{cases} \vec{V}(t) = (\vec{V}_m)_{t_i}^{t_f} \\ \vec{a}(t) = (\vec{a}_m)_{t_i}^{t_f} \end{cases}$  where  $t = \frac{t_f + t_i}{2}$

**Note :** the velocity, acceleration, and speed for two and three dimensions can be written as shown in the following table.

Table II. 1: Formulas for Position, Velocity, Acceleration, and Speed

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{r} = \overline{OM} = x\vec{i} + y\vec{j}$	$\vec{r} = \overline{OM} = x\vec{i} + y\vec{j} + z\vec{k}$
Velocity	$\vec{V} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j}$	$\vec{V} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$
acceleration	$\vec{a} = \dot{\vec{V}} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$	$\vec{a} = \dot{\vec{V}} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$
Speed	$ \vec{V}  = \sqrt{V_x^2 + V_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$	$ \vec{V}  = \sqrt{V_x^2 + V_y^2 + V_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

**7. Curvilinear Abscissa, Velocity, and Acceleration**

If the trajectory of a mobile point M is known, you can:

- Orient it in an arbitrary direction.
- Choose an origin point  $M_0$ , fixed, on this trajectory.
- Choose a graphical unit. The algebraic value of the arc  $(\widehat{M_0M})$  is **the curvilinear abscissa “s”** of point M.

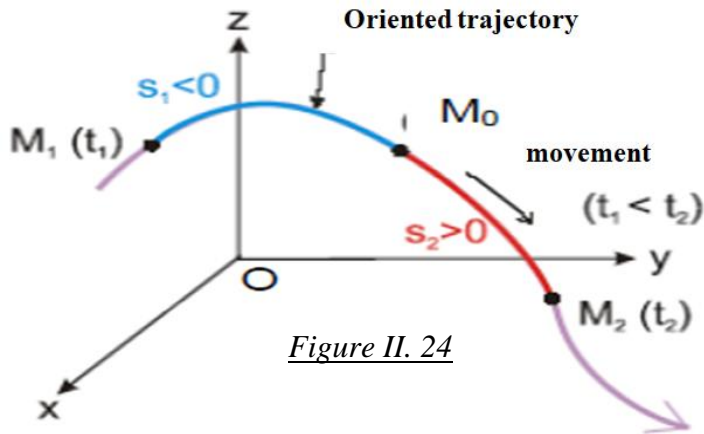


Figure II. 24

**Example:**

1) distances on a road map, are determined using curvilinear abscissas. The origin is the kilometer point zero, and the unit is the kilometer. The curvilinear speed and acceleration are defined, respectively, by the relationships:

$$V(t) = \frac{dS(t)}{dt} \quad (II. 47)$$

$$a(t) = \frac{dV(t)}{dt} \quad (II. 48)$$

2) circular motion on a trajectory with radius R:

$$s = R\theta \quad ; \quad V = R \frac{d\theta}{dt} = R\dot{\theta} \quad ; \quad a = R \frac{d^2\theta}{dt^2} = R\ddot{\theta}$$

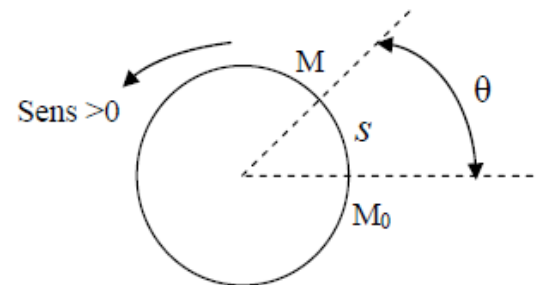


Figure II. 25

**8. Intrinsic (Tangential and Normal) components of acceleration**

➤ **Definition**

In certain cases, to determine the acceleration at a point M, its intrinsic components are used, with algebraic projections (Figure II.26):

- **a<sub>t</sub>** along a tangential axis (MT) with the unit vector  $\vec{u}_t$  oriented in the direction of motion.
- **a<sub>n</sub>** along a normal axis (MN) with the unit vector  $\vec{u}_n$  oriented towards the concave side of the trajectory.

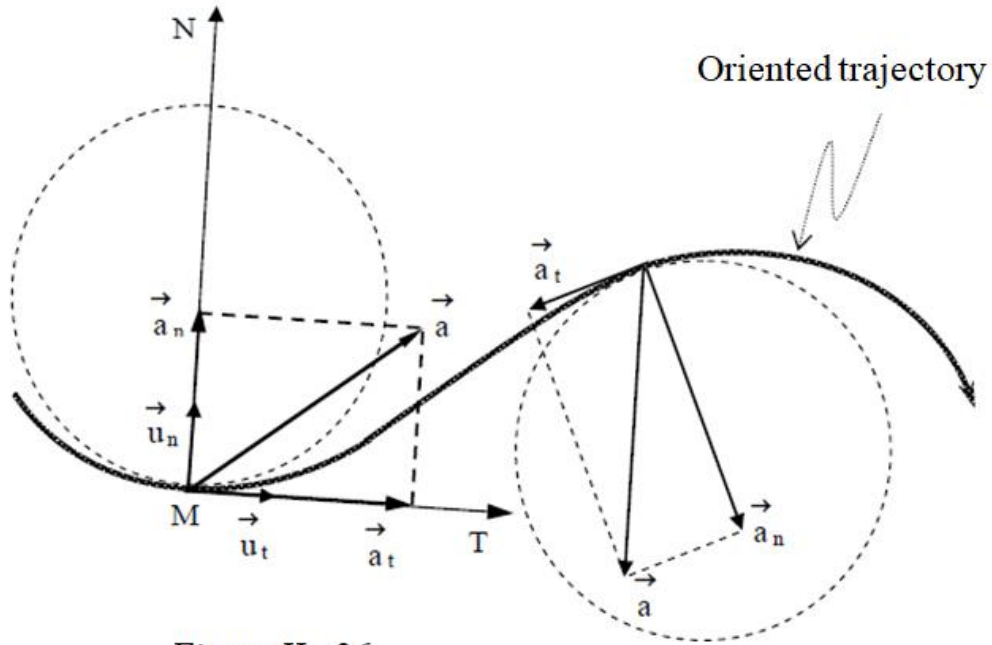


Figure II. 26

Hence the expression:

$$\vec{a} = \vec{a}_t + \vec{a}_n = a_t \vec{u}_t + a_n \vec{u}_n \quad (\text{II.49})$$

Where  $a_t$  and  $a_n$  are, respectively, the **tangential** and **normal** components of acceleration.

**Note:**

The unit vectors  $\vec{u}_t$  and  $\vec{u}_n$  form an orthonormal basis called the Frenet basis. It is a projection basis (or coordinate system) linked to the position M of the mobile object. In physics, this notion should not be confused with that of a reference frame, which is related to an observer.

➤ **Expressions of the tangential and normal components of acceleration:**

The velocity vector is tangential; it is expressed in the Frenet coordinate system as:

$$\vec{V} = V \vec{u}_t$$

where V is the magnitude of velocity. Let's differentiate this expression with respect to time to find the acceleration:

$$\vec{a} = \frac{dV}{dt} \vec{u}_t + V \frac{d\vec{u}_t}{dt}$$

Note that:

$$\frac{d\vec{u}_t}{dt} = \frac{d\vec{u}_t}{ds} \cdot \frac{ds}{dt}$$

Let us remember that:

$$\frac{ds}{dt} = V$$

The vectors of the Frenet basis form an orthonormal basis, and their derivatives satisfy several relationships. In particular, we will admit:

$$\frac{d\vec{u}_t}{ds} = \frac{1}{\rho} \vec{u}_n$$

Here,  $\rho(s)$  is called the *radius of curvature* of the trajectory at the considered point. If the trajectory is sufficiently smooth, there is always one and only one circle tangent to it;  $\rho$  then represents its radius. As a consequence, we obtain the following expression for acceleration:

$$\vec{a} = \frac{dV}{dt} \vec{u}_t + \frac{V^2}{\rho} \vec{u}_n = a_t \vec{u}_t + a_n \vec{u}_n \quad (\text{II. 50})$$

### Interpretation:

- ✓  $a_t = \frac{dV}{dt}$  indicates that the tangential component is related to the *change in the magnitude velocity*, if the object moves faster or slower.
- ✓  $a_n = \frac{V^2}{\rho}$  the presence of  $\rho$  signifies that the normal component give a curvature in the trajectory. Consequently, *the direction of the velocity vector varies*.

### Notes:

- Accelerated rectilinear motion: there is no change in the direction of the velocity vector. In this case, the radius of curvature  $\rho$  of the trajectory is *infinite*, and consequently,  $a_n=0$ . "Varied" means that  $a_t \neq 0$ .
- Uniform circular motion: the object is moving along a circular trajectory with a radius  $R = \rho$ . Therefore,  $a_n = \frac{V^2}{R} \neq 0$ . "Uniform" indicates that  $a_t = 0$ .
- Let  $\vec{r}(t)$  be a vector-valued function that denotes the position of an object as a function of time. Then  $\vec{a}(t) = \ddot{\vec{r}}(t)$  is the acceleration vector. The tangential and normal components of acceleration  $\vec{a}_T, \vec{a}_N$  are given by the formulas:

$$a_T = \vec{a} \cdot \vec{u}_T = \frac{\vec{V} \cdot \vec{a}}{|\vec{V}|}$$

$$a_N = \vec{a} \cdot \vec{u}_N = \frac{|\vec{V} \wedge \vec{a}|}{|\vec{V}|} = \sqrt{a^2 - a_T^2}$$

These components are related by the formula:  $\vec{a}(t) = a_T \vec{u}_T + a_N \vec{u}_N$

Here  $\vec{u}_T$  and  $\vec{u}_N$  is the unit tangent and the unit normal vectors to the curve defined by  $\vec{r}(t)$ .

➔ **Additional Information:**

**a- The radius of curvature at a point P on the trajectory:**

Let P' be a point close to P, and O the point of intersection of the normal's to the trajectory at P and P' (figure II. 27). The radius of curvature,  $\rho$ , is equal to the limit of the distance (OP) as P' approaches P.

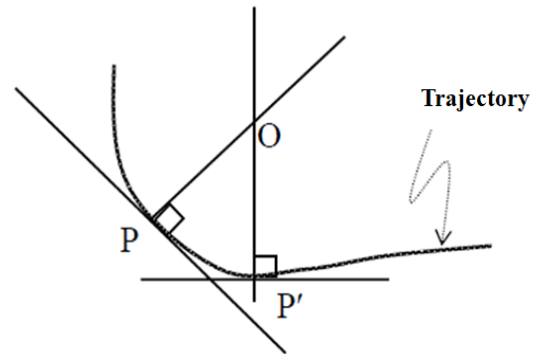


Figure II. 27

**b- Nature of motion:**

- $\rho = \text{constant}$ : The motion is circular.
- $a_t = 0$ : The motion is uniform.
- $a_t = \text{constant}$ : The motion is uniformly varied.
- $a_t > 0$ : The motion is accelerated (uniformly if  $a_t$  is constant).
- $a_t < 0$ : The motion is decelerated or retarded (uniformly if  $a_t$  is constant).

**Exercise:** Show that the motion described by the following parametric equations:

$$x(t) = \cos t^2 \quad (m)$$

$$y(t) = \sin t^2 \quad (m)$$

is uniformly accelerated circular motion in the Cartesian coordinate system (O, x, y).

**Answer:** Note that  $x^2 + y^2 = (\cos t^2)^2 + (\sin t^2)^2 = 1$ . The trajectory is thus a circle centered at O with a radius  $R = 1m$ . The velocity vector has components:

$$\vec{V} \begin{cases} V_x = \frac{dx}{dt} = -2t \sin t^2 \quad (m \cdot s^{-1}) \\ V_y = \frac{dy}{dt} = 2t \cos t^2 \quad (m \cdot s^{-1}) \end{cases}$$

And the modulus is:

$$|\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2} = 2t \quad (m/s)$$

The acceleration intrinsic components are:

$$\vec{a} \begin{cases} a_t = \frac{dV}{dt} = 2 \quad (m/s^{-2}) \\ a_n = \frac{V^2}{R} = 4t^2 \quad (m/s^{-2}) \end{cases}$$

In conclusion, the tangential component of acceleration is constant and positive ( $a_t = 2m/s$ ). Thus, the motion is uniformly accelerated.

### II.4 Motion in polar coordinates

#### 1. Definition

Some planar motions are more effectively analyzed in a different coordinate system than the Cartesian coordinates. Polar coordinates are more natural for circular and elliptical trajectories. This coordinate system is suitable for studying planar motions with rotational symmetry. In this section, we will introduce polar coordinates and define new unit vectors for analysing vectors. To specify the position of point M, we could equally state as (x,y) or as (r,θ). The radial coordinate r is the direct distance from the origin O called the pole, and the angular coordinate θ the angle from the positive x axis. Therefore, the position of any point M in the plane containing (Ox) can be determined by:

- The polar radius:  $\vec{r}(t) = |\overline{OM}(t)|$  (II.51)
- The polar angle:  $\theta(t) = (\overline{Ox}, \overline{OM})$  (II.52)

These values can vary with time. Note that the radius r and the angle θ (as defined in Figure II.28) are both positive.

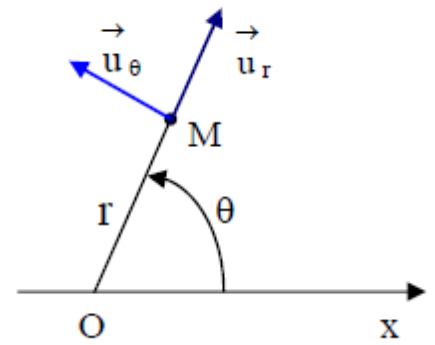


Figure II. 28

#### 2. Basis vectors:

In this system, we use the basis formed by the unit vectors:

- $\vec{u}_r$ , which has the direction of  $\overline{OM}$ ,
- $\vec{u}_\theta$ , it is just  $\vec{u}_r$  rotated by an angle of  $\pi/2$  ( $90^\circ$ ) in the anticlockwise direction.

The basis  $(\vec{u}_r, \vec{u}_\theta)$  is associated with the point M, which means that the directions of the unit vectors can varied over time, they are shown in figure II.29. We can write the polar vectors  $\vec{u}_r, \vec{u}_\theta$  in terms of  $\vec{i}$  and  $\vec{j}$ :

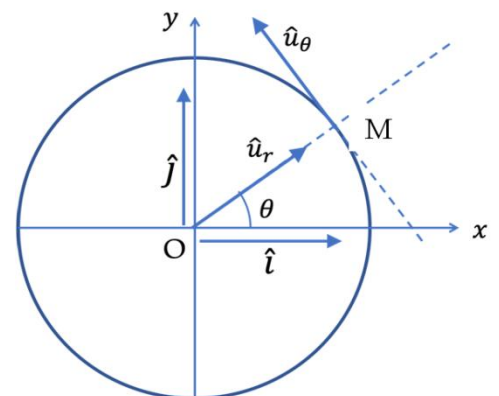


Figure II. 29

- $\vec{u}_r = \cos \theta \vec{i} + \sin \theta \vec{j}$  (II.53)
- $\vec{u}_\theta = -\sin \theta \vec{i} + \cos \theta \vec{j}$  (II.54)

Similar to  $\vec{i}$  and  $\vec{j}$ , the radial and tangential vectors  $\vec{u}_r$  and  $\vec{u}_\theta$  are orthogonal and hence, they also form a basis for Oxy plane. Since, their lengths are also equal to 1, they are a pair of orthonormal vectors:

$$|\vec{u}_r| = |\vec{u}_\theta| = 1 \quad \text{and} \quad \vec{u}_r \cdot \vec{u}_\theta = 0$$

Their derivatives verify a several number of relations, in particular, it can be assumed:

$$\frac{d\vec{u}_r}{dt} = \frac{d\theta}{dt} \vec{u}_\theta = \dot{\theta} \vec{u}_\theta \quad ; \quad \frac{d\vec{u}_\theta}{dt} = -\frac{d\theta}{dt} \vec{u}_r = -\dot{\theta} \vec{u}_r \quad (\text{II. 55})$$

**Notes :**

- Since  $\vec{u}_r$  depends on the direction of the position, which is a time-dependent. However, the Cartesian unit vectors  $\vec{i}$  and  $\vec{j}$  are independent of position:

$$\frac{d\vec{i}}{dt} = 0 \quad \text{and} \quad \frac{d\vec{j}}{dt} = 0$$

- The derivative  $d\theta/dt$  gives us the rate at which the angular variable changes, and therefore, it is called the *angular velocity*. It is denoted by the small “omega”,  $\omega$ .

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

**3. Relation with Cartesian coordinates:**

The polar coordinates  $r$  and  $\theta$  of point M are related to the Cartesian coordinates  $x$  and  $y$  by the following relationships (figure II. 30):

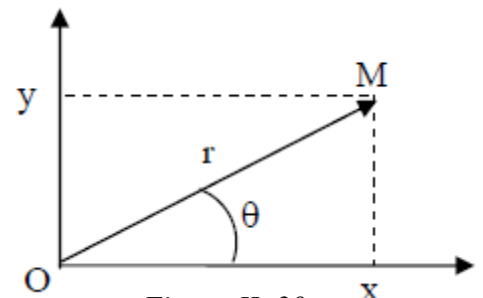


Figure II. 30

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad (\text{II.56})$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left( \frac{y}{x} \right) \end{cases} \quad (\text{II. 57})$$

**4. Position, velocity and acceleration vectors**

❖ **Position**

The definitions of  $\vec{r}(t)$  and  $\vec{u}_r$  allow us to write:

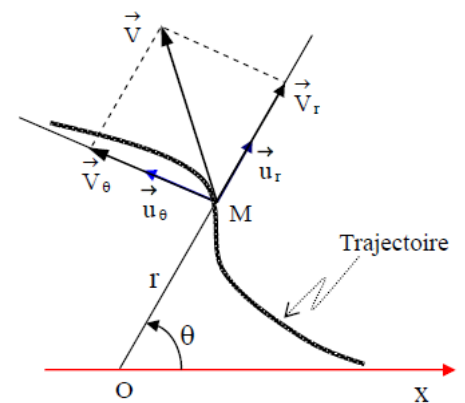


Figure II. 31

$$\overrightarrow{OM} = \vec{r}(t) = r\overrightarrow{u}_r \quad (\text{II.58})$$

### ❖ Velocity

Velocity is the derivative of position vector:

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(r\overrightarrow{u}_r)}{dt} = \frac{dr}{dt}\overrightarrow{u}_r + r\frac{d\overrightarrow{u}_r}{dt} = \dot{r}\overrightarrow{u}_r + r\dot{\theta}\overrightarrow{u}_\theta = V_r\overrightarrow{u}_r + V_\theta\overrightarrow{u}_\theta$$

The velocity components can be identified as following:

$$\vec{V} \begin{cases} V_r = \frac{dr}{dt} = \dot{r} & \text{radial component} \\ V_\theta = r\frac{d\theta}{dt} = r\dot{\theta} & \text{transverse component} \end{cases} \quad (\text{II.59})$$

$$(\text{II.60})$$

### ➡ Case of Circular Motion:

When motion happens in a circle of radius R. Then, the radial variable is fixed in time. This gives :

$$r = R = Ct \Rightarrow \frac{dr}{dt} = 0 \Rightarrow \vec{V} \begin{cases} V_r = \frac{dr}{dt} = 0 \\ V_\theta = r\frac{d\theta}{dt} = R\omega \end{cases}$$

Therefore, we will get the velocity as:

$$\vec{V} = R\omega\overrightarrow{u}_\theta$$

That is velocity is tangential, and directed in the  $\overrightarrow{u}_\theta$  direction for the counterclockwise motion, and in the  $-\overrightarrow{u}_\theta$  direction for a clockwise motion, with magnitude,  $V=|R\omega|$ , i.e., the speed v.

### ❖ Acceleration

Acceleration is derivative of velocity. Therefore:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\dot{r}\overrightarrow{u}_r + r\dot{\theta}\overrightarrow{u}_\theta) = \ddot{r}\overrightarrow{u}_r + \dot{r}\frac{d\overrightarrow{u}_r}{dt} + \dot{r}\dot{\theta}\overrightarrow{u}_\theta + r\ddot{\theta}\overrightarrow{u}_\theta + r\dot{\theta}\frac{d\overrightarrow{u}_\theta}{dt} = \ddot{r}\overrightarrow{u}_r + \dot{r}\dot{\theta}\overrightarrow{u}_\theta + \dot{r}\dot{\theta}\overrightarrow{u}_\theta + r\ddot{\theta}\overrightarrow{u}_\theta - r\dot{\theta}^2\overrightarrow{u}_r = (\ddot{r} - r\dot{\theta}^2)\overrightarrow{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\overrightarrow{u}_\theta = a_r\overrightarrow{u}_r + a_\theta\overrightarrow{u}_\theta$$

By identification, we get:

$$\vec{a} \begin{cases} a_r = (\ddot{r} - r\dot{\theta}^2) & \text{radial component} \\ a_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta}) & \text{tangential component} \end{cases} \quad (\text{II.61})$$

$$(\text{II.62})$$

**Note:**

In the case of circular motions,  $a_r$  and  $a_\theta$  are also called (negative of) centripetal (normal) ( $a_N$ ) and tangential ( $a_T$ ) accelerations respectively.

**➡ Case of Circular Motion:**

For a circular motion,  $r$  is fixed making its derivative equal to zero.

$$r = R = Cte \Rightarrow \frac{dr}{dt} = 0 \Rightarrow V_r = \frac{dr}{dt} = 0$$

Therefore,

$$\vec{a} \begin{cases} a_r = -R\dot{\theta}^2 = -R\omega^2 \\ a_\theta = r\ddot{\theta} = R\frac{d\omega}{dt} \end{cases}$$

**Note:**

Notice the negative sign in the radial part  $a_r$ ; this indicates that the radial acceleration is pointed towards the center, which is why  $R\omega^2$  is the centripetal acceleration,  $a_N$ . If we substitute  $\omega = \frac{V}{R}$  where  $V$  is the speed, then:

$$a_N = -a_r = R\omega^2$$

The tangential component  $a_\theta$  is the tangential acceleration  $a_T$ , which, for a counterclockwise motion, is positive for increasing speed and negative for decreasing speed. It can be also written using the rate of change of speed  $V$ :

$$a_\theta = R \left( \frac{d\omega}{dt} \right) = \frac{dR\omega}{dt} = \frac{dV}{dt} = a_T$$

**Example:** In planar polar coordinates, an object's position at time  $t$  is given as:

$$\begin{cases} r = e^t \quad (m) \\ \theta = \sqrt{8}t \quad (rad) \end{cases}$$

(a) Find radial velocity, tangential velocity and speed of particle at  $t = 0s$

(b) Find radial acceleration, tangential acceleration and magnitude of acceleration at  $t = 0s$

**Solution:**

(a) radial velocity and tangential velocity:

$$\vec{V} \begin{cases} V_r = \frac{dr}{dt} = \dot{r} = e^t & \text{radial velocity} & \text{at } t = 0s \Rightarrow V_r = 1m/s \\ V_\theta = r \frac{d\theta}{dt} = r\dot{\theta} = \sqrt{8}e^t & \text{tangential velocity} & \text{at } t = 0s \Rightarrow V_\theta = \sqrt{8}m/s \end{cases}$$

▪ Speed at  $t=0s$  :  $|\vec{V}| = \sqrt{V_r^2 + V_\theta^2} = \sqrt{(1)^2 + (\sqrt{8})^2} = \sqrt{9} = 3m/s$

(b) radial acceleration and tangential acceleration:

$$r = e^t \Rightarrow \dot{r} = e^t \Rightarrow \ddot{r} = e^t, \theta = \sqrt{8}t \Rightarrow \dot{\theta} = \sqrt{8} \Rightarrow \ddot{\theta} = 0$$

$$\vec{a} \begin{cases} a_r = (\ddot{r} - r\dot{\theta}^2) = e^t - 8e^t & \text{radial acceleration at } t = 0s \Rightarrow a_r = -7m/s^2 \\ a_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta}) = 2\sqrt{8}e^t & \text{tangential acceleration at } t = 0s \Rightarrow a_\theta = 2\sqrt{8}m/s^2 \end{cases}$$

▪ Magnitude of acceleration at  $t=0s$ :

$$|\vec{a}| = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(-7)^2 + (2)^2} = \sqrt{49 + 32} = \sqrt{81} = 9m/s^2$$

## II.5 Motion in cylindrical coordinates

### 1. Definition

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions. In the cylindrical coordinate system, a point M in space (Figure II. 32) is represented by the ordered triple

$(r, \theta, z)$ , where:

- $(r, \theta)$  are the polar coordinates of the point's projection "m" in the Oxy-plane .
- $z$  is the usual  $z$ -coordinate in the Cartesian coordinate system.

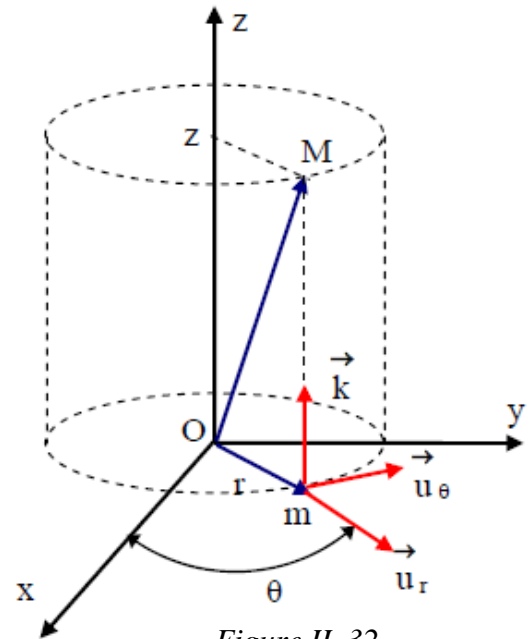


Figure II. 32

In this system, we use the basis formed by the unit vectors  $(\vec{u}_r, \vec{u}_\theta, \vec{k})$  where:

- $(\vec{u}_r, \vec{u}_\theta)$  are the unit polar vectors in the Oxy-plane .

- $\vec{k}$  is the usual unit vector in the Cartesian coordinate system.

Cylindrical coordinates are an extension of polar coordinates to the three-dimensional case.

### ➔ Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates  $(x,y,z)$  and the cylindrical coordinates  $(r,\theta,z)$  of a point are related as follows.

- These equations are used to convert from cylindrical coordinates to rectangular coordinates:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad (\text{II. 63})$$

- These equations are used to convert from rectangular coordinates to cylindrical coordinates:

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left( \frac{y}{x} \right) \\ z = z \end{cases} \quad (\text{II. 64})$$

**Note that:**  $r \in [0, +\infty[$  ,  $\theta \in [0, 2\pi]$  ,  $z \in ]-\infty, +\infty[$

In this system, the position  $\vec{OM}$ , velocity  $\vec{V}$  and acceleration  $\vec{a}$  vectors are defined by the polar components of their projections onto the plane (Oxy), supplemented by their axial components. Therefore, we get for:

## 2. Position, velocity and acceleration vectors

### ❖ Position vector

$$\vec{OM} = r\vec{u}_r + z\vec{k} \quad (\text{II.65})$$

### ❖ Velocity vector

$$\begin{aligned} \vec{V} &= \frac{d\vec{OM}}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt} + \frac{dz}{dt}\vec{k} = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta + \frac{dz}{dt}\vec{k} \\ &= V_r\vec{u}_r + V_\theta\vec{u}_\theta + V_z\vec{k} \end{aligned} \quad (\text{II. 66})$$

$$\text{Where: } \vec{V} \begin{cases} V_r = \frac{dr}{dt} = \dot{r} & \text{radial component} \\ V_\theta = r \frac{d\theta}{dt} = r\dot{\theta} & \text{transverse component} \\ V_z = \frac{dz}{dt} = \dot{z} & \text{axial component} \end{cases} \quad (\text{II. 67})$$

❖ **Acceleration vector**

$$\vec{a} = \frac{d\vec{V}}{dt} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{u}_\theta + \ddot{z}\vec{k} = a_r\vec{u}_r + a_\theta\vec{u}_\theta + a_z\vec{k} \quad (\text{II. 68})$$

$$\text{Where: } \vec{a} \begin{cases} a_r = (\ddot{r} - r\dot{\theta}^2) & \text{radial component} \\ a_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta}) & \text{transverse component} \\ a_z = \ddot{z} & \text{axial component} \end{cases} \quad (\text{II. 69})$$

**Example:** Convert the rectangular, or cartesian, coordinates (1, -3, 5) to cylindrical coordinates.

**Solution:**

Use the equations (II. 63) from Conversion between Cylindrical and Cartesian Coordinates to translate from rectangular to cylindrical coordinates:

$$r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (-3)^2} = \sqrt{10}$$

we apply the formula to find  $\theta$ . In this case,  $y$  is negative and  $x$  is positive, which means we must select the value of  $\theta$  between  $3\pi/2$  and  $2\pi$ :

$$\tan \theta = \frac{y}{x} = \frac{-3}{1} = -3 \Rightarrow \theta = \arctang(-3) = 5.03 \text{ rad}$$

In this case, the  $z$  coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5$$

The point with rectangular coordinates (1, -3, 5) has cylindrical coordinates approximately equal to :  $(r, \theta, z) = (\sqrt{10}, 5.03, 5)$

### II.6 Motion in Spherical coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, the location of a point in space is described using two distances ( $r$  and  $z$ ) and an angle measure ( $\theta$ ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

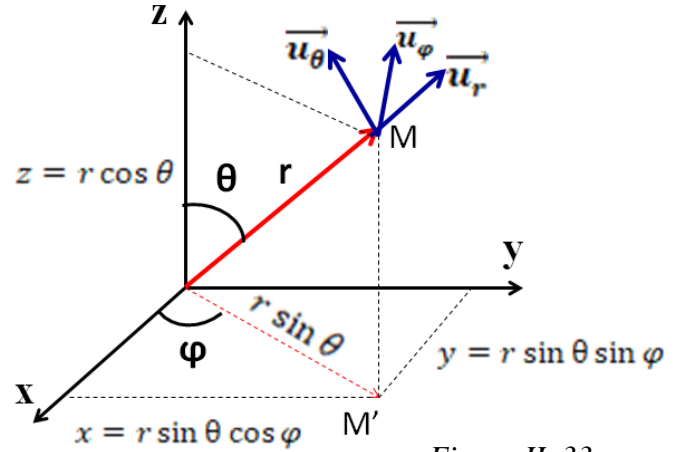


Figure II. 33

#### 1. Definition:

In the spherical coordinate system, a point M in space (Figure II. 33) is represented by the ordered triple  $(r, \theta, \varphi)$  where:

- $r$  is the distance between M and the origin O ( $r \neq 0$ );
- $\theta$  (the Greek letter teta) is the angle formed by the positive z-axis and line segment OM, where O is the origin and  $-\pi \leq \theta \leq \pi$ .
- $\varphi$  (the Greek letter phi) is the same angle used to describe the location in cylindrical coordinates;  $0 \leq \varphi \leq 2\pi$ .

In this system, we use the basis formed by the unit vectors  $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$

We can write the spherical vectors  $\vec{u}_r$ ,  $\vec{u}_\theta$  and  $\vec{u}_\varphi$  in terms of  $\vec{i}, \vec{j}$  and  $\vec{k}$  :

- $\vec{u}_r = \sin \theta \cos \varphi \vec{i} + \sin \theta \sin \varphi \vec{j} + \cos \theta \vec{k}$
- $\vec{u}_\theta = \cos \theta \cos \varphi \vec{i} + \cos \theta \sin \varphi \vec{j} - \sin \theta \vec{k}$
- $\vec{u}_\varphi = -\sin \varphi \vec{i} + \cos \varphi \vec{j}$

Similar to  $\vec{i}, \vec{j}$  and  $\vec{k}$  the vectors  $\vec{u}_r$ ,  $\vec{u}_\theta$  and  $\vec{u}_\varphi$  are orthogonal and hence, they also form a basis for spherical coordinates. Since, their lengths are also equal to 1, they are orthonormal vectors:

$$|\vec{u}_r} = |\vec{u}_\theta} = |\vec{u}_\varphi} = 1$$

$$\vec{u}_r \cdot \vec{u}_\theta = \vec{u}_\theta \cdot \vec{u}_\varphi = \vec{u}_\varphi \cdot \vec{u}_r = 0$$

$\vec{u}_r$ ,  $\vec{u}_\theta$  and  $\vec{u}_\varphi$  are successively deduced by performing a circular permutation, (figure II.34).

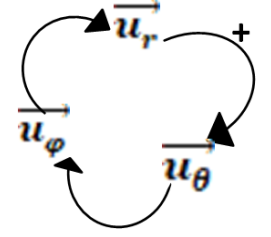


Figure II. 34

$$\vec{u}_r \wedge \vec{u}_\theta = \vec{u}_\varphi, \vec{u}_\theta \wedge \vec{u}_\varphi = \vec{u}_r, \vec{u}_\varphi \wedge \vec{u}_r = \vec{u}_\theta$$

Their derivatives of unit vectors can be assumed:

$$\frac{d\vec{u}_r}{dt} = \dot{\theta}\vec{u}_\theta + \dot{\varphi} \sin \theta \vec{u}_\varphi \quad (\text{II. 70})$$

$$\frac{d\vec{u}_\theta}{dt} = -\dot{\theta}\vec{u}_r + \dot{\varphi} \cos \theta \vec{u}_\varphi \quad (\text{II. 71})$$

$$\frac{d\vec{u}_\varphi}{dt} = -\dot{\varphi}(\sin \theta \vec{u}_r + \cos \theta \vec{u}_\theta) \quad (\text{II. 72})$$

➔ **Converting among Spherical and Rectangular Coordinates**

Rectangular coordinates  $(x,y,z)$ , and spherical coordinates  $(r,\theta,\varphi)$  of a point are related as follows:

- These equations are used to convert from spherical coordinates to rectangular coordinates

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad (\text{II. 73})$$

- These equations are used to convert from rectangular coordinates to spherical coordinates.

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \varphi = \tan^{-1} \left( \frac{y}{x} \right) \end{cases} \quad (\text{II. 74})$$

**2. Position, velocity and acceleration vectors**

❖ **Position**

$$\vec{OM} = r\vec{u}_r \quad (\text{II. 75})$$

❖ Velocity

The velocity is the derivative of the position vector, using equations (II. 69) and (II. 74), we get:

$$\begin{aligned}\vec{V} &= \frac{d\vec{OM}}{dt} = \frac{d(r\vec{u}_r)}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt} \\ &= \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + r\dot{\varphi}\sin\theta\vec{u}_\varphi = V_r\vec{u}_r + V_\theta\vec{u}_\theta + V_\varphi\vec{u}_\varphi\end{aligned}\quad (\text{II. 76})$$

Where:

$$\vec{V} \begin{cases} V_r = \dot{r} \\ V_\theta = r\dot{\theta} \\ V_\varphi = r\dot{\varphi}\sin\theta \end{cases}$$

❖ Acceleration

The acceleration is the derivative of the velocity vector, using equations (II. 69, 70, 71) and (II. 75), we get:

$$\begin{aligned}\vec{a} &= \frac{d\vec{V}}{dt} = (\ddot{r} - r\dot{\theta}^2 + r\ddot{\varphi}\sin^2\theta)\vec{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} + \dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta\cos\theta)\vec{u}_\theta + \\ &(2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta + \dot{r}\dot{\theta}\sin\theta)\vec{u}_\varphi = a_r\vec{u}_r + a_\theta\vec{u}_\theta + a_\varphi\vec{u}_\varphi\end{aligned}\quad (\text{II. 77})$$

Where:

$$\vec{a} \begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 + r\ddot{\varphi}\sin^2\theta \\ a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} + \dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta\cos\theta \\ a_\varphi = 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta + \dot{r}\dot{\theta}\sin\theta \end{cases}$$

**II.7 Circular Motion**

An object is obviously in a circular motion if it is moving in a circle. We will study uniform and non-uniform motion in a circle. In a uniform motion, speed is constant and in a non-uniform motion, motion may be speeding up or slowing down.

❖ **Position:**

Since the trajectory is a circle, the object can be located using polar coordinates:

$$\overrightarrow{OM} \begin{cases} r(t) = R = \text{constante} \\ \theta = \theta(t) \end{cases} \quad (\text{II. 78})$$

And:  $\overrightarrow{OM} = R\overrightarrow{u}_r$

❖ **Velocity:**

The components are obtained from the general expressions in polar coordinates:

$$\vec{V} = \frac{dr}{dt} \overrightarrow{u}_r + r \frac{d\theta}{dt} \overrightarrow{u}_\theta = R \frac{d\theta}{dt} \overrightarrow{u}_\theta = R\omega \overrightarrow{u}_\theta = V\overrightarrow{u}_\theta \quad (\text{II. 79})$$

- ✓ **V**: is the tangential speed when the trajectory is oriented in the direction of  $\theta$ .
- ✓  **$\omega = d\theta/dt$** : is the angular velocity of the motion. It is related to tangential speed by the equation  $V = R\omega$ , and its unit is radians per second (rad/s).

In the case where the trajectory is in the plane (O, x, y), noting that:  $\overrightarrow{u}_\theta = \vec{k} \wedge \overrightarrow{u}_r$ , the relation (II.78) becomes:

$$\vec{V} = R\omega \overrightarrow{u}_\theta = R\omega(\vec{k} \wedge \overrightarrow{u}_r) = (\omega\vec{k}) \wedge (R\overrightarrow{u}_r) = \vec{\omega} \wedge \overrightarrow{OM}$$

Thus, in circular motion, the angular velocity is represented by a vector with a magnitude of  $|d\theta/dt|$ , and the triad  $(\overrightarrow{OM}, \vec{V}, \vec{\omega})$  is right-handed (see Figure II.36).

❖ **Acceleration:**

Let's use the general expression of the acceleration in polar coordinates:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\overrightarrow{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\overrightarrow{u}_\theta \quad (\text{II. 80})$$

Taking into account that  $r = R$ , we obtain:

$$\vec{a} = (-R\dot{\theta}^2)\overrightarrow{u}_r + (R\ddot{\theta})\overrightarrow{u}_\theta = -R\omega^2\overrightarrow{u}_r + R\alpha\overrightarrow{u}_\theta \quad (\text{II. 81})$$

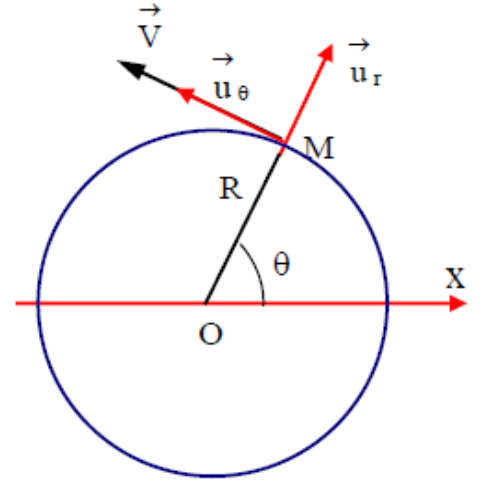


Figure II. 35

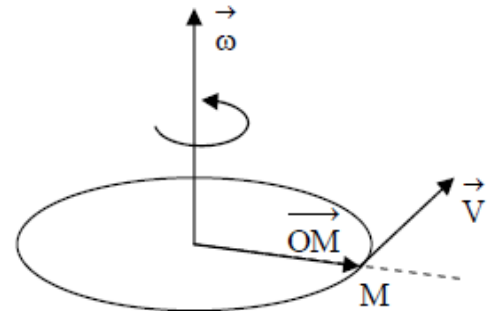


Figure II. 36

- $\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$  is the angular acceleration. It is related to the transverse component of the acceleration vector by the equation:  $a_\theta = R\alpha$  and its unit is radians per second squared (rad/s<sup>2</sup>).

❖ **Transition from angular acceleration to angular velocity and polar angle:**

By definition,

$$\alpha = \frac{d\omega}{dt} \Rightarrow \int d\omega = \int \alpha dt$$

Consequently, if we know the angular velocity  $\omega_0$  of an object at a specific time  $t_0$ , and the time-dependent expression of its angular acceleration  $\alpha(t)$ , we can determine its angular velocity at any time  $t$  by writing:

$$\omega(t) = \omega_0 + \int_{t_0}^t \alpha dt \quad (\text{II. 82})$$

Similarly, we can pass from angular velocity to polar angle by writing:

$$\omega = \frac{d\theta}{dt} \Rightarrow \int d\theta = \int \omega dt$$

Thus, if we know the position  $\theta_0$  of an object at a specific time  $t_0$ , and the time-dependent expression of its angular velocity  $\omega(t)$ , we can determine its position at any time  $t$  by writing:

$$\theta(t) = \theta_0 + \int_{t_0}^t \omega dt \quad (\text{II. 83})$$

**Example:**

An object undergoes circular motion with an angular acceleration  $\alpha = 2 \text{ rad/s}^2$ . Knowing that at time  $t = 0\text{s}$ ,  $\omega_0 = 1 \text{ rad/s}$ , and  $\theta_0 = 2 \text{ rad}$ . Determine the other characteristics of the motion, namely  $\omega(t)$  and  $\theta(t)$ .

**Solution**

Using the previous relationships, we have:

$$\omega(t) = \omega_0 + \int_0^t \alpha dt = 1 + \int_0^t 2 dt = 2t + 1 \text{ (rad/s)}$$

$$\theta(t) = \theta_0 + \int_0^t \omega dt = 2 + \int_0^t (2t + 1) dt = t^2 + t + 2 \text{ (rad)}$$

❖ **Case of uniform circular motion:**

During uniform circular motion, the angular rate of rotation and speed will be constant

This is the circular motion of an object that, at  $t = 0s$ , is at point  $M_0(R, \theta_0)$ , and is rotating at a constant angular velocity:

$$\omega = \frac{d\theta}{dt} = cte$$

$$Circular \Rightarrow R = cte$$

$$Uniform \Rightarrow \begin{cases} V = cte \\ \omega = cte \end{cases}$$

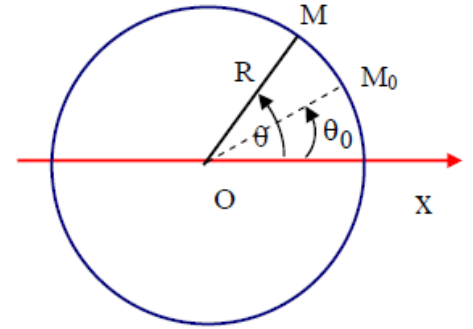


Figure II. 37

This motion is periodic, meaning that the object passes through any point on the trajectory at equal time intervals. It's characterized by:

- The period T, which is the time needed to complete one full revolution.
- The frequency f, which is the number of revolutions per unit time.

There is a simple relationship between angular velocity and period. The first revolution is completed after the period T. Thus:

$$\theta(T) = \theta_0 + \int_0^T \omega dt = \theta_0 + \omega \int_0^T dt = \theta_0 + \omega T$$

After one complete revolution, the angle  $\theta_0$  has increased by  $2\pi$ , which allows writing:

$$\theta(T) - \theta_0 = 2\pi = \omega T$$

This gives the relation:

$$\omega = \frac{2\pi}{T} \quad (II. 84)$$

Since the object completes one revolution per time period, the frequency is:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad (II. 85)$$

## II.8 Relative Motion

### 1. Introduction

The concept of reference frames was first introduced to discuss relative motion in one or more dimensions. When we say an object has a certain velocity, then this velocity is with respect to some frame that is known as the reference frame. In everyday life, when we measure the velocity of an object, the reference frame is taken to be the ground or the earth.

For example, if you are travelling in a train and the train is moving at a speed of 100 km/hr, then your speed according to another passenger sitting on that train is zero. According to him, you are not moving. But if someone observes you from outside the train, standing on the ground, according to him, you are moving at 100 km/hr as you are on the train and the train is moving at 100 km/hr.

Here, the motion observed by the observer depends on the location (frame) of the observer. This type of motion is called **Relative Motion**.

In the previous study, we have considered fixed reference frames, whose basis vectors do not change with time. However, two reference frames can be in relative motion, either because their origins are moving relative to each other, or because the relative orientation of the two reference frames changes over time, or both. In such cases, we aim to study the characteristics of motion with respect that one reference frame is known in the other reference frame.

Consider a mobile point M and the following two Cartesian coordinate systems:

- $\mathfrak{R}(O, x, y, z)$ , is referred to as the absolute frame, assumed to be fixed,
- $\mathfrak{R}'(O', x', y', z')$ , is the relative frame, in arbitrary motion relative to  $\mathfrak{R}$ ,

## 2. Absolute Motion

The motion of object M considered with respect to the absolute frame  $\mathfrak{R}(O, x, y, z)$ , is characterized by the following:

### ❖ The position vector

$$\vec{r} = \overline{OM} = x\vec{i} + y\vec{j} + z\vec{k} \quad (\text{II.86})$$

### ❖ Absolute Velocity

$$\vec{V}_a(t) = \left. \frac{d\overline{OM}}{dt} \right|_{\mathfrak{R}} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \quad (\text{II.87})$$

### ❖ Absolute Acceleration

$$\vec{a}_a(t) = \left. \frac{d\vec{V}_a}{dt} \right|_{\mathfrak{R}} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} \quad (\text{II.88})$$

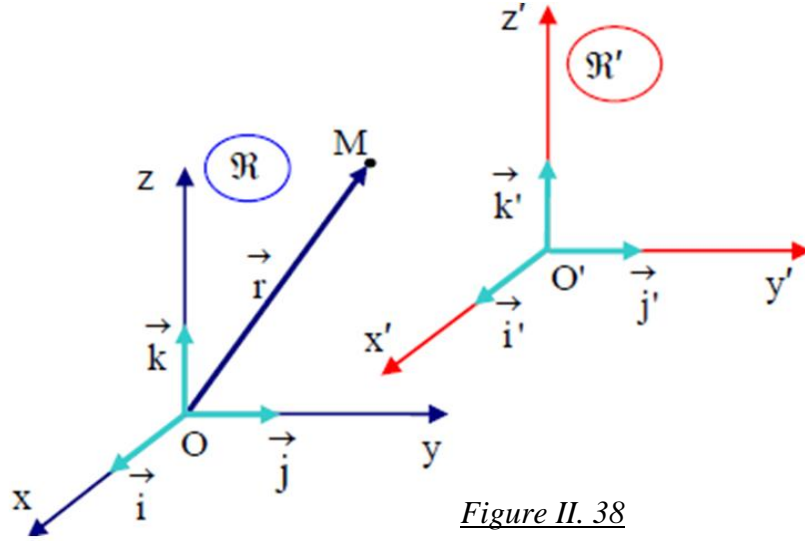


Figure II. 38

**Note:**

The derivatives are performed in  $\mathfrak{R}$ , where the basis  $(\vec{i}, \vec{j}, \vec{k})$  remains constant.

**3. Relative Motion**

The relative frame  $\mathfrak{R}'(O', x', y', z')$ , is characterized by the following:

❖ **The position vector**

$$\vec{r}' = \overline{O'M} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}' \quad (\text{II.89})$$

❖ **Relative velocity**

$$\vec{V}_r(t) = \left. \frac{d\overline{O'M}}{dt} \right|_{\mathfrak{R}'} = \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}' \quad (\text{II.90})$$

❖ **Relative acceleration**

$$\vec{a}_r(t) = \left. \frac{d\vec{V}_r}{dt} \right|_{\mathfrak{R}'} = \frac{d^2x'}{dt^2}\vec{i}' + \frac{d^2y'}{dt^2}\vec{j}' + \frac{d^2z'}{dt^2}\vec{k}' \quad (\text{II.91})$$

**Note:**

The derivatives are performed in  $\mathfrak{R}'$ , in which the basis  $(\vec{i}', \vec{j}', \vec{k}')$  remains constant.

**4. Composition of Velocity Vectors**

The absolute velocity of point M is:

$$\vec{V}_a(t) = \left. \frac{d\overline{OM}}{dt} \right|_{\mathfrak{R}}$$

Using the Chasles' relation, we get:

$$\overline{OM} = \overline{OO'} + \overline{O'M}$$

Hence:  $\vec{r} = \vec{r}_0 + \vec{r}'$  where:  $\vec{r}' = \overrightarrow{O'M} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$

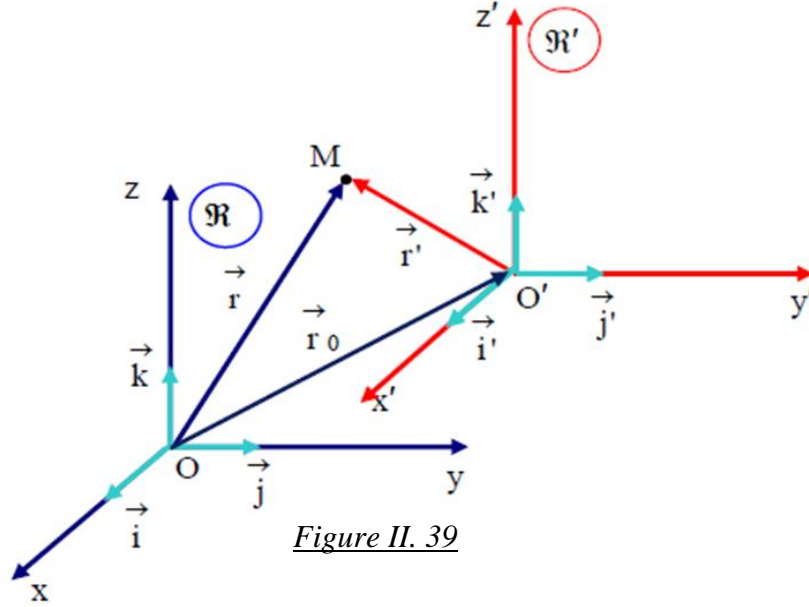


Figure II. 39

If we differentiate with respect to time, considering that the basis  $(\vec{i}', \vec{j}', \vec{k}')$  can vary in  $\mathfrak{R}$ , we obtain:

$$\begin{aligned} \vec{V}_a(t) &= \left. \frac{d\overrightarrow{OM}}{dt} \right|_{\mathfrak{R}} = \frac{d\vec{r}_0}{dt} + \frac{dx'}{dt}\vec{i}' + x' \frac{d\vec{i}'}{dt} + \frac{dy'}{dt}\vec{j}' + y' \frac{d\vec{j}'}{dt} + \frac{dz'}{dt}\vec{k}' + z' \frac{d\vec{k}'}{dt} \\ \vec{V}_a(t) &= \left( \frac{d\vec{r}_0}{dt} + x' \frac{d\vec{i}'}{dt} + y' \frac{d\vec{j}'}{dt} + z' \frac{d\vec{k}'}{dt} \right) + \left( \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}' \right) = \vec{V}_e + \vec{V}_r \quad (\text{II. 92}) \end{aligned}$$

$\vec{V}_e$ : is called the *entrainment velocity* and represents the velocity of frame  $\mathfrak{R}'$  with respect to frame  $\mathfrak{R}$ . More precisely, it is the absolute velocity of a point A fixed in reference frame  $\mathfrak{R}'$  and coinciding with the position of M at the considered time. Its expression includes two terms:

- ✓  $\frac{d\vec{r}_0}{dt} = \vec{V}_e(O')$  represents the translation velocity of the origin  $O'$  with respect to  $\mathfrak{R}$ .
- ✓  $x' \frac{d\vec{i}'}{dt} + y' \frac{d\vec{j}'}{dt} + z' \frac{d\vec{k}'}{dt}$  represents the change in orientation of the moving frame  $\mathfrak{R}'$ .

➔ **Theorem of Velocity Composition:**

The absolute velocity vector is equal to the sum of the entrainment and relative velocity vectors:

$$\vec{V}_a = \vec{V}_e + \vec{V}_r$$

**5. Composition of Acceleration Vectors**

If we differentiate the absolute velocity vector over time, we obtain the absolute acceleration vector  $\vec{a}_a$  defined in frame  $\mathfrak{R}$ :

$$\begin{aligned} \vec{a}_a(t) &= \left. \frac{d\vec{V}_a}{dt} \right|_{\mathfrak{R}} = \frac{d}{dt} \left( \frac{d\vec{r}_0}{dt} + \frac{dx'}{dt} \vec{i}' + x' \frac{d\vec{i}'}{dt} + \frac{dy'}{dt} \vec{j}' + y' \frac{d\vec{j}'}{dt} + \frac{dz'}{dt} \vec{k}' + z' \frac{d\vec{k}'}{dt} \right) \\ &= \left( \frac{d^2\vec{r}_0}{dt^2} + x' \frac{d^2\vec{i}'}{dt^2} + y' \frac{d^2\vec{j}'}{dt^2} + z' \frac{d^2\vec{k}'}{dt^2} \right) + \left( \frac{d^2x'}{dt^2} \vec{i}' + \frac{d^2y'}{dt^2} \vec{j}' + \frac{d^2z'}{dt^2} \vec{k}' \right) \\ &\quad + 2 \left( \frac{dx'}{dt} \frac{d\vec{i}'}{dt} + \frac{dy'}{dt} \frac{d\vec{j}'}{dt} + \frac{dz'}{dt} \frac{d\vec{k}'}{dt} \right) \quad (\text{II. 93}) \end{aligned}$$

This expression reveals three terms:

- $\vec{a}_e = \left( \frac{d^2\vec{r}_0}{dt^2} + x' \frac{d^2\vec{i}'}{dt^2} + y' \frac{d^2\vec{j}'}{dt^2} + z' \frac{d^2\vec{k}'}{dt^2} \right)$  : represents the **entrainment acceleration**, is the acceleration of a point A, fixed in the moving reference frame  $\mathfrak{R}'$ , as observed from the absolute reference frame  $\mathfrak{R}$ . It takes into account the second derivative of the position vector of the point coinciding with A with respect to time.
- $\vec{a}_r = \left( \frac{d^2x'}{dt^2} \vec{i}' + \frac{d^2y'}{dt^2} \vec{j}' + \frac{d^2z'}{dt^2} \vec{k}' \right)$  : **Relative acceleration** represents the acceleration of a point M as observed from the moving reference frame  $\mathfrak{R}'$ .
- $\vec{a}_c = 2 \left( \frac{dx'}{dt} \frac{d\vec{i}'}{dt} + \frac{dy'}{dt} \frac{d\vec{j}'}{dt} + \frac{dz'}{dt} \frac{d\vec{k}'}{dt} \right)$  : **Coriolis acceleration** represents an additional acceleration component caused by the change in the orientation of the moving reference frame  $\mathfrak{R}'$ . It arises due to the relative motion between the reference frames and is often encountered in systems with rotation.

➤ **Theorem of Composition of Accelerations:**

The absolute acceleration vector is equal to the sum of the entrainment, relative, and Coriolis acceleration vectors:

$$\vec{a}_a = \vec{a}_e + \vec{a}_r + \vec{a}_c$$

➤ **Particular case:**

If the moving reference frame  $\mathfrak{R}'$  translates with respect to  $\mathfrak{R}$ :

$$\text{In this case: } \vec{i}' = \vec{i}, \quad \vec{j}' = \vec{j}, \quad \vec{k}' = \vec{k}$$

$$\text{And: } \left. \frac{d\vec{i}'}{dt} \right|_{\mathfrak{R}} = \left. \frac{d\vec{j}'}{dt} \right|_{\mathfrak{R}} = \left. \frac{d\vec{k}'}{dt} \right|_{\mathfrak{R}} = 0$$

Then: the velocity and acceleration expression written as:

$$\vec{V}_a = \vec{V}_e + \vec{V}_r \Rightarrow \vec{V}_e = 0 \Rightarrow$$
$$\vec{V}_a = \vec{V}_r$$

$$\vec{a}_a = \vec{a}_e + \vec{a}_r + \vec{a}_c \Rightarrow \vec{a}_e = 0 \text{ and } \vec{a}_c = 0 \Rightarrow$$
$$\vec{a}_a = \vec{a}_r$$

These reference frames are called **Galilean reference frames**.

#### ❖ Galilean reference frames

Galilean reference frames are a type of inertial reference frame used in classical mechanics, based on the principles of Galilean relativity. In these frames, the laws of physics behave uniformly and consistently, regardless of the frame's constant velocity or lack of acceleration. This concept is fundamental to classical mechanics and forms the basis for analyzing motion and interactions between objects in non-accelerating reference frames.

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**Frequently Asked Questions about Kinematics Physics**

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**Q1) What is kinematics in physics?**

Kinematics in physics is the study of the motion of objects and systems without reference to any forces that caused the motion.

**Q2) What is the importance of kinematics?**

Kinematics is important for understanding how objects move given changes in position and velocity over time without studying the causal forces involved. Building a solid understanding of how objects move in space will then help us understand how forces are applied to various objects.

**Q3) What are the 5 formula of kinematics?**

The formulas for kinematics include five equations: the equation for velocity without position  $v=v_0+at$ ; the equation for displacement  $\Delta x=v_0t+\frac{1}{2}at^2$ ; the equation for position without acceleration  $x=x_0+\frac{1}{2}(v_0+v)t$ ; the equation for velocity without time  $v^2=v_0^2+2a\Delta x$ ; the equation for distance  $d=vt$ .

**Q4) How kinematics is used in everyday life?**

Kinematics is used in everyday life for explaining motion without reference to the forces involved. Some examples of kinematics include measuring the distance of a walking trail, understanding how we can a car's velocity to calculate its acceleration, and seeing the effects of gravity on falling objects.

**Q5) Who invented kinematics?**

Kinematics was invented by various physicists and mathematicians throughout history, including Isaac Newton, Galileo Galilei, and Franz Reuleaux.

**Q6) What is Rectilinear Motion?**

When we require only one coordinate axis and time to describe a particle's motion, it is said to be in linear motion or rectilinear motion. Examples of linear motion are a parade of soldiers, a train moving along a straight line, and many more.

**Q7) What are types of rectilinear motion?**

There are three types of rectilinear motion and they are:

- ✓ Uniform rectilinear motion
- ✓ Uniformly accelerated rectilinear motion
- ✓ Rectilinear movement with non-uniform acceleration

**Q8) Give some examples of rectilinear motion.**

Following are the rectilinear motion examples:

- ✓ The use of elevators in public places is an example of rectilinear motion.
- ✓ Gravitational forces acting on objects resulting in free fall is an example of rectilinear motion.
- ✓ Kids sliding down from a slide, is a rectilinear motion.
- ✓ The motion of planes in the sky is a rectilinear motion.

**Q9) Define distance and displacement**

The total length covered in a journey is known as the distance and the shortest path between the initial and final point of the journey is known as displacement.

**Q10) Define speed and velocity**

Speed is the rate of change in distance, while velocity is the rate of change in displacement.

**Q11) What is the difference between absolute motion and relative motion?**

The object is said to be in absolute motion when viewed from a fixed point of observation (there is no change in the position of an observer).

When there is no fixed point of view, the motion of an item is called relative motion (the position of the observer changes with respect to time).

**Q12) Why is motion always relative?**

Motion is always within a certain frame of reference; therefore, motion is always relative.

**Q13) Give an example of relative motion.**

Assume you're jogging side by side with a friend. Between you and your friend, there is no relative motion. If you pass by someone who is standing stationary, you will have relative motion with respect to that individual, but not compared to your jogging friend. As a result, motion is relative.

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**EXERCISES**


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**Exercise 1:**

A particle is in motion and is accelerating. The functional form of the velocity is:

$$V(t) = 20t - 5t^2 \quad (\text{m/s})$$

- 1- Find the functional form of the acceleration.
- 2- Find the instantaneous velocity at  $t = 1, 2, 3,$  and  $5$  s.
- 3- Find the instantaneous acceleration at  $t = 1, 2, 3,$  and  $5$  s.
- 4- Interpret the results of (3) in terms of the directions of the acceleration and velocity vectors.

**Solution:**

- 1- The acceleration is the derivative of velocity with respect to time:

$$a(t) = \frac{dV}{dt} = 20 - 10t$$

- 2- The instantaneous velocity at different times: Given velocity function:

$$V(t) = 20t - 5t^2 \quad (\text{m/s})$$

- At  $t=1\text{s}$ :  $V(1)=20(1)-5(1)^2=15$  m/s
- At  $t=2\text{s}$ :  $V(2)=20(2)-5(2)^2=20$  m/s
- At  $t=3\text{s}$ :  $V(3)=20(3)-5(3)^2=15$  m/s
- At  $t=5\text{s}$ :  $V(5)=20(5)-5(5)^2=-25$  m/s

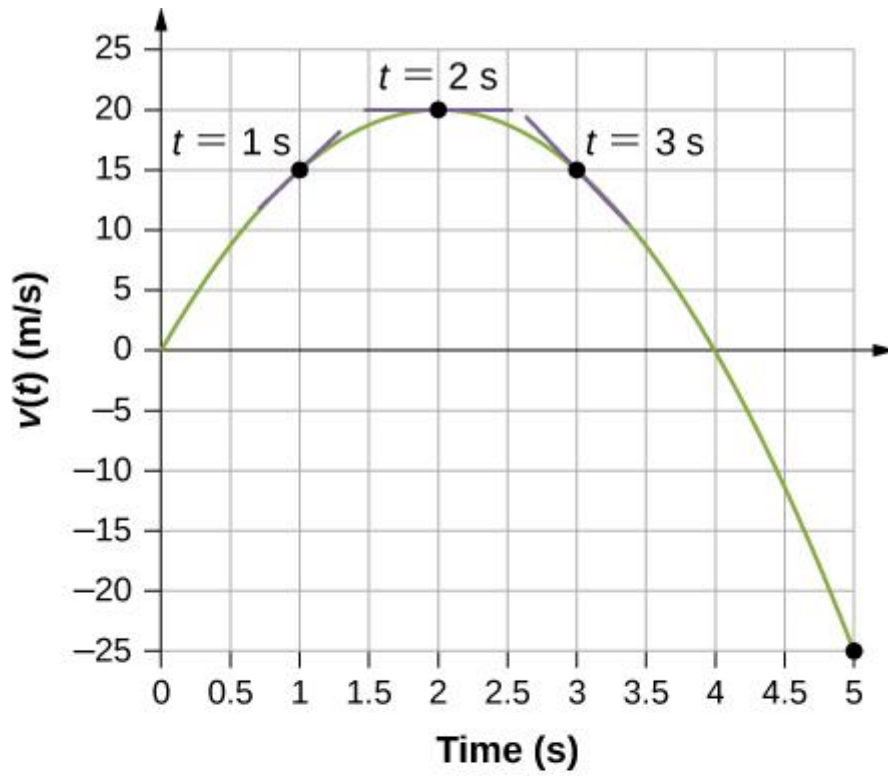
- 3- The instantaneous acceleration at different times:

$$a(t) = 20 - 10t \quad (\text{m/s}^2)$$

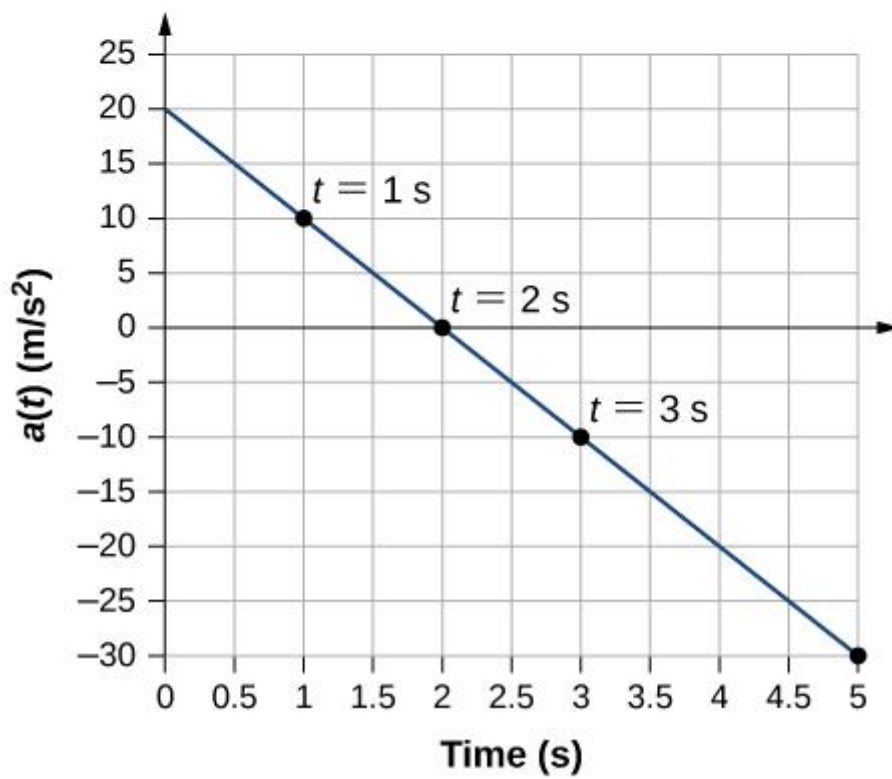
- At  $t=1\text{s}$ :  $a(1)=20-10(1)=10= 10$  m/s<sup>2</sup>
- At  $t=2\text{s}$ :  $a(2)=20-10(2)=0a(2)= 0$  m/s<sup>2</sup>
- At  $t=3\text{s}$ :  $a(3)=20-10(3)= -10$  m/s<sup>2</sup>
- At  $t=5\text{s}$ :  $a(5)=20-10(5)=-30= -30$  m/s<sup>2</sup>

We can see these results graphically in the following Figure.

**Figure:** (a) Velocity versus time. Tangent lines are indicated at times 1, 2, and 3 s. The slopes of the tangent lines are the accelerations. At  $t = 3$  s, velocity is positive. At  $t = 5$  s, velocity is negative, indicating the particle has reversed direction. (b) Acceleration versus time. Comparing the values of accelerations given by the black dots with the corresponding slopes of the tangent lines (slopes of lines through black dots) in (a), we see they are identical.



(a) Velocity



(b) Acceleration

4- **Interpretation of results:**

- At  $t = 1\text{s}$ , velocity  $V(1\text{s})=15\text{ m/s}$  is positive and acceleration is positive, so both velocity and acceleration are in the same direction. The particle is moving faster.
- At  $t = 2\text{s}$ , velocity has increased to  $V(2\text{s})=20\text{ m/s}$ , where it is maximum, which corresponds to the time when the acceleration is zero. We see that the maximum velocity occurs when the slope of the velocity function is zero, which is just the zero of the acceleration function.
- At  $t = 3\text{s}$ , velocity is  $V(3\text{s})=15\text{ m/s}$  and acceleration is negative. The particle has reduced its velocity and the acceleration vector is negative. The particle is slowing down.
- At  $t = 5\text{s}$ , velocity is  $V(5\text{s})=-25\text{ m/s}$  and acceleration is increasingly negative. Between the times  $t = 3\text{ s}$  and  $t = 5\text{ s}$  the particle has decreased its velocity to zero and then become negative, thus reversing its direction. The particle is now speeding up again, but in the opposite direction.

This interpretation shows the relationship between the signs of velocity and acceleration, indicating whether the particle is speeding up or slowing down and in which direction it's moving.

**Exercise 2:**

The position of an object is given by:  $x = 2t^3 + 5t^2 + 5$  (where  $t$  is in seconds and  $x$  in meters).

Find:

1. The velocity and acceleration as functions of time.
2. Position, velocity and acceleration at time  $t=2\text{s}$ .
3. The average velocity and average acceleration between the two instants  $t=2\text{s}$  and  $t=3\text{s}$ .

**Exercise 3:**

A particle moves in a parabolic path defined by the vector-valued function:

$$\vec{r} = t^2\vec{i} + \sqrt{5 - t^2}\vec{j}, \text{ where } t \text{ measures time in seconds.}$$

1. Find the velocity, acceleration, and speed as functions of time.
2. Sketch the curve along with the velocity vector at time  $t=1\text{s}$ .

**Solution**

1. velocity, acceleration, and speed

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{v} \begin{cases} V_x = \frac{dx}{dt} = 2t \\ V_y = \frac{dy}{dt} = -\frac{t}{\sqrt{5-t^2}} \end{cases}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} \begin{cases} a_x = \frac{dV_x}{dt} = 2 \\ a_y = \frac{dV_y}{dt} = -\frac{5}{(5-t^2)^{\frac{3}{2}}} \end{cases}$$

$$|\vec{v}| = \sqrt{V_x^2 + V_y^2} = \sqrt{(2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2} = \sqrt{4t^2 + \frac{t^2}{5-t^2}} = \sqrt{\frac{21t^2 - 4t^4}{5-t^2}} \text{ (m/s)}$$

2. The graph of  $\vec{r} = t^2\vec{i} + \sqrt{5-t^2}\vec{j}$  is a portion of a parabola (see figure)

When  $t=1$ ,  $\vec{r}(t = 1s) = 1\vec{i} + \sqrt{5-1}\vec{j} = 1\vec{i} + 2\vec{j}$

Thus the particle would be located at the point (1, 2)

when  $t=1$ .

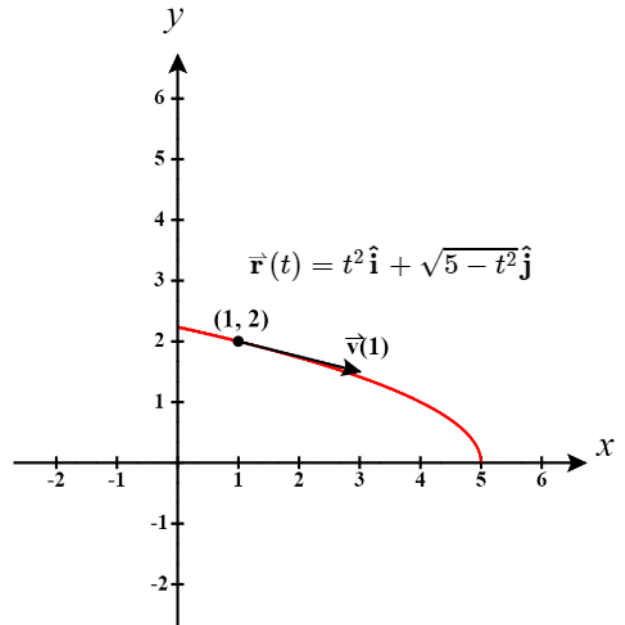
The velocity vector at  $t=1$  is :

$$\vec{v}(t = 1s) \begin{cases} V_x = 2 \\ V_y = -\frac{1}{2} \end{cases}$$

and the acceleration vector at  $t=1$  is:

$$\vec{a}(t = 1s) \begin{cases} a_x = 2 \\ a_y = -\frac{5}{8} \end{cases}$$

Notice that the velocity vector is tangent to the path, as is always the case.



**Exercise 4:**

A particle moves in a path defined by the vector-valued function:

$$\vec{r} = (t^2 - 3t)\vec{i} + (2t - 4)\vec{j} + (t + 2)\vec{k}$$

where  $t$  measures time in seconds and where distance is measured in meter. Find the velocity, acceleration, and speed as functions of time.

**Exercise 5:**

A particle moves according to the following equation:

$$\vec{r} = (t^2 + t)\vec{i} + (3t - 2)\vec{j} + (2t^3 - 4t^2)\vec{k}$$

Determine the vectors and magnitudes of velocity and acceleration at the instant  $t=2s$ .

**Exercise 6:**

The position of a particle in (oxy) plan as a function of time in SI units is:

$$\vec{r} \begin{cases} x = t \\ y = \frac{1}{2}t^2 \end{cases}$$

- 1- Find the trajectory equation then plot it.
- 2- Write expressions for its velocity and its acceleration and deduce the magnitude of the both as functions of time.
- 3- Find the intrinsic components (tangential and normal) of acceleration.

**Solution:**

- 1- The trajectory equation:  $y = f(x) \Leftrightarrow y = \frac{1}{2}x^2$ , the trajectory is a parabola.
- 2- Velocity and acceleration:

$$\vec{v} \begin{cases} V_x = \frac{dx}{dt} = 1 \\ V_y = \frac{dy}{dt} = t \end{cases} \Rightarrow |\vec{v}| = \sqrt{1 + t^2}$$

$$\vec{a} \begin{cases} a_x = \frac{dV_x}{dt} = 0 \\ a_y = \frac{dV_y}{dt} = 1 \end{cases} \Rightarrow |\vec{a}| = 1 \text{ (m.s}^{-2}\text{)}$$

- 3- The intrinsic components (tangential and normal) of acceleration:

- Tangential acceleration  $a_T$  :

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{t}{\sqrt{1 + t^2}}$$

$$a_T = \frac{t}{\sqrt{1 + t^2}}$$

- Normal acceleration  $a_N$  :

$$\vec{a} = \vec{a}_T + \vec{a}_N = a_T \vec{u}_T + a_N \vec{u}_N$$

$$a^2 = a_T^2 + a_N^2 \Rightarrow a_N = \sqrt{a^2 - a_T^2} = \sqrt{1 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2} = \frac{1}{\sqrt{1+t^2}}$$

$$a_N = \frac{1}{\sqrt{1+t^2}}$$

**Exercise 7:**

The motion of an object is described by the following parametric equations with respect to time:

$$\begin{cases} x = 2t \\ y = 4t(t - 1) \end{cases}$$

- 1- Represent and indicate the nature of the trajectory in the (Oxy) plane.
- 2- Determine the components of the velocity vector as a function of t. Deduce its magnitude.
- 3- Show that the motion has constant acceleration. Determine its tangential and normal components. Calculate the radius of curvature of the trajectory at t=1s. At what time are the velocity and acceleration vectors perpendicular to each other?
- 4- Is there a moment when the two vectors are parallel?

**Exercise 8:**

A material point M is in motion as defined by the equations:

$$\begin{cases} x = t^2 \\ y = (t - 1)^2 \end{cases}$$

- 1- Find the equation of the trajectory in Cartesian coordinates, then represent it Graphically.
- 2- What is the instant t when the magnitude of velocity is minimum?
- 3- Find the point M(x,y) at which the velocity magnitude is equal 0.10 m/s. Calculate the intrinsic components (tangential and normal) of acceleration in this case.
- 4- What is the value of the radius of curvature when the velocity is minimum?

**Exercise 9:**

A particle moving with an initial velocity  $V_0 = 50\vec{j}$  (m/s) undergoes acceleration  $\vec{a} = (35 + 2t^3)\vec{i} + (4 - t^2)\vec{j}$  (m/s<sup>2</sup>). What are the particle's position and velocity after t=3s, assuming that it starts at the origin?

**Solution:**

In the problem we are given the acceleration at all times, the initial velocity and also the initial position. We know that at t = 0, the velocity components are  $V_x = 0$  and  $V_y = 50$  m/s

and the coordinates are  $x = 0$  and  $y = 0$ .

From the acceleration  $a$  we do know something about the velocity. Since the acceleration is the time derivative of the velocity:

$$\vec{a} \begin{cases} a_x = \frac{dV_x}{dt} \Rightarrow dV_x = a_x dt \\ a_y = \frac{dV_y}{dt} \Rightarrow dV_y = a_y dt \end{cases}$$

the velocity is the indefinite integral, of the acceleration. Having learned our calculus well, we immediately write:

$$\vec{V} \begin{cases} V_x = \int a_x dt \Rightarrow V_x = \int (35 + 2t^3) dt \\ V_y = \int a_y dt \Rightarrow V_y = \int (4 - t^2) dt \end{cases}$$

$$\vec{V} \begin{cases} V_x = 35t + \frac{t^4}{2} + C_1 \\ V_y = 4t - \frac{1}{3}t^3 + C_2 \end{cases}$$

Of course, when we do the integration, we get constants  $C_1$  and  $C_2$  which have not been determined. We can determine them using the rest of the information in the problem.

Since  $V_x = 0$  and  $V_y = 50$  m/s at  $t = 0$ s and we get:

$$\begin{cases} \text{At } t = 0, V_x = 0 \Rightarrow 0 = 35(0) + \frac{0^4}{2} + C_1 \Rightarrow C_1 = 0 \\ \text{At } t = 0, V_y = 50 \Rightarrow 50 = 4(0) - \frac{1}{3}0^3 + C_2 \Rightarrow C_2 = 50 \end{cases}$$

$$\vec{V} \begin{cases} V_x = 35t + \frac{t^4}{2} \\ V_y = 4t - \frac{1}{3}t^3 + 50 \end{cases}$$

So the velocity as a function of time is  $\vec{V} = \left(35 + \frac{t^4}{2}\right)\vec{i} + \left(4t - \frac{1}{3}t^3 + 50\right)\vec{j}$  where  $t$  is in seconds and the result is in m/s. We can find the position as a function of time in the same way.

Since:

$$\vec{V} = \frac{d\vec{r}}{dt} \Rightarrow d\vec{r} = \vec{V} dt$$

$$\vec{V} \begin{cases} V_x = \frac{dx}{dt} \Rightarrow dx = V_x dt \\ V_y = \frac{dy}{dt} \Rightarrow dy = V_y dt \end{cases}$$

then  $r$  is the anti-derivative (integral) of  $V$ . We get:

$$\vec{r} \begin{cases} x = \int V_x dt \Rightarrow x = \int (35t + \frac{t^4}{2}) dt \\ y = \int V_y dt \Rightarrow y = \int (4t - \frac{1}{3} t^3 + 50) dt \end{cases}$$

$$\vec{r} \begin{cases} x = 35 \frac{t^2}{2} + \frac{t^5}{10} + C'_1 \\ y = 2t^2 - \frac{t^4}{12} + 50t + C'_2 \end{cases}$$

and once again we need to solve for the constants,  $C'_1$  and  $C'_2$

$$\begin{cases} At t = 0, x = 0 \Rightarrow 0 = 35 \left(\frac{0}{2}\right) + \frac{0^5}{10} + C'_1 \Rightarrow C'_1 = 0 \\ At t = 0, y = 0 \Rightarrow 0 = 2(0^2) - \frac{1}{12} 0^4 + C'_2 \Rightarrow C'_2 = 0 \end{cases}$$

$$\vec{r} \begin{cases} x = 35 \frac{t^2}{2} + \frac{t^5}{10} \\ y = 2t^2 - \frac{t^4}{12} + 50t \end{cases}$$

So the position as a function of time is:  $\vec{r} = \left(35 \frac{t^2}{2} + \frac{t^5}{10}\right) \vec{i} + \left(2t^2 - \frac{t^4}{12} + 50t\right) \vec{j}$

Now we can answer the questions, we want to know the value of  $\vec{r}$  (the particle's position) at  $t=3s$ :

$$\vec{r} \begin{cases} x = 35 \frac{3^2}{2} + \frac{3^5}{10} = 181 \text{ m} \\ y = 2(3)^2 - \frac{3^4}{12} + 50(3) = 161 \text{ m} \end{cases}$$

The components of the velocity at  $t = 3s$  are:

$$\vec{V} \begin{cases} V_x = 35(3) + \frac{3^4}{2} = 146 \text{ m} \cdot \text{s}^{-1} \\ V_y = 4(3) - \frac{1}{3} 3^3 + 50 = 53 \text{ m} \cdot \text{s}^{-1} \end{cases}$$

**Note:** Here we have been careful to include the proper (SI) units in the final answers because coordinates and velocities must have units.

**Exercise 10:**

A point particle moves according to the following equations:

$$\begin{cases} V_x = \frac{t}{t^2 + 1} \\ V_y = 2t \end{cases}$$

Find:

- 1- The equations of motion  $x(t)$  and  $y(t)$ , given that at:  $t=0$ ,  $x_0=0$  and  $y_0=1$ .
- 2- The equation of the trajectory.
- 3- The coordinates of the acceleration vector  $\vec{a}$ .

**Exercise 11:**

A point particle moves in the  $(xoy)$  plane according to the equations:

$$\begin{cases} x(t) = a \sin(\omega t) \\ y(t) = b \cos(\omega t) \end{cases}$$

$a$ ,  $b$ , and  $\omega$  are positive constants. Find:

- 4- The equation of the trajectory.
- 5- Expressions for velocity and acceleration.
- 6- The moment's  $t$  for which velocity and acceleration are maximum.

**Exercise 12:**

The Cartesian coordinates of particle is given by the following equations:

$$\begin{cases} x = ae^{-\theta} \cos \theta \\ y = ae^{-\theta} \sin \theta \end{cases} \quad \theta = \omega t, \quad \omega = cte$$

- 1- Find the trajectory equation in polar coordinates.
- 2- Find expressions for the velocity, acceleration and their modulus in the polar coordinates
- 3- Find the intrinsic components (tangential and normal) of acceleration.

**Solution:**

- 1- The trajectory equation in polar coordinates:  $r = f(\theta)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(ae^{-\theta} \cos \theta)^2 + (ae^{-\theta} \sin \theta)^2} = ae^{-\theta} \sqrt{(\cos \theta)^2 + (\sin \theta)^2}$$

$$r = f(\theta) = ae^{-\theta}$$

2- The velocity and acceleration in the polar coordinates:

Position vector :  $\vec{r} = r\vec{u}_r$

$$r = ae^{-\theta} \Rightarrow \dot{r} = -a\omega e^{-\theta} \Rightarrow \ddot{r} = a\omega^2 e^{-\theta}$$

$$\theta = \omega t \Rightarrow \dot{\theta} = \omega \Rightarrow \ddot{\theta} = 0$$

Velocity :

$$\vec{V} = \frac{d(r\vec{u}_r)}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta = V_r\vec{u}_r + V_\theta\vec{u}_\theta$$

$$\vec{V} \begin{cases} V_r = \frac{dr}{dt} = \dot{r} = -a\omega e^{-\theta} \\ V_\theta = r\frac{d\theta}{dt} = r\dot{\theta} = a\omega e^{-\theta} \end{cases}$$

Modulus of Velocity (Speed):  $|\vec{V}| = \sqrt{V_r^2 + V_\theta^2} = \sqrt{(-a\omega e^{-\theta})^2 + (a\omega e^{-\theta})^2} = \sqrt{2} a\omega e^{-\theta}$

Acceleration :  $\vec{a} \begin{cases} a_r = (\ddot{r} - r\dot{\theta}^2) = a\omega^2 e^{-\theta} - a\omega^2 e^{-\theta} = 0 \\ a_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta}) = -2a\omega^2 e^{-\theta} \end{cases}$

Modulus of Acceleration :  $|\vec{a}| = \sqrt{a_r^2 + a_\theta^2} = 2a\omega^2 e^{-\theta}$

3- The intrinsic components (tangential and normal) of acceleration:

- Tangential acceleration  $a_T$  :

$$a_T = \frac{d|\vec{V}|}{dt} = -\sqrt{2} a\omega^2 e^{-\theta}$$

$$a_T = -\sqrt{2} a\omega^2 e^{-\theta}$$

- Normal acceleration  $a_N$  :

$$\vec{a} = \vec{a}_T + \vec{a}_N = a_T\vec{u}_T + a_N\vec{u}_N$$

$$a^2 = a_T^2 + a_N^2 \Rightarrow a_N = \sqrt{a^2 - a_T^2} = \sqrt{(2a\omega^2 e^{-\theta})^2 - (-\sqrt{2}a\omega^2 e^{-\theta})^2}$$

$$a_N = \sqrt{2} a\omega^2 e^{-\theta}$$

### Exercise 13:

A particle is represented in polar coordinates by the equations:

$$\begin{cases} r = r_0 e^{\frac{t}{a}} \\ \theta = \frac{t}{a} \end{cases}$$

where  $r_0$  and  $a$  are positive constants.

- 1- Give the expression of the velocity vector of this particle.
- 2- Show that the angle between  $(\vec{V}, \vec{u}_\theta)$  is constant. What is its value?
- 3- Provide the expression of the acceleration vector.
- 4- Show that the angle between the acceleration vector and the unit vector normal  $(\vec{a}, \vec{u}_N)$  is constant. What is its value?
- 5- Calculate the radius of curvature of the trajectory.

**Exercise 14:**

A particle is moving in polar coordinates according to the parametric equations:

$$\begin{cases} r = r_0 (1 - \sin(\omega t)) \\ \theta = \omega t \end{cases}$$

- 1- Plot the trajectory of the motion in polar coordinates  $r = f(\theta)$ . (Note: Create a table showing the changes of  $r$  as function  $\theta$ , then plot the trajectory)
- 2- Find the polar components of the velocity and acceleration vectors, and then deduce the corresponding Cartesian components.
- 3- Find the magnitudes of velocity and acceleration, and deduce the tangential and normal components of the acceleration vector.
- 4- Find the radius of curvature of the trajectory as a function of time.
- 5- Find the length of the path between the initial ( $t_1 = 0$ s) and final ( $t_1 = \frac{2\pi}{\omega}$  s) moments.

**Exercise 15:**

A point particle M is in motion as defined by the equations:

$$\begin{cases} x(t) = R \cos(\omega t) \\ y(t) = R \sin(\omega t) \\ z(t) = V_0 t \end{cases}$$

in a system of orthonormal axes (Oxyz);  $R$ ,  $v_0$ , and  $\omega$  are positive constants.

- 1- Write the equation of the trajectory on the xOy plane and determine its nature.
- 2- Calculate the coordinates of the velocity vector  $\vec{V}$  and acceleration  $\vec{a}$ .
- 3- Since the trajectory is oriented in the direction of increasing  $z$ , find the angle between the velocity vector and the (oz) axis.

- 4- Provide the expression for the curvilinear abscissa  $S(t)$  of the particle at time  $t$ , taking  $s=0$  at  $t=0$ , in terms of  $R$ ,  $v_0$ , and  $\omega$ .

**Exercise 16:**

The motion of a point particle in cylindrical coordinates is given by the following equations:

$$\begin{cases} r(t) = 2re^{\omega t} \\ \theta(t) = \omega t \\ z(t) = 2\sqrt{2}e^{\omega t} \end{cases}$$

$r$  and  $\omega$  are positive constants. Determine:

- 1- The cylindrical components of velocity and acceleration, as well as their magnitudes.
- 2- The Cartesian components of velocity and acceleration.
- 3- The intrinsic components (tangential and normal) of acceleration. Deduce the radius of curvature.

# Chapter III

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## Dynamics of Material Point

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*In the previous chapter, dedicated to kinematics, we provided a geometric description of motion without concern for the agents causing it. In this section, we delve into dynamics, which is the branch of mechanics that deals with the causes of motion. It enables us to identify the causes of a known motion and predict motion based on given causes. In essence, it involves establishing relationships between the two.*

## Introduction

The dynamics of particles deals with the study of forces acting on a body and its effects, when the body is in motion. It is further divided into Kinematics and kinetics.

**Kinematics:** The study of motion of body without considering the forces which cause the motion of the body (see Chapter II).

**Kinetics:** The study of motion of body with considering the external forces which cause the motion of the body.

### III.1 Concept of Force

Motion is the result of the interaction between the particle and its environment. This interaction, called **force**, is characterized by the properties of the particle (mass, charge, dipole moment, etc.) and by the nature of the environment in which it is placed. We have seen that in physics, force is intuitively perceived as a quantity that reflects interactions between objects. It is a cause capable of producing or modifying the motion of a body, or causing its deformation. Forces can be classified into contact forces or action-at-a-distance forces.

➤ There is a **contact force** when it represents an interaction between two physically contacting bodies. Contact forces include, for example:

- **Frictional forces**: Frictional forces arise when two contacting bodies have relative motion with respect to each other. They always oppose the motion of the considered body.
- **Tension forces** exerted on a body: These are forces that pull on an element of a body, such as the tension exerted by a wire or a spring.

➤ Action-at-a-**distance forces**: These are forces that can manifest even if there is no physical contact between the two interacting bodies. These forces operate through vector fields, such as:

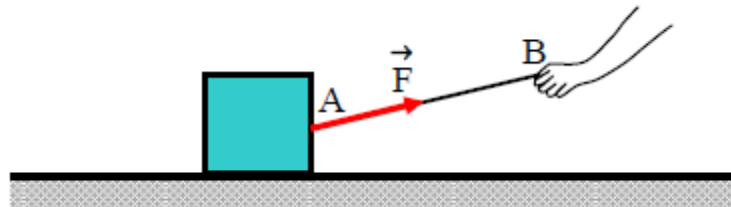
- **Gravitational forces**: These are attractive forces that act between bodies and are due to their masses. The weight of a body and the forces exchanged by celestial bodies are primarily gravitational forces.
- **Electric forces**: They act between two objects carrying electric charges. They can be either attractive or repulsive.
- **Magnetic forces**: They act between magnets, between magnets and certain materials (especially iron), or between two conductors carrying an electric current. They can also be attractive or repulsive

### III. 1.1 Force Vector

Force can be represented by a vector  $\vec{F}$  with four properties:

- **Direction:** The straight line along which the action is exerted (as in the case of the wire in Figure III.1).
- **Magnitude:** The intensity or strength of the force associated with an appropriate unit.
- **Point of Application:** The point on the object where the force is exerted (point A in this case, Figure III.1).
- **Sense:** The direction in which the force is applied (from A to B, as seen in Figure III.1).

Forces are additive, meaning that if  $N$  forces act simultaneously on an object, the resulting motion of the object is the same as if it were subjected to the action of a single force equal to the vector sum of the  $N$  forces. This sum is referred to as the **resultant** of the  $N$  forces.



*Figure III. 1*

### III.1.2 Fundamental Interactions

There is four fundamental interactions:

- **Gravitational Interaction:** as an attractive force between all particles. This force appears in most phenomena described by astronomy and geology (the motion of celestial bodies, the rise of tides, objects attracted to the Earth, etc...).
- **Electromagnetic Interaction:** This interaction occurs between electric charges in all phenomena involving electricity and/or magnetism.
- **Strong Interaction:** This is the force that acts between nucleons, the constituents of an atom's nucleus. It enables particles composed of quarks, such as protons and neutrons, to remain stable. The strong interaction operates at very short distances and is responsible for the cohesion of the nucleus.
- **Weak Interaction:** This applies to all types of matter particles (quarks, electrons, neutrinos, etc.). Neutrinos, in particular, which are electrically neutral and not made up of quarks, are sensitive only to weak and gravitational interactions. The weak interaction is observed in certain types of nuclear reactions, such as radioactivity.

III.2 Principle of Inertia

III.2.1 Experiment

Let's consider Figure III.2, which depicts a collision between two particles carried out under "zero" friction conditions on a horizontal plane. To study the motions, a chronophotography was performed (breaking down movements using a series of photographs taken at regular time intervals), which is reproduced in the following figure.

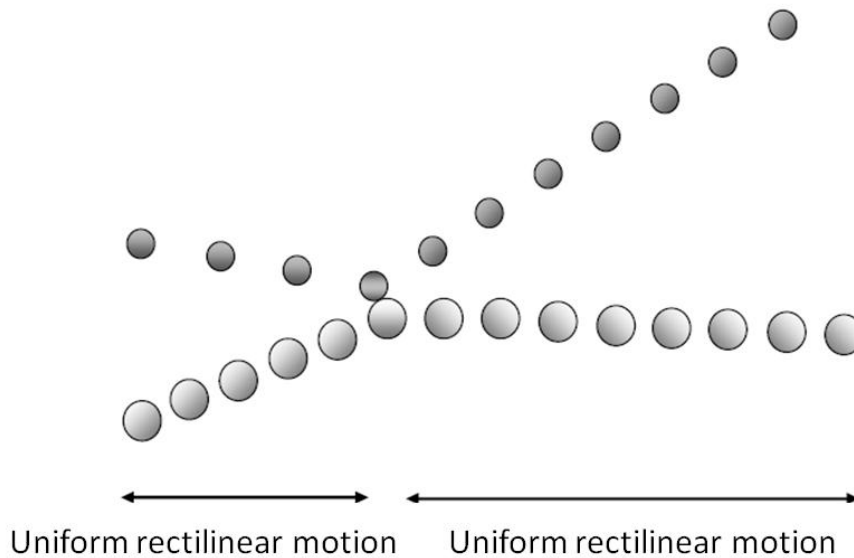


Figure III. 2

We can observe that the trajectory of the center of each particle is linear, and its velocity is constant (the distances between the particles are of the same length) both before and after the collision. The forces applied to each of them in these two zones are gravity (weight) and the contact force with the ground (Figure III.3).

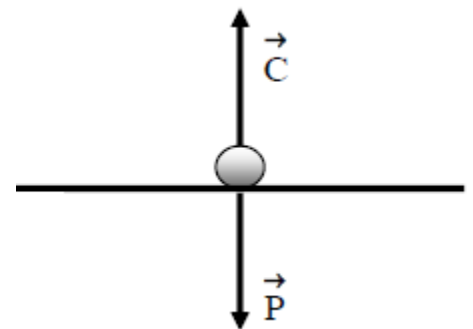


Figure III. 3

Since the motion is maintained on the horizontal plane, this means that neither of the two forces is stronger than the other, and their vector sum is zero. Let's denote the contact force experienced by one of the two objects from the other during the collision as  $\vec{F}_C$ , as shown in Figure III.4. The force balance applied to it is:

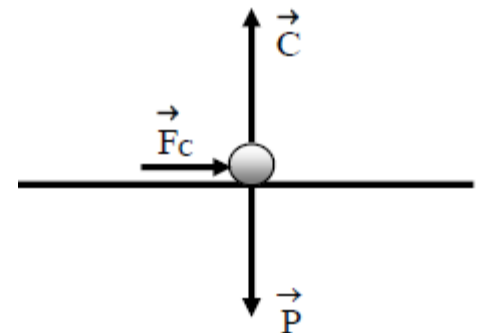


Figure III. 4

$$\underbrace{\vec{P} + \vec{C}}_0 + \vec{F}_C = \vec{F}_C$$

Notice that, the net force isn't zero during the collision, and the direction of motion has changed. Also, note that when these objects are placed on the ground, they remain stationary until an external action comes into play to move them.

**Conclusion:** These particles maintain their state of rest or uniform rectilinear motion as long as no external force intervenes to alter them.

### III.2.2 Principle of Inertia

**"In the absence of external forces, an object at rest remains at rest, and an object in motion continue to move in uniform rectilinear motion."**

The absence of external forces means that the vector sum of the applied forces is zero.

This property of all bodies to resist changes in velocity is called **inertia**. Anybody possessing this property has zero acceleration. Such a body is said to have inertia.

**Note:** Many common phenomena are manifestations of inertia. The movements of passengers caused by vehicles during acceleration, braking, and turning are examples of this.

### III.2.3 Isolated Body in mechanics

In the absence of external forces, a body is said to be free or mechanically isolated. A system is defined as a collection of bodies distinguished from the rest of the universe. A system is isolated when the resultant of the external forces applied to it is zero.

**Note:** In some contexts, systems for which the sum of the forces acting on them is zero are referred to as "pseudo-isolated," reserving the term "isolated" for cases where they are not subjected to any force.

### III.2.4 Galilean Reference Frame (or Inertial Reference Frame)

We have seen that a free particle, thus possessing inertia, must have zero acceleration. However, acceleration depends on the reference frame used. Therefore, a particle is only free with respect to a reference frame in which it does not experience acceleration.

**An inertial reference frame (or Galilean reference frame) is a reference frame in which a body not subjected to any force or subject to forces with a net result of zero will be at rest or in uniform rectilinear motion.**

To define this, let's consider the kinematic state of a free particle with respect to a moving reference frame. Its relative acceleration  $\vec{a}_r$  is given by:

$$\vec{a}_r = \vec{a}_a - \vec{a}_e - \vec{a}_c \quad (\text{III.1})$$

$\vec{a}_a, \vec{a}_e$  and  $\vec{a}_c$  being the absolute, entrainment, and Coriolis accelerations respectively. If the frame is in uniform rectilinear motion,

- ✓ The characteristic of being rectilinear implies:  $\vec{a}_c = 0$  and  $\vec{a}_e = \frac{d^2\vec{r}_0}{dt^2}$
- ✓ The characteristic of being uniform implies:  $\frac{d^2\vec{r}_0}{dt^2} = 0$  thus  $\vec{a}_e = 0$

Since the particle is isolated:

$$\vec{a}_a = 0$$

This leads to:

$$\vec{a}_r = \vec{a}_a = 0$$

Therefore, a free particle does not acceleration in a fixed reference frame or in uniform rectilinear motion. The latter is hence Galilean. In the majority of common experiments, a reference frame attached to the Earth can be considered an inertial reference frame.

**Note:** In reality, the Earth is not truly an inertial reference frame due to its orbital motion around the Sun and its own rotation around its axis. However, in the first motion, its acceleration towards the Sun is around  $4,4 \cdot 10^{-3} \text{ m/s}^2$ , and in the second motion, a point on the equator has an additional acceleration towards the center of the Earth of approximately  $3,37 \cdot 10^{-2} \text{ m/s}^2$ . These two accelerations, being small compared to gravity, can be neglected.

**Illustration:** Let's imagine an object placed on the bed of a truck in uniform rectilinear motion. The object remains still with respect to the vehicle as long as the vehicle's motion maintains its uniform rectilinear characteristic. However, if the bed is sufficiently smooth, the object might slide when the truck quickly executes a turn. In this case, the reference frame tied to the truck experiences curvilinear motion, and the principle of inertia no longer applies. Consequently, the object would not retain its state of rest with respect to the truck.

### II.2.5 Concept of Mass

Up until this point in the course, we understand that the motion of a particle is described by its position, velocity, and acceleration vectors. However, everyday examples show that these kinematic parameters are insufficient to fully account for the behavior of a material point. For instance, consider a person attempting to quickly move a wooden crate, initially at rest, over a fixed distance. They would have to accelerate as much as possible to increase the speed rapidly and reach the goal in the shortest time. It is clearly that if they place a second crate on top of the first, it would be more challenging to accelerate, and hence alter the motion's speed. Thus, the greater amount of matter to be moved, the greater resistance to changing its speed, or in other words, its inertia will be greater.

This necessitates introducing a physical quantity that measures a body's ability to resist the motion we want to impose. Recognizing that a body's mass is proportional to the quantity of matter it comprises, it follows that mass is larger when its inertia is greater. Therefore, the mass can be seen as a measure of inertia. This implies that an object with a larger mass will be more difficult to accelerate, decelerate, or change its direction.

- **Mass:** is a fundamental measure of the amount of matter in the object. It is denoted by “m”. The SI unit of mass is Kilograms (Kg). It is a scalar quantity.

### III.3 Linear Momentum

#### III.3.1 Definitions

In certain situations, to study the state of a body, it is necessary to introduce information about the object itself and its motion. We've seen that, in addition to its kinematic characteristics, motion can be influenced by the mass of the object. The concept of **momentum** provides a quantitative distinction between the motions of two particles with the same velocity but different masses. *Momentum is the quantity that combines a kinematic property of motion, velocity, and mass.*

The momentum  $\vec{\mathcal{P}}$  of a particle is defined as the product of its mass and its velocity vector.

$$\vec{\mathcal{P}} = m\vec{V} \quad (\text{III.2})$$

- **Momentum:** Momentum can be defined as **mass in motion**. All objects have mass; so if an object is moving, then it has momentum, it has its mass in motion. It depends upon the variables mass and velocity. In terms of an equation, the momentum of an object is equal to the mass of the object times the velocity of the object. Momentum = mass • velocity.

For a system consisting of N particles, its total linear momentum  $\vec{\mathcal{P}}_t$  is defined as the vector sum of the linear momenta of each particle.

$$\vec{\mathcal{P}}_t = \sum_{i=1}^N \vec{\mathcal{P}}_i \quad (\text{III.3})$$

Hence, linear momentum is a vector with the same direction and sense as velocity and its SI unit is kg.m/s.

Impulse  $d\vec{\mathcal{P}}$  is the change momentum produced in a brief interval of time.

#### III.3.2 Conservation of Momentum

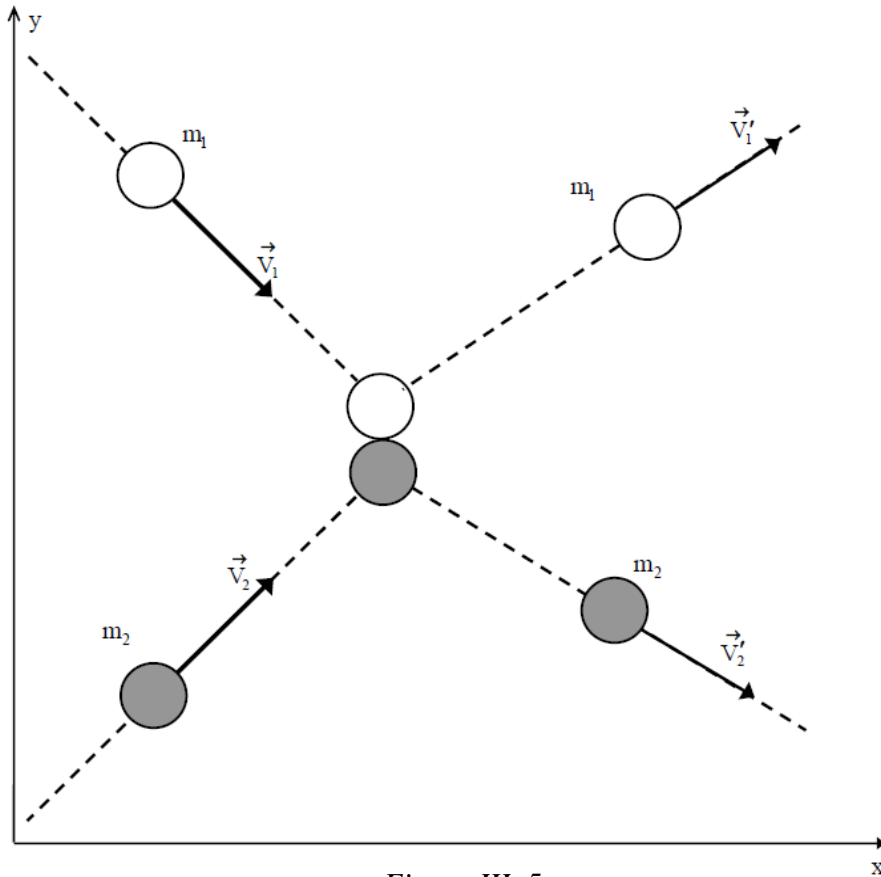
##### a. Case of one Particle:

An isolated body possesses inertia and, consequently, moves with a constant velocity, implying that it conserves its momentum.

**Principle of conservation of momentum:** A free particle moves with a constant momentum.

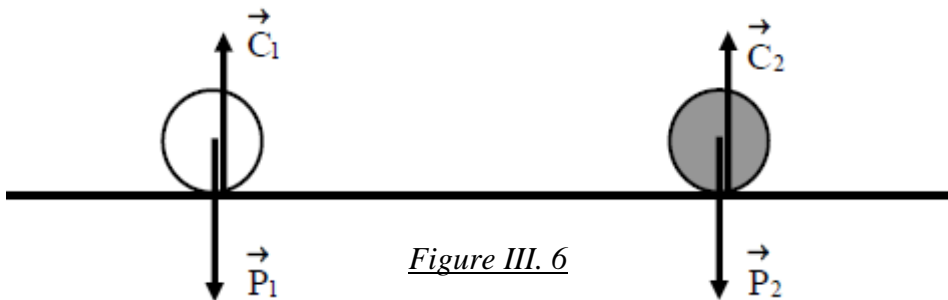
**b. System of Two Particles:**

Figure III.5 depicts a collision between two particles conducted under "zero" friction conditions on a horizontal plane.



*Figure III. 5*

Let's consider the system formed by the two particles. In order to apply the principle of conservation of momentum to it, the resultant of the acting external forces must be zero. Let's verify if this condition is met. The forces applied to the two particles are gravity (weight) and the contact force with the horizontal plane (figure III. 6).



*Figure III. 6*

Since the motion is maintained on the horizontal plane, this means that neither of the two forces prevails over the other, and their vector sum is zero. The balance of external forces is thus:

$$\sum \vec{F}_{ext} = \underbrace{\vec{P}_1 + \vec{C}_1}_0 + \underbrace{\vec{P}_2 + \vec{C}_2}_0 = 0$$

**Note:** This relationship also holds during the collision since the force exerted by  $m_1$  on  $m_2$  and the force applied by  $m_2$  on  $m_1$  are internal forces within the system formed by the two masses.

With the resultant of external forces acting on the system being zero, we can apply the principle of conservation of linear momentum to it:

$$\underbrace{m_1 \vec{V}_1 + m_2 \vec{V}_2}_{\vec{P}_t} = \underbrace{m_1 \vec{V}'_1 + m_2 \vec{V}'_2}_{\vec{P}'_t} \quad (\text{III.4})$$

The components equations of conservation momentum are for each:

$$m_1 V_{1x} + m_2 V_{2x} = m_1 V'_{1x} + m_2 V'_{2x} \quad (\text{III.5})$$

$$m_1 V_{1y} + m_2 V_{2y} = m_1 V'_{1y} + m_2 V'_{2y} \quad (\text{III.6})$$

Graphically, the conservation of momentum principle can be illustrated by Figure III.7.

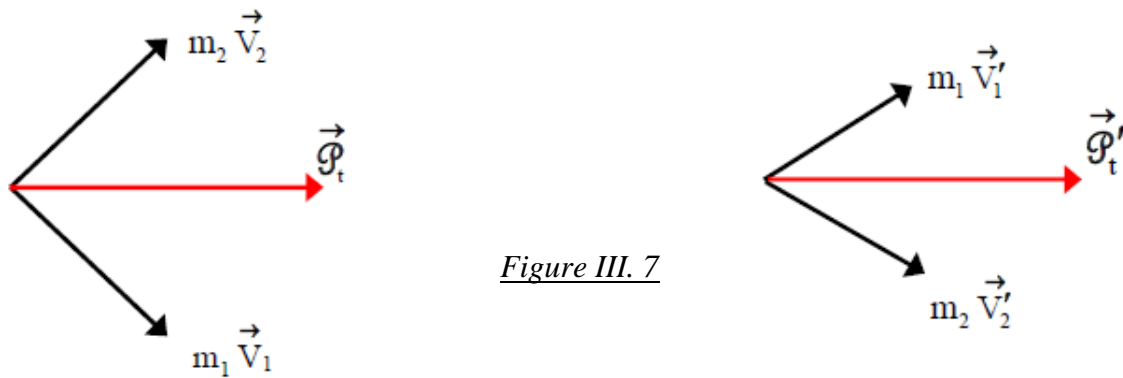


Figure III.7

**Principle of Conservation Momentum:** This principle can be extended to a system consisting of any number of particles: **The total momentum of an isolated system of multiple particles is constant.**

$$\text{If: } \sum \vec{F}_{ext} = 0 \quad \text{Then: } \vec{P}_t = \sum \vec{P}_i = \text{constante} \quad (\text{III.7})$$

**Notes:**

- 1) The momentum of each particle may vary in some cases, but the total momentum remains constant.
- 2) The principle of conservation momentum is valid even at the scale of elementary particles of matter.

For the example of the collision between two particles that we studied, we can write:

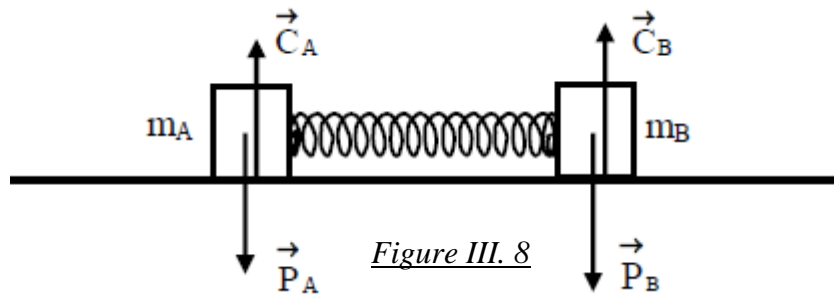
$$\vec{\mathcal{P}}_t = \vec{\mathcal{P}}_t' \Leftrightarrow \vec{\mathcal{P}}_1 + \vec{\mathcal{P}}_2 = \vec{\mathcal{P}}_1' + \vec{\mathcal{P}}_2' \quad (\text{III. 8})$$

$$\Leftrightarrow \vec{\mathcal{P}}_1' - \vec{\mathcal{P}}_1 = -(\vec{\mathcal{P}}_2' - \vec{\mathcal{P}}_2) \quad (\text{III. 9})$$

$$\Leftrightarrow \Delta\vec{\mathcal{P}}_1 = -\Delta\vec{\mathcal{P}}_2 \quad (\text{III. 10})$$

- 3) This means that the momentum lost by one of the two particles is gained by the other. Therefore, there is an exchange of momentum during the collision.

**Example:** Two blocks A and B, with masses  $m_A$  and  $m_B$ , are connected by a negligible-mass spring and are resting on a horizontal plane. The spring is stretched by moving the two blocks away from each other, and then they are released from their initial positions. Determine the relationship between their velocities, assuming the absence of friction (Figure III. 8).



If we consider the system composed of both blocks, and taking into account that the tension of the spring is an internal force, we will have:

$$\sum \vec{F}_{ext} = \underbrace{\vec{P}_A + \vec{C}_A}_0 + \underbrace{\vec{P}_B + \vec{C}_B}_0 = 0$$

As a result, the total momentum is conserved:

$$(\vec{\mathcal{P}}_t)_{t=0s} = (\vec{\mathcal{P}}_t')_{t>0s}$$

Thus:

$$m_A \vec{V}_{A0} + m_B \vec{V}_{B0} = 0 = m_A \vec{V}'_A + m_B \vec{V}'_B$$

where:

$$\vec{V}'_A = -\frac{m_B}{m_A} \vec{V}'_B$$

### III.4 Laws of motion (Newton's Laws)

When a particle/body is at rest or moving in a straight line (rectilinear motion) or in a curved line (curvilinear motion), the particle/body obeys certain laws of motion. These laws are called **Newton's law of motion**. These laws are also called the principles of motion, or principles of Dynamics. Three laws form the foundation of classical mechanics:

#### III.4.1 First Law:

It is the principle of inertia (cf. §III.2.4). Everybody continues to be in its state of rest or of uniform motion in a straight line unless and until it is acted upon some external force to change that state. It is also called the law of inertia, and consists of the following two parts:

- 1) A body at rest continues in the same state, unless acted upon by some external force. It appears to be self-evident, as a train at rest on a level track will not move unless pulled by an engine. Similarly, a book lying on a table remains at rest, unless it is lifted or pushed.
- 2) A body moving with a uniform velocity continues its state of uniform motion in a straight line, unless it is compelled by some external force to change its state. It cannot be exemplified because it is, practically, impossible to get rid of the forces acting on a body.

#### III.4.2 Second Law:

In Newtonian mechanics, the time derivative of the momentum vector of a body is equal to the sum of the forces acting on it. Mathematically, this is expressed by the relation:

$$\vec{F} = \frac{d\vec{P}}{dt} \quad (\text{III. 11})$$

where  $\vec{P}$  is the momentum of the particle and  $\vec{F}$  is the resultant of the forces it experiences.

This equation, known as the **“fundamental relation of dynamics”**, holds as long as the velocity is much less than that of light. Notice that it associates a kinetic term, the derivative of the momentum vector, with a dynamic term, the sum of the forces applied to the material point. This allows for predicting motion given the forces and determining their resultant for a given motion.

- **Particular case of a body with constant mass:** Since mass is constant, the derivative of momentum is carried out as follows:

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{V})}{dt} = \underbrace{\frac{dm}{dt}}_0 \vec{V} + m \frac{d\vec{V}}{dt} = m \frac{d\vec{V}}{dt} = m\vec{a} \quad (\text{III. 12})$$

Hence, the **fundamental relation of dynamics (F.R.D.)** for a body with constant mass is given by:

$$\vec{F} = m\vec{a} \quad (\text{III. 13})$$

**Notes:**

- $\vec{F}$  represents the resultant of external forces acting on the considered body;
- Since this is a vector relationship, the components for each:

$$F_x = ma_x \quad ; \quad F_y = ma_y \quad ; \quad F_z = ma_z$$

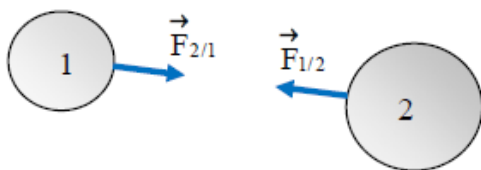
- $\vec{a} = \frac{\vec{F}}{m}$  has the same direction as the resultant force.

**We can conclude that:**

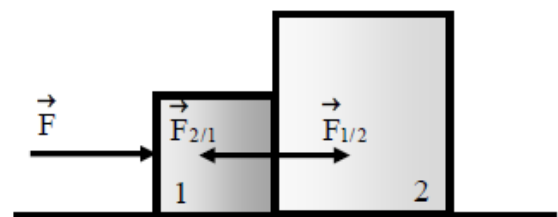
- 1) For a given body, greater force produces greater acceleration and the lesser force produces the lesser acceleration.
- 2) The acceleration is zero, if there is no external force on the body.
- 3) We know that,  $1\text{N} = 1\text{kg}\cdot\text{m}/\text{s}^2$ , that is, the unit force (Newton) is a force, which produce unit acceleration ( $\text{m}/\text{s}^2$ ) on a unit mass (1kg).

**III.4.3 Third Law:**

To every **action**, there is always an equal and opposite **reaction**. When two bodies interact, they exert opposite forces on each other in direction but equal in magnitude and direction. In the examples of Figures III.9 and III.10, the force  $\vec{F}_{1/2}$  exerted by body 1 on body 2 is equal in magnitude and opposite in direction to the force  $\vec{F}_{2/1}$  exerted by body 2 on body 1.



*Figure III. 9*



*Figure III. 10*

**Examples:**

This law appears to be self-evident as when a bullet is fired from a gun, the bullet moves out with a great velocity, and the reaction of the bullet, in the opposite direction, gives an unpleasant shock to the man holding the gun. Similarly, when a swimmer tries to swim, he pushes the water backwards and the reaction of the water pushes the swimmer forward.

**Properties:**

- Both forces act along the line, called the line of action, joining the two bodies.
- We call the force exerted by one of the two bodies the **action**, and the force exerted by the other the **reaction**. However, it can be considered that either of the two forces is the action, and the other is the reaction.
- The two forces are of the same nature and do not act on the same body. Thus, the force commonly referred to as the reaction (to gravity) of the ground on a supported body is not so in the sense of Newton's third law. In fact, these forces are not of the same nature and both apply to the body placed on the ground.
- It is impossible to find a force that acts in isolation; every force is associated with a reaction.

**Note:**

Consider the free fall of a ball A with mass  $m_A$  under the influence of gravitational force  $\vec{F}_g$ .

Since its acceleration is  $a_A = F_g/m_A$ , its displacement during a time interval  $t$  is  $\Delta y_A = a_A \frac{t^2}{2}$ .

A exerts on the Earth a force of the same magnitude, generating an acceleration of the Earth  $a_T = F_g/m_T$ , where  $m_T$  is the mass of the planet. The displacement associated with this acceleration

during  $t$  is thus  $\Delta y_T = a_T \frac{t^2}{2}$ . The ratio of A's displacement to that of the Earth is:  $\frac{\Delta y_A}{\Delta y_T} = \frac{a_A}{a_T} =$

$\frac{m_T}{m_A} \approx 10^{24}$ . This result shows that  $\Delta y_A \gg \Delta y_T$ .

**III.4.4 Validity of Newton's Laws:**

Newton's laws are valid in any Galilean reference frame. To demonstrate this, consider two inertial reference frames:

- $\mathfrak{R}(x, y, z)$ , assumed to be fixed (absolute frame).
- $\mathfrak{R}'(\acute{x}, \acute{y}, \acute{z})$ , in uniform rectilinear motion with respect to  $\mathfrak{R}$  (relative frame).

If a particle is subjected to forces with a resultant  $\vec{F}$ , we can apply Newton's second law in frame  $\mathfrak{R}$ :

$$\vec{F} = m\vec{a}_a \quad (\text{III. 14})$$

where  $\vec{a}_a$  is the absolute acceleration. In  $\mathfrak{R}'$ , the particle's acceleration is:

$$\vec{a}_r = \vec{a}_a - \vec{a}_e - \vec{a}_c$$

where  $\vec{a}_e$  and  $\vec{a}_c$  are, respectively, the entrainment, and Coriolis accelerations respectively. Since  $\mathfrak{R}'$  is in uniform rectilinear motion,

- The rectilinear implies:  $\vec{a}_c = 0$  and  $\vec{a}_e = \frac{d^2\vec{r}_0}{dt^2}$
- The uniform implies:  $\frac{d^2\vec{r}_0}{dt^2} = 0$  so  $\vec{a}_e = 0$

Consequently,  $\vec{a}_r = \vec{a}_a$  and:

$$\vec{F} = m\vec{a}_r \quad (\text{III. 15})$$

In conclusion, Newton's second law is valid in any Galilean reference frame.

### III.5 Body Movements - Law of Force

In this section, we will consider the general laws, known as laws of forces, established for various interactions.

#### III.5.1 Weight

##### a. Definition

We define the **weight** of a body at a given location as the attractive force, denoted as  $\vec{P}$ , that the Earth exerts on it.

The experimental study of the fall of bodies with different masses, regardless of their initial velocities and neglecting air resistance, leads to the following conclusions:

- The motion of free fall near the Earth is characterized by constant acceleration.
- All bodies fall in a given location with the same acceleration, denoted as  $\vec{g}$ , and referred to as the *acceleration due to gravity*.

Using Newton's second law, we can define the weight of a body with mass  $m$ :

$$\vec{P} = m\vec{g} \quad (\text{III. 16})$$

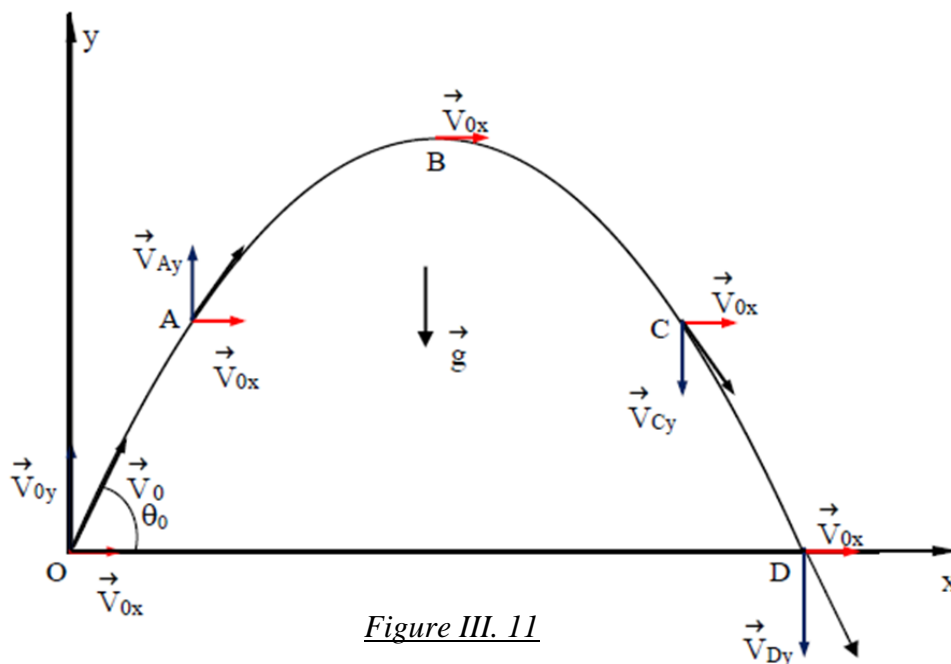
**We conclude:** The weight of an object is defined as the force of gravity on the object and may be calculated as the mass times the acceleration of gravity,  $P = mg$ . Since the weight is a force, its SI unit is the Newton. Weight = mass • acceleration due to gravity.

**Notes:**

- In general, any function that depends on position is called a "field." Specifically, a vector field is a vector associated with each point in space. Hence, the term "gravitational field" is used to refer to  $\vec{g}$ .
- $\vec{g}$  approximately points toward the center of the Earth, with its magnitude ranging from 9.78 m/s<sup>2</sup> to 9.83 m/s<sup>2</sup> depending on location (stronger at the poles, weaker near the equator). It's approximately 9.81 m/s<sup>2</sup> in Algeria. These variations are due to the Earth's slightly flattened shape and the effect of centrifugal force associated with Earth's rotation, which is not strictly gravitational.

**b. Application - Projectile Motion in a Gravitational Field:**

Let's consider a projectile moving in a gravitational field. We choose a Cartesian coordinate system with the vertical upward axis (Oy), as shown in Figure III.11. As a result,



*Figure III. 11*

If the initial velocity of the particle lies in the (Oxy) plane and if its initial position coincides with the origin O, we write:

$$\begin{aligned}x_0(0s) &= 0, & y_0(0s) &= 0 \\ \vec{V}_0(0s) &= V_0 \cos \theta_0 \vec{i} + V_0 \sin \theta_0 \vec{j} \\ V_{0x} &= V_0 \cos \theta_0; & V_{0y} &= V_0 \sin \theta_0\end{aligned}$$

Where  $V_0$  is the magnitude of the initial velocity and  $\theta_0$  is the launch angle. The equations of motion are as follows:

$$\begin{cases} a_x = 0 & \text{(III. 17)} \\ a_y = -g & \text{(III. 18)} \end{cases}$$

The velocity is obtained through a first integration:

$$\begin{cases} V_x(t) = V_{0x} + \int_0^t a_x dt = V_0 \cos \theta_0 & \text{(III. 19)} \\ V_y(t) = V_{0y} + \int_0^t a_y dt = V_0 \sin \theta_0 - g \int_0^t dt = -gt + V_0 \sin \theta_0 & \text{(III. 20)} \end{cases}$$

We can then integrate the velocity to find the position:

$$\begin{cases} x(t) = x_0 + \int_0^t V_x dt = 0 + \int_0^t (V_0 \cos \theta_0) dt = (V_0 \cos \theta_0)t \\ y(t) = y_0 + \int_0^t V_y dt = 0 + \int_0^t (-gt + V_0 \sin \theta_0) dt = -\frac{1}{2}gt^2 + (V_0 \sin \theta_0)t \end{cases}$$

Let's consider:

$$\begin{cases} x(t) = (V_0 \cos \theta_0)t & \text{(III. 21)} \\ y(t) = -\frac{1}{2}gt^2 + (V_0 \sin \theta_0)t & \text{(III. 22)} \end{cases}$$

We can eliminate time from these equations and express the y-coordinate in terms of x:

$$t = \frac{x}{V_0 \cos \theta_0} \quad \text{(III. 23)}$$

By substituting this expression into (III.22), we obtain:

$$y = \frac{-g}{2(V_0 \cos \theta_0)^2} x^2 + \frac{V_0 \sin \theta_0}{V_0 \cos \theta_0} x \quad \text{(III. 24)}$$

Thus:

$$y = \frac{-g}{2(V_0 \cos \theta_0)^2} x^2 + \tan \theta_0 x \quad \text{(III. 25)}$$

Y is thus a quadratic function of x, and the trajectory of the particle is a parabola.

The trajectory of a projectile has two particular characteristics:

- **The maximum height:** this is the ordinate,  $B_y$ , of the apex of the trajectory (point B in Figure III .11). It is characterized by:

$$V_{By} = -gt + V_0 \sin \theta_0 = 0 \Rightarrow t_B = \frac{V_0 \sin \theta_0}{g}$$

With relation (III.22), we get:

$$y_B = -\frac{(V_0 \sin \theta_0)^2}{2g} + \frac{(V_0 \sin \theta_0)^2}{g} = \frac{(V_0 \sin \theta_0)^2}{2g} \quad (\text{III. 26})$$

- **The range:** this is the horizontal distance between the firing point and the point of impact. In other words, it is the horizontal component of the corresponding displacement vector.

In our case, since the projectile is launched from the ground which is perfectly horizontal, the range is equal to  $x_D$  (abscissa of point D in Figure III.11) and is obtained by solving equation (III.25) for  $y = 0$ . Note that the latter can be expressed in the form:

$$y = \frac{-g}{2(V_0 \cos \theta_0)^2} x^2 + \tan \theta_0 x = 0 \Leftrightarrow x \left( \frac{-g}{2(V_0 \cos \theta_0)^2} x + \tan \theta_0 \right) = 0 \quad (\text{III. 27})$$

The solution is either  $x = 0$  (the starting point), or the one that interests us:

$$x_D = \tan \theta_0 \frac{2(V_0 \sin \theta_0)^2}{g} = \frac{2V_0^2 \sin \theta_0 \cos \theta_0}{g} = \frac{V_0^2 \sin 2\theta_0}{g} \quad (\text{III. 28})$$

Figure III.12 illustrates the relationship between the range and the launch angle (Parabolic trajectories of a projectile associated with different launch angles, for a firing velocity of 10 m/s).

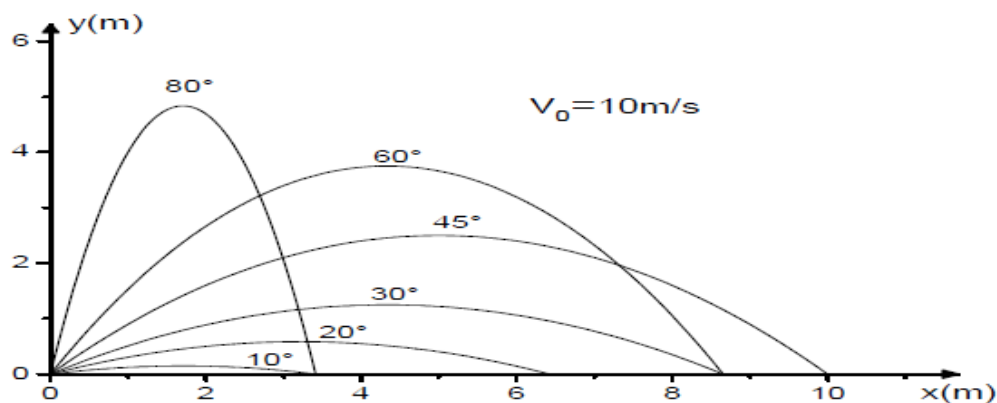


Figure III. 12

### III.5.2 Universal Law of Gravitation

#### a. The Universal Law of Gravitation

Newton's Universal Law of Gravitation forms the basis of the theory that explains a wide variety of phenomena, ranging from the motion of planets to the fall of objects and the ebb and flow of tides. Newton's Universal Law of Gravitation is expressed as follows: Between two material particles with masses  $m_1$  and  $m_2$ , placed at a distance  $r$  from each other, there is an attractive force of intensity:

$$F_g = G \frac{m_1 m_2}{r^2} \quad (\text{III. 29})$$

$G$  is called *the universal gravitational constant* or Cavendish constant. It was first measured by H. Cavendish using a torsion balance in 1771. It is a proportionality constant that depends on the chosen system of units. In the SI system:  $G = 6,726 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$ .

**Notes:**

- If  $\vec{F}_{1/2}$  is the gravitational force exerted by object 1 on object 2, and  $\vec{F}_{2/1}$  is the force exerted by object 2 on object 1, then  $\vec{F}_{1/2} = -\vec{F}_{2/1}$  (3<sup>rd</sup> law of Newton) (Figure III. 13).

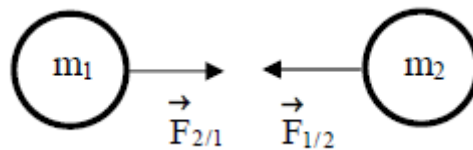


Figure III. 13

- The universal law of gravitation follows the principle of superposition: if there are more than two bodies, it is necessary to consider the presence of all the attractive forces exerted on the bodies. Each body experiences an attractive force that results from the gravitational forces exerted by the neighboring bodies.

**Example:** case of three non-aligned bodies.

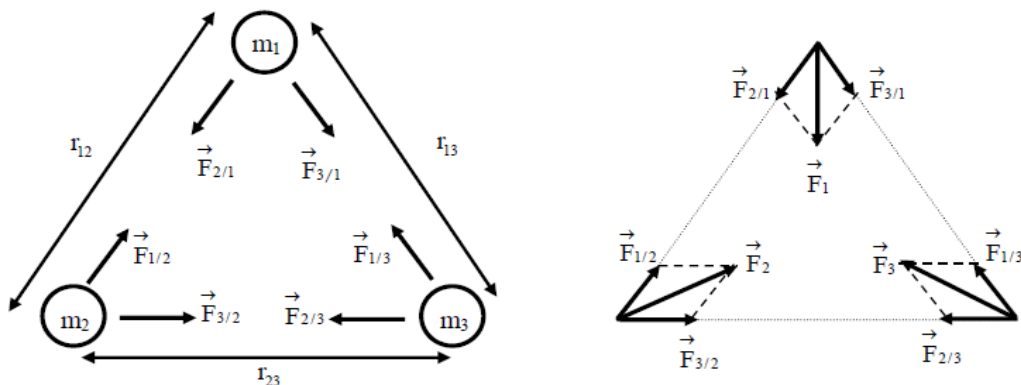


Figure III. 14

Figure III.14 depicts the forces exerted on each of the bodies and their associated resultants.

Thus, the resultant of the forces exerted by  $m_2$  and  $m_3$  on  $m_1$  is:

$$\vec{F}_1 = \vec{F}_{2/1} + \vec{F}_{3/1}$$

▪ The gravitational force exerted by the Earth on an object can be expressed in the form:

$$F_g = mg(r) = m \underbrace{\left( G \frac{M_T}{r^2} \right)}_{g(r)} \quad (\text{III. 30})$$

▪ The gravitational force near the Earth, meaning the attractive force exerted by the Earth on an object, is called weight. Using Newton's universal law of gravitation and weight, we have:

$$mG \frac{M_T}{R_T^2} = mg_0 \quad (\text{III. 31})$$

So :

$$g_0 = G \frac{M_T}{R_T^2} \quad (\text{III. 32})$$

Where:

- $M_T$  is the mass of the Earth ( $5,98 \cdot 10^{24}$  kg);
- $m$  is the mass of the object in kilograms;
- $R_T$  is the radius of the Earth ( $6,37 \cdot 10^6$  m);
- $g_0$  is the acceleration due to gravity ( $9,8 \text{ m/s}^2$  near the surface of the Earth).

### b. Application 1: Kepler's Third Law

The universal law of gravitation was established by Newton, relying in particular on Kepler's laws, including the third law: “*The square of the period of revolution of a planet in the solar system is proportional to the cube of the semi-major axis of its orbit*”.

$$T^2 = Kr^3 \quad (\text{III. 33})$$

- $k$  is a proportionality constant expressed in  $\text{s}^2/\text{m}^3$ . It is the same for all planets orbiting the Sun.
- $r$  is the mean radius of the planet's orbit. In the case of an elliptical orbit, it represents the semi-major axis of the trajectory; for large planets, the trajectory is nearly circular and  $r$  corresponds to the radius of the circle.

The constant  $k$  is determined by applying Newton's second law to a large planet with uniform circular motion.

$$\sum \vec{F}_{ext} = \vec{F}_g = m\vec{a}$$

which, in magnitudes:

$$G \frac{m_s m}{r^2} = m \frac{V^2}{r} \quad (\text{III. 34})$$

where  $m_s$  is the mass of the Sun and  $m$  is the mass of the planet. As velocity is related to the period of revolution by:

$$V = \omega r = \frac{2\pi}{T} r \quad (\text{III. 35})$$

this leads to:

$$G \frac{m_s}{r} = \frac{4\pi^2}{T^2} r^2$$

Where:

$$T^2 = \frac{4\pi^2}{Gm_s} r^3 \quad (\text{III. 36})$$

By identifying with Kepler's third law (equation III.33), we obtain the following expression for the constant  $k$ :

$$k = \frac{4\pi^2}{Gm_s} \quad (\text{III. 37})$$

**Example:** With the Sun having a mass  $m_s = 1,99.10^{30}$  kg and the Earth's orbit around the Sun having a mean radius  $r_T = 1,496.10^{11}$  m, we get:

$$\mathbf{k = 2,9734.10^{-19} \text{ s}^2 \text{ m}^3 \quad \text{and} \quad \mathbf{T=3,1678.10^7 \text{ s}}$$

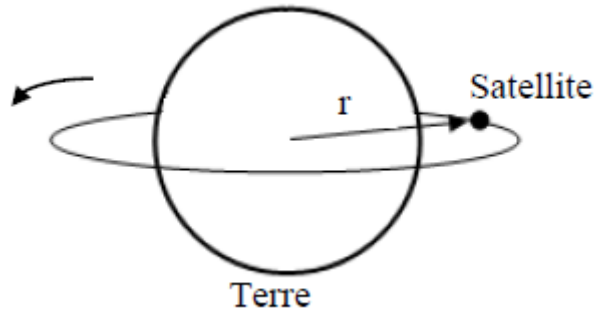
**Note:** For two planets 1 and 2 with respective revolution periods  $T_1$  and  $T_2$ , moving on orbits with mean radius  $r_1$  and  $r_2$ , we obtain the relation:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3} \quad (\text{III. 38})$$

**c. Application 2: The Geostationary Satellite :** A geostationary satellite is a satellite that an observer on Earth always sees in the same spot in the sky (it appears motionless relative to the ground). This characteristic is important for telecommunications and broadcasting. The satellite orbits the Earth at the same angular velocity as the Earth's rotation on its axis. As a result, its orbital period around the planet is the same as the planet's rotation period. Consequently, the geostationary satellite follows a uniform circular motion in a terrestrial

frame with a period of  $T = 24$  hours, and it lies within the Earth's equatorial plane. Recall that the period is related to the angular velocity by:

$$\omega = \frac{2\pi}{T} \quad (\text{III. 39})$$



*Figure III. 15*

Newton's second law gives, in magnitudes:

$$\begin{cases} F_g = G \frac{m_T m}{r^2} \\ a = \frac{V^2}{r} = \omega^2 r \end{cases} \Rightarrow G \frac{m_T m}{r^2} = m \omega^2 r \quad (\text{III. 40})$$

where  $m_T$  is the mass of the Earth ( $5,98 \cdot 10^{24}$  kg), and  $m$  is the mass of the satellite. So,  $r$ :

$$r = \left( G \frac{m_T}{\omega^2} \right)^{1/3} \quad (\text{III. 41})$$

Numerical application gives:

$$r = 4, 21 \cdot 10^7 \text{ m} \quad ; \quad V = \frac{2\pi r}{T} = 3,08 \cdot 10^3 \text{ m/s}$$

### III.5.3 Contact Forces or Binding Forces

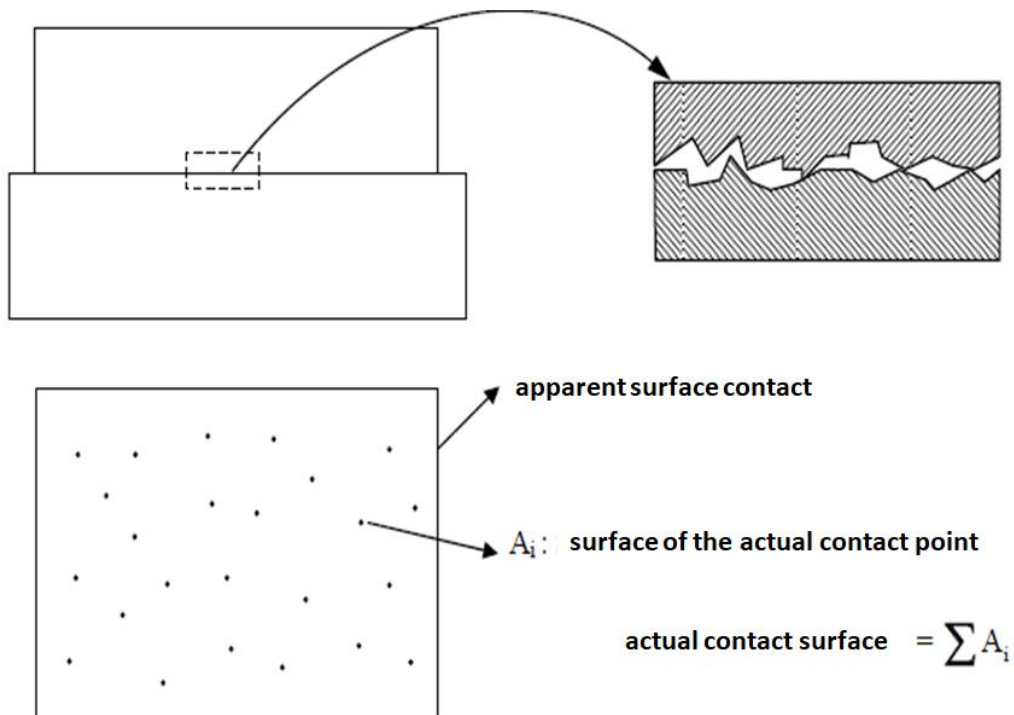
#### Introduction

These are forces that occur between two bodies in physical contact with each other. This definition involves three elements: *two objects* brought into contact through a *surface*.

Note that at the atomic scale; even very smooth surfaces are not truly flat. In reality, they have irregularities all over, as indicated in the representation in Figure III.16 where the contacting surfaces are all uneven. Thus, instead of having contact everywhere, there are highly localized

microcontacts. It is important to distinguish between the apparent surface and the surface that is actually in contact. The latter is much smaller than the former. As a result, the contact force doesn't apply uniformly across the entire surface but only on the asperities. In these areas, molecules are very close to each other, and the atoms of both bodies mutually interact. The forces acting on them are quite complex to analyze: Coulombic forces, van der Waals forces, etc...

Several factors influence these forces: the nature of the materials involved, surface finish, interstitial matter (contaminants, adsorbed molecules, wear debris, dust, etc.), temperature, and the level of surface contamination. As it is practically impossible to model contact forces while accounting for all microscopic interactions, they are determined on a global level through experimental methods.



*Figure III. 16*

### a. Contact Forces - Friction Forces

Let's consider an object placed on a horizontal plane. It is in equilibrium under the action of two forces  $\vec{P}$  and  $\vec{C}$  which must be equal and opposite. Let's analyze the behavior of this object and the evolution of the contact forces (plane/object) when a gradually increasing horizontal pulling force  $\vec{F}$  is applied to it. We will observe two types of friction: **static friction** and **kinetic (dynamic) friction**.

**a.1 Static Friction**

**First Case: Adhesion**

The body is in equilibrium, and with  $\vec{P}$  being vertical and  $\vec{F}$  horizontal, the application of Newton's second law highlights a contact force  $\vec{C}$  such that:

$$\sum \vec{F}_{ext} = \vec{P} + \vec{c} + \vec{F} = 0$$

Let's consider:

$$\vec{c} = -\vec{P} - \vec{F}$$

The geometric projection of this equation onto the (Ox) axis gives:

$$\vec{c}_x = -\vec{F} \quad (\text{III. 42})$$

$\vec{c}_x$  is an adhesive force that opposes  $\vec{F}$  and any potential rightward movement of the object. Moreover, this force is tangent to the contact surface, and by definition, it is a friction force. Since the object doesn't move,  $\vec{c}_x$  is referred to as

**static friction.**

**Second Case: Breaking Equilibrium**

If  $\vec{F}$  becomes large enough, there's a limiting value  $|\vec{F}|_{limite}$  from which equilibrium is disrupted, and the object starts to slide on the surface. This value is associated with a limiting adhesion  $|\vec{C}_{0x}| = |\vec{F}|_{limite}$  which allows us to define the coefficient of static friction  $\mu_s$ :

$$\mu_s = \frac{|\vec{C}_{0x}|}{|\vec{C}_{0y}|} \quad (\text{III. 43})$$

The coefficient of static friction is characterized by the following properties:

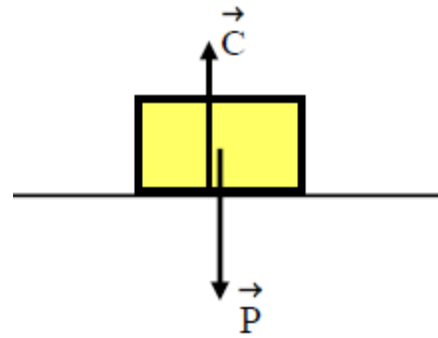


Figure III. 17

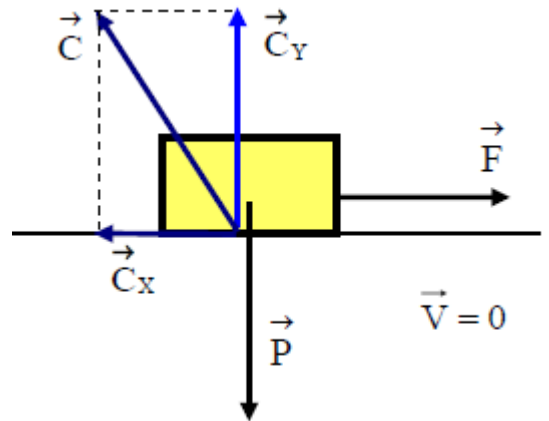


Figure III. 18

- ✓ This dimensionless coefficient expresses the proportionality between the modulus of the limiting friction force,  $|\vec{C}_{0x}|$ , and that of the normal force,  $|\vec{C}_{0y}|$ .
- ✓  $\mu$  depends on the nature of the contacting surfaces.
- ✓ It is determined experimentally.
- ✓ It doesn't depend on the contact area.

**a.2 Dynamic Friction:**

This is the force that opposes motion when the object placed on the plane is already in sliding motion (Figure III.19). The fundamental law of dynamics is written as:

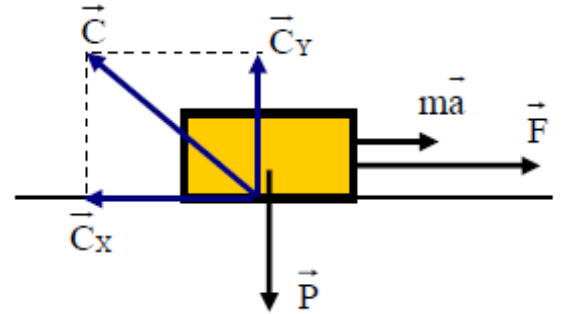


Figure III. 19

$$\sum \vec{F}_{ext} = \vec{P} + \vec{c} + \vec{F} = m\vec{a} \quad \text{Which implies:} \quad \vec{c} = m\vec{a} - \vec{P} - \vec{F}$$

$\vec{C}$  has a tangential component,  $\vec{c}_x$ , which opposes the motion of the body and, by definition, is a dynamic friction force. Just as in the static case, we introduce a coefficient of dynamic (or kinetic) friction:

$$\mu_d(\text{or } \mu_g) = \frac{|\vec{c}_x|}{|\vec{c}_y|} \quad (\text{III. 44})$$

The coefficient of dynamic friction is characterized by the following properties:

- ✓ Its values are determined experimentally.
- ✓  $\mu_d < \mu_s$ .
- ✓  $\mu_d$  is nearly independent of velocity.
- ✓  $\mu_d$  depends only on the nature of the contacting surfaces.

**Additional Notes:**

- Impurities between the two contact surfaces are often more significant sources of friction than the interlocking of surface asperities.
- Friction is weakly dependent on the surface, as atomic-scale roughness means that only a very small percentage of the total surface of both objects is actually in contact (the real contact surface area is much smaller than the apparent contact surface, see Figure III.16).
- In reality, the origin of friction involves a multitude of parameters covering a wide range of physical phenomena: surface roughness, elasticity, plasticity, adhesion, lubrication, temperature, wear, surface chemistry, humidity, etc...

**Exercise**

A block with a mass of  $m = 1 \text{ kg}$  is placed on a rough surface inclined at an angle  $\theta$  relative to the horizontal (Figure III.20). By gradually increasing the value of  $\theta$ , the following observations were made:

- The block started to slide when the angle reached  $\theta = 20^\circ$ .
- Its acceleration was  $3,2 \text{ m/s}^2$  for  $\theta = 30^\circ$ .

Determine the values of the coefficients of static and dynamic friction.

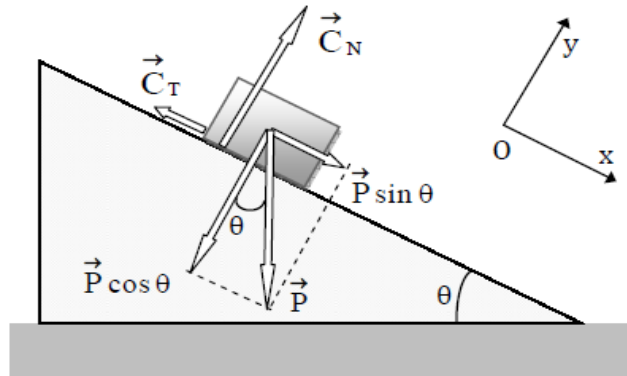


Figure III. 20

**Solution:**

Applying Newton's second law gives:

$$\vec{P} + \vec{c} = m\vec{a}$$

Let's project this equation onto the two Cartesian coordinate axes:

$$\sum (F_{ext})_x = P \sin \theta - C_T = ma \quad (\text{III. 45})$$

$$\sum (F_{ext})_y = -P \cos \theta + C_N = 0 \quad (\text{III. 46})$$

Equation (III.46) and the definition of the coefficient of friction allow us to write:

$$C_T = \mu C_N = \mu P \cos \theta \Rightarrow P \sin \theta - \mu P \cos \theta = ma$$

We isolate  $\mu$ :

$$\mu = \frac{P \sin \theta - ma}{P \cos \theta} \quad (\text{III. 47})$$

**First case:** Equilibrium breaking ( $a = 0 \text{ m/s}^2$ ;  $\mu = \mu_s$ ;  $\theta = 20^\circ$ ):  $\mu_s = \text{tg} \theta = 0,364$ .

**Second case:** Uniformly accelerated motion ( $a = 3.2 \text{ m/s}^2$ ;  $\mu = \mu_d$ ;  $\theta = 30^\circ$ ):

$$\mu_d = \frac{g \sin \theta - a}{g \cos \theta} = 0.2$$

## b. Viscous Friction

### b.1 Viscous Resistance

When a plate is moved in a direction perpendicular to its largest surface, a force is felt that opposes the motion. This force is known as air resistance. A force of the same nature can also be experienced when the plate is exposed to an air flow, such as wind. Thus, we can place objects in a wind tunnel and study the properties of air resistance. Experimentally, we can highlight the parameters influencing air resistance:

- **Area:** If we double the object's area, the measured force also doubles. Therefore, air resistance is proportional to the area.
- **Velocity:** Increasing the velocity of the air flow results in an increased force exerted by the air.
- **Density:** At higher altitudes, air density decreases. This means that there are fewer air particles in the same volume. Consequently, if the number of particles decreases, resistance decreases. Air resistance is therefore proportional to its density.
- **Shape of the object:** Depending on the shape of the object, air resistance will be greater or smaller. For a spherical object, air resistance will be less significant than for a flat plate.

When an object moves through a fluid (gas or liquid), the medium generally exerts a resistance force opposite to the object's velocity. Here, we must distinguish between two types of resistance:

- **Viscous resistance:** This is caused by the viscosity of the fluid, a physical property of the fluid that manifests when the fluid flows near a surface or an object, or equivalently, when a surface or object moves within the fluid. The viscous resistance force on the moving object is proportional to its velocity and is approximately opposite to it.

$$\vec{f}_V = -k\vec{V} \quad (\text{III.48})$$

$k$  is a coefficient that depends on the object's geometry and the viscosity of the fluid, which in turn depends on pressure and temperature.  $\vec{f}$  is exactly opposite to the velocity only if the object has rotational symmetry around the axis of velocity. For all other objects, the resistance force is only approximately opposite to the velocity  $\vec{V}$  and can cause rotational movements, etc.

- **Turbulent resistance:** If the object's velocity surpasses a certain limit (dependent on the object's shape and the medium it's moving through), the fluid flow around the object becomes

turbulent, characterized by the formation of vortices. The appearance of these vortices results in pressure drops behind the moving object, which manifest as a resistance force called drag force. This force is generally proportional to the square of the object's velocity.

$$|\vec{f}_t| = kV^2 \quad (\text{III. 49})$$

In this case as well, the force is only approximately opposite to the velocity's direction. It is exactly opposite only if the object exhibits rotational symmetry around the direction of its velocity. In this case, the drag force can be written as:

$$\vec{f}_t = -kV^2 \frac{\vec{V}}{|\vec{V}|} \quad (\text{III. 50})$$

The total resistance force (viscous + drag) is then:

$$\vec{f} = -k\vec{V} - kV^2 \frac{\vec{V}}{|\vec{V}|} \quad (\text{III. 51})$$

**Note:** In most cases considered within the scope of this course, where velocities are low, the resistance force reduces to the form:

$$\vec{f} = -k\vec{V}$$

### b.2 Example of Horizontal Viscous Friction

Let's consider the assumed horizontal motion of an object submerged in a fluid and subjected to a constant horizontal force  $\vec{F}$ . The fluid exerts a frictional force on the object in the form of:  $\vec{f} = -k\vec{V}$

The other forces acting on the object are its weight  $\vec{P}$ , the buoyant force  $\vec{A}$ , and the applied force  $\vec{F}$ . Since the motion is maintained on a horizontal plane, this means that  $\vec{P}$  and  $\vec{A}$  balance each other, and by applying Newton's second law, we have:

$$\vec{P} + \vec{A} + \vec{f} + \vec{F} = m\vec{a} \Leftrightarrow -k\vec{V} + \vec{F} = m \frac{d\vec{V}}{dt}$$

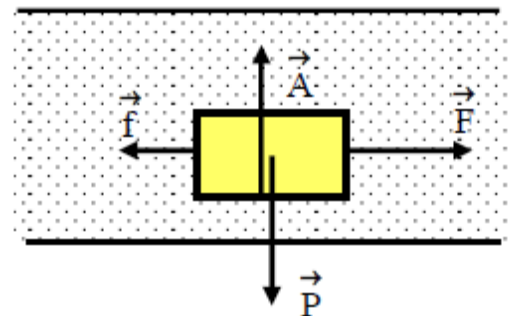


Figure III. 21

Where:

- $\vec{F}$  is the applied force.
- $k$  is the coefficient of viscous friction.
- $\vec{V}$  is the velocity vector of the object.
- $m$  is the mass of the object.
- $\vec{a}$  is its acceleration vector.

This equation describes the dynamic behavior of the object subjected to both the applied force and the viscous friction force as it moves horizontally in the fluid. The acceleration of the object depends on the net force acting on it, which includes the applied force minus the opposing force due to viscous friction.

Algebraically, we will have:

$$-kV + F = m \frac{dV}{dt}$$

Therefore, it follows:

$$\frac{dV}{V - \frac{F}{K}} = -\frac{K}{m} dt$$

By integrating, we obtain:

$$\log\left(V - \frac{F}{K}\right) = -\frac{K}{m}t + cte$$

Taking the exponential:

$$V - \frac{F}{K} = B e^{-\frac{K}{m}t}$$

$B$  is an integration constant that can be determined, for example, using initial conditions. If the initial velocity is zero, then:

$$B = -\frac{F}{K}$$

As a result, we obtain:

$$V(t) = \frac{F}{K} (1 - e^{-\frac{K}{m}t})$$

Let's define:

$$V_L = \frac{F}{K} \quad ; \quad \tau = \frac{m}{K}$$

$\tau$  will be considered as a time constant. This leads to:

$$V(t) = V_L \left(1 - e^{-\frac{t}{\tau}}\right) \quad (\text{III.52})$$

Thus, the velocity increases from its initial value to an asymptotic value  $V_L$ , called the terminal velocity (Figure III.22). We also observe an interesting behavior: a heavy object takes more time to reach its terminal velocity compared to a light mobile object.

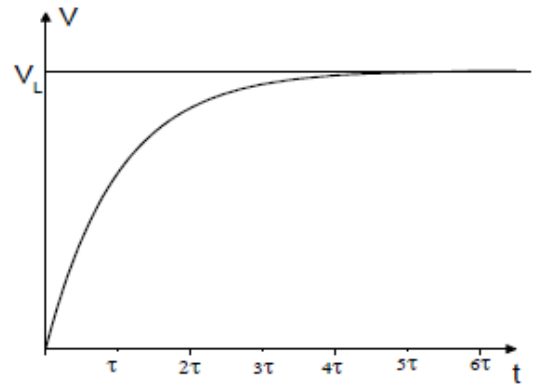


Figure III. 22

### III.5.4 Elastic Forces

Let's consider the mass-spring system shown in Figure III.23. A mass is suspended from the lower end of a spring with an unstressed length  $l_0$ , hanging from the ceiling, causing the spring to experience an elongation  $\Delta l = l - l_0$  to reach equilibrium.

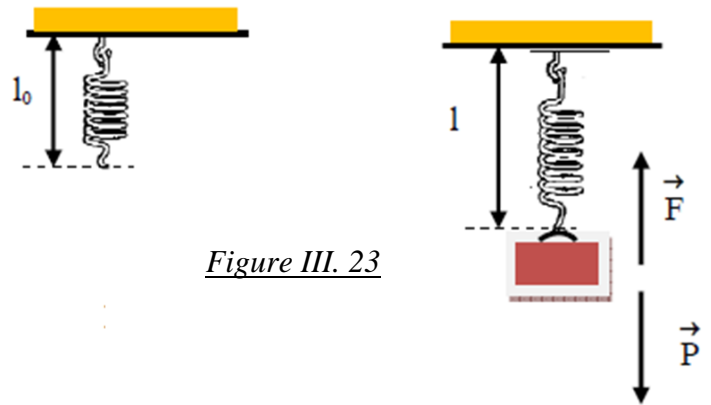


Figure III. 23

According to Newton's second law, the weight must be balanced by a force  $\vec{F}$  exerted by the spring on the mass in the opposite direction of elongation:

$$\vec{P} + \vec{F} = \vec{0} \Rightarrow \vec{P} = -\vec{F} \Rightarrow |\vec{P}| = |\vec{F}|$$

$$= mg$$

By varying the mass and measuring the corresponding elongation  $\Delta l$ , one can plot the graph of the change in the magnitude of  $\vec{F}$  as a function of  $\Delta l$ . This results in a straight-line segment with a slope  $K$  (Figure III.24).

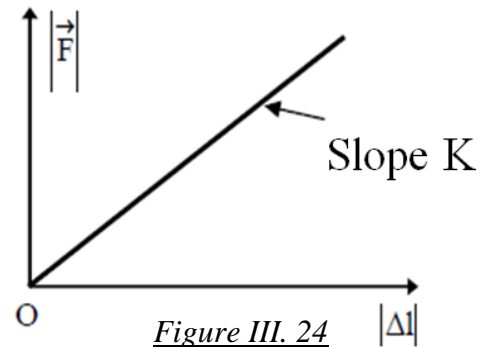


Figure III. 24

A similar experiment can be conducted with a spring in compression, where a mass is placed on its upper end, as shown in Figure III.25:

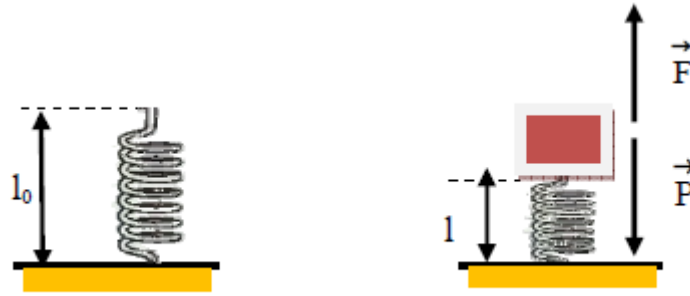


Figure III. 25

In this case as well, the weight must be balanced by a force  $\vec{F}$  exerted by the spring on the mass in the opposite direction of its compression. By varying the mass and measuring the corresponding compression  $\Delta l$ , one can plot the graph of the change in the magnitude of  $\vec{F}$  as a function of  $\Delta l$ . This also results in a straight-line segment with a slope  $K$ . It follows that in both cases, the force  $\vec{F}$  is related to the spring deformation by:

$$|\vec{F}| = K|\Delta l| \quad (\text{III.53})$$

This linear dependence of force on deformation is known as Hooke's law. The constant  $K$  is called the "spring constant" or "stiffness constant" of the spring, with its unit being Newton per meter (N/m). Since  $\vec{F}$  is directed opposite to the spring's deformation, it's appropriate to write:

$$\vec{F} = -K\Delta\vec{l} \quad (\text{III.54})$$

Where:

$$\Delta\vec{l} = (l - l_0)\vec{u}$$

Here,  $\vec{u}$  is the unit vector oriented in the direction of the spring's elongation. Thus, when a spring is deformed by an external action, it exerts a reaction force described by equation (III.54) on the agent causing the deformation.

### III.6 Angular Momentum

Linear momentum has proven to be very useful in the study of translational motion. Angular momentum is the physical quantity that plays a similar role in the case of rotational motion; it is also known as *rotational momentum*.

#### III.6.1 Angular Momentum of a Particle

##### a. Definition:

Consider a particle with mass  $m$  located at a point identified by the position vector  $\vec{r}$  and moving with velocity  $\vec{V}$ . Its angular momentum  $L$ , with respect to the origin  $O$ , is defined by:

$$\vec{L}_{/O} = \vec{r} \wedge \vec{\mathcal{P}} \quad (\text{III. 55})$$

Given that  $\vec{\mathcal{P}}$  is the linear momentum given by:  $\vec{\mathcal{P}} = m\vec{V}$

By its definition, the angular momentum is a vector perpendicular to the plane containing the vectors  $\vec{r}$  and  $\vec{\mathcal{P}}$ , and it is oriented in a way such that the triad  $(\vec{r}, \vec{\mathcal{P}}, \vec{L})$  is right-handed. Its magnitude is:

$$|\vec{L}_{/O}| = |\vec{r} \wedge \vec{\mathcal{P}}| = |\vec{r}| \cdot |\vec{\mathcal{P}}| \cdot \sin(\vec{r}, \vec{\mathcal{P}})$$

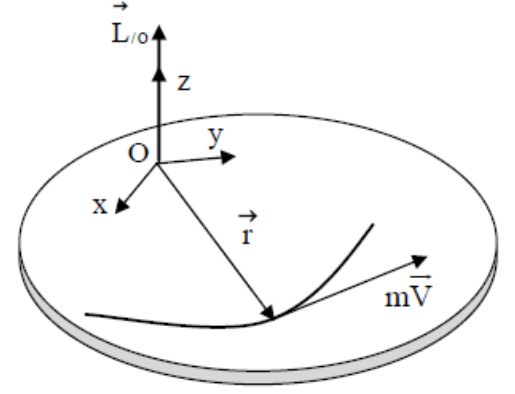


Figure III. 26

Here,  $\vec{L}_{/O}$  represents the angular momentum with respect to the origin O,  $\wedge$  denotes the cross product operation, and  $(\vec{r}, \vec{\mathcal{P}})$  is the angle between vectors  $\vec{r}$  and  $\vec{\mathcal{P}}$ .

This expression describes the magnitude of the angular momentum vector and involves the cross product of the position vector  $\vec{r}$  and the linear momentum vector  $\vec{\mathcal{P}}$ . The resulting angular momentum vector points along the axis of rotation according to the right-hand rule.

**b. Special Cases**

- In the case of circular motion with a radius r centered at the origin, the position vector is always perpendicular to the direction of the velocity vector. Therefore:

$$\vec{L}_{/O} = m\vec{r} \wedge \vec{V} \implies |\vec{L}_{/O}| = mrV = mr^2\omega$$

Since the angular momentum has the same direction as the angular velocity vector  $\vec{\omega}$ , it becomes evident that:

$$\vec{L}_{/O} = mr^2\vec{\omega} \quad (\text{III. 56})$$

- For arbitrary curvilinear motion in a plane, we introduce the radial and tangential components of velocity in a polar coordinate system with the origin at O:

$$\vec{V} = \vec{V}_r + \vec{V}_\theta$$

and we have:

$$\vec{L}_{/O} = m\vec{r} \wedge \vec{V} = m\vec{r} \wedge (\vec{V}_r + \vec{V}_\theta) = m\vec{r} \wedge \vec{V}_r + m\vec{r} \wedge \vec{V}_\theta$$

Considering that  $\vec{V}_r$  is parallel to  $\vec{r}$ , we obtain:

$$\vec{L}_{/O} = m\vec{r} \wedge \vec{V}_\theta$$

And in scalar form:

$$|\vec{L}_{/O}| = mrV_\theta = mr^2 \frac{d\theta}{dt} \quad (\text{III. 57})$$

### III.6.2 Angular Momentum Theorem for a Particle

The derivative of the angular momentum is obtained by applying the rules of differentiation for product functions:

$$\frac{d\vec{L}_{/O}}{dt} = \frac{d(\vec{r} \wedge \vec{\mathcal{P}})}{dt} = \frac{d\vec{r}}{dt} \wedge \vec{\mathcal{P}} + \vec{r} \wedge \frac{d\vec{\mathcal{P}}}{dt} \quad (\text{III. 58})$$

We note that:

$$\frac{d\vec{r}}{dt} \wedge \vec{\mathcal{P}} = \vec{V} \wedge m\vec{V} = 0$$

and, according to the second law of Newton:

$$\vec{r} \wedge \frac{d\vec{\mathcal{P}}}{dt} = \vec{r} \wedge \vec{F}$$

The derivative of angular momentum is then reduced to the form:

$$\frac{d\vec{L}_{/O}}{dt} = \vec{r} \wedge \vec{F} \quad (59)$$

where  $\vec{F}$  is the resultant of external forces applied to the particle. From the definition of the moment of  $\vec{F}$  with respect to the origin:

$$\vec{\mathcal{M}}_{/O}(\vec{F}) = \vec{r} \wedge \vec{F}$$

It follows that:

$$\frac{d\vec{L}_{/O}}{dt} = \vec{\mathcal{M}}_{/O}(\vec{F}) \quad (\text{III. 60})$$

Hence, **the theorem of angular momentum for a particle:** *the time derivative of the angular momentum of a moving particle is equal to the resultant moment of external forces, with respect to the reference point used for angular momentum.*

#### Notes:

- The theorem of angular momentum is analogous to Newton's second law.
- In Cartesian coordinates, we obtain the following expressions:

$$\begin{aligned} \vec{L}_{/O} &= m(yV_z - zV_y)\vec{i} + m(zV_x - xV_z)\vec{j} + m(xV_y - yV_x)\vec{k} \\ \vec{\mathcal{M}}_{/O}(\vec{F}) &= m(yF_z - zF_y)\vec{i} + m(zF_x - xF_z)\vec{j} + m(xF_y - yF_x)\vec{k} \end{aligned}$$

### III.6.3 Conservation of Angular Momentum - Central Forces

The analysis of the relationship:

$$\frac{d\vec{L}_{/O}}{dt} = \vec{\mathcal{M}}_{/O}(\vec{F}) = \vec{r} \wedge \vec{F} \quad (\text{III. 61})$$

shows that the time derivative of angular momentum is zero in two cases:

- The particle is isolated, meaning that the resultant of external forces is zero. As a result, the angular momentum of a free particle is constant.
- $\vec{F}$  is constantly parallel to  $\vec{r}$  and, as a result, its direction consistently passes through the reference point used for angular momentum. By definition, a force whose direction consistently passes through the same fixed point is a **central force**, and  $\vec{F}$  is then central. In this case, the reference point is called the center of forces. Thus, the angular momentum with respect to the center of forces is constant if the force is central. The converse is also true: if angular momentum is constant, the force is central.

### III.6.4 Application: Derivation of Kepler's Second Law

In the context of a planet's motion around the Sun, the planet is subject to the gravitational force alone, which is central as it is constantly directed toward the Sun. Its angular momentum with respect to the center of the Sun is obviously constant, that is:

$$\vec{L} = m\vec{r} \wedge \vec{V} = \text{constante} \quad (\text{III. 62})$$

Based on the Figure III.27, the surface  $dS$  delimited by  $\vec{r}(t)$ ,  $\vec{r}(t+dt)$  and  $d\vec{r}$  is given by:

$$dS = \frac{1}{2} |\vec{r} \wedge d\vec{r}|$$

Therefore,

$$\frac{dS}{dt} = \frac{1}{2} \left| \vec{r} \wedge \frac{d\vec{r}}{dt} \right| = \frac{1}{2} |\vec{r} \wedge \vec{V}|$$

Given the relation (III.62), it follows:

$$\frac{dS}{dt} = \frac{1}{2m} |m\vec{r} \wedge \vec{V}| = \frac{|\vec{L}|}{2m} = C \quad (\text{constante}) \quad (\text{III. 63})$$

The quantity  $\frac{dS}{dt}$  is called areal velocity. By integrating it over two instants  $t_1$  and  $t_2$ , we obtain the area swept during that time interval:

$$S_{12} = C \int_{t_1}^{t_2} dt = C(t_2 - t_1)$$

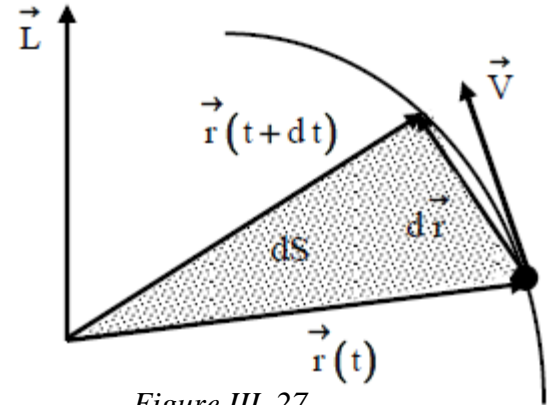


Figure III. 27

**Consequences:**

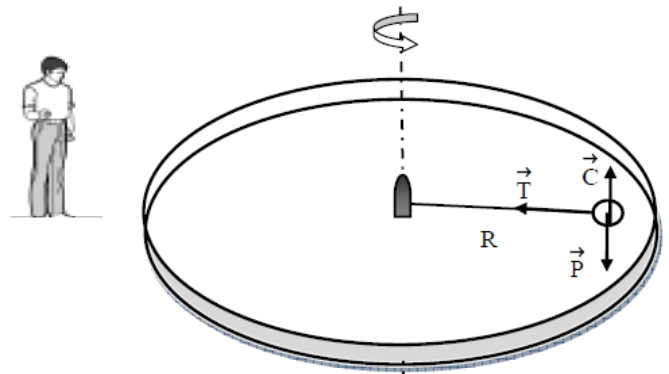
1. The area swept between two instants is proportional to the corresponding time interval. This is Kepler's second law (law of areas).
2. The plane  $(\vec{r}, \vec{V})$  is fixed because  $\vec{r} \wedge m\vec{V} = \text{constante}$ . Therefore, the trajectory of a planet in an ideal framework (neglecting interactions with other celestial bodies) lies in a single plane.

**III.7 Pseudo-forces or Inertial Forces**

We have seen that Newton's second law,  $\vec{F} = m\vec{a}$ , holds in any inertial reference frame. In this section, we will explore the implications of using a non-inertial reference frame.

**III.7.1 Case of Circular Motion**

Let's consider a ball with weight  $\vec{P}$  connected by an inextensible string to the axis of a rotating platform with a constant angular velocity (Figure III.28). After a certain time, this ball starts undergoing circular motion with the same angular velocity. This situation is perceived differently by an observer at rest and an observer on the rotating platform.



*Figure III. 28*

For the observer (inertial) on the ground, the ball undergoing the weight  $\vec{P}$ , the contact force  $\vec{C}$  with the platform, and the tension  $\vec{T}$  in the string, is in uniform circular motion. They apply Newton's second law:

$$\underbrace{\vec{P} + \vec{C}}_0 + \vec{T} = m\vec{a} \quad \text{which gives} \quad |\vec{T}| = m \frac{V^2}{R}$$

Here, R is the distance from the ball to the center of the platform.

For the observer (non-inertial) on the rotating platform (Figure III.29), the ball is not moving relative to them, and they conclude that the ball is in equilibrium under the influence of two radial forces,  $\vec{F}_{in}$  and  $\vec{T}$ , and two vertical forces,  $\vec{P}$  and  $\vec{C}$ . In this case, Newton's second law can be written as:

$$\underbrace{\vec{P} + \vec{C}}_0 + \vec{T} + \vec{F}_{in} = 0 \quad \text{which gives:} \quad \vec{F}_{in} = -\vec{T}$$

$\vec{F}_{in}$  is a force directed outward. It is not associated with any material object causing it and is called an inertial force.

By comparing the two results, we can conclude that:

$$\vec{F}_{in} = -\vec{T} = -m\vec{a} \quad \text{and} \quad |\vec{F}_{in}| = m \frac{V^2}{R}$$

In the case of circular motion, this force is known as the **centrifugal force**.

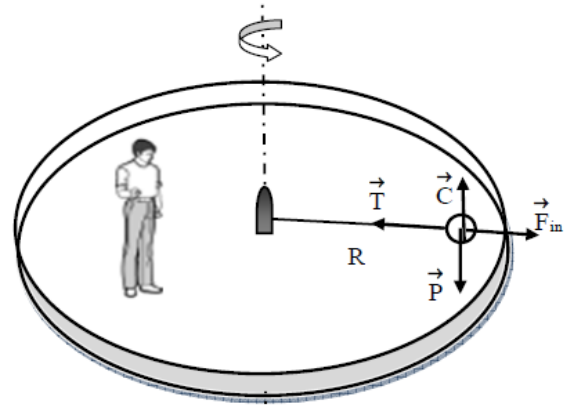


Figure III. 29

### III.7.2 Case of Linear Motion

Let's consider a simple pendulum consisting of a mass  $m$  suspended by an inextensible string from the ceiling of a vehicle undergoing uniform acceleration, with a horizontal acceleration  $\vec{a}$ . This pendulum forms an angle  $\theta$  with the vertical, and the analysis of its inclination is not the same for an observer at rest and for one inside the moving vehicle.

For the observer (inertial) on the ground (Figure III.30), the forces acting on the pendulum are the weight  $\vec{P}$  and the tension  $\vec{T}$  in the string. They apply Newton's second law, recognizing that the pendulum is being accelerated by the vehicle with acceleration  $\vec{a}$  :

$$\vec{P} + \vec{T} = m\vec{a}$$

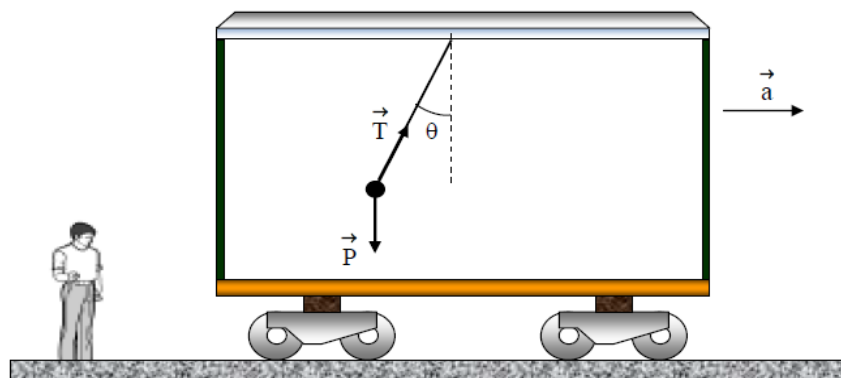
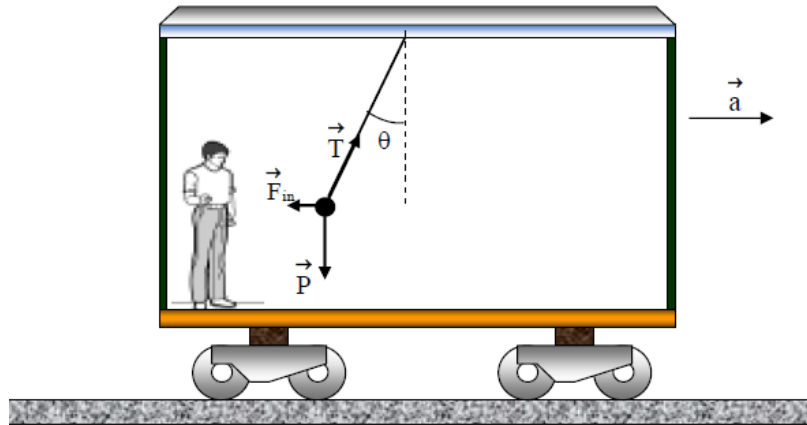


Figure III. 30



*Figure III. 31*

The observer (non-inertial) inside the vehicle (Figure III.31) sees the pendulum at rest and concludes that the weight, unable to balance the tension in the string, should be supplemented by a third force  $\vec{F}_{in}$ . They formulate Newton's second law accordingly:

$$\vec{P} + \vec{T} + \vec{F}_{in} = 0$$

By comparing these two equations, they deduce that:

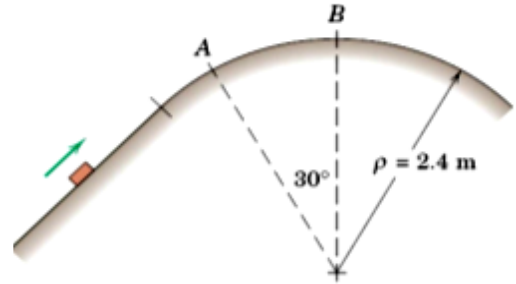
$$\vec{F}_{in} = -m\vec{a}$$

This force is of non-material origin, as there is no physical body causing it. It is also an inertial force.

EXERCISES

**Exercise 1:**

If the 2kg block passes over the top B of the circular portion of the path with a speed of 3.5 m/s, calculate the magnitude of the normal force exerted by the path on the block. Determine the maximum speed which the block can have at A without losing contact with the path. (Use the intrinsic coordinates)



**Solution:**

1. Normal force exerted by the path on the block:

The fundamental law of dynamics is written as:

$$\sum \vec{F}_{ext} = \vec{P} + \vec{N} = m\vec{a}$$

By projection in the intrinsic coordinates:

$$\sum \vec{F}_N = \vec{P} + \vec{N} = m\vec{a}_N$$

$$P - N = ma_N = m \frac{V^2}{\rho} \Rightarrow N = m(g - \frac{V^2}{\rho})$$

$$N_B = 2 \left( 9.81 - \frac{3.5^2}{2.4} \right) = 9.41N$$

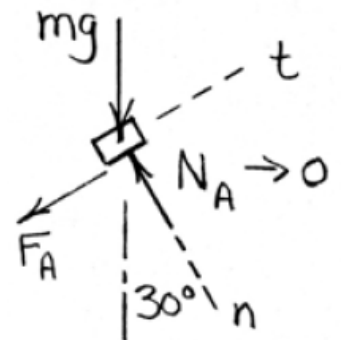
2- Maximum speed which the block can have at A without losing contact with the path:

Loss a contact at A  $\Rightarrow N_A = 0$

$$P \cos \theta - 0 = ma_N = m \frac{V^2}{\rho}$$

$$mg \cos 30^\circ = m \frac{V^2}{\rho}$$

$$V^2 = \rho \cdot g \cdot \cos 30^\circ \Rightarrow V = \sqrt{\rho g \cos 30^\circ} = \sqrt{2.4 \times 9.81 \times 0.86} = 4.5 \text{ m/s}$$



**Exercise 2:**

If the 2kg block passes over the top B of the circular portion of the path with a speed of 3.5 m/s, calculate the magnitude of the normal force exerted by the path on the block. Determine the

maximum speed which the block can have at A without losing contact with the path. (Use the Polar coordinates).

**Solution:**

The same steps of the first exercise but the projection are in Polar coordinates in this case:

1. Normal force exerted by the path on the block:

The second Newton's law of dynamics is written as:

$$\sum \vec{F}_{ext} = \vec{P} + \vec{N} = m\vec{a}$$

$$r = 2.4 \text{ m} = Cte \Rightarrow \dot{r} = 0 \text{ and } \ddot{r} = 0$$

$$\dot{\theta} = \frac{V}{r}$$

$$\sum \vec{F}_r = \vec{P} + \vec{N} = m\vec{a}_r$$

$$\sum F_r = m(\ddot{r} - r\dot{\theta}^2) = -m\frac{V^2}{r}$$

$$-P + N_B = -m\frac{V^2}{r} \Rightarrow N_B = -m\frac{V^2}{r} + P = -m\frac{V^2}{r} + mg = m\left(g - \frac{V^2}{r}\right)$$

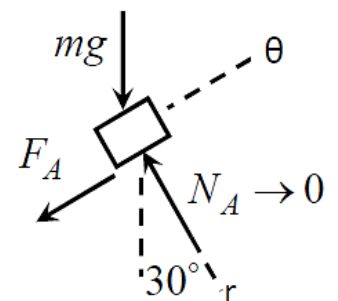
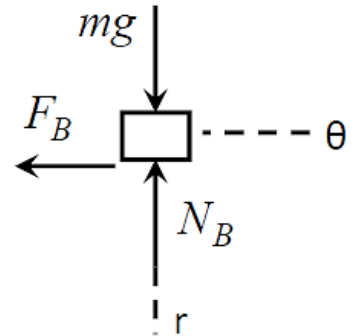
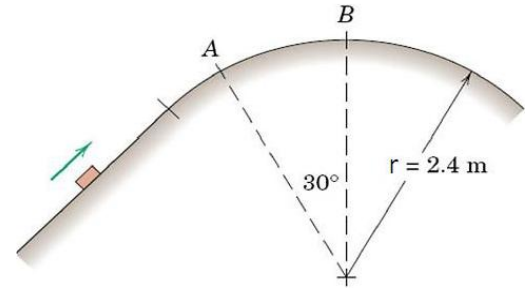
$$N_B = 2\left(9.81 - \frac{3.5^2}{2.4}\right) = 9.41 \text{ N}$$

2- Maximum speed which the block can have at A without losing contact with the path:

Loss of contact at A :  $\Rightarrow N_A = 0$

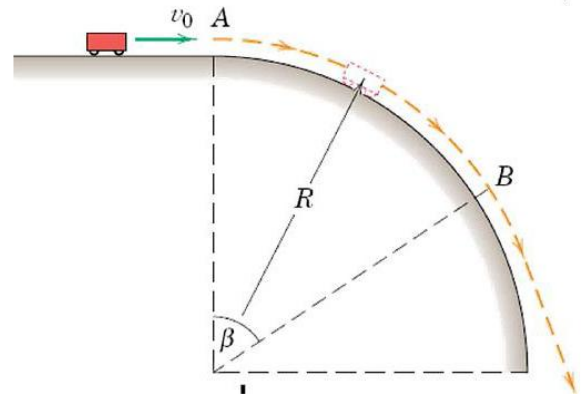
$$\sum F_r = -m\frac{V^2}{r} \Rightarrow -mg \cos \theta = -m\frac{V^2}{r}$$

$$V^2 = r \cdot g \cdot \cos 30^\circ \Rightarrow V = \sqrt{r g \cos 30^\circ} = \sqrt{2.4 \times 9.81 \times 0.866} = 4.5 \text{ m/s}$$



**Exercise 03:**

A small vehicle enters the top A of the circular path with a horizontal velocity  $V_0$  and gathers speed as it moves down the path. Determine an expression for the angle  $\beta$  which locates the point where the vehicle leaves the path and becomes a projectile. Evaluate your expression for  $V_0 = 0$ . Neglect friction.



**Solution:**

At the first we should calculate velocity, normal force and then deduce the angle:

$$\sum F_{\theta} = ma_{\theta}; \quad mg \sin \theta = ma_{\theta} \Rightarrow a_{\theta} = g \sin \theta$$

$$\int VdV = \int a_{\theta} ds ; \quad \int_{V_0}^V VdV = \int_0^{\theta} g \sin \theta (Rd\theta)$$

$$V^2 = V_0^2 + 2gR(1 - \cos \theta)$$

$$\sum F_r = ma_r \Rightarrow -mg \cos \theta + N = -m \frac{V^2}{r}$$

$$N = mg \cos \theta - m \frac{V_0^2}{R} - 2mg(1 - \cos \theta) = mg(3 \cos \theta - 2 - \frac{V_0^2}{gR})$$

$$\text{When } N = 0, \text{ So } 3 \cos \beta = 2 + \frac{V_0^2}{gR}$$

$$\beta = \cos^{-1} \left( \frac{2}{3} + \frac{V_0^2}{3gR} \right)$$

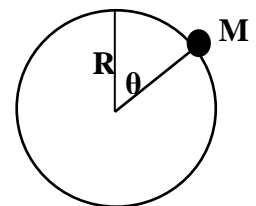
$$\text{For } V_0 = 0 \Rightarrow \beta = \cos^{-1} \left( \frac{2}{3} \right) = 48.2^{\circ}$$

**Exercise 04:**

Body M slides with a friction coefficient  $f$  starting from the top of a sphere with radius  $R$ . (See the figure).

1-Write down the differential equations of the body's motion during its slide? (Projection on the polar and intrinsic coordinates axes).

2-Calculate the velocity at which the particle leaves the surface of the sphere, neglecting friction and initial velocity?

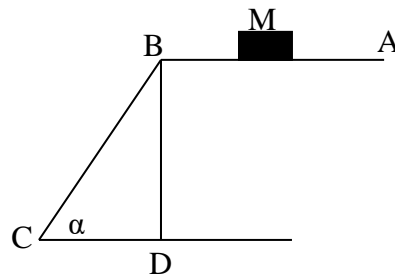


**Exercise 5:**

A body  $P$  with mass  $m$  is moving along a straight line (D) and subjected to a force  $F = F_0 e^{-kt}$ . At time  $t = 0$ ,  $P$  is located at  $x_0=0$  and has an initial velocity  $V_0=0$ . Find the velocity and position of the body at time  $t$ .

**Exercise 06:**

A body with a mass of  $m = 500\text{g}$  is moving on a track ABC, consisting of a rough straight part  $AB = 2\text{m}$  with a friction coefficient  $f$ , and a smooth inclined plane  $BC$  that forms an angle  $\alpha = 20^\circ$  with the horizontal and has a height  $BD = 1\text{m}$  (see the figure). The body is launched from point A with an initial velocity  $V_0 = 2\text{m/s}$  towards point B where it reaches a velocity  $V_B = 0\text{m/s}$ . The acceleration due to gravity is given as  $g = 9.81 \text{ m/s}^2$ . Determine:



1- **Part AB:** the acceleration  $a_1$ , the time  $\Delta t$  for the AB segment, the friction force  $F_f$ , and the friction coefficient  $f$ .

2- **Part BC:** the reaction force  $R$  from the plane on the body, the acceleration  $a_2$ , and the velocity  $V_c$  at point C.

**Exercise 7:**

A water droplet, treated as a point mass with mass  $m$ , falls vertically under the influence of its weight without any initial velocity. It experiences frictional forces proportional to its velocity  $V$ , given by the expression:  $\vec{F} = -k \vec{V}$ , where  $k$  is a positive constant. Write the differential equation of motion. Show that the droplet reaches a terminal velocity  $V_1$ . Determine this terminal velocity in terms of  $m$ ,  $k$ , and  $g$ . Find the expression for velocity as a function of time,  $V(t)$ . Calculate the time of fall when the droplet reaches a velocity with a relative precision of approximately  $10^{-2}$ . Given:

$$m=1.00 \times 10^{-6} \text{ kg}, \quad g=9.81 \text{ m/s}^2, \quad V_1=5.00 \times 10^{-3} \text{ m/s}$$

# Chapter IV

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## Work and Energy

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*In the previous chapter, dedicated to dynamics, we established the relationships that exist between motion and the forces that cause it. In particular, we saw that understanding the forces acting on a particle and the initial conditions (position and velocity) can allow us to predict its motion. However, we do not always know all the forces at play, and even if we do, it sometimes happens that the equations to be solved are challenging to handle. In such situations, we can turn to concepts like work and energy, which are the subject of this chapter.*

## IV. Work of a Force

### IV.1 Definitions

If a particle undergoes an infinitesimal displacement  $d\vec{r}$  under the influence of a force  $\vec{F}$ , the latter performs an infinitesimal work  $dW$  defined by:

$$dW = \vec{F} \cdot d\vec{r} \quad (IV.1)$$

If the particle is moved from point A to another point B (figure IV.1), the total work is:

$$W(\vec{F})_A^B = \int_A^B \vec{F} \cdot d\vec{r} \quad (IV.2)$$

Note that:

$$\vec{F} \cdot d\vec{r} = |\vec{F}| \cdot |d\vec{r}| \cdot \cos(\vec{F}, d\vec{r}) = F_t \cdot |d\vec{r}| = F_t \cdot ds \quad (IV.3)$$

where  $F_t$  is the tangential component of  $\vec{F}$  and  $s$  is the curvilinear coordinate. This leads to:

$$W(\vec{F})_A^B = \int_{s_A}^{s_B} F_t \cdot ds \quad (IV.4)$$

The work  $W$  is expressed in Joules (1 J = 1 N·m).

#### Notes:

- If the force  $\vec{F}$  is perpendicular to the displacement, its work is zero.
- $W$  is an algebraic quantity: if  $W$  is positive, the work is called **motive work**; otherwise, it is referred to as **resistive work**.
- The work generally depends on the path taken between A and B.

#### Example:

In the case of figure IV.2, representing three different paths, most often we have:

$$W_1(\vec{F})_A^B \neq W_2(\vec{F})_A^B \neq W_3(\vec{F})_A^B \quad (IV.5)$$

### IV.2 Using Cartesian Coordinates

By expressing the dot product in terms of the components of  $\vec{F}$  and  $d\vec{r}$ , we have:

$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz \quad (IV.6)$$

This leads to:

$$W(\vec{F})_A^B = \int_A^B \vec{F} \cdot d\vec{r} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz \quad (IV.7)$$

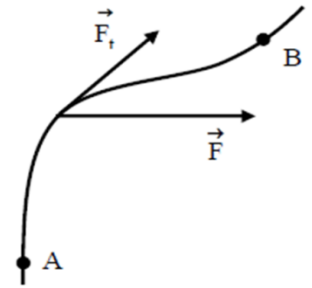


Figure IV. 1

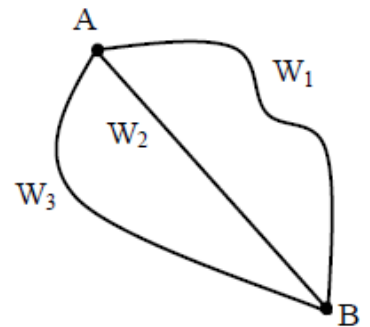


Figure IV. 2

**Note:**

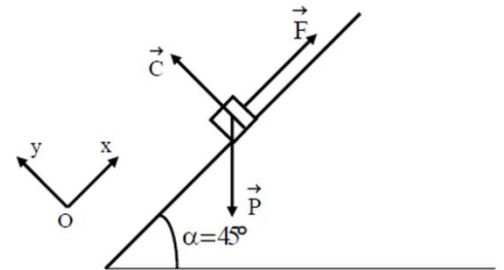
If the force  $\vec{F}$  is constant in magnitude and direction, the expression for  $W(\vec{F})_A^B$  becomes simpler:

$$\begin{aligned} W(\vec{F})_A^B &= \int_A^B \vec{F} \cdot d\vec{r} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz \\ &= F_x(x_B - x_A) + F_y(y_B - y_A) + F_z(z_B - z_A) \end{aligned} \quad (\text{IV. 8})$$

This result shows that the work then depends only on the initial and final positions and not on the path taken. The work done by weight (gravity) is a good illustration of this.

**Exercise:**

A body with a mass of  $m = 1 \text{ kg}$  is pulled with a force  $\vec{F}$  over a distance of  $10 \text{ m}$  along an inclined plane at an angle of  $45^\circ$  to the horizontal (figure IV.3). Neglecting friction and considering the motion to be accelerated with an acceleration  $a = 1 \text{ m/s}^2$ , calculate the works done by the forces applied to the body.



*Figure IV. 3*

**Answer:**

Applying Newton's second law yields:

$$\vec{F} + \vec{P} + \vec{C} = m\vec{a} \quad (\text{IV. 9})$$

Projecting this equation onto the two Cartesian coordinate axes:

$$\left( \sum F_{ext} \right)_x = F - P \sin \alpha = ma \quad (\text{IV. 10})$$

$$\left( \sum F_{ext} \right)_y = C - P \cos \alpha = 0 \quad (\text{IV. 11})$$

Equation (IV.10) allows us to write:

$$F = ma + P \sin \alpha = m(a + g \sin 45^\circ) = 8.07N$$

The work done by  $\vec{F}$  is then:

$$\begin{aligned}
 W(\vec{F})_A^B &= \int_A^B \vec{F} \cdot d\vec{r} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy = F_x(x_B - x_A) + F_y(y_B - y_A) \\
 &= F_x(x_B - x_A) + 0(y_B - y_A) = 8.07 \cdot 10 = 80.7 \text{ J}
 \end{aligned}$$

The work done by  $\vec{P}$  is given by:

$$W(\vec{P})_A^B = P_x(x_B - x_A) + P_y(y_B - y_A) = -P \sin \alpha (x_B - x_A) - P \cos \alpha (0) = -70.7 \text{ J}$$

Since friction is negligible, the contact force  $\vec{C}$  is perpendicular to the plane, and hence perpendicular to the displacement, resulting in zero work.

### IV.3 Notion of Power

To vertically lift a body to a constant height  $h$  at a constant speed, a force  $\vec{F}$  needs to be applied such that:

$$\vec{F} + \vec{P} = \vec{0} \quad (\text{IV. 12})$$

Hence:

$$|\vec{F}| = |\vec{P}| = mg \quad (\text{IV. 13})$$

And:

$$W(\vec{F}) = mgh \quad (\text{IV. 14})$$

Thus, since time does not appear in this expression, the value of the work is the same whether the displacement takes one second or a year. To account for the speed at which this work is done, power is defined as the rate of change of work with respect to time:

$$p = \frac{dW}{dt} \quad (\text{IV. 15})$$

Expanding the elemental work gives:

$$p = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{V} \quad (\text{IV. 16})$$

The average power developed between two instants  $t_i$  and  $t_f$  is:

$$p_m = \frac{W|_{t_i}^{t_f}}{\Delta t} \quad (\text{IV. 17})$$

Power is measured in Watts (W).

## IV.4 Kinetic Energy

### IV.4.1 Definition

Consider a particle moving under the influence of a resultant force  $\vec{F}$  between two points A and B. Since Newton's second law applies along the path from A to B, it can be used in the expression for work. Indeed,

$$W(\vec{F})_A^B = \int_A^B \vec{F} \cdot d\vec{r} \quad (\text{IV. 18})$$

Given that:

$$\begin{cases} \vec{F} = m\vec{a} \\ d\vec{r} = \vec{v}dt \end{cases}$$

it follows:

$$W(\vec{F})_A^B = m \int_A^B \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_{V_A}^{V_B} \vec{v} \cdot d\vec{v} = \frac{1}{2} m(V_A^2 - V_B^2) \quad (\text{IV. 19})$$

Which implies:

$$W(\vec{F})_A^B = \Delta \left( \frac{1}{2} mV^2 \right)_A^B \quad (\text{IV. 20})$$

By definition, the quantity:

$$E_C = \frac{1}{2} mV^2 \quad (\text{IV. 21})$$

is called **kinetic energy**; it is measured in Joules (J).

### IV.4.2 Theorem of Kinetic Energy

The relation:

$$W(\vec{F})_A^B = \Delta \left( \frac{1}{2} mV^2 \right)_A^B = \Delta E_C |_A^B \quad (\text{IV. 22})$$

Expresses **the kinetic energy theorem**, which states: *When a body moves between two points A and B under the influence of a resultant force  $\vec{F}$ , the work done by this force, regardless of the path taken or the nature of the forces, is equal to the change in kinetic energy of the body.*

#### Notes:

- ✓ Kinetic energy is always positive. If the body is subjected to several forces with a resultant  $\vec{F} = \sum_i \vec{F}_i$ , then:

$$\Delta E_C |_A^B = \sum_i W_i(\vec{F}_i)_A^B = W_1(\vec{F}_1)_A^B + W_2(\vec{F}_2)_A^B + W_3(\vec{F}_3)_A^B + \dots$$

- ✓  $\vec{P} = m\vec{v}$  being the momentum, it follows:

$$\mathcal{P}^2 = m^2V^2 \Rightarrow E_C = \frac{\mathcal{P}^2}{2m} \quad (\text{IV.23})$$

- ✓ If  $\vec{F}$  is perpendicular to the displacement, its work is zero, and consequently, the kinetic energy remains constant.

### IV.5 Conservative Forces and Potential Energy

#### IV.5.1 Conservative Forces

A force  $\vec{F}$  is said to be conservative if it satisfies one of these two equivalent conditions:

- **First condition:** The work done by the force  $\vec{F}$  between two points A and B does not depend on the path taken. This path-independence implies, for figure IV.4:

$$W_1(\vec{F})_A^B = W_2(\vec{F})_A^B = W_3(\vec{F})_A^B \quad (\text{IV.24})$$

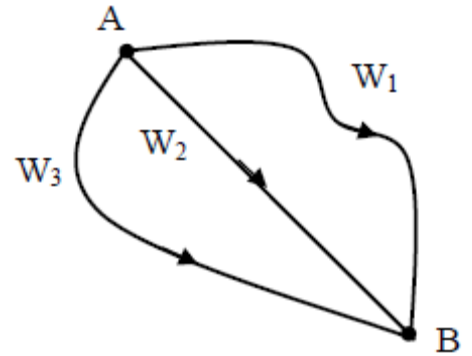


Figure IV. 4

Which means:

$$\int_A^B \vec{F} \cdot d\vec{r}_1 = \int_A^B \vec{F} \cdot d\vec{r}_2 = \int_A^B \vec{F} \cdot d\vec{r}_3 \quad (\text{IV25})$$

- **Second condition:** The total work done along a closed path (a round trip) is zero. This is because of the work's path independence:

$$\int_A^B \vec{F} \cdot d\vec{r}_1 = \int_A^B \vec{F} \cdot d\vec{r}_2 \Rightarrow \int_A^B \vec{F} \cdot d\vec{r}_1 - \int_A^B \vec{F} \cdot d\vec{r}_2 = 0 \quad (\text{IV26})$$

which can be written as:

$$\int_A^B \vec{F} \cdot d\vec{r}_1 + \int_B^A \vec{F} \cdot d\vec{r}_2 = 0 \quad \text{Then: } W_1(\vec{F})_A^B + W_2(\vec{F})_B^A = 0 \quad (\text{IV.27})$$

#### Notes:

- The work done by a conservative force, being independent of the path, must depend only on the position and not on velocity or time.
- Examples of conservative forces include elastic, gravitational, and electric forces.

## IV.5.2 Concept of Potential Energy

### a. First Experiment

Let's consider a mass-spring system as shown in figure IV.5, without friction on the ground. We move the mass in a way that compresses the spring by a length  $a$ , and then we let the system go on its own. The mass starts oscillating with an amplitude of  $a$ , centered around point O. The left part of the figure depicts characteristic configurations of this motion over one cycle. At the endpoints, i.e., at  $x = \pm a$ , the mass has no kinetic energy because its velocity is zero. However, it starts moving back toward the equilibrium position at  $x = 0$  with increasing velocity, meaning it gains kinetic energy. Since energy is conserved, this energy comes from the transformation of a form of energy related to the state (compression or stretching) of the mass-spring system. This energy is called **potential energy** (symbol  $E_p$ ).

Beyond the equilibrium point, the mass undergoes a deceleration until it reaches the next extreme point, at which it loses all its kinetic energy. In fact, the system regains potential energy, which it then uses to return the mass to the equilibrium position. This movement is sustained by a succession of transformations from potential energy to kinetic energy and from kinetic energy to potential energy. This is true when friction is absent.

It's worth noting that the force responsible for this motion is the elastic force, which is conservative (gravity and contact force cancel each other out). Consequently, the associated potential energy is referred to as elastic potential energy. Furthermore, the experiment is reproducible, meaning that when it is repeated, the same results are obtained. Additionally, if the amplitude " $a$ " is changed, the potential energy will quantitatively vary. This indicates that potential energy is dependent on the position occupied by the mass

### a. Second Experiment

Let's consider the free fall of an object dropped from a height  $h$  above the ground without any initial velocity (figure IV.6). The object starts moving under the influence of its weight, a conservative force exerted by the Earth. Initially, the object has no kinetic energy since its velocity is zero. Therefore, due to the system's state of the Earth-object, the object possesses gravitational potential energy that it can transform into kinetic energy. Since the experiment is

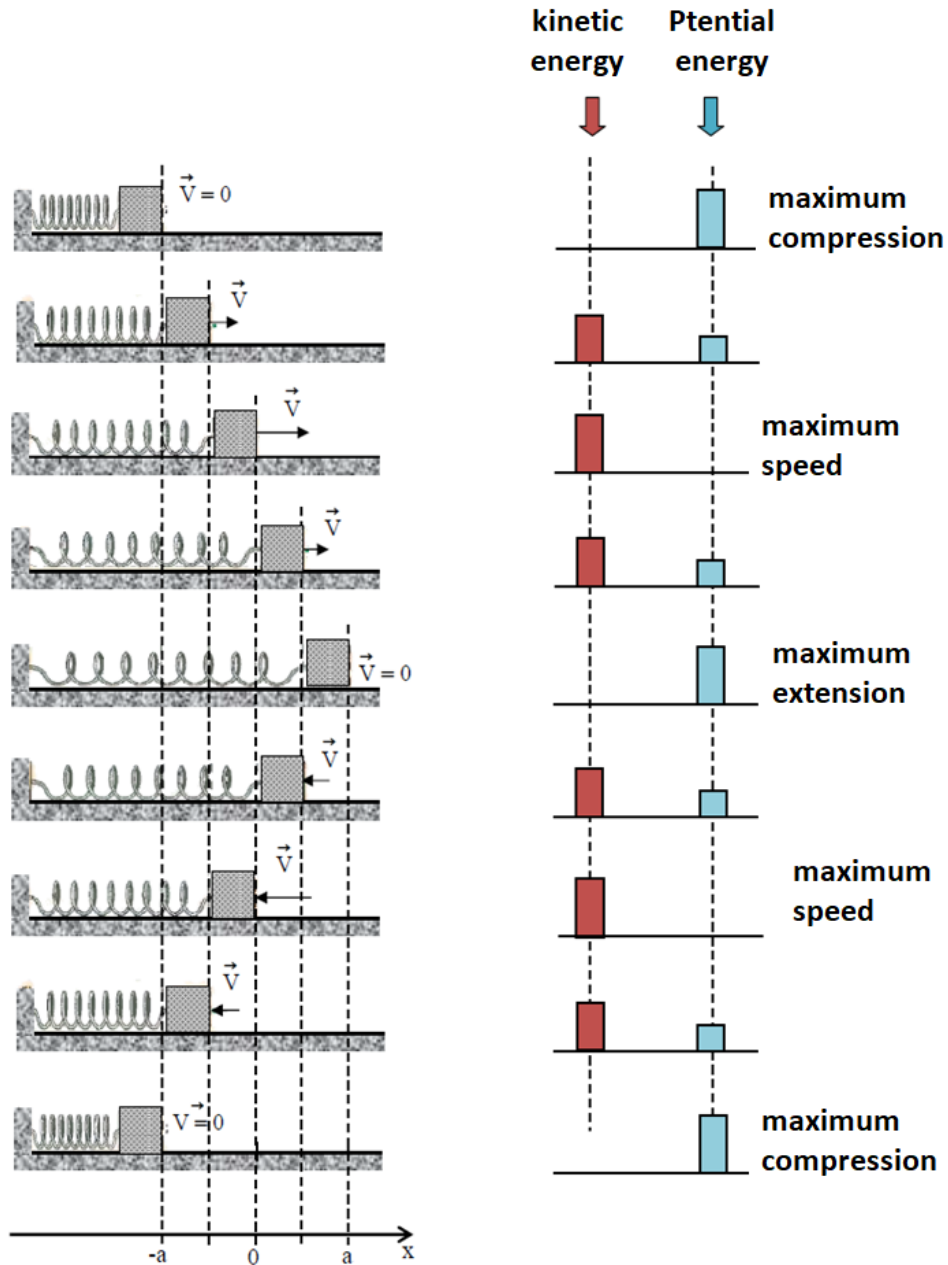


Figure IV. 5

reproducible, this means that the object regains its potential energy each time it returns to its initial position. By changing the falling height and studying the obtained kinetic energy, it becomes evident that this potential energy depends on the object's position. Now, let's consider the case where the same object is thrown upwards from the ground with a velocity  $\vec{V}_0$ , thus having a kinetic energy of  $\frac{1}{2}mV^2$ . Upon reaching a certain height  $h$ , its velocity becomes zero (having lost all its kinetic energy).

It then undergoes a free fall similar to the one described earlier. Thus, between the ground and the point of reversal, the object has converted its kinetic energy into potential energy, which it reconverts into kinetic energy as it falls back down.

### IV.5.3 Total Mechanical Energy

We have seen that the potential energy  $E_P$  of an object has an energy related to the system's state responsible for the conservative force it experiences. This energy, dependent on the object's position, can be transformed into kinetic energy  $E_C$ , which the object regains when returning to its initial state. Thus, the system can convert potential energy into kinetic energy and vice versa. Let's consider the displacement of a particle under the influence of a conservative force:

- if the object loses an amount of potential energy  $\Delta E_P (< 0)$ , it gains an amount of kinetic energy  $\Delta E_C (> 0)$ ;
- if the object loses an amount of kinetic energy  $\Delta E_C (< 0)$ , it gains an amount of potential energy  $\Delta E_P (> 0)$ .

Considering the signs of the two quantities, in all cases, we have the equality:

$$\Delta E_P = -\Delta E_C \quad (\text{IV. 28})$$

This allows us to write:

$$\Delta E_P + \Delta E_C = 0 \Leftrightarrow \Delta(E_P + E_C) = 0 \quad (\text{IV. 29})$$

Thus:

$$E_P + E_C = \text{constante} \quad (\text{IV. 30})$$

The quantity  $E_T = E_P + E_C$  represents the total **mechanical energy (or dynamic energy)** of the particle.

**Theorem:** *The total mechanical energy of a particle subjected to conservative forces is constant.*

### IV.5.4 Determination of Potential Energy

Consider an object moving between two positions,  $A_0 (\vec{r}_0)$  and  $A (\vec{r})$ , under the influence of a conservative force  $\vec{F}$ . The theorem of mechanical energy allows us to write:

$$\Delta E_P|_{A_0}^A = -\Delta E_C|_{A_0}^A \quad (\text{IV. 31})$$

By virtue of the kinetic energy theorem,

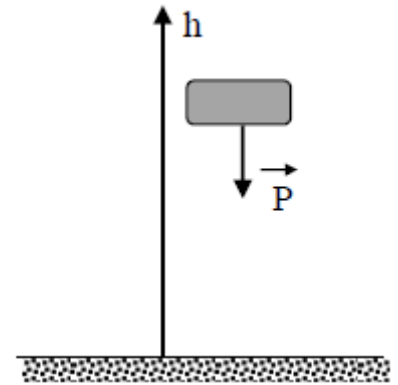


Figure IV. 6

$$\Delta E_P|_{A_0}^A = -\Delta E_C|_{A_0}^A = -W(\vec{F})_{A_0}^A = -\int_{A_0}^A \vec{F} \cdot d\vec{r} \quad (\text{IV.32})$$

Hence:

$$E_P(A) - E_P(A_0) = -\int_{A_0}^A \vec{F} \cdot d\vec{r} \quad (\text{IV.33})$$

Thus, it becomes clear that to determine the potential energy at any point A, the potential energy at a reference point  $A_0$  must be known. Often,  $A_0$  is chosen conventionally at the location where the force  $\vec{F}$  is zero (when possible), and the value  $E_P(A_0)$  is arbitrarily assigned as zero.

**a. Elastic Potential Energy**

Let's return to the mass-spring system from section IV.5.2a. With the deformation of the spring represented by  $x$ , the elastic force applied by the spring is:

$$\vec{F} = -kx\vec{i} \quad (\text{IV.34})$$

This force becomes zero at the origin ( $x = 0$ ), which we'll use as a reference point. Consequently,

$$E_P(x) - \underbrace{E_P(0)}_0 = -\int_0^x F \cdot dx = -\int_0^x (-kx) \cdot dx = \frac{1}{2}kx^2 \quad (\text{IV.35})$$

In general, the potential energy associated with an elastic force  $\vec{F} = -k\vec{\Delta l}$ , in the case of deformation  $\vec{\Delta l}$ , is given by:

$$E_P(\Delta l) = \frac{1}{2}k(\Delta l)^2 \quad (\text{IV.36})$$

**b. Gravitational Potential Energy**

Consider an object with mass  $m$  moving under the influence of the gravitational force  $\vec{F}_g$  exerted by the Earth on it (figure IV.7), which is given by:

$$\vec{F}_g = -G \frac{M_T m}{r^2} \vec{u}$$

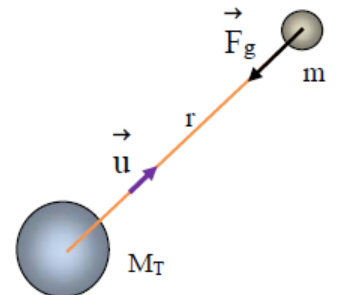


Figure IV. 7

It's evident that this force becomes zero at infinity, which we consider as the reference:

$$E_P(r) = - \int_{\infty}^r \vec{F}_g \cdot d\vec{r} = - \int_{\infty}^r \left( -G \frac{M_T m}{r^2} \right) \vec{u} \cdot d\vec{r} \quad (\text{IV. 37})$$

Note that:

$$\vec{u} \cdot d\vec{r} = |\vec{u}| \cdot |d\vec{r}| \cdot \cos(\vec{u}, d\vec{r}) = dr \quad (\text{IV. 38})$$

Thus:

$$E_P(r) = -GM_T m \int_{\infty}^r \left( -\frac{1}{r^2} \right) dr = -G \frac{M_T m}{r} \quad (\text{IV. 39})$$

### c. Gravitational Potential Energy Near the Earth (or Gravitational Potential Energy)

Consider an object with mass  $m$  moving under its weight between two points A and B. The change in its potential energy is obtained by writing:

$$\Delta E_P|_A^B = -\Delta E_C|_A^B = -W(\vec{P})_A^B \quad (\text{IV. 40})$$

The work done by  $\vec{P}$  is given by:

$$W(\vec{P})_A^B = \int_{x_A}^{x_B} P_x dx + \int_{y_A}^{y_B} P_y dy = P_x(x_B - x_A) + P_y(y_B - y_A) \quad (\text{IV. 41})$$

In a Cartesian coordinate system with the vertical ascending axis (Oy), the components of  $\vec{P}$  are:  $P_x = 0$ ;  $P_y = -mg$ . Consequently,

$$\Delta E_P|_A^B = -W(\vec{P})_A^B = mg(y_B - y_A) \quad (\text{IV. 42})$$

This result holds for any points A and B, allowing us to write generally:

$$\Delta E_P = mg\Delta y \quad (\text{IV. 43})$$

By convention,  $E_P$  is assigned the value of zero at  $y=0$ . The potential energy can then be expressed as:

$$E_P(y) = mgy \quad (\text{IV. 44})$$

It's possible to choose the reference level for potential energy ( $E_p=0$ ) at any altitude. In this case, an upward vertical axis is introduced, with the origin situated at this reference level, and the potential energy due to gravity is written as:

$$E_p(h) = mgh \quad (\text{IV.45})$$

where  $h$  is the coordinate of the position on this axis.

**Example:**

Consider an object moving along a trajectory in the shape of a quarter-circle with a radius  $R$ , situated in the vertical plane ( $O x y$ ). We aim to express the gravitational potential energies of the object for the two positions  $M$  and  $N$  and the two reference levels  $O$  and  $O'$  in figure IV.8.

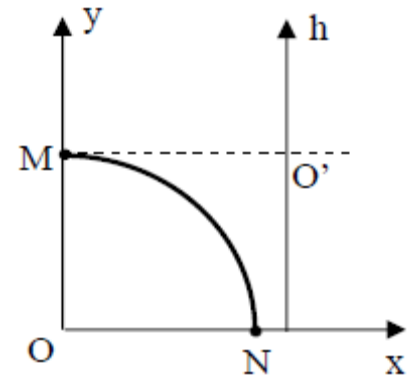


Figure IV. 8

In the following table, the potential energy for the two positions  $M$ ,  $N$  is shown for both reference levels  $O$  and  $O'$ :

Table IV.1

Position	Reference : O	Reference : O'
M	$mgR$	0
N	0	$-mgR$

**d. Study of some examples**

For a particle in motion under the influence of conservative forces, the total energy is constant. To calculate it, one must find a position where the velocity is known.

❖ **Particle in an Elastic Force Field**

Let's return to the mass-spring system from section IV.5.2a. With the deformation of the spring represented by  $x$ , the potential energy of the mass is:

$$E_p(x) = \frac{1}{2} kx^2 \quad (\text{IV.46})$$

Its total energy is then:

$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}mV^2 \quad (\text{IV. 47})$$

Knowing that the velocity is zero at the point of maximum elongation,  $x = a$ , we obtain:

$$E_T = \frac{1}{2}ka^2 \quad (\text{IV. 48})$$

Thus, at any point with abscissa  $x$ :

$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}mV^2 = \frac{1}{2}ka^2 \quad (\text{IV. 49})$$

Hence, the kinetic energy at this point is:

$$\begin{aligned} E_C(x) &= \frac{1}{2}mV^2 = E_T - E_P \\ &= \frac{1}{2}k(a^2 - x^2) \end{aligned} \quad (\text{IV. 50})$$

Consequently, the velocity at a point with abscissa  $x$  is given by the relation:

$$V = \sqrt{\frac{k}{m}(a^2 - x^2)} \quad (\text{IV. 51})$$

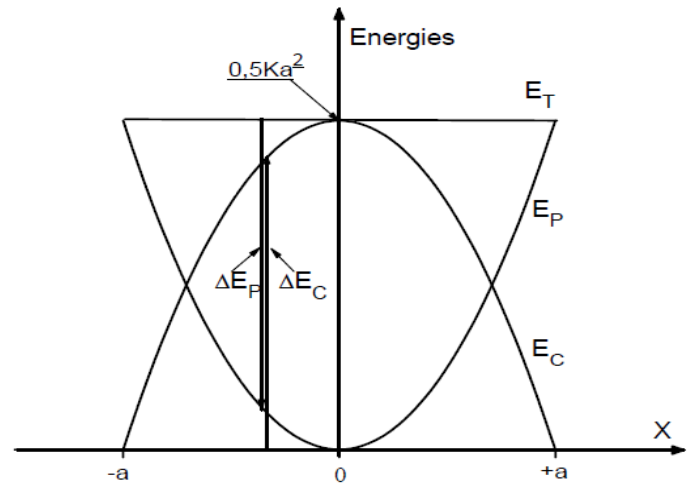


Figure IV. 9

The energy diagram in figure IV.9 clearly illustrates that any decrease in potential energy is accompanied by an equal increase in kinetic energy.

❖ **Motion of a Projectile Launched Upwards from the Ground - Escape Velocity**

Let  $\vec{V}_0$  be the initial velocity of a projectile launched upwards. Its gravitational potential energy is given by the expression:

$$E_P(r) = -G \frac{M_T m}{r} \quad (\text{IV. 52})$$

where  $r$  is the distance from the center of the Earth,  $m$  is the mass of the projectile, and  $M_T$  is the mass of the Earth. Its total mechanical energy is then:

$$E_T = -G \frac{M_T m}{r} + \frac{1}{2}mV^2 = \text{constante} \quad (\text{IV. 53})$$

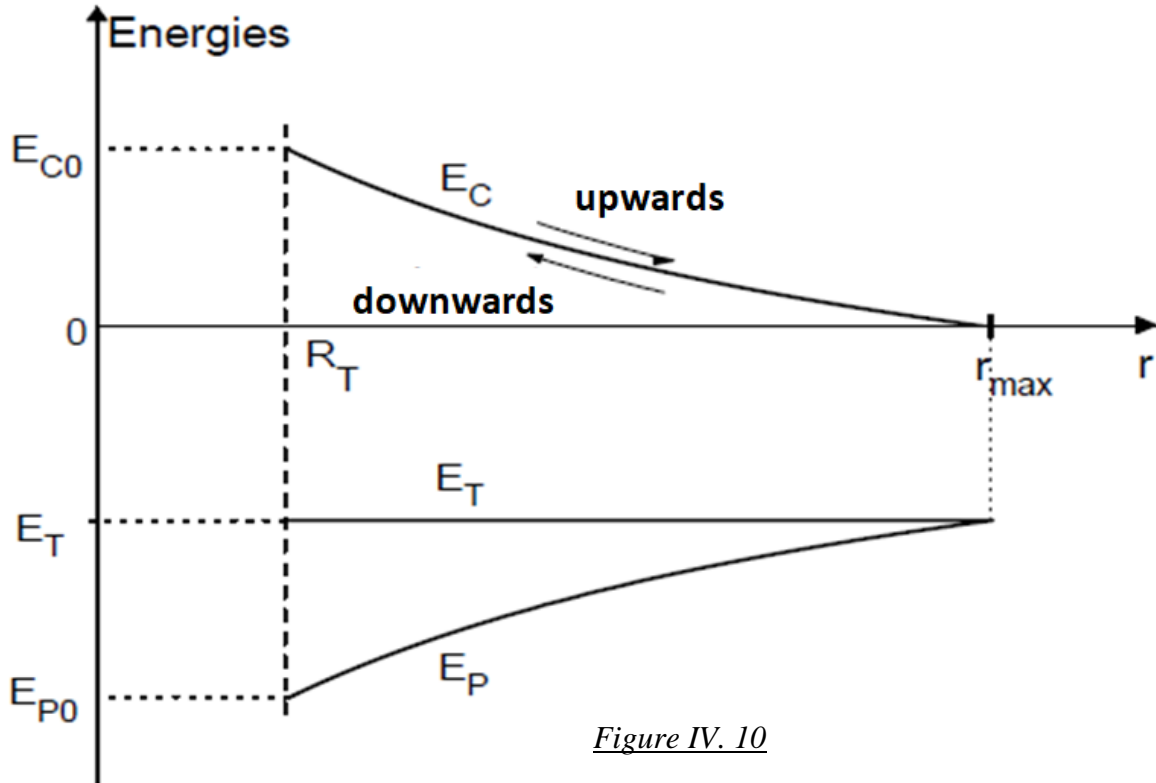
Using the initial conditions  $r = R_T$  (the radius of the Earth) and  $V=V_0$ , we have:

$$E_T = -G \frac{M_T m}{r} + \frac{1}{2} m V^2 = -G \frac{M_T m}{R_T} + \frac{1}{2} m V_0^2 \quad (\text{IV. 54})$$

The kinetic energy at any point in the trajectory is then:

$$E_C = \frac{1}{2} m V^2 = G \frac{M_T m}{r} - G \frac{M_T m}{R_T} + \frac{1}{2} m V_0^2 \quad (\text{IV. 55})$$

The diagram of the different energies is represented in figure IV.10.



The distance " $r_{\max}$ " corresponds to the point of maximum altitude reached before entering a free fall that brings the object back towards the Earth. For the projectile to be able to escape the Earth's gravitational pull, it must be admitted that if it were solely under the effect of Earth's gravity, its initial velocity would allow it to go towards infinity where its kinetic energy would be zero. Its total mechanical energy is then given by:

$$E_T = -G \frac{M_T m}{\underbrace{r_\infty}_0} + \frac{1}{2} m \underbrace{V_\infty^2}_0 = -G \frac{M_T m}{R_T} + \frac{1}{2} m V_0^2 = 0 \quad (\text{IV. 56})$$

Hence, the escape velocity of the projectile is:

$$V_0 = \sqrt{2G \frac{M_T}{R_T}} \quad (\text{IV.57})$$

### IV.5.5 Conservative Forces and Potential Energy

Let's start by considering the motion of a particle constrained to move under the influence of a conservative force  $\vec{F}$ . In this case, the change in its potential energy between two points  $A_0$  and  $A_1$  is given by the relation:

$$\Delta E_P|_{A_0}^A = - \int_{A_0}^A \vec{F} \cdot d\vec{r} \quad (\text{IV.58})$$

For an infinitesimal displacement, this relation becomes:

$$dE_P = -\vec{F} \cdot d\vec{r} = -F_x dx - F_y dy - F_z dz \quad (\text{IV.59})$$

Moreover, recall that if a function  $f(x, y, z)$  is continuously differentiable, meaning that its partial derivatives  $\partial f/\partial x$ ,  $\partial f/\partial y$ , and  $\partial f/\partial z$  exist and are continuous, its gradient, denoted as  $\overrightarrow{\text{grad}}f$ , is defined as the vector whose components are the partial derivatives of that function:

$$\overrightarrow{\text{grad}}f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \quad (\text{IV.60})$$

The difference:

$$df = f(x + dx, y + dy, z + dz) - f(x, y, z) \quad (\text{IV.61})$$

is called the differential  $df$  of the function  $f$ , which can be expressed as the dot product:

$$df = \overrightarrow{\text{grad}}f \cdot d\vec{r} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad (\text{IV.62})$$

Thus, the differential of potential energy can be written as:

$$dE_P = \frac{\partial E_P}{\partial x} dx + \frac{\partial E_P}{\partial y} dy + \frac{\partial E_P}{\partial z} dz \quad (\text{IV.63})$$

By comparing equations (IV.59) and (IV.63), we obtain by identification:

$$F_x = -\frac{\partial E_P}{\partial x} ; F_y = -\frac{\partial E_P}{\partial y} ; F_z = -\frac{\partial E_P}{\partial z} \quad (\text{IV.64})$$

This means that the force  $\vec{F}$  is derived from the potential  $E_P(\vec{r})$  or is related to its gradient through the relation:

$$\vec{F} = -\overrightarrow{\text{grad}}E_P = -\vec{\nabla}E_P$$

In polar coordinates, the gradient is written as:

$$\overrightarrow{\text{grad}} = \frac{\partial}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{u}_\theta \quad (\text{IV. 65})$$

Hence:

$$\vec{F}(r, \theta) = -\frac{\partial E_P}{\partial r}\vec{u}_r - \frac{1}{r}\frac{\partial E_P}{\partial \theta}\vec{u}_\theta \quad (\text{IV. 66})$$

#### IV.6 Total Mechanical Energy and Non-Conservative Forces

Recall that for a particle moving under the influence of conservative forces,

$$\begin{cases} \Delta E_T = \Delta E_P + \Delta E_C = 0 \\ \Delta E_C = \sum W(\vec{F}_C) \end{cases} \Rightarrow \Delta E_P + \sum W(\vec{F}_C) = 0 \quad (\text{IV. 67})$$

represents the work done by **conservative forces**.

If the particle is subject to both conservative and non-conservative forces:

$$\begin{cases} \Delta E_T = \Delta E_P + \Delta E_C \\ \Delta E_C = \sum W(\vec{F}_C) + \sum W(\vec{F}_{NC}) \end{cases} \Rightarrow \underbrace{\Delta E_P + \sum_0 W(\vec{F}_C)}_0 + \sum W(\vec{F}_{NC}) \quad (\text{IV. 68})$$

represents the work done by **non-conservative forces**.

**Theorem:** *The change in total mechanical energy of a body is equal to the work done by the non-conservative forces acting on it.*

As examples of non-conservative forces, we mention:

- Contact forces (except elastic force);
- Magnetic forces;
- Any forces that explicitly depend on time or velocity.

**Note:**

The non-conservative force that frequently appears in exercises is the contact force denoted as  $\vec{C}$ . Between two points A and B along the trajectory followed by the body on which it acts, its work is given by:

$$W(\vec{C})_A^B = W(\vec{C}_{\parallel})_A^B + \underbrace{W(\vec{C}_{\perp})_A^B}_0 \quad (\text{IV. 69})$$

Thus:

$$W(\vec{C})_A^B = \int_A^B \vec{C}_{\parallel} \cdot d\vec{r} = - \int_{S_A}^{S_B} |\vec{C}_{\parallel}| \cdot ds \quad (\text{IV. 70})$$

In the specific case where  $|\vec{C}_{\parallel}|$  remains constant during the motion, we have:

$$W(\vec{C})_A^B = -|\vec{C}_{\parallel}| \int_{S_A}^{S_B} ds = -|\vec{C}_{\parallel}|(S_B - S_A) \quad (\text{IV. 71})$$

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**EXERCISES**


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**Exercise 1:**

A particle subjected to a force:  $\vec{F} = (2x + 1)\vec{i} + 2y\vec{j}$  moves from point O(0,0) to point B(2,4). Calculate the work done by the force as the particle moves along the following paths:

1. The path OAB, knowing that A(2,0).
2. The path OCB, knowing that C(0,4).
3. The path OB. What can be deduced from this?

**Exercise 2:**

A point particle M is moving under the influence of a force  $\vec{F} = (y^2 - x^2)\vec{i} + 3xy\vec{j}$ .

1. Is the force  $\vec{F}$  derived from a potential (Is the force  $\vec{F}$  conservative)?
2. Calculate the work done by the force as the particle moves from the origin (0,0) to the point (2,4) along the following paths:
  - a. Along the x-axis, from point (0,0) to point (2,0), then parallel to the y-axis to point (2,4).
  - b. Along the y-axis, from point (0,0) to point (0,4), then parallel to the x-axis to point (2,4).
  - c. Along a straight line segment connecting the points (0,0) and (2,4).
  - d. Along the trajectory  $y=x^2$ .

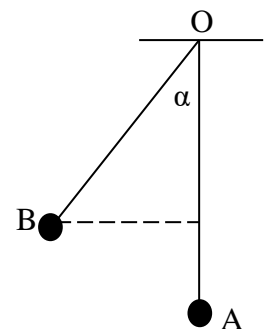
**Exercise 3:**

A ball with a mass  $m=50\text{g}$ , attached to an inextensible string of length  $l=40\text{cm}$ , is displaced from its equilibrium position by an angle  $\alpha=60^\circ$  and then released without initial velocity.

1. Calculate:

- The work done by the weight between the two positions A and B.
- The work done by the tension force of the string  $\vec{T}$ .

2. Explain the variation in kinetic and potential energy. What can be deduced about the mechanical energy  $E_T$  total?

**Exercise 4:**

Consider a satellite of mass  $m$  orbiting around the Earth of mass  $M$  at a distance  $r$  from the center of the Earth. Assuming that its trajectory is circular:

1. Provide the expression for the potential energy corresponding to the gravitational force between the satellite and the Earth, and specify the chosen reference point for potential energy.
2. Give the total mechanical energy in terms of  $G$ ,  $M$ ,  $m$ , and  $r$ .
3. Show that circular trajectories satisfy Kepler's third law:  $\omega^2 r^3 = GM$ . Where  $\omega$  is the angular velocity.
4. If a satellite appears stationary in the sky, calculate its height, its velocity, and its total energy.

Given:  $M=5.98 \times 10^{24}$  kg,  $R_T=6400$  km,  $m=68$  kg, and  $G=6.67 \times 10^{-11}$  N m<sup>2</sup> kg<sup>-2</sup>.

**Solution:**

1. The force between the Earth and the satellite is written as:

$$\vec{F} = -G \frac{Mm}{r^2} \vec{u}$$

$F$  is a force derived from a potential, therefore:

$$W = - \int_r^\infty \vec{F} \cdot d\vec{l} = - \int_r^\infty \left( -G \frac{Mm}{r^2} \right) \vec{u} \cdot d\vec{l} = -GMm \int_r^\infty \frac{dr}{r^2} = -G \frac{Mm}{r}$$

$$\text{and: } W = -\Delta E_P = E_P(r) - E_P(\infty)$$

By considering :  $E_P(\infty) = 0 \Rightarrow E_P(r) = -G \frac{Mm}{r}$

2. Total energy: as

$$F = G \frac{Mm}{r^2} = m \frac{V^2}{r} \Rightarrow mV^2 = G \frac{Mm}{r} \Rightarrow E_C = \frac{1}{2} mV^2 = \frac{1}{2} \frac{GmM}{r}$$

So :

$$E_T = E_P + E_C = -G \frac{Mm}{r} + \frac{1}{2} \frac{GmM}{r} = -G \frac{Mm}{2r}$$

3. We have:  $F = G \frac{Mm}{r^2} = m \frac{V^2}{r} = mr\omega^2 \Rightarrow \omega^2 r^3 = GM$

4. If the satellite remains stationary  $\Rightarrow$  It has the same period as the Earth:  $T=24\text{h}=86400\text{s}$

$$F = G \frac{Mm}{r^2} = mr\omega^2 = mr \frac{4\pi^2}{T^2} \Rightarrow r = (R_T + h) = \left( GM \frac{T^2}{4\pi^2} \right)^{1/3} = 4.2 \cdot 10^7 \text{ m}$$

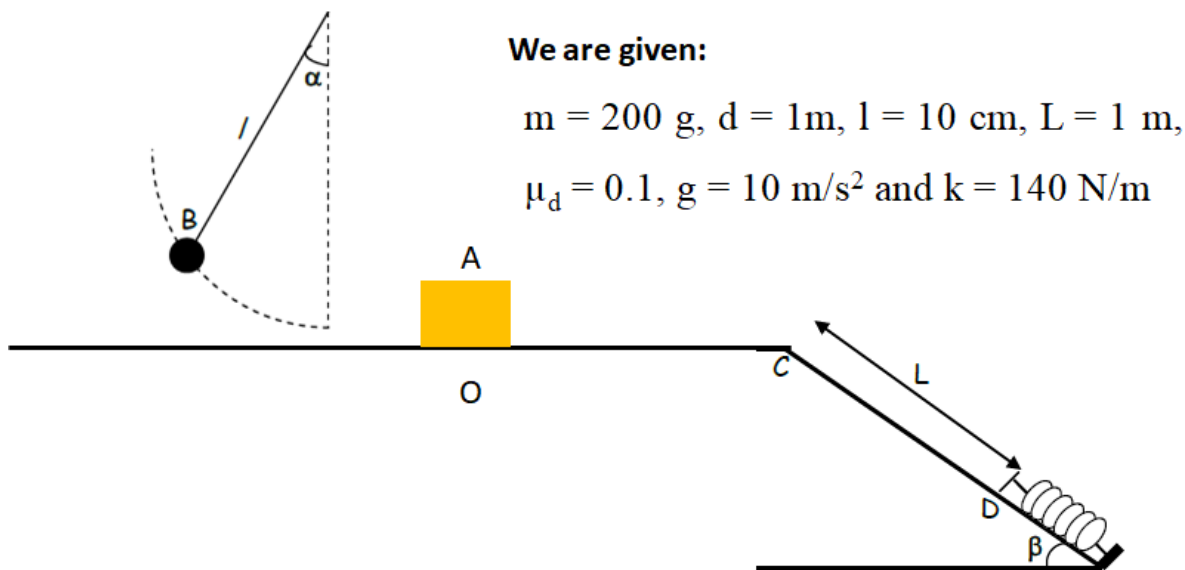
$$\Rightarrow h = 3.6 \cdot 10^7 \text{ m}$$

$$V = \frac{2\pi}{T} r = 3052.77 \text{ m/s}$$

$$E_T = -G \frac{Mm}{2r} = -3.2 \cdot 10^8 \text{ J}$$

**Exercise 5:**

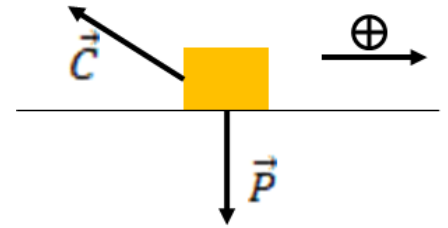
A ball B with mass  $m$ , attached to an inextensible string of length  $l$ , is displaced from its equilibrium position by an angle  $\alpha$  and then released without initial velocity. As it passes through the vertical position, the ball collides with a body A of the same mass and comes to a stop. Body A slides on a track OCD (without friction) as shown in Figure. The segment OC =  $d$  is a rough horizontal plane with a coefficient of dynamic friction  $\mu_d$ . The portion CD =  $L$ , which is perfectly smooth, is inclined at an angle  $\beta = 30^\circ$  with respect to the horizontal.



- 1- Draw the forces exerted on body A at a position between O and C.
- 2- Calculate the acceleration of body A between O and C. Deduce the nature of the motion.
- 3- Provide the expression for the velocity of ball B just before it touches body A.
- 4- Using the conservation of the system's momentum, determine the velocity of body A after the interaction.
- 5- Express the velocity of body A at point C in terms of  $g$ ,  $l$ ,  $d$ ,  $\alpha$ , and  $\mu_d$ .
- 6- By what angle  $\alpha_m$  should ball B be displaced for body A to arrive at C with zero velocity?
- 7- From point C, body A encounters segment CD with zero velocity. It reaches a perfect spring with an unstressed length  $l_0$  and a spring constant  $k$ .
  - Illustrate the forces acting on A during the compression of the spring.
  - What is the value of the maximum compression of the spring?

**Solution:**

- 1- Forces exerted on body A at a position between O and C:
- 2- Calculate the acceleration of body A between O and C:



$$\vec{P} + \vec{C} = m\vec{a} \Rightarrow \begin{cases} (ox): -C_x = ma \\ (oy): C_y = P = mg \end{cases} \Rightarrow a = -\mu_d g = -1 \text{ m} \cdot \text{s}^{-2}$$

$$a = -1 \text{ m} \cdot \text{s}^{-2} = \text{Cte} \Rightarrow \text{non - uniform rectilinear motion}$$

- 3- The expression for the velocity of ball B just before it touches body A:

Without friction  $\Rightarrow E_{ti} = E_{tf} \Rightarrow \frac{1}{2}mV_B^2 = mgl(1 - \cos \alpha)$

$$V_B^2 = 2gl(1 - \cos \alpha) \Rightarrow V_B = \sqrt{2gl(1 - \cos \alpha)}$$

- 4- Conservation of momentum  $\Rightarrow \vec{P}_B = \vec{P}_A \Leftrightarrow \vec{V}_B + 0 = 0 + \vec{V}_A$

$$V_A = V_B = \sqrt{2gl(1 - \cos \alpha)}$$

- 5- Velocity at point C :

$$\Delta E_T = W(\vec{C}_x) \Rightarrow \frac{1}{2}mV_C^2 - \frac{1}{2}mV_A^2 = -C_x \cdot OC = -\mu_d mgd$$

$$\text{So: } V_C = \sqrt{2gl(1 - \cos \alpha) - 2\mu_d gd}$$

- 6- angle  $\alpha_m$  should ball B be displaced for body A to arrive at C with zero velocity:

$$V_C = 0 \Rightarrow \cos \alpha_m = 1 - \frac{\mu_d \cdot d}{l} \Rightarrow \alpha_m = \frac{\pi}{2}$$

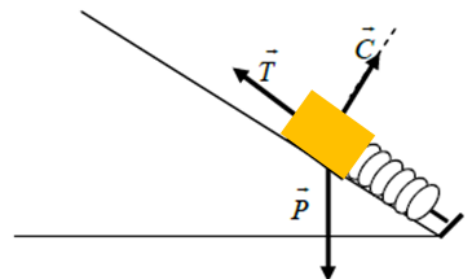
- 7- a. Forces (see figure )

- b. Maximum compression :

$$E_{T1} = mgh = mg(L + x) \sin \beta \text{ and } E_{T2} = \frac{1}{2}kx^2$$

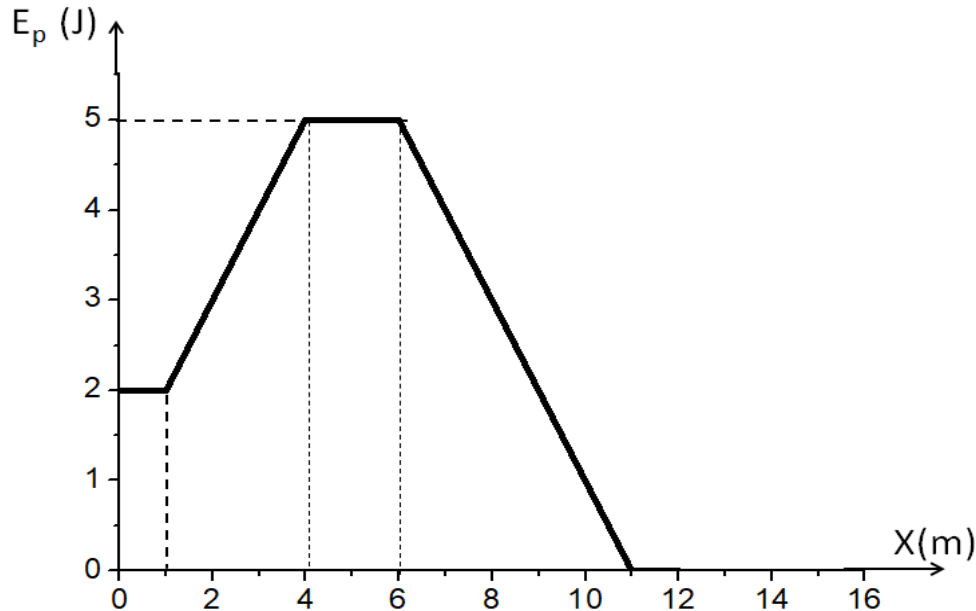
No friction so :  $E_{T1} = E_{T2}$

$$\frac{1}{2}kx^2 - mgx \sin \beta - mgL \sin \beta = 0 \Rightarrow 70x^2 - x - 1 = 0 \Rightarrow x = 12.7 \text{ cm}$$



**Exercise 6:**

A particle with a mass of  $m = 40\text{g}$  undergoes rectilinear motion along an  $x$ -axis. It is subjected to a conservative force  $\vec{F} = F_x \vec{i}$ . The potential energy  $E_p(x)$  varies with position  $x$  as shown in the graph below.



1 - This particle passes through the origin  $O$  with a momentum  $P_0 = 0.8 \text{ kg}\cdot\text{m/s}$ , moving towards positive abscissas.

a) - Calculate its total mechanical energy.

b) - What is the work done by the force  $\vec{F}$  when the mass moves from the origin  $O$  to the point with  $x=12 \text{ m}$ .

c) - Plot the curve  $F_x(x)$  for  $x$  between 0 and 12 m. Using the graph of  $F_x(x)$ , recover the result from question b).

d) - What is the velocity of the mass  $m$  when it passes through the point with  $x=3 \text{ m}$ ? At what other point does it have the same velocity?

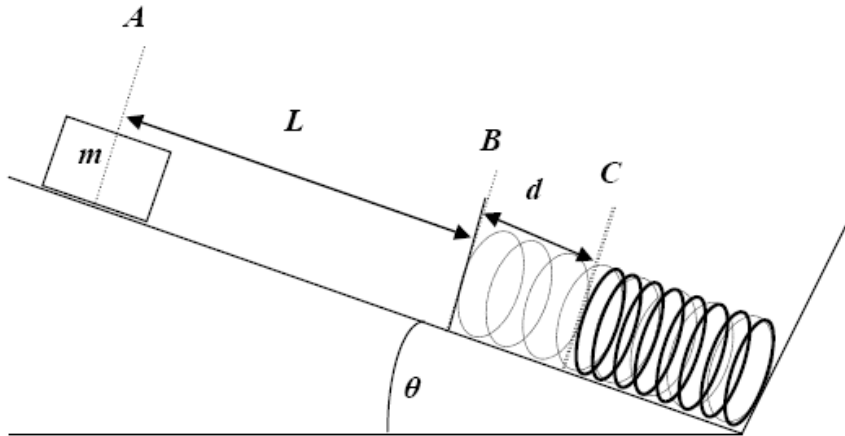
2 - What is the minimum momentum  $P_{\min}$  it must have at the origin in order to reach the point with  $x=12 \text{ m}$ ?

**Exercise 7:**

A block with mass  $m$  is released from rest at the top (position A) of an inclined plane that makes an angle  $\theta$  with the horizontal. The block slides without friction and compresses a spring with a spring constant  $k$  at the bottom of the inclined plane. Let  $L$  be the initial distance between the

block and the spring (when it's in position B and not compressed). At the moment of impact, the spring is compressed by a length  $d$  (position C) before it returns to its original length. Friction between the mass and the ground is negligible.

(Note: No numerical application is required in this exercise)



- 1- Show that the elastic potential energy  $E_{pe}$  of the spring in terms of its elongation  $x$  is given by  $E_{pe} = \frac{1}{2}kx^2$  (specify the origin of this energy).
- 2- Recall the theorem of total mechanical energy. What can be said about the mechanical energy for the system under study?
- 3- Calculate the total energies at points A and C.
- 4- Deduce the expression of the spring constant  $k$  in terms of  $m$ ,  $\theta$ ,  $L$ , and  $d$ .
- 5- If now the contact between the block and the inclined plane is characterized by a coefficient of friction  $\mu_g$ , what is the expression for the maximum height reached by the mass  $M$  when it is released from point C with no initial velocity (the spring is compressed by a length  $d$ )?

### Solution:

1- The elastic force is derived from a potential  $\vec{F}_e = -kx\vec{i}$ , so we have:

$$dE_p = -\vec{F} \cdot d\vec{l} \Rightarrow E_p = \int dE_p = \int (-kx\vec{i}) \cdot (dx\vec{i}) = \int kx dx = \frac{1}{2}kx^2 + C$$

$$\text{if: } E_p(0) = 0 \Rightarrow C = 0 \Rightarrow E_p = \frac{1}{2}kx^2$$

2- If there are no frictional forces, therefore:

$$W = \Delta E_C = -\Delta E_P \Rightarrow \Delta E_C + \Delta E_P = \Delta E_T = 0 \Rightarrow E_T = Cte$$

3- Total energy at A and C:

If we choose point C as the origin of gravitational potential energies:

$$E_{TA} = mg(L + d) \sin \theta \quad \text{and} \quad E_{TC} = \frac{1}{2} kd^2$$

$$4- E_{TA} = E_{TC} \Rightarrow mg(L + d) \sin \theta = \frac{1}{2} kd^2 \Rightarrow k = \frac{2mg}{d^2} (L + d) \sin \theta$$

5- Now there are frictional forces, therefore:  $\Delta E_T = W(\vec{C})$

$$\Delta E_T = E_{TA} - E_{TC} = mgD \sin \theta - \frac{1}{2} kd^2 \quad \text{and} \quad W(\vec{C}) = \int_C^A \vec{C}_x \cdot d\vec{l} = - \int_C^A C_x \cdot dx = -C_x D$$

- Calcul of  $C_x$ :

$$\vec{P} + \vec{C} = m\vec{a} \Rightarrow C_y = P_y = mg \cos \theta \quad \text{and} \quad C_x = \mu_g C_y = \mu_g mg \cos \theta$$

Therefore :

$$mgD \sin \theta - \frac{1}{2} kd^2 = -\mu_g mg \cos \theta \cdot D \Rightarrow$$

$$D = \frac{kd^2}{2mg(\sin \theta + \mu_g \cos \theta)}$$

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