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Mémoire

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Stability of solution for a class of wave equation : Analytical results

Option :AFA

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I dedicate this modest work:

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All the members of my family (my dear parents in the world, my husband Abd Elouaheb, my granddaughter Houria, my dear brother Salah Eddine and my dear sisters: Houda, Ahlem, Imane, Bouchra and souhila).

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Those who have been by my side in difficult times.

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those who have been by my side in difficult times.

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RÉSUMÉ

Dans ce mémoire, en premier chapitre, nous présentons une classe d'équations: les équations elliptiques, paraboliques et hyperboliques, Nous exposons également la forme formelle de chaque type d'équation.

Dans le deuxième chapitre, nous présentons des théories et définitions importantes, et nous étudions également le comportement asymptotique d'un type spécifique d'équations aux dérivées partielles hyperbolique de type Kirchhoff, en utilisant la méthode de l'énergie, avec des conditions aux limites de Dirichlet homogènes. Nous prouvons ensuite que la solution est globale et stable dans deux cas: premièrement, la stabilité est obtenue grâce à une fonction exponentielle où l'énergie se dissipe lorsque celle-ci tend vers l'infini, et deuxièmement, la stabilité se présent sous la forme d?une fonction puissance.

Dans le dernier chapitre, nous répétons le travail précédent en remplaçant les constantes exponentielles par des variables exponentielles, ce qui aboutit aux mêmes résultats.

Mots clés :

- Equation hyperbolique.
- Méthode d'énergie.
- Variables exponentielles.
- Constantes exponentielles.
- Terme source.

ABSTRACT

In the first chapter of memory, we present a variety of important partial differential equations, which include the following types of equations: elliptic, parabolic, and hyperbolic equations. We also write down the formal form of each type of equation.

In the second chapter, we present important theories and definitions, and we also study the asymptotic behavior pectific type of hyperbolic partial defferential eqations of the kirchhoff-type, using the energy method, with the homogeneous Dirichlet boundary conditions. We then prove that the solution is global and stable under two cases: firstly, stability is achived through an expenential function where the energy dissipates as it goes to infinity, and secondly, stability is in the form of a pawer function.

In the final chapter, we repeat the previous work by replacing the exponential constants with exponential variables, resulting in the same outcomes.

Key words:

- Hyperbolic equation.
- Energy methode.
- Exponential variables.
- Exponential constants.
- Source term.

ملخص

ندرس في هذه المذكرة :

في الفصل الاول من المذكرة نقوم بعرض جملة من انواع المعادلات التفاضلية الجزئية المهمة والتي تتمثل في المعادلات: التكافئية الزائدية و الناقصية وكتابة الشكل القانوني لكل صنف.

في الفصل الثاني نقوم بعرض النظريات والتعاريف المهمة و كذلك نقوم بدراسة السلوك التقاربي لنوع من المعادلات التفاضلية الجزئية الزائدية من صنف كرشوف بطريقة الطاقة وفق شروط حدية ابتدائية لديريكلي متجانسة ثم نقوم بالبرهنة على ان الحل عالمي وانه مستقر وفق حالتين: الحالة الاولى تكون مستقرة وفق دالة اسية وان الطاقة تضمحل لما t يؤول الى زائد مالانهاية . والحالة الثانية يكون فيها الاستقرار على شكل دالة قوة لكثير حدود.

وفي الفصل الاخير نقوم بنفس العمل السابق مع استبدال الثوابت الاسية بالمتغيرات الاسية لنتحصل في النهاية على نفس النتائج.

كلمات مفتاحية:

- معادلة زائدية.
- طريقة الطاقة.
- المتغيرات الاسية.
- الثوابت الاسية.
- حد المنبع.

LIST OF SYMBOLS

	Sets and numbers
❀ $\mathbb{R}^n$	The euclidean space.
❀ Ω	a bounded regular domain in $\mathbb{R}^n$.
❀ $[a, b]$	a closed interval of $\mathbb{R}^n$.
❀ $H_0^1(\Omega)$	$u \in H^1(\Omega), u(0) = u(1) = 0$.
❀ $L^p(\Omega)$	Space of p-th order integrable functions on Ω.
❀ $L^{p'}(\Omega)$	Dual space of $L^p(\Omega)$.
❀ $L^\infty(\Omega)$	L infinity space.
❀ $W^{1,p}(I)$	Sobolev space.
❀ B^*	Dual space of B.
❀ $C_c^1(I)$	Space of all first derivative function has a complect support.
❀ $\partial\Omega$	The boundary of Ω.
	Functions
❊ $\lvert . \rvert$	Absolute value.
❊ $\\| \cdot \\|$	Norm.
❊ $\\|f\\|_2$	Norm of $L^2(\Omega)$.
❊ $\\|u\\|_{L^p([a,b],X)}$	Norm of $L^2([a, b] : X)$ defined by : $\int_a^b (\\|u(t)\\|_X^p)^{1/p}$.
❊ sup	Superior value.
❊ inf	Inferior value.
❊ Δ	Laplacien.
❊ ∇u	Devergence.
	Abbreviation Meaning
● PDE	Partial differential equation.
● ODE	Ordinary differential equation.

CONTENTS

INTRODUCTION

The study of partial differential equations (PDE) started in a century eighteen (18) action Euler, Alembert, Lagrange and Laplace. This equations are used to mathematically formulate, and thus aid the solution of, physical. The importance of a differential eqution as a thecnique for determining a function is that if we know the function and possibly some of its derivatives at a particular point.

In chapter 01:

*classification of partial differential equation

*canonical forms In this chapter werite general forms the partial differential equation and used (Δ) for prove la classiffication the this equation rite canonical forme for all class the equation.

In chapter 02: we are going to solve a class of parabolic equation, We prove the existence and uniquess of the solution, also, we study the behavion of the solution also used the kirchhoff- type method.

In chapter 03: We do the same previous work with exponential constants replaced by exponential variables to get the same results in the end

CHAPTER 1

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATION AND CANONICAL FORMS

1.1 second-order partial differential equation

The most general case of second-order linear partial differential equation (PDE) in two independent variables is given by

$$A\frac{\partial^{2u}}{\partial^2 x} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \tag{1.1}$$

Where the coefficients A, B and C are functions of x and y and do not vanish simultaneously, because in that case the second-order PDE degenerates to one of first order.

Further the coefficients D, E and F are also assumed to be functions of x and y. We shall assume that the function $u(x, y)$ and the coefficients are twice continuously differentiable in some domain Ω. The classification of second-order PDE depends on the form of the leading part of the equation consisting of the second-order terms. So, for simplicity of notation, we combine the lower ordre terms and rewrite the above equation in the flowing form

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = \phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \tag{1.2}$$

or using the short-hand notations for partial derivatives,

$$A(x, y)u_{x\,x} + B(x, y)u_{xy} + C(x, y)u_{yy} = \phi(x, y, u, u_x, u_y). \tag{1.3}$$

As we shall see, there are fundanmentally theree types of PDE_s-hyperbolic, and elliptic PDE_S. From the physical point of view, these PDE_S respectively represents the wave propagation, the time-dipendent diffusion processes and the steady state or equilibrium processes. Thus, hyperbolic equations model the transport of some physical quantity, such as fluids or waves. Parabolic problems descrde evolutionary phenomena that lead to a steady state described by an elleptic equation.and elleptic equations are associated to a special state of a system, in principle corresponding to the minimum of the invergy.

Mathematically, these classification of second-order PDE_S is based upon the possibility of redesing equation (1.3) by coordinate transformation to canonical or standard from at a point. It may be noted that, for the purposes of classification, it is not necessary to restrict considuration to linear equation. It is applicable to quasilinear second-order PDE as well.

Aquasilinear second-ordre PDE is linear in the second derivatives only.

The type of second-ordre PDE (1.2) at a point (x_0, y_0) depents on the sign of the discriminant defined as

$$\Delta(x_0,y_0) = \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B\left(x_0,y_0\right)^2 \text{-}4A\left(x_0,y_0\right)C\left(x_0,y_0\right). \tag{1.4}$$

The classification of second-order linear PDE_s is given by the following. If $\Delta\left(x_0 y_0\right) \succ 0$, the equation is hyperbolic, $\Delta\left(x_0, y_0\right) = 0$ the equation is parabolic, and $\Delta\left(x_0, y_0\right) \prec 0$ the equation is elliptic. In order to illustrate the significance of the discriminant Δ and thus the classification of the PDE(1.3). We try to reduce the given equation(1.3) to a canonical form. To do this, we transform the independent variables x and y to the new independent variables ξ and η through the change of variables

$$\xi = \xi\left(x, y\right), \qquad \eta = \eta\left(x, y\right). \tag{1.5}$$

Where both ξ and η are twice continuously differentiable and that the Jacobian

$$J = \frac{\partial\left(\xi, \eta\right)}{\partial\left(x, y\right)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0, \tag{1.6}$$

in the region under consideration. The nonvanishing of the Jacobian of the transformation ensure that a one to one transformation exists betwrrn the new and old variables. This simply means that the new independent variables can serve as new coordinate variables without any apply the chain rule to compute the terms of the equation (1.3) in terms of ξ and η as follows:

$$\begin{aligned} u_x &= \omega_\xi \xi_x + \omega_\eta \eta_x, \\ u_y &= \omega_\xi \xi_y + \omega_\eta \eta_y, \\ u_{xx} &= \omega_{\xi\xi}\xi_x^2 + 2\omega_{\xi\eta}\xi_x\eta_x + \omega_{\eta\eta}\eta_x^2 + \omega_\xi \xi_{xx} + \omega_\eta \eta_{xx}, \\ u_{yy} &= \omega_{\xi\xi}\xi_y^2 + 2\omega_{\xi\eta}\xi_y\eta_y + \omega_{yy}\eta_y^2 + \omega_\xi \xi_{yy} + \omega\eta_{yy}, \\ u_{xy} &= \omega_{\xi\xi}\xi_x\xi_y + \omega_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + \omega_{\eta\eta}\eta_x\eta_y + \omega_\xi \xi_{xy} + \omega_\eta \eta_{xy}, \end{aligned} \tag{1.7}$$

substituting these expressions into equation (1.3) we obtain the transformed PDE as

$$a\omega_{\zeta\zeta} + b\omega_{\zeta\eta} + c\omega_{\eta\eta} = \phi\left(\zeta, \eta, \omega, \omega_\zeta, \omega_\eta\right). \tag{1.8}$$

Where Φ becomes ϕ and the new coefficients of the higher order terms a, b and c are expressed via the original coefficients and the change of variables formulas as follows :

$$\begin{aligned} a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\ b &= 2A\xi_x\eta_y + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y, \\ c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned} \tag{1.9}$$

As this stage the form of the EDP (1.8) a is no simpler than that of the original EDP(1.2), but this is to be expected because so far the choice of the new vaeiable ξ and has been that equation(1.9) can be written in matrix form as

$$\begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} + \begin{vmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{vmatrix} J^2,$$

where J is Jacobian of the change of variables given by (1.6). Expanding the determinant and multiplying by the factor, -4 to obtain

$$b^2 - 4ac = J^2\left(B^2 - 4AC\right) \Rightarrow \delta = J^2\Delta. \tag{1.10}$$

Where $\delta = b^2 - 4ac$ is the discriminant of the equation(1.8). This shows that the discriminant of (1.2) has the same sign as the discriminant of the transformed equation (1.8) and therfore it is clear that any real nonsingular ($J \neq 0$) transformation does not change the type of PDE. Note that the discriminant involoves only the coefficients of second-order derivatives of the corresponding PDE.

1.2 Canonical forms

Let use now try to contruct transformation, which will make one, or possibly two of the coefficients of the leading second-order terms of equation(1.8) vanish, thus reducing the equation to asimpler form called canonical form. For convenience, we reproduce below the original PDE

$$A(x;y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = \Phi(x,y,u,u_x,u_y)\,, \tag{1.11}$$

and the corresponding transformrd PDE

$$a(\xi,\eta)\,\omega_{\xi\xi} + b(\xi,\eta)\,\omega_{\eta\eta} + c(\xi,\eta)\,\omega_{\eta\eta} = \phi(\xi,\eta,\omega,\omega_\zeta,\omega_\eta)\,. \tag{1.12}$$

We again mention here that for the(1.2) or(1.8) to remain a second-order PDE, the coefficients A, B and C or a, b and c do not vanish simultaneously. By definition, a PDE is hyperbolic if the discriminant $\Delta = b^2 - 4ac \succ 0$. Since the sign of dicriminant is invariant under the change of coordinater see equatio(1.10), it follows that for a hyperboulic PDE, we should have $b^2 - 4ac \succ 0$. The simplest case of satisfying this condition is $a = c = 0$. So, if we try to chose the new variables ξ and η such that the cofficients a and c vanish, we get the following canonical form of hyperbolic euqation :

$$\omega_{\xi\eta} = \psi(\xi,\eta,\omega,\omega_\xi,\omega_\eta)\,. \tag{1.13}$$

Where $\psi = \frac{\phi}{b}$, this form is called the first canonical form of the hyperbolic eqution, we also have another simple case for which $b^2 - 4ac \succ 0$ condition is satified. This the case when $b = 0$ and $c = -a$. Is this case(1.10) reduces to

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi(\alpha,\beta,\omega,\omega_\alpha,\omega_\beta)\,, \tag{1.14}$$

which is the second canonical form of the hyperbolicequatoin. By definition, a PDE is parabolic if the discriminant $\Delta = b^2 - 4ac = 0$. It follows that for a parabolic PDE, we should have $b^2 - 4ac = 0$. Thesimplest case of satisfying this condition is a (or c) $= 0$. In this case another necessary requirement $b = 0$ will follow automatically (sinec $b^2 - 4ac$).

So if we try to chose the new variables ξ and η such that the coefficients a and b vanish, we get the following canonical form of parabolic eqution :

$$\omega_{\eta\eta} = \psi\left(\xi, \eta, \omega, \omega_{\zeta}, \omega_{\eta}\right), \tag{1.15}$$

where $\psi = \frac{\phi}{c}$.

By definition, a PDE is elliptic if the discriminant $\Delta = b^2 - 4ac \prec 0$. It follows that for a elliptic PDE, we should have $b^2 - 4ac \prec 0$. The simplesr case of satisfying this condition is $b = 0$ and $c = a$. Soif we tey to chose the new variables ξ and η such that b vanishes and $c = a$, we get the following canonical form of elliptic eqution:

$$\omega_{\xi\xi} + \omega_{\eta\eta} = \psi\left(\xi, \eta, \omega, \omega_{\xi}, \omega_{\eta}\right), \tag{1.16}$$

where $\psi = \frac{\phi}{a}$.

In summary eqution(1.8) can be reduced to a canonical form if the coordinate fransformation $\xi = \xi\left(x, y\right)$ and $\eta = \eta\left(x, y\right)$ can be selected such that:

- $a = c = 0$ corresponds to the first canonical form of hyperbolic PDE given by (1.15)
- $b = 0, c = -a$ corresponds to the second canonical foem of hyperbolic PDE given by (1.14)
- $a = b = 0$ corresponds to the canonical form of parabolic PDE given by (1.15)
- $b = 0, c = a$ corresponds to the canonical form of elliptic PDE given by (1.16)

1.2.1 Hyperbolic equtions

For a hyperbolic PDE the discriminant $(\Delta = b^2 - 4ac) \succ 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variablrs ξ and η such that the coefficients a and c vanish in(1.8). Thus form (1.9) we have

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0, \tag{1.17}$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0. \tag{1.18}$$

Divising eqution(1.7) and(1.8) throughout by ξ_y^2 and η_y^2 respectively to obtain

$$A\left(\frac{\xi_x}{\xi_y}\right)^2+B\left(\frac{\xi_x}{\xi_x}\right)+C=0, \tag{1.19}$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2+B\left(\frac{\eta_x}{\eta_y}\right)+C=0. \tag{1.20}$$

Equation (1.10) is a quadratic equation for $\left(\frac{\xi_x}{\xi_y}\right)$ whose roots are given by

$$\mu_1(x,y)=\frac{-B-\sqrt{B^2-4AC}}{2A},$$
$$\mu_2(x,y)=\frac{-B+\sqrt{B^2-4AC}}{2A}.$$

The roots of the equation (1.20)can also be found in an identical manner, so as only two distinct roots are possible between the two equation (1.19) and (1.20). Here we may consider μ_1 as the root of (1.19) and μ_2 as that of (1.20) That is :

$$\mu_1(x,y)=\frac{\xi_x}{\xi_y}=\frac{-B-\sqrt{B^2-4AC}}{2A}, \tag{1.21}$$

$$\mu_2(x,y)=\frac{\eta_x}{\eta_y}=\frac{-B-\sqrt{B^2-4AC}}{2A}. \tag{1.22}$$

The above equations lead to the following two first-order diferential aquation

$$\xi_x-\mu_1(x,y)\,\xi_Y=0, \tag{1.23}$$

$$\eta_x-\mu_2(x,y)\,\eta_y=0. \tag{1.24}$$

These are the equations that define the new coordinate variables ξ and η that are necessary to make $a=c=0$ in(1.8).

As the total derivative of ξ along the coordinate $\xi(x,y)=$ constant, $d\xi=0$. It follows that

$$d\xi=\xi_x dx=\xi_y dy=0,$$

and hence the slope of such curves is given by

$$\frac{dy}{dx}=-\frac{\xi_x}{\xi_y}.$$

We also have a similar result along coordinate line $\eta(x,y)=$ constant, i.e,

$$\frac{dy}{dx}=-\frac{\eta_x}{\eta_y}.$$

Using these results equation (1.19)can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.25}$$

This is called the characteristic polynomial of the PDE (1.2) and its roots are given by

$$\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A} = \lambda_1(x,y), \tag{1.26}$$

$$\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A} = \lambda_2(x,y). \tag{1.27}$$

The required variables ξ and η are determined by the respective solutions of the two ordinary differential equations (1.26) and (1.27), known as the characteristic equation of the PDE (1.2). They are ordinary differential equation for families of curves in the xy-plane along wich ξ = constant and η =constant. Clearly these families of curves depend on the coefficients A, B and C in the original PDE (1.2).

Integration of equation(1.26) leads to the family of curvilinear coordinates $\xi(x,y) = c_1$ while the integration of (1.27) gives another family of curvilinear coordinates $\eta(x,y) = c_2$, where c_1 and c_2 are arbitrary constants of integration. These two families of curvilinear

coordinates $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are called characteristic curves of the hyperbolic equation (1.3) or simply the characteristics of the equation. Hence second-order hyperbolic eqution have twe families of characteristic curves. The fact that $\Delta \succ 0$ means that the characteristic are real curves in xy-plane.

If the coefficients A,B and C are constants it is easy to integrate euations (1.26)and (1.27) to obtain the expressions for change of variables formulas for reducing a hyperbolic PDE to the canonical from. Thus integration of (1.26) produces

$$y = \frac{B+\sqrt{B^2-4AC}}{2A}x + c_1 \text{ and } y = \frac{B-\sqrt{B^2-4AC}}{2A}x + c_2. \tag{1.28}$$

Or

$$y = \frac{B+\sqrt{B^2-4AC}}{2A}x = c_1 \text{ and } y = \frac{B-\sqrt{B^2-4AC}}{2A}x = c_2. \tag{1.29}$$

Thus when the coefficients A, B and C are two constants the two families of characteristic curves associated with PDE reduces to two distinct families of parallel straight lines.

Since the families of curves ξ =constant and η =constant are the characteristic curves, the change of variables are given by the following equations :

$$\xi = y - \frac{B - \sqrt{B^2 - 4AC}}{2A}x = y - \lambda_1 x, \tag{1.30}$$

$$\eta = y - \frac{B + \sqrt{B^2 - 4AC}}{2A}x = y - \lambda_2 x. \tag{1.31}$$

The first canonical form of the hyperbolic is :

$$\omega_{\xi\eta} = \psi\left(\xi, \eta, \omega, \omega_\xi, \omega_\eta\right), \tag{1.32}$$

where $\psi = \frac{\phi}{b}$ and b is calculated from (1.9)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_x + \xi_y\eta_y\right) + 2C\xi_y\eta_y, \\ &= 2A\left(\frac{B^2 - \left(B^2 - 4AC\right)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C, \\ &= 4C - \frac{B^2}{A} = \frac{\Delta}{A}. \end{aligned} \tag{1.33}$$

Each of the familiies $\xi\left(x, y\right) =$ constant and $\eta\left(x, y\right) =$ constant forms an envelop of the domain of the xy-plane in which th PDE is hyperbolic.

The transformation $\xi = \xi\left(x, y\right)$ and $\eta = \eta\left(x, y\right)$ can be regarded as a mapping from the xy-plane to the $\xi\eta{-}plane$. And the curves along which ξ and η are constant in the xy-plane brcome coordinates lines in the $\xi\eta - plane$. Sinec these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other worde, charactreristic curves of a hyperbolic PDE are those curves for the PDE. Must be referred as coordinate curves in order that it take on canonical form. We now determine the Jacobian of transformation definded by (1.30)and (1.31). We have

$$J = \begin{vmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 1 \end{vmatrix} = \lambda_2 - \lambda_1.$$

We know that $\lambda_1 = \lambda_2$ only if $B^2 - 4AC = 0$. However, for an hyperbolic PDE, $B^2 - 4AC \neq 0$. Hence Jacobian is nonsingular for the given transformation. A consequence of $\lambda_1 \neq \lambda_2$ is that at no point can the particular curves from each family share a common trangent line. It is easy to show that the hyperbolic PDE has a second canonical form.

The following linear change of variables

$$\alpha = \xi + \eta \qquad\qquad \beta = \xi - \eta,$$

converts (1.32) in to

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi\left(\alpha, \beta, \omega, \omega_{\alpha}, \omega_{\beta}\right), \tag{1.34}$$

which is the seconde canonical form of the hyperbolic equtions

1.2.2 Parabolic equations

For a parabolic PDE the discriminant $\Delta = B^2 - 4AC = 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ξ and η such that the coefficients a and b vanish in (1.8). Thus, from (1.9) we have

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2.$$

Dividing the above eqution throughhout by ξ_y^2 to obtain

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C + 0. \tag{1.35}$$

As the total derivative of ξ along the coordinate line $\xi\left(x, y\right)$ =constant, $d\xi = 0$. It follows that

$$d\xi = \xi_x dx + \xi_y dy = 0,$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

Using this result, equation (1.35) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.36}$$

This is called the characteristic polynomial of the PDE (1.2). Since $B^2 - 4AC = 0$ in this case the characteristi (1.35) has only root, given by

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda\left(x, y\right). \tag{1.37}$$

Hence we see that for a parabolic PDE there is only one family of real characteristic curves.

The required varibles ξ is determined by the ordinary differential equation (1.37), known as the characteristic equations of thr PDF (1.2). This is an ordinary differential equation for families of curves in the xy-plane along which $\xi = const$. To determine the second transformation variablr η, we set $b = 0$ in (1.9) so that

$$\begin{aligned} 2A\xi_x\eta_x + B\left(\xi_x\eta_x + \xi_y\eta_y\right) + 2C\xi_y\eta_y &= 0, \\ 2A\frac{\xi_x}{\xi_y}\eta_x + B\left(\frac{\xi_x}{\xi_y}\eta_y + \eta_x\right) + 2C\eta_y &= 0, \\ 2A\left(\frac{-B}{2A}\right)\eta_x + B\left[\left(\frac{-B}{2A}\right)\eta_y + \eta_x\right] + 2C\eta_y &= 0, \\ -B\eta_x - \frac{B^2}{2A}\eta_y + B\eta_x + 2C\eta_y &= 0, \\ \left(B^2 - 4AC\right)\eta_y &= 0. \end{aligned}$$

Since $B^2 - 4AC = 0$ for a parabolic PDF, η_y could be an arbitrary function of (x, y) and consequently the transformation variable η can be chosen arbitrary, as long as the change of coordinates formulas define a non-degenerate transfoemation. lf the corfficients A, B and C are constants, it is easy to integrate equation (1.37) to obtain the expressions for change of variables formulas for reducing a parabolic PDE to the canonical foem. thus integration of (1.37) produces

$$y = \frac{B}{2A}x + c_1, \tag{1.38}$$

or

$$y - \frac{B}{2A}x = c_1, \tag{1.39}$$

sincr the families of curves $\zeta =$ const are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{B}{2A}x, \tag{1.40}$$

$$\eta = x, \tag{1.41}$$

where we have set $\eta = x$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -\frac{B}{2A} & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$$

Naw, we have from(1.9)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y, \\ &= 2A\left(-\frac{B}{2A}\right) + B + 0 = 0. \end{aligned}$$

In these new coordinate variables given by(1.39) and (1.41), equation(1.8) reduces to following canonical form:

$$\omega_{\eta\eta} = \psi\left(\xi, \eta, \omega, \omega_\xi, \omega_\eta\right), \tag{1.42}$$

where$\psi = \frac{\varphi}{\xi}$. As the choice of η is arbitrary the form taken by ψ will depend on the choice of η. We have from (1.9)

$$c = A\eta_x^2 + B\eta_\chi\eta_\gamma + C\eta_\gamma^2 = A. \tag{1.43}$$

Equation (1.8) may also assume the form

$$\omega_{\xi\xi} = \psi(\xi, \eta, \omega_\xi, \omega_\eta), \tag{1.44}$$

if we choose $c = 0$ instead of $a = 0$.

1.2.3 Elliptic equations

For an elliptic PDE the discriminant $\Delta = B^2 - 4AC \prec 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ξ and η to produce $b = 0$ and $a = 0$, or $b = 0$ and $a - c = 0$.

Then, from (1.9) we abtain the following equation :

$$A(\xi_\chi^2 - \eta_\chi^2) + B(\xi_\chi\xi_\gamma - \eta_\chi\eta_\gamma) + C(\xi_\gamma^2 6 - \eta_\gamma^2) = 0, \tag{1.45}$$

$$2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = 0. \tag{1.46}$$

For hyperbolic and parabolic PDE_S, ξ and η are satisfied by equations that are not coupled each other(see (1.17) and (1.35)). However, equation (1.46) are coupled since both unknowns ξ and η appear in both equations. In an attempt to separate them, we add the first of these equation to complex number$i$ times the second to give

$$A\left(\xi_x + i\eta_x^2\right) + B\left(\xi_x + i\eta_y\right)\left(\xi_x + i\eta_y\right) + C\left(\xi_y + i\eta_y\right)^2 = 0.$$

Dividing the above equation throughout by $(\xi_y + i\eta_y)^2$to obtain

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi^x + i\eta_y}{\xi^x + i\eta_y}\right) + C = 0. \tag{1.47}$$

This equation can be solved for two possible values of the ratio

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{-B \pm \sqrt{B - 4AC}}{2A} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{1.48}$$

Clearly, these twe roots are complex conjugates and are given by

$$\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}, \tag{1.49}$$

$$\frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A}. \tag{1.50}$$

Where $\beta(x, y)$ is the complex conjugate of $\alpha(x, y)$. They are given by:

$$\alpha(x, y) = \xi(x, y) - i\eta(x, y), \tag{1.51}$$

$$\beta(x, y) = \xi(x, y) - i\eta(x, y). \tag{1.52}$$

We will now proceed in a purely formal fashion. As the total derivative of α along the coordinate line $\alpha(x, y)$ =constant, $d\alpha = 0$, it follows that

$$d\alpha = \alpha_x dx + \alpha_y dy = 0,$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = \frac{\alpha_x}{\alpha_y},$$

we also have a similar result along coordinate line $\beta(x, y)$ =constant, i.e,

$$\frac{dy}{dx} = -\frac{\beta_x}{\beta_y}.$$

From the forgoing discussion it follows that :

$$\frac{dy}{dx} = \lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A}, \tag{1.53}$$

$$\frac{dy}{dx} = \lambda_2 = \frac{B + i\sqrt{4AC - B^2}}{2A}. \tag{1.54}$$

Equations (1.53) and (1.54) are called the characteristic equation of the PDE (1.3).

Clearly, the solution of this differential eqution are necessarily complex-valued and as consequence there are no real characteristic axist for an elliptic EDP. The complex variables α and β are determined by the respective solution of the two ordinary differential equations (1.53) and (1.54). Integration of equation (1.53) leads to the family of curvilinear coordinates $\alpha(x, y) = c_1$, where the integration of (1.54) gives another family of curvilinear coordinates $\beta(x, y) = c_1$ where c_1 and c_2 are complex constants of integration. Since α and β are complex function the characteristic curves of the elliptic equation (1.3) are not real. Now the real and imaginary parts of α and β give the required trans formation variables ξ and η. Thus, we have

$$\xi = \frac{\alpha + \beta}{2}, \qquad \eta = \frac{\alpha - \beta}{2}. \tag{1.55}$$

With the choice of coordinate variables(1.55),equation (1.8) reduces to following canonical form.

$$\omega_{\xi\xi} + \omega_{\eta\eta} = \psi(\xi, \eta, \omega, \omega_\xi, \omega_\eta), \tag{1.56}$$

where $\psi = \frac{\phi}{\alpha}$.

NOTE: It may be noted that the quasilinear second-order equation in two independent variables can also be classified in a similar way according to rule analogous to those developed above for semilinear equations. However, since A, B, and C are now functions of u_x, u_y and u its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.

CHAPTER 2

GLOBAL EXISTENCE AND STABILITY OF SOLUTION FOR A P-KIRCHHOFF TYPE HYPERBOLIC EQUIATION WITH DAMPING AND SOURCE TERMS

In this chapter, we consider the following value problem

$$\begin{cases} u_{tt} - M\left(\int_\Omega \mid \nabla u \mid^p dx\right) \Delta pu + \mid u_t \mid^{m-2} u_t = \mid u \mid^{r-2} u, (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$ and

$$M(s) = a + bs,$$

with positive parameters $a, b,\ \Delta_P u = \operatorname{div}\left(\mid \nabla u \mid^{P-2} \nabla u\right), p \geq 2$.

We bigin this section with some notations and definitions. Denote by $\parallel . \parallel_p$, the $Lp(\Omega)$ norme of a Lebesgue function $u \in L^p(\Omega)$ for $P \geq 1$. We use $W_0^{1,p}(\Omega)$ to denote the Well-Known Sobolev space such that both u and $\mid \nabla u \mid$are in $W_0^{1,p}(\Omega)$ equipped with the norm $\parallel u \parallel_{W_0^{1,P}(\Omega)} = \parallel \nabla u \parallel_p$.

Lemme 2.1 *Let s be a number with $2 \leq s \leq +\infty$ if $n \leq p$ and $2 \leq s \leq \frac{pn}{n-p}$ if $n > p$.*

Then there is a constant c_* depanding on Ω and s such that

$$\parallel u \parallel_s \leq c_* \parallel \nabla u \parallel_p, \forall u \in W_0^{1,p}(\Omega).$$

Théorème 2.1 *Suppose that* $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ *and*

$$2p < r \leq p^*,$$

where

$$p^* = \begin{cases} \frac{pn}{n-p}, \text{ if } n > p, \\ +\infty \text{ if } n \leq p \end{cases}$$

Then problem (2.1) *has a unique weak solution such that*

$$u \in L^\infty\left((0,T), W_0^{1,p}(\Omega)\right),$$
$$u_t \in L^\infty\left((0,T),\ L^2(\Omega)\right) \cap L^m(\Omega \times (0,T)),$$
$$u_{tt} \in L^2\left((0,T),\ W^{-1,\,p'}(\Omega)\right).$$

2.1 Global existance

In this section, we state and prove our result, we define the potential energy functional and the Nehari's functional, by the following

$$E(t) = E(u(t)) = \frac{1}{2} \parallel u_t(t) \parallel_2^2 + \frac{a}{p} \parallel \nabla u(t) \parallel_p^p + \frac{b}{2p} \parallel \nabla u(t) \parallel_p^{2p} - \frac{1}{r} \parallel u(t) \parallel \quad (2.2)$$

$$J(t) = J(u(t)) = \frac{a}{p} \parallel \nabla u(t) \parallel_p^p + \frac{b}{2p} \parallel \nabla u(t) \parallel_p^{2p} - \frac{1}{r} \parallel u(t) \parallel_r^r. \quad (2.3)$$

$$I(t) = I(u(t)) = a \parallel \nabla u(t) \parallel_p^p + b \parallel \nabla u(t) \parallel_p^{2p} - \parallel u(t) \parallel_r^r \quad (2.4)$$

We can considiring $a = b = 1$, and this does not change the general result of (2.1).

Lemme 2.2 *Under the assumptiobs of theorem 2.1, we have*

$$E'(t) = - \parallel u_t(t) \parallel_m^m \leq 0, \quad t \in [0,T]. \quad (2.5)$$

and

$$E(t) \leq E(0).$$

Preuve. We miltiply the first equation of(2.1) by u_t and integrating over the domain Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2} \parallel u_t(t) \parallel_2^2 + \frac{1}{p}\int_\Omega \mid \nabla u \mid^p dx + \frac{1}{2p}\left(\int_\Omega \mid \nabla u \mid^p dx\right)^2\right) - \frac{1}{r} \parallel u(t) \parallel_r^r = - \parallel u_t(t) \parallel_m^m,$$

then

$$E'(t) = - \parallel u_t(t) \parallel_m^m \leq 0.$$

Integratin(2.5)g over $(0,t)$,we obtain $E(t) \leq E(0)$. ■

Lemme 2.3 *Assume that the assumption of theorem 2.1 hold,*

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1, \tag{2.6}$$

where

$$\beta_1 := \alpha c_*^r \left(\frac{pr}{r-p} E(0)\right)^{\frac{r-p}{p}}, \beta_2 := (1-\alpha) c_*^r \left(\frac{2pr}{r-2p} E(0)\right)^{\frac{r-2p}{2p}}$$

with $0 < \alpha < 1$, c_* *is the best embedding constant of* $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, *then* $I(t) > 0$, *for all* $t \in [0,T]$.

Preuve. *B*y cntinuity, there exists T_*, such that

$$I(t) \geq 0, \quad for\ all\ t \in [0,T_*]. \tag{2.7}$$

Now, we have for all $t \in [0,T_*]$:

$$\begin{aligned} J(t) = J(u(t)) &= \frac{1}{p} \parallel \nabla u(t) \parallel_p^p + \frac{1}{2p} \parallel \nabla u(t) \parallel_p^{2p} - \frac{1}{r} \parallel ut \parallel_r^r \\ &\geq \frac{1}{p} \parallel \nabla u(t) \parallel_p^p + \frac{1}{2p} \parallel \nabla u(t) \parallel_p^{2p} - \frac{1}{r} \left(\parallel \nabla u(t) \parallel_p^p + \parallel \nabla u(t) \parallel_p^{2p} - I(t) \right) \\ &\geq \frac{r-p}{pr} \parallel \nabla u(t) \parallel_p^p + \frac{r-2p}{2pr} \parallel \nabla u(t) \parallel_p^{2p} + \frac{1}{r} I(t), \end{aligned}$$

using (2.7), we obtian

$$\frac{r-p}{pr} \parallel \nabla u(t) \parallel_p^p + \frac{r-2p}{2pr} \parallel \nabla u(t) \parallel_p^{2p} \leq J(t), \text{ for all } t \in [0,T_*]. \tag{2.8}$$

By the definition of E, we get

$$\parallel \nabla u(t) \parallel_p^p \leq \frac{pr}{r-p} E(t) \leq \frac{pr}{r-p} E(0) \tag{2.9}$$

and

$$\parallel \nabla u(t) \parallel_p^{2p} \leq \frac{2pr}{r-2p} E(t) \leq \frac{2pr}{r-2p} E(0). \tag{2.10}$$

On the other hand, we have

$$\| u(t) \|_r^r = \alpha \| u(t) \|_r^r + (1-\alpha) \| u(t) \|_r^r .$$

By the embedding of $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, we obtain

$$\begin{aligned} \| u(t) \|_r^r &\leq \alpha c_*^r \| \nabla u(t) \|_p^r + (1-\alpha) c_*^r \| \nabla u(t) \|_p^r \\ &\leq \alpha c_*^r \| \nabla u(t) \|_p^{r-p} \times \| \nabla u(t) \|_p^p + (1-\alpha) c_*^r \| \nabla u(t) \|_p^{r-2p} \times \| \nabla u(t) \|_p^{2p} \end{aligned}$$

By(2.9) and(2.10), we get

$$\| u(t) \|_r^r \leq \beta_1 \| \nabla u(t) \|_p^p + \beta_2 \| \nabla u(t) \|_p^{2p}, \text{ for all } t \in [0, T_*] . \tag{2.11}$$

Since $\beta_1 + \beta_2 < 1$, then

$$\| u(t) \|_r^r < \| \nabla u(t) \|_p^p + \| \nabla u(t) \|_p^{2p}, \text{ for all } t \in [0, T_*] .$$

This implies that

$$I(t) > 0, \ for\ all\ t \in [0, T_*] .$$

By repeating the above procedure, we can extend T_* to T. ■

Théorème 2.2 *Under the assumptions of lemma2.3, the local solition of* (2.1) *is global.*

Preuve. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2} \| u_t(t) \|_2^2 + \frac{1}{p} \| \nabla u(t) \|_p^p + \frac{1}{2p} \| \nabla u(t) \|_p^{2p} - \frac{1}{r} \| u(t) \|_r^r \\ &\geq \frac{1}{2} \| u_t(t) \|_2^2 + \frac{r-p}{pr} \| \nabla u(t) \|_p^p + \frac{r-2p}{2pr} \| \nabla u(t) \|_p^{2p} . \end{aligned}$$

So that

$$\| u_t(t) \|_2^2 + \| \nabla u(t) \|_p^p \leq C\ E\ (t) . \tag{2.12}$$

By Lemma 2.2, we obtain

$$\| u_t(t) \|_2^2 + \| \nabla u(t) \|_p^p \leq C\ E\ (0) . \tag{2.13}$$

This implies that the local solution is global in time. ■

2.2 Stability of solution

In this section our main result is established based in Komornik's integral in- equality [**7**].

For this, we need the following Lemma:

Lemme 2.4 *Suppose that the assumptions of Lemma2.3 and* $m > p$*, hold, then ther exists a positive constant* c *such that*

$$\int_\Omega | u \ (t) |^m \, dx \leq c \ E(t) . \tag{2.14}$$

Preuve. By using (2.9) we obtain

$$\begin{aligned}\int_\Omega | u \ (t) |^m \, dx &= \| u \ (t) \|_m^m \leq c_*^m \ \| \nabla u(t) \|_p^m \\ &\leq c_*^m \ \| \nabla u(t) \|_p^{m-p} \times \| \nabla u(t) \|_p^p \\ &\leq c_*^m \ \| \nabla u(t) \|_p^{m-p} \times \frac{rp}{r-p} E(t) \leq c \ E(t) .\end{aligned}$$

■

Now, we state our main result :

Théorème 2.3 *Let the assumptions of Lemma2.3, then, there exists constants* $C, \zeta > 0$*, such that*

$$\begin{aligned}&E(t) \leq \frac{C}{(1+t)^{\frac{2}{m-2}}}, for \ all \ t \geq 0 \ if m > 2. \\ &E(t) \leq Ce^{-\zeta t}, for \ all \ t \geq 0 \ if m = 2.\end{aligned}$$

Preuve. Multiplying first equation of(2.1) by $u(t) \ E^q(t) \, (q > 0)$, and integrating over $\Omega \times (S, T)$, we obtain

$$\int_S^T \int_\Omega E^q(t) \left[u(t) \, u_{tt}(t) - u(t) \left(M \left(\int_\Omega | \nabla u |^p \, dx \right) \Delta_p u + | u_t |^{m-2} \, u_t \right) \right] dxdt$$
$$= \int_S^T E^q(t) \int_\Omega | u(t) |^r \, dxdt$$

So that

$$\int_S^T \int_\Omega E^q(t)\left[(u(t)\,u_t(t))_t - |\, u_t(t)\,|^2 + |\,\nabla u(t)\,|^p + \|\,\nabla u(t)\,\|_p^p |\,\nabla u(t)\,|^p + u(t)\,|\,u_t\,|^{m-2}\,u_t\right] dxdt = \int_S^T E^q(t)\int_\Omega |\,u(t)\,|^r\,dxdt$$

We add and subtract the term

$$\int_S^T E^q(t)\int_\Omega \left[\beta_1 \,|\,\nabla u(t)\,|^p + \beta_2 \,\|\,\nabla u(t)\,\|_p^p + (2+\beta_1+\beta_2)\,|\,u_t(t)\,|^2\right] dxdt$$

and use(2.11), to get

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega \left[|\,\nabla u(t)\,|^p + |\,u_t(t)\,|^2\right] dxdt \\
&(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left[\|\,\nabla u(t)\,\|_p^p |\,\nabla u(t)\,|^p + |\,u_t(t)\,|^2\right] dxdt \\
&+\int_S^T E^q(t)\int_\Omega \left[(u(t)\,u_t(t))_t - (3-\beta_1-\beta_2)\,|\,u_t(t)\,|^2\right] dxdt \\
&+\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|\,u_t(t)\,|^{m-2}\,dxdt \\
&= -\int_S^T E^q(t)\int_\Omega \left[\beta_1\,|\,\nabla u(t)\,|^p + \beta_2\,\|\,\nabla u(t)\,\|_p^p |\,\nabla u(t)\,|^p - |\,u(t)\,|^r\right] dxdt \le 0
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\gamma\int_S^T E^q(t)\int_\Omega \left[\frac{1}{p}\,|\,\nabla u(t)\,|^p + \frac{1}{2p}\,\|\,\nabla u(t)\,\|_p^p |\,\nabla u(t)\,|^p + \frac{|\,u_t(t)\,|^2}{2} - \frac{|\,u(t)\,|^r}{r}\right] dxdt \\
&\le (1-\beta_1)\int_S^T E^q(t)\int_\Omega \left[\frac{1}{p}\,|\,\nabla u(t)\,|^p + \frac{|\,u_t(t)\,|^2}{2}\right] dxdt \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left[\frac{1}{2p}\,\|\,\nabla u(t)\,\|_p^p |\,\nabla u(t)\,|^p + \frac{|\,u_t(t)\,|^2}{2}\right] dxdt \qquad (2.15)
\end{aligned}$$

(2.16)

2.2. Stability of solution

where $\gamma = Min\left(\left(1-\beta_1\right),\left(1-\beta_2\right)\right)$. By (2.15), (2.16)and definition of $E\left(t\right)$, we get

$$\gamma\int_S^T E^{q+1}\left(t\right)dt \leq -\int_S^T E^q\left(t\right)\int_\Omega \left(u\left(t\right)u_t\left(t\right)\right)_t dxdt$$

$$+\left(3-\beta_1-\beta_2\right)\int_S^T E^q\left(t\right)\int_\Omega \mid u_t\left(t\right)\mid^2 dxdt$$

$$-\int_S^T E^q\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)\mid u_t\left(t\right)\mid^{m-2} dxdt. \tag{2.17}$$

Using the definition of $E\left(t\right)$ and the following expression

$$\frac{d}{dt}\left(E^q\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)dx\right) = qE^{q-1}\left(t\right)\frac{d}{dt}E\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)dx$$

$$+\ E^q\left(t\right)\int_\Omega \left(u\left(t\right)u_t\left(t\right)\right)_t dx.$$

Inequality(2.17), becomes

$$\gamma\int_S^T E^{q+1}\left(t\right)dt \leq q\int_S^T E^{q-1}\left(t\right)\frac{d}{dt}E\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)dx - \int_S^T \frac{d}{dt}\left(E^q\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)dx\right)dt \tag{2.18}$$

$$-\int_S^T E^q\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)\mid u_t\left(t\right)\mid^{m-2} dxdt$$

$$+\left(3-\beta_1-\beta_2\right)\int_S^T E^q\left(t\right)\int_\Omega \mid u\left(t\right)\mid^2 dxdt. \tag{2.19}$$

In the sequel, we denote by c the various constants.

We estimate the terms in the right-hand side of (2.19) as follow:

By (2.5)and Young's inequality, we obtain

$$q\int_S^T E^{q-1}\left(t\right)\frac{d}{dt}E\left(t\right)\int_\Omega u\left(t\right)u_t\left(t\right)dx$$

$$\leq q\int_S^T E^{q-1}\left(t\right)\left(-E'\left(t\right)\right)\int_\Omega \left[\frac{1}{p}\mid u\left(t\right)\mid^p + \frac{p-1}{p}\mid u_t\left(t\right)\mid^{\frac{p}{p-1}}\right]dxdt \tag{2.20}$$

Since,$1 \leq \frac{p}{p-1} < 2$, by the embedding of $L^2(2) \hookrightarrow L_{\frac{p}{p-1}}(\Omega)$, we have

$$q \int_S^T E^{q-1}(t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) \, dx$$
$$\leq q \int_S^T E^{q-1}(t) \left(-E'(t)\right) \int_\Omega \left[\frac{1}{p} \mid u(t) \mid^p + c\frac{p-1}{p} \mid u_t(t) \mid^2\right] dxdt$$

Thus, by (2.12), we find

$$\begin{aligned} &q \int_S^T E^{q-1}(t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) \, dx \\ &\leq c \int_S^T E^q(t) \left(-E'(t)\right) dt \\ &\leq cE^{q+1}(S) - cE^{q+1}(T) \\ &\leq cE^q(0) E(S) \leq cE(S). \end{aligned} \tag{2.21}$$

for the second term, we have

$$\begin{aligned} &-\int_s^T \frac{d}{dt} \left(E^q(t) \int_\Omega u(t) u_t(t) \, dx \right) dxdt \\ &\leq \mid E^q(t) \int_\Omega u(S) u_t(S) \, dx - E^q(t) \int_\Omega u(T) u_t(T) \, dx \mid \\ &\leq E^q(t) \left| \int_\Omega u(x,S) u_t(x,S) \, dx \right| + E^q(t) \mid \int_\Omega u(x,T) u_t(x,T) \, dx \mid \\ &\leq cE^{q+1}(S) + cE^{q+1}(T) \\ &\leq cE^q(0) E(S) \leq cE(S). \end{aligned} \tag{2.22}$$

$$\tag{2.23}$$

For the third term, we use the following Young inequality:

$$XY \leq \frac{\epsilon}{\lambda_1} X^{\lambda 1} + \frac{1}{\lambda_2 \epsilon^{\frac{\lambda_2}{\lambda_1}}} Y^{\lambda_2}, X, Y \geq 0, \epsilon > 0$$

and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$, with $\lambda_1 = m, \lambda_2 = \frac{m}{m-1}$.

By (2.5) and Lemma 2.4, we have

$$
\begin{aligned}
-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\mid u_t(t)\mid^{m-2}dxdt
&\le \int_S^T E^q(t)\left(\epsilon c\int_\Omega \mid u_t(t)\mid^m dx + c_\epsilon\int_\Omega \mid u_t(t)\mid^m dx\right)dt\\
&\le \epsilon c\int_S^T E^q(t)\int_\Omega \mid u_t(t)\mid^m dxdt + c_\varepsilon\int_S^T E^q(t)\left(-E'(t)\right)dt\\
&\le \epsilon c\int_S^T E^{q+1}(t)\,dt + c_\epsilon E(S)
\end{aligned}
\tag{2.24}
$$

For the last term of (2.19), we have

$$
\begin{aligned}
(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega \mid u(t)\mid^2 dxdt &\le c\int_S^T E^q(t)\left(\int_\Omega \mid u(t)\mid^2 dx\right)^{\frac{2}{m}}dt\\
&\le c\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m}}dt.
\end{aligned}
\tag{2.25}
$$

By Young's inequality with $\lambda_1 = (q+1)/q$ and $\lambda_2 = q+1$, we have

$$
\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m}}dt \le \epsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon\int_S^T \left(-E'(t)\right)^{\frac{2(q+1)}{m}}dt.
$$

We take $q = \frac{m}{2} - 1$,to find

$$
\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m}}dt \le \epsilon c\int_S^T E^{q+1}(t)\;\,dt + c_\varepsilon\int_S^T \left(-E'(t)\right)dt
$$

This implies

$$
\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m}}dt \le \epsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon\;E(S)\,. \tag{2.26}
$$

Substituting (2.26) into(2.25) ,we obtain

$$
(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega \mid u(t)\mid^2 dxdt \le \epsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon\;E(S)\,. \tag{2.27}
$$

By insert (2.21), (2.23), (2.24) and (2.27) in (2.19), we arrive at

$$\gamma \int_S^T E^{\frac{m}{2}}(t)\,dt \leq \epsilon c \int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon\ E(S).$$

Choosing ϵ small enough for that

$$\int_S^T E^{\frac{m}{2}}(t)\,dt \leq c\ E(S).$$

By taking T goes to ∞, we get

$$\int_S^T E^{\frac{m}{2}}(t)\,dt \leq c\ E(S).$$

By Komrnik's integral inequality yields the result. ■

2.2. Stability of solution

CHAPTER 3

GLOBAL EXISTENCE AND STABILITY OF SOLUTION FOR A P-KIRCHHOFF TYPE HYPERBOLIC EQUATION WITHE VARIABLE EXPONENTS

We consider the following boundary value problem :

$$\begin{cases} u_{tt} - M\left(\int\limits_{\Omega} |\nabla u\ |^{p}\, dx\right)\Delta_p u + \mid u_t \mid^{m(x)-2} u_t = \mid u\ \mid^{r(x)\ -\ 2} u, (x,t) \in \Omega \times (0\ ,T)\,, \\ u\,(x,t) = 0, (x,t) \in \partial\Omega \times (0,T)\,, \\ u\,(x,0) = u_0\,(x)\,, u_t\,(x\ ,0) = u_1\ (x)\ \ x \in \Omega\ , \end{cases} \tag{3.1}$$

whereΩ is a bounded domain in $\mathbb{R}^n$, $n \geq\ 1$ with smooth boundary $\partial\Omega$ and $M\ (s) = a + bs$ with positive parameters $a, b, \triangle_p u\ \ = \text{div}\ \left(|\nabla u|^{p-2}\,\nabla u\right)$, with $p \geq 2$. $r\,(.)$ and $m\,(.)$ are given measurable functions on Ω.

3.1 Premilinaires:

We bgin this section with some notations and definitions. denote by $\|.\|_p$, the$L^p\,(\Omega)$ norm of a lebesgue function $u \in L^P\,(\Omega)$. We use $W_0^{1,p}\,(\Omega)$ to the well-known sobolev space such that u and $|\nabla u|$ are in $L^p\,(\Omega)$ equipped with the norm $\|u\|_{W_0^{1,p}}\,(\Omega) = \|\nabla u\|_p$.

Let $q : \Omega \to [1, +\infty]$ be a mesurable function, where Ω is adomain of $\mathbb{R}^n$. We define the lebesque space withe a variale exponent $q(.)$ by :

$$L = \{v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega, l_{q(.)}(\lambda v) < +\infty, for \text{ some } \lambda > 0\},$$

where $l_{q(.)}(v) = \int\limits_{\Omega} |v \ (x)|^{q(x)} \, dx.$

The set $L^{1,q}(\Omega)$ equipped withe the norm (Luxemburg's norm)

$$\| v\|_{q(.)}(\Omega) := \inf_{x \in \Omega} \left\{ \lambda \geq 0 : \int\limits_{\Omega} \left| \frac{v(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\},$$

$L^{q(.)}(\Omega)$ is a banach space [8].

Next, we define the variable-exponent Sobolev space $W^{1,q(.)}(\Omega)$ as follows :

$$W^{1,q(.)}(\Omega) \ : \ = \{ \ v \in L^{q(.)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{q(.)}(\Omega)$$

this is a Banach space withe respect to the norm

$$\|v\|_{w^{1,q(.)}(\Omega)} = \|v\|_{q(.)} + \|\nabla v\|_{q(.)}.$$

Furthemore, we set $W^{1,q(.)}(\Omega)$ to be the closure of $C_0^{\infty}(\Omega)$ in the space$W^{1,q(.)}(\Omega)$. Let use note that the space$W^{1,q(.)}(\Omega)$ has a differenet definition in the case of variable exponents.

However, under the log-Hölder continuity condition, both definitions areeqivalent [**8**]. The space $W^{-1,q(.)}(\Omega)$, dual of $W_0^{1,\acute{q}(.)}$ (Ω), is defined in the same way as the classical sobolev spaces, where $\frac{1}{q(.)} + \frac{1}{\acute{q}(.)} = 1$.

Lemme 3.1 *If*

$$1 \ \leq \ q \ : \ = \ ess \ \inf_{x \in \Omega} q \ (x) \ \leq \ q \ (x) \leq \ \ q_2 : \ = ess \ \sup_{x \in \Omega} q \ (x) < \infty \ ,$$

then we have

$$min \ \left\{ \|u\|^{q_1}_{q \ (.)} \ , \|u\|^{q_2}_{q \ (.)} \right\} \leq l_{q \ (.)}(u) \leq \max \left\{ \|u\|^{q_1}_{q \ (.)}, \|u\|^{q_2}_{q \ (.)} \right\} \ ,$$

for any $u \in L^{q \ (.)} \ (\Omega)$

Lemme 3.2 *(Hölder's Ineqality) Supposse that* $p, \ q, \ s \ \geq 1$ *are measurable functions defined on* Ω *such that*

$$\frac{1}{s \ (y)} = \frac{1}{p \ (y)} + \frac{1}{q \ (y)} \ , \ for \ a \ , e \ . \ y \in \Omega$$

for the existence of the local soulution of problem (3.1)*, we refer the reader to* ***[13]****. Their result is given in the following theorem :*

Théorème 3.1 *suppose that* $r, m \in C\left(\overline{\Omega}\right)$ *with*

$$\begin{aligned} &2 \leq \quad r_1 \leq r\ \ (x) \leq\ r_2 <\ 2\frac{n-1}{n-2}\ \textit{if}\ n \geq 3 \\ &r\,(x)\ \ \geq\ \ 2\ , \qquad\qquad\qquad\qquad \textit{if}\ \ n = 1, 2\ , \end{aligned}$$

and

$$\begin{aligned} &2 \leq\ \ m_1 \leq\ \ m\ \ (x)\ \ \leq\ \ m_2 < \frac{2n}{n-2} \qquad\qquad \textit{if}\ n\ \geq 3 \\ &m\ \ (x)\ \ \geq\ \ 2\ , \qquad\qquad\qquad\qquad\qquad \textit{if}\ n = 1, 2\ , \\ &r_1\ :\ \ = ess \underset{x\,\in\,\Omega}{inf}\ r\ \ (x)\ \ ,\ r_2 :\ \ = ess \underset{x\,\in\,\Omega}{\sup}\ r\ \ (x)\ \ , \\ &m_1 :\ \ = ess \underset{x\,\in\,\Omega}{inf}\ m\ \ (x)\ \ ,\ m_2 :\ \ = ess \underset{x\,\in\,\Omega}{\sup}\, m\ \ (x)\ \ . \end{aligned}$$

We also assume that $m\,(.)$ *and* $r\,(.)$*satisfy the log-Hölder continuitycondition :*

$$|q\,(x) - q\,(y)| \leq\ \ -\frac{A}{\log|x-y|}\ \ ,\ \ \ for\ a\ .e\ .x\ ,y \in \Omega\ ,withe\ \ |\ x-y|\ \ < \delta\ , \tag{3.2}$$

$A > 0,\ 0 < \delta < 1.$

Then, for any $(u_0,\ u_1) \in\ W_0^{1,p}(\Omega) \times\ L^2(\Omega)$*, problem*(3.1) *has a unique weak local solution*

$$\begin{aligned} &u \in\ \ L^{\infty}\left((0,T\ ), W_0^{1,p}(\Omega)\right)\ \ , \\ &u_t \in\ \ \ L^{\infty}\left((0,T\ ), L^2(\Omega)\right) \cap L^{m\ (.)}\left(\Omega \times (0,T\ )\right)\ \ , \\ &u_{tt} \in\ \ \ L^2\left((0,T\ ), W_0^{-1,p}(\Omega)\right)\ \ , \end{aligned}$$

3.2 Global existance

In the order to state and prouve our result, we define the potential energy functional and the Nehari's functional, respectively by the following

$$E(t) = E(u(t)) = \frac{1}{2}\left\|u_t\,(t)\right\|_2^2 + \frac{a}{p}\left\|\nabla u\,(t)\right\|_p^p + \frac{b}{2p}\left\|\nabla u\,(t)\right\|_p^{2p} - \int_{\Omega} \frac{1}{r\,(x)}\left|u\ \ (t)\right|^{r(x)} dx \tag{3.3}$$

$$I(t) = I(u(t)) = a\left\|\nabla u(t)\right\|_p^p + b\left\|\nabla u\,(t\ )\right\|_p^{2p} - \int_{\Omega} \left|u\,(t)\right|^{r(x)} dx. \tag{3.4}$$

We can considering $a = b = 1$, and this does not change the general result.

Lemme 3.3 *Under the assumption of theorem 3.1, we have*

$$\acute{E}\ (t\ ) = -\int |u_t\ \ (t\ )|^{m(x)}\, dx \leq 0\ , \qquad\qquad t \in\ \ [0, T]\ \ . \tag{3.5}$$

and

$$E\ (t\ )\ \ \leq\ \ \ E\ (0\ )$$

Preuve. We multiply the first equation of (3.1) by u_t and integrating over the domain Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\parallel u_t\|_2^2 + \frac{1}{P}\int_\Omega |\ \nabla u\ \ (t\ )|^p\, dx + \frac{1}{2p}\left(\int_\Omega |\ \nabla u\ \ (t\ )|^p\, dx\right)^2 - \int_\Omega \frac{1}{r\ \ (x)}\, |u\ |^{r(x)}\, dx\right)$$
$$= \ -\int_\Omega |\ u_t\ \ (t\ )\ |^{m(x)}\ dx\ ,$$

then

$$\grave{E}(t) = -\int_\Omega |u_t\,(t)|^{m(x)}\, dx \leq 0$$

Intergratying (3.5) over $(0, t)$, we obtain

$$E(t\ )\ \ \leq\ \ \ E\ (0\ )\ .$$

■

Lemme 3.4 *Assume that the assumptions of thorem 3.1 and $r_1 > 2p$, hold,*

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1, \tag{3.6}$$

where

$$\beta_1 := max\left\{\alpha c_*^{r_1}\left(\frac{pr_1}{r_1 - p}E\ \ (0)\right)^{\frac{r_1 - p}{p}}\ ,\ \alpha c_*^{r_2}\left(\frac{pr_1}{r_1 - p}E\,(0)\right)^{\frac{r_2 - p}{p}}\right\}\ ,$$
$$\beta_2 := max\left\{(1-\alpha)\, c_*^{r_1}\left(\frac{2pr_1}{r_{1-2p}}E\ \ (0)\right)^{\frac{r_1 - 2p}{2p}}, (1-\alpha)\, c_*^{r_2}\left(\frac{2pr_1}{r_{1-2p}}E\ \ (0)\right)^{\frac{r_2 - 2p}{2p}}\right\}\ ,$$

withe $0 <\ \ \alpha\ \ < 1$, c_* *is the best embedding constant of* $W_0^{1,p}(\Omega) \to L^{r\,(\cdot)}(\Omega)$*, then* $I\ \ (t\ ) > 0$*, for all* $t \in [0, T]$.

Preuve. By continuity, there existsT_*, such that

$$I\ (t) \geq \ \ 0, \qquad for\ all\ t \in [\ 0, T]\ . \tag{3.7}$$

Now, we have for all $t \in [\ 0, T]$:

$$\begin{aligned} J\ (t)\ &= J\ (u\ (t\ )) = \frac{1}{p} \|\nabla u\ \ (t\ )\|_p^p + \frac{1}{2p} \parallel \nabla u\ \ (t\ ) \parallel_p^{2p} - \int_\Omega \frac{1}{r\ (x)} \mid u\ \ (t\ ) \mid^{r\ (x)} dx \\ &\geq \frac{1}{p} \parallel \nabla u\ \ (t\ ) \parallel_p^p + \frac{1}{2p} \parallel \nabla u\ \ (t\ ) \parallel_p^{2p} - \frac{1}{r_1} \left(\parallel \nabla u\ \ (t\ ) \parallel_p^p + \parallel \nabla u\ \ (t\ ) \parallel_p^{2p} - I\ \ (t\ ) \right) \\ &\geq \frac{r_1 - p}{pr_1} \parallel \nabla u\ \ (t\ ) \parallel_p^p + \frac{r_1 - 2p}{pr_2} \parallel \nabla u\ \ (t\ ) \parallel_p^{2p} + \frac{1}{r_1} I\ \ (t\ ) \end{aligned}$$

using(3.7), we obtain

$$\frac{r_1 - p}{pr_1} \parallel \nabla u\ \ (t\ ) \parallel_p^p + \frac{r_1 - 2p}{pr_2} \|\nabla u\ \ (t\ )\|_p^{2p} \leq J\ \ (t\ )\ \ , \qquad for\ all\ \ t \in \ \ [0, T_*]\ . \tag{3.8}$$

By Lemma 3.3, we get

$$\parallel \nabla u\ \ (t\ )\|_p^p \leq \frac{pr_1}{r_1 - p} E\ \ (t\ ) \leq \frac{pr_1}{r_1 - p} E\ \ (0\ ) \tag{3.9}$$

$$\parallel \nabla u\ \ (t\ )\|_p^{2p} \leq \frac{2pr_1}{r_1 - 2p} E\ \ (t\ ) \leq \frac{2pr_1}{r_1 - 2p} E\ \ (0\ ). \tag{3.10}$$

On the other hand, by lemma3.1, we have

$$\begin{aligned} \int_\Omega |u\ \ (t\ )|^{r\ (x)} dx\ &\leq\ Max \left\{ \|u\ \ (t\ )\|_{r\ (.)}^{r_1}\ , \|u\ \ (t\ )\|_{r\ (.)}^{r_2} \right\} \\ &= \alpha\ Max \left\{ \|u\ \ (t\ )\|_{r(.)}^{r_1}\ , \|u\ \ (t\ )\|_{r(.)}^{r_2} \right\} \\ &\quad + (1 - \alpha)\ \ Max \left\{ \|u\ \ (t\ )\|_{r\ (.)}^{r_1}\ , \|u\ \ (t\ )\|_{r\ (.)}^{r_2} \right\}\ . \end{aligned}$$

By the embedding of $W_0^{1,P}\ \ (\Omega) \to L^{r\ (.)}(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega |u\ \ (t\ )|^{r(x)} dx &\leq \alpha\ Max \left\{ c_*^{r_1} \|\nabla u\ \ (t\ )\|_p^{r_1}, c_*^{r_2} \|\nabla u\ \ (t\ )\|_p^{r_2} \right\} \\ &\quad + (1 - \alpha)\ \ Max\ \left\{ c_*^{r_1} \|\nabla u\ \ (t\ )\|_p^{r_1}, c_*^{r_2} \|\nabla u\ \ (t\ )\|_p^{r_2} \right\} \\ &\leq \alpha\ Max\ \left\{ c_*^{r_1} \|\nabla u\ \ (t\ )\|_p^{r_1 - p}, c_*^{r_2} \|\nabla u\ \ (t\ )\|_p^{r_2 - p} \right\} \times \|\nabla u\ \ (t\ )\|_p^p \\ &\quad + (1 - \alpha)\ \ Max\ \left\{ c_*^{r_1} \|\nabla u\ \ (t\ )\|_p^{r_1 - 2p}, c_*^{r_2} \|\nabla u\ \ (t\ )\|_p^{r_2 - 2p} \right\} \times \|\nabla u\ \ (t\ )\|_p^{2p} \end{aligned}$$

By (3.9) and (3.10), we get

$$\int_{\Omega} | u \ (t\ ) |^{r(x)} dx \leq \beta_1 \parallel \nabla u \ (t\ ) \parallel_p^p + \beta_2 \parallel \nabla u \ (t\ ) \parallel_p^{2p}, \quad for\ all\ t \in \ [0, T_*] \ . \tag{3.11}$$

Since $\beta_1 + \beta_2 < 1, then$

$$\int_{\Omega} |u \ (t)| \, dx < \|\nabla u \ (t)\|_p^p + \|\nabla u \ (t)\|_p^{2p} \ , \qquad for\ all\ t \in [0, T_*] \ .$$

This implies that

$$I \ (t\ ) > 0 \qquad for\ all\ t \in [0, T_*]$$

By repeating the above procedure, we can extend T_* to T ■

Théorème 3.2 *Under the assumptions of lemma3.4, thelocal solution of* (3.1) *is global .*

Preuve. We have

$$\begin{aligned} E\ (u\ (t\ )) &= \frac{1}{2} \|u_t \ (t\ )\|_2^2 + \frac{1}{p} \|\nabla u \ (t\ )\|_p^p + \frac{1}{2p} \|\nabla u \ (t\ )\|_p^{2p} - \int_{\Omega} \frac{1}{r\,(x)} |u \ (t\ )|^{r\ (x)} dx \ . \\ &\geq \frac{1}{2} \|u_t \ (t\ )\|_2^2 + \frac{r_1 - p}{pr_1} \|\nabla u \ (t\ )\|_p^p + \frac{r_1 - 2p}{2pr_1} \|\nabla u \ (t\ )\|_p^{2p} . \end{aligned}$$

So that

$$\|u_t \ (t\ )\|_2^2 + \|\nabla u \ (t\ )\|_p^p \leq C \ E\ (t\ ) \ . \tag{3.12}$$

By lemma3.3, we obtain

$$\| \ u_t \ (t\ )\|_2^2 + \|\nabla u \ (t\ )\|_p^p \leq C \ E\ (0\ ) \ . \tag{3.13}$$

This implies that the local soulition is global in time. ■

3.3 Stability solution

In this section our main result is based a komrnik's inequality [7], as in [**5**].

For this, we need the following lemma :

Lemme 3.5 *suppose that the assumptions of Lemma 3.4 and* $m_1 \ > p$*, hold, then there exists a positive constant* c *such that*

$$\int_{\Omega} |u \ (t\ )|^{m(x)} dx \leq cE \ (t\ ) \ . \tag{3.14}$$

Preuve. We have

$$\int_{\Omega} |u\ (t\ )|^{m\ (x)}\, dx = Max\left\{\|u\ (t\ )\|_{m\ (.)}^{m_1}\ ,\|u\ (t\ )\|_{m\ (.)}^{m_2}\right\}$$
$$\leq Max\ \left\{c_*^{m_1}\|\nabla u\ (t\ )\|_p^{m_1}, c_*^{m_2}\|\nabla u\ (t\ )\|_p^{m2}\right\}$$
$$\leq Max\ \left\{c_*^{m_1}\|\nabla u\ (t\ )\|_p^{m_1-p}, c_*^{m_2}\|\nabla u\ (t\ )\|_p^{m2-p}\right\}\times\|\nabla u\ (t\ )\|_p^p$$

By using (3.9), we obtain

$$\int_{\Omega} |u\ (t\ )|^{m(x)}\, dx \leq cE\ (t\ )\ .$$

Now, we state our main result : ■

Théorème 3.3 *Let the assumptions of Lemma3.4, then, ther exists constants* $C,\zeta>0$, *such that*

$$E\ (t) \leq \frac{C}{(1+t)^{\frac{2}{m_2-2}}}\ ,\quad for\ \ all\ \ t\geq\ 0\ if\ \ m_2>2$$
$$E\ (t)\leq Ce^{-\zeta t}\ , for\ all\quad t\ \geq\ 0\ if\ \ m_2=2$$

Preuve. Multiplying first equation of by(3.1) $u\ (t\ )\, E^q\,(t\ )\ \ (q<0)$ and integrating over $\Omega\times(\ S\ ,T)$, we obtain

$$\int_S^T\int_{\Omega} E^q\,(t\ )\left[u\ (t\ )\,u_{tt}\,(t\ )-u\ (t\ )\left(M\ \left(\int_{\Omega}|\nabla u\ |^x\,dx\right)\triangle_p u+|u_t\ |^{m(x)-2}\,u_t\right)\right]dxdt$$
$$=\int_S^T E^q\,(t)\int_{\Omega}|u_t|^{r\ (x)}\,dxdt$$

So that

$$\int_S^T\int_{\Omega} E^q\,(t)\left[(u\,(t)\,u_t\ \ (t))_t-|u_t\,(t)|^2+|\nabla u\,(t)|^p+\|\nabla u\,(t)\|_p^p\,\|\nabla u\,(t)\|^p+u\,(t)\,|u_t|^{m(x)-2}\,u_t\right]dxdt$$
$$=\int_S^T E^q\,(t)\int_{\Omega}|u_t|^{r(x)}\,dxdt$$

We add and substract the term

$$\int_S^T E^q\,(t)\int_{\Omega}\left[\beta_1\,|\nabla u\ \ (t\ )|^p+\beta_2\,\|\nabla u\ \ (t\ )\|_P^p\,|\nabla u\ \ (t\ )|^p+(2+\beta_1+\beta_2)\,|u\ \ (t\ )|^2\right]dxdt$$

and use(3.11), to get

$$
\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega \left[|\nabla u(t)|^p+|u_t(t)|^2\right]dxdt\\
&\quad+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left[\|\nabla u(t)\|_p^p|\nabla u(t)|^p+|u_t(t)|^2\right]dxdt\\
&\quad+\int_S^T E^q(t)\int_\Omega \left[(u(t)u_t(t))_t-(3-\beta_1-\beta_2)|u_t(t)|^2\right]dxdt\\
&\quad+\int_S^T E^q(t)\int_\Omega (u(t)u_t(t))\,|u_t(t)|^{m(x)-2}\,dxdt\\
&\quad=-\int_S^T E^q(t)\int_\Omega \left[\beta_1|\nabla u(t)|^p+\beta_2\|\nabla u(t)\|\,|\nabla u(t)|^p-|u(t)|^{r(x)}\right]dxdt\le 0
\end{aligned}
\tag{3.15}
$$

It is clear that

$$
\begin{aligned}
&\gamma\int_S^T E^q(t)\int_\Omega \left[\frac{1}{p}|\nabla u(t)|^p+\frac{1}{2p}\|\nabla u(t)\|_P^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}-\frac{|u(t)|^{r(x)}}{r(x)}\right]dxdt\\
&\quad\le(1-\beta_1)\int_S^T E^q(t)\int_\Omega \left[\frac{1}{p}|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxd\\
&\quad+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left[\frac{1}{2p}\|\nabla u(t)\|_P^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt
\end{aligned}
\tag{3.16}
$$

where $\gamma = Min\left((1-\beta_1),(1-\beta_2)\right)$. By (3.15),(3.16) and definition of $E(t)$, we get

$$
\begin{aligned}
\gamma\int_S^T E^{q+1}(t)\,d(t)\le&-\int_S^T E^q(t)\int_\Omega (u(t)\,u_t(t))_t\,dxdt\\
&+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt\\
&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t|^{m(x)-2}\,dxdt
\end{aligned}
\tag{3.17}
$$

3.3. Stability solution

Using the definition of$E\ (t\ )$ *and* the following expression

$$\frac{d}{dt}\left(E^q\ (t\ )\int_\Omega u\ (t\ )\ u_t\ (t\ )\,dx\right) = q\ E^{q-1}\ (t\ )\frac{d}{dt}E\ (t\ )\int_\Omega u\ (t\ )\ u_t\ (t\ )\,dx + E^q\ (t\ )\int_\Omega (u\ (t\ )\ u_t\ (t\ ))_t\,dx.$$

Inequality(3.17), becomes

$$\begin{aligned}\gamma\int_S^T E^{q+1}\ (t\ )\ d\ (t\ ) \le q &\int_S^T E^{q-1}(t\ )\frac{d}{dt}E\ (t\ )\int_\Omega u\ (t\ )\ u_t\ (t\ )\,dx\\ &-\int_S^T \frac{d}{dt}\left(E^q\ (t\ )\int_\Omega u\ (t\ )\ u_t\ (t\ )\ dx\right)dt\\ &-\int_S^T E^q\ (t\ )\int_\Omega u\ (t\ )\ u_t\ (t\ )\ |u_t\ (t\ )|^{m(x)-2}\,dxdt\\ &+(3-\beta_1-\beta_2)\int_S^T E^q\ (t\ )\int_\Omega |u_t\ (t\ )|^2\,dxdt \end{aligned} \tag{3.18}$$

We denote by c the various constants.

We estimate the terms in the right-hand side of (3.18) as follow:

By (3.5) and young's inequality, we obtain

$$\begin{aligned}&q\int_S^T E^{q-1}(t\ )\frac{d}{dt}E(t\ )\int_\Omega u(t\ )\ u_t(t\ )\,dx\\ &\le\ q\int_S^T E^{q-1}(t\ )\left(-\acute{E}\ (t\ )\right)\int_\Omega\left[\frac{1}{p}|u\ (t\ )|^p+\frac{p-1}{p}|u_t\ (t\ )|^{\frac{p}{p\ -1}}\right]dxdt\end{aligned} \tag{3.19}$$

Since, $1\le\frac{p}{p\ -1}<2$, by the embedding of $L^2\ (\Omega)\to L^{\frac{p}{p-1}}\ (\Omega)$, we have

$$\begin{aligned}&q\int_S^T E^{q\ -\ 1}\ (t\ )\frac{d}{dt}E\ (t\ )\int_\Omega u\ (t\ )\ u_t\ (t)\,dx\\ &\le\ q\int_S^T E^{q-1}\ (t\ )\left(-\acute{E}\ (t\ )\right)\int_\Omega\left[\frac{1}{p}|u\ (t\ )|^p+c\frac{p-1}{p}|u_t\ (t\ )|^2\right]dxdt\end{aligned}$$

Thus, by(3.12), we find

$$
\begin{aligned}
q\int_S^T E^{q-1}(t)\,\frac{d}{dt}E(t)\int_\Omega u(t)\ u_t(t)\,dx \\
&\le\ c\int_S^T E^q\ (t)\left(-\acute{E}\ (t)\right)dt \\
&\le cE^{q+1}\ (S) -\ cE^{q+1}\ (T) \\
&\le\ cE^q(0)\,E(S) \le cE(S). \qquad (3.20)
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&-\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega u\ (t)\ u_t\ (t)\ dx\right)dxdt \\
&\le \left|E^q\ (t)\int_\Omega u\ (S)\ u_t\ (S)\,dx -\ E^q\ (t)\int_\Omega u\ (T)\ u_t\ (T)\,dx\right| \\
&\le E^q\ (t)\left|\int_\Omega u\ (x\ ,S)\ u_t\ (x\ ,S)\,dx\right| +\ E^q(t)\left|\int_\Omega u\ (x\ ,T)\ u_t(x,T)\,dx\right| \\
&\le cE^{q+1}(S) + cE^{q+1}(T) \\
&\le cE^q(0)\,E(S) \le cE(S). \qquad (3.21)
\end{aligned}
$$

For the third term, we use the following Young inequality :

$$
XY \le \frac{\varepsilon}{\lambda_1}X^{\lambda_1} + \frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2}, X, Y \ge 0\ ,\ \varepsilon > 0\ and\ \ \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1\ ,
$$

withe $\lambda_1(x) = m\ (x)$, $\lambda_2(x) = \frac{m\ (x)}{m\ (x)\ -1}$.

By (3.5) and Lemma3.5, we have

$$-\int_S^T E^q(t)\int_\Omega u(t)\ u_t(t)\,|u_t\ (t)|^{m(x)-2}\ dxdt$$
$$\leq \int_S^T E^q\ (t)\left(\varepsilon c\int_\Omega |u\ (t)|^{m(x)}\,dx + c_\varepsilon\int_\Omega |u_t\ (t)|^{m(x)}\,dx\right)dt$$
$$\leq \varepsilon c\int_S^T E^q\ (t)\int_\Omega |u\ (t)|^{m(x)}\,dxdt\ + c_\varepsilon\int_S^T E^q\ (t)\left(-\acute{E}\ (t)\right)dt$$
$$\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{3.22}$$

For the laste term of(3.18), we have

$$(3-\beta_1-\beta_2)\int_S^T E^q\ (t)\int_\Omega |u_t\ (t)|^2\,dxdt$$
$$\leq (3-\beta_1-\beta_2)\int_S^T E^q\ (t)\left[\int_{\Omega-}|u_t(t)|^2\,dx + \int_{\Omega+}|u_t(t)|^2\,dx\right]dt$$
$$\leq c\int_S^T E^q\ (t)\left[\left(\int_{\Omega-}|u_t\ (t)|^{m_2}\,dx\right)^{\frac{2}{m_2}} + \left(\int_{\Omega+}|u_t\ (t)|^{m_1}\,dx\right)^{\frac{2}{m_1}}\right]dt$$
$$\leq c\int_S^T E^q\ (t)\left[\left(\int_\Omega |u_t\ (t)|^{m(x)}\,dx\right)^{\frac{2}{m_2}} + \left(\int_\Omega |u_t\ (t)|^{m(x)}\,dx\right)^{\frac{2}{m_1}}\right]dt\ .$$

This implies

$$(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt$$
$$\leq c\int_S^T E^q\ (t)\left(-\acute{E}\ (t)\right)^{\frac{2}{m_2}}dt + c\int_S^T E^q\ (t)\left(-\acute{E}\ (t)\right)^{\frac{2}{m_1}}dt\ . \tag{3.23}$$

First, if we use Young's inequality withe $\lambda_1\ = (q+1)\diagup q$ and $\lambda_2\ = q+1$, we have

$$\int_S^T E^q\ (t)\left(-\acute{E}\ (t)\right)^{\frac{2}{m_2}}dt\ \leq \varepsilon c\int_S^T E^{q+1}\ (t)\,dt\ \ + c_\varepsilon\int_S^T\left(-\acute{E}\ (t)\right)^{\frac{2(q+1)}{m_2}}dt\ .$$

We take $q = \frac{m_2}{2} - 1$ to find

$$\int_S^T E^q(t)\left(-\acute{E}(t)\right)^{\frac{2}{m_2}} dt \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon \int_S^T \left(-\acute{E}(t)\right) dt .$$

This implies

$$\int_S^T E^q(t)\left(-\acute{E}(t)\right)^{\frac{2}{m_2}} dt \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) . \tag{3.24}$$

On the other hand, we have

$$\int_S^T E^q(t)\left(-\acute{E}(t)\right)^{\frac{2}{m_1}} dt \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) . \tag{3.25}$$

Indeed, ·if $m_1 = 2$ then

$$\int_S^T E^q(t)\left(-\acute{E}(t)\right)^{\frac{2}{m_1}} dt \le cE(S) \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) .$$

·if $m_1 > 2$ then,we use the Young's inequality $\lambda_1 = \frac{m_1}{m-2}$ and $\lambda_2 = \frac{m_1}{2}$, to obtain

$$\begin{aligned}\int_S^T E^q(t)\left(-\acute{E}(t)\right)^{\frac{2}{m_1}} dt &\le \varepsilon c \int_S^T E^{q\frac{m_1}{m_1-2}}(t)\,dt + c_\varepsilon \int_S^T \left(-\acute{E}(t)\right) dt \\ &\le \varepsilon c \int_S^T E^{q\frac{m_1}{m_1-2}}(t)\,dt + c_\varepsilon E(S) .\end{aligned}$$

We notice that $q\frac{m_1}{m_1-2} = q + 1 + \frac{m_1-m_2}{m_1-2}$, then

$$\begin{aligned}\int_S^T E^q(t)\left(-\acute{E}(t)\right)^{\frac{2}{m_1}} dt &\le \varepsilon c\,(E(S))^{\frac{m_1-m_2}{m_1-2}} \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) \\ &\le \varepsilon c\,(E(0))^{\frac{m_1-m_2}{m_1-2}} \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) \\ &\le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) .\end{aligned}$$

3.3. Stability solution

We substituting (3.24) and(3.25) in(3.23), we obtain

$$(3-\beta_1-\beta_2)\int_S^T E^q\ (t\ )\int_\Omega |u_t\ (t\ )|^2\, dxdt \quad \leq \varepsilon c\int_S^T\ E^{q+1}\ (t\ )\, dt + c_\varepsilon E\ (S\ )\ . \tag{3.26}$$

By insert (3.20), (3.21), (3.22) and (3.23) in (3.18), we arrive at

$$\gamma\int_S^T E^{\frac{m_2}{2}}\ (t\ )\, dt\ \leq\ \varepsilon c\int_S^T\ E^{\frac{m_2}{2}}\ (t\ )\, dt\ + c_\varepsilon E\ (S\ )\ .$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m_2}{2}}\ (t\ )\, dt\ \leq cE\ (S\ )\ .$$

By taking T goes to ∞, we get

$$\int_S^T E^{\frac{m_2}{2}}\ (t\ )\, dt\ \leq cE\ (S\ )\ .$$

By Komornik's integral inequality yields the result. ■

CONCLUSION

In this work, we notice that the last result related to the behavior of the soulution resulted from the fact that the initial energy of the problem is positive. The question is what happens if the initial energy is negative and this requires future ressearch.

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