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جامعة 20 أوت 1955 -سكيكدة
كلية العـلوم
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Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: ***Applied functional analysis***

On the existence of nontrivial solutions of a system of nonlinear partial differential equations

Presented by:

Esma Abada

Publicly discussed:

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In front of the Jury:

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Université 20 Août 1955 -Skikda
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Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option: ***Analyse fonctionnelle appliquée***

Sur l'existence de solutions non-triviales d'un système des équations aux dérivées partielles non linéaires

Présentée par :

Esma Abada

Soutenue publiquement :

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Devant les Jury:

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Année Universitaire: 2022/2023

Dedication

In the name of **ALLAH**, the most gracious, the most merciful.
I dedicate this Thesis to Someone Who took me by the hand when I was little and Who Guided me on the path to happiness, She is The best mother in the world and the most Important and First Person in my life: **My darling Salima**, This thesis would not have been accomplished without her support.
To My Father and my teacher **El Hasseine**, for their constant support, encouragement and assistance, Without him, I would not have completed my path in mathematics.
To My Dear sisters Fatima and Takoua, and all my family. I will never be able to testify enough to them all my love and gratitude for always supporting me.
To all my friends and colleagues, especially Nadjla, Chaima and Faiza, who are not just friends or colleagues, but as my sisters.
To all those who are present in my heart but absent in my phrases.

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On the existence of nontrivial solutions of a system of nonlinear partial differential equations

Abstract

The purpose of this work is to study some nonlinear fractional differential problems with boundary conditions in a bounded domain in order to generalize some results from the classical differential problems to the fractional case, which has a diversity of applications in various fields, manly physics, engineering, mathematical biology, signal processing, and image processing. In this study, under some suitable conditions on the nonlinearities, we apply the Leray-Schauder degree and the Schauder fixed point theorem to establish the existence of solutions; also, the Banach principle of contraction and the absurd reasoning are applied to prove the uniqueness of solutions. The first problem is a coupled semilinear fractional Laplacian system in a fractional Sobolev space, and the second problem is a semilinear equation involving the distributional Riesz fractional gradient in a Bessel potential space. The third problem is a nonlocal nonlinear equation related to the distributional Riesz fractional derivative in a Bessel-potential space. Finally, the fourth problem is a nonlinear problem involving left and right Riemann-Liouville fractional derivatives in a new fractional space of Sobolev type, the study of this problem is illustrated with an example to affirm the validity of methods used.

Keywords: Leray-Schauder degree, fixed point theorem, fractional laplacian, distributional Riesz fractional derivative, weak solution, nonlinear problem, fractional Sobolev spaces, Sobolev space with left Riemann-Liouville fractional derivative .

Sur l'existence de solutions non-triviales d'un système des équations aux dérivées partielles non linéaires

Résumé

Le but de ce travail est d'étudier quelques problèmes différentiels fractionnaires non-linéaires avec des conditions aux limites dans un domaine borné afin de généraliser certains résultats des problèmes différentiels classiques au cas fractionnaire qui a une diversité d'applications dans divers domaines tels que la physique, l'ingénierie, la mathématique biologie, traitement du signal et d'image, etc. Dans cette étude, nous appliquons sous certaines conditions appropriées sur les non-linéarités le degré de Leray-Schauder et le théorème du point fixe de Schauder pour obtenir l'existence de solutions ; aussi le principe de contraction de Banach et le raisonnement par absurde sont appliqués pour prouver l'unicité des solutions. Le premier problème est un système de Laplacien fractionnaire semi-linéaire dans un espace de Sobolev fractionnaire, et le second problème est une équation semi-linéaire avec le gradient fractionnaire distributionnel de Riesz dans un espace de potentiel de Bessel. Le troisième problème est une équation non-linéaire non locale liée à la dérivée fractionnaire distributionnelle de Riesz dans un espace de potentiel de Bessel. Enfin, le quatrième problème est un problème non-linéaire contient des dérivées fractionnaires gauche et droite de Riemann-Liouville dans un nouvel espace fractionnaire de type Sobolev, l'étude de ce problème est illustrée par un exemple pour affirmer la validité des méthodes utilisées.

Mots-clefs : Degré de Leray-Schauder, théorème du point fixe, Laplacien fractionnaire, dérivée fractionnaire de Riesz distributionnelle, solution faible, problème non linéaire, espaces de Sobolev fractionnaires, espace de Sobolev avec dérivée fractionnaire de Riemann-Liouville à gauche.

البحث على وجود حلول غير تافهة لجملة معادلات تفاضلية جزئية غير خطية

ملخص

الغرض من هذا العمل هو دراسة بعض المسائل التفاضلية الجزئية غير الخطية ذات الشروط الحدودية في مجال محدود من أجل تعميم بعض نتائج المسائل التفاضلية الكلاسيكية الى الحالة الكسرية التي لها تطبيقات متنوعة في مجالات مختلفة ، مثل الفيزياء والهندسة، علم الأحياء الرياضياتي ، معالجة الإشارات والصور ، إلخ.

نطبق في هذه الدراسة تحت شروط معينة تفرض على العناصر الغير خطية درجة ليراي شودر و نظرية النقطة الثابتة لـ شودر لبرهان وجود الحلول ؛ كما يتم تطبيق مبدأ تقلص باناخ والاستدلال بالخلف لإثبات وحدانية الحلول. المشكلة الأولى هي جملة لابلاسيان الكسري شبه الخطي في فضاء سوبولاف الكسري، المشكلة الثانية هي معادلة نصف خطية متعلقة بمشتقة ريس التوزيعية الكسرية في فضاء بيسال بالقوة. المشكلة الثالثة هي معادلة غير خطية محلية تتعلق بمشتق كسوري توزيعي لريس في فضاء بيسال للقوة. أخيرًا ، المشكلة الرابعة هي مشكلة غير خطية تحتوي على مشتقات ريمان-ليوفيل الكسرية اليسرى واليمنى في فضاء كسري جديد من نوع سوبوليف ، وقد تم توضيح دراسة هذه المشكلة بمثال لتأكيد صحة الطرق المستخدمة.

الكلمات المفتاحية درجة ليراي شودر، نظرية النقطة الثابتة، لابلاسيان الكسري، مشتق ريس الكسري التوزيعي، حل ضعيف ، مسألة غير خطية، فضاءات سوبولاف الكسرية، فضاءات سوبولاف مع مشتق ريمان ليوفيل الكسري الأيسر.

THE RELEVANT PUBLICATIONS WITH THE THESIS

1. Abada E, Lakhal H, Maouni M. Topological degree method for fractional Laplacian system. *Bulletin of Mathematical Analysis and Applications.* 2021; 13(2): 10-19.

2. Abada E, Lakhal H, Maouni M. Existence and uniqueness of solution for a nonlinearfractional problem involving the distributional Riesz derivative. *Mathematical Methods in the Applied Sciences.* 2022; 45(10): 6181-6193.

3. Abada E, Lakhal H, Saadi C. Application of Shauder fixed point theorem to a nonlocal fractional elliptic problem with nonlinearities. *Submitted for publication in international journal.*

4. Lakhal H, Abada E, Saadi C. Study of a nonlinear equation involving Left-right Riemann-Liouville fractional derivative. *Submitted for publication in international journal.*

NOTATION

$\rightarrow$ The strong convergence.

$\rightharpoonup$ The weak convergence.

$\hookrightarrow$ The continuous embedding.

Δ^s The fractional Laplace operator of order s.

D^s The distributional Riesz fractional gradient of order s.

$\langle\cdot,\cdot\rangle$ The scalar product.

$meas(\Omega)$ Measure of Ω.

$u \in AC[a,b]$ u fonction absolument continue sur $[a,b]$.

$W^{m,p}(\Omega)$ is the Sobolev space.

$p.v.$ is the abbriviation for "in the principal value sense".

$a.e.$ is the abbriviation for " almost everywhere".

Γ is the usual Gamma function.

$div^s v = D^s \cdot D^s v$ is the fractional divergence of v.

B_R is the open ball with center 0 and radius R.

$H^1[a,b] = W^{1,2}[a,b]$ is the Sobolev space.

CONTENTS

INTRODUCTION

In the last few years, partial differential equations have made large strides in research. This progress is mainly due to an eminent factor, which is the diversity of applications for partial differential equations in various fields, mainly physics and engineering. The development of this type of equation was not only in the classical case but also in the fractional case.

Recently, fractional calculus has become one of the most important mathematical fields that attracts the attention of researchers. This discipline investigates the properties of fractional derivatives and fractional spaces, and it is a generalization of integral and differential calculus. The history of fractional calculus started in 1695 with an idea of a derivative of any arbitary order that was mentioned in a letter written by Leibniz to Guillaume de l'Hôpital. Fractional calculus is constructed gradually by many mathematicians, for example Abel, Liouville, Heaviside Grünewald, and Riemann. For more information about the history of this field; see previous work [49, 61, 56]. One of the primary reasons why fractional calculus dominates various topics in mathematics papers is its applications in various fields, such as: mechanics, engineering, physics, biology, signal and image processing, economics, dynamic systems, and other sciences; see for example, [38, 35, 72, 70, 75, 30].

Among the various applied mathematical research subjects, fractional equations involving fractional derivatives have taken an important and valuable place due to their many applications in various scientific fields such as mechanics, physics, image processing, electrochemistry, mathematical biology, and viscoelasticity, etc. For example, see [17, 38, 50, 52, 69, 71, 75, 44, 51] and the references therein, as well as the variety of definitions of fractional derivatives provided by researchers in this field, such as: the Hilfer derivative, the Caputo derivative, the Marchaud derivative, the Katugampola derivative, the AtanganaBaleanu derivative, the Davidson derivative, the Caputo Fabrizio derivative, etc.; see previous work [7, 9, 16, 24, 27, 44]. Other well-known definitions that we will use in the problems in this thesis include the fractional Laplacian operator, the distributional Riesz fractional derivative, and the Riemann-Liouville derivative.

In the past few years, nonlinear fractional elliptic problems have taken a valuable place in fractional calculus. This importance is due to the desire of mathematicians to strengthen the construction of fractional calculus and expand the results from the classical case of partial differential equation to the fractional case in both theoretical and numerical study [58, 57, 43, 45, 76, 65, 22], it is also due to the possible applications that can be obtained in science and engineering; see previous work [4, 5, 25, 41, 46].
The fundamental object in nonlinear fractional elliptic equation is the fractional Laplacian. The fractional Laplacian of u is defined as

$$(-\Delta)^s u(x) = c_{n,s}\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy,$$

where $c_{n,s}$ is a normalization constant and p.v. denotes the Cauchy principal value.

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\hat{u}(\xi))(x),$$

where $\mathcal{F}$ denotes the Fourier transform.
This nonlocal operator is a fractional order generalization of the Laplacian operator; one means of recognizing the difference between the two operators is that the classical Laplacian describes a function's local behavior, whereas the fractional Laplacian considers non-local interactions between different parts of the function. Furthermore, the fractional Laplacian operator has many interesting properties, such as symmetry, positivity, and continuity, making it an effective tool for studying a wide range of mathematical problems, and it has many applications in various fields; see previous work [8, 23, 46].
The above benefits of studying nonlinear fractional elliptic equations and systems prompted us to choose the first problem in this thesis, which is the following system:

$$\begin{cases} (-\Delta)^s u(x) + g_1(x, u(x), v(x)) = f_1(x) & \text{in } \Omega, \\ (-\Delta)^s v(x) + g_2(x, u(x), v(x)) = f_2(x) & \text{in } \Omega, \\ u = v = 0, & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary, with $s \in$

$(0,1)$, $f=(f_1,f_2)\in L^2(\Omega)\times L^2(\Omega)$ and $g_1,g_2:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are satisfying the Carathéodory conditions and some assumptions.

Shieh and Spector (2015) introduced a new kind of fractional derivative named the distributional Riesz fractional derivative. This derivative has the important property that it is the foundational object in the constructions of weak formulations of partial differential equations, which makes it more fundamental than the fractional Laplacian; see [66, 67].
From the definition of this derivative, the authors in [47] could introduce a nonlocal operator $-D^s\cdot(AD^s(.))$ in the duality sense, which is a generalization of the fractional Laplacian.
The non-local operator $-D^s\cdot(AD^su)$ can be applied to any multi-dimensional system, for example, a thin material that is usually viewed as 2D, a normal 3D crystalline solid, or a multi-dimensional Lévy flight.
This operator is the main object in the second problem studied in this thesis, which is the following equation:

$$\begin{cases} -D^s\cdot(A(x)D^su(x))=f(x,u(x)) & \text{in } \Omega,\\ u=0 & \text{on } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{2}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set with a Lipschitz boundary, $s\in(0,1)$ and $n>2s$, with some suitable conditions on the nonlinear term f and the matrix $A(x)$.

The following equation is the third class of fractional problems studied in this thesis; it is a class of nonlocal nonlinear fractional Laplacian equations where the distributional Riesz fractional gradient is the founder object in the associated variational formulation.

$$\begin{cases} -M(\int_\Omega u(x)dx)D^s\cdot(D^su(x))=f(x,u(x)) & \text{in } \Omega,\\ u=0 & \text{on } \mathbb{R}^n\backslash\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $s\in(0,1)$ with $n>2s$, $M:\mathbb{R}\to\mathbb{R}^+$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$. are satisfying certain hypotheses.
The study of nonlocal nonlinear fractional elliptic equations appears in many papers in recent years, including the fractional equations of Kirchhoff type; see previous work [48, 73, 32, 33].

In [42] the authors introduced a new fractional Sobolev space using Riemann-Liouville fractional derivatives and established in their paper the important properties of this space that helped them apply the results to a fractional concave-convex problem. They show for this problem the existence of weak solutions using the variational principle and the Mountain Pass Theorem.

This intriguing space inspired us to investigate the following problem related to the Riemann-Liouville fractional derivative:

$$\begin{cases} {}_xD_b^s({}_aD_x^s u(x)) = \varphi(x) - \psi(x,u) & \text{in } (a,b), \\ P_s(u) = 0 & \text{on } \partial(a,b), \end{cases} \tag{3}$$

where $s \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)$, $\varphi \in L^2([a,b])$ and $\psi(x,k) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. $P_s(u)$ represent the boundary condition of the problem which depend of the behavior of $s \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)$, that is:

$$P_s(u) = \begin{cases} \lim_{x \to a^+} {}_aI_x^{1-s} u(x) = 0, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ u(a) = u(b) = 0, & \text{if } s \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Moreover, ${}_aD_x^s u(x), {}_xD_b^s u(x)$ are the left and right Riemann-Liouville fractional derivatives of u.
Several works with the left and right Riemann-Liouville fractional derivatives appeared in the fractional field, we refer to [12, 13, 36].

Topological theory is one of the most crucial theories that are flourishing domains of research in nonlinear analysis like Schauder and Banach fixed point theorems, Brouwer degree, and Leray-Schauder degree. In 1930, Schauder gave a fixed point theorem in an infinite dimensional Banach space. After that, in 1934, Leray and Schauder introduced in a famed paper a degree of compact perturbations of the identity in Banach spaces of infinite dimension; this degree is named the Leray-Schauder degree, which has the same finality as the degree of Brouwer that is defined only in finite dimensional spaces. The Schauder fixed point theorem is one of the topological results that can be proved using the Leray-Schauder degree. Schauder fixed point theorem and Leray-Schauder degree method have many important applications in mathematical problems such as nonlinear fractional partial differential problems; for more information about this theory, we refer the readers to [54, 28, 21, 29, 34].

In this thesis, we are interested in the existence of solutions for the three above fractional differential problems using the Leray-Schauder degree method and the Schauder fixed point theorem, and we used absurd reasoning under a certain assumption on the nonlinear functions to prove the uniqueness of solutions for the first problem, and we apply the Banach principle of contraction to prove the uniqueness of solutions for the second problem and the fourth problem in a particular case.

PLAN OF THESIS

This thesis consists of the following four chapters:

- The first chapter is divided into two sections. The first section is devoted to certain preliminary notions, theorems, and properties on the L^P spaces, fractional spaces, fractional Laplacian, and the distributional Riesz fractional gradient. After that, we give in the second section some important theorems and properties on the topological degree theory and the fixed point theory.

- In the second chapter, we present a study of the existence of solutions for a nonlinear fractional Laplacian system with Dirichlet boundary conditions and some suitable conditions on the nonlinearities. This study is based on the application of the Leray-Schauder degree method.

- The third chapter deals with a nonlinear elliptic equation involving a nonlocal operator that is defined using the distributional Riesz fractional gradient. We prove the existence of weak solutions using the Leray-Schauder degree theory and their uniqueness using the Banach fixed point theorem.

- The problem studied in the fourth chapter is a class of nonlocal quasilinear fractional elliptic problems with Dirichlet boundary conditions using the Schauder fixed point theorem.

- The fifth chapter is dedicated to a study of the Riemann-Liouville fractional derivative problem in a new fractional Sobolev space using an application of the Leray-Schauder degree theory.

CHAPTER

1 PRELIMINARIES

In this chapter, we recall briefly some preliminary notions of base and the principal mathematical results in functional analysis, which we will use through this thesis. In particular, we cite firstly the theory of L^p spaces and fractional Sobolev spaces with some definitions and properties of fractional Laplacian and distributional Riesz fractional derivatives, and secondly, we introduce the following topological theories: topological degree and Schauder fixed point theorem, the presentation of definitions and theorems in this chapter is without demonstrations.

In all the rest of this chapter, Ω represents an open subset of $\mathbb{R}^n$ with the Lebesgue mesure dx.
First, we recall the following spaces:

$C(\Omega)$ the space of continuous functions.

$$C(\Omega) = \{f : \Omega \to \mathbb{R} \text{ with } f \text{ continuous }\}.$$

$C^m(\Omega)$ the space of the real functions m times continuously differentiable on Ω.

$$C^m(\Omega) = \{v \in C(\Omega) : D^\alpha v \in C(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \leq m\}.$$

Let K be a compact subset of Ω, we introduce the following space that contains continuous functions with compact support.

$$C_c(\Omega) = \{f \in C(\Omega) \text{ such that } f(x) = 0 \quad \text{for all } x \in \Omega \backslash K\}.$$

We note $C_c^\infty(\Omega)$ the space of infinitely differentiable functions with compact support.

1.1 Functional spaces

In this part, we give some important definitions and results of functional spaces that we will use in the next chapters.

We start it with the L^p spaces.

1.1.1 The L^p spaces

Firstly, we define the L^p spaces for $1 \leq p < \infty$.

Definition 1.1.1. *[14] Let $p \in \mathbb{R}$ with $1 \leq p < \infty$; we pose*

$$L^p(\Omega) = \Big\{f : \Omega \to \mathbb{R} : f \ \textit{ measurable function and } \int_\Omega |f(x)|^p dx < \infty\Big\},$$

with the norm

$$\|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p dx\Big)^{\frac{1}{p}}. \tag{1.1}$$

Particular case

The $L^2(\Omega)$ space is a particular case ($p = 2$) of the space $L^p(\Omega)$. It is a Hilbert space, equipped with the following inner product:

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)\, dx.$$

Now, we define the space of measurable functions, which is bounded almost everywhere, noted $L^\infty(\Omega)$.

Definition 1.1.2. *[14] We pose*

$$L^\infty(\Omega) = \Big\{f : \Omega \to \mathbb{R} : f \ \textit{ measurable and } \exists C \textit{ a constant, such that}$$

$$|f(x)| \leq C \quad a.e. \ on\, \Omega\Big\},$$

with the norm

$$\|f\|_{L^\infty(\Omega)} = \inf\Big\{C : \ |f(x)| \leq C \quad a.e. \ on\, \Omega\Big\}.$$

Remark 1.1.1. *If $f \in L^\infty(\Omega)$, we have*

$$|f(x)| \leqslant \|f\|_{L^\infty(\Omega)} \ a.e. \ on \ \Omega.$$

Next, we define the following space of locally integrable functions on Ω:

$$L^p_{Loc} = \left\{f : \Omega \to \mathbb{R} \text{ measurable: for all } K \subset \Omega \text{ compact, } \int_K |f(x)|^p dx < \infty\right\}.$$

In addition, we have

$$L^p(\Omega) \subset L^p_{Loc}(\Omega).$$

Proposition 1.1.1. *[14]*

1. *For $1 \leqslant p \leqslant \infty, (L^p(\Omega), \|\cdot\|_p)$ is a Banach space.*
2. *For $1 \leqslant p < \infty, (L^p(\Omega), \|\cdot\|_p)$ is a separable space.*
3. *For $1 < p < \infty, (L^p(\Omega), \|\cdot\|_p)$ is a reflexive space.*

Theorem 1.1.1 (Minkowski inequality)**.** *[14] Let $f, g \in L^p(\Omega)$ and $p \geqslant 1$, then*

$$f + g \in L^p(\Omega) \ and \ \|f+g\|_p \leqslant \|f\|_p + \|g\|_p.$$

Theorem 1.1.2 (Dominate convergence theorem of Lebesgue)**.** *[14] Let (f_n) be a sequence of functions of $L^1(\Omega)$. We suppose that*

1. *$f_n(x) \to f(x)$ a.e. on Ω.*
2. *There exists a function $g \in L^1(\Omega)$ such that,*

$$\forall n \in \mathbb{N}, |f_n(x)| \leqslant g(x) \quad a.e. \ on\, \Omega.$$

Hence, $f \in L^1(\Omega)$ and $\lim_{n\to+\infty} \|f_n - f\|_{L^1(\Omega)} = 0$.

Theorem 1.1.3. *[15] Let (f_n) a sequence of functions of $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that $\lim_{n\to+\infty} \|f_n - f\|_p = 0$. Then there exists a subsequence (f_{n_k}) and a function $g \in L^p(\Omega)$ such that*

1. $f_{n_k}(x) \to f(x)$ *a.e. on* Ω.
2. $\forall k \in \mathbb{N}, |f_{n_k}(x)| \leqslant g(x)$ *a.e. on* Ω.

Theorem 1.1.4 (Tonelli). *[14] We suppose that*

1. $$\int_{\Omega_2} |f(x,y)|dy < \infty \quad \textit{for almost every } x \in \Omega_1.$$
2. $$\int_{\Omega_1} dx \int_{\Omega_2} |f(x,y)|dy < \infty.$$

Therefore, $f \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.1.5 (Fubini). *[14] We suppose that* $f \in L^1(\Omega_1 \times \Omega_2)$. *Hence, for almost every* $x \in \Omega_1$,

$$y \mapsto f(x,y) \in L^1(\Omega_2) \quad \textit{and} \quad x \mapsto \int_{\Omega_2} f(x,y)dy \in L^1(\Omega_1)$$

The same, for almost every $y \in \Omega_2$,

$$x \mapsto f(x,y) \in L^1(\Omega_1) \quad \textit{and} \quad y \mapsto \int_{\Omega_1} f(x,y)dx \in L^1(\Omega_2)$$

Moreover, we have

$$\int_{\Omega_1} dx \int_{\Omega_2} f(x,y)dy = \int_{\Omega_2} dy \int_{\Omega_1} f(x,y)dx = \iint_{\Omega_1 \times \Omega_2} f(x,y)dxdy.$$

Notation: Let $1 \leqslant p \leqslant \infty$; we denote by q the conjugate exponent of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.1.6 (Hölder inequality). *[14] Let* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *with* $1 \leqslant p \leqslant \infty$. *Therefore,* $f \cdot g \in L^1(\Omega)$, *and*

$$\int |f(x)g(x)|dx \leqslant \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Remark 1.1.2. *In the space* $L^2(\Omega)$, *the Cauchy-Schwarz inequality is given as follows:*

$$\Big|\int f(x)g(x)dx\Big| \leqslant \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Theorem 1.1.7. *[14] Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, with $1 \leqslant p \leqslant \infty$. Hence, for almost everything $x \in \mathbb{R}^n$, the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$. We pose*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Therefore, $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

We note $\tilde{f}(x) = f(-x)$, and we give the following proposition.

Proposition 1.1.2. *[14]*
Let $f \in L^1(\mathbb{R}^n), g \in L^p(\mathbb{R}^n)$ and $h \in L^q(\mathbb{R}^n)$. Thus, we have

$$\int (f * g)h = \int g(\tilde{f} * h).$$

Definition 1.1.3. *[59] Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that the following function*

$$\begin{aligned} f : \Omega \times \mathbb{R}^m &\to \mathbb{R} \\ (x, u) &\mapsto f(x, u) \end{aligned}$$

satisfies the Carathéodory conditions if $u \mapsto f(x, u)$ is continuous for almost every $x \in \Omega$ and $x \mapsto f(x, u)$ is measurable for every $u \in \Omega$.

Given any f satisfying the Carathéodory conditions and a function $u : \Omega \to \mathbb{R}^m$, we can define another function by composition

$$\mathcal{F}(u)(x) := f(x, u(x)). \tag{1.2}$$

The composition operator $\mathcal{F}$ is called a Nemytskii operator. The theorem that follows is about the boundedness and continuity of the Nemytskii operator from $L^p(\Omega)$ to $L^q(\Omega)$.

Theorem 1.1.8. *[59] Let $\Omega \subset \mathbb{R}^n$ be a domain, and let*

$$\begin{aligned} f : \Omega \times \mathbb{R}^m &\to \mathbb{R} \\ (x, u) &\mapsto f(x, u) \end{aligned}$$

satisfy the Carathéodory conditions. In addition, let $p \in (1, \infty)$ and $g \in L^q(\Omega)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) be given, and let f satisfy

$$|f(x, u)| \leq C|u|^{p-1} + g(x).$$

Therefore, the Nemytskii operator $\mathcal{F}$ defined by (1.2), is a bounded and conti nuous map from $L^p(\Omega)$ to $L^q(\Omega)$.

Theorem 1.1.9 (Lax-Milgram theorem). *[15] Let l be a continuous linear form on Hilbert space H and a is a continuous and coercive bilinear form, then there is one and only one function $u \in H$ such that:*

$$a(u, v) = l(v), \quad \forall v \in H.$$

Moreover, if the bilinear form a is symmetric, then u is the only element of H that minimizes the functional $J : H \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}a(v, v) - l(v), \quad \forall v \in H,$$

i.e.

$$J(u) = \min_{v \in H} J(v) \text{ and } J(u) < J(v) \text{ if } u \neq v.$$

1.1.2 Fractional Sobolev spaces

Fractional Sobolev spaces are an important topic that has the capability to extend many classical Sobolev spaces results to the nonlocal and fractional setting. These spaces have served as the basis for much recent research on fractional differential equations.
We start this subsection by defining the $W^{s,p}$ spaces and giving their important properties and inequalities.

The $W^{s,p}$ fractional Sobolev spaces

We will give the definition of $W^{s,p}(\Omega)$ for any real $s > 0$ and for any $p \geq 1$.

Definition 1.1.4 ($0 < s < 1$). *[26]*
For $s \in (0, 1)$ and any $p \geq 1$, the space $W^{s,p}(\Omega)$ is defined as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}},$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$$

is the Gagliardo semi-norm of u.

Proposition 1.1.3. *[26] Let Ω be an open subset of $\mathbb{R}^n$ and $0 < s < 1$, then we have*

1. *For $1 \leqslant p < \infty, W^{s,p}(\Omega)$ is a Banach space.*
2. *For $1 \leqslant p < \infty, W^{s,p}(\Omega)$ is a separable space.*
3. *For $1 < p < \infty, W^{s,p}(\Omega)$ is a reflexive space.*

Proposition 1.1.4. *[26] Let $p \geq 1$ and $0 < s \leq s' < 1$. Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Therefore,*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)},$$

where $C = C(n, s, p) \geq 1$ is a positive constant.
Particularly,

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proposition 1.1.5. *[26] Let $p \geq 1$ and $0 < s < 1$. Let Ω be an open subset of $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary and $u : \Omega \to \mathbb{R}$ be a measurable function. Therefore,*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

where $C = C(n, s, p) \geq 1$ is a positive constant.
Particularly,

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proposition 1.1.6. *[26] Let Ω be a Lipschitz open subset of $\mathbb{R}^n$, $s \in (0, 1)$ and $p > 1$, then*

1. *If $sp < n$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leqslant np/(n - sp)$.*
2. *If $n = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.*
3. *If $sp > n$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and, more precisely,*

$$W^{s,p}(\Omega) \hookrightarrow C^{0,s-n/p}(\Omega).$$

Theorem 1.1.10. *[26] [Compact embeddings] Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$, $0 \leq s < 1$, $p > 1$, and $n \geq 1$, then*

1. *If $sp < n$, then the embedding of $W^{s,p}(\Omega)$ into $L^k(\Omega)$ is compact for every $k < np/(n - sp)$.*

2. *If $sp = n$, then the embedding of $W^{s,p}(\Omega)$ into $L^q(\Omega)$ is compact for every $q < \infty$.*

3. *If $sp > n$, then the embedding of $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}(\Omega)$ is compact for $\lambda < s - n/p$.*

Definition 1.1.5 ($s > 1$)**.** *[26] For $s > 1$ and it is not an integer, we write $s = m + \sigma$, where m is an integer and $\sigma \in (0,1)$. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined as follows:*

$$W^{s,p}(\Omega) := \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha \text{ s.t. } |\alpha| = m\},$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}}. \tag{1.3}$$

Theorem 1.1.11. *[26] $W^{s,p}(\Omega)$ is a Banach space with the norm* (1.3)*.*

Remark 1.1.3. *If $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.*

Proposition 1.1.7. *[26] Let $p \geq 1$ and $s, s' > 1$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$. Then, if $s' \geq s$, we have*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Theorem 1.1.12. *[26] For any $s > 0$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.*

Remark 1.1.4. *We note $W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}}}$ (the completion of the space of infinitly differentiable functions with compact support), according to the previous theorem 1.1.12, we have*

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n),$$

however, when Ω is a subset of $\mathbb{R}^n$, $W_0^{s,p}(\Omega) \neq W^{s,p}(\Omega)$ i.e., in general $W_0^{s,p}(\Omega)$ is not dense in $W^{s,p}(\Omega)$.

Remark 1.1.5. *For $s < 0$ and $p \in (1, \infty)$, we can define $W^{s,p}(\Omega)$ as the dual space of $W^{-s,q}(\Omega)$ where $1/p + 1/q = 1$. Notice that, in this case, the space $W^{s,p}(\Omega)$ is actually a space of distributions on Ω, since it is the dual of a space having $C_c^\infty(\Omega)$ as density subset.*

The $H^{s,p}$ fractional Sobolev space, $0 < s < 1$

In this part, we define the $W^{s,p}(\mathbb{R}^n)$ fractional Sobolev space in the case $(p = 2)$, for $0 < s < 1$, that is noted $H^{s,p}(\mathbb{R}^n)$.

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\},$$

which equipped with the inner product and norm

$$\langle u, v \rangle_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx, \tag{1.4}$$

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s(\mathbb{R}^n)} \right)^{\frac{1}{2}}, \tag{1.5}$$

where

$$[u]_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

Theorem 1.1.13. *[26] $H^s(\mathbb{R}^n)$ equipped with the inner product $< \cdot, \cdot >_{H^s(\mathbb{R}^n)}$ is a Hilbert space.*

Fractional Laplacian operator $(-\Delta)^s$

There are several definitions of the fractional Laplacian [26], including the following: as an integral in the sens of the Cauchy principal value in the real space

Definition 1.1.6. *[26]*

$$\begin{aligned}(-\Delta)^s u(x) &= c(n,s) p.v. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \\ &= c(n,s) \lim_{\varepsilon \to 0^+} \int_{\complement B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \; x \in \mathbb{R}^n,\end{aligned} \tag{1.6}$$

$\forall u \in \mathscr{S}$, $\forall s \in (0,1)$

$$c(n,s) := \pi^{-(2s+n/2)} \frac{\Gamma(s+n/2)}{\Gamma(-s)}, \tag{1.7}$$

and $\mathscr{S}$ is the Schwarz space.

Lemma 1.1.1. *[26] Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator defined by (1.6). Then, for any $u \in \mathscr{S}$,*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n,s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad \forall x \in \mathbb{R}^n.$$

The $D^{s,2}(\Omega)$ fractional Sobolev space, $0 < s < 1$

Next, we denote by $D^{s,2}(\Omega)$ the completion of the space $C_c^\infty(\Omega)$ compared to the H^s norm, i.e., $D^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\|_{H^s}}$.

Theorem 1.1.14. *[26] Let Ω be a bounded Lipschitz open set we have*

$$D^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^n), \ \ \textit{such that}\, u = 0 \ \textit{in} \ \mathbb{R}^n \backslash \Omega\},$$

Now, we give the inner product in $D^{s,2}(\Omega)$ and norm inherited from $H^s(\mathbb{R}^n)$

$$\langle u, v\rangle_{D^{s,2}(\Omega)} = C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx,$$

$$\|u\|_{D^{s,2}(\Omega)} = \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx \right)^{\frac{1}{2}}.$$

Theorem 1.1.15. *[26]*

1. *$D^{s,2}(\Omega)$ equipped with the inner product $< \cdot, \cdot >_{D^{s,2}(\Omega)}$ is a Hilbert space.*

2. *$D^{s,2}(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^n)$.*

Proposition 1.1.8. *[26] Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and $s \in (0,1)$ such that $n > 2s$. Let $u : \Omega \to \mathbb{R}$ be a measurable function compactly supported. Then, there exists a positive constant $c_{emb} > 0$ depending on n, s and Ω such that*

$$\|u\|_{L^2(\Omega)} \le c_{emb} \|u\|_{D^{s,2}(\Omega)}.$$

Theorem 1.1.16. *[55] Let $s \in (0,1)$, $n \ge 1$, $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded open set and $\mathscr{T}$ be a bounded subset of $L^2(\Omega)$. Suppose that*

$$\sup_{f \in \mathscr{T}} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}} dxdy < +\infty.$$

Then $\mathscr{T}$ is precompact in $L^2(\Omega)$.

The compact embedding theorem 1.1.16 gives that $D^{s,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$.

The distributional Riesz fractional derivative D^s

Shieh and Spector (2015) introduced the following definition of distributional Riesz fractional derivative: We note $\mathcal{C}(n,s)=c(n,s)^{\frac{1}{2}}$, with $c(n,s)$ is the constant defined by (1.7).

Definition 1.1.7. *For $u\in L^p(\mathbb{R}^n), p\in(1,\infty)$,*

$$D_j^s u=\frac{\partial^s u}{\partial x_j^s}=\frac{\partial}{\partial x_j}\left(I_{1-s}*u\right),\quad 0<s<1,\quad j=1,\ldots,n,$$

where $\frac{\partial}{\partial x_j}$ is taken in the distributional sense, for every $v\in C_c^\infty(\mathbb{R}^n)$,

$$\left\langle\frac{\partial^s u}{\partial x_j^s},v\right\rangle=-\left\langle\left(I_{1-s}*u\right),\frac{\partial v}{\partial x_j}\right\rangle=-\int_{\mathbb{R}^n}\left(I_{1-s}*u\right)\frac{\partial v}{\partial x_j}dx,$$

with I_{1-s} denoting the Riesz potential of order $1-s$:

$$\left(I_{1-s}*u\right)(x)=\mathcal{C}(n,1-s)\int_{\mathbb{R}^n}\frac{u(y)}{|x-y|^{n-1+s}}dy,$$

with

$$\mathcal{C}(n,s)=2^s\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n+s+1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}.$$

Theorem 1.1.17. *[66] For $u\in C_c^\infty\left(\mathbb{R}^N\right)$,*

$$D^s u=I_{1-s}*Du.$$

Definition 1.1.8. *[47, 66, 67, 68, 19] For sufficiently regular functions u and vector G the fractional gradient (D^s) and the fractional divergence $(D^s\cdot)$ are written in integral form respectively by*

$$\begin{aligned}D^s u(x):&=\mathcal{C}(n,s)\lim_{\epsilon\to0}\int_{\mathbb{R}^n}\frac{zu(x+z)}{|z|^{n+s+1}}\chi_\epsilon(0,z)dz\\&=\mathcal{C}(n,s)\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+s}}\frac{x-y}{|x-y|}dy,\end{aligned}$$

and

$$\begin{aligned}D^s\cdot G(x):&=\mathcal{C}(n,s)\lim_{\epsilon\to0}\int_{\mathbb{R}^n}\frac{z\cdot G(x+z)}{|z|^{n+s+1}}\chi_\epsilon(0,z)dz\\&=\mathcal{C}(n,s)\int_{\mathbb{R}^n}\frac{G(x)-G(y)}{|x-y|^{n+s}}\cdot\frac{x-y}{|x-y|}dy,\end{aligned}$$

where $\chi_\epsilon(x,z)$ is the characteristic function of the set $\{(x,z):|z-x|>\epsilon\}$ for $\epsilon>0$.

Proposition 1.1.9. *[47, 60] distributional Riesz fractional derivative verifies the following properties.*

1. *For* $u \in C_c^\infty(\mathbb{R}^n)$
$$(-\Delta)^s u = -D^s \cdot D^s u,$$

2. *For* $s \to 1$ *: if* $Du \in L^p(\mathbb{R}^n)^n \cap L^q(\mathbb{R}^n)^n$, $1 < q < p$, *then* $D^s u \underset{s\to 1}{\longrightarrow} Du$ *in* $L^p(\mathbb{R}^n)^n$.

Remark 1.1.6. *The fractional divergence of* $v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *is defined as follows:*
$$div^s v = D^s \cdot D^s v.$$

The following proposition gives the fractional integration by parts formula.

Proposition 1.1.10. *Let* $s \in (0,1)$, *we have that*

$$\int_{\mathbb{R}^n} u \; div^s v \; dx = -\int_{\mathbb{R}^n} v \cdot D^s u \; dx,$$

for all $u \in C_c^\infty(\mathbb{R}^n)$ *and* $v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

The fractional spaces $X^{s,p}$ and $L^{s,p}$

Let $1 < p < \infty$ and $s \in (0,1)$. If $u \in C_c^\infty(\mathbb{R}^n)$, we define the the following space,
$$X^{s,p}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{X^{s,p}(R^n)}},$$
where the norm

$$\|u\|^p_{X^{s,p}(\mathbb{R}^n)} = \|u\|^p_{L^p(\mathbb{R}^n)} + \|D^s u\|^p_{L^p(\mathbb{R}^n)}.$$

Let us introduce the space $L^{s,p}(\mathbb{R}^n)$, we first recall the Bessel potentials g_s, for $s \in \mathbb{R}$. The Bessel potentials g_s are defined by (see [53])

$$g_s(x) := \frac{1}{(4\pi)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{\frac{-\pi|x|^2}{\delta}} e^{\frac{-\delta}{4\pi}} \delta^{\frac{s-n}{2}} \frac{d\delta}{\delta},$$

Then the Bessel potential spaces $L^{s,p}(\mathbb{R}^n)$ are defined as follows:

$$L^{s,p}(\mathbb{R}^n) := g_s(L^p(\mathbb{R}^n)),$$

in the sense that every $u \in L^{s,p}(\mathbb{R}^n)$ can be written as

$$u = g_s * f,$$

for some $f \in L^{s,p}(\mathbb{R}^n)$.

For $s \in (0,1)$, we consider the Bessel Potential space $L^{s,2}(\mathbb{R}^n)$(see [66]). We have from remark 2.3 and theorem 1.7 in [66] that

$$H^s(\mathbb{R}^n) = L^{s,2}(\mathbb{R}^n) = X^{s,2}(\mathbb{R}^n),$$

with

$$\|u\|^2_{L^{s,2}(\mathbb{R}^n)} := \|u\|^2_{L^2(\mathbb{R}^n)} + \|D^s u\|^2_{L^2(\mathbb{R}^n)}.$$

The norms $\|u\|^2_{L^{s,2}(\mathbb{R}^n)}$ and $\|D^s u\|^2_{L^2(\mathbb{R}^n)}$ are equivalent according to the following proposition.

Proposition 1.1.11. *[66] Let $1 < p < \infty$ and $s \in (0,1)$, and define $\frac{1}{p^*} := \frac{1}{p} - \frac{s}{n}$. If $\frac{1}{q} > \frac{1}{p^*}$ and $q \geq 1$, then there exists $C_{emb} = C_{emb}(\Omega, n, s, p)$ such that*

$$\left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}} \leq C_{emb} \|D^s u\|_{L^p(\mathbb{R}^n)},$$

for all $u \in L^{s,p}(\mathbb{R}^n)$.

We consider the space $L^{s,2}_0(\Omega)$. Namely, we set

$$L^{s,2}_0(\Omega) = \overline{\{u \in C^\infty_c(\mathbb{R}^n), supp(u) \subset \Omega\}}^{\|\|_{L^{s,2}(\mathbb{R}^n)}},$$

also, we can set

$$L^{s,2}_0(\Omega) = \left\{u \in L^{s,2}(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\right\},$$

which equipped with the inner product

$$\langle u, v\rangle_{L^{s,2}_0(\Omega)} = \int_{\mathbb{R}^n} D^s u \cdot D^s v dx,$$

and with the following equivalent norms

$$\|u\|^2_{L^{s,2}_0(\Omega)} = \|D^s u\|^2_{L^2(\mathbb{R}^n)} = \frac{c^2_{n,s}}{2}[u]^2_{s,\mathbb{R}^n},$$

is a Hilbert space.

Proposition 1.1.12. *[47, 66] Let $1 < p < \infty$ and $s \in (0,1)$ be such that $sp < n$. Then there exists a constant $C = C(n,p,s) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|D^s u\|_{L^p(\mathbb{R}^n)},$$

for all $u \in L^{s,p}(\mathbb{R}^n)$, where $p^* = \frac{np}{n-sp}$.

Proposition 1.1.13. *[47] For $0 < s \leq 1$ and $1 \leq q \leq 2^*$, where $2^* = \frac{2n}{n-2s}$. Then, by the Sobolev-Poincaré inequalities, we have the embeddings*

$$L_0^{s,2}(\Omega) \hookrightarrow L^q(\Omega),$$

Those embeddings are compact for $1 \leq q < 2^*$.

1.2 Topological degree and Fixed point theorems

1.2.1 Topological degree

The topological degree is a tool developed for nonlinear applications which plays the same role as the determinant for linear applications. The topological degree which indicates by its non-nullity that the following problem

$$\text{Find } x \in \Omega \text{ such that } f(x) = y. \qquad (\mathcal{P})$$

has at least one solution, and it is a "stable" solution. Obviously, this degree will depend on f and y, but also of the set on which we seek the solutions to $(\mathcal{P})$.
We begin by giving the existence and uniqueness of a map, which is the topological degree, in finite dimension, is called the Brouwer degree, after that we give it in infinite dimension, which is called the Leray-Schauder degree.
Brouwer degree

The Brouwer degree is a topological degree in the finite dimensional case. Before defining the degree of Brouwer we recall the following definition

Definition 1.2.1. *[28] Let Ω be a bounded open subset of $\mathbb{R}^n$ and $F \in C\left(\bar{\Omega}, \mathbb{R}^n\right) \cap C^1(\Omega, \mathbb{R}^n), y_0 \in \Omega$ is called regular point if $J_F(y_0) \neq 0$ (or $J_F(y_0) = \det DF(y_0)$ with $\mathrm{DF}(y_0) = \left(\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j}(y_0)\right)$, Otherwise, x_0 is called critical point or singular point. Let us designate by*

$$S_F(\Omega) = \{y_0 \in \Omega : J_F(y_0) = 0\},$$

the set of singular points of F on Ω.

Definition 1.2.2. *[28] Let Ω be an open and bounded subset of $\mathbb{R}^N$ and let $F \in C\left(\bar{\Omega}, \mathbb{R}^n\right) \cap C^1(\Omega, \mathbb{R}^n)$. Assume that $y_0 \in \mathbb{R}^N \backslash F(\partial\Omega)$ and y_0 is a regular value of F. Then we define the Brouwer degree of F as*

$$\deg(F, \Omega, y_0) = \sum_{x \in F^{-1}(y_0) \cap \Omega} \operatorname{sgn} J_{F(x)} \cdot \tag{1.8}$$

The sum in (1.8) *is finite. Indeed, otherwise the set*

$$F^{-1}(y_0) \cap \Omega := \{x \in \Omega : F(x) = y_0\}.$$

The following theorem indicates the existence and uniqueness of Brouwer's topological degree with its properties.

Theorem 1.2.1. *[34] Let $n \geq 1$ and $\mathcal{A}$ be the set of triples (f, Ω, y) where Ω is a bounded open subset of $\mathbb{R}^n, y \in \mathbb{R}^n$ and $f : \bar{\Omega} \to \mathbb{R}^n$ is continuous such that $y \notin f(\partial\Omega)$ There is one and only one app $d : \mathcal{A} \to \mathbb{Z}$ which satisfies the following properties:*

- ***Normalization*** *if Ω is a bounded open set of $\mathbb{R}^n$ and $y \in \Omega$ then $d(Id, \Omega, y) = 1$.*
- ***Additivity*** *if Ω is a bounded open set of $\mathbb{R}^n$, $y \in \mathbb{R}^n$, $f : \bar{\Omega} \to \mathbb{R}^n$ is continuous and Ω_1, Ω_2 are disjoint open sets included in Ω such that $y \notin f(\Omega \backslash (\Omega_1 \cup \Omega_2))$, then $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, (\Omega)_2, y)$.*
- ***Invariance by homotopy*** *if Ω is a bounded open set of $\mathbb{R}^n, h : [0,1] \times \bar{\Omega} \to \mathbb{R}^n$ and $y : [0,1] \to \mathbb{R}^n$ are continuous and for all $t \in [0,1]$, $y(t) \notin h(t, \bar{\Omega})$, then $d(h(0, \cdot), \Omega, y(0)) = d(h(1, \cdot), \Omega, y(1))$.*

d is called the Brouwer topological degree.

Leray-Schauder degree

The Leray-Schauder degree is a topological degree in the infinitely dimensional case. The following theorem gives the existence of the Leray-Schauder degree with its principal properties.

Theorem 1.2.2. *[34, 28] Let E be a Banach space and $\mathcal{A}$ the set of triplets $(\mathrm{Id} - f, \Omega, y)$ where Ω is a bounded open subset of E , $y \in E$ and $f : \bar{\Omega} \to \mathbb{R}^n$ is compact which means ($\{f(x), x \in \bar{\Omega}\}$ is relatively compact and f is continuous), with $y \notin (\mathrm{Id} - f)(\partial\Omega)$, then there exists a map $d : \mathcal{A} \to \mathbb{Z}$ satisfy the following properties:*

- *(Normality) if $y \in \Omega$ therefore $d(\mathrm{Id}, \Omega, y) = 1$.*

- *(Additivity)* $d(\mathrm{Id}-f,\Omega,y)=d\left(\mathrm{Id}-f,\Omega_1,y\right)+d\left(\mathrm{Id}-f,\Omega_2,y\right)$ *if* $\Omega_1\cup\Omega_2\subset\Omega,\Omega_1\cap\Omega_2=\emptyset$ *and* $y\notin\left\{x-f(x),x\in\bar{\Omega}\backslash\Omega_1\cup\Omega_2\right\}$.

- *(Homotopy invariance) if* h *is a compact map from* $[0,1]\times\bar{\Omega}$ *into* E *(which is equivalent to say* h *is continuous and* $\{h(t,x),t\in[0,1],x\in\bar{\Omega}\}$*is relatively compact in* E *),* $y\in C([0,1],E)$ *and* $y(t)\notin\{x-h(t,x),x\in\partial\Omega\}$ *(for all* $t\in[0,1]$*), then*

$$d(\mathrm{Id}-h(t,\cdot),\Omega,y(t))=d(\mathrm{Id}-h(0,\cdot),\Omega,y(0)),$$

for all $t\in[0,1]$.

d *is named the Leray-Schauder degree.*

Proposition 1.2.1. *[28] The Leray-Schauder topological degree verifies the following properties:*

- *If* $d(\mathrm{Id}-f,\Omega,y)\neq 0$ *then there exists* $x\in\Omega$ *such that* $x-f(x)=y$.
- *For all* $z\in E, d(\mathrm{Id}-f,\Omega,y)=d(\mathrm{Id}-f-z,\Omega,y-z)$.
- *Let* $(\mathrm{Id}-f,\Omega,y)\in\mathcal{A}$ *and* $r=\mathrm{dist}(y,\mathrm{Id}-f)(\partial\Omega))>0$. *If* $g:\bar{\Omega}\to\mathbb{R}^n$ *compact and* $z\in\mathbb{R}^n$ *are such as* $\sup\limits_{\partial\Omega}(\|g-f\|)+\|y-z\|<r$, *then* $d(\mathrm{Id}-f,\Omega,y)=d(\mathrm{Id}-g,\Omega,z)$.
- $d(Id-f,\Omega,\cdot)$ *is constant on connected components of* $E\backslash(\mathrm{Id}-f)(\partial\Omega)$.
- *For all* $z\in E, d(\mathrm{Id}-f,\Omega,y)=d((\mathrm{Id}-f)(\cdot-z),z+\Omega,y)$.

1.2.2 Fixed point theorems

Theorem 1.2.3 (Brouwer Fixed Point Theorem)**.** *[28] Let* $\mathcal{K}$ *be a nonempty, convex, closed and bounded subset of* $\mathbb{R}^N$. *Assume that* $F:\mathcal{K}\to\mathcal{K}$ *is continuous. Then* F *has a fixed point in* $\mathcal{K}$.

Now, we give the Brouwer fixed point theorem version in a particular case, where $\mathcal{K}$ is a ball in $\mathbb{R}^n$ with center 0 and radius noted R and $\|.\|$ denoted a norm in $\mathbb{R}^N$.

Theorem 1.2.4. *[34] Let* $N\geq 1,R>0$ *and* $f\in C\left(B_R,B_R\right)$ *with* $B_R=\left\{x\in\mathbb{R}^N,\|x\|\leq R\right\}$ *Then* f *admits a fixed point, i.e. there exists* $x\ inB_R$ *such that* $f(x)=x$.

The following theorem is a generalization of the Brouwer Fixed Point Theorem into the infinite dimensional setting.

Theorem 1.2.5. *[28] [Schauder Fixed Point Theorem] Let $\mathcal{K}$ be a nonempty, closed, convex and bounded subset of a normed linear space X. Assume that $F \in \mathscr{C}(\mathcal{K}, X)$ and $F(\mathcal{K}) \subset \mathcal{K}$. Then there is a fixed point of F in $\mathcal{K}$.*

The Schauder Fixed Point Theorem version is given as follows in a specific case, where $\mathcal{K}$ is a ball included in E a Banach space, with centre 0 and radius noted R.

Theorem 1.2.6. *[34] Let $(E, \|.\|_E)$ be a Banach space, $R > 0$, $B_R = \{x \in E, \|x\|_E \leq R\}$ and f a compact map from B_R into B_R (i.e. f continues and $\{f(x), x \in B_R\}$ relatively compact in E. Then f admits a fixed point, i.e. there exists $x \in B_R$ such that $f(x) = x$.*

Definition 1.2.3. *[62] Let X be a metric space, with metric d. If $f : X \to X$ and if there is a constant $c < 1$ such that*

$$d(f(x), f(y)) \leq c\, d(x, y),$$

for all $x, y \in X$, then f is a contraction of X into X.

Theorem 1.2.7 (Banach fixed point theorem)**.** *[62] Let X be a complete metric space and f be a contraction of X into X, then there exists a unique $x \in X$ such that $f(x) = x$.*

CHAPTER 2

LERAY-SCHAUDER DEGREE FOR A NONLINEAR FRACTIONAL LAPLACIAN SYSTEM

In this chapter, we consider a coupled semilinear fractional elliptic system with Dirichlet boundary conditions using some suitable conditions on the nonlinearities, and we investigate the existence and uniqueness of weak solutions for this problem in a fractional Sobolev space. Our study into the existence of solutions is based on the Leray-Schauder degree method, while the uniqueness of solutions is investigated using an additional assumption. This chapter exposes results published in [1].

2.1 Motivation for the problem

Over the last few decades, many researchers have worked to expand the field of fractional calculus in order to develop advanced mathematical models that have the ability to describe more complex systems, see the previous work [70].

In particular, the current work investigates the existence of weak solutions in fractional Sobolev space for a semilinear fractional elliptic system of non-local equations involving the fractional Laplacian with Dirichlet boundary conditions involving the fractional Laplacian. This problem is a fractional version of the problem in [40], which is concerned with establishing the existence of weak solutions in the classical Sobolev space for a semilinear elliptic system of local equations involving the classical Laplacian with Dirichlet boundary conditions.

Fractional elliptic systems involving fractional Laplacian with nonlinearities has been the subject of many papers, as the work of W. Dai et al. (2017) who studied the Liouville type theorems for the following problem on a half space.

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n_+, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n_+, \\ u(x) = v(x) = 0, & x \notin \mathbb{R}^n_+. \end{cases}$$

Mana Manassés de Souza (2020) studied the existence and multiplicity of solutions for the following fractional elliptic system involving concave-convex nonlinearities.

$$(-\Delta)^\sigma u_i + a_i(x)u_i = f_i\left(x, u_1, \ldots, u_m\right) \quad \text{for } x \in \mathbb{R}^n \quad \text{and } i = 1, \ldots, m$$

where $\sigma \in (0,1), n \geq 1, (-\Delta)^\sigma$ denotes the fractional Laplacian of order σ , $a_i(x)$ are continuous.
H. Fan et al.(2016) studied the multiplicity of positive solutions for the following fractional elliptic system with critical nonlinearities.

$$\begin{cases} (-\Delta)^{\frac{s}{2}}u = \lambda|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ (-\Delta)^{\frac{s}{2}}v = \mu|v|^{q-2}v + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded set in $\mathbb{R}^N$ with smooth boundary, $N > s$ with $s \in (0,2)$ fixed, $1 < q < 2$, $\lambda, \mu > 0, \alpha, \beta > 1$ satisfy $\alpha + \beta = 2^*_s = \frac{2N}{N-s}, 2^*_s$ is the fractional Sobolev critical exponent, and $(-\Delta)^{\frac{s}{2}}$ is the fractional Laplacian.

The idea of using the fractional Laplacian in the place of the classical Laplacian in many problems is very interesting mostly because of the nonlocal property of this operator, this idea has been used in many model of applications in different fields such as physics, mechanics, mathematical biology, electrochemistry,...etc. We state the following three models:

Quasi-geostrophic [20]

$$\partial_t\theta + u \cdot \nabla\theta + \kappa(-\Delta)^{\alpha/2}\theta = f.$$

Cahn-Hilliard[6]

$$\partial_t u + (-\Delta)^{\alpha/2} \left(-\varepsilon^2 \Delta u + f(u) \right) = 0.$$

Porous Medium [6]

$$\partial_t u + (-\Delta)^{\alpha/2} \left(|u|^{m-1} \operatorname{sign} u \right) = 0.$$

The topological degree method was effectively applied to study the existence of solutions for different nonlinear problem, see [54, 28]. We will use the Leray-Schauder degree to study the existence of weak solutions for our problem.

The rest of this chapter is organized as follows: in the second section we give the mathematical formulation of our problem with some assumptions and preliminaries. The study of existence of solutions using the Leray-Schauder degree is presented in section 3.

2.2 Mathematical formulation of the problem

We consider the following semilinear fractional Laplacian problem:

$$\begin{cases} (-\Delta)^s u(x) + g_1(x, u(x), v(x)) = f_1(x), & \text{in } \Omega, \\ (-\Delta)^s v(x) + g_2(x, u(x), v(x)) = f_2(x), & \text{in } \Omega, \\ u = v = 0, & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary, $(-\Delta)^s$ is the fractional Laplacian defined in definition 1.1.6, with $s \in (0,1), n > 2s, f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$ and we cite the following hypothesis on the nonlinearities:

(h$_1$) $g_1, g_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are satisfying the Carathéodory conditions and satisfy the following growth conditions:
there exist $a, b \in L^2(\Omega)$ and $K_1, K_2, r_1, r_2 \in \mathbb{R}_+^*$ such that

$$|g_1(x, m, p)| \leq a(x) + K_1|m| + K_2|p|, \ \forall m, p \in \mathbb{R} \ \text{ and a.e. } x \in \Omega.$$
$$|g_2(x, m, p)| \leq b(x) + r_1|m| + r_2|p|, \ \forall m, p \in \mathbb{R} \ \text{ and a.e. } x \in \Omega.$$

(h$_2$) Sign condition

$$\begin{cases} g_1(x, m, p)m & \geq 0, \ \forall m, p \in \mathbb{R} \ \text{ and a.e. } x \in \Omega. \\ g_2(x, m, p)p & \geq 0, \ \forall m, p \in \mathbb{R} \ \text{ and a.e. } x \in \Omega. \end{cases}$$

Remark 2.2.1. *From the hypothesis* (h_1)*, the Nemytski operators* G *and* F *respectively defined as:*
$G(u,v)(x) = g_1(x,u(x),v(x))$ *and* $F(u,v)(x) = g_2(x,u(x),v(x))$ *are continuous and bounded from* $L^2(\Omega) \times L^2(\Omega)$ *into* $L^2(\Omega)$.

2.3 Existence of weak solutions for the main problem

Our purpose in this section is to prove the existence of weak solutions for the problem (2.1) by satisfying the Leray-Shauder degree conditions. For this purpose, firstly, we give the following definition of weak solutions, according to [11].

Definition 2.3.1. *[11] We say that* $(u,v) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$ *is a weak solution for the problem* (2.1)*, if for every* $(\psi_1,\psi_2) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$*, the system*

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}}dydx = \int_\Omega f_1(x)\psi_1 dx \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_\Omega g_1(x,u(x),v(x))\psi_1 dx, \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx = \int_\Omega f_2(x)\psi_2 dx \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_\Omega g_2(x,u(x),v(x))\psi_2 dx, \end{cases}$$

holds.
Where, $C(n,s) = \frac{1}{2}c(n,s)$*, with* $c(n,s)$ *is the constant defined by* (1.7).

Remark 2.3.1. *We obtain the precedent weak formulation by multiplying the first and second equations of the problem* (2.1) *by* $\psi_1 \in C_c^\infty(\Omega)$ *and* $\psi_2 \in C_c^\infty(\Omega)$ *respectively, then integrating over* $\mathbb{R}^n$*, using the Definition 1.1.6 of fractional Laplacian, the symmetry of the integrand, and some calculus.*
We can also prove this weak formulation using the definition of fractional Laplacian in Lemma 1.1.1, and we apply the integration by parts formula proved in Lemma 5.6 of the paper [64].

Next, we state the following theorem that gives the existence of weak solution for the problem (2.1).

Theorem 2.3.1. *Under the assumptions* (h_1) *and* (h_2)*, the problem* (2.1) *has at least one weak solution* $(u,v) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.

Before proving the main result, we provide some lemmas that we will need in our proof.

Auxiliary results

The outcome of the following lemma is necessary to formulate a new problem that is equivalent to our main problem.

Lemma 2.3.1. *For $(u, v) \in L^2(\Omega) \times L^2(\Omega)$, $(f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$, $t \in [0, 1]$ and under the assumption (h_1), the following problem has a unique weak solution $(\varphi, \phi) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.*

$$\begin{cases} (-\Delta)^s\varphi(x) + tg_1(x, u(x), v(x)) = tf_1(x) & \textit{in } \Omega, \\ (-\Delta)^s\phi(x) + tg_2(x, u(x), v(x)) = tf_2(x) & \textit{in } \Omega, \\ \varphi = \phi = 0 & \textit{on } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{2.2}$$

Proof. For all $(u, v) \in L^2(\Omega) \times L^2(\Omega)$, we have $g_1(., u, v), g_2(., u, v) \in L^2(\Omega)$ from the assumption (h_1).
To prove the existence and uniqueness of weak solution for the problem (2.2) we will use the Lax-Milgram theorem.

Weak formulation of the problem (2.2)

We can conclude from the same ideas in Remark 2.3.1 that the problem (2.2) is weakly formulated as follows:

$$\begin{cases} \text{Find } (\varphi, \phi) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(x)-\varphi(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_1(x)\psi_1 dx \\ \qquad\qquad - \int_\Omega tg_1(x, u(x), v(x))\psi_1 dx, \forall\psi_1 \in D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(\phi(x)-\phi(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_2(x)\psi_2 dx \\ \qquad\qquad - \int_\Omega tg_2(x, u(x), v(x))\psi_2 dx, \forall\psi_2 \in D^{s,2}(\Omega). \end{cases}$$

Then, we pose

$$\begin{cases} \text{Find } (\varphi, \phi) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ a_1(\varphi, \psi_1) = l_1(\psi_1), \quad \forall\psi_1 \in D^{s,2}(\Omega), \\ a_2(\phi, \psi_2) = l_2(\psi_2), \quad \forall\psi_2 \in D^{s,2}(\Omega), \end{cases} \tag{2.3}$$

with

$$a_1(.,.) : D^{s,2}(\Omega) \times D^{s,2}(\Omega) \longrightarrow \mathbb{R}$$
$$(\varphi, \psi_1) \mapsto a_1(\varphi, \psi_1) = C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(x)-\varphi(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx,$$

$$
\begin{aligned}
l_1 : D^{s,2}(\Omega) &\longrightarrow \mathbb{R}\\
\psi_1 &\mapsto l_1(\psi_1) = \textstyle\int_\Omega t f_1(x)\psi_1 dx - \int_\Omega t g_1(x,u(x),v(x))\psi_1 dx,
\end{aligned}
$$

$$
\begin{aligned}
a_2(.,.) : D^{s,2}(\Omega)\times D^{s,2}(\Omega) &\longrightarrow \mathbb{R}\\
(\phi,\psi_2) &\mapsto a_2(\phi,\psi_2) = C(n,s)\textstyle\iint_{\mathbb{R}^{2n}} \frac{(\phi(x)-\phi(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx,
\end{aligned}
$$

$$
\begin{aligned}
l_2 : D^{s,2}(\Omega) &\longrightarrow \mathbb{R}\\
\psi_2 &\mapsto l_2(\psi_2) = \textstyle\int_\Omega t f_2(x)\psi_2 dx - \int_\Omega t g_2(x,u(x),v(x))\psi_2 dx.
\end{aligned}
$$

Let us prove that the bilinear forms $a_1(.,.)$ and $a_2(.,.)$ are continuous and coercive in $D^{s,2}(\Omega)\times D^{s,2}(\Omega)$. In addition, we prove that the linear forms l_1 and l_2 are continuous.
We have

$$
\begin{aligned}
|a_1(\varphi,\psi_1)| &= \left| C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\varphi(x)-\varphi(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx \right|\\
&\leq C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)||\psi_1(x)-\psi_1(y)|}{|x-y|^{n+2s}} dydx\\
&\leq C(n,s)\|\varphi\|_{D^{s,2}(\Omega)}\|\psi_1\|_{D^{s,2}(\Omega)},
\end{aligned}
$$

which shows the continuity of the bilinear form $a_1(.,.)$, where we have used the Cauchy-Schwarz inequality.

Moreover,

$$
\begin{aligned}
a_1(\psi_1,\psi_1) =& C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\psi_1(x)-\psi_1(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx\\
&= C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\psi_1(x)-\psi_1(y))^2}{|x-y|^{n+2s}} dydx\\
&= C(n,s)\|\psi_1\|^2_{D^{s,2}(\Omega)},
\end{aligned}
$$

thus, $a_1(.,.)$ is coercive.

For the linear form l_1 we use the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|l_1(\psi_1)| &= \left|\int_\Omega t f_1(x)\psi_1 dx - \int_\Omega t g_1(x,u(x),v(x))\psi_1 dx\right|\\
&\leq \|f_1\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)} + \|g_1(.,u,v)\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)}\\
&\leq \left[\|f_1\|_{L^2(\Omega)} + \|g_1(.,u,v)\|_{L^2(\Omega)}\right]\|\psi_1\|_{L^2(\Omega),}
\end{aligned}
$$

then, the proposition (1.1.8), gives that

$$|l_1(\psi_1)| \leq c_{emb}\left[\|f_1\|_{L^2(\Omega)} + \|g_1(.,u,v)\|_{L^2(\Omega)}\right]\|\psi_1\|_{D^{s,2}(\Omega)},$$

hence, l_1 is continuous.

Similar to the calculus of $a_1(.,.)$, we can obtain that $a_2(.,.)$ is continuous and coercive.
The Cauchy-Schwarz inequality implies that

$$\begin{aligned}|a_2(\phi,\psi_2)| &= \left|C(n,s)\iint_{\mathbb{R}^{2n}}\frac{(\phi(x)-\phi(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx\right|\\ &\leq C(n,s)\iint_{\mathbb{R}^{2n}}\frac{|\phi(x)-\phi(y)||\psi_2(x)-\psi_2(y)|}{|x-y|^{n+2s}}dydx\\ &\leq C(n,s)\|\phi\|_{D^{s,2}(\Omega)}\|\psi_2\|_{D^{s,2}(\Omega)},\end{aligned}$$

which shows the continuity of the bilinear form $a_2(.,.)$.

Furthermore,

$$\begin{aligned}a_2(\psi_2,\psi_2) &= C(n,s)\iint_{\mathbb{R}^{2n}}\frac{(\psi_2(x)-\psi_2(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx\\ &= C(n,s)\iint_{\mathbb{R}^{2n}}\frac{(\psi_2(x)-\psi_2(y))^2}{|x-y|^{n+2s}}dydx\\ &= C(n,s)\|\psi_2\|^2_{D^{s,2}(\Omega)},\end{aligned}$$

hence, $a_2(.,.)$ is coercive.

For the linear form l_2 we use the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|l_2(\psi_2)| &= \left|\int_\Omega tf_2(x)\psi_2dx - \int_\Omega tg_2(x,u(x),v(x))\psi_2dx\right|\\ &\leq \|f_2\|_{L^2(\Omega)}\|\psi_2\|_{L^2(\Omega)} + \|g_2(.,u,v)\|_{L^2(\Omega)}\|\psi_2\|_{L^2(\Omega)}\\ &\leq \left[\|f_2\|_{L^2(\Omega)} + \|g_2(.,u,v)\|_{L^2(\Omega)}\right]\|\psi_2\|_{L^2(\Omega),}\end{aligned}$$

then, the proposition (1.1.8), gives that

$$|l_2(\psi_2)| \leq c_{emb}\left[\|f_2\|_{L^2(\Omega)} + \|g_2(.,u,v)\|_{L^2(\Omega)}\right]\|\psi_2\|_{D^{s,2}(\Omega)},$$

hence, l_2 is continuous.

Consequently, we may apply the Lax-Milgram theorem, and we conclude that the problem (2.2) has a unique weak solution $(\varphi,\phi) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.

□

The previous lemma ensures that we can define the following map, noted A by

$$A : [0,1] \times L^2(\Omega) \times L^2(\Omega) \longrightarrow D^{s,2}(\Omega) \times D^{s,2}(\Omega)$$

$$(t,u,v) \mapsto A(t,u,v) = (\varphi, \phi),$$

where (φ, ϕ) is a weak solution to the problem (2.2).

The proof of the existence of weak solutions for the problem (2.1) is equivalent to prove the existence of solution to the following problem

$$\begin{cases} \text{Find } (u,v) \in L^2(\Omega) \times L^2(\Omega), \\ (u,v) = A(1,u,v). \end{cases} \tag{2.4}$$

Priori estimate

Lemma 2.3.2. *Under the assumption (h_2), there exists $R > 0$, $\forall (u,v) \in L^2(\Omega) \times L^2(\Omega)$ such that:*
If we suppose that

$$\begin{cases} A(t,u,v) = (u,v), \\ t \in [0,1], (u,v) \in L^2(\Omega) \times L^2(\Omega), \end{cases}$$

we get the following priori estimate of the solution

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} < R + 1.$$

Proof. Let $A(t,u,v) = (\varphi, \phi) = (u,v)$, we obtain

$$\begin{cases} \text{Find } (u,v) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_1(x)\psi_1 dx \\ \qquad - \int_\Omega tg_1(x,u(x),v(x))\psi_1 dx, \quad \forall \psi_1 \in D^{s,2}(\Omega), \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_2(x)\psi_2 dx \\ \qquad - \int_\Omega tg_2(x,u(x),v(x))\psi_2 dx, \quad \forall \psi_2 \in D^{s,2}(\Omega). \end{cases} \tag{2.5}$$

Let us choose $(\psi_1, \psi_2) = (u,v)$ in the equation of (2.5) and obtain

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx = t\int_\Omega f_1(x)u(x)dx \\ \qquad -t\int_\Omega g_1(x,u(x),v(x))u(x)dx, \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dydx = t\int_\Omega f_2(x)v(x)dx \\ \qquad -t\int_\Omega g_2(x,u(x),v(x))v(x)dx. \end{cases}$$

Applying the assumption (h_2) and the Cauchy-Schwarz inequality, we have

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx & \leq \|f_1\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}, \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}dydx & \leq \|f_2\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \end{cases}$$

and Proposition 1.1.8 implies that

$$\begin{cases} \frac{C(n,s)}{c_{emb}^2}\|u\|_{L^2(\Omega)}^2 & \leq \|f_1\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}, \\ \frac{C(n,s)}{c_{emb}^2}\|v\|_{L^2(\Omega)}^2 & \leq \|f_2\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \end{cases} \tag{2.6}$$

Adding the two inequalities of (2.6), we get

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} \leq \frac{c_{emb}^2}{C(n,s)}\|f_1\|_{L^2(\Omega)} + \frac{c_{emb}^2}{C(n,s)}\|f_2\|_{L^2(\Omega)}.$$

If we set $R = \frac{c_{emb}^2}{C(n,s)}\|f_1\|_{L^2(\Omega)} + \frac{c_{emb}^2}{C(n,s)}\|f_2\|_{L^2(\Omega)}$, we obtain

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} < R+1.$$

□

We note

$$B_{R+1} = \{(u,v) \in L^2(\Omega)\times L^2(\Omega); \|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} < R+1\}.$$

Lemma 2.3.3. *Under the assumption (h_1), the following set*

$$\left\{A(t,u,v), t\in[0,1], (u,v)\in \bar{B}_{R+1}\right\}$$

is relatively compact in $L^2(\Omega)\times L^2(\Omega)$.

Proof. Let $(t_n)_{\in\mathbb{N}} \subset [0,1]$ and $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \bar{B}_{R+1}$. So using the assumption (h_1), the sequences $\{g_1(.,u_n,v_n)\}_{n\in\mathbb{N}}$ and $\{g_2(.,u_n,v_n)\}_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$.

Firstly, setting $A(t_n,u_n,v_n) = (\varphi_n,\phi_n)$, we get

$$\begin{cases} \text{Find } (\varphi,\phi) \in D^{s,2}(\Omega)\times D^{s,2}(\Omega), \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\varphi(x)-\varphi(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}}dydx = \int_\Omega tf_1(x)\psi_1 dx \\ \qquad\qquad - \int_\Omega tg_1(x,u(x),v(x))\psi_1 dx \ \forall\psi_1 \in D^{s,2}(\Omega), \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\phi(x)-\phi(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx = \int_\Omega tf_2(x)\psi_2 dx \\ \qquad\qquad - \int_\Omega tg_2(x,u(x),v(x))\psi_2 dx \ \forall\psi_2 \in D^{s,2}(\Omega). \end{cases} \tag{2.7}$$

Let us choose $(\psi_1, \psi_2) = (u, v)$ in the equation of (2.9), apply the Cauchy-Schwarz inequality and obtain

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|\varphi_n(x)-\varphi_n(y)|^2}{|x-y|^{n+2s}} dydx \leq \|f_1\|_{L^2(\Omega)} \|\varphi_n\|_{L^2(\Omega)} \\ \qquad\qquad\qquad\qquad + \|g_1(.,u_n,v_n)\|_{L^2(\Omega)} \|\varphi_n\|_{L^2(\Omega)}, \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{n+2s}} dydx \leq \|f_2\|_{L^2(\Omega)} \|\phi_n\|_{L^2(\Omega)} \\ \qquad\qquad\qquad\qquad + \|g_2(.,u_n,v_n)\|_{L^2(\Omega)} \|\phi_n\|_{L^2(\Omega)}. \end{cases}$$

Then, using the fact that $\{g_1(.,u_n,v_n)\}_{n\in\mathbb{N}}$ and $\{g_2(.,u_n,v_n)\}_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$ and the proposition 1.1.8, we obtain

$$\begin{cases} \|\varphi_n\|^2_{D^{s,2}(\Omega)} & \leq \frac{c_{emb}}{C(n,s)} [\|f_1\|_{L^2(\Omega)} + M_1] \|\varphi_n\|_{D^{s,2}(\Omega)}, \\ \|\phi_n\|^2_{D^{s,2}(\Omega)} & \leq \frac{c_{emb}}{C(n,s)} [\|f_2\|_{L^2(\Omega)} + M_2] \|\phi_n\|_{D^{s,2}(\Omega)}. \end{cases} \tag{2.8}$$

By simplification and adding the two inequalities of (2.8), we arrive at

$$\|(\varphi_n, \phi_n)\|_{D^{s,2}(\Omega)\times D^{s,2}(\Omega)} \leq \frac{c_{emb}}{C(n,s)} [\|f_1\|_{L^2(\Omega)} + M_1] + \frac{c_{emb}}{C(n,s)} [\|f_2\|_{L^2(\Omega)} + M_2].$$

Then,

$$\|(\varphi_n, \phi_n)\|_{D^{s,2}(\Omega)\times D^{s,2}(\Omega)} \leq K,$$

where $K = \frac{c_{emb}}{C(n,s)} [\|f_1\|_{L^2(\Omega)} + M_1] + \frac{c_{emb}}{C(n,s)} [\|f_2\|_{L^2(\Omega)} + M_2]$.

Therefore $\{(\varphi_n, \phi_n)\}_{n\in\mathbb{N}}$ is bounded in $D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.

Next, using the Compact Embedding Theorem 1.1.16, we deduce that there is a subsequence of $\{(\varphi_{n_k}, \phi_{n_k})\}_{k\in\mathbb{N}}$ which converges to (φ, ϕ) strongly in $L^2(\Omega) \times L^2(\Omega)$.

We conclude that $\left\{A(t,u,v), t \in [0,1], (u,v) \in \bar{B}_{R+1}\right\}$ is relatively compact in $L^2(\Omega) \times L^2(\Omega)$ □

Lemma 2.3.4. *Under the assumption* (h_1)*, A is continuous from* $[0,1] \times L^2(\Omega) \times L^2(\Omega)$ *into* $L^2(\Omega) \times L^2(\Omega)$.

Proof. Let $\{(t_n, u_n, v_n)\}_{n\in\mathbb{N}} \subset [0,1] \times L^2(\Omega) \times L^2(\Omega)$ be a sequence which converges to (t,u,v) in $[0,1] \times L^2(\Omega) \times L^2(\Omega)$ when $n \to +\infty$.

We pose for all $n \in \mathbb{N}$ that $A(t_n, u_n, v_n) = (\varphi_n, \phi_n)$ and $A(t,u,v) = (\varphi, \phi)$, we obtain

$$\begin{cases} \text{Find } (\varphi_n, \phi_n) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi_n(x)-\varphi_n(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega t_n f_1(x)\psi_1 dx \\ \qquad\qquad - \int_\Omega t_n g_1(x, u_n(x), v_n(x))\psi_1 dx \ \forall \psi_1 \in D^{s,2}(\Omega), \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\phi_n(x)-\phi_n(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega t_n f_2(x)\psi_2 dx \\ \qquad\qquad - \int_\Omega t_n g_2(x, u_n(x), v_n(x))\psi_2 dx \ \forall \psi_2 \in D^{s,2}(\Omega). \end{cases} \tag{2.9}$$

$$\begin{cases} \text{Find } (\varphi,\phi)\in D^{s,2}(\Omega)\times D^{s,2}(\Omega),\\ C(n,s)\iint_{\mathbb{R}^{2n}}\frac{(\varphi(x)-\varphi(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}}dydx=\int_\Omega tf_1(x)\psi_1dx\\ \qquad\qquad -\int_\Omega tg_1(x,u(x),v(x))\psi_1dx \quad \forall\psi_1\in D^{s,2}(\Omega),\\ C(n,s)\iint_{\mathbb{R}^{2n}}\frac{(\phi(x)-\phi(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx=\int_\Omega tf_2(x)\psi_2dx\\ \qquad\qquad -\int_\Omega tg_2(x,u(x),v(x))\psi_2dx \quad \forall\psi_2\in D^{s,2}(\Omega). \end{cases} \tag{2.10}$$

Making the difference between the two previous systems, we get

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}}\frac{((\varphi_n-\varphi)(x)-(\varphi_n-\varphi)(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}}dydx=(t_n-t)\int_\Omega f_1(x)\psi_1dx\\ \qquad\qquad +\int_\Omega(tg_1(x,u(x),v(x))-t_ng_1(x,u_n(x),v_n(x)))\psi_1dx,\\ C(n,s)\iint_{\mathbb{R}^{2n}}\frac{((\phi_n-\phi)(x)-(\phi_n-\phi)(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx=(t_n-t)\int_\Omega f_2(x)\psi_2dx\\ \qquad\qquad +\int_\Omega(tg_2(x,u(x),v(x))-t_ng_2(x,u_n(x),v_n(x)))\psi_2dx,\\ \forall(\psi_1,\psi_2)\in D^{s,2}(\Omega)\times D^{s,2}(\Omega). \end{cases} \tag{2.11}$$

Let us choose $(\psi_1,\psi_2)=(\varphi_n-\varphi,\phi_n-\phi)$ in the equation of (2.11), apply the Cauchy-Schwarz inequality and Proposition 1.1.8 and obtain

$$\begin{cases} \|\varphi_n(x)-\varphi(x)\|_{L^2(\Omega)}\le\frac{c_{emb}^2}{C(n,s)}[\|tg_1(x,u(x),v(x))-t_ng_1(x,u_n(x),v_n(x))\|_{L^2(\Omega)}\\ \qquad\qquad +|t_n-t|\|f_1\|_{L^2(\Omega)}],\\ \|\phi_n(x)-\phi(x)\|_{L^2(\Omega)}\le\frac{c_{emb}^2}{C(n,s)}[\|tg_2(x,u(x),v(x))-t_ng_2(x,u_n(x),v_n(x))\|_{L^2(\Omega)}\\ \qquad\qquad +|t_n-t|\|f_2\|_{L^2(\Omega)}]. \end{cases}$$

We have that $(t_n)_{n\in\mathbb{N}}$ converges to t and $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ converges to (u,v) in $L^2(\Omega)\times L^2(\Omega)$ when $n\to+\infty$ and from the assumption (h_1), we deduce that $\{g_1(.,u_n,v_n)\}_{n\in\mathbb{N}}$, $\{g_2(.,u_n,v_n)\}_{n\in\mathbb{N}}$ converge respectively to $g_1(.,u,v)$, $g_2(.,u,v)$.

Therefore, $\{(\varphi_n,\phi_n)\}_{n\in\mathbb{N}}$ converges to (φ,ϕ) in $L^2(\Omega)\times L^2(\Omega)$.

We conclude that A is continuous from $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ to $L^2(\Omega)\times L^2(\Omega)$. □

Proof of Theorem 2.3.1

Now, we prove Theorem 2.3.1. We have according to the previous lemmas:

1. There is no solution (u,v) for the equation $A(t,u,v)=(u,v)$ in the boundary of the following ball

   $$B_{R+1}=\{(u,v)\in L^2(\Omega)\times L^2(\Omega);\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)}<R+1\},$$

   and that's for all $t\in[0,1]$.

2. The set $\left\{A(t,u,v), t\in[0,1], (u,v)\in \bar{B}_{R+1}\right\}$ is relatively compact in $L^2(\Omega)\times L^2(\Omega)$.

3. The map A is continuous from $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ to $L^2(\Omega)\times L^2(\Omega)$.

These results allow us to define the Leray-Schauder degree $d(Id - A(t,\cdot,\cdot),$ $B_{R+1}, 0)$.
Next, the homotopy invariance property of the degree implies

$$d(Id - A(t,\cdot,\cdot), B_{R+1}, 0) = d(Id - A(0,\cdot,\cdot), B_{R+1}, 0) = d(Id, B_{R+1}, 0),$$

then, using the normality property of the degree, we obtain

$$d(Id - A(t,\cdot,\cdot), B_{R+1}, 0) = 1 \neq 0, \text{ for all} t \in [0,1].$$

Therefore, there exists $(u,v)\in B_{R+1}$ such that

$$Id(u,v) - A(1,u,v) = 0,$$

equivalently, there exists $(u,v)\in B_{R+1}$

$$A(1,u,v) = (u,v),$$

which proves that the problem (2.1) has at least one weak solution $(u,v)\in D^{s,2}(\Omega)\times D^{s,2}(\Omega)$.

2.4 Uniqueness of solutions

In this section, we prove the uniqueness of solution in a particular case using the absurd reasoning.

We assume that

(h$_3$) The functions g_1, g_2 verify the following assumption. There exists $c_1, c_2 \in \mathbb{R}_+$ such that

$$\begin{cases} (g_1(x,s_1,p_1) - g_1(x,s_2,p_2))(s_2 - s_1) & \leq c_1|s_1 - s_2|^2, \\ \quad \forall s_1, s_2, p_1, p_2 \in \mathbb{R} \text{ and a.e. } x\in\Omega. \\ (g_2(x,s_1,p_1) - g_2(x,s_2,p_2))(p_2 - p_1) & \leq c_2|p_1 - p_2|^2, \\ \quad \forall s_1, s_2, p_1, p_2 \in \mathbb{R} \text{ and a.e. } x\in\Omega. \end{cases}$$

Next, we state the following theorem which give the existence and the uniqueness of solution.

Theorem 2.4.1. *Under the assumptions* (h_1), (h_2) *and* (h_3), *and if we have* $min\{c_1, c_2\} < \frac{C(n,s)}{c_{emb}^2}$, *the problem* (2.1) *has a unique weak solution* $(u, v) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.

Proof. We have already shown the existence of solutions, we pass to the uniqueness.
Let $(u, v), (\tilde{u}, \tilde{v}) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$ two different weak solutions of (2.1), then we obtain

$$\begin{cases} \text{Find } (u, v) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_1(x)\psi_1 dx \\ \qquad\qquad - \int_\Omega tg_1(x, u(x), v(x))\psi_1 dx, \quad \forall \psi_1 \in D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_2(x)\psi_2 dx \\ \qquad\qquad - \int_\Omega tg_2(x, u(x), v(x))\psi_2 dx, \quad \forall \psi_2 \in D^{s,2}(\Omega). \end{cases}$$

and

$$\begin{cases} \text{Find } (\tilde{u}, \tilde{v}) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(\tilde{u}(x)-\tilde{u}(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_1(x)\psi_1 dx \\ \qquad\qquad - \int_\Omega tg_1(x, \tilde{u}(x), \tilde{v}(x))\psi_1 dx, \quad \forall \psi_1 \in D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{(\tilde{v}(x)-\tilde{v}(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega tf_2(x)\psi_2 dx \\ \qquad\qquad - \int_\Omega tg_2(x, \tilde{u}(x), \tilde{v}(x))\psi_2 dx, \quad \forall \psi_2 \in D^{s,2}(\Omega). \end{cases}$$

Making the difference between the two previous systems, we get

$$\begin{cases} C(n, s) \iint_{\mathbb{R}^{2n}} \frac{((u-\tilde{u})(x)-(u-\tilde{u})(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}} dydx \\ = \int_\Omega (g_1(x, \tilde{u}(x), \tilde{v}(x)) - g_1(x, u(x), v(x)))\psi_1 dx, \quad \forall \psi_1 \in D^{s,2}(\Omega), \\ C(n, s) \iint_{\mathbb{R}^{2n}} \frac{((v-\tilde{v})(x)-(v-\tilde{v})(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}} dydx \\ = \int_\Omega (g_2(x, \tilde{u}(x), \tilde{v}(x)) - g_2(x, u(x), v(x)))\psi_2 dx, \quad \forall \psi_2 \in D^{s,2}(\Omega). \end{cases}$$

Let us choose $(\psi_1, \psi_2) = (u - \tilde{u}, v - \tilde{v})$ and get

$$\begin{cases} \|u - \tilde{u}\|^2_{D^{s,2}(\Omega)} \\ \quad = \dfrac{1}{C(n, s)} \displaystyle\int_\Omega (g_1(x, \tilde{u}(x), \tilde{v}(x)) - g_1(x, u(x), v(x)))(u(x) - \tilde{u}(x))dx, \\ \|v - \tilde{v}\|^2_{D^{s,2}(\Omega)} \\ \quad = \dfrac{1}{C(n, s)} \displaystyle\int_\Omega (g_2(x, \tilde{u}(x), \tilde{v}(x)) - g_2(x, u(x), v(x)))(v(x) - \tilde{v}(x))dx, \end{cases}$$

now, we use the assumption (h_3) and the proposition 1.1.8, we obtain

$$\begin{cases} \|u-\tilde{u}\|^2_{D^{s,2}(\Omega)} & \leq \dfrac{1}{C(n,s)}\displaystyle\int_\Omega c_1|u(x)-\tilde{u}(x)|^2dx, \\ \|v-\tilde{v}\|^2_{D^{s,2}(\Omega)} & \leq \dfrac{1}{C(n,s)}\displaystyle\int_\Omega c_2|v(x)-\tilde{v}(x)|^2dx, \end{cases}$$

hence,

$$\begin{cases} \|u-\tilde{u}\|^2_{D^{s,2}(\Omega)} & \leq \dfrac{c_1c^2_{emb}}{C(n,s)}\|u-\tilde{u}\|^2_{D^{s,2}(\Omega)}, \\ \|v-\tilde{v}\|^2_{D^{s,2}(\Omega)} & \leq \dfrac{c_2c^2_{emb}}{C(n,s)}\|v-\tilde{v}\|^2_{D^{s,2}(\Omega)}, \end{cases}$$

next,

$$\begin{cases} 1 & \leq \dfrac{c_1c^2_{emb}}{C(n,s)}, \\ 1 & \leq \dfrac{c_2c^2_{emb}}{C(n,s)}, \end{cases}$$

then,

$$\begin{cases} \frac{C(n,s)}{c^2_{emb}} & \leq c_1, \\ \frac{C(n,s)}{c^2_{emb}} & \leq c_2, \end{cases}$$

It follows from the above inequalities a contradiction with the hypothesis $min\{c_1,c_2\} < \frac{C(n,s)}{c^2_{emb}}$, then we conclude that the problem (2.1) has a unique weak solution.

□

CHAPTER

3

ON A NONLINEAR ELLIPTIC EQUATION INVOLVING THE DISTRIBUTIONAL RIESZ FRACTIONAL DERIVATIVE

In this chapter, we consider a nonlinear elliptic equation involving a nonlocal operator defined using the distributional Riesz derivative that was defined with his important properties in the preliminaries. We show that this problem has at least one weak solution u. In our proof, we use the Leray Schauder degree method to prove the existence of weak solutions and the Banach fixed point theorem to prove the uniqueness of weak solutions in a particular case. This chapter presents some results published in [2].

3.1 Motivation for the problem

In recent years, many researchers have considered problems related to the distributional Riesz fractional derivative, where they defined it and gave its important properties in their papers, we refer to [47, 60, 63, 66, 67, 68]. In our work, we study the existence of solutions in a Bessel Potential space for the previous problem in the nonlinear case.
We studied in this chapter the existence of weak solutions in a Bessel potential space $L_0^{s,2}(\Omega)$ using the Leray-Schauder degree for the following problem that

contains the distributional Riesz gradient.

$$\begin{cases} -D^s \cdot (A(x)D^s u(x)) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases}$$

here $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary, D^s is the distributional Riesz fractional derivative (Definition 1.1.7), $s \in (0,1)$ with $n > 2s$, $A(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a measurable, bounded, positive definite matrix and $f : \Omega \times \mathbb{R} \to \mathbb{R}$.
The weak solutions for the previous problem are the solutions for the following problem.

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^n} A(x)D^s u(x) \cdot D^s v(x)dx = \int_\Omega f(x, u(x))v(x)dx, \ \forall v \in L_0^{s,2}(\Omega), \end{cases}$$

this problem was considered by Shieh and Spector (2015) in the linear case as follows:

$$\begin{cases} \text{Find } u \in H_0^s(\Omega), \\ \int_{\mathbb{R}^n} A(x)D^s u(x) \cdot D^s v(x)dx = \int_\Omega g(x)v(x)dx, \ \forall v \in H_0^s(\Omega). \end{cases}$$

they studied for it the existence and uniqueness of solutions using Lax-Milgram theorem.

Shieh and Spector (2015, 2018) show that the ditributinal Riesz derivative is an interesting object for the study of fractional partial differential equation from its ability to expand many results in the classical Sobolev spaces to the fractional case, see [66, 67].

The distributional Riesz fractional gradient is more fundamental than the fractional Laplacian because of its important property that it is the foundational object in the construction of weak formulations of partial differential equations unlike the fractional Laplacian which does not have this property. Distributional Riesz fractional derivative satisfy basic physical invariance requirements, see the paper of Silhavy (2020) who presented a novel approach to fractional gradient analysis.

As is well known in harmonic analysis, the only higher dimensional operator that is translational and rotational invariant is the Riesz operator. Moreover, the operators that can be used independently of any dimension and coordinate system are only fractional gradients defined with the Riesz operator. The interest of the matrix A is that it introduces anisotropy, which often occurs in the case of physical systems, like crystals or materials. Combining the above benefits, gives that we can apply this to any multidimensional systems which may differ along different vectorial directions, whether it is a thin material that is usually viewed as 2d, or a normal 3d crystalline solid, or a multidimensional Lévy flight. In general, problems similar to our second problem can be used to model the behavior of crystalline solid materials, which may help in the development and creation of new materials for a variety of applications. The function $u(x)$ can describe the displacement of the material at a given point x in the space. The matrix $A(x)$ represents the stiffness or elastic properties of the material at that point, which is dependent on its location. The distributional fractional Riesz operator D^s indicates the non-local behavior of the material, which takes into account its history and memory. The function $f(x, u(x))$ can represent the external forces acting on the material, as well as any non-linear effects caused by material deformation; see for example [37, 68].

We study also in this chapter the uniqueness of solutions for the problem (**??**) in a particular case using the Banach fixed point theorem.

The organization of this chapter is as follows. In section 2, we present some preliminaries and we pose our problem. Section 3 contains the main result of this paper and some auxiliary results with their proof. Section 4 contains the uniqueness result in a particular case.

3.2 Preleminaries and position of the problem

Let Ω be an open subset of $\mathbb{R}^n$, the following nonlocal operator $-D^s \cdot (AD^s(.))$ is defined in the duality sense which is a generalization of the fractional Laplacian for the distributional Riesz derivative, with a measurable, bounded, positive definite matrix $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and for $u, v \in H_0^s(\Omega)$, extended by zero outside Ω :

Definition 3.2.1. *[47]*

$$\langle -D^s \cdot (AD^s u)\,, v\rangle := \int_{\mathbb{R}^n} A(x) D^s u \cdot D^s v dx.$$

The operator $-D^s \cdot (AD^s(.))$ was defined as follows:

$$-D^s \cdot (AD^s(.)) : L_0^{s,2}(\Omega) \longrightarrow \left(L_0^{s,2}(\Omega)\right)'$$

$$u \mapsto -D^s \cdot (AD^s u)\,,$$

when $AD^s u \in \left[L^2(\mathbb{R}^n)\right]^n$.

We study in this chapter the following problem involving the nonlocal operator $-D^s \cdot (AD^s(.))$:

$$\begin{cases} -D^s \cdot (A(x)D^s u(x)) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{3.1}$$

Where $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary, $s \in (0,1)$ with $n > 2s$, $A(x) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a measurable, bounded, positive definite matrix and $f : \Omega \times \mathbb{R} \to \mathbb{R}$.
We will use the following assumptions:

$$\begin{cases} (\mathrm{h}_1)\ A : \Omega \to \mathbb{R}^{n\times n} \text{ with coefficients bounded, measurable and strictly ellipt-} \\ \text{ic, such that } \alpha|z|^2 \leq A(x)z \cdot z \text{ and } A(x)z \cdot z^* \leq \beta|z|\,|z^*| \text{ for some } \alpha, \beta > 0, \\ \text{ for all } x \in \Omega \text{ and all } z, z^* \in \mathbb{R}^n. \\ (\mathrm{h}_2)\ f \text{ satisfies the Caratheodory conditions.} \\ (\mathrm{h}_3)\ \exists C_1 \geq 0 \text{ and } d \in L^2(\Omega) \text{ such that } |f(x,m)| \leq d(x) + C_1|m|,\ \forall m \in \mathbb{R}. \\ (\mathrm{h}_4)\ \lim\limits_{m \to \pm\infty} \frac{f(x,m)}{m} = 0,\ \forall x \in \Omega. \end{cases}$$

3.3 Existence of weak solutions for the problem (3.1)

This section contains the main result of this paper and some auxiliary results with its proof.

First, we give the weak formulation of the problem (3.1):

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^n} A(x)D^s u(x) \cdot D^s v(x)dx = \int_\Omega f(x,u(x))v(x)dx,\ \forall v \in L_0^{s,2}(\Omega). \end{cases} \tag{3.2}$$

It is a consequence of the fractional integration by parts formula relating the distributional Riesz gradient(Proposition 1.1.10), for more details; see [68, 19].
Therefore, a solution of the problem (3.2) is a weak solution of the problem (3.1).
The following theorem gives the existence of weak solutions for the problem (3.1).

Theorem 3.3.1. *Under the assumptions* $(h_1), (h_2), (h_3)$ *and* (h_4)*, the problem* (3.1) *has at least one weak solution* $u \in L_0^{s,2}(\Omega)$*.*

New formulation of the problem (3.2)

In this subsection, we give a new problem equivalent to the problem (3.2). First, we define the following operator noted T by

$$T : [0,1] \times L^2(\Omega) \longrightarrow L_0^{s,2}(\Omega)$$

$$(t,u) \mapsto T(t,u) = w,$$

where w is a solution to the following linear problem.

$$\begin{cases} \text{Find } w \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^n} A(x)\, D^s w(x) \cdot D^s v(x) dx = \\ \qquad\qquad t \int_\Omega f(x,u(x)) v(x) dx, \ \forall v \in L_0^{s,2}(\Omega). \end{cases} \tag{3.3}$$

Lemma 3.3.1. *For $u \in L^2(\Omega)$, $t \in [0,1]$ and using the assumptions (h_1) and (h_3), the map T associates each $(t,u) \in [0,1] \times L^2(\Omega)$ with a unique function $w \in L_0^{s,2}(\Omega)$.*

Proof. To prove that the map T associates each $(t,u) \in [0,1] \times L^2(\Omega)$ with a unique function $w \in L_0^{s,2}(\Omega)$, we prove the existence and uniqueness of solution $w \in L_0^{s,2}(\Omega)$ for the problem (3.3) using the Lax-Milgram theorem. We have

$$\begin{cases} \text{Find } w \in L_0^{s,2}(\Omega), \\ a(w,v) = l(v), \forall v \in L_0^{s,2}(\Omega), \end{cases} \tag{3.4}$$

with

$$a(.,.) : L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega) \longrightarrow \mathbb{R}$$

$$(w,v) \mapsto a(w,v) = \int_{\mathbb{R}^n} A(x)\, D^s w(x) \cdot D^s v(x) dx,$$

$$l : L_0^{s,2}(\Omega) \longrightarrow \mathbb{R}$$

$$v \mapsto l(v) = t \int_\Omega f(x,u(x)) v(x) dx.$$

Let us prove that the bilinear form $a(.,.)$ is continuous and coercive.

Using the assumption (h_1) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |a(w,v)| &= \left| \int_{\mathbb{R}^n} A(x)\, D^s w(x) \cdot D^s v(x) dx \right| \\ &\leq \beta \int_{\mathbb{R}^n} |D^s w(x)|\, |D^s v(x)|\, dx \\ &\leq \beta \|D^s w\|_{L^2(\mathbb{R}^n)} \|D^s v\|_{L^2(\mathbb{R}^n)} \\ &\leq \beta \|w\|_{L_0^{s,2}(\Omega)} \|v\|_{L_0^{s,2}(\Omega)}, \end{aligned}$$

therefore, the bilinear form $a(.,.)$ is continuous.

We have

$$a(v,v) = \int_{\mathbb{R}^n} a(x,u(x))\,|D^s v|^2\,dx$$

Applying the assumption (h_1), we get

$$a(v,v) \geq \alpha \int_{\mathbb{R}^n} |D^s v|^2\,dx = \alpha \|D^s v\|^2_{L^2(\mathbb{R}^n)} = \alpha \|v\|^2_{L^{s,2}_0(\Omega)},$$

which gives that the bilinear form $a(.,.)$ is coercive.

In addition, we prove that the linear form l is continuous.

Using the Cauchy-Schwarz inequality, the assumption (h_3) and the proposition 1.1.11, we obtain

$$\begin{aligned}|l(v)| &= \left| t\int_\Omega f(x,u(x))v(x)dx \right| \\ &\leq \int_\Omega (d(x) + C_1\,|u(x)|)\,|v(x)|\,dx \\ &\leq \|d\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + C_1\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq (C_{emb}\|d\|_{L^2(\Omega)} + C_{emb}C_1\|u\|_{L^2(\Omega)})\|v\|_{L^{s,2}_0(\Omega)},\end{aligned}$$

hence, the linear form l is continuous.

Therefore, the problem (3.3) has a unique weak solution $w \in L^{s,2}_0(\Omega)$. We conclude that the map T associates each $(t,u) \in [0,1] \times L^2(\Omega)$ with a unique function $w \in L^{s,2}_0(\Omega)$. □

u is a solution for the problem (3.2) if and only if u is a solution to the following problem

$$\begin{cases} \text{Find } u \in L^2(\Omega), \\ u = T(1,u). \end{cases} \tag{3.5}$$

We will prove using the Leray-Schauder degree that the problem (3.5) has at least one solution.

Priori Estimate, Compactness and Continuity Results

This subsection is devoted to present some auxiliary results about the conditions of the Leray-Schauder degree method.

Lemma 3.3.2 (Priori estimate)**.** *Under the assumptions* (h_1)*,* (h_3) *and* (h_4)*, there exists* $R>0$*,* $\forall u \in L^2(\Omega)$ *such that*

$$\left.\begin{cases} T(t,u)=u \\ t\in[0,1], u\in L^2(\Omega) \end{cases}\right\} \Rightarrow \|u\|_{L^2(\Omega)} < R.$$

Proof. Let $t\in[0,1]$ and $T(t,u)=w=u$, which means that

$$\begin{cases}\text{Find } u\in L_0^{s,2}(\Omega), \\ \int_{\mathbb{R}^n} A(x)D^s u(x)\cdot D^s v(x)dx = t\int_\Omega f(x,u(x))v(x)dx,\ \forall v\in L_0^{s,2}(\Omega).\end{cases}$$

Let us choose $v=u$ in this equation and using the assumption (h_1), we obtain

$$\alpha\|u\|^2_{L_0^{s,2}(\Omega)} \leq \int_\Omega |f(x,u(x))u(x)|dx,$$

we will deduce from this inequality that there exists $R>0$ such that $\|u\|_{L^2(\Omega)}<R$.
Let us assume that this R don't exist, so there exists a sequence $(u_n)_{n\in\mathbb{N}}\subset L_0^{s,2}(\Omega)$ such that

$$\|u_n\|_{L^2(\Omega)} \geq n \text{ and } \alpha\left\|u_n\right\|^2_{L_0^{s,2}(\Omega)} \leq \int_\Omega \left|f\left(x,u_n(x)\right)u_n(x)\right|\mathrm{d}x,$$

and we will arrive to a contradiction.

We pose $v_n=\frac{u_n}{\|u_n\|_{L^2(\Omega)}}$. So, we have $\|v_n\|_{L^2(\Omega)}=1$ and

$$\alpha\left\|v_n\right\|^2_{L_0^{s,2}(\Omega)} \leq \int_\Omega \left|\frac{f\left(x,u_n\right)}{\left\|u_n\right\|_{L^2(\Omega)}}v_n\right|\mathrm{d}x,$$

in addition, $|f(x,s)|\leq|d|+C_1|s|$, it gives

$$\begin{aligned}\alpha\left\|v_n\right\|^2_{L_0^{s,2}(\Omega)} &\leq \int_\Omega \frac{|d|+C_1\left|u_n\right|}{\left\|u_n\right\|_{L^2(\Omega)}}\left|v_n\right|\mathrm{d}x \\ &\leq \int_\Omega \frac{|d|\left|v_n\right|}{\left\|u_n\right\|_{L^2(\Omega)}}\ \mathrm{d}x + C_1\int_\Omega \left|v_n\right|^2\ \mathrm{d}x \\ &\leq \|d\|_{L^2(\Omega)}+C_1.\end{aligned}$$

Therefore, $(v_n)_{n\in\mathbb{N}}$ is bounded in $L_0^{s,2}(\Omega)$, then it has a subsequence,

$$\lim_{n\to+\infty} v_n = v \text{ in } L^2(\Omega).$$

Consequently $\|v\|_{L^2} = 1$ (which implies that $v \neq 0$). We have also by Theorem 1.1.3

$$\lim_{n\to+\infty} v_n = v \text{ a.e.},$$
$$|v_n| \leq H \quad \text{with} \quad H \in L^2(\Omega).$$

Finally, using Proposition 1.1.11, we get

$$\frac{\alpha}{C_{emb}^2} = \frac{\alpha}{C_{emb}^2} \|v_n\|^2_{L^2(\Omega)} \leq \alpha \|v_n\|^2_{L_0^{s,2}(\Omega)} \leq \int_\Omega \frac{|f(x,u_n)|}{\|u_n\|_{L^2(\Omega)}} |v_n| \, \mathrm{d}x,$$

we pose

$$X_n = \int_\Omega \frac{|f(x,u_n)\|v_n|}{\|u_n\|_{L^2(\Omega)}} \mathrm{d}x,$$

and we prove that $\lim_{n\to+\infty} X_n = 0$, which is impossible because X_n is minored by the constant $\frac{\alpha}{C_{emb}^2}$ which is positive.
Let us prove that

$$\lim_{n\to+\infty} \frac{|f(x,u_n)| \, |v_n|}{\|u_n\|_{L^2(\Omega)}} = 0 \text{ a.e. with domination } \Big(\text{in } L^1(\Omega)\Big),$$

we will obtain using the dominated convergence theorem that $\lim_{n\to+\infty} X_n = 0$.
First, we prove the domination

$$\frac{|f(x,u_n)|}{\|u_n\|_{L^2(\Omega)}} \leq \frac{|d| + C_1 |u_n|}{\|u_n\|_{L^2}} \leq |d| + C_1 |v_n| \leq |d| + C_1 H.$$

So

$$\left| \frac{f(x,u_n)}{\|u_n\|_{L^2(\Omega)}} v_n \right| \leq (|d| + C_1 H) \, H \in L^1(\Omega).$$

Now, we prove the convergence almost everywhere.
We have $v_n \to v$ a.e. so $\exists F$ such as $\text{meas}(F^c) = 0$ and

$$\lim_{n\to+\infty} v_n(x) = v(x) \; \forall x \in F.$$

First case: if $v(x) > 0$; $\lim_{n\to+\infty} v_n(x) = v(x)$ and $\lim_{n\to+\infty} \|u_n\|_{L^2(\Omega)} = +\infty$, then

$$\lim_{n\to+\infty} u_n(x) = \lim_{n\to+\infty} v_n(x) \|u_n\|_{L^2(\Omega)} = +\infty.$$

$$\frac{f(x,u_n(x))}{\|u_n\|_{L^2(\Omega}} v_n(x) = \frac{f(x,u_n(x)) \, u_n(x)}{u_n(x) \, \|u_n\|_{L^2(\Omega)}} v_n(x) = \frac{f(x,u_n(x))}{u_n(x)} (v_n(x))^2,$$

$$\lim_{n\to+\infty} \frac{f(x,u_n(x))}{u_n(x)} (v_n(x))^2 = 0.$$

We have used here the assumption (h$_4$); $\lim_{s\to+\infty} \frac{f(x,s)}{s} = 0$.

Second case: if $v(x) < 0$; we have the same $\lim_{n\to+\infty} \frac{f(x,u_n(x))}{\|u_n\|_{L^2(\Omega)}} v_n(x) = 0$, because $\lim_{s\to-\infty} \frac{f(s)}{s} = 0$.

Third case: if $v(x) = 0$, we obtain

$$\begin{aligned}\left|\frac{f(x,u_n(x))}{\|u_n\|_{L^2(\Omega)}} v_n(x)\right| &\leq \frac{|d(x)| + C_1|u_n(x)|}{\|u_n\|_{L^2(\Omega)}} |v_n(x)| \\ &\leq (|d(x)| + C_1|v_n(x)|)\,|v_n(x)|,\end{aligned}$$

hence

$$\lim_{n\to+\infty} \left|\frac{f(x,u_n(x))}{\|u_n\|_{L^2(\Omega)}} v_n(x)\right| = 0.$$

We conclude from the three cases that

$$\lim_{n\to+\infty} \frac{f(x,u_n)}{\|u_n\|_{L^2(\Omega)}} v_n = 0 \text{ a.e.}$$

Consequently, we have proved that $\lim_{n\to+\infty} X_n = 0$, but we have $X_n \geq \frac{\alpha}{C_{emb}^2}$ for all $n \in \mathbb{N}^\star$, which is a contradiction. □

We deduce that there isn't any solution u for the equation $T(t,u) = u$ in the boundary of $B_R = \{u \in L^2(\Omega); \|u\|_{L^2(\Omega)} < R\}$ and that's for all $t \in [0,1]$.

Lemma 3.3.3. *Under the assumptions* (h_1) *and* (h_3)*, the following set*

$$\left\{T(t,u), t \in [0,1], u \in \bar{B}_{R+1}\right\}$$

is relatively compact in $L^2(\Omega)$*.*

Proof. Let $R > 0$, We suppose that $\|u\|_{L^2(\Omega)} \leq R$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} A(x)D^s w(x)\cdot D^s v(x)dx &= t\int_\Omega f(x,u(x))v dx \\ &\leq \|d\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + C_1\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq (C_{emb}\|d\|_{L^2(\Omega)} + C_{emb}C_1R)\|v\|_{L_0^{s,2}(\Omega)}.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^n} A(x)D^s w(x)\cdot D^s v(x)dx \leq \bar{R}\|v\|_{L_0^{s,2}(\Omega)} \forall v \in L_0^{s,2}(\Omega), \tag{3.6}$$

where, $\bar{R} = C_{emb}\|d\|_{L^2(\Omega)} + C_{emb}C_1 R$.
Let us take $v = w$ in the inequality (3.6) and using the assumption (h_1), we conclude that

$$\|w\|_{L_0^{s,2}(\Omega)} \leq \frac{\bar{R}}{\alpha} = \tilde{R},$$

then,

$$\|T(t,u)\|_{L_0^{s,2}(\Omega)} \leq \tilde{R}.$$

Therefore, the set $\left\{T(t,u), t \in [0,1], u \in \bar{B}_R\right\}$ is bounded in $L_0^{s,2}(\Omega)$ and from the proposition 1.1.13, $L_0^{s,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we deduce that $\left\{T(t,u), t \in [0,1], u \in \bar{B}_R\right\}$ is relatively compact in $L^2(\Omega)$. □

Lemma 3.3.4. *Under the assumptions* (h_1), (h_2) *and* (h_3), T *is continuous from* $[0,1] \times L^2(\Omega)$ *into* $L^2(\Omega)$.

Proof. Let $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ such as $t_n \to t$ when $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ such as $u_n \to u$ in$L^2(\Omega)$ when $n \to +\infty$.

We want to prove that $A(t_n, u_n)$ converges to $A(t,u)$ in $L^2(\Omega)$.

We pose $w_n = A(t_n, u_n)$ and $w_= A(t,u)$.

For proving that w_n converges to w in $L^2(\Omega)$, we need to cross the limit in the following equation

$$\begin{cases} \text{Find } w_n \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^n} A(x)\, D^s w_n(x) \cdot D^s v(x)dx = t_n \int_\Omega f(x, u_n(x))v(x)dx, \ \forall v \in L_0^{s,2}(\Omega). \end{cases}$$

$(w_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L_0^{s,2}(\Omega)$, therefore $(w_n)_{n\in\mathbb{N}}$ has a subsequence converge weakly to $\bar{w}$ in $L_0^{s,2}(\Omega)$ and strongly in $L^2(\Omega)$.
We have also $u_n \to u$ a.e. and $\exists M \in L^2(\mathbb{R}^n) : |u_n| \leq M$ a.e.
Now, let us prove that $D^s w_n \to D^s\bar{w}$ weakly in $(L^2(\mathbb{R}^n))^n$ in the sense of distribution.

$$\begin{aligned} \lim_{n\to+\infty} \left\langle \frac{\partial^s w_n}{\partial x_j^s}, v \right\rangle &= \lim_{n\to+\infty} \int_{\mathbb{R}^n} I_{1-s} * w_n \frac{\partial v}{\partial x_j} dx \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^n} w_n I_{1-s} * \frac{\partial v}{\partial x_j} dx \\ &= \int_{\mathbb{R}^n} \bar{w} I_{1-s} * \frac{\partial v}{\partial x_j} dx \\ &= \int_{\mathbb{R}^n} I_{1-s} * \bar{w} \frac{\partial v}{\partial x_j} dx \\ &= \left\langle \frac{\partial^s \bar{w}}{\partial x_j^s}, v \right\rangle. \end{aligned}$$

We conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^n} AD^s w_n \cdot D^s v \, dx = \int_{\mathbb{R}^n} AD^s \bar{w} \cdot D^s v \, dx.$$

For the last term of the equation of the problem, we get using the assumption (h_2), $\lim_{n\to+\infty} f(x,u_n) = f(x,u)$ a.e., in addition using the assumption (h_3) and the Dominate Convergence theorem, we obtain

$$\lim_{n\to+\infty} f(x,u_n) = f(x,u) \text{ in } L^2(\Omega),$$

and

$$\lim_{n\to+\infty}\int_{\Omega} f(x,u_n) v dx = \int_{\Omega} f(x,u) v dx.$$

By crossing the limit in the equation of the problem, we obtain that

$$\int_{\mathbb{R}^n} A(x) D^s \bar{w}(x) \cdot D^s v(x) dx = t \int_{\Omega} f(x,u(x)) v dx.$$

Therefore,

$$\bar{w} = T(t,u) = w.$$

□

Proof of Theorem 3.3.1

Now, we can prove Theorem 3.3.1 by applying the Leray-Schauder degree method.

Proof. We have according to Lemma 3.3.2 that there is no solution of the equation $Id(u) - T(t,u) = 0$ in the boundary of the ball B_R and Lemma 3.3.3 gives us that $\left\{T(t,u), t \in [0,1], u \in \bar{B}_R\right\}$ is relatively compact in $L^2(\Omega)$. Furthermore, according to Lemma 3.3.4, the homotopy T is continuous from $[0,1] \times L^2(\Omega)$ into $L^2(\Omega)$. Consequently, we can define the degree $d(Id - T(t,\cdot), B_R, 0)$ and with the homotopy invariance property we have

$$d(Id - T(t,\cdot), B_R, 0) = d(Id - T(0,\cdot), B_R, 0) = d(Id, B_R, 0) = 1 \neq 0,$$

for all $t \in [0,1]$.
Therefore, there exists $u \in B_R$ such that

$$Id(u) - T(1,u) = 0,$$

which is equivalent to

$$T(1,u) = u,$$

and proves that the problem (3.1) has at least one weak solution $u \in L_0^{s,2}(\Omega)$.
□

3.4 Uniqueness of solutions

In this section, we prove the uniqueness of weak solution $u \in L_0^{s,2}(\Omega)$ of the problem (3.1) in a particular case where the function f satisfies the following assumption
(h_5) $\exists C > 0,$ for a.e.$x \in \Omega$ and $\forall p_1, p_2 \in \mathbb{R}$

$$\left\| f\left(x, p_1\right) - f\left(x, p_2\right) \right\|_{L^2(\Omega)} \leq C \left\| p_1 - p_2 \right\|_{L^2(\Omega)}.$$

We define the following operator

$$B : L^2(\Omega) \longrightarrow L_0^{s,2}(\Omega)$$

$$u \mapsto B(u) = w,$$

where w is a solution to the problem (3.3) (for $t = 1$).

Lemma 3.4.1. *Under the assumptions* (h_1) *and* (h_5) *and for* $CC_{emb}^2 < \alpha$, *the operator* B *admits a unique fixed point.*

Proof. We use in this proof an application of the Banach fixed point theorem, so we need to prove that B is a contraction.
For all $u_1, u_2 \in L^2(\Omega)$, we pose $B(u_1) = w_1$ and $B(u_2) = w_2$, then we obtain the two following problems

$$\begin{cases} \text{Find } w_1 \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^n} A\left(x\right) D^s w_1(x) \cdot D^s v(x) dx = \int_\Omega f(x, u_1(x)) v(x) dx, \ \forall v \in L_0^{s,2}(\Omega). \end{cases}$$

$$\begin{cases} \text{Find } w_2 \in L_0^{s,2}(\Omega), \\ \displaystyle\int_{\mathbb{R}^n} A\left(x\right) D^s w_2(x) \cdot D^s v(x) dx = \int_\Omega f(x, u_2(x)) v(x) dx, \ \forall v \in L_0^{s,2}(\Omega). \end{cases}$$

We make the difference between the two previous equations, take $v = w_1 - w_2$, apply Cauchy-Schwarz, use the assumption (h_1) and obtain

$$\left\| w_1 - w_2 \right\|_{L_0^{s,2}(\Omega)}^2 \leqslant \frac{1}{\alpha} \left\| f\left(x, u_1\right) - f\left(x, u_2\right) \right\|_{L^2(\Omega)} \left\| w_1 - w_2 \right\|_{L^2(\Omega)}.$$

Using Proposition 1.1.11 and the assumption (h_5), we get

$$\left\| w_1 - w_2 \right\|_{L^2(\Omega)} \leqslant \frac{CC_{emb}^2}{\alpha} \left\| u_1 - u_2 \right\|_{L^2(\Omega)}. \tag{3.7}$$

Consequently, for $CC_{emb}^2 < \alpha$, the operator B is a contraction. Therefore, B admits a unique fixed point, and we conclude that the problem (3.1) has a unique weak solution. □

CHAPTER

4

STUDY OF A NONLOCAL FRACTIONAL ELLIPTIC PROBLEM WITH NONLINEARITIES

In this chapter, we study the existence of weak solutions in the fractional space $L_0^{s,2}$ for a class of nonlocal nonlinear fractional elliptic problems with Dirichlet boundary condition. The proof of the existence of weak solutions in $L_0^{s,2}$ for our problem is based on the application of the Shauder fixed point theorem. This chapter presents some results published in [3].

4.1 Motivation for the problem

We are interested in this chapter in the existence of weak solutions for the following nonlocal problem

$$\begin{cases} -M(\int_\Omega u(x)dx)D^s \cdot (D^s u(x)) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases}$$

Where Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $s \in (0, 1)$ with $n > 2s$, $M : \mathbb{R} \to \mathbb{R}^+$ is a continuous function and the nonlinear function $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

This problem is the fractional version of the following problem, which was studied by Chipot and Lovat; see [18].

$$\begin{cases} -a\left(\int_\Omega u(x)dx\right)\Delta u = f \quad \text{in} \quad \Omega \\ u(x) = 0 \quad \text{on} \quad \Gamma \end{cases}$$

Here Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 1$ with smooth boudary Γ, a is some function from $\mathbb{R}$ into $(0, +\infty)$.
The following problem is the parabolic version of the previous problem, which was studied by the same authors

$$\begin{cases} u_t - a\left(\int_\Omega u(x,t)dx\right)\Delta u = f \quad \text{in} \quad \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ u(x,0) = u_0(x) \quad \text{on} \quad \Omega. \end{cases}$$

T represents an arbitrary time. We can use u the solution of this problem to describe the density of bacteria, or in general, a population susceptible to spreading. a is the coefficient of diffusion, which depends on the entire population in the domain rather than the local density; the global of medium state guides those moves.

Many researchers have recently become interested in nonlinear nonlocal fractional elliptic problems, such as the work of Xiang M et al. (2015) used the Fountain theorem and the dual Fountain theorem to study a nonlocal nonlinear fractional elliptic problem of Kirchhoff type, and the work of Fiscella A. et al. (2019) provided the existence of two solutions for a nonlocal fractional elliptic problem with critical nonlinearities using variational methods.

The significance of our problem stems from its ability to generalize many results from classical Sobolev spaces to the fractional case.

The organization of this chapter is as follows. In section 2, we present our problem and his weak formulation. Section 3 contains the existence results using the Schauder fixed point theorem.

4.2 Mathematical formulation for the problem

We consider the following nonlinear fractional elliptic problem

$$\begin{cases} -M\left(\int_\Omega u(x)dx\right)D^s \cdot (D^s u(x)) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{4.1}$$

Where Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $s \in (0,1)$ with $n > 2s$, $M : \mathbb{R} \to \mathbb{R}^+$ is a continuous function and the nonlinear function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, satisfy the following hypothesis:

$$
\begin{cases}
(\mathrm{h}_1)\ \exists\alpha,\beta>0:\ \alpha\le M(t)\le\beta,\ \forall t\in\mathbb{R}.\\
(\mathrm{h}_2)\ f \text{ satisfies the Caratheodory conditions.}\\
(\mathrm{h}_3)\ \text{Growth condition}\\
\qquad \exists C_1\ge 0 \text{ and } d\in L^2(\Omega) \text{ such that } |f(x,s)|\le d(x)+C_1|s|^\gamma,\ \forall s\in\mathbb{R},\\
\text{where } \gamma\in(0,1).
\end{cases}
$$

Next, using the fractional integration par parts formula relating the distributional Riesz fractional gradient, the problem (4.1) is weakly formulated as follows:

$$
\begin{cases}
\text{Find } u\in L_0^{s,2}(\Omega),\\
M\left(\int_\Omega u(x)dx\right)\int_{\mathbb{R}^n} D^s u(x)\cdot D^s v(x)dx=\int_\Omega f(x,u(x))v(x)dx,\ \forall v\in L_0^{s,2}(\Omega),
\end{cases}
\tag{4.2}
$$

we refer to the papers [68, 19].

4.3 Existence results using the Schauder fixed point theorem

This section is devoited to study the existence of weak solutions for the problem (4.1) using an application of Schauder fixed point theorem.
First of all, we start this section by giving the theorem of existence of weak solutions

Theorem 4.3.1. *Under the assumptions* $(h_1),(h_2)$ *and* (h_3)*, the problem* (4.1) *has at least one weak solution* $u\in L_0^{s,2}(\Omega)$*.*

This study is divided into three main steps:

Step 1

In this step, we are interested in the following problem in order to formulate a fixed point problem equivalent to the problem (4.1) using a certain operator.
For $\bar{u}\in L_0^{s,2}(\Omega)$, we have the following linear weak problem:

$$
\begin{cases}
\text{Find } u\in L_0^{s,2}(\Omega),\\
M\left(\int_\Omega \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s u(x)\cdot D^s v(x)dx\\
\qquad\qquad =\int_\Omega f(x,\bar{u}(x))v(x)dx,\ \forall v\in L_0^{s,2}(\Omega).
\end{cases}
\tag{4.3}
$$

We will prove the existence and uniqueness of solutions for the previous problem, as well as define the following operator:

$$T : L^2(\Omega) \longrightarrow L_0^{s,2}(\Omega)$$

$$\bar{u} \mapsto T(\bar{u}) = u,$$

where u is a solution to the linear problem (4.3).

Lemma 4.3.1. *For $\bar{u} \in L^2(\Omega)$and using the hypothesis (h_1) and (h_3), the problem (4.3) has a unique solution $u \in L_0^{s,2}(\Omega)$.*

Proof. To prove the existence and uniqueness of solution $u \in L_0^{s,2}(\Omega)$, we will use the Lax-Milgram theorem. We have

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ a(u,v) = l(v), \forall v \in L_0^{s,2}(\Omega). \end{cases} \tag{4.4}$$

With

$$a(.,.) : L_0^{s,2}(\Omega) \times L_0^{s,2}(\Omega) \longrightarrow \mathbb{R}$$

$$(u,v) \mapsto a(u,v) = M\left(\int_\Omega \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s u(x) \cdot D^s v(x)dx,$$

$$l : L_0^{s,2}(\Omega) \longrightarrow \mathbb{R}$$

$$v \mapsto l(v) = \int_\Omega f(x,\bar{u}(x))v(x)dx.$$

Now we prove that the bilinear form $a(.,.)$ is continuous and coercive. Moreover, we prove that the linear form l is continuous.

The continuity of $a(.,.)$

Using the assumption (h_1) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |a(u,v)| &= \left| M\left(\int_\Omega \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s u(x) \cdot D^s v(x)dx \right| \\ &\le \beta \int_{\mathbb{R}^n} |D^s u(x)|\,|D^s v(x)|\,dx \\ &\le \beta \|D^s u\|_{L^2(\mathbb{R}^n)} \|D^s v\|_{L^2(\mathbb{R}^n)} \\ &\le \beta \|u\|_{L_0^{s,2}(\Omega)} \|v\|_{L_0^{s,2}(\Omega)}, \end{aligned}$$

therefore, the bilinear form $a(.,.)$ is continuous.

The coercivity of $a(.,.)$

Applying the assumption (h_1), we get

$$\begin{aligned} a(v,v) &= M\left(\int_\Omega \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s v(x) \cdot D^s v(x)dx \\ &\ge \alpha \int_{\mathbb{R}^n} |D^s v|^2\,dx = \alpha \|D^s v\|^2_{L^2(\mathbb{R}^n)} = \alpha \|v\|^2_{L_0^{s,2}(\Omega)}, \end{aligned}$$

which gives that the bilinear form $a(.,.)$ is coercive.

The continuity of $l(.)$

Using the Cauchy-Schwarz, Minkowski and Holder inequalities, the assumption (h_3) and Proposition 1.1.11, we obtain

$$\begin{aligned}
|l(v)| &= \left|\int_\Omega f(x,\bar{u}(x))v(x)dx\right| \\
&\leq \left(\int_\Omega |d(x)+C_1|\bar{u}(x)|^\gamma|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\leq \left[\left(\int_\Omega |d(x)|^2\,dx\right)^{\frac{1}{2}} + C_1\left(\int_\Omega |\bar{u}(x)|^{2\gamma}\,dx\right)^{\frac{1}{2}}\right]\|v\|_{L^2(\Omega)} \\
&\leq \left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}\|\bar{u}\|^\gamma_{L^2(\Omega)}\right]\|v\|_{L^2(\Omega)} \\
&\leq C_{emb}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}\|\bar{u}\|^\gamma_{L^2(\Omega)}\right]\|v\|_{L_0^{s,2}(\Omega)},
\end{aligned}$$

hence, the linear form l is continuous.

We conclude that the problem (4.3) has a unique solution $u \in L_0^{s,2}(\Omega)$. □

Remark 4.3.1. *The problem* (4.1) *has at least one weak solution $u \in L_0^{s,2}(\Omega)$ if and only if the following fixed point problem has at least one solution*

$$\begin{cases} \text{Find } u \in L^2(\Omega), \\ u = T(u). \end{cases} \tag{4.5}$$

Later, we prove The existence of solutions for this fixed point problem using the Schauder fixed point theorem.

Step 2

This step is devoted to give some auxiliary results which are the conditions of the Schauder fixed point theorem.

Lemma 4.3.2. *Under the hypotesis (h_1) and (h_3), the operator T maps the ball $\bar{B}_R = \{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq R\}$ into itself.*

Proof . *For each $\bar{u} \in \bar{B}_R$, we will prove that $T(\bar{u}) \in \bar{B}_R$.*

Noting $u = T(\bar{u})$, we obtain

$$\begin{cases} \text{Find } u \in L_0^{s,2}(\Omega), \\ M\left(\int_\Omega \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s u(x)\cdot D^s v(x)dx \\ \qquad\qquad = \int_\Omega f(x,\bar{u}(x))v(x)dx, \ \forall v \in L_0^{s,2}(\Omega). \end{cases}$$

Let us choose $v = u$ in the equation of the previous problem, we get

$$M\left(\int_\Omega \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s u(x)\cdot D^s u(x)dx = \int_\Omega f(x,\bar{u}(x))u(x)dx,$$

using Cauchy-Schwarz inequality and the hypothesis (h_1) and (h_3),

$$\alpha\|u\|^2_{L_0^{s,2}(\Omega)} \leq \left(\int_\Omega \left|d(x)+C_1\left|\bar{u}(x)\right|^\gamma\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |u(x)|^2\,dx\right)^{\frac{1}{2}},$$

applying the Minkowski inequality, we get

$$\alpha\|u\|^2_{L_0^{s,2}(\Omega)} \leq \left[\left(\int_\Omega |d(x)|^2\,dx\right)^{\frac{1}{2}} + C_1\left(\int_\Omega |\bar{u}(x)|^{2\gamma}\,dx\right)^{\frac{1}{2}}\right]\|u\|_{L^2(\Omega)},$$

using Holder inequatlity, we obtain

$$\alpha\|u\|^2_{L_0^{s,2}(\Omega)} \leq \left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}\|\bar{u}\|^\gamma_{L^2(\Omega)}\right]\|u\|_{L^2(\Omega)},$$

next, using the proposition 2.3, we get

$$\|u\|_{L^2(\Omega)} \leq \frac{C^2_{emb}}{\alpha}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}\|\bar{u}\|^\gamma_{L^2(\Omega)}\right],$$

then, we have

$$\|u\|_{L^2(\Omega)} \leq \frac{C^2_{emb}}{\alpha}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}R^\gamma\right],$$

here, we used the assumption that $\bar{u}\in\bar{B}_R$.
Now, we choose R large enough such that

$$\frac{C_{emb}}{\alpha}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}R^\gamma_{L^2(\Omega)}\right] \leq R,$$

hence

$$\|T(\bar{u})\|_{L^2(\Omega)} \leq R.$$

Lemma 4.3.3. *Under the assumptions (h_1) and (h_3), $\left\{T(w), w\in\bar{B}_R\right\}$ is relatively compact in $L^2(\Omega)$.*

Proof . *Let $\{\bar{u}_n\}_{n\in\mathbb{N}}\subset\bar{B}_R$. Setting that $T(\bar{u}_n)=u_n$, we get*

$$\begin{cases} \text{Find } u_n \in L_0^{s,2}(\Omega), \\ M\left(\int_\Omega \bar{u}_n(x)dx\right)\int_{\mathbb{R}^n} D^s u_n(x)\cdot D^s v(x)dx \\ \qquad\qquad = \int_\Omega f(x,\bar{u}_n(x))v(x)dx,\ \forall v\in L_0^{s,2}(\Omega). \end{cases}$$

Let us choose $v = u_n$ in the equation of the previous problem, we get

$$M\left(\int_\Omega \bar{u}_n(x)dx\right)\int_{\mathbb{R}^n} D^s u_n(x)\cdot D^s u_n(x)dx = \int_\Omega f(x,\bar{u}_n(x))u_n(x)dx,$$

using Cauchy-Schwarz, Minkowski and Holder inequatlities and the hypothesis (h_1) and (h_3), we obtain

$$\alpha\|u_n\|^2_{L_0^{s,2}(\Omega)} \le C_{emb}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}\|\bar{u}_n\|^\gamma_{L^2(\Omega)}\right]\|u_n\|_{L_0^{s,2}(\Omega)},$$

then

$$\|u_n\|_{L_0^{s,2}(\Omega)} \le \frac{C_{emb}}{\alpha}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}R^\gamma\right].$$

Consequently,

$$\|u_n\|_{L_0^{s,2}(\Omega)} \le \bar{R},$$

with

$$\bar{R} = \frac{C_{emb}}{\alpha}\left[\|d\|_{L^2(\Omega)} + C_1(meas(\Omega))^{\frac{1-\gamma}{2}}R^\gamma\right],$$

therefore, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L_0^{s,2}(\Omega)$, and from the proposition 1.1.13 , $L_0^{s,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we deduce that there is a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ which converges to u strongly in $L^2(\Omega)$. We conclude that $\left\{T(w), w\in\bar{B}_R\right\}$ is relatively compact in $L^2(\Omega)$.

Lemma 4.3.4. *Assume that the assumptions (h_1), (h_2) and (h_3) are satisfied, the operator T is continuous from $L^2(\Omega)$ into $L^2(\Omega)$.*

Proof . *Let $\{\bar{u}_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ such that $\lim_{n\to+\infty}\bar{u}_n = \bar{u}$ in $L^2(\Omega)$. Next, we are going to prove that $T(\bar{u}_n)$ converges to $T(\bar{u})$ in $L^2(\Omega)$.*

We pose $u_n = T(\bar{u}_n)$ and $u = T(\bar{u})$.

For proving that $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $L^2(\Omega)$, we need to cross the limit in the following equation:

$$\begin{cases} \text{Find } u_n \in L_0^{s,2}(\Omega), \\ M\left(\int_\Omega \bar{u}_n(x)dx\right)\int_{\mathbb{R}^n} D^s u_n(x)\cdot D^s v(x)dx \\ \qquad\qquad = \int_\Omega f(x,\bar{u}_n(x))v(x)dx,\ \forall v\in L_0^{s,2}(\Omega). \end{cases}$$

$(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L_0^{s,2}(\Omega)$, therefore $(u_n)_{n\in\mathbb{N}}$ has a subsequence converges weakly to w in $L_0^{s,2}(\Omega)$ and strongly in $L^2(\Omega)$.

For the first term of the equation of the previous problem, we have

- $\lim_{n\to+\infty} \bar{u}_n = \bar{u}$ *in* $L^2(\Omega)$, *and using the continuity of the function* M, *we arrive at*

$$\lim_{n\to+\infty} M\left(\int_\Omega \bar{u}_n(x)dx\right) = M\left(\int_\Omega \bar{u}(x)dx\right).$$

- $D^s u_n \to D^s w$ *weakly in* $(L^2(\mathbb{R}^n))^n$ *in the sense of distribution.*

$$\begin{aligned}
\lim_{n\to+\infty}\left\langle \frac{\partial^s u_n}{\partial x_j^s}, v\right\rangle &= \lim_{n\to+\infty}\int_{\mathbb{R}^n} I_{1-s} * u_n \frac{\partial v}{\partial x_j} dx \\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^n} u_n I_{1-s} * \frac{\partial v}{\partial x_j} dx \\
&= \int_{\mathbb{R}^n} w I_{1-s} * \frac{\partial v}{\partial x_j} dx \\
&= \int_{\mathbb{R}^n} I_{1-s} * w \frac{\partial v}{\partial x_j} dx \\
&= \left\langle \frac{\partial^s w}{\partial x_j^s}, v\right\rangle.
\end{aligned}$$

Hence, $\dfrac{\partial^s u_n}{\partial x_j^s} \to \dfrac{\partial^s w}{\partial x_j^s}$ *weakly, for* $j = 1, \ldots, n$.

From the two previous point, we conclude that

$$\begin{aligned}
&\lim_{n\to+\infty} M\left(\int_\Omega \bar{u}_n(x)dx\right)\int_{\mathbb{R}^n} D^s u_n(x)\cdot D^s v(x)dx \\
&\quad = M\left(\int_{\mathbb{R}^n} \bar{u}(x)dx\right)\int_{\mathbb{R}^n} D^s u(x)\cdot D^s v(x)dx.
\end{aligned}$$

Next, for the second term $f(x,\bar{u}_n)$, *we get using the assumption* (h_2), $\lim_{n\to+\infty} f(x,u_n) = f(x,u)$ *a.e., in addition using the assumption* (h_3) *and the Dominate Convergence theorem, we obtain*

$$\lim_{n\to+\infty} f(x,\bar{u}_n) = f(x,\bar{u}) \text{ in } L^2(\Omega),$$

and

$$\lim_{n\to+\infty}\int_\Omega f(x,\bar{u}_n)v dx = \int_\Omega f(x,\bar{u})v dx.$$

We have proved that $T(\bar{u}_n)$ *converges to* $T(\bar{u})$ *in* $L^2(\Omega)$. *In conclusion, the operator* T *is continuous from* $L^2(\Omega)$ *into* $L^2(\Omega)$.

Step 3

In This step, we apply Schauder fixed point theorem using the previous auxiliary results to prove the main result.

Proof of Theorem 4.3.1

It follows from the Lemmas 4.3.2, 4.3.3 and 4.3.4, that

1. The operator T maps the ball $\bar{B}_R = \{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq R\}$ into itself.
2. $\Big\{T(w), w \in \bar{B}_R\Big\}$ is relatively compact in $L^2(\Omega)$.
3. The operator T is continuous from $L^2(\Omega)$ into $L^2(\Omega)$.

Then we can apply the Schauder fixed point theorem and conclude that the fixed point problem

$$\begin{cases} \text{Find } u \in L^2(\Omega), \\ u = T(u). \end{cases} \tag{4.6}$$

admits at least one solution $u \in L^2(\Omega)$ which is equivalent to deduce that there exists at least one weak solution $u \in L_0^{s,2}(\Omega)$ for the problem (4.1).

CHAPTER

5

STUDY OF LEFT AND RIGHT RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATION

The aim of this chapter is to study the existence and uniqueness of solutions for a semilinear fractional problem involving Riemann-Liouville fractional derivatives in a new fractional function space that is introduced by Cesar E et al. in the paper [42]. This study is based on the Leray-Schauder degree theory to obtain the existence of solutions. For the uniqueness, we apply the contraction principle of Banach. We illustrate this study with an example. This chapter presents some results published in [39].

The organization of this chapter is as follows. In section 2, we give some preliminaries and the mathematical formulation of our problem with his weak formulation. Section 3 contains the existence results using the Leray-Schauder degree method. Section 4 is devoted to study the uniqueness of soltions using the Banach principle of contraction. Finally, in section 5, we present an example to prove the validity of our results.

5.1 Preliminaries and position of the problem

Preliminairies

We will introduce some fractional spaces and results that are presented in [10, 42, 31], we will use them in this chapter.

Definition 5.1.1. *Let $s \in (0,1)$, the left and right Riemann-Liouville fractional integrals of u for $u \in L^1[a,b]$ are defined as:*

$$ {}_aI_x^s u(x) = \frac{1}{\Gamma(s)}\int_a^x (x-\alpha)^{s-1}u(\alpha)d\alpha, \quad {}_xI_b^s u(x) = \frac{1}{\Gamma(s)}\int_x^b (\alpha-x)^{s-1}u(\alpha)d\alpha $$

Definition 5.1.2. *${}_aD_x^s u(x), {}_xD_b^s u(x)$ are the left and right Riemann-Liouville fractional derivatives of u, respectively, are defined if $u \in AC[a,b]$, as follows:*

$$ {}_aD_x^s u(x) = \frac{d}{dx}{}_aI_x^{1-s}u(x), \quad \text{and } {}_xD_b^s u(x) = -\frac{d}{dx}{}_xI_b^{1-s}u(x), $$

where $x \in [a,b]$ and $s \in (0,1)$.

Theorem 5.1.1 (Integration by parts for fractional derivatives)**.** *Let $s \in (0,1)$ and for $p \geq 1, f \in L^p[a,b]$ and $g \in AC[a,b]$. Then:*

$$ \int_a^b f(x){}_aD_x^s g(x)dx = \int_a^b g(x){}_xD_b^s f(x)dx + g(b){}_xI_b^{1-s}f(b^-), $$

and

$$ \int_a^b f(x){}_xD_b^s g(x)dx = \int_a^b g(x){}_aD_x^s f(x)dx + g(a){}_aI_x^{1-s}f(a^+). $$

Definition 5.1.3. *Let $u \in L^1(a,b)$ and $s \in (0,1)$. If there is $v \in L^1_{loc}(a,b)$ such that:*

$$ \int_a^b u(x){}_xD_b^s w(x)dx = \int_a^b v(x)w(x)dx, \quad \forall w \in C_0^\infty[a,b], \tag{5.1} $$

then v is called the weak left fractional derivative of u which we denote by:

$$ {}_a\mathcal{D}_x^s u = v, $$

Theorem 5.1.2. *Let $s \in (0,1)$, The weak fractional derivative ${}_a\mathcal{D}_x^s$ is unique and linear.*

•Let $s \in (0,1)$, we define the fractional left derivative space of Sobolev type as:

$$E_L^s[a,b] = \{u \in L^2[a,b] : {}_a\mathcal{D}_x^s u \in L^2[a,b]\}$$

endowed with the inner product

$$\langle u, v\rangle_{E_L^s[a,b]} = \int_a^b u(x)v(x)dx + \int_a^b {}_a\mathcal{D}_x^s u(x) {}_a\mathcal{D}_x^s v(x)dx.$$

and the norm

$$\|u\|_{E_L^s[a,b]} = \left(\int_a^b |u(x)|^2 dx + \int_a^b |{}_a\mathcal{D}_x^s u(x)|^2 dx\right)^{1/2}. \tag{5.2}$$

is a Hilbert space. We denote by $E_{L,0}^s[a,b]$ as a closure of $C_c^\infty[a,b]$ in $E_L^s[a,b]$ endowed with the norme of $E_L^s[a,b]$, that is:

$$E_{L,0}^s[a,b] = \overline{C_c^\infty[a,b]}^{\|.\|_L}.$$

Theorem 5.1.3. *Let $s \in (0,1)$, $u \in E_L^s[a,b]$ if and only if $u \in L^2[a,b]$, ${}_aI_x^{1-s}u \in H^1[a,b]$ and*

$$ {}_a\mathcal{D}_x^s u(x) = \frac{d}{dx} {}_aI_x^{1-s} u(x) \qquad a.e.\ x \in (a,b),$$

where $\dfrac{d}{dx}$ is understood in weak sense.

Proposition 5.1.1 (Fractional Poincaré inequality). *Let $s \in (0,1)$, for all $u \in E_{L,0}^s[a,b]$ we have*

$$\|u\|_{L^2[a,b]} \leq \frac{(b-a)^s}{\Gamma(s+1)} \|{}_a\mathcal{D}_x^s u\|_{L^2[a,b]}. \tag{5.3}$$

From the Fractional Poincaré inequality, the norm of $E_{L,0}^s[a,b]$ is defined by $\|u\|_{E_{L,0}^s[a,b]} = \left(\int_a^b |{}_a\mathcal{D}_x^s u(x)|^2 dx\right)^{1/2}$ which is equivalent to the norme $\|.\|_{E_L^s[a,b]}$.

Embedding results

Theorem 5.1.4.

- *For $s \in (0, \frac{1}{2})$, the embedding $E_{L,0}^s[a,b] \hookrightarrow L^q[a,b]$ is compact for every $q \in [1, 2_s^*)$.*

- *For* $s \in \left(\frac{1}{2}, 1\right)$ *and let* $(u_n)_{n\in\mathbb{N}} \in E^s_{L,0}[a,b]$ *such that* $u_n \to u$ *in* $E^s_{L,0}[a,b]$*. Then*
$$u_n \to u \text{ in } C[a,b],$$
that is $\lim_{n\to\infty} \|u - u_n\|_\infty = 0.$

Theorem 5.1.5. *Let* $s \in (0,1)$*, then*
$E^s_{L,0}[a,b] = \Big\{u \in L^2[a,b] : \ {}_a\mathcal{D}^s_x u \in L^2[a,b] \ \text{ and } \ \lim_{x\to a^+} {}_aI^{1-\alpha}_x u(x) = 0 \ \text{ a.e. } x > a\Big\}.$
Also, if $s \in \left(0, \frac{1}{2}\right)$ *we have:*

$$E^s_{L,0}[a,b] = \left\{u \in L^2[a,b] : {}_a\mathcal{D}^s_x u \in L^2[a,b] \text{ and } \lim_{x\to a^+} {}_aI^{1-\alpha}_x u(x) = 0 \ \forall x > a\right\}.$$

And if $s \in \left(\frac{1}{2}, 1\right)$ *we have:*

$$E^s_{L,0}[a,b] = \left\{u \in L^2[a,b] : \quad {}_a\mathcal{D}^s_x u \in L^2[a,b], \ u(a) = u(b) = 0\right\}.$$

Theorem 5.1.6. *Let* $s \in (0,1)$*,* $E^s_{L,0}[a,b]$ *is a closed subspace of* $E^s_L[a,b]$*.*

We consider the following Rieman-Liouville fractional differential equation:
$$\begin{cases} {}_xD^s_b({}_aD^s_x u(x)) = \varphi(x) - \psi(x,u) & \text{in } (a,b), \\ P_s(u) = 0 & \text{on } \partial(a,b). \end{cases} \tag{5.4}$$

Where $s \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)$, $\varphi \in L^2([a,b])$ and $\psi(x,k) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. $P_s(u)$ represent the boundary condition of the problem which depend of the behavior of $s \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)$, that is:

$$P_s(u) = \begin{cases} \lim_{x\to a^+} {}_aI^{1-s}_x u(x) = 0, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ u(a) = u(b) = 0, & \text{if } s \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Remark 5.1.1. *The case* $s = \frac{1}{2}$ *is sill an open problem.*

Now we give the weak formulation of the problem (5.4), see [42].

$$\begin{cases} \text{Find } u \in E^s_{L,0}[a,b], \\ \int_a^b \mathcal{D}^s_{a,x} u(x) \mathcal{D}^s_{a,x} w(x) dx = \int_a^b \varphi(x) w(x) dx - \int_a^b \psi(x,u) w(x) dx, \quad \forall w \in E^s_{L,0}[a,b]. \end{cases} \tag{5.5}$$

Remark 5.1.2. *The precedent weak formulation is a consequence of the theorem of integration by parts for fractional derivatives and Theorem 5.1.3, for more details, see [42].*

We will use the following assumptions:

(h_1) Growth condition
$|\psi(x,k)| \leq a(x) + h|k|, \quad \forall k \in \mathbb{R}, \quad \text{a.e. } x \in [a,b],$
such there exist $a \in L^2([a,b])$, and $h \in \mathbb{R}^+$.

(h_2) Sign condition:
$\psi(x,k)k \geq 0, \quad \forall k \in \mathbb{R}$ and a.e $x \in [a,b]$.

5.2 Existence results

In this section, we give the theorem of the existence of weak solutions to our problem (5.4), we reformulate our problem, and show the Leray-Schauder degree conditions to prove the result.
The following theorem gives the existence of solutions to the weak formulation (5.5).

Theorem 5.2.1. *Under the hypotheses* (h_1) *and* (h_2)*, the problem* (5.4) *has a weak solution* $u \in E^s_{L,0}([a,b])$.

To reformulate our problem, we need to define the following operator: $T : [0,1] \times L^2([a,b]) \to E^s_{L,0}([a,b])$, such that $T(t,\overline{u}) = u$, where u is a solution to the following linear problem:

$$\begin{cases} \text{Find } u \in E^s_{L,0}[a,b], \\ \int_a^b \mathcal{D}^s_{a,x}u(x)\mathcal{D}^s_{a,x}w(x)dx = t\int_a^b \varphi(x)w(x)dx - t\int_a^b \psi(x,\overline{u})w(x)dx, \\ \forall w \in E^s_{L,0}[a,b] \end{cases} \quad (5.6)$$

We must prove the existence and the uniqueness of a solution to this problem in order to obtain that T is a well-defined operator, and this is implied by the following theorem:

Theorem 5.2.2. *Thanks to hypothesis* (h_1) *the problem* (5.6) *has unique solution* $u \in E^s_{L,0}([a,b])$.

Proof. To prove Theorem 5.2.2, we use the Lax-Milgram theorem.

We pose

$$a(u,w) : E^s_{L,0}([a,b]) \times E^s_{L,0}([a,b]) \to \mathbb{R}$$
$$(u,w) \mapsto a(u,w) = \int_a^b \mathcal{D}^s_{a,x}u(x)\mathcal{D}^s_{a,x}w(x)dx,$$

and

$$l(w) : E^s_{L,0}([a,b]) \to \mathbb{R}$$
$$w \mapsto l(w) = t\int_a^b \varphi(x)w(x)dx - t\int_a^b \psi(x,\overline{u})w(x)dx \quad \forall t \in [0,1],$$

Step 01 We will prove the bilinear form $a(.,.)$ is coercive.
For all $E^s_{L,0}([a,b])$

$$\begin{aligned} a(w,w) &= \int_a^b \mathcal{D}^s_{a,x}w(x)\mathcal{D}^s_{a,x}w(x)dx, \\ &= \|w\|^2_{E^s_{L,0}([a,b])}, \end{aligned}$$

thus, a(.,.) is coercive.

Step 02 We will prove the bilinear form $a(.,.)$ is continuous.
For all $u, w \in E^s_{L,0}([a,b])$, we have

$$|a(u,w)| = |\int_a^b \mathcal{D}^s_{a,x}u(x)\mathcal{D}^s_{a,x}w(x)dx|,$$

Then, applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned} |a(u,w)| &\leq \|\mathcal{D}^s_{a,x}u\|_{L^2([a,b])}\|\mathcal{D}^s_{a,x}w\|_{L^2([a,b])} \\ &= \|u\|_{E^s_{L,0}([a,b])}\|w\|_{E^s_{L,0}([a,b])}. \end{aligned}$$

Therefore, $a(.,.)$ is continuous.

Step 03 We will prove the linear form $l(.)$ is continuous.
For all $w \in E^s_{L,0}([a,b])$, we have

$$\begin{aligned} |l(w)| &= |t\int_a^b \varphi(x)w(x)dx - t\int_a^b \psi(x,\overline{u})w(x)dx| \\ &\leq |\int_a^b \varphi(x)w(x)dx| + |\int_a^b \psi(x,\overline{u})w(x)dx|. \end{aligned}$$

Hence, from Proposition 5.1.1, the Cauchy-Schwarz inequality and hypothesis (h_1), we obtain

$$\begin{aligned}|l(w)| &\leq \left(\frac{(b-a)^s}{\Gamma(s+1)}\right)\|\varphi\|_{L^2([a,b])}\|w\|_{E^s_{L,0}([a,b])}\\ &+\left(\frac{(b-a)^s}{\Gamma(s+1)}\right)\left(\|a\|_{L^2([a,b])}+h\|\overline{u}\|_{L^2([a,b])}\right)\|w\|_{E^s_{L,0}([a,b])}\\ &\leq \left(\frac{(b-a)^s}{\Gamma(s+1)}\right)\left(\|a\|_{L^2([a,b])}+h\|\overline{u}\|_{L^2([a,b])}+\|\varphi\|_{L^2([a,b])}\right)\|w\|_{E^s_{L,0}([a,b])},\end{aligned}$$

hence, $l(.)$ is continuous.

• As a result, we may apply the Lax-Milgram theorem, and we conclude that the problem (5.6) has a unique solution $u \in E^s_{L,0}([a,b])$. □

We observe that to prove the existence of weak solutions for the problem (5.4) is equivalent to show the existence of solutions for the following problem.

$$\begin{cases}\text{Find } u \in L^2([a,b]),\\ u = T(1,u).\end{cases} \tag{5.7}$$

For this purpose, we will give some results that satisfy the Leray Schauder degree conditions.

Leray-Schauder degree conditions

Lemma 5.2.1. *(Priori estimate) Let ψ satisfy the assumption (h_2), we will show that*
$\exists R > 0$ such that

$$\begin{cases}T(t,u) = u\\ t\in[0,1], u \in L^2([a,b])\end{cases} \Rightarrow \|u\|_{L^2([a,b])} < R+1.$$

Proof. Let $T(t,u) = u$, $\forall t \in [0,1]$, and we apply the hypothesis (h_2) to get

$$\begin{aligned}\int_a^b {}_a\mathcal{D}^s_x u(x)\,{}_a\mathcal{D}^s_x u(x)dx &= \int_a^b t\varphi(x)u(x)dx - \int_a^b t\psi(x,u)udx,\\ &\leq \int_a^b \varphi(x)u(x)dx.\end{aligned}$$

By Cauchy-schwarz inequality, we arrive at

$$\int_a^b |{}_a\mathcal{D}^s_x u(x)|^2 dx \leq \|\varphi\|_{L^2([a,b])}\|u\|_{L^2([a,b])}.$$

Moreover, from the Poincaré inequality, implies that

$$\|u\|_{L^2([a,b])} \leq \left(\frac{(b-a)^s}{\Gamma(s+1)}\right)^2 \|\varphi\|_{L^2([a,b])}.$$

We note

$$R = \left(\frac{(b-a)^s}{\Gamma(s+1)}\right)^2 \|\varphi\|_{L^2([a,b])},$$

then,

$$\|u\|_{L^2([a,b])} < R + 1. \tag{5.8}$$

We conclude from (5.8) that there are no solutions to the equation $T(t,u) = u, \forall t \in [0,1]$ in the boundary of the ball $B_{R+1} = \{u \in L^2([a,b]) : \|u\|_{L^2([a,b])} < R+1\}$. □

Lemma 5.2.2. *Let the assumption* (h_1) *be satisfied,* $T\colon [0,1] \times L^2([a,b]) \to L^2([a,b])$ *is continuous.*

Proof. Let $\{t_n, \overline{u}_n\}_{n\in\mathbb{N}} \subset [0,1] \times L^2(\Omega)$ which converges to $(t,\overline{u})$ in $[0,1] \times L^2(\Omega)$ when $n \to +\infty$. We will show that $T(t_n,\overline{u}_n)$ converges to $T(t,\overline{u})$, putting $T(t_n,\overline{u}_n) = u_n$ and $T(t,\overline{u}) = u$, we have

$$\int_a^b {}_a\mathcal{D}_x^s u_n(x) . {}_a\mathcal{D}_x^s w(x) dx = \int_a^b t_n \varphi(x) w(x) dx - \int_a^b t_n \psi(x,\overline{u}_n) w(x) dx, \quad \forall w \in E_{L,0}^s([a,b]),$$

and

$$\int_a^b {}_a\mathcal{D}_x^s u(x) . {}_a\mathcal{D}_x^s w(x) dx = \int_a^b t \varphi(x) w(x) dx - \int_a^b t \psi(x,\overline{u}) w(x) dx, \quad \forall w \in E_{L,0}^s([a,b]),$$

substracting the last two equations, we obtain

$$\int_a^b \left({}_a\mathcal{D}_x^s u_n - {}_a\mathcal{D}_x^s u\right) . {}_a\mathcal{D}_x^s w dx = \int_a^b (t_n - t)\varphi w dx - \int_a^b \left(t\psi(x,\overline{u}) - t_n\psi(x,\overline{u}_n)\right) w dx, \forall w \in E_{L,0}^s([a,b]).$$

We take $w(x) = u_n(x) - u(x)$ and apply the Cauchy-Schwarz inequality and Proposition 5.1.1, we obtain

$$\|u_n - u\|_{L^2([a,b])} \leq \left(\frac{(b-a)^s}{\Gamma(s+1)}\right)^2 \left(|t_n - t| \|\varphi\|_{L^2([a,b])} + \|t\psi(.,\overline{u}) - t_n\psi(.,\overline{u}_n)\|_{L^2([a,b])}\right).$$

Then, our hypothesis $\overline{u}_n \to \overline{u}$ in $L^2([a,b])$ implies that

$$\begin{cases} \overline{u}_n \to \overline{u} & \text{a.e on } [a,b] \\ |\overline{u}_n| < \mathcal{H} & \text{a.e on } [a,b], \text{ with } \mathcal{H} \in L^2([a,b]), \end{cases}$$

from the assumption (h_1) and the Dominate convergence theorem, $\psi(x,\overline{u}_n) \to \psi(x,\overline{u})$ in $L^2([a,b])$ and we have $(t_n)_{n\in\mathbb{N}}$ converges to t when $n \to +\infty$. Therefore $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^2([a,b])$. So T is continuous from $[0,1] \times L^2([a,b])$ into $L^2([a,b])$. □

Lemma 5.2.3. *Under the assumption (h_1), the following set*

$$\{T(t,\overline{u}), t \in [0,1], \overline{u} \in \overline{B}_{R+1}\}$$

is relatively compact in $L^2([a,b])$.

Proof. Let $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ and $(\overline{u}_n)_{n\in\mathbb{N}} \subset \overline{B}_{R+1}$, using the Cauchy-Schwarz inequality, we get

$$\Big|\int_a^b {}_a\mathcal{D}_x^s u_n(x) . {}_a\mathcal{D}_x^s u_n(x) dx\Big| = \Big|\int_a^b t_n \varphi(x) u_n(x) dx - \int_a^b t_n \psi(x,\overline{u}_n) u_n(x) dx\Big|$$
$$\|u_n\|^2_{E^s_{L,0}([a,b])} \leq \Big(\|\varphi\|_{L^2([a,b])} + \|\psi(.,\overline{u}_n)\|_{L^2([a,b])}\Big) \|u_n\|_{L^2([a,b])}.$$

From Proposition 5.1.1, we obtain

$$\|u_n\|_{E^s_{L,0}([a,b])} \leq \frac{(b-a)^s}{\Gamma(s+1)} \Big(\|\varphi\|_{L^2([a,b])} + \|\psi(.,\bar{u}_n)\|_{L^2([a,b])}\Big). \tag{5.9}$$

Using the hypothesis (h_1), the sequence $(\psi(.,\overline{u}_n))_{n\in\mathbb{N}}$ is bounded in $L^2([a,b])$. In addition, from the last inequality, $(u_n)_{n\in\mathbb{N}}$ is bounded in $E^s_{L,0}([a,b])$, so we have that $u_n \rightharpoonup u$ in $E^s_{L,0}([a,b])$, $E^s_{L,0}([a,b])$ is compactly embedded in $L^2([a,b])$. We conclude that there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges to u in $L^2([a,b])$. □

Now, we can prove Theorem 5.2.1.

Proof. From Lemmas 5.2.1, 5.2.2 and 5.2.3, we conclude that $d(I_d - T(t,.), B_{R+1}, 0)$ is well-defined.
By the homotopy invariance property, we have

$$\begin{aligned} d(I_d - T(1,.), B_{R+1}, 0) &= d(I_d - T(0,.), B_{R+1}, 0) \\ &= d(I_d, B_{R+1}, 0) = 1 \neq 0, \end{aligned}$$

Therefore, there exists $u \in B_{R+1}$ such that

$$u - T(1,u) = 0 \Leftrightarrow u = T(1,u).$$

Hence, we have shown that u is a solution of (5.4).

□

5.3 Uniqueness results

Let assume that ψ is Lipschitz continuous with respect to the second variable, that is, there exists a constant $C > 0$ such that for almost every $x \in [a, b]$ and for any $k_1, k_2 \in L^2([a, b])$

$$(h_3) \qquad \|\psi(x, k_1) - \psi(x, k_2)\|_{L^2([a,b])} \le C\|k_1 - k_2\|_{L^2([a,b])}. \qquad (5.10)$$

Thus, in order to prove that T admits a unique fixed point, it is sufficient to prove that T is a contraction.
We take $B_t(u) = T(t, u)$ for all $t \in [0, 1]$.

Lemma 5.3.1. *Thanks to assumption* (h_3)*, The operator* B_t *is a contraction from* $L^2([a, b])$ *to* $L^2([a, b])$ *for all* $t \in [0, 1]$*.*

Proof. For all $\bar{u}_1, \bar{u}_2 \in L^2([a, b])$, we pose $B_t(\bar{u}_1) = u_1$ and $B_t(\bar{u}_2) = u_2$, $\forall t \in [0, 1]$, and get

$$\int_a^b {}_a\mathcal{D}_x^s u_1 {}_a\mathcal{D}_x^s w dx = t\int_a^b \varphi(x) w dx - t\int_a^b \psi(x, \overline{u}_1) w dx, \qquad (5.11)$$

and

$$\int_a {}_a\mathcal{D}_x^s u_2 {}_a\mathcal{D}_x^s w dx = t\int_a^b \varphi(x) w dx - t\int_a^b \psi(x, \overline{u}_2) w dx, \qquad (5.12)$$

By taking the difference between the equations (5.11) and (5.12), we take $w = u_1 - u_2$, use Cauchy-Schwarz inequality and obtain

$$\|u_1 - u_2\|^2_{E^s_{L,0}([a,b])} \le \|\psi(., \overline{u}_2) - \psi(., \overline{u}_1)\|_{L^2([a,b])}\|u_1 - u_2\|_{L^2([a,b])},$$

From Proposition 5.1.1, and hypothesis (h_3), we get

$$\|u_1 - u_2\|_{L^2([a,b])} \le C\left(\frac{(b-a)^s}{\Gamma(s+1)}\right)^2 \|\overline{u}_1 - \overline{u}_2\|_{L^2([a,b])}.$$

Consequently, if $C\left(\frac{(b-a)^s}{\Gamma(s+1)}\right)^2 < 1$, B_t is a contraction, $\forall t \in [0, 1]$. □

As a result of the previous lemma, we have if $C\left(\frac{(b-a)^s}{\Gamma(s+1)}\right)^2 < 1$, B_t is a contraction, $\forall t \in [0, 1]$, so we may apply the Banach fixed point theorem, and we conclude that B_1 admits a unique fixed point $u \in L^2([a, b])$. Hence the problem (5.4) admits a unique weak solution.

5.4 Example

To validate our theoretical results, we present an example to illustrate Lemmas 5.2.1, 5.2.2, 5.2.3, and 5.3.1

In this example, if we take, $s = \dfrac{1}{4}$, $\varphi(x) = sin(x)$ and $\psi(x, u(x)) = \frac{1}{2}u(x)$, we have ψ satisfied the assumptions (h_1), (h_2) and (h_3). So our problem is considered as follows:

$$\begin{cases} {}_xD_1^{\frac{1}{4}}({}_0D_x^{\frac{1}{4}}u(x)) = sin(x) - \dfrac{1}{2}u(x) & \text{in } (0,1), \\ \lim\limits_{x\to 0^+} {}_0I_x^{1-\frac{1}{4}}u(x) = 0 & \text{on } \partial(0,1), \end{cases} \tag{5.13}$$

where $a = 0, b = 1$ and $\Gamma(5/4) \approx 0.90640248$.
We have

$$E_L^{\frac{1}{4}}[0,1] = \{u \in L^2[0,1] : {}_0\mathcal{D}_x^{\frac{1}{4}}u \in L^2[0,1] \text{and} \lim_{x\to 0^+} {}_0I_x^{1-\frac{1}{4}}u(x) = 0, \forall x > a\}$$

and

$$E_{L,0}^{\frac{1}{4}}[0,1] = \overline{C_0^\infty[0,1]}^{\|\cdot\|_L}.$$

Using the assumption (h_2) and the example (5.13), we find from Lemma 5.2.2 that

$$\|u\|_{L^2[a,b]} \le R \text{ with } \quad R \approx 0.63559578.$$

Thus

$$B_{R+1} = \{u \in L^2([a,b]) : \|u\|_{L^2([a,b])} < 1.63559578\}.$$

Indeed, for the uniqueness of u, we have used the Banach fixed point theorem, which is satisfied for the example 5.13 as follows,

$$\|u_1 - u_2\|_{L^2([a,b])} \le 0.60859424 \ \|\overline{u}_1 - \overline{u}_2\|_{L^2([a,b])}.$$

We remark from the above inequality that $0.60859424 < 1$. Then, B_t is a contraction, so Problem (5.13) admits only one weak solution $u \in E_{L,0}^{\frac{1}{4}}[a,b]$.

CONCLUSION

The focus of this work was on nonlinear fractional differential problems, which can generalize many results from classical Sobolev spaces to fractional Sobolev spaces, where our problems contain two important nonlocal operators: fractional Laplacian and distributional Riesz fractional gradient, as well as the left and right Riemann-Liouville fractional derivatives.

First, we have given the weak formulation for each of our problems. Next, we used the Lax-Milgram theorem to prove the uniqueness of solutions for the linear problem associated with our nonlinear weak problem. This result helped us to define an operator that is used to formulate a new problem whose existence is equivalent to the existence of solutions for our nonlinear weak problem. Under some hypotheses in nonlinear terms, we applied two topological methods:

- We satisfied the conditions of the Leray-Schauder degree theorem, where we constructed a ball that contains the solutions in such a way that these solutions are not in the boundary of the ball. We also proved that the operator defined is compact, and using the homotopy invariance and normality properties, we obtained our results for the first, second, and fourth problems.

- For the third problem, we satisfied the conditions of the Schauder fixed point theorem for our operator that is compact and maps the closure of the ball B_R into itself to prove the existence of weak solutions.

The different studies in this thesis are in the following spaces:

- Fractional Sobolev space $D^{s,2}(\Omega)$ relating to the fractional Laplacian.
- Bessel potential space $L_0^{s,2}(\Omega)$ relating to the distributional Riesz fractional gradient.
- The left Riemann-Liouville fractional Sobolev space $E^s_{L,0}[a,b]$ relating to the left Riemann-Liouville fractional derivative.

These spaces, with their nice properties (Poincaré inequalities, embedding theorems, etc.), helped us obtain our results and generalize important problems from the classical case to the fractional case.

We concluded that topological methods are efficient methods not only for nonlinear differential problems but also for fractional differential problems.

For the uniqueness of solutions, we have used the Banach principle of contraction and absurd reasoning under certain assumptions on the nonlinearities.

The study of fractional problems is extensive for generalizing other problems with significant applications in other sciences.

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