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ON THE EXISTENCE OF PERIODIC SOLUTIONS OF A SECOND ORDER ITERATIVE DIFFERENTIAL EQUATION

R. KHEMIS, A. BOUAKKAZ and S. CHOUAF

Abstract. In this work, we consider a class of second order iterative differential equations. Using Schauder's fixed point theorem and the Green's functions method, the existence of periodic solutions is proved after establishing the equivalence of our problem with a certain integral equation. Finally, we end this article with a simple conclusion recapitulating the guiding idea of our approach. Obtained findings complement some previous publications in the literature.

1. Introduction

In this work, we are interested in the following class of second order iterative differential equations:

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)+p(t)\frac{d}{\mathrm{d}t}x(t)+q(t)x(t)\\&=\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}C_{\ell,m}(t)\big(x^{[m]}(t)\big)^{\ell}+\frac{\mathrm{d}}{\mathrm{d}t}g\left(t,x^{[1]}(t),x^{[2]}(t),\dots,x^{[n]}(t)\right)+h(t),\end{aligned}\tag{1}$$

where $x^{[2]}(t)=x(x(t))$, $x^{[m]}(t)=x^{[m-1]}(x(t))$, $p,q,C_{\ell,m},h\in\mathcal{C}(\mathbb{R},\mathbb{R})$, $m=\overline{1,n}$, $\ell=\overline{1,\infty}$, and $g\in\mathcal{C}^1\left(\mathbb{R}^{n+1},\mathbb{R}\right)$.

This kind of equation, which can be regarded as a special type of retarded differential equations with lags that depend on the time and the state, appears in many models that describe miscellaneous natural phenomena, such as insect population dynamics models in ecology, models of hematopoiesis in hematology, epidemiological models in epidemiology, two-body equations of motion in classical electrodynamics, and so on (see [**7, 9, 10, 14, 17**]).

Until relatively recently, there was a little interest in the theory and applications of iterative differential equations which have distinctive characteristics and do not possess a well-developed theory. This is due to many reasons, perhaps the most obvious one is the fact that the iterative terms in these equations, obstacle the

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application of habitual methods and yield some difficulties during their study. However, in the last few years, there has been a great resurgence of interest in this topic where several important achievements have been made. For first order iterative differential equations, the interested reader can consult the articles [**2, 7, 18, 19**] and references therein. For second order ones, see [**3, 4, 6, 11, 20**], while virtually no significant progress has been achieved regarding higher order iterative differential equations except the works of the first author and his collaborators on third-order ones [**5, 8, 13**].

So, it is highly desirable to investigate in this direction which poses a special challenge for us and especially since the above involved papers motivated us to try to improve and extend some existing results in order to participate in the development of this emerging theory.

In this manuscript, we are to consider the existence of periodic solutions of equation (1). By employing Schauder's fixed point theorem as well as some properties of an obtained Green's function, we establish a set of sufficient conditions that guarantee the achievement of this aim.

Now, we feel the need to highlight some of the key aspects of our contributions which lie in three points:

– Several researches and documents have revealed that the delay in many real phenomena actually depends on the time and the state. Equation (1) involves iterative terms resulting from many delays of this kind.

– Up to now, there have been few works that focus on the study of second order iterative differential equations. So, our results enrich and complement some earlier works to some extent (see [**1, 3, 11, 12**]).

– The approach used here has proven to be an efficacious tool to deal with other iterative problems.

The plan of our paper is as follows: In the next section, we present some notations and materials needed to establish our main findings. We also state and prove some preliminary results. The third section is consecrated to discuss the existence of periodic solutions of equation (1). Finally, we draw a brief conclusion in the last section.

2. PRELIMINARIES

Let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t)\} \qquad \text{for } T > 0.$$

Endowing P_T with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

$(P_T, \|\cdot\|)$ is a Banach space and for every L, $M \geq 0$,

$$P_T(L, M) = \{x \in P_T : \|x\| \leq L,\ |x(t_2) - x(t_1)| \leq M|t_2 - t_1|,\ t_1, t_2 \in \mathbb{R}\},$$

is a closed, convex, and bounded subset of P_T.

Furthermore, we need the following hypotheses:
p, q, $C_{\ell,m}$, $m = \overline{1,n}$, $\ell = \overline{1,\infty}$, and h are continuous and T-periodic real-valued functions such that

$$\begin{aligned} p(t+T) &= p(t), & q(t+T) &= q(t), \\ C_{\ell,m}(t+T) &= C_{\ell,m}(t), & h(t+T) &= h(t), \end{aligned} \tag{2}$$

and

$$\int_0^T p(s)\mathrm{d}s > 0, \qquad \int_0^T q(s)\mathrm{d}s > 0. \tag{3}$$

The function $g(t, x_1, x_2, \dots, x_n)$ is supposed T-periodic in t and globally Lipschitz in $x_1, \dots, x_n$, i.e.,

$$g(t+T, x_1, x_2, \dots, x_n) = g(t, x_1, x_2, \dots, x_n), \tag{4}$$

and there exist n positive constants c_1, $c_2, \dots, c_n$ such that

$$|g(t, x_1, x_2, \dots, x_n) - g(t, y_1, y_2, \dots, y_n)| \le \sum_{m=1}^{n} c_m \|x_m - y_m\|. \tag{5}$$

Lemma 2.1 ([**15**])**.** *Suppose that* (2) *and* (3) *hold and*

$$\frac{R_1 \left[\exp\left(\int_0^T p(u)\mathrm{d}u\right) - 1\right]}{Q_1 T} \ge 1, \tag{6}$$

where

$$R_1 = \max_{t\in[0,Tt]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\mathrm{d}u\right)}{\exp\left(\int_0^T p(u)\mathrm{d}u\right) - 1} q(s)\mathrm{d}s \right|$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\mathrm{d}u\right)\right)^2 R_1^2.$$

Then there are continuous and T-periodic functions a and b such that

$$b(t) > 0, \qquad \int_0^T a(u)\mathrm{d}u > 0, \qquad a(t) + b(t) = p(t),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t) + a(t)b(t) = q(t)$$

for all $t \in \mathbb{R}$.

Lemma 2.2 ([**16**])**.** *Suppose that conditions of Lemma 2.1 hold and $\phi \in P_T$. Then the equation*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) + p(t)\frac{\mathrm{d}}{\mathrm{d}t} x(t) + q(t)x(t) = \phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)\mathrm{d}s,$$

where

$$
\begin{aligned}
G(t,s) &= \frac{\int_t^s \exp\left[\int_t^u b(v)\mathrm{d}v + \int_u^s a(v)\mathrm{d}v\right]\mathrm{d}u}{\left[\exp(\int_0^T a(u)\mathrm{d}u) - 1\right]\left[\exp(\int_0^T b(u)\mathrm{d}u - 1\right]} \\
&\quad + \frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\mathrm{d}v + \int_u^{s+T} a(v)\mathrm{d}v\right]\mathrm{d}u}{\left[\exp(\int_0^T a\,(u)\,\mathrm{d}u) - 1\right]\left[\exp(\int_0^T b(u)\mathrm{d}u) - 1\right]}.
\end{aligned}
\tag{7}
$$

Corollary 2.3 ([**16**])**.** *Green's function G defined by* (7), *satisfies the following properties*

$$
\begin{aligned}
&G(t,t+T) = G(t,t),\\
&G(t+T,s+T) = G(t,s),\\
&\frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - \frac{\exp\int_t^s b(v)\mathrm{d}v}{\exp\left(\int_0^T b(v)\mathrm{d}v\right) - 1}.
\end{aligned}
\tag{8}
$$

Theorem 2.4 ([**18**])**.** *Let M be a closed convex compact subset of a Banach space $(X, \|\cdot\|)$. Suppose that $\mathcal{H}\colon M \to M$ is continuous. Then there exists $x \in M$ such that $x = \mathcal{H}x$.*

Lemma 2.5. *Suppose* (2)–(4) *and* (6) *hold. If $x \in P_T(L,M) \cap \mathcal{C}^2(\mathbb{R},\mathbb{R})$, then x is a solution of* (1) *if and only if*

$$
\begin{aligned}
x(t) = &\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T} C_{\ell,m}(s)\left(x^{[m]}(s)\right)^{\ell} G(t,s)\mathrm{d}s\\
&+ \int_t^{t+T}\left[E(t,s) - a(s)G(t,s)\right] g(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s))\mathrm{d}s\\
&+ \int_t^{t+T} G(t,s)h(s)\mathrm{d}s,
\end{aligned}
$$

where

$$
E(t,s) = \frac{\exp\int_t^s b(v)\mathrm{d}v}{\exp\left(\int_0^T b(v)\mathrm{d}v\right) - 1}.
\tag{9}
$$

Proof. Suppose that $x \in P_T(M,L) \cap \mathcal{C}^2(\mathbb{R},\mathbb{R})$ is a solution of (1). From Lemma 2.2, we have

$$
\begin{aligned}
x(t) = &\int_t^{t+T} G(t,s)\Big[\sum_{m=1}^{n}\sum_{\ell=1}^{\infty} C_{\ell,m}(s)\left(x^{[m]}(s)\right)^{\ell}\Big]\mathrm{d}s\\
&+ \int_t^{t+T} G(t,s)\left[\frac{d}{\mathrm{d}s} g\left(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right)\right]\mathrm{d}s\\
&+ \int_t^{t+T} G(t,s)h(s)\mathrm{d}s.
\end{aligned}
$$

An integration by parts gives

$$\begin{aligned}&\int_t^{t+T} G(t,s)\Big[\frac{\mathrm{d}}{\mathrm{d}s}g\left(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\Big]\mathrm{d}s\\ &= G(t,s)g\left(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\Big|_t^{t+T}\\ &\quad-\int_t^{t+T}\Big[\frac{\mathrm{d}}{\mathrm{d}s}G(t,s)\Big]g\left(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\mathrm{d}s.\end{aligned}$$

It follows from Corollary 2.3 that

$$\Big[G(t,s)g\left(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\Big]_t^{t+T}=0$$

and

$$\begin{aligned}&\int_t^{t+T} G(t,s)\Big[\frac{d}{\mathrm{d}s}g\left(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\Big]\mathrm{d}s\\ &=\int_t^{t+T} g\left(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\left[E(t,s)-a(s)G(t,s)\right].\end{aligned}$$

Consequently,

$$\begin{aligned}x(t)&=\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T} C_{\ell,m}(s)\left(x^{[m]}(s)\right)^{\ell}G(t,s)\mathrm{d}s\\ &\quad+\int_t^{t+T}\left[E(t,s)-a(s)G\left(t,s\right)\right]g\left(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[n]}(s)\right)\mathrm{d}s\\ &\quad+\int_t^{t+T}G(t,s)h(s)\mathrm{d}s.\end{aligned}$$

It is not hard to obtain the converse implication. Indeed, the derivation of the last integral equation completes the proof. □

Lemma 2.6 ([**16**])**.** *Let* $A=\int_0^T p(u)\mathrm{d}u$ *and* $B=T^2\exp\left(\frac{1}{T}\int_0^T\ln\left(q(u)\right)\mathrm{d}u\right)$. *If*

$$A^2\geq 4B, \tag{10}$$

then

$$\min\Big\{\int_0^T a(u)\mathrm{d}u,\int_0^T b(u)\mathrm{d}u\Big\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right):=M_1$$

and

$$\max\Big\{\int_0^T a(u)\mathrm{d}u,\int_0^T b(u)\mathrm{d}u\Big\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right):=M_2.$$

Corollary 2.7 ([**16**])**.** *Functions* G *and* E *satisfy*

$$0<\alpha_1\leq G(t,s)\leq\alpha_2\quad\text{and}\quad|E(t,s)|\leq\beta, \tag{11}$$

where

$$\alpha_1=\frac{T}{\left(\mathrm{e}^{M_2}-1\right)^2},\quad\alpha_2=\frac{T\exp\int_0^T p(u)\mathrm{d}u}{\left(\mathrm{e}^{M_1}-1\right)^2},\quad\text{and}\quad\beta=\frac{\exp\int_0^T b(v)\mathrm{d}v}{\exp\left(\int_0^T b(v)\mathrm{d}v\right)-1}.$$

Lemma 2.8 ([**19**])**.** *For any* φ, $\psi \in P_T(L,M)$,

$$\left\|\varphi^{[r]} - \psi^{[r]}\right\| \leq \sum_{j=0}^{r-1} M^j \left\|\varphi - \psi\right\|, \qquad r = 1, 2, \ldots$$

Lemma 2.9 ([**18**])**.** *It holds*

$$P_T(L,M) = \{x \in P_T,\ \|x\| \leq L,\ |x(t_2)-x(t_1)| \leq M|t_2-t_1| \text{ for all } t_1, t_2 \in [0,T]\}.$$

We use in the sequel the following notations:

$$\lambda = \max_{t\in[0,Tt]} |b(t)|, \qquad \lambda_1 = \max_{t\in[0,T]} |a(t)|, \qquad \gamma = \exp\Big(\int_0^T b(v)\mathrm{d}v\Big),$$

$$\Gamma = T\lambda\beta, \rho = \max_{t\in[0,Tt]} |g(t,0,\ldots,0)|, \qquad \varsigma = \rho + L\sum_{m=1}^{n}\sum_{j=0}^{m-1} c_m M^j,$$

$$\delta = \left(\Big[\exp\Big(\int_0^T a(v)\mathrm{d}v\Big) - 1\Big]\Big[\exp\Big(\int_0^T b(v)\mathrm{d}v\Big) - 1\Big]\right)^{-1},$$

$$\mu = T\mathrm{e}^{2M_2}\delta\left[T\lambda\gamma\left(2\mathrm{e}^{2M_2}+1\right) + \mathrm{e}^{M_2} + 1\right],$$

$$P_T(L) = \{x \in P_T,\ \|x\| \leq L\},$$

and we suppose that

$$T\Bigg(\alpha_2 L^{\ell-1}\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\sum_{j=0}^{m-1} \ell L_{C_{\ell,m}} M^j + (\alpha_2\lambda_1 + \beta)\sum_{m=1}^{n}\sum_{j=0}^{m-1} c_m M^j\Bigg) < \infty, \tag{12}$$

$$\alpha_2 T \sum_{m=1}^{n}\sum_{\ell=1}^{\infty} L_{C_{\ell,m}} L^{\ell} + (\beta + \alpha_2\lambda_1)\,T\varsigma + \alpha_2 T L_h \leq L, \tag{13}$$

and

$$\sum_{m=1}^{n}\sum_{l=1}^{\infty} L^{\ell} L_{C_{\ell,m}}(2\alpha_2+\mu) + \varsigma(2\beta+\Gamma) + L_h(2\alpha_2+\mu) + \lambda_1\varsigma(2\alpha_2+\mu) \leq M. \tag{14}$$

Lemma 2.10 ([**4**])**.** *For any* $t_1, t_2 \in [0,T]$,

$$\int_{t_1}^{t_1+T} |G_2(t_2,s) - G_2(t_1,s)|\mathrm{d}s \leq \mu\,|t_2 - t_1|.$$

Lemma 2.11. *For any* $t_1, t_2 \in [0,T]$, *we have*

$$\int_{t_1}^{t_1+T} |E(t_2,s) - E(t_1,s)|\mathrm{d}s \leq \Gamma|t_2 - t_1|.$$

Proof. Let $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned}
&\int_{t_1}^{t_1+T} \left|E\left(t_2, s\right) - E\left(t_1, s\right)\right| \mathrm{d}s \\
&= \int_{t_1}^{t_1+T} \frac{\left|\exp\left(\int_{t_2}^{s} b(v)\mathrm{d}v\right) - \exp\left(\int_{t_1}^{s} b(v)\mathrm{d}v\right)\right|}{\left(\exp\left(\int_0^T b(v)\mathrm{d}v\right) - 1\right)} \mathrm{d}s \\
&= \frac{\int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} b(v)\mathrm{d}v\right)\left|1 - \exp\left(\int_{t_1}^{t_2} b(v)\mathrm{d}v\right)\right| \mathrm{d}s}{\left(\exp\left(\int_0^T b(v)\mathrm{d}v\right) - 1\right)}.
\end{aligned}$$

By applying the mean value theorem to the function $f(t) = \exp\left(\int_t^{t_2} b(v)\mathrm{d}v\right)$ over the interval $[t_1, t_2]$, there exists $\xi \in\,]t_1, t_2[$ such that

$$\begin{aligned}
|f(t_2) - f(t_1)|t &= \left|\exp\right| \int_{t_2}^{t_2} b(v)\mathrm{d}v\Big) - \exp\Big(\int_{t_1}^{t_2} b(v)\mathrm{d}v\Big)\Big| \\
&= \Big|1 - \exp\Big(\int_{t_1}^{t_2} b(v)\mathrm{d}v\Big)\Big| = |f'(\xi)||t_2 - t_1| \\
&= b(\xi)\exp\Big(\int_{\xi}^{t_2} b(v)\mathrm{d}v\Big)|t_2 - t_1|.
\end{aligned}$$

So,

$$\begin{aligned}
&\int_{t_1}^{t_1+T} \exp\Big(\int_{t_2}^{s} b(v)\mathrm{d}v\Big)\Big|1 - \exp\Big(\int_{t_1}^{t_2} b(v)\mathrm{d}v\Big)\Big|\mathrm{d}s \\
&= \int_{t_1}^{t_1+T} \exp\Big(\int_{t_2}^{s} b(v)\mathrm{d}v\Big) b\Big(\xi\Big)\exp\Big(\int_{\xi}^{t_2} b(v)\mathrm{d}v\Big)\left|t_2 - t_1\right| \mathrm{d}s \\
&\leq \sup_{s\in[0,Tt]} |b(s)|\left|t_2 - t_1\right| \int_{t_1}^{t_1+T} \exp\Big(\int_{\xi}^{t_2} b(v)\mathrm{d}v\Big)\exp\Big(\int_{t_2}^{s} b(v)\mathrm{d}v\Big)\mathrm{d}s \\
&= \sup_{s\in[0,Tt]} |b(s)|\left|t_2 - t_1\right| \int_{t_1}^{t_1+T} \exp\Big(\int_{\xi}^{t_2} b(v)\mathrm{d}v + \int_{t_2}^{s} b(v)\Big)\mathrm{d}s \\
&= \sup_{s\in[0,Tt]} |b(s)|\left|t_2 - t_1\right| \int_{t_1}^{t_1+T} \exp\Big(\int_{\xi}^{s} b(v)\mathrm{d}v\Big)\mathrm{d}s \\
&\leq \sup_{s\in[0,Tt]} |b(s)|\left|t_2 - t_1\right| \exp\Big(\int_0^T b(v)\mathrm{d}v\Big) \int_{t_1}^{t_1+T} \mathrm{d}s \\
&= \lambda T \exp\Big(\int_0^T b(v)\mathrm{d}v\Big)\left|t_2 - t_1\right|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{t_1}^{t_1+T} \left|E\left(t_2,s\right)-E\left(t_1,s\right)\right| \mathrm{d}s \\
&= \frac{\int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} b(v)\mathrm{d}v\right)\left|1-\exp\left(\int_{t_1}^{t_2} b(v)\mathrm{d}v\right)\right| \mathrm{d}s}{\left(\exp\left(\int_0^T b(v)\mathrm{d}v\right)-1\right)} \\
&= \lambda T \frac{\exp\left(\int_0^T b(v)\mathrm{d}v\right)}{\left(\exp\left(\int_0^T b(v)\mathrm{d}v\right)-1\right)} \left|t_2-t_1\right| \\
&= \lambda T\beta \left|t_2-t_1\right| = \Gamma \left|t_2-t_1\right|,
\end{aligned}$$

which completes the proof. □

3. Existence of periodic solutions

In this section, we state and prove the existence result.

By virtue of Lemma 2.5, we define an operator $\mathcal{H}\colon P_T\left(L,M\right)\to P_T$ by

$$\begin{aligned}
(\mathcal{H}\varphi)(t) = &\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{\ell} G(t,s)\mathrm{d}s \\
&+ \int_t^{t+T} \left[E(t,s)-a(s)G\left(t,s\right)\right] g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\mathrm{d}s \\
&+ \int_t^{t+T} G(t,s)h(s)\mathrm{d}s.
\end{aligned} \tag{15}$$

Clearly, $\left(\mathcal{H}\varphi\right)(t+T) = \left(\mathcal{H}\varphi\right)(t)$ which shows that the operator $\mathcal{H}$ is well defined. Furthermore, fixed points of the operator $\mathcal{H}$ are solutions of equation (1) and vice versa.

Remark. Thanks to Arzelà-Ascoli theorem, we can prove the compactness of $P_T(L,M)$.

Theorem 3.1. *Let $C_{\ell,m}\in P_T\left(L_{C_{\ell,m}}\right)$ and $h\in P_T(L_h)$. Suppose that conditions* (2)*–*(6)*,* (10)*, and* (12)*–*(14) *hold. Then equation* (1) *has at least one periodic solution in $P_T(L,M)$.*

Proof. For establishing the existence result, we use Schauder's fixed point theorem and for the sake of clarity, the proof is done in two steps.

Step 1: We show that $\mathcal{H}$ is a continuous operator on $P_T(L,M)$.

Let φ, $\psi \in P_T(L,M)$. We have

$$\begin{aligned}
&|(\mathcal{H}\varphi)(t)-(\mathcal{H}\psi)(t)|\\
&\le \sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T}|C_{\ell,m}(s)|\,G(t,s)\left|\big(\varphi^{[m]}(s)\big)^{\ell}-\big(\psi^{[m]}(s)\big)^{l}\right|\mathrm{d}s\\
&\quad+\int_t^{t+T}|E(t,s)|\,\Big|g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\\
&\quad-g\big(s,\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\big)\Big|\,\mathrm{d}s\\
&\quad+\int_t^{t+T}|a(s)|\,G\,(t,s)\,\Big|g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\\
&\quad-g\big(s,\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\big)\Big|\,\mathrm{d}s.
\end{aligned}$$

From (5) and Lemma 2.8, we obtain

$$\begin{aligned}
&\Big|g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)-g\big(s,\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\big)\Big|\\
&\le \sum_{m=1}^{n}\sum_{j=0}^{m-1}c_m M^j\,\|\varphi-\psi\|\,.
\end{aligned}\tag{16}$$

From the mean value theorem, it follows that

$$\left|\big(\varphi^{[m]}(s)\big)^{\ell}-\big(\psi^{[m]}(s)\big)^{\ell}\right|\le \ell\,|\eta_m(s)|^{\ell-1}\left\|\varphi^{[m]}-\psi^{[m]}\right\|\tag{17}$$

for some constant $\eta_m(s)$ such that $\varphi^{[m]}(s)\le\eta_m(s)\le\psi^{[m]}(s)$.

Taking into account (16), and (17) and using Corollary 2.7, we get the following estimate

$$\begin{aligned}
|(\mathcal{H}\varphi)(t)-(\mathcal{H}\psi)(t)|\le{}&\alpha_2\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\sum_{j=0}^{m-1}\ell TL^{\ell-1}L_{C_{\ell,m}}M^j\,\|\varphi-\psi\|\\
&+(\alpha_2\lambda_1+\beta)\,T\sum_{m=1}^{n}\sum_{j=0}^{m-1}c_mM^j\,\|\varphi-\psi\|\,,
\end{aligned}$$

which yields

$$\begin{aligned}
&\|(\mathcal{H}\varphi-(\mathcal{H}\psi\|=\sup_{t\in[0,Tt]}|(\mathcal{H}\varphi)\,(t)-(\mathcal{H}\psi)(t)|\\
&\le T\Bigg(\alpha_2L^{\ell-1}\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\sum_{j=0}^{m-1}\ell L_{C_{\ell,m}}M^j+(\alpha_2\lambda_1+\beta)\sum_{m=1}^{n}\sum_{j=0}^{m-1}c_mM^j\Bigg)\,\|\varphi-\psi\|\,.
\end{aligned}$$

Using (12), we infer that $\mathcal{H}$ is continuous.

Step 2: The main concern now is to prove that $\mathcal{H}$ maps the set $P_T(L,M)$ into itself.
First, we prove that $\|\mathcal{H}\varphi\| \leq L$ for all $\varphi \in P_T(L,M)$.
Let $\varphi \in P_T(L,M)$. Using (5), we obtain

$$\left| g(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)) \right| \leq \varsigma. \tag{18}$$

In view of Corollary 2.7 and (18), we get

$$\begin{aligned}
|(H\varphi)(t)| &= \sum_{m=1}^{n} \sum_{\ell=1}^{\infty} \int_{t}^{t+T} |C_{\ell,m}(s)| \left| \left(\varphi^{[m]}(s)\right)^{\ell} \right| G(t,s) \mathrm{d}s \\
&\quad + \int_{t}^{t+T} |E(t,s)| \left| g\big(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \right| \mathrm{d}s \\
&\quad + \int_{t}^{t+T} |a(s)| \, G\left(t,s\right) \left| g\big(s, \varphi^{[1]}\left(s\right), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \right| \mathrm{d}s \\
&\quad + \int_{t}^{t+T} G(t,s) \, |h(s)| \, \mathrm{d}s \\
&\leq \left(\beta + \alpha_2 \lambda_1\right) T\varsigma + \alpha_2 T \bigg(L_h + \sum_{m=1}^{n} \sum_{\ell=1}^{\infty} L_{C_{\ell,m}} L^{\ell} \bigg).
\end{aligned}$$

Thus, from (13), we find that

$$|(\mathcal{H}\varphi)(t)| \leq L \qquad \text{for all } \varphi \in P_T(L,M). \tag{19}$$

Now, we show that $|(\mathcal{H}\varphi)(t_2) - \left(\mathcal{H}\varphi\right)(t_1)| \leq M \, |t_2 - t_1|$ for all $\varphi \in P_T(L,M)$ and $t_1, t_2 \in \mathbb{R}$.

For $\varphi \in P_T(L,M)$ and $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$, we have

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{\ell} G\left(t_2,s\right) \mathrm{d}s - \int_{t_1}^{t_1+T} C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{\ell} G\left(t_1,s\right) \mathrm{d}s \right| \\
&= \bigg| \int_{t_2}^{t_1} C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{\ell} G\left(t_2,s\right) \mathrm{d}s + \int_{t_1+T}^{t_2+T} C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{\ell} G\left(t_2,s\right) \mathrm{d}s \\
&\quad + \int_{t_1}^{t_1+T} \Big[C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{\ell} \Big] \left(G\left(t_2,s\right) - G\left(t_1,s\right)\right) \mathrm{d}s \bigg| \\
&\leq \int_{t_2}^{t_1} |C_{\ell,m}(s)| \left| \big(\varphi^{[m]}(s)\big)^{\ell} \right| G\left(t_2,s\right) \mathrm{d}s + \int_{t_1+T}^{t_2+T} |C_{\ell,m}(s)| \left| \big(\varphi^{[m]}(s)\big)^{\ell} \right| G\left(t_2,s\right) \mathrm{d}s \\
&\quad + \int_{t_1}^{t_1+T} |C_{\ell,m}(s)| \left| \big(\varphi^{[m]}(s)\big)^{\ell} \right| \left| G\left(t_2,s\right) - G\left(t_1,s\right) \right| \mathrm{d}s.
\end{aligned}$$

Using Corollary 2.7 and Lemma 2.10, we arrive at

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{l} G\left(t_2,s\right) \mathrm{d}s - \int_{t_1}^{t_1+T} C_{\ell,m}(s) \big(\varphi^{[m]}(s)\big)^{l} G\left(t_1,s\right) \mathrm{d}s \right| \\
&\leq 2\alpha 2 L^{\ell} L_{C_{\ell,m}} \left| t_2 - t_1 \right| + L^{\ell} L_{C_{\ell,m}} \mu \left| t_2 - t_1 \right| = L^{\ell} L_{C_{\ell,m}} \left(2\alpha_2 + \mu\right) \left| t_2 - t_1 \right|.
\end{aligned} \tag{20}$$

Arguing as before, we also get

$$\begin{aligned}
&\Big|\int_{t_2}^{t_2+T} E\left(t_2,s\right) g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s \\
&\qquad -\int_{t_1}^{t_1+T} E\left(t_1,s\right) g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\leq \int_{t_2}^{t_1} \left|E\left(t_2,s\right)\right| \Big|g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\Big|\,\mathrm{d}s \\
&\qquad +\int_{t_1+T}^{t_2+T} \left|E(t_2,s)\right| \Big|g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\Big|\,\mathrm{d}s \\
&\qquad +\int_{t_1}^{t_1+T} \left|E(t_2,s)-E\left(t_1,s\right)\right| \Big|g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\Big|\,\mathrm{d}s.
\end{aligned}$$

Lemma 2.11 and (18) give

$$\begin{aligned}
&\Big|\int_{t_2}^{t_2+T} E(t_2,s)g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s \\
&\qquad -\int_{t_1}^{t_1+T} E(t_1,s)g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\leq \varsigma(2\beta+\Gamma)\left|t_2-t_1\right|.
\end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned}
&\Big|\int_{t_2}^{t_2+T} G(t_2,s)h\left(s\right)\mathrm{d}s-\int_{t_1}^{t_1+T} G(t_1,s)h(s)\mathrm{d}s\Big| \\
&\leq \int_{t_2}^{t_1} G(t_2,s)\left|h(s)\right|\mathrm{d}s+\int_{t_1+T}^{t_2+T} G(t_2,s)\left|h(s)\right|\mathrm{d}s \\
&\qquad +\int_{t_1}^{t_1+T} \left|G(t_2,s)-G(t_1,s)\right|\left|h(s)\right|\mathrm{d}s.
\end{aligned}$$

Using Corollary 2.7 and Lemma 2.10, we obtain

$$\Big|\int_{t_2}^{t_2+T} G(t_2,s)h(s)\mathrm{d}s-\int_{t_1}^{t_1+T} G(t_1,s)h(s)\mathrm{d}s\Big| \leq L_h\left(2\alpha_2+\mu\right)\left|t_2-t_1\right|. \tag{22}$$

We have also

$$\begin{aligned}
&\Big|\int_{t_2}^{t_2+T} a(s)G(t_2,s)g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s \\
&-\int_{t_1}^{t_1+T} a(s)G(t_1,s)g\big(s,\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big|
\end{aligned}$$

$$\begin{aligned}&\leq \int_{t_2}^{t_1} |a(s)|\, G(t_2,s) \left| g\big(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big)\right| \mathrm{d}s \\ &+ \int_{t_1+T}^{t_2+T} |a(s)|\, G(t_2,s) \left| g\big(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big)\right| \mathrm{d}s \\ &+ \int_{t_1}^{t_1+T} |a(s)|\, |G(t_2,s) - G(t_1,s)| \left| g\big(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big)\right| \mathrm{d}s.\end{aligned}$$

By means of Corollary 2.7, Lemma 2.10, and (18), we find

$$\begin{aligned}&\Bigg| \int_{t_2}^{t_2+T} a(s)G(t_2,s)g\big(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big)\mathrm{d}s \\ &\quad - \int_{t_1}^{t_1+T} a(s)G(t_1,s)g\big(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big)\mathrm{d}s \Bigg| \\ &\leq \lambda_1 \varsigma (2\alpha_2 + \mu)\, |t_2 - t_1|\,.\end{aligned} \tag{23}$$

According to (20)–(23), we get

$$\begin{aligned}&|(H\varphi)(t_2) - (H\varphi)\,(t_1)| \\ &\leq \Bigg(\sum_{m=1}^{n} \sum_{\ell=1}^{\infty} L^{\ell} L_{C_{\ell,m}} (2\alpha_2 + \mu) \Bigg) |t_2 - t_1| + \varsigma\,(2\beta + \Gamma)\, |t_2 - t_1| \\ &\quad + L_h\,(2\alpha_2 + \mu)\, |t_2 - t_1| + \lambda_1 \varsigma\,(2\alpha_2 + \mu)\, |t_2 - t_1|\,.\end{aligned}$$

So,

$$\begin{aligned}&|(H\varphi)(t_2) - (H\varphi)\,(t_1)| \\ &\leq \Bigg[\sum_{m=1}^{n} \sum_{\ell=1}^{\infty} L^{\ell} L_{C_{\ell,m}}\,(2\alpha_2 + \mu) + \varsigma\,(2\beta + \Gamma) \\ &\qquad + L_h\,(2\alpha_2 + \mu) + \lambda_1 \varsigma\,(2\alpha_2 + \mu) \Bigg] \times |t_2 - t_1|\,.\end{aligned}$$

In light of (14) and Lemma 2.9, we get

$$|(H\varphi)(t_2) - (H\varphi)\,(t_1)| \leq M\, |t_2 - t_1| \quad \text{for all } \varphi \in P_T(L,M) \text{ and } t_1, t_2 \in \mathbb{R}. \tag{24}$$

From (19) and (24), we conclude that $\mathcal{H}\,(P_T(L,M)) \subset P_T(L,M)$.

As a consequence of these two steps, Remark 3 and Schauder's fixed point theorem, we conclude that operator $\mathcal{H}$ has at least one fixed point in $P_T(L,M)$ which is a solution of equation (1). □

4. CONCLUSION

The existing results in the literature reveal that despite the great deal of interest being shown in the investigation of iterative differential equations over recent decades, their theory remains an emerging one and has not been yet well-developed.

The present work has aimed to contribute to the investigation of such equations and more precisely, we have probed into the existence of periodic solutions for a second order iterative differential equation by following the steps below.

Firstly, based on some earlier publications, we have introduced a new second order iterative differential problem and set up an appropriate subset of the Banach space of continuous periodic functions. Secondly, we have established an equivalence between this problem and a certain integral equation with a Green's function type kernel. Finally, with the aid of Schauder's fixed point theorem, some properties of the obtained kernel and the Arzelà-Ascoli theorem, we have proved that the corresponding integral operator admits at least one fixed point which is a solution of the original problem.

References

1. Agarwal R. P., Bohner M. and Wan-Tong L., *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, Marcel Dekker Inc, New York, 1984.

2. Berinde V., *Existence and approximation of solutions of some first order iterative differential equations*, Miskolc Math. Notes. **11** (2010), 13–26.

3. Bouakkaz A., Ardjouni A. and Djoudi A., *Existence of positive periodic solutions for a second-order nonlinear neutral differential equation by the Krasnoselskii's fixed point theorem*, Nonlinear Dyn. Syst. Theory **17** (2017), 230–238.

4. Bouakkaz A., Ardjouni A. and Djoudi A., *Periodic solutions for a second order nonlinear functional differential equation with iterative terms by Schauder fixed point theorem*, Acta Math. Univ. Comen. **87** (2018), 223–235.

5. Bouakkaz A., Ardjouni A., Khemis R. and Djoudi A., *Periodic solutions of a class of third-order functional differential equations with iterative source terms*, Bol. Soc. Mat. Mex. **26** (2020), 443–458.

6. Bouakkaz A. and Khemis R., *Positive periodic solutions for a class of second-order differential equations with state-dependent delays*, Turkish J. Math. **44** (2020), 1412–1426.

7. Bouakkaz A. and Khemis R., *Positive periodic solutions for revisited Nicholson's blowflies equation with iterative harvesting term*, J. Math. Anal. Appl. **494** (2021), Art. ID 124663.

8. Cheraiet S., Bouakkaz A. and Khemis R., *Bounded positive solutions of an iterative three-point boundary-value problem with integral boundary conditions*, J. Appl. Math. Comput. **65** (2021), 597–610.

9. Cooke K. L., *Functional differential equations: some models and perturbation problems*, in: Differential Equations and Dynamical Systems, Academic Press, New York, 1967, 167–183.

10. Driver R. D., *Delay-differential Equations and an Application to a Two-body Problem of Classical Electrodynamics*, Thesis Ph.D., University of Minnesota, 1960.

11. Guerfi A. and Ardjouni A., *Periodic solutions for second order totally nonlinear iterative differential equations*, J. Anal. **30** (2022), 353–367.

12. Khemis R., Ardjouni A. and Djoudi A., *Existence of periodic solutions for a second-order nonlinear neutral differential equation by the Krasnoselskii's fixed point technique*, Matematiche **72** (2017), 145–156.

13. Khemis R., Ardjouni A., Bouakkaz A. and Djoudi A., *Periodic solutions of a class of third-order differential equations with two delays depending on time and state*, Comment. Math. Univ. Carolin. **60** (2019), 379–399.

14. Li Y. and Kuang Y., *Periodic solutions in periodic state-dependent delay equations and population models*, Proc. Amer. Math. Soc. **130** (2002), 1345–1353.

15. Liu Y. and Ge W., *Positive periodic solutions of nonlinear Duffing equations with delay and variable coefficients*, Tamsui Oxf. J. Math. Sci. **20** (2004), 235–255.

16. Wang Y., Lian H. and Ge W., *Periodic solutions for a second order nonlinear functional differential equation*, Appl. Math. Lett. **20** (2007), 110–115.
17. Yang D. and Zhang W., *Solutions of equivariance for iterative differential equations*, Appl. Math. Lett. **17** (2004), 759–765.
18. Zhao H. Y. and Fečkan M., *Periodic solutions for a class of differential equations with delays depending on state*, Math. Commun. **23** (2018), 29–42.
19. Zhao H. Y. and Liu J., *Periodic solutions of an iterative functional differential equation with variable coefficients*, Math. Methods Appl. Sci. **40** (2017), 286–292.
20. Zhao H. Y. and Liu J., *Periodic solutions of a second-order functional differential equation with state-dependent argument*, Mediterr. J. Math. **15** (2018), 214.

R. Khemis, Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), 20 August 1955 University of Skikda, Algeria,
e-mail: `kbra28@yahoo.fr; r.khemis@univ-skikda.dz`

A. Bouakkaz, Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), 20 August 1955 University of Skikda, Algeria,
e-mail: `ahlemkholode@yahoo.com; a.bouakkaz@univ-skikda.dz`

S. Chouaf, Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), 20 August 1955 University of Skikda, Algeria,
e-mail: `safachouaf10@gmail.com; s.chouaf@univ-skikda.dz`