الجمهورية الجزائرية الديمقراطية الشعبية

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

وزارة التعليم العالي والبحث العلمي

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

20 août 1955 University – Skikda
Faculty of Sciences
Department de Mathematics

جامعة 20 أوت 1955 - سكيكدة
كلية العلوم
قسم الرياضيات

Master's Thesis

Field : Mathematics and Computer Science
Program : Mathematics
Option : AFA

Subject

A note on homogenization of the hyperbolic-parabolic equations in domains with small holes

Presented by:
M'sallaoui Meriem Ines

Publicly defended on: 02/07/2025

Jury Committee:

Maouni Messaoud	**Pr,**	**Skikda University**	**Chair**
Karek Chafia	**M.C.B,**	**Skikda University**	**Supervisor**
Faghmous Chadia	**M.C.B,**	**Skikda University**	**Examiner**

Academic Year: 2024/2025

Acknowledgements

First of all, we thank Allah for helping, guiding and giving me the ability to complete this memory.Then, I would like to extend my deepest gratitude to my doctor Karek Chafia for her unwavering support and encouragement throught these years, as well as for all the efforts she has made and the valuable knowledge she has imparted to me. May she always be granted continued success and guidance. Also, special thanks should be addressed to my professor Maouni Messaoud for every word of knowledge he has taught me and for his constant, encouragement and support.

Finally, I would like to address our gratitude for all teachers of our department for their advice during these years.

DEDICATION

I DEDICATE MY GRADUATION TO MY PAARENTS, IN GRATITUDE FOR THEIR UNWAVERING SUPPORT, ENCOURAGEMENT,

AND FOR PROVIDING EVERYTHING I NEEDED THROUGHT MY ACADEMIC JOURNEY MAY GOD

PRESERVE THEM AND GRANT THEM A LONG LIFE.

I ALSO DEDICATE THIS ACHIEVEMENT TO MY SIBLINGS, MY FAMILY , AND TO EVERYONE WHO BELIVED ME

AND STOOD BY MY SIDE EVEN WITH JUST A WORD OF ENCOURAGEMENT.

Abstract

This memory concerns the asymptotic behavior of the hyperbolic-parabolic problems in periodically perforated domains with small holes and Dirichlet conditions on the boundary of the holes. We focus on the homogenezation of these equations. Using the time-dependent unfolding method, we obtain some homogenization and corrector results.

Key words : hyperbolic-parabolic problem, periodic unfolding, perforated domain, homognisation.

Résumé

Ce mémoire porte sur le comportement asymptotique d'un probléme hyperbolique-parabolique dans un domaine perforé périodiquement avec des petit trous. Nous nous intressons sur l' homogénéisation de ces équations avec des conditions de Neumann et de Dirichlet sur le bords des trous. En utilisant la méthode de l'éclatement périodique dépendant du temps, nous obtenons des résultats d' homogénéisation et des résultats correcteurs.

Mots clés : probléme hyperbolique-parabolique éclatement périodique, domaine perforé, homogénéisation.

ملخص

في هذه المذكرة قمنا بدراسة السلوك التقاربي لمسألة من النوع القطع المكافئ الزائدي في مجالات مسامية بشكل دوري تحتوي على ثقوب صغيرة. نركز في هذا العمل على تجانس هذه المعادلات تحت شروط حدية من نوع نيومان وديريكليه. و ذلك باستخدام طريقة الإنفجار الدوري المتعلقة بالزمن، تمكنا من الحصول على نتائج دقيقة للتجانس بالإضافة إلى نتائج تصحيحية.

كلمات مفتاحية: مشكلة قطع مكافئ زائدي ، انتشار دوري، مجال مثقب، تجانس.

Notation

Symbole	**Signification**
∇u	The gradient of u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)$
Δu	$= div(\nabla u)$
$\mathcal{D}(\Omega)$	The set of continuous functions whose support is a compact set of $\mathbb{R}^N$ contained in Ω
$L^p(\Omega)$	The Lebesgue space, $\begin{cases} \{u : \Omega \longmapsto \mathbb{R} \text{ mesurable } ; \int_\Omega \lvert u\rvert^p < +\infty\} & \text{if } 1 \leq p \leq +\infty, \\ \{u : \Omega \longmapsto \mathbb{R}, \text{ u mesurable and such that there exists } C \in \mathbb{R}; \text{ with } \lvert u\rvert \leq C\} & \text{if } p = +\infty \end{cases}$
$L^p_{loc}(\Omega)$	$= \{u \in L^p(\omega), \text{ for any open bounded set } \omega \text{ with } \overline{\omega} \subset \Omega\}$
$L^{p'}(\Omega)$	The dual space of $L^p(\Omega)$
$L^p(\Omega; X)$	The set of measurable functions: $u : x \in \Omega \to u(x) \in X$ such that $\lVert u(x)\rVert_X \in L^p(\Omega)$.
$H^1(\Omega)$	$\left\{u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \dots, N\right\}$
$H^1_0(\Omega)$	The space of function in $H^1(\Omega)$ will vanish on the boundary in the sens of the trace.
$H^{-1}(\Omega)$	the dual space of $H^1_0(\Omega)$

CONTENTS

INTRODUCTION

The theory of homogenization plays a fundamental role in the analysis of partial differential equations (***PDEs***) with rapidly oscilating coefficients or defined in domains with fine structures. It provides powerful tools to approximate complex heterogeneous media by effective homogeneus models, which are often easier to analyse and simulate numerically. This is particularly relevant in the modeling of physical phenomena such as heat conduction, fluid flow, or wave propagation in compsite or perforated materials.

In homogenization we have a lot of methods like energy method, multiscal method, periodic unfolding method ..., we use here the periodic unfolding method.

This method was initially introduced by Cioranescu, Damlamian, and Griso [3] for fixed domains(see [4] for further details). It was subsequently extended to perforated domains by Cioranescu, Donato, and Zaki [10]. This method was later adapted for two-component domains separated by a periodic interface by Donato, Nguyen, and Tardieu [13]. Recently, Cioranescu, Damlamian, Donato, Griso, and Zaki [6] provided a comprehensive overview of the periodic unfolding method for perforated domains, addressing both cases where the unit hole is compact within the open unit cell and when this compactness cannot be achieved.

Concerning the time-dependent periodic unfolding method for fixed domains, our reference is Gaveau [15], which provides a list of elementary results without proofs.

However, to address time-dependent problems in perforated domains, we adapt the results from [6] to accommodate time-dependent functions. Specifically, we focus on treating perforated domains with small holes.

The aim of this work is to study the asymptotic behavior of the following hyperbolic-parabolic system

$$\begin{cases} u''_{\varepsilon,\delta}(x,t) + u'_{\varepsilon\delta}(x,t) - div(A^{\varepsilon}(x)\nabla u_{\varepsilon,\delta}(x,t)) = f_{\varepsilon,\delta}(x,t) \quad \text{in} \quad \Omega^*_{\varepsilon,\delta}\times]0,T[\\ u_{\varepsilon,\delta}(x,t) = 0 \quad \text{on} \quad \partial\Omega^*_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x) \ , \ u'_{\varepsilon,\delta}(x,0) = u^1_{\varepsilon,\delta}(x) \quad \text{in} \quad \Omega^*_{\varepsilon,\delta} \end{cases} \tag{1}$$

For the initial data, we always assume $u^0_{\varepsilon,\delta} \in H^1_0(\Omega^*_{\varepsilon,\delta})$, $u^1_{\varepsilon,\delta} \in L^2(\Omega)$ and $f_{\varepsilon,\delta} \in L^2(0,T;L^2(\Omega^*_{\varepsilon,\delta}))$

The problem (1) models many kinds of phenomena arising in electricity and mag- netism, in the theory of elasticity, in hydrodynamics and in vibration theory [17] and [18].

Our aime here is to apply the periodic infolding methode due to cioranescu, Damlamian and Griso [[3], [4]] and extended to perfated domains in Cioranescu, Damlamian, Donato, Griso and Zaki [7].

This memory is organized as flows:

In the first chapter: we define first the periodic unfolding for one in perforated domains and some properties and also we define the unfolding operator in domains depending on two parameters and some properties .

In the second chapter:we define the time periodic unfolding for one and two parametres in perforated domains and some properties.

In the last chapter: we study the homoginization of hyperbolic-parabolic problems in the perforated domain Ω_{ε} and $\Omega_{\varepsilon,\delta}$.

CHAPTER

1

THE PERIODIC UNFOLDING METHOD IN PERFORATED DOMAINS

In this chapter we recall the definition and some properties of the periodic unfolding operators in perforated domains $\mathcal{T}_\varepsilon$ for the classical homogenization .

1.1 The periodic unfolding operator $\mathcal{T}_\varepsilon$

In this section, we introduce the case of perforated domains introduced by cioranescu at all in the following we denote:

Definition 1.1 *[3] Let $\varphi \in L^p(\Omega_\varepsilon), p \in [1, +\infty]$. We define the function:*

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \tag{1.1}$$

for every $x \in \mathbb{R}^N$ and $y \in Y$.

Remark 1.1 *Notice that the oscillations due to perforation are shifted into second variable* $\boldsymbol{y}$ *which belongs to the fixed domain* $\boldsymbol{Y}$, *while the first variable* $\boldsymbol{x}$ *belongs to* $\mathbb{R}^{\boldsymbol{N}}$.
One see immediatly the interst of the unfolding operator. Indeed when trying to pass to the limit in sequance defined on $\boldsymbol{\Omega}_\varepsilon$ *one needs first, while using standard methods,to extend it to a fixed domain. with* $\boldsymbol{\mathcal{T}}_\varepsilon$, *such extensions are more necessary.*
The main properties given in[7] for fixed domain can easily be adapted for the perforated ones without any major difficlty in the proofs. These properties are listed in the proposition below. To do so, let us first define the following domain: $\hat{\boldsymbol{\Omega}}_\varepsilon = \left(\boldsymbol{int} \bigcup_{\xi \in \boldsymbol{\Lambda}_\varepsilon} (\boldsymbol{\xi} + \boldsymbol{Y}) \right)$ *where:*

$$\boldsymbol{\Lambda}_\varepsilon = \{\varepsilon \in \mathbb{Z}^N, \varepsilon(\boldsymbol{\xi} + \overline{\boldsymbol{Y}}) \cap \boldsymbol{\Omega} \neq \emptyset\} = \boldsymbol{\Omega} \setminus \hat{\boldsymbol{\Omega}}_\varepsilon.$$

$\boldsymbol{\Xi}_\varepsilon = \{\boldsymbol{\xi} \in \mathbb{Z}^N, \varepsilon(\boldsymbol{\xi} + \boldsymbol{Y}) \subset \boldsymbol{\Omega}\}$
The set $\hat{\boldsymbol{\Omega}}_\varepsilon$ *is the smallest finite union of* $\boldsymbol{\varepsilon Y}$ *cells containing* $\boldsymbol{\Omega}$

Proposition 1.1 *[3] The unfolding operator* $\boldsymbol{\mathcal{T}}_\varepsilon$ *has the following properties :*

1. $\boldsymbol{\mathcal{T}}_\varepsilon$ *is a linear operator.*
2. $\boldsymbol{\mathcal{T}}_\varepsilon(\varphi)\left(\boldsymbol{x}, \left\{\frac{\boldsymbol{x}}{\varepsilon}\right\}_{\boldsymbol{Y}}\right) = \varphi(\boldsymbol{x}) \ \ \forall \varphi \in \boldsymbol{L}^p(\boldsymbol{\Omega}_\varepsilon)$, $\boldsymbol{x} \in \mathbb{R}^N$.
3. $\boldsymbol{\mathcal{T}}_\varepsilon(\varphi\psi) = \boldsymbol{\mathcal{T}}_\varepsilon(\varphi)\boldsymbol{\mathcal{T}}_\varepsilon(\psi) \ \ \forall \varphi, \psi \in \boldsymbol{L}^p(\boldsymbol{\Omega}_\varepsilon)$.
4. *Let* $\varphi \in \boldsymbol{L}^p(\boldsymbol{Y})$ *or* $\boldsymbol{L}^p$ *be periodic function set* $\phi(\frac{\boldsymbol{x}}{\varepsilon})$ *Then* $\boldsymbol{\mathcal{T}}_\varepsilon(\varphi_\varepsilon)(\boldsymbol{x}, \boldsymbol{y}) = \varphi(\boldsymbol{y})$. *a,e. in* $\hat{\boldsymbol{\Omega}}_\varepsilon$.
5. *one has the integration formula:* $\int_{\boldsymbol{\Omega}} \varphi \, \boldsymbol{dx} = \frac{1}{|\boldsymbol{Y}|} \int_{\boldsymbol{\Omega} \times \boldsymbol{Y}} \boldsymbol{\mathcal{T}}_\varepsilon(\varphi) \, \boldsymbol{dx} \, \boldsymbol{dy} \, \forall \varphi \in \boldsymbol{L}^1(\boldsymbol{\Omega}_\varepsilon)$.
6. *For every* $\varphi \in \boldsymbol{L}^2(\boldsymbol{\Omega}_\varepsilon), \boldsymbol{\mathcal{T}}_\varepsilon(\varphi)$ *belongs to* $\boldsymbol{L}^2(\mathbb{R}^N \times \boldsymbol{Y})$. *It also belongs to* $\boldsymbol{L}^2(\hat{\boldsymbol{\Omega}}_\varepsilon \times \boldsymbol{Y})$.
7. *For every* $\varphi \in \boldsymbol{L}^2(\boldsymbol{\Omega}_\varepsilon)$ $\|\boldsymbol{\mathcal{T}}_\varepsilon(\varphi)\|_{\boldsymbol{L}^2(\mathbb{R}^N \times \boldsymbol{Y})} = \sqrt{|\boldsymbol{Y}|}\|\varphi\|_{\boldsymbol{L}^2(\boldsymbol{\Omega}_\varepsilon)}$.
8. $\boldsymbol{\nabla}\boldsymbol{\mathcal{T}}_\varepsilon\varphi(\boldsymbol{x}, \boldsymbol{y}) = \varepsilon\boldsymbol{\mathcal{T}}(\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi)(\boldsymbol{x}, \boldsymbol{y})$ *for every* $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\Omega} \times \boldsymbol{Y}$.
9. *If* $\varphi \in \boldsymbol{H}^1(\hat{\boldsymbol{\Omega}}_\varepsilon)$, *then* $\boldsymbol{\mathcal{T}}_\varepsilon(\varphi)$ *is in* $\boldsymbol{L}^2(\mathbb{R}^N, \boldsymbol{H}^1(\boldsymbol{Y}))$.
10. *One has the estimate* $\|\boldsymbol{\nabla}\boldsymbol{\mathcal{T}}_\varepsilon(\varphi)\|_{\boldsymbol{L}^2(\mathbb{R}^N \times \boldsymbol{Y})} = \varepsilon\sqrt{|\boldsymbol{Y}|}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi\|_{(\boldsymbol{L}^2(\boldsymbol{\Omega}_\varepsilon))^N}$.

Proof 1.1 1. *Let* $\alpha, \beta \in \mathbb{R}$ *and* φ, ψ $L^2(\Omega_\varepsilon)$

$$\begin{aligned}\mathcal{T}_\varepsilon\left(\alpha\varphi + \beta\psi\right) &= \varphi\left(\alpha\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)\right) + \psi\left(\beta\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)\right)\\ &= \alpha\,\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) + \beta\,\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \alpha\,\mathcal{T}_\varepsilon(\varphi) + \beta\,\mathcal{T}_\varepsilon(\psi)\end{aligned}$$

$\mathcal{T}_\varepsilon$ *is a linear operator .*

2. $\mathcal{T}_\varepsilon(\varphi)\left(x, \left\{\frac{x}{\varepsilon}\right\}\right) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon\left\{\frac{x}{\varepsilon}\right\}\right) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + x - \varepsilon\left[\frac{x}{\varepsilon}\right]\right) = \varphi(x).$

3. *Let* $\varphi, \psi \in L^2(\Omega_\varepsilon)$, *have by definition1.1*

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi\psi) &= \varphi\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)\\ &= \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)\psi\left(\varepsilon\left[\frac{x}{y}\right] + \varepsilon y\right)\\ &= \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)\end{aligned}$$

4. *by definition for all* $\varphi_\varepsilon = \varphi(\frac{x}{\varepsilon}), \varphi \in L^2(Y)$, *we have*

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi_\varepsilon)(x,y) &= \varphi\left(\varepsilon\left[\frac{\left\{\frac{x}{\varepsilon}\right\}}{\varepsilon}\right] + \varepsilon\left\{\frac{\left\{\frac{x}{\varepsilon}\right\}}{\varepsilon}\right\}\right)\\ &= \varphi\left\{\frac{x}{\varepsilon}\right\} = \varphi(y)\end{aligned}$$

5. *According the definition 1.1 we have*

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y) &= \frac{1}{|Y|}\int_{\hat{\Omega}_\varepsilon\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dx\,dy\\ &= \frac{1}{|Y|}\sum_{\xi\in\Xi_\varepsilon}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dx\,dy\end{aligned}$$

on each set $(\varepsilon\xi+\varepsilon Y)\times Y$ *with* $\xi \in \Xi_\varepsilon$, *the function* $\mathcal{T}_\varepsilon(\varphi)(x,y)dx\,dy = \varphi(\varepsilon\xi+\varepsilon y)$ *is constant in x , it is consequence of 1.1.So for each integral in the sum of the right* *member, we have:*

$$\begin{aligned}
\int_{(\varepsilon\xi+\varepsilon Y)\times Y} \mathcal{T}_\varepsilon(\varphi)(x,y)\,dx\,dy &= |\varepsilon\xi+\varepsilon Y| \int_Y \varphi(\varepsilon\xi+\varepsilon y)\,dy \\
&= \varepsilon^N |Y| \int_Y \varphi(\varepsilon\xi+\varepsilon y)\,dy \qquad (1.2)\\
&= |Y| \int_{(\varepsilon\xi+\varepsilon Y)} \varphi(x)\,dx.
\end{aligned}$$

we suppose $x = \varepsilon\xi + \varepsilon y$, *this implies* $dx = \varepsilon^N\,dy \Rightarrow dy = \dfrac{1}{\varepsilon^N}\,dx$ *by summing in* Ξ, *the right member be* $\int_{\hat{\Omega}_\varepsilon} \varphi(x)\,dx$, *so we get*
$\sum_{\xi\in\Xi_\varepsilon} \varepsilon^N |Y| \int_{(\varepsilon\xi+\varepsilon Y)} \varphi(x)\dfrac{1}{\varepsilon^N}\,dx = \int_{\hat{\Omega}_\varepsilon} \varphi(x)\,dx$.
Now we remplace this expression in (1.2) *we get the result.*

6. *For all* $\varphi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned}
\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N\times Y)} &= \left(\int |\mathcal{T}_\varepsilon(\varphi)|^2 dx\,dy\right)^{\frac{1}{2}} \\
&= \left(1\int_{\Omega\times Y} |\varphi|^2\,dx\,dy\right)^{\frac{1}{2}} \\
&= \left(\int 1 dy\right)^{\frac{1}{2}} \left(\int |\varphi|^2\right)^{\frac{1}{2}} \\
&= \sqrt{|Y|}\|\varphi\|_{L^2(\Omega_\varepsilon)}
\end{aligned}$$

7. *For all* $\varphi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned}
\nabla_y \mathcal{T}_\varepsilon(\varphi)(x,y) &= \nabla_y \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right) \\
&= \varepsilon\nabla_x \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right) \\
&= \varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)(x,y)
\end{aligned}$$

8. *We apply proposition 1.1, we obtain:*

$$\begin{aligned}
\|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon\times Y))^N} &= \|\varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)\|_{(L^2(\Omega\times Y))^N} \\
&= \varepsilon\sqrt{|Y|}\|\nabla_x\varphi\|_{L^2(\Omega_\varepsilon))^N}.
\end{aligned}$$

1.2 unfolding operator in domains depending on two parameters

In this chapter we recall the definition and some of its properties of the unfoling operator $\mathcal{T}_{\varepsilon,\delta}$ depending on two small parameters ε and δ as introduced in [5].

Definition 1.2 *Let $p \in [1, +\infty[$. For $\varphi \in L^p(\Omega)$ the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ is the function*

$\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^{\mathbb{N}})$ *defined by:*

$$\mathcal{T}_{\varepsilon,\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x, \delta z) \ \text{if} \ (x,z) \in \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y \\ 0 \quad \text{otherwise.} \end{cases}$$

where is $\mathcal{T}_\varepsilon$ the operator for fixed domaines as introduced in. To go futher , let us introduce what is called a perforated domain with small holes denpted here $\Omega^*_{\varepsilon,\delta}$. Let B be an open set, such that $B \subset\subset Y$ and denote $Y^*_\delta = \frac{Y}{\delta \overline{B}}$, δB is the holes in Y set:$B_{\varepsilon,\delta} = \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + \delta B)$, $\Omega_{\varepsilon,\delta} = \{x \in \Omega | \{\frac{x}{\varepsilon}\} \in Y_\delta\}$ Then $\Omega^*_{\varepsilon,\delta}$ is defined as $\Omega^*_{\varepsilon,\delta} = \{x \in \Omega$ such that $\{\frac{x}{\varepsilon}\}_Y \in Y^*_\delta\}$ where $\delta \to 0$. this definition means that $\Omega^*_{\varepsilon,\delta}$, is a domain ε periodically perforated by holes $\varepsilon\delta B$,

Remark 1.2 *As shown in , it turns out that operator $\mathcal{T}_{\varepsilon,\delta}$ is well-adapted for domains with small holes when dealing with functions which vanish on the boundry of $\Omega^*_{\varepsilon,\delta}$. It is precisely the case we treat in this work. We will deal with functions belongin in particular, to $H^1_0(\Omega^*_{\varepsilon,\delta})$. The extensions of these functions by zero to the whole of Ω , belong to $H^1_0(\Omega)$. consequently in the sequel, we will not distinguish the elements of $H^1_0(\Omega^*_{\varepsilon,\delta})$ and their extensions from $H^1_0(\Omega)$*

Definition 1.3 *Let $p \in]1, +\infty[$. The local average $M^\varepsilon_Y : L^p(\Omega) \mapsto L^p(\Omega)$, is defined for any ϕ in $L^p(\Omega)$ by $M^\varepsilon_Y(\phi)(x) = \int_{(1/\delta)Y} \mathcal{T}_{\varepsilon,\delta}\phi(x,z)\, dz$.*

note that $\mathcal{T}_{\varepsilon,\delta}(M^\varepsilon_Y(\phi)) = M^\varepsilon_Y(\phi) \quad$ on $\Omega \times \frac{1}{\delta}Y$.

Also, if $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$, such that $v_\varepsilon \to v$ strongly in $L^p(\Omega)$, then $M^\varepsilon_Y(v_\varepsilon) \to v \quad$ strongly in $L^p(\Omega)$.

Proposition 1.2 *:*

(1) *For any* $v, w \in L^p(\Omega), \mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w).$

(2) *For any* $u \in L^1(\Omega), \delta^N \int_{\Omega\times\mathbb{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)|\, dx\, dz \leq \int_\Omega |u|\, dx.$

(3) *For any* $u \in L^2(\Omega)$, $\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leq \dfrac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}.$

(4) *For any* $u \in L^1(\Omega), \left|\int_\Omega u\, dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u)\, dx\, dz\right| \leq \int_{\Lambda_\varepsilon} |u|\, dx.$

(5) *Let* $u \in H^1(\Omega)$. *Then* $\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \dfrac{1}{\varepsilon,\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))$, *in* $\Omega \times \dfrac{1}{\delta}Y.$

(6) *suppose* $N \geqslant 3$ *and let* $\omega \subset \mathbb{R}^N$ *be open and bounded . The following estimates hold:*
$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \dfrac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$
$\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))\|^2_{L^2(\Omega;L^2_*(\mathbb{R}^N))} \leq \dfrac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$
$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega\times\omega)} \leq \dfrac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{\frac{2}{N}}\|\nabla u\|^2_{L^2(\Omega)} + 2|\omega|\|u\|^2_{L^2(\Omega)},$
where C is the sobolev-poincare-wirtinger constant for $H^1(Y)$.

(7) *Suppose* $N \geqslant 3$ *and let* $\omega_{\varepsilon,\delta}$ *be a sequence in* $H^1(\Omega)$ *which is uniformly bounded as both* ε *and* δ *approach* 0. *Then there exists* W *in* $L^2(\Omega; L^2_*(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^2(\Omega\times\mathbb{R}^N)$ *such that, up to a subsequence,* $\dfrac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(\omega_{\varepsilon,\delta}) - \mathcal{M}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup W$ *weakly in* $L^2(\Omega; L^2_*(\mathbb{R}^N))$, *and*
$\dfrac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\omega_{\varepsilon,\delta})) \rightharpoonup \nabla_z W$ *weakly in* $L^2(\Omega\times\mathbb{R}^N)$.
Furthermore if, $\lim \sup_{\varepsilon,\delta\to(0^+,0^+)} \dfrac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty$, *then one can choose the subsequence above and some* $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ *such that* $\dfrac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(\omega_{\varepsilon,\delta}) \rightharpoonup U$ *weakly in* $L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$.

Definition 1.4 *A sequence* $\{v_{\varepsilon,\delta}\}$ *in* $L^1(\Omega)$ *satisfies the unfolding criterion for integrals* $(u.c.i)$ *if*
$\int_\Omega v_{\varepsilon,\delta}\, dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})\, dx\, dz \to 0,$
for every sequence $(\varepsilon, \delta) \to (0^+, 0^+)$ *this property is denoted*

$$\int_\Omega v_{\varepsilon,\delta}\, dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})\, dx\, dz.$$

Proposition 1.3 ([[5]$\boldsymbol{u.c.i}$) *if* $\boldsymbol{v_\varepsilon}$ *is a sequence in* $\boldsymbol{L^1(\Omega)}$ *satisfying* $\displaystyle\int_{\Lambda_\varepsilon} |\boldsymbol{u}_{\varepsilon|}\, \boldsymbol{dx} \to \boldsymbol{0}$
then it satisfies:

$$\int_\Omega \boldsymbol{v}_{\varepsilon,\delta}\, \boldsymbol{dx} \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \boldsymbol{\delta}^N \int_{\Omega\times\mathbb{R}^N} \boldsymbol{\mathcal{T}}_{\varepsilon,\delta}(\boldsymbol{v}_{\varepsilon,\delta})\, \boldsymbol{dx}\, \boldsymbol{dz}.$$

Remark 1.3 *As observed in [5], for any* $\boldsymbol{\psi} \in \boldsymbol{D}(\boldsymbol{\Omega})$, *one has* $\|\boldsymbol{\mathcal{T}}_{\varepsilon,\delta(\psi)-\psi}\|_{L^\infty(\Omega\times\frac{1}{\delta}Y)} \to \boldsymbol{0}$.

CHAPTER

2

TIME-DEPENDENT UNFOLDING OPERATOR IN PERFORATED DOMAINS

2.1 The time-dependent unfolding operator in perforated domains

In this section, we adapt the unfolding operator in [7] to time-dependent functions we present the unfolding operator $\boldsymbol{\mathcal{T}_\varepsilon^*}$ which maps functions defined on the oscillating domain $\boldsymbol{\Omega_\varepsilon^* \times (0,T)}$ into functions defined on the fixed domain $\boldsymbol{\Omega \times Y^* \times (0,T)}$. This avoids the use of any extension operator. We also give some properties by extending the previous ones (see [7] for more details and [4] for the classical unfolding theory).
For any $\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{n}}$, we use $[\boldsymbol{z}]_{\boldsymbol{y}}$ to denote the unique integer combination $\sum\limits_{j=1}^{n} \boldsymbol{k_j b_j}$ of the period such that $\boldsymbol{z} - [\boldsymbol{z}]_{\boldsymbol{Y}} \in \boldsymbol{Y}$ set:

$$\{\boldsymbol{z}\}_{\boldsymbol{Y}} = \boldsymbol{z} - [\boldsymbol{z}]_{\boldsymbol{Y}} \in \boldsymbol{Y} \text{ a.e. } \boldsymbol{For\ z} \in \mathbb{R}^{\boldsymbol{n}}. \tag{2.1}$$

Then for each $z \in \mathbb{R}^n$, we have:

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}\right) \text{ a.e. } \boldsymbol{for}\ x \in R^n$$

Let us first recall the unfolding operator $\mathcal{T}_\varepsilon$ for the fixed domain $\Omega \times (0, T)$ introduced in [11], where the properties of $\mathcal{T}_\varepsilon$, are shown without proofs.

Definition 2.1 *For $p \in [1, +\infty)$ and $q \in [1, \infty)$, let ϕ be in$L^q(0, T; L^p(\Omega))$. The unfolding operator $\mathcal{T}_\varepsilon^* : L^q(0, T; L^p(\Omega)) \to L^q(0, T; L^p(\Omega \times Y))$ is defined as follows:*

$$\mathcal{T}_\varepsilon^*(\phi)(x, y, t) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right), \text{ a.e., for } (x, y, t) \in \hat{\Omega}_\varepsilon \times Y \times (0, T), \\ 0, \quad \text{a.e., for } (x, y, t) \in \Lambda_\varepsilon \times Y \times (0, T). \end{cases} \tag{2.2}$$

In a similar way , we extend the unfolding operator for the perforated domain Ω_ε^* to the following unfolding operator $\mathcal{T}_\varepsilon^*$ the perforated domain $\Omega_\varepsilon^* \times (0, T)$.

Definition 2.2 *For $p \in [1, +\infty)$ and $q \in [1, +\infty]$ let ϕ be in $L^q(0, T; L^p(\Omega))$ the unfoldin operator*
$\mathcal{T}_\varepsilon^* : L^q(0, T; L^p(\Omega)) \to L^q(0, T; L^p(\Omega \times Y))$ is defined as follows:

$$\mathcal{T}_\varepsilon^*(x, y, t) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y, t\right) \text{ a.e. for } (x, y, t) \in \hat{\Omega}_\varepsilon \times Y^* \times (0, T), \\ 0 \text{ a.e. for } (x, y, t) \in \Lambda_\varepsilon \times Y^* \times (0, T). \end{cases}$$

From this definition, the following properties are imediate:

(i) for $w, v \in L^q(0, T; L^p(\Omega_\varepsilon^*)), \mathcal{T}_\varepsilon^*(vw) = \mathcal{T}_\varepsilon^*(v)\mathcal{T}_\varepsilon^*(w)$

(ii) For $\psi \in L^q(\Omega_\varepsilon^*)$ and $\varphi \in L^q(0, T)$

$$\mathcal{T}_\varepsilon^*(\psi\varphi) = \varphi\mathcal{T}_\varepsilon^*(\psi), \tag{2.3}$$

(iii) for $\phi \in L^q(0, T; W^{1,p}(\Omega_\varepsilon^*)), \nabla_y(\mathcal{T}_\varepsilon^*(\phi)) = \varepsilon\mathcal{T}_\varepsilon^*(\nabla\phi)$

Remark 2.1 *concerning $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$, we have the following:*

(i) $\mathcal{T}_\varepsilon^*(\omega|_{\Omega_\varepsilon^* \times (0,T)}) = \mathcal{T}_\varepsilon(\omega)|_{\Omega \times Y^* \times (0,T)}$,

(ii) $\mathcal{T}^*_\varepsilon(\psi) = \mathcal{T}_\varepsilon(\widetilde{\psi})|_{\Omega\times Y^*\times(0,T)}$,

where ω and ψ are defined on $\Omega\times(0,T)$ and Ω^*_ε, respectively. in definition2.1 and 2.2 if ϕ is independent of t, then $\mathcal{T}_\varepsilon$ and $\mathcal{T}^*_\varepsilon$ are the classical unfolding operators defined in [3] and [2], respectively .

For simplicity, we always write $\mathcal{T}^*_\varepsilon(\phi)$ instead of $\mathcal{T}^*_\varepsilon(\phi|_{\Omega^*_\varepsilon\times(0,T)})$ for any function ϕ defined in $\Omega\times(0,T)$.

Next we list some properties of the unfolding operator $\mathcal{T}^*_\varepsilon$ used in this work.

Proposition 2.1 *for $p\in[1,+\infty)$ and $q\in[1,\infty)$. the operator $\mathcal{T}^*_\varepsilon$ is linear and continuous from $L^q(0,T;L^p(\Omega^*_\varepsilon))$ to $L^q(0,T;L^p(\Omega\times Y^*))$. let $\phi\in L^q(0,T,L^1(\Omega^*_\varepsilon))$ and $\omega\in L^q(0,T,L^q(\Omega^*_\varepsilon))$ for a.e. $t\in(0,T)$ we have:*

(i) $\dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y}\mathcal{T}^*_\varepsilon(\phi)(x,y,t)\,dx\,dy\,dt = \int_{\hat{\Omega}^*_\varepsilon}\phi(x,t)\,dx\,dt = \int_{\Omega^*_\varepsilon}\phi(x,t)\,dx\,dt - \int_{\Lambda^*_\varepsilon}\phi(x,t)\,dx\,dt$

(ii) $\|\mathcal{T}^*_\varepsilon(\omega)\|_{L^p(\Omega\times Y^*)} = |Y|^{\frac{1}{p}}\|\omega\|_{L^p(\hat{\Omega^*_\varepsilon})} \le |Y|^{\frac{1}{p}}\|\omega\|_{L^p(\Omega^*_\varepsilon)}$.

Proposition 2.2 *For $q\in[1,+\infty]$, let ϕ_ε be in $L^q(0,T;L^1(\Omega^*_\varepsilon))$ and satisfy*

$$\int_0^T\int_{\Lambda^*_\varepsilon}|\phi_\varepsilon|\,dx\,dt \to 0$$

Then

$$\int_0^T\int_{\Omega^*_\varepsilon}\phi_\varepsilon\,dx\,dt - \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}^*_\varepsilon(\phi_\varepsilon)\,dx\,dy\,dt \to 0.$$

As usual , this convergence is denoted by :

$$\int_0^T\int_{\Omega^*_\varepsilon}\phi_\varepsilon\,dx\,dt \backsimeq \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}^*_\varepsilon(\phi_\varepsilon)\,dx\,dy\,dt.$$

Morever, we have the following convergences:

Proposition 2.3 *(i) For $p,q\in(1,+\infty]$ let $\phi_\varepsilon\in L^q(0,T;L^p(\Omega^*_\varepsilon))$ and $\psi\in L^{q'}(0,T;L^{p'}(\Omega^*_\varepsilon))$, $\left(\dfrac{1}{p}+\dfrac{1}{p'}=1, \dfrac{1}{q}+\dfrac{1}{q'}=1\right)$ such that:*

$$\|\phi_\varepsilon\|_{L^q(0,T;L^p(\Omega^*_\varepsilon))} \le C \text{ and } \|\psi\|_{L^{q'}(0,T;L^{p'}(\Omega^*_\varepsilon))} \le C,$$

then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi \, dx \, dt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \mathcal{T}_\varepsilon^*(\psi) \, dx \, dy \, dt.$$

(ii) *For* $p, q \in (1, +\infty]$ *let* $\phi_\varepsilon \in L^p(0,T; L^p(\Omega_\varepsilon^*))$ *and* $\psi_\varepsilon \in L^q(0,T; L^{p_0}(\Omega_\varepsilon^*))$ $\left(\frac{1}{p} + \frac{1}{p_0} < 1, \frac{1}{q} + \frac{1}{q'} = 1\right)$ *such that:*

$$\|\phi_\varepsilon\|_{L^q(0,T,L^p(\Omega_\varepsilon^*))} \leq C \text{ and } \|\psi_\varepsilon\|_{L^{q'}(0,T;L^{p_0}(\Omega_\varepsilon^*))} \leq C,$$

then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi_\varepsilon \, dx \, dt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \, dx \, dy \, dt.$$

Proposition 2.4 *(some convergence properties)*

(i) *For* $p, q \in [1, \infty)$, *let* $\omega \in L^q(0,T; L^p(\Omega))$. *Then:*

$$\mathcal{T}_\varepsilon^*(\omega) \to \omega \text{ strongly in } L^q(0,T; L^p(\Omega \times Y^*)).$$

(ii) *For* $p, q \in [1, \infty)$, *let* $\{\omega_\varepsilon\}$ *be a sequence in* $L^q(0,T; L^p(\Omega))$ *such that:*

$$\omega_\varepsilon \to \omega \text{ strongly in } L^q(0,T; L^p(\Omega)).$$

then

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \to \omega \text{ strongly in } L^q(0,T; L^p(\Omega \times Y^*)).$$

(iii) *For* $p \in [1, \infty)$ *and* $q \in (1, \infty)$, *let* $\{\omega_\varepsilon\}$ *be a sequence in* $L^q(0,T; L^p(\Omega_\varepsilon^*))$ *such that:* $\|\omega_\varepsilon\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq C$, *if*

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \to \hat{\omega} \text{ weakly in } L^q(0,T; L^p(\Omega \times Y^*)),$$

then we have

$$\hat{\omega_\varepsilon} \to \theta \mathcal{M}.(\hat{\omega}) \text{ weakly in } L^q(0,T; L^P(\Omega)).$$

For $q = \infty$, *the weak convergence above are replaced by the weak** *convergence,respectively.*

Proposition 2.5 *[[12],[14]] Let* $p \in]1, +\infty[$ *and* $\{\varphi_\varepsilon\}$ *be a sequence in the space*

$L^\infty(0,T;W_0^{1,p}(\Omega))$ *such that*

$$\|\nabla\varphi\|_{L^\infty(0,T;L^p(\Omega))} \leq C.$$

Then there exist $\varphi \in L^\infty(0,T;W_0^{1,p}(\Omega))$ *and* $\hat{\varphi} \in L^\infty(0,T;W_{per}^{1,p}(\Omega))$ *such that up to a subsequence :*

- *(i)* $\mathcal{T}_\varepsilon^*(\varphi_\varepsilon) \to \varphi$ *weakly* in* $L^\infty(0,T;W_{per}^{1,p}(Y))$
- *(ii)* $\mathcal{T}_\varepsilon^*(\nabla\varphi_\varepsilon) \to \nabla_x\varphi + \nabla_y\hat{\varphi}$ *weakly* in* $L^\infty(0,T;L^p(\Omega\times Y))$

Theorem 2.1 *For* $p \in (1,+\infty)$, *let* $\{w_\varepsilon\}$ *be a sequence in* $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^n))$ *such that:*

$$\|w_\varepsilon\|_{L^p(0,T;L^p(\Omega))}\leq C \ \text{ and } \ \|\frac{\partial w_\varepsilon}{\partial t}\|_{L^p(0,T;W^{1,p}(\Omega))} \leq C,$$

Then there exist $w \in L^p(0,T;W_0^{1,p}(\Omega))$ *with* $\dfrac{\partial w}{\partial t} = 0$ *and* $\hat{w} \in L^\infty(0,T;W_{per}^{1,p}(\Omega))$ *with* $\mathcal{M}_{Y^*}(\hat{w}) \equiv 0$ *such that , up to a subsequence:*

(i) $w_\varepsilon \rightharpoonup w$ *weakly* in* $L^p(0,T;L^p(\Omega;W^{1,p}(\Omega)))$,

(ii) $\mathcal{T}_\varepsilon^*(\nabla w) \rightharpoonup \nabla w + \nabla\hat{w}$ *weakly* in* $L^p(0,T;L^p(\Omega\times\mathbb{R}^n))$,

(iii) $\mathcal{T}_\varepsilon^*(\dfrac{\partial w_\varepsilon}{\partial t}) \rightharpoonup \dfrac{\partial w}{\partial t}$ *weakly* in* $L^p(0,T;L^p(\Omega\times\mathbb{R}^n))$,

(iv) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w$ *strongly in* $L^q(0,T;W^{1,p}(\Omega))$,

(v) $\|w_\varepsilon - w\|_{L^p(0,T;W_0^{1,p}(\Omega))} \to 0.$
where q *is any number in* $(1,+\infty)$.
We end this section by recalling the definition of the mean value operator $\mathcal{M}_Y$ *and that of the local average* $\mathcal{M}_Y^\varepsilon$ *and give some of their properties that will be useful in the sequel.*

Definition 2.3 *Let* $p,q \in [1,+\infty]$. *The mean value operator* $\mathcal{M}_Y : L^q(0,T;L^p(\Omega\times Y)) \to L^q(0,T;L^p(\Omega))$ *is defined by :*

$$\mathcal{M}_Y(u)(x,t) = \frac{1}{|Y|}\int_Y u(x,y,t)\,dy,$$

For every $u \in L^q(0,T;L^p(\Omega\times Y))$

Definition 2.4 *Let $p, q \in [1, +\infty]$, the local average operator* $\mathcal{M}_Y^\varepsilon : L^q(0,T;L^p(\Omega)) \to L^q(0,T;L^p(\Omega))$ *is defined by:*

$$\mathcal{M}_Y^\varepsilon(u)(x,t) = \frac{1}{|Y|}\int_Y \mathcal{T}_\varepsilon(u)(x,y,t)\,dy$$

For every $u \in L^q(0,T;L^p(\Omega))$.

Remark 2.2 *In connection, some of the properties of $\mathcal{T}_\varepsilon$ (in the case of dependence on time) can be derived directly for those of the unfolding operator for fixed domains frome [3] with the time t as a mere parameter.*
As a consequence , we have the following result .

Proposition 2.6 *Let $p \in [1, \infty[$ and $q \in [1, \infty]$for $\varphi \in L^q(0,T;L^P(\Omega))$, one has*

$$\mathcal{T}_\varepsilon(\mathcal{M}_Y^\varepsilon(\varphi))(x,y,t) = \mathcal{M}_Y(\mathcal{T}_\varepsilon(\varphi))(x,t) = \mathcal{T}_Y^\varepsilon(\varphi)(x,t) \text{ in } \Omega \times [0,T].$$

Let$\{\omega_\varepsilon\}$ be a sequence in $L^q(0,T,L^p(\Omega))$ such that

$$\omega_\varepsilon \to w \text{ strongly in } L^q(0,T;L^p(\Omega)).$$

Then

$$\mathcal{M}_Y^\varepsilon(\omega_\varepsilon) \to \mathcal{M}_Y(\omega) = w \text{ strongly in } L^q(0,T;L^p(\Omega)).$$

For any $\varphi \in L^q(0,T;L^p(\Omega))$

$$\|\mathcal{M}_Y(\varphi)\|_{L^q(0,T;L^P(\Omega))} \leq |Y|^{\frac{1-p}{p}} \|\varphi\|_{L^q(0,T;L^p(\Omega))}$$

2.2 Time-dependent unfolding operator in domains with two parameters

In this section, we extend the operator $\mathcal{T}_{\varepsilon,\delta}$, defined in the previous section to time-dependent functions by adapting what is done in [12]. We start by defining the unfolding operator for time-dependent functions in the domain $\Omega^*_{\varepsilon,\delta}\times]0,T[$, depending ε and δ
In what follows , we have $(\varepsilon,\delta) \to (0,0)$ through any sequence and subsequence.

Definition 2.5 *Let $p \in [1,+\infty[$ and $q \in L^q(0,T;L^p(\Omega))$ the unfolding operator $\mathcal{T}_{\varepsilon,\delta} : L^q(0,T;L^p(\Omega)) \to L^q(0,T;L^p(\Omega \times Y))$ is definded as*

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(x,\delta z,t) \text{ if } (x,z,t) \in \Omega_\varepsilon^* \times \dfrac{1}{\delta}Y \times]0,T[\\ 0 \ \text{ otherwise.} \end{cases}$$

that is,

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t) = \begin{cases} \varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon\delta z, t\right) \text{ if } (x,z,t) \in \Omega_\varepsilon^* \times \dfrac{1}{\delta}Y \times]0,T[, \\ 0 \quad \text{otherwise.} \end{cases}$$

As mentioned above , for $\delta = 1$ we are in presence of the unfolding operator for fixed domains introduced in [7]

Remark 2.3 *From now on, if a function does not depend on t, by $\mathcal{T}_{\varepsilon,\delta}(\varphi)$ we simply mean the operator introduced in definition 1.2.*
Being defined by means of the operator $\mathcal{T}_\varepsilon$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$, inherits most of the general properties of it. In particular, the following proposition is straightforward:

Proposition 2.7 *Let $p \in [1,+\infty[$ and $q \in [1,+\infty]$*

(1) $\mathcal{T}_{\varepsilon,\delta}$ is linear and continuous from $L^q(0,T;L^p(\Omega))$ to $L^q(0,T;L^p(\Omega \times \mathbb{R}^n))$.

(2) $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$ for every $v,w \in L^q(0,T;L^p(\Omega))$.

(3) $\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi)) = \varepsilon\delta\, \mathcal{T}_{\varepsilon,\delta}(\nabla\varphi)$ in $\Omega \times \dfrac{1}{\delta}Y \times]0,T[$ for all $\varphi \in L^q(0,T;H_0^1(\Omega))$.

Theorem 2.2 *Let $p \in [1,+\infty[$ and $q \in [1,+\infty[$.*

- *Let $\varphi \in L^q(0,T;L^p(\Omega))$*

$$\begin{aligned} \frac{\delta^N}{|Y|}\int_{\Omega\times\mathbb{R}^n} \mathcal{T}_{\varepsilon,\delta}\varphi(x,z,t)\,dx\,dz &= \int_{\Omega_\varepsilon^*} \varphi(x,t)\,dx \\ &= \int_\Omega \varphi(x,t)\,dx - \int_{\Lambda_\varepsilon} \varphi(x,t)\,dx. \end{aligned}$$

For a.e. $t \in]0, T[$

The continuity of the operator $\mathcal{T}_{\varepsilon,\delta}$, *as follows:*

$$\|\mathcal{T}_{\varepsilon,\delta}(\varphi)_{L^q(0,T;L^p(\Omega))}\| \leq \left(\frac{|Y|}{\delta^N}\right)^{\frac{1}{p}} \|\varphi\|_{L^q(0,T;L^p(\Omega))}.$$

- *Let* $\varphi \in L^q(0,T;H^1(\Omega))$ *and* $N \geq 3$. *Then for a.e.* $t \in]0,T[$

$$\|\nabla_z(\mathcal{T}_{\varepsilon\delta}(\varphi))\|_{L^p(\Omega\times\frac{1}{\delta}Y)} \leq \left(\frac{\varepsilon|Y|}{\delta^{\frac{N}{p}-1}}\right)^{\frac{1}{p}} \|\nabla\varphi\|_{L^p(\Omega)}.$$

Proposition 2.8 *Let* $p \in [1,+\infty]$ *and* $\varphi_\varepsilon \in L^q(0,T;L^1(\Omega))$ *satisfying*

$$\int_0^T \int_{\Lambda_\varepsilon} \varphi_\varepsilon \, dx \, dt \to 0$$

Then

$$\int_0^T \int_\Omega \varphi_\varepsilon \, dx \, dt \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon) \, dx \, dz \, dt.$$

Proposition 2.9 *Let* $p, q \in [1,+\infty]$. *Let* $\{\varphi_\varepsilon\}$ *be a sequence in* $L^q(0,T;L^p(\Omega))$ *and* $\{\psi_\varepsilon\}$ *be a sequence in* $L^{q'}(0,T;L^{p0}(\Omega))$ *, such that*

$$\|\varphi_\varepsilon\|_{L^q(0,T;L^p(\Omega))} \leq C \;\; \text{and} \;\; \|\psi_\varepsilon\|_{L^{q'}(0,T;L^{p0}(\Omega))} \leq C,$$

where $\dfrac{1}{p} + \dfrac{1}{p_0} < 1$ *and* $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. *Then*

$$\int_0^T \int_{\Omega_\varepsilon^*} \varphi_\varepsilon\psi_\varepsilon \, dx \, dt \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega\times\frac{1}{\delta}Y} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon\psi_\varepsilon) \, dx \, dz \, dt.$$

The next two propsitions extend to time-dependent functions some properties given in [5] Theorem 2.11].

Proposition 2.10 *Let* $u \in L^q(0,T;H^1(\Omega))$ *for* $q \in [1,+\infty[$ *one has estimates*

$$\|\mathcal{T}_{\varepsilon,\delta}\left(u - \mathcal{M}_Y^\varepsilon(u)\right)\|_{L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N))} \leq \frac{C\varepsilon|Y|^{\frac{1}{p}}}{\delta^{\frac{N}{p}-1}} \|\nabla u\|_{L^q(0,T;L^p(\Omega))},$$

and for ω an open and bounded subset of $\mathbb{R}^N$

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|_{L^q(0,T;L^p(\Omega\times\omega))} \leq \frac{2C\varepsilon|Y|^{\frac{1}{p}}}{\delta^{\frac{N}{p}-1}}\|\nabla u\|_{L^q(0,T;L^p(\Omega))} + 2|\omega||Y|^{\frac{1-p}{p}}\|u\|_{L^q(0,T;L^p(\Omega))},$$

Where C is the Sobolev-poincare-Wirtinger constant for $H^1(Y)$.

Theorem 2.3 *Let* $p \in [1,+\infty[, q \in [1,+\infty]$. $N \geq 3$, $\{w_{\varepsilon\delta}\}$ *be a sequence in* $L^q(0,T;H^1(\Omega))$ *which is unformly bounded with respect to* ε *and* δ *as* $(\varepsilon,\delta) \to (0,0)$ *Then up to a subsequence, there exists* W *in* $L^q(0,T;L^{p^*}(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^q(0,T;L^p(\Omega\times\mathbb{R}^N))$ *such that*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\left(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - \mathcal{M}^{\varepsilon}_{Y}(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\right) \rightharpoonup W \text{ weakly in } L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N))),$$

and

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\left(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\right) \rightharpoonup \nabla_z W \text{ weakly in } L^q(0,T;L^p(\Omega\times\mathbb{R}^N)). \tag{2.4}$$

furthemore if

$$K^* = \lim_{(\varepsilon,\delta)\to(0^+,0^+)}\sup\frac{\delta^{\frac{N}{p}-1}}{\varepsilon} < +\infty, \tag{2.5}$$

then one can choose the subsequence above and some U *in* $L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N)))$ *with*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \text{ weakly in } L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N))). \tag{2.6}$$

CHAPTER

3

HOMOGENIZATION OF THE HYPERBLOIC-PARABPLIC PROBLEM

3.1 homogenization of hyperbolic-parabolic problem in domains Ω_ε

In this chapter , we study a class of hyperbloic-parabolic problems in periodically perforated domains with a homogeneus Neuman condition on the boundary of holes. We focus on the homogenization of equation(3.1). The proof is based on the periodic unfolding method in perforated domains.

We consider the hyperbolic-parabolic equation with homogenious Dirichlet-Neuman boundary in the perforated domain , namely:

$$\begin{cases} u''_\varepsilon + u'_\varepsilon - \text{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon, \text{ on } \Omega^*_\varepsilon \times (0,T), \\ u_\varepsilon = 0, \quad \text{in } \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon . n_\varepsilon = 0, \quad \text{in } \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u^0_\varepsilon, \, u'_\varepsilon(x,0) = u^1_\varepsilon, \text{ in } \Omega^*_\varepsilon, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open and bounded set with lipchitz continious boundary, S_ε is a set of ε-periodic holes of sise ε and $\Omega^*_\varepsilon = \Omega \backslash S_\varepsilon$, n_ε is the outward unit normal vector field defined on ∂S_ε.

For the unitial data , we always assume $u^0_\varepsilon \in H^1_0(\Omega)$, $u^1_\varepsilon \in L^2(\Omega^*_\varepsilon)$ and $f_\varepsilon \in L^2(0,T,L^2(\Omega^*_\varepsilon))$.

The coefficient matrix A^ε satisfies the following assumptions:

$$\begin{cases} A^\varepsilon \in M(\alpha,\beta,\Omega), \\ A^\varepsilon \text{symetric}, \end{cases} \tag{3.2}$$

where $M(\alpha,\beta,\Omega)$ is the classical set of the $n \times n$ matrix-valued functions(see [8]).

We need to introduce some necessary asumptions. In what follow , we suppose that the coefficient matrix A^ε satisfies :

$$\mathcal{T}\varepsilon(A^\varepsilon) \to A \text{ strongly in } (L^1(\Omega \times Y))^{n\times n}, \tag{3.3}$$

where $\mathcal{T}_\varepsilon$ is the unfolding operator in fixed domains [3]. These assumptions recover the classical periodic coefficient case mentioned above.

For the initial data, we make the following assumptions:

$$\begin{cases} (i) u^0_\varepsilon \rightharpoonup u^0 \text{ weakly in } H^1_0(\Omega), \\ (ii) \tilde{u}^1_\varepsilon \to u^1 \text{ strongly in } L^2(\Omega), \\ (iii) \widetilde{f}_\varepsilon \rightharpoonup f \text{ weakly in } L^2(0,T;L^2(\Omega^*_\varepsilon)). \end{cases} \tag{3.4}$$

set

$$\mathcal{W}_\varepsilon = \left\{ v_\varepsilon | v_\varepsilon \in L^2(0,T;V^\varepsilon), v'_\varepsilon \in L^2(0,T;L^2(\Omega^2_\varepsilon)) \right\} \tag{3.5}$$

with

$$\boldsymbol{V^{\varepsilon} = \{v \in H^1(\Omega_{\varepsilon}^{*}) | v = 0 \text{ on } \partial\Omega\}}$$

with the norm defined by

$$\|\boldsymbol{v_{\varepsilon}}\|_{\mathcal{W}_{\varepsilon}} = \|\boldsymbol{v_{\varepsilon}}\|_{\boldsymbol{L^2(0,T;V^{\varepsilon})}} + \|\boldsymbol{v'_{\varepsilon}}\|_{\boldsymbol{L^2(0,T;L^2(\Omega_{\varepsilon}^{*}))}}.$$

Variational formulation

We multiply the equation of problem(3.1) by a test function $\boldsymbol{v}$ where $\boldsymbol{v} \in \boldsymbol{V^{\varepsilon}}$ and we integer on $\boldsymbol{\Omega_{\varepsilon}^{*}}$, we get

$$\int_{\Omega_{\varepsilon}^{*}} \boldsymbol{u''_{\varepsilon} v dx} + \int_{\Omega}^{*} \boldsymbol{u'_{\varepsilon} dx} - \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{div(A^{\varepsilon} \nabla u_{\varepsilon}) v dx} = \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{f_{\varepsilon} v dx} \tag{3.6}$$

Now, we apply the Green's formula on third integral in above equation we get :

$$-\int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \Delta u_{\varepsilon} v dx} = \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} \nabla v dx} - \int_{\partial\Omega \cup \partial S_{\varepsilon}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} v.n_{\varepsilon} d\Gamma}$$

by Green's formula for the second integral we get:

$$-\int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \Delta u_{\varepsilon} v dx} = \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} \nabla v dx} - \int_{\partial S \varepsilon} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} v.n_{\varepsilon} dx} + \int_{\partial\Omega} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} v.n_{\varepsilon}}$$

the last integral equal to zero because: $\int_{\partial S_{\varepsilon}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} v.n_{\varepsilon} d\Gamma = 0}$ and $\boldsymbol{v = 0}$ on $\boldsymbol{\partial\Omega}$ are essential condtions.

Then

$$-\int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} v dx} = \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} \nabla v dx} \tag{3.7}$$

Substituting the formula (3.7) into (3.6)we get:

$$\langle \boldsymbol{u''_{\varepsilon}, v} \rangle_{\boldsymbol{(V^{\varepsilon})', V^{\varepsilon}}} + \langle \boldsymbol{u'_{\varepsilon}, v} \rangle_{\boldsymbol{(V^{\varepsilon})', V^{\varepsilon}}} + \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{A^{\varepsilon} \nabla u_{\varepsilon} \nabla v dx} = \int_{\Omega_{\varepsilon}^{*}} \boldsymbol{f_{\varepsilon} v dx}$$

The variational formulation of problem (3.1) is to fined $\boldsymbol{u}_\varepsilon \in \boldsymbol{\mathcal{W}}_\varepsilon$ such that

$$\begin{cases} \langle \boldsymbol{u}''_\varepsilon, \boldsymbol{v}\rangle_{(\boldsymbol{V}^\varepsilon)', \boldsymbol{V}^\varepsilon} + \langle \boldsymbol{u}'_\varepsilon, \boldsymbol{v}\rangle_{(\boldsymbol{V}^\varepsilon)', \boldsymbol{V}^\varepsilon} + \int_{\boldsymbol{\Omega}^*_\varepsilon} \boldsymbol{A}^\varepsilon \nabla \boldsymbol{u}_\varepsilon \nabla \boldsymbol{v}\, \boldsymbol{dx} = \int_{\boldsymbol{\Omega}^*_\varepsilon} \boldsymbol{f}_\varepsilon \boldsymbol{v}\, \boldsymbol{dx} \\ \text{in } \boldsymbol{\mathcal{D}}'(\boldsymbol{0}, \boldsymbol{T}) \text{ for all } \boldsymbol{v} \in \boldsymbol{V}^\varepsilon \\ \boldsymbol{u}_\varepsilon(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{u}^0_\varepsilon, \boldsymbol{u}'_\varepsilon(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{u}^1_\varepsilon \text{ in } \boldsymbol{\Omega}^*_\varepsilon. \end{cases} \tag{3.8}$$

For every fixed $\boldsymbol{\varepsilon}$, classical results provide that problem (3.6) has a unique solution $\boldsymbol{u}_\varepsilon$ such that:

$\boldsymbol{u}_\varepsilon \in \mathbb{C}^0([\boldsymbol{0}, \boldsymbol{T}; \boldsymbol{V}^\varepsilon]) \cap \mathbb{C}^1([\boldsymbol{0}, \boldsymbol{T}]; \boldsymbol{L}^2(\boldsymbol{\Omega}^*_\varepsilon))$.

Homogenization of hyperbolic-parabolic equation

In this section we state our homogenization result of problem (3.1), the main result of the homogenization give by the next theorem.

Theorem 3.1 *Let $\boldsymbol{A}^\varepsilon$ be a matrix satisfying (3.2) and (3.3). Suppose that $\boldsymbol{u}_\varepsilon$ is the solution of problem (3.1) with (3.4). Then there exist $\mathbf{u} \in \boldsymbol{L}^2(\boldsymbol{0}, \boldsymbol{T}; \boldsymbol{H}^1_0(\boldsymbol{\Omega}))$ with*
$\boldsymbol{u}' \in \boldsymbol{L}^2(\boldsymbol{0}, \boldsymbol{T}; \boldsymbol{L}^2(\boldsymbol{\Omega}))$ and
$\hat{\boldsymbol{u}} \in \boldsymbol{L}^2(\boldsymbol{\Omega}, \boldsymbol{H}^1_{per}(\boldsymbol{\Omega} \times \boldsymbol{Y}^*))$ with $\boldsymbol{\mathcal{M}}_{\boldsymbol{Y}^*}(\hat{\boldsymbol{u}}) = \boldsymbol{0}$, such that

$$\begin{cases} \boldsymbol{\mathcal{T}}^*_\varepsilon \longrightarrow \boldsymbol{u} \text{ strongly in } \boldsymbol{L}^2(\boldsymbol{0}, \boldsymbol{T}; \boldsymbol{L}^2(\boldsymbol{\Omega}, \boldsymbol{H}^1(\boldsymbol{Y}^*))), \\ \boldsymbol{\mathcal{T}}^*_\varepsilon(\boldsymbol{u}'_\varepsilon) \rightharpoonup \boldsymbol{u}' \text{ in } \boldsymbol{L}^2(\boldsymbol{0}, \boldsymbol{T}; \boldsymbol{L}^2(\boldsymbol{\Omega} \times \boldsymbol{Y}^*)), \\ \boldsymbol{\mathcal{T}}^*_\varepsilon(\nabla \boldsymbol{u}_\varepsilon) \rightharpoonup \nabla \boldsymbol{u} + \nabla_{\boldsymbol{y}} \hat{\boldsymbol{u}} \text{ weakly in } \boldsymbol{L}^2(\boldsymbol{0}, \boldsymbol{T}; \boldsymbol{L}^2(\boldsymbol{\Omega} \times \boldsymbol{Y}^*)), \\ \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{L^2(0,T;L^2(\Omega^*_\varepsilon)} \longrightarrow \boldsymbol{0}. \end{cases} \tag{3.9}$$

The pair $(u, \hat{u})$ *with* $\mathcal{M}_Y^*(u) = 0$ *is the unique solution of the following problem:*

$$\begin{cases} \theta \int_0^T \int_\Omega u\Psi\varphi^{''} dx\, dt + \theta \int_0^T \int_\Omega u\Psi\varphi^{'} dx\, dt \\ + \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla\Psi + \nabla_y \Phi)\varphi dx\, dy\, dt = \theta \int_0^T \int_\Omega f\Psi\varphi dx\, dt \\ \text{for any } \Psi \in H_0^1(\Omega), \Phi \in L^2(\Omega; H_{per}^1(Y^*)) \text{ and } \varphi \in \mathcal{D}(0,T), \\ u = 0 \text{ on } \Omega \times (0,T), \\ u(x,0) = u^0, u^{'}(x,0) = u^1 \text{ in } \Omega. \end{cases} \tag{3.10}$$

We also have

$$\hat{u} = \sum_{j=1}^n \frac{\partial u}{\partial x_j} \chi_j. \tag{3.11}$$

with $\chi_j \in L^\infty(\Omega; H_{per}^1(Y^*))(j = 1, ..., n)$ *being the solution of the cell problem*

$$\begin{cases} -div_y(A\nabla_y(\chi_j + y_j)) = 0 \text{ in } Y^*, \\ A\nabla_y(\chi_j + y_j).n_1 = 0, \text{ on } \partial S, \\ \mathcal{M}_Y^*(\chi_j)(x, .) = 0, \ \chi_j(x, .)\ Y - periodic. \end{cases}$$

Moreover, u *is the unique solution of the following homogenized hyperbolic-parabolic equation*

$$\begin{cases} u^{''} + u^{'} - div(A^0\nabla u) = \theta^{-1} f, \text{ in } \Omega \times (0,T), \\ u = 0, \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u^0, u^{'}(x,0) = u^1, \text{ in } \Omega, \end{cases} \tag{3.12}$$

Where the homogenized matrix $A^0 = (a_{ij}^0)_{1\le i,j\le n}$ *is defined by*

$$a_{ij}^0(x) = \mathcal{M}_{Y^*}(a_ij + \sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k}). \tag{3.13}$$

In addition, we have the following convergence:

$$\begin{cases} (i) \widetilde{u_\varepsilon} \rightharpoonup \theta u \text{ weakly* in } L^\infty(0,T; L^2(\Omega)), \\ (ii) A^\varepsilon \widetilde{\nabla u_\varepsilon} \rightharpoonup \theta A^0 \nabla u \text{ weakly* in } L^\infty(0,T; L^2(\Omega)). \end{cases} \tag{3.14}$$

We would like to mention that the effective matrix filed $\boldsymbol{A^0}$ depends on $\boldsymbol{x}$, while the classical matrix is constant .

Proof

The proof of theorem (3.1) is fondamentally based on periodic unfolding method. Our starting point is following variational of problem (3.1). Find $\boldsymbol{u_\varepsilon \in L^2(0,T;V^\varepsilon)}$ with $\boldsymbol{u' \in L^2(0,T;L^2(\Omega_\varepsilon^*))}$ and
$\boldsymbol{u''_\varepsilon \in L^2(0,T;(V^\varepsilon)')}$ such that (3.8)
. For every $\boldsymbol{\varepsilon}$, we have the following uniforme estimates:

$$\|\boldsymbol{u_\varepsilon}\|_{L^\infty(0,T;V^\varepsilon)} + \|\boldsymbol{u'_\varepsilon}\|_{L^\infty(0,T,L^2(\Omega_\varepsilon^*))} \leq \boldsymbol{C} \tag{3.15}$$

In view of (3.15) and Theorem (2.1), we get there exist $\boldsymbol{u \in L^\infty(0,T.H_0^1(\Omega))}$ with $\boldsymbol{u' \in L^\infty(0,T;L^2(\Omega))}$ and $\boldsymbol{\hat{u} \in L^\infty(0,T;L^2(\Omega,H^1_{per}(Y^*)))}$ with $\boldsymbol{\mathcal{M}_{Y^*}(\hat{u}) = 0}$, such that, up to a subsequence(still denoted by $\boldsymbol{\varepsilon}$), the convergence in (3.9) hold . From propostion (2.4)(iii), we further get that

$$\begin{cases} \boldsymbol{\hat{u} \rightharpoonup \theta \mathcal{M}_{Y^*}(u)} \ \text{ weakly}^* \text{ in } \ \boldsymbol{L^\infty(0,T;L^2(\Omega))}, \\ \boldsymbol{A^\varepsilon \widetilde{\nabla u_\varepsilon} \rightharpoonup \theta \mathcal{M}_{Y^*}[A(\nabla u + \nabla_y \hat{u})]} \text{ weakly}^* \text{ in } \boldsymbol{L^\infty(0,T;L^2(\Omega))}. \end{cases} \tag{3.16}$$

Now let $\boldsymbol{\Psi, \Phi \in \mathcal{D}(\Omega)}$ and $\boldsymbol{\psi \in H^1_{per}(Y^*)}$. set

$$\boldsymbol{v_\varepsilon(x) = \Psi(x) + \varepsilon\phi(x)\psi^\varepsilon(x)} \text{ with } \boldsymbol{\psi^\varepsilon(x) = \psi(\frac{x}{\varepsilon})}. \tag{3.17}$$

Then

$$\begin{aligned} \boldsymbol{\nabla_x v_\varepsilon} &= \boldsymbol{\nabla_x \left[\Psi(x) + \varepsilon\phi(x)\psi^\varepsilon(x)\right]} \\ &= \boldsymbol{\nabla\Psi(x) + \varepsilon\nabla_x \left[\phi(x)\psi(\frac{x}{\varepsilon})\right]} \\ &= \boldsymbol{\nabla\Psi + \varepsilon\nabla_x\phi\psi(\frac{x}{\varepsilon}) + \varepsilon \times \frac{1}{\varepsilon}\phi(x)\nabla_y\psi(\frac{x}{\varepsilon})} \\ &= \boldsymbol{\nabla\Psi + \varepsilon\psi^\varepsilon\nabla\phi + \phi(\nabla_y\psi)(\frac{\cdot}{\varepsilon})} \end{aligned}$$

By propostion (2.4) (ii), we have

$$\begin{cases} \mathcal{T}^*_\varepsilon(v_\varepsilon) \to \Psi \text{ strongly in } L^2(\Omega \times Y), \\ \mathcal{T}^*_\varepsilon(\phi\psi^\varepsilon) \to \Phi \text{ strongly in } L^2(\Omega \times Y) \text{ with } \Phi = \phi(x)\psi(y), \\ \mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \to \nabla\Psi + \nabla_y\Phi \text{ strongly in } L^2(\Omega \times Y). \end{cases} \tag{3.18}$$

Let $\varphi(t) \in \mathcal{D}(0,T)$, we take $v_\varepsilon\varphi$ a test function in the problem (3.1) we get

$$\int_0^T \int_{\Omega^*_\varepsilon} u''_\varepsilon v_\varepsilon \varphi \, dx\, dt + \int_0^T \int_{\Omega^*_\varepsilon} u'_\varepsilon v_\varepsilon \varphi \, dx\, dt + \int_0^T \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi \, dx\, dt = \int_0^T \int_{\Omega^*_\varepsilon} f_\varepsilon v_\varepsilon \varphi \, dx\, dt \tag{3.19}$$

We pass at the limit in problem(3.19) term by term

We start by the firt term

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^*_\varepsilon} u''_\varepsilon v_\varepsilon \varphi \, dx\, dt = \lim_{\varepsilon \to 0} \int_0^T u_\varepsilon v_\varepsilon \varphi'' \, dx\, dt \tag{3.20}$$

By the unfolding operator $\mathcal{T}^*_\varepsilon$ and we use the definition 2.2 (i)(ii) we get

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^*_\varepsilon} u_\varepsilon v_\varepsilon \varphi \, dx\, dt = \lim_{\varepsilon \to 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(u_\varepsilon) \mathcal{T}^*_\varepsilon(v_\varepsilon) \mathcal{T}^*_\varepsilon(\varphi') \, dx\, dy\, dt$$

by convergence (3.18) and (3.9) and, by Fubini theorem we get

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^*_\varepsilon} u_\varepsilon v_\varepsilon \varphi'' dx\, dt = \lim_{\varepsilon \to 0} \frac{|Y^*|}{|Y|} \int_0^T \int_\Omega u\Psi\varphi'' dx\, dt$$

such that $\theta = \dfrac{|Y^*|}{|Y|}$, then we get

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^*_\varepsilon} u'' v_\varepsilon \varphi \, dx\, dt = \theta \int_0^T \int_\Omega u\Psi\varphi'' \, dx\, dt. \tag{3.21}$$

For the second integral:

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^*_\varepsilon} u'_\varepsilon v_\varepsilon \varphi \, dx\, dt = -\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^*_\varepsilon} u_\varepsilon v_\varepsilon \varphi' \, dx\, dt$$

We use the unfolding operator $\mathcal{T}_\varepsilon^*$ and by the definition (2.2) (i)(ii) we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon \varphi \, dx\, dt = -\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(u_\varepsilon)\mathcal{T}_\varepsilon^*(v_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi')\, dx\, dy\, dt$$

By Fubini theorem and by convergence (3.18) and,(3.9) we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon' \varphi \, dx\, dt = -\theta\int_0^T\int_\Omega u\Psi\varphi'\, dx\, dt \tag{3.22}$$

For the third term we use the unfolding operator $\mathcal{T}_\varepsilon^*$ and, by definition 2.2 (i) (ii) we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon \varphi \, dx\, dt = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon)\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi)\, dx\, dy\, dt$$

By (3.3) and by convergence (3.9) and (3.18) we have

$$\begin{cases} \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \to \nabla u + \nabla_y \hat{u} \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega\times Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \to \nabla\Psi + \nabla\Phi \text{ strongly in } L^2(0,T;L^2(\Omega\times Y^*)), \\ \mathcal{T}_\varepsilon^*(A^\varepsilon) \to A \text{ strongly in } (L^1(\Omega\times Y^*))^{n\times n}. \end{cases}$$

We get

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega_\varepsilon^*} \mathcal{T}_\varepsilon^*(A^\varepsilon)\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi)\, dx\, dy\, dt \\ &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u + \nabla_y\hat{u})(\nabla\Psi + \nabla_y\Phi)\varphi\, dx\, dy\, dt \end{aligned}$$

This imply :

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon \varphi \, dx\, dt = \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\, dx\, dy\, dt \tag{3.23}$$

For the laste integral in equation (3.19) we use the unfolding operator $\mathcal{T}_\varepsilon^*$ and by definition 2.2 (i)(ii), we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx\, dt = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f_\varepsilon)\mathcal{T}_\varepsilon^*(v_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi)\, dx\, dy\, dt$$

By (3.4) and (3.19) we get

$$\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(f_\varepsilon)\mathcal{T}_\varepsilon^*(v_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi)\,dx\,dydt=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}f\Psi\varphi\,dx\,dy\,dt$$

By Fubini theorem we obtain:

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi\,dx\,dt=\theta\int_0^T\int_\Omega f\Psi\varphi\,dx\,dt \tag{3.24}$$

Now we can pass at the limit in expression (3.19) using (3.21), (3.22), (3.23) and (3.24) we get

$$\begin{aligned}&\theta\int_0^T\int_\Omega u\Psi\varphi''\,dx\,dt-\theta\int_0^T\int_\Omega u\Psi\varphi'\,dx\,dt\\&+\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\,dx\,dy\,dt\\&=\theta\int_0^T\int_\Omega f\Psi\varphi\,dx\,dt.\end{aligned} \tag{3.25}$$

setting $\Psi=0$ we get

$$\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\hat{u})(\nabla_y\Phi)\varphi dx\,dy\,dt=0$$

We obtain

$$div_y A(\nabla u+\nabla_y\hat{u})=0.$$

Since u is independent of y and $\mathcal{M}_{Y^*}=0$ we obtain (3.11) Moreover we have the following identity

$$\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\hat{u})\nabla\Psi\,dx\,dy=|Y^*|A^0\nabla u\nabla\Psi \tag{3.26}$$

sabstituting (3.11) and (3.26) into (3.10), we get that

$$u^{''}+u^{'}-div(A^0\nabla u)=f\ \text{ in }\ \Omega\times(0,T).$$

Wich gives the equation (3.12) and $u^{''}\in L^2(0,T;H^{-1}(\Omega))$. Hence from classical results, we have $u\in\mathbb{C}^0([0,T];L^2(\Omega))$ and $u^{'}\in\mathbb{C}^0([0,T];H^{-1}(\Omega))$.
Now, in order to chek the initial conditions, let v_ε be given (3.17) and $\varphi\in\mathbb{C}^\infty([0,T])$ with

$\varphi(0)=1$ and $\varphi(T)=0$. choosing $v_\varepsilon\varphi$ as test function in the variational formulation (3.8) we have

$$\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\,dx\,dt-\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi\,dx\,dt=\int_0^T\langle u'',v_\varepsilon\varphi\rangle_{(V^\varepsilon)',V^\varepsilon}\,dx\,dt \\ -\int_0^T\langle u',v_\varepsilon\varphi\rangle_{(V^\varepsilon)',V^\varepsilon}\,dx\,dt \tag{3.27}$$

We integrate by parts (for the variable t) the first integral in the other side of the equation (3.27) we get

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon\varphi\,dx\,dt&=\int_{\Omega_\varepsilon^*}\left[\varphi u_\varepsilon' v_\varepsilon\right]_0^T\,dx-\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'\varphi' v_\varepsilon\,dx\,dt\\ &=\int_{\Omega_\varepsilon^*}\left[\varphi(T)u_\varepsilon'(T)v_\varepsilon-\varphi(0)u_\varepsilon'(0)v_\varepsilon\right]\,dx-\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'\varphi' v_\varepsilon\,dx\,dt\\ &=-\int_{\Omega_\varepsilon^*} u_\varepsilon'(x,0)v_\varepsilon(x)\,dx\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'\varphi' v_\varepsilon\,dx\,dt\end{aligned} \tag{3.28}$$

substituting this formula into (3.27) we get

$$\int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\varphi\,dx\,dt-\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi\,dx\,dt=-\int_{\Omega_\varepsilon^*} u_\varepsilon'(x,0)v_\varepsilon(x)\,dx \\ +\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'\varphi' v_\varepsilon\,dx\,dt-\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon\varphi\,dx\,dt \tag{3.29}$$

by similar method above, passing to the limit in the expretion (3.27), using (3.9), (3.21), (3.22), (3.23), (3.24) we get

$$-\frac{1}{|Y|}\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\,dx\,dt+\theta\int_0^T\int_\Omega f\Psi\varphi\,dx\,dt \\ =-\theta\int_\Omega u^1\Psi\,dx-\theta\int_0^T\int_\Omega u'\Psi\varphi'\,dx\,dt-\theta\int_0^T\int_\Omega u'\Psi\varphi\,dx\,dt$$

Also, using the integration by parts for the forth integral we get

$$-\frac{1}{|Y|}\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\,dx\,dt+\theta\int_0^T\int_\Omega f\Psi\varphi\,dx\,dt \\ =-\theta\int_\Omega u^1\Psi\,dx+\theta\int_\Omega u'(x,0)\Psi\,dx+\theta\int_0^T\int_\Omega u''\Psi\varphi\,dx\,dt-\theta\int_0^T\int_\Omega u'\Psi\varphi\,dx\,dt$$

combining this equation with (3.25) we have

$$u'(x,0) = u^1$$

Now for the second intial condition choosing $\varphi \in \mathbb{C}^\infty([0,T])$ with $\varphi(0) = \varphi(T) = \varphi'(T) = 0$, $\varphi'(0) = 1$ and taking $v_\varepsilon\varphi$ as test function in the variational formulation (3.8) we have

$$\begin{aligned} &\int_0^T \int_\Omega f_\varepsilon v_\varepsilon \varphi \, dx\, dt - \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi \, dx\, dt \\ &= \int_0^T \langle u'', v_\varepsilon \varphi\rangle_{(V^\varepsilon)', V^\varepsilon} \, dx\, dt - \int_0^T \langle u', v_\varepsilon\varphi\rangle_{(V^\varepsilon)', V^\varepsilon} \, dx\, dt \end{aligned} \tag{3.30}$$

The integration by parts gives

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} dx\, dt &= \int_{\Omega_\varepsilon^*} [\varphi u' v_\varepsilon]_0^T - \int_0^T \int_{\Omega_\varepsilon^*} u'_\varepsilon v_\varepsilon \varphi' \, dx\, dt \\ &= -\int_0^T \int_{\Omega_\varepsilon^*} u'_\varepsilon v_\varepsilon \varphi' \, dx\, dt \end{aligned}$$

We use it for the second time integration by parts we get

$$-\int_0^T \int_{\Omega_\varepsilon^*} u'_\varepsilon v_\varepsilon \varphi' \, dx\, dt = \int_{\Omega_\varepsilon^*} u_\varepsilon(x,0) v_\varepsilon \, dx + \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi'' \, dx\, dt$$

by sabstituting this expretion into (3.30) we get

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx\, dt - \frac{1}{|Y|} \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx\, dt \\ &= \int_{\Omega_\varepsilon^*} u_\varepsilon(x,0) v_\varepsilon \, dx + \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi'' \, dx\, dt - \int_0^T \int_{\Omega_\varepsilon^*} u'_\varepsilon v_\varepsilon \varphi \, dx\, dt \end{aligned} \tag{3.31}$$

By similar method above passing to the limit into expretion (3.31), using (3.9), (3.3) and (3.21), (3.22),(3.23), (3.24) we have

$$\begin{aligned} -\frac{1}{|Y|} \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi)\varphi \, dx\, dy\, dt + \theta \int_0^T \int_\Omega f \Psi \varphi \, dx\, dt \\ = \theta \int_\Omega u^0 \Psi \, dx + \theta \int_0^T \int_\Omega u \Psi \varphi'' \, dx\, dt - \theta \int_0^T \int_\Omega u' \Psi \varphi \, dx\, dt \end{aligned}$$

By integration by parts two time for the forth integral we get

$$-\frac{1}{|Y|}\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi\,dx\,dy\,dt+\theta\int_0^T\int_\Omega f\Psi\varphi\,dx\,dt$$
$$=\theta\int_\Omega u^0\Psi\,dx-\theta\int_\Omega u(x,0)\Psi dx+\theta\int_0^T\int_\Omega u''\Psi\varphi\,dx\,dt-\theta\int_0^T\int_\Omega u'\Psi\varphi\,dx\,dt.$$

combining this equation with (3.27) we have

$$u(x,0)=u^0.$$

3.2 Homogenization of hyperbolic-parabolic problem in the domains $\Omega_{\varepsilon,\delta}$

In this chapter, we suppose thas $N\geq 3$ and that ε and $\delta=\delta(\varepsilon)$ are such that (2.5) holds, that is, there exists the following limit and is finite:

$$k^*=\lim_{\varepsilon\to 0}\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}<+\infty, \tag{3.32}$$

We also denote by $M(\alpha,\beta,\Omega)$ the set of $N\times N$ matrices $A=(aij)_{1\leq i,j\leq N}$ in $(L^\infty(\Omega))^{N\times N}$ such that

(i) $(A(x)\lambda,\lambda)\geq\alpha|\lambda|^2$,

(ii) $|A(x)\lambda|\leq\beta|\lambda|$,

for any $\lambda\in\mathbb{R}^N$ and almost everywhere on Ω, where $\alpha,\beta\in\mathbb{R}$ such that $0<\alpha<\beta$. We want to study the asymptotic behavior as $\varepsilon\to 0$, of the problem

$$\begin{cases} u''_{\varepsilon,\delta}(x,t)+u'_{\varepsilon\delta}(x,t)-div(A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t))=f_{\varepsilon,\delta}(x,t) \quad \text{in } \Omega^*_{\varepsilon,\delta}\times]0,T[\\ u_{\varepsilon,\delta}(x,t)=0 \quad \text{on } \partial\Omega^*_{\varepsilon,\delta}\times]0,T[,\\ u_{\varepsilon,\delta}(x,0)=u^0_{\varepsilon,\delta}(x)\,,\ u'_{\varepsilon,\delta}(x,0)=u^1_{\varepsilon,\delta}(x) \ \text{in} \ \ \Omega^*_{\varepsilon,\delta}. \end{cases} \tag{3.33}$$

We suppose that the data satisfy the following assuptions

$$\begin{cases} \boldsymbol{A^\varepsilon \in M(\alpha,\beta,\Omega), A^\varepsilon} \text{ symmetric,} \\ \boldsymbol{f_{\varepsilon,\delta} \in L^2(0,T;L^2(\Omega^*_{\varepsilon,\delta}))}, \\ \boldsymbol{u^0_{\varepsilon,\delta} \in H^1_0(\Omega^*_{\varepsilon,\delta})}, \\ \boldsymbol{u^1_{\varepsilon,\delta} \in L^2(\Omega)}. \end{cases} \tag{3.34}$$

Moreover, we assume that

$$\begin{cases} \boldsymbol{u^0_{\varepsilon,\delta} \rightharpoonup u^0} \quad \text{weakly in} \quad \boldsymbol{L^2(\Omega)}, \\ \boldsymbol{u^1_{\varepsilon,\delta} \rightharpoonup u^1} \quad \text{weakly in} \quad \boldsymbol{L^2(\Omega)}, \\ \boldsymbol{f_{\varepsilon,\delta} \rightharpoonup f} \quad \text{weakly in} \quad \boldsymbol{L^2(0,T;L^2(\Omega))}. \end{cases} \tag{3.35}$$

The set

$$\boldsymbol{\mathcal{W}_{\varepsilon,\delta} = \{v_{\varepsilon,\delta} \in L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta})) : v'_{\varepsilon,\delta} \in L^2(0,T;L^2(\Omega^*_{\varepsilon,\delta}))\}}, \tag{3.36}$$

is equipped with the norm

$$\boldsymbol{\|v_{\varepsilon,\delta}\|_{\mathcal{W}_{\varepsilon,\delta}} = \|v_{\varepsilon\delta}\|_{L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta}))} + \|v'_{\varepsilon,\delta}\|_{L^2(0,T;L^2(\Omega^*_{\varepsilon,\delta}))}}.$$

Variational formulation

We multiply (3.33) by a test function $\boldsymbol{v}$ where $\boldsymbol{v \in V^\varepsilon}$ we get :

$$\boldsymbol{\int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta} v \, dx + \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta} \, dx - \int_{\Omega^*_{\varepsilon,\delta}} (div A^\varepsilon \nabla u_{\varepsilon,\delta}) v \, dx = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v \, dx}.$$

we integrate on $\boldsymbol{\Omega^*_{\varepsilon,\delta}}$ and by Green's formula we get :

$$\boldsymbol{- \int_{\Omega^*_{\varepsilon,\delta}} div(A^\varepsilon \nabla u_{\varepsilon,\delta}) v \, dx = \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v \, dx - \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} v n_{\varepsilon,\delta} \, d\Gamma}. \tag{3.37}$$

then:

$$-\int_{\Omega^*_{\varepsilon,\delta}} div(A^\varepsilon \nabla u_{\varepsilon,\delta}) v\, dx = \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v\, dx \tag{3.38}$$

we replace (3.38) in (3.37) we get :

$$\int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta} v\, dx + \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta} v\, dx + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v\, dx = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v\, dx \tag{3.39}$$

The variational formulation of problem(3.33) is : find $u_{\varepsilon,\delta} \in \mathcal{W}_{\varepsilon,\delta}$ such that for all $v \in H^1_0(\Omega^*_{\varepsilon,\delta})$

$$\begin{cases} \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta} v\, dx + \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta} v\, dx + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v\, dx = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v\, dx \ \text{ in } \mathcal{D}'(0,T) \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x) \ \ , u'_{\varepsilon,\delta}(x,0) = u^1_{\varepsilon,\delta}(x) \ \Omega^*_{\varepsilon,\delta} \end{cases} \tag{3.40}$$

Classical results [[16],[8]] provide for every fixed ε and δ the existence and uniqueness of a solution of problem (3.40) such that

$$u_{\varepsilon,\delta} \in \mathbb{C}^0([0,T]; H^1_0(\Omega^*_{\varepsilon,\delta})) \cap \mathbb{C}^1(0,T; L^2(\Omega^*_{\varepsilon,\delta}))$$

and satisfies the estimate

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;H^1_0(\Omega^*_{\varepsilon,\delta}))} + \|u'_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,\delta}))} \le C, \tag{3.41}$$

where C is independent of ε and δ.

Remark 3.1 *In the following, we identify functions in $H^1_0(\Omega^*_{\varepsilon,\delta})$ with their zero extension to $H^1_0(\Omega)$ so that we can right (3.41) as*

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;H^1_0(\Omega))} + \|u'_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{3.42}$$

where C is independent of ε and δ. We adapt here for the evolution problem some arguments introduced in [5] . Let us introduce the functional space

$$K_B = \left\{ \Phi \in L^2(0,T; L^{2^*}(\mathbb{R}^N)) : \nabla\Phi \in L^2(0,T.L^2(\mathbb{R}^N)), \Phi \textit{ is constant on } B \right\}. \tag{3.43}$$

We also need the following lemmas from [5] in order to pass to the limit in equation (3.40).

Lemma 3.1 *[5]. Let $N \geq 3$. Then, for every $\delta_0 > 0$, the set*

$$\cup_{0<\delta<\delta_0}\{\phi \in H^1_{per}(Y) : \phi = 0 \text{ on } \delta B\},$$

is dense in $H^1_{per}(Y)$.

Lemma 3.2 *[5]. Let $v \in \mathcal{D}(\mathbb{R}^N) \cap K_B$ (i.e.,$v = v(B)$ is constant on B) and set*

$$w_{\varepsilon,\delta}(x) = v(B) - v\left(\frac{1}{\delta}\{\frac{x}{\varepsilon}\}_Y\right)$$

for $x \in \mathbb{R}^N$. Then

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \text{ weakly in } H^1(\Omega). \tag{3.44}$$

Remark 3.2 *1. from the definition of $w_{\varepsilon,\delta}$ above, one has*

$$\mathcal{T}_{\varepsilon,\delta}(x,z) = v(B) - v(z) \text{ in } \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y,$$

and consequently (see [5]),

$$\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) = -\frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})) = -\frac{1}{\varepsilon\delta}\nabla_z v \text{ in } \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y. \tag{3.45}$$

2. Let $\{w_{\varepsilon,\delta}\}$ be a sequence satisfying (3.44). We have,

$$\mathcal{T}_\varepsilon(w_{\varepsilon,\delta}) \to v(B) \text{ strongly in } L^2(\Omega \times Y) \tag{3.46}$$

Indeed, it was shown in [5] that $\{w_{\varepsilon,\delta}\}$ is bounded in $H^1(\Omega)$ so that together with (3.44) and Rellich compatness theorem, one has $w_{\varepsilon,\delta} \to v(B)$ strongly in $L^2(\Omega)$; that is ,

$$\|w_{\varepsilon,\delta} - v(B)\|_{L^2(\Omega)} \to 0.$$

We state now a homogenization theorem for system (3.33):

Theorem 3.2 *Under assumption* (3.34) *and* (3.35), *suppose that as* $\varepsilon \to 0$, *there is a matrix field a such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,z) \to A(x,z) \ \text{a.e. in } \Omega \times Y, \tag{3.47}$$

and as both $\varepsilon, \delta \to 0$, *there exists a matrix field* A^0 *such that*

$$\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(x,z) \to A^0(x,z) \ \text{a.e. in } \ \Omega \times (\mathbb{R}^N \backslash B). \tag{3.48}$$

Let $u_{\varepsilon,\delta}$ *be the solution of*(3.40). *Then there exists* u *in* $L^\infty(0,T;H^1_0(\Omega))$ *and* $\hat{u}$ *in* $L^\infty(0,T;L^2(\Omega;H^1_{per}(Y))$
such that

$$\begin{cases} (i) u_{\varepsilon,\delta} \rightharpoonup u \ \text{weakly* in } \ L^\infty(0,T;H^1_0(\Omega)), \\ (ii) u'_{\varepsilon,\delta} \rightharpoonup u' \ \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (iii) \mathcal{T}_\varepsilon(u_{\varepsilon,\delta}) \rightharpoonup u \ \text{weakly* in } \ L^\infty(0,T;L^2(\Omega;H^1(Y))), \\ (iv) \mathcal{T}_\varepsilon(u'_{\varepsilon,\delta}) \rightharpoonup u' \ \text{weakly* in } L^\infty(0,T;L^2(\Omega \times Y)), \\ (v) \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y \hat{u} \ \text{weakly* in } L^\infty(0,T;L^2(\Omega \times Y)). \end{cases} \tag{3.49}$$

and $U \in L^2(0,T;L^2(\Omega, L^2_{loc}(\mathbb{R}^N)))$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \ \text{weakly in } L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N))), \tag{3.50}$$

with U *vanishing on* $\Omega \times B \times]0,T[$ *and* $U - k^* u \in L^2(0,T;L^2(\Omega;K_B))$
(K_B *being defined by* (3.43)).
The couple $(u, \hat{u})$ *satisfies the limit equation*

$$\int_Y A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t)) \nabla_y \phi(y) dy = 0, \tag{3.51}$$

for a.e. $x \in \Omega$, *a.e.* $t \in]0,T[$ *and for* $\phi \in H^1_{per}(Y)$. *while the function* U *obeys*

$$\int_{\mathbb{R}^N \backslash B} A^0(x,z) \nabla_z U(x,z,t) \nabla_z v(z) dz = 0. \tag{3.52}$$

for a.e. $x \in \Omega$, *a.e.* $t \in]0, T[$ *and for all* $v \in K_B$ *with* $v_B = 0$.

The ordered triplet $(u, \hat{u}, U)$ *satisfies the limit equation*

$$
\begin{aligned}
&\langle u''(.,t), \psi \rangle_{(H_0^1(\Omega))', H_0^1(\Omega)} + \langle u'(.,t), \psi \rangle_{(H_0^1(\Omega))', H_0^1(\Omega)} \\
&+ \int_{\Omega \times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t)) \nabla \psi(x) dx\, dy \\
&- k^* \int_{\Omega \times \partial B} A^0(x,z) \nabla_z U(x,z,t) \nu_B \psi(x) dx\, d\sigma_z \\
&= \int_\Omega f(x,t) \psi(x) dx, \quad \textit{for a.e. } t \in]0,T[\textit{ and for all } \psi \in H_0^1(\Omega), \\
&u(x,0) = u^0 \ , u'(x,0) = u^1 \textit{ in } \Omega.
\end{aligned}
\tag{3.53}
$$

where ν_B *is the inward normal to* ∂B *and* $d\sigma_z$ *its surface measure.*

In what follows, we will use the notation $m_Y(.)$ *for the average over* Y *defined as*

$$
m_Y(v) = \frac{1}{|Y|} \int_Y v(y)\, dy, \quad \forall v \in L^1(Y)
$$

The result below describes now the homogenized problem in the variable (x, t) *in* $\Omega \times]0, T[$.
To this aim, let us consider the correctors $\hat{\chi}_j, j = 1, ..., N$ *solutions of the cell problem; they are the same for domains without holes (see[[1],[8]]).*

$$
\begin{aligned}
&\hat{\chi}_j \in L^\infty(\Omega; H^1_{per}(Y)), \\
&\int_Y A \nabla(\hat{\chi}_j - y_j) \nabla \varphi\, dy = 0 \ \textit{ a.e. } x \in \Omega, \forall \varphi \in H^1_{per}(Y) \\
&m_Y(\hat{\chi}_j) = 0,
\end{aligned}
\tag{3.54}
$$

where A *is given by* (3.47).

We condider also the cell problem corresponding to the holes B *defining the corrector* θ *for smal holes, introduced in [5],*

$$
\begin{aligned}
&\theta \in L^\infty(\Omega; K_B), \quad \theta(x, B) \equiv 1, \\
&\int_{\mathbb{R}^N \setminus B} {}^t A^0(x,z) \nabla_z \theta(x,z) \nabla_z \Psi(z) dz = 0 \\
&\textit{a.e. for } x \in \Omega, \forall \Psi \in K_B \textit{ with } \Psi(B) = 0.
\end{aligned}
\tag{3.55}
$$

corollary 3.1 *Under assuption*(3.34) *and* (3.35), $u \in H_0^1(\Omega)$ *is the unique solution of* *the limit problem*

$$\begin{cases} u'' + u' - div(\mathcal{A}^{hom}\nabla u) + (k^*)^2\Theta u = f \ \textit{in} \ \Omega\times]0,T[, \\ u = 0 \ \textit{in} \ \partial\Omega\times]0,T[, \\ u(x,0) = u^0 \ , u'(x,0) = u^1 \ \textit{in} \ \Omega. \end{cases} \tag{3.56}$$

where the homogenized matrix field is

$$\mathcal{A}^{hom} = m_Y(a_{ij} + \sum_{k=1}^{N} a_{ik}\frac{\partial\hat{\chi}_j}{\partial y_k}). \tag{3.57}$$

and

$$\Theta = \int_{\partial B} {}^t A^0 \nabla_z \theta \nu_B \, d\sigma_z. \tag{3.58}$$

Remark 3.3 *as shown in [5], Θ can be interpreted as the local capacity of B. (see also ([[11],[9]]). Moreover, from (3.55) it is easily seen that Θ is non -negative, i.e.,*

$$\Theta(x) = \int_{\mathbb{R}^N\setminus B} A_0(x,z)\nabla_z\theta(x,z)\nabla_z\theta(x,z)\,dz \geq 0,$$

that is essential for the existence of the solution of the homogenized system (3.56).

proof of theorem

We prove the results in serval steps .

step 1

The existence of $u \in L^\infty(0,T;H_0^1(\Omega))$ such that up to subsequences, convergences (3.49)(i)-(ii) hold, follows from estimate(3.41) while the existence of $\hat{u} \in L^\infty(0,T;L^2(\Omega;H^1_{per}(Y)))$ and such that convergences (3.49)(iii)-(v) hold, follows from (3.42) and theorem 2.3 there exists a function W in $L^2(0,T;L^2(\Omega;L^{2^*}(\mathbb{R}^N)))$ with $\nabla_z W \in L^2(0,T;L^2(\Omega\times\mathbb{R}^N))$ such that (up to a subsequence)

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) - \mathcal{M}^{\varepsilon}_Y\left(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\right) \rightharpoonup W \text{ weakly in } L^2(0,T;L^2(\Omega;L^{2^*}(\mathbb{R}^N))). \tag{3.59}$$

Moreover, in view of (2.5), again by Theorem 2.3 there exists $\boldsymbol{U}$ such that (up to a subsequence) (3.50) holds.

step 2

Let us check the properties of the function $\boldsymbol{U}$. From(i) and (ii) of (3.49) we have by compactness,

$$\boldsymbol{u}_{\varepsilon,\delta} \to \boldsymbol{u} \text{ strongly in } \quad \boldsymbol{L^2(0,T;L^2(\Omega))} \tag{3.60}$$

so that from proposition 2.6 and theorem 2.3,

$$\frac{\boldsymbol{\delta^{\frac{N}{2}-1}}}{\boldsymbol{\varepsilon}}\boldsymbol{\mathcal{M}^{\varepsilon}_{Y}(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}} \to \boldsymbol{k^{*}u} \text{ strongly in } \quad \boldsymbol{L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N)))}. \tag{3.61}$$

Thus, from (3.50) , (3.59) and (3.61) we conclude that

$$\boldsymbol{U = W + k^{*}u} \text{ and } \boldsymbol{\nabla_z U = \nabla_z W}$$

Moreover, by using (2.4) of theorem 2.3, we have

$$\boldsymbol{\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})} = \frac{\boldsymbol{\delta^{\frac{N}{2}-1}}}{\boldsymbol{\varepsilon}}\boldsymbol{\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))1_{\frac{1}{\delta}Y}} \rightharpoonup \boldsymbol{\nabla_z U} \quad \boldsymbol{w-L^2(0,T;L^2(\Omega\times\mathbb{R}^N))}. \tag{3.62}$$

Also, from definition 2.5

$$\boldsymbol{\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0} \text{ in } \boldsymbol{\Omega \times B \times]0,T[},$$

and thus from (3.50), definition 2.4 and (3.61)

$$\boldsymbol{U = u = 0} \text{ in } \boldsymbol{\Omega \times B \times]0,T[}. \tag{3.63}$$

this means that

$$\boldsymbol{W = U - k^{*}u} \text{ in } \boldsymbol{L^2(0,T;L^2(\Omega;K_B))}.$$

step 3

Let us prove the first limit equation.

Let $\psi \in \mathcal{D}(\Omega)$ and $\phi \in C^1_{per}(Y)$ vanisging in a neighborhood of $y = 0$, and set

$$v_\varepsilon(x) = \varepsilon\psi(x)\phi^\varepsilon(x) \text{ with } \phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon}). \tag{3.64}$$

then

$$\begin{aligned} \nabla v_\varepsilon(x) &= \nabla(\varepsilon\psi(x)\phi^\varepsilon(x)) \\ &= \varepsilon\nabla\psi(x)\phi^\varepsilon(x) + \psi(x)\nabla\phi^\varepsilon(x), \end{aligned} \tag{3.65}$$

then

$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \to \psi\nabla_y\phi \text{ strongly in } L^2(\Omega \times Y). \tag{3.66}$$

Taking v_ε as a test function in (3.40), multiplyin by $\varphi \in \mathcal{D}(0, T)$, and integrating over $]0, T[$, we get

$$\begin{aligned} &\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(t)\,dx\,dt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\,dx\,dt \\ &+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt = \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt. \end{aligned} \tag{3.67}$$

We have v_ε defind in (3.64) , this equation can be rewitten as

$$\begin{aligned} &\varepsilon\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)\psi(x)\phi^\varepsilon(x)\varphi''(t)\,dx\,dt - \varepsilon\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}\psi(x)\phi^\varepsilon(x)\varphi'(t)\,dx\,dt \\ &+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt = \varepsilon\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt. \end{aligned} \tag{3.68}$$

We first use the unfolding operator $\mathcal{T}_\varepsilon$ to pass to the limit in the third term of the left-hand side of this equation using (3.65) and (3.66), we have

$$\begin{cases} \mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \varphi \text{ weakly}^* \text{ in } L^\infty(0,T;L^p(\Omega;W^{1,p}(Y))). \\ \mathcal{T}_\varepsilon(\nabla v_\varepsilon) \to \psi \nabla_y \phi \text{ strongly in } L^2(\Omega \times Y). \\ \mathcal{T}_\varepsilon(A^\varepsilon) \to A(x,y) \text{ a.e., in } \Omega \times Y. \\ \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y \hat{u} \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega \times Y)). \end{cases}$$

then we get

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega\times Y} A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t) \nabla v_\varepsilon(x) \varphi(t)\, dx\, dt \\ &= \lim_{\varepsilon\to 0} \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon) \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \mathcal{T}_\varepsilon(\nabla v_\varepsilon) \mathcal{T}_\varepsilon(\varphi)\, dx\, dy\, dt \\ &= \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u + \nabla_y \hat{u}) \psi \nabla_y \phi(y) \varphi(t)\, dx\, dy\, dt \end{aligned} \tag{3.69}$$

On the other hand, the first term on the first and second term on the left-hand side of (3.68) as well as the term on the right-hand side goes to zero as $\varepsilon \to 0$, which implies

$$\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t) \nabla v_\varepsilon(x) \varphi(t)\, dx\, dt = 0,$$

so that:

$$\frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u + \nabla_y \hat{u}) \psi \nabla_y \phi(y) \varphi(t)\, dx\, dy\, dt = 0.$$

By lemma 3.1, we obtain (3.51) which describes the asymtotic behavior of the problem based on the oscillations in the coefficients of (3.33) Now, to take into account the effect of the perforations, let us use $w_{\varepsilon,\delta}\psi$ as a test function in(3.40), where $w_{\varepsilon,\delta}$ is the function defined in lemma 3.2 and for $\psi \in \mathcal{D}(\Omega)$. Thus, we have

$$\begin{aligned}\langle u''_{\varepsilon,\delta}(x,t), w_{\varepsilon,\delta}(x)\psi(x)\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))',H_0^1(\Omega^*_{\varepsilon,\delta})} + \langle u'_{\varepsilon,\delta}(x,t), w_{\varepsilon,\delta}(x)\psi(x)\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))',H_0^1(\Omega^*_{\varepsilon,\delta})} \\ + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}\psi(x)\, dx + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x)\nabla\psi(x)\, dx \\ = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\, dx.\end{aligned}$$

Let $\varphi \in \mathcal{D}(0,T)$ and multiply the integrands in this equation and integrate over $]0,T[$,

$$\begin{aligned}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}\psi(x)\varphi''(t)\, dx\, dt - \int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\varphi'(t)\, dx\, dt \\ + \int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\, dx\, dt \\ + \int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x)w_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)\, dx\, dt \\ = \int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}\psi(x)\varphi(t)\, dx\, dt.\end{aligned} \tag{3.70}$$

For the first term on the left-hand side of this equation, we apply the operator $\mathcal{T}_\varepsilon$. Thus, from definition 2.2 with Remark 3.2(2) and (3.49)(iii) , we obtain,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\varphi''(t)\, dx\, dt \\ = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} \mathcal{T}_\varepsilon(u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\psi)\varphi''(t)\, dx\, dy\, dt \\ = \frac{v(B)}{|Y|}\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u(x,t)\psi(x)\varphi''(t)\, dx\, dy\, dt.\end{aligned} \tag{3.71}$$

For the second term on the left-hand side of this equation, we apply the operator $\mathcal{T}_\varepsilon$. Thus, from definition 2.2 with Remark 3.2(2) and (3.49)(iii) , we obtain,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\varphi'(t)\, dx\, dt \\ = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} \mathcal{T}_\varepsilon(u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\psi)\varphi'(t)\, dx\, dy\, dt \\ = \frac{v(B)}{|Y|}\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u(x,t)\psi(x)\varphi'(t)\, dx\, dy\, dt.\end{aligned} \tag{3.72}$$

For the third term on the left-hand side of equation(3.70), we use the operator $\mathcal{T}_{\varepsilon,\delta}$. Then, Remark 1.3 toghether with (3.32),(3.48),(3.62), (3.64), and proposition 2.7 (2), proposition 2.8 and Remark 3.2, yield

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt\\
&=\lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\psi(x))\varphi(t)\,dx\,dz\,dt\\
&=\lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\frac{1}{\delta}\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi(t)\,dx\,dz\,dt\\
&=\lim_{\varepsilon\to 0}(-\frac{\delta^{\frac{N}{2}-1}}{\varepsilon|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}))\nabla_z v\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi(t)\,dx\,dz\,dt)\\
&=-\frac{k^*}{|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v\psi(x)\varphi(t)\,dx\,dz\,dt,\\
&=-\frac{k^*}{|Y|}\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v\psi(x)\varphi(t)\,dx\,dz\,dt
\end{aligned}$$

so that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt\\
&=-\frac{k^*}{|Y|}\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v\psi(x)\varphi(t)\,dx\,dz\,dt
\end{aligned}\tag{3.73}$$

For the forth term on the left-hand side of (3.70), we use $\mathcal{T}_\varepsilon$. from definition 2.2(2), proposition2.1(i), 2.4 , 2.5(ii), definition 2.5 together with remark 1.3, (3.47), passing to the limit gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)\,dx\,dt\\
&=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\varphi(t)\,dx\,dt\\
&=\frac{v(B)}{|Y|}\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt.
\end{aligned}\tag{3.74}$$

For the term on the right-hand side of (3.70), we use $\mathcal{T}_\varepsilon$. from definition 2.5, Remark 3.2(2), and (3.35), definition 2.2(2) and passing to the limit, yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}\psi(x)\varphi(t)\,dx\,dt.\\
&=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(f_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\psi)\varphi(t)\,dx\,dy\,dt \qquad (3.75)\\
&=\frac{v(B)}{|Y|}\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt.
\end{aligned}$$

Thus, combining (3.71)-(3.75), the limit equation of (3.70)

$$\begin{aligned}
&v(B)\int_0^T\int_{\Omega\times Y} u(x,t)\psi(x)\varphi''(t)\,dx\,dy\,dt - v(B)\int_0^T\int_{\Omega\times Y} u(x,t)\psi(x)\varphi'(t)\,dx\,dy\,dt\\
&-k^*\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt\\
&+v(B)\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y \hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt\\
&=v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt. \qquad (3.76)
\end{aligned}$$

Which is true for all $\varphi\in\mathcal{D}(0,T)$, $\psi\in H^1_0(\Omega)$ and $v\in K_B$. So , we obtain (3.52) for $v\in K_B$ such that $v(B)=0$.

If $v(B)\neq 0$, by applying Stok's formula and (3.52), we have

$$\begin{aligned}
&\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt\\
&=v(B)\int_0^T\int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B\psi(x)\varphi(t)\,dx\,d\sigma_z\,dt.
\end{aligned}$$

which used in (3.76) gives the first equation of problem (3.53).

step 4

It remains now to check the limit initial condition. Let $v_\varepsilon = w_{\varepsilon,\delta}\psi$ where $w_{\varepsilon,\delta}$ given by lemma 3.2 and $\psi\in\mathcal{D}(\Omega)$. Let $\varphi\in C^\infty([0,T])$ with $\varphi(0)=1$ and $\varphi(T)=0$. Take $v_\varepsilon\varphi$ as a test function in (3.40). Using the initial condition in (3.40) and by integration by parts, we have

$$\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v_\varepsilon(x)\varphi(t)\,dx\,dt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon \varphi(t)\,dx\,dt$$
$$= \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt + \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt.$$

Using the initial condition in (3.33) and by integration by parts of this term $\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt$, we have

$$\begin{aligned}
\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt &= \int_{\Omega^*_{\varepsilon,\delta}} (u'_{\varepsilon,\delta}(x,t)\varphi(t))|_0^T v_\varepsilon(x)\,dx \\
&\quad - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\,dx\,dt \\
&= - \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,0)v_\varepsilon(x)\,dx \\
&\quad - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\,dx\,dt \\
&= - \int_{\Omega^*_{\varepsilon,\delta}} u^1_{\varepsilon,\delta}(x)v_\varepsilon(x)\,dx \\
&\quad - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\,dx\,dt.
\end{aligned}$$

In view of (3.72)-(3.75) and (3.35), passin to the limit in this equation yields

$$\begin{aligned}
& v(B)\int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt \\
& + k^* \int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt \\
& - v(B)\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt \\
& = -v(B)\int_\Omega u^1(x)\psi(x)\,dx - v(B)\int_0^T \int_\Omega u'(x,t)\psi(x)\varphi'(t)\,dx\,dt \\
& + v(B)\int_0^T \int_\Omega u'(x,t)\psi(x)\varphi(t)\,dx\,dt \\
& = -v(B)\int_\Omega u^1(x)\psi(x)\,dx + v(B)\int_\Omega u'(x,0)\psi(x)\,dx \\
& + v(B)\int_0^T \int_\Omega u''(x,t)\psi(x)\varphi(t)\,dx\,dt + v(B)\int_0^T \int_\Omega u'(x,t)\psi(x)\varphi(t)\,dx\,dt.
\end{aligned}$$

Combining this equation with (3.76) yields

$$-\int_{\Omega} u^1(x)\psi(x)\,dx + \int_{\Omega} u'(x,0)\psi(x)\,dx = 0, \quad \forall\psi \in \mathcal{D}(\Omega), \tag{3.77}$$

Which implies $u'(x,0) = u^1(x)$. For the first initial condition, let us now choose $\varphi \in C^\infty([0,T])$ with $\varphi(0) = \varphi(T) = \varphi'(T) = 0$ and $\varphi'(0) = 1$. Let us take again $v_\varepsilon\varphi$ as a test function in (3.40).

Using the initial conditions in (3.40) and by integration by parts, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x,z)\varphi(t)\,dx\,dt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)v_\varepsilon(x,z)\varphi(t)\,dx\,dt \\
&\quad = \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t)v_\varepsilon(x,z)\varphi(t)\,dx\,dt + \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x,z)\varphi(t)\,dx\,dt.
\end{aligned}$$

Using the initial condition in (3.33) and by integration by parts, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t)v_\varepsilon(x,z)\varphi(t)\,dx\,dt \\
&= \int_{\Omega^*_{\varepsilon,\delta}} (u'_{\varepsilon,\delta}(x,t)\varphi(t))|_0^T v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\,dx\,dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} (u_{\varepsilon,\delta}(x,t)\varphi'(t))|_0^T v_\varepsilon(x)\,dx + \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(t)\,dx\,dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,0)v_\varepsilon(x)\,dx + \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(x,t)\,dx\,dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u^0_{\varepsilon,\delta}(x,0)v_\varepsilon(x)\,dx + \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(x,t)\,dx\,dt,
\end{aligned}$$

In view of (3.72)-(3.75), the initial condition in (3.40) together with (3.35), passing to the limit in this equation yields

$$\begin{aligned}
&v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt \\
&+k^*\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt \\
&-v(B)\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt \\
&=-v(B)\int_\Omega u^0(x)\psi(x)\,dx - v(B)\int_0^T u(x,t)\psi(x)\varphi(t)\,dx\,dt \\
&=-v(B)\int_\Omega u^0(x)\psi(x)\,dx + v(B)\int_\Omega u(x,0)\psi(x)\,dx \\
&+v(B)\int_0^T u''(x,t)\psi(x)\varphi(t)\,dx\,dt + v(B)\int_0^T u'(x,t)\psi(x)\varphi(t)\,dx\,dt.
\end{aligned} \tag{3.78}$$

Combining (3.78) with (3.76)

$$-\int_\Omega u^0(x)\psi(x)\,dx + \int_\Omega u(x,0)\psi(x)\,dx \quad \forall\psi\in\mathcal{D}(\Omega),$$

which implies $u(x,0)=u^0(x)$. This cncludes the proof

CONCLUSION

In this memory, we have treated the asymptotic behavior of hyperbolic-parabolic problem with boundary condition in perforated domain with small holes, we used the periodic unfolding method we get that the triplet $(\boldsymbol{u}, \hat{\boldsymbol{u}}, \boldsymbol{U})$ with $\boldsymbol{\mathcal{M}}(\hat{\boldsymbol{u}}) = \mathbf{0}$ is unique solution of problem (3.33), this implie that all convergence in theorem 3.2, and we get the following homogeneous problem:

$$\begin{cases} \boldsymbol{u}'' + \boldsymbol{u}' - \boldsymbol{div}(\boldsymbol{\mathcal{A}}^{hom}\nabla \boldsymbol{u}) + (\boldsymbol{k}^*)^2\Theta \boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega\times]\mathbf{0}, \boldsymbol{T}[, \\ \boldsymbol{u} = \mathbf{0} \text{ in } \partial\Omega\times]\mathbf{0}, \boldsymbol{T}[, \\ \boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{u}^{\mathbf{0}} \ , \boldsymbol{u}'(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{u}^{\mathbf{1}} \text{ in } \Omega \end{cases}$$

BIBLIOGRAPHY

[1] A. Bensoussan, J.-L. Lions, G. Papanicolaou; Asymptotic analysis for periodic structures, North-Holland, Amsterdam 1978.

[2] S. Brahim-Otsmarl, G.A. Francfort and F. Murat, Correctors for the homogenization

[3] D. Cioranescu, A. Damlamian, G. Griso; Periodic unfolding and homogenization, C.R. Acad.Sci. Paris, Ser. I, 335 (2002), 99-104.

[4] D. Cioranescu, A. Damlamia.n and G. Griso, The periodic unfolding method in homogenization, SIAM J. lVlath. Anal. 40(4), (2008) 1585-1620.

[5] D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei; The periodic unfolding method for perforated domains and Neumann sieve models, J. Math. Pures Appl. 89 (2008),248-277.

[6] D.Cioranescu, A.Damlamian, P. Donato, G. Griso and R. Zaki, The periodic unfolding method in domains with holes, SIAM J. math. Anal. 44(2), (2012) 718-760.of the wave and heat equations, J. Math. Pures Appl. 7L, (1992) 797-231.

[7] D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki, The periodic un-folding method in domains with holes, SIAM J. Math. Anal. 44(2), (2012) 718-760.

[8] D. Cioranescu, P. Donato; An Introduction to Homogenization, Oxford University Press 1999.

[9] D. Cioranescu, P. Donato, F. Murat, E. Zuazua; Homogenization and corrector for the wave equation in domains with small holes, Annali Scuola Norm. Sup. Pisa, Sc. Fis. Mat. (4), 17 (1991), 251-293.

[10] D. Cioranescu, P. Donato, R. Zaki; The periodic unfolding method in perforated domains, Portugaliae Math. (N.S), 63, 4 (2006), 467-496.

[11] D. Cioranescu, F. Murat; Un terme etrange venu d?ailleurs, in Nonlinear Partial Differential Equations and their Applications, College de France Seminar, I and II, Vol. 60, 98-138 and Vol 70, 154-178, Pitman Boston 1982. English translation: A strange term coming from nowhere, in Topics in Mathematical Modelling in Composite Materials, Springer New York 1997, 45-94.

[12] P. Donato, Z. Yang; The periodic unfolding method for the wave equation in domains withholes, Advances in Math. Sci. and Appl., 22, 2 (2012), 521-551.

[13] P. Donato, K. H. Le Nguyen and R. Tardieu, The periodic unfolding method for a class of imperfect transmission problems, J. Math. Sci. 176(6), (2011) 891-927.

[14] F. Gaveau; Homogeneisation et correcteurs pour quelques problemes hyperboliques, PhD Thesis, Universite Pierre et Marie Curie, Paris, 2009.

[15] F. Gaveau, Homogenization et correcteurs pour quelques problems hyperboliques, Dissertation for the doctoral Degree,Paris, Paris 6(2009).

[16] J.-L. Lions, E. Magenes; Problemes aux limites non homogenes et applications, Vol. 1, Dunod,Paris, 1968.

[17] E.Sanchez-Palencia. Non-Hmogeneous Media and Vibration Theory. Lecture Notes in Physics, 127, Springer Verlag, Berlin, 1980.

[18] V. A. Steklov. Fundamental Problems in Mathematical Physics. Nauka, Moscow, 1983.