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Faculty of Science

Department of Mathematics

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from :

1. **Lakhchine Lynda**
2. **Lakhchine Meriem**

Supervision Fromm: ATOUI HALIM **MAA** U.SKIKDA

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president : HAMDI ZAKARIA MCB U. SKIKDA
Examiner: BEDRANI YACINE MAA U. SKIKDA

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(ELHAMDOULILAH)

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Do not forget the special thanks to our parents

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Dedication

Idedicate this dissertation to my adorable andcaffectinate family,

parents for theirv love , patince, confidence ,and their financial support through the many years of my education.

To my dear brothers

To my dearest sisters

To my best frind ,

jo my derayaunt habiba

lynda

I dedicate this modest work which is the fruit of

A long path of study and work

To my dear father and my mother

To all my brothers and all my sisters

To all my friends and close friends

Without forgetting our teachers,

Who helped and guided us during all our years of study

meriem

Contents

Abstract

In this work we study a transmission problem for one dimensional wave equations with nonlinear weights and under the effects of time-varying delay. Using Kato's technique we prove existence and uniqueness of solutions and we construct a suitable Lyapunov functional to understand the asymptotic behavior of solutions.

Keywords: Wave equation, transmission problem, time-varying delay term.

Résumé

Dans ce travail, nous étudions un problème de transmission pour des équations d'onde unidimensionnelles avec des poids non linéaires et sous les effets d'un retard variant dans le temps. En utilisant la technique de Kato, nous prouvons l'existence et l'unicité des solutions et nous construisons une fonctionnelle de Lyapunov appropriée pour comprendre le comportement asymptotique des solutions.

Mots clés: Équation d'onde, problème de transmission, terme de retard variant dans le temps.

Introduction

The control of partial differential equations under the effects of time delay has become recently an active area of research, and a related problems have attracted a great attention in the literature. It is shown that for a stable system, any small delay may destabilize it, which leads to add more conditions or terms to fix that problem see [8, 7, 9, 18].

In fact, transmission problems have several applications in real life (physics, biology,...). For instance, we find some applications of this theory in the metallurgical industry and smart materials technology, see [3, 16] and the references therein We mention that our study in this work is related to the propagation of waves over a body of two different types of materials: one is the elastic part and the other part that has time delay effect.

Global existence and asymptotic stability for a transmission problem for wave equation was studied by Benseghir in [4] with constant weights and constant time delay. Zitouni et al.[19] extend the results in [4] under some assumptions to the case of time-varing delay see also [20].

The purpose of stabilization consists to ensure the decay and to give an estimate of the decay rate of the energy of solutions towards 0. More precisely, we are interested to determine the asymptotic behavior of the energy denoted by $E(t)$ and to give an estimate of the decay rate of the energy. There are several types of stabilization; here the energy is denoted by $E(t)$,

1) If $E(t) \to 0$ as $t \to \infty$ then we have a **"Strong stabilization"**.

2) If $E(t) \leq c(ln(t))^{-\delta}, c, \delta > 0$ then we have a **"Logarithmic stabilization"**.

3) If $E(t) \leq ct^{-\delta}, c, \delta > 0$ then we have a **"Polynomial stabilization"**.

4) If $E(t) \leq ce^{-\delta t}c, \delta > 0$ then we have a **"Uniform stabilization"**.

The present work is devoted to the study of the global existence and asymptotic behaviour in time of solutions to transmission wave equations. Notice that this problem has been studied in [14], our aim here is to understand and to give more details. See also [2] for the one dimensional wave equation with non-constant delay and nonlinear weights.

This work consists of essentially of three chapters: In the first chapter we summarizes some basic concepts needed in the following chapters, note that all results and theorems are made without proofs. Next We investigate in the second chapter the existence and the uniqueness properties of solutions for the initial boundary value problem and we prove the global existence of its solutions in some Sobolev spaces by means of the semigroup theory. Finally, To prove decay estimates, we introduce suitable energie and Lyapounov functionals.

Chapter 1

Preliminary

In this chapter, we give a brief introduction to basic topological and functional spaces, together with some important results from functional analysis theory. For more details,see for instance [5, 6, 17].

1.1 General spaces

Definition 1.1. *(Vector space) A vector space E over $\mathbb{K}$ is set E together with two maps $+ : E \times E \to E$(addition) and $K \times E \to E$ (scalar multiplication) such that the pair $(E, +)$ is an abelian group and the follouing properties are satisfied*

1. *$\forall x, y, z \in E$ one has $(x + y) + z = x + (y + z)$.*
2. *There exists an element 0 such that $x + 0 = x$ for all $x \in E$.*
3. *$\forall x \in E$ there exuts an element $(-x) \in E$ such that $x + (-x) = 0$.*
4. *$\forall x, y \in E$, one has $x + y = y + x$, furthermore, the scalar multiplication satisfies the follouing properties.*
5. *$\forall x \in E$,and $\lambda, \mu \in \mathbb{K}$ on has $(\lambda + \mu) \times x = (\lambda x) + (\mu \times x)$ and $(\lambda\mu) \times x = \lambda(\mu x)$.*

6. $\forall \lambda \in \mathbb{K}$ *and* $x, y \in E$ *one has* $\lambda(x+y) = (\lambda x) + (\lambda y)$.

Definition 1.2. *(Normed vector space) Let* E *be a vector space over the field* $F = \mathbb{R}$ *or* $F = \mathbb{C}$*, a norm on* E *is a real valued function* $\|\cdot\|$ *with the folluoing properties:*

1. *Zero vector:* $\|x\| = 0$ *if and only if* $x = 0$.

2. *Scalar factors:* $\|\lambda x| = |\lambda|\|x\|$, $\forall \lambda \in F, \forall x \in E$.

3. *Triangle inequality:* $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$.

4. $\|x\| \geq 0$ *for all* $x \in E$.

Definition 1.3. *(Banach space) Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm.*

Example 1.1. *We define the norm* $\|\cdot\|_\infty$ *on* $\mathbb{R}^n (or\, \mathbb{C}^n)$ *by,*

$$\|(x_1, x_2, ...x_n)\|_\infty = \max\{|x_1|, |x_2, \ldots, |x_n|\}.$$

Notice that $\mathbb{R}^n$ *equipped with this norm is a finite dimensional Banach space.*

Definition 1.4. *Let* H *be a complex vector space, an inner product on* H *is a function* $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ *such that for all* $x, y, z \in H$,

1. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle; x \to \langle x, z \rangle$ *is linear.*

2. $< x, y >= \overline{< y, x >}$,

3. $\|x\|^2 =< x, x >\geq 0$ *with equality* $< x, x >= 0$ *iff* $x = 0$.

Theorem 1.1. *(Cauchy-Schwarz inequality) Let* $(H, < .,. >)$ *be an inner prudect space then for all* $x, y \in H$ $| < x, y > | \leq \|x\| \times \|y\|$ *and equality holds iff* x *and dependent* y *are linearly.*

Corollary 1.1. *Let* $(H, < .,. >)$ *be an inner product space and* $\|x\| = \sqrt{< x, x >}$ *then* $\|.\|$ *is a norm on* H *moreover* $< .,. >$ *is continuous on* $H \times H$*, were* H *is viwed as the normed space* $(H, \|.\|)$*.*

Definition 1.5. *(Hilbert space) A Hilbert space is an inner product space* $(H, < .,. >)$ *such that the induced hilberation norm is complete.*

1.2 Functional spaces

In the following we suppose that (Ω, S, μ) is a measure space.

1.2.1 Lebesgue spaces $L^p(\Omega)$

Definition 1.6. *(L^p spaces) For* $1 \leq p \leq \infty$*, we call* $L^p(\Omega)$ *the space of measurable functions* f *on* Ω *such that*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < +\infty \quad for \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \quad for \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space. In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Theorem 1.2. *(Younge's inequality): Suppose* $1 < p < \infty$ *and* p *the conjugate of* p*, then* $ab \leq \frac{a^2}{p} + \frac{bp}{p}$ *for all* $a \geq 0$ *and* $b \geq 0$*.*

Proof. Fix $b > 0$ and define a function $f : (0, \infty) \to \mathbb{R}$ by

$$f(a) = \frac{a^p}{p} + \frac{bp}{p} - ab$$

thus $f'(a) = a^{p-1} - b$ hence f is decreasing on the interval $(0.b^{\frac{1}{p-1}})$ an f is increasing on the interval $(b^{\frac{1}{(p-1)}}, \infty)$. thus $f(a) \geq 0$ for all $a \in (0, \infty)$, which implies the desicred inequality. □

Theorem 1.3. *(Holder's inequality) Suppose that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$*, where* $1 \leq p < \infty$ *and* p' *the conjugate of* p*, then* $\|fh\|_1 \leq \|f\|_p \|h\|_{p'}$.

Theorem 1.4. *(Menkowski's inequality) suppose is a measure space* $1 \leq p \leq \infty$ *and* $f, g \in L^p(\Omega)$*, then* $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

1.2.2 Sobolev spaces $W^{k,p}(\Omega)$

Now, we will introduce the Sobolev spaces.

Definition 1.7. *The Sobolev space* $W^{k,p}(\Omega)$ *is defined to be the subset of* L^p *such that function* f *and its weak derivatives up to some order* k *have a finite* L^p *norm, for given* $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \ \forall \alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} , \ for\ p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \ for\ p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \, . \, \|_{W^{k,p}}$ is a Banach space.

Definition 1.8. *Whene $p = 2$, we denote by*

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$).

Theorem 1.5. *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. We then have*

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$$

Moreover, the set $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

1.3 Semigroups

A linear operator $A : E \mapsto F$ is a transformation which maps lineary E in F, that is

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v), \quad \forall u, v \in E \quad and \quad \alpha, \beta \in \mathbb{C}$$

Definition 1.9.

- *A linear operator $A : E \mapsto F$ is said to be bounded there exists $C \geq 0$ such that*

  $$\|Au\|_F < C\|u\|_E \quad \forall u \in E$$

  The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$

- *A bounded operator* $A \in \mathcal{L}(E,F)$ *is said to be compact if for each sequence* $(u)_{n\in\mathbb{N}} \in E$ *with* $\|u_n\|_E = 1$ *for each* $n \in \mathbb{N}$, *the sequence* $(Au_n)_{n\in\mathbb{N}}$ *has a subsequence which converges in* F.

- *An unbounded linear operator T from E into F is a pair (T,D(T)), consisting of a subspace* $D(A) \subset E$ *(called the domain of* A *) and a linear transformation*

$$T : D(A) \subset E \to F$$

In the case when $E = F$ *then we say £(A,D(A))£ is an unbounded linear operator on E. If* $D(A) = E$ *then* $A \in \mathcal{L}(E,F)$.

Definition 1.10. *A family* $(S(t))_{t\geq 0}$ *of bounded linear operations in* X *is called a semigroup if :*

1. $S(I) = 0$ *(I is called identity operator on X).*

2. $S(t+s) = S(t)S(s)\ \forall t, s \geq 0$.

Definition 1.11. *A semigroup of bounded linear operators* $(S(t)_{t\geq 0})$ *is called :*

1. *uniformly continuous semegrop if* $\lim \|S(t) - T\|_{l(H)} = 0$.

2. *strong continuous semigroup (in short a Co-semigroup) if for each* $u \in H.S(t)u$ *is continuous in* t *on* $[0, \infty[$.

Definition 1.12. *for a semigroup* $(S(t))_{t\geq 0}$*,we define an linear operator* A *which domain* $D(A)$ *consisting of points* u *such that the limit:*

$$Au := \lim_{t\to 0_+} \frac{(S(t)u - u)}{t}$$

then A *is called the infinitesimal generator of the semigroup in X.*

Theorem 1.6. *Let $(S(t))_{t\geq 0}$ be a Co-semigroup then there escist a constant $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\|_{l(x)} \leq M \exp(\omega t), \forall t \geq 0$$

In the above theorem, if $\omega = 0$ then the corresponding semigroup uniformly bounded more over,if $M = 1$ then $(S(t))_{t\geq 0}$ is said to be a Co-semigroup of contraction.

Definition 1.13. *A linear operator $(A.D(A))$ on H,is said to be dissipative if $R\langle Au, u\rangle_H \geq 0, \forall u \in D(A)$.*

Definition 1.14. *A linear operator $(A, D(A))$ on X, is said to be m-dissipative if:*

1. *A is dissipative operator.*
2. *A is maximal i,e $\exists\lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$.*

Theorem 1.7. *(Lumer-Phillips) Let A be a linear operator with dense domain D(A) in a Banach space X.*

1. *If A is dissipative and there exists $\lambda_0 > 0$ such that the range $R(\lambda_0 I - A) = X$, then A generates a C_0semigroup of contractions on X.*
2. *If A is the infinitesimal generator of a $C_0 - semigrop$ of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.*

Now we consider the abstract problem;

$$\begin{cases} U_t = AU, t > 0 \\ U(0) = U_0 \end{cases}$$

where A is the infinitesimal generator of $C_0 - semigroupS(t)$ over a hilbert space H.

Theorem 1.8. *(Hill-Yoshida) let $(A, D(A))$ be a linear operator on H . Assume that A is the infinitesimal generator of Co-semigroup of contractions $(S(t))_{t\geq 0}$*

1. *for $U_0 \in D(A)$,the problem(1) admets a uniqe strong solution:*

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H)$$

2. *for $U_0 \in H$,the problem(1) admets a unique weak solution $U(t) \in C^0(\mathbb{R}^0, H)$.*

Chapter 2

Existence and uniqueness of solution

In this chapter we prove the existence and uniqunesse of solution for the system. We start by intoducing some important notations and assumpsions needed in order to acheive our studies.

2.1 Problematic and notations

We begin the prsent section by discussioning the following most relevant concepts marking our main problem:

$$\begin{aligned} &u_{tt}(x,t) - au_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 \text{ in } \Omega\times]0,\infty[\\ &v_{tt}(x,t) - bv_{x,x}(x,t) = 0 \text{ in }]L_1, l_2[\times]0,\infty[\end{aligned} \tag{2.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\bigcup]L_2, L_3[$ and a,b are positive constants.

Under the transmission condition

$$\begin{aligned} &u(L_i,t) = v(L_i,t), i = 1,2 \\ &au_x(L_i,t) = bv_x(L_i,t), i = 1,2 \end{aligned} \tag{2.2}$$

the boundary conditions

$$u(0,t) = u(L_3,t) = 0, \tag{2.3}$$

and initial conditions

$$\begin{aligned} & u(x,0)=u_0(x), u_t(x,0)=u_1(x) \text{ on } \Omega \\ & u_t(x,t-\tau(0))=f_0(x,t-\tau(0)) \text{ in } \Omega]0,\tau(0)[\\ & v(x,0)=v_0(x), v_t(x,0)=v_1(x)) \text{ in }]L_1,L_2[\end{aligned} \tag{2.4}$$

where the functions $(u_0, u_1, v_0, v_1, f_0)$ belongs to a suitable spaces here $0<\tau(t)$ is the time-varying delay and $\mu_1(t)$ and $\mu_2(t)$ are nonlinear weights. We assume as in [12] that

$$\tau(t)\in W^{2,\infty}([0,t]), \forall T>0 \tag{2.5}$$

and that there exist positive constants τ_0, τ_1 and d such that

$$0<\tau_0\leq \tau_t \leq \tau_1, \tau'(t)\leq d<1, \forall t>0 \tag{2.6}$$

We need the following hypothesis:

$(H1)$ $\mu_1:\mathbb{R}_+\to]0,\infty[$ is a non increasing function of class $\mathbb{C}^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right|\leq M_1, \forall t\geq 0, \tag{2.7}$$

where $M_1>0$ is a constant.

$(H2)$ $\mu_2:\mathbb{R}_+\to\mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, such that

$$|\mu_2(t)|\leq \beta\mu_1(t), \tag{2.8}$$

$$|\mu_2'(t)|\leq M_2\mu_1(t), \tag{2.9}$$

for some $0<\beta<\sqrt{1-d}$ and $M_2>0$.

We introduce the new variable

$$z(x,\rho,t)=u_t(x,t-\tau(t)\rho), (x,\rho)\in\Omega\times]0,1[,\ t>0 \tag{2.10}$$

Take the derivative of z with respect to t and ρ respectively, we get

$$z_t(x,\rho,t) = \frac{\partial}{\partial t}[u_t(x,t-\tau(t)\rho)] = u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho), \tag{2.11}$$

and

$$z_\rho(x,\rho,t) = \frac{\partial}{\partial \rho}[u_t(x,t-\tau(t)\rho)] = u_{tt}(x,t-\tau(t)\rho)(-\tau(t)). \tag{2.12}$$

Multipling (2.11) by $\tau(t)$ and (2.12) by $(1-\tau'(t)\rho)$, we get

$$\tau(t)z_t(x,\rho,t) = \tau(t)(1-\tau'(t)\rho)u_{tt}(x,t-\tau(t)\rho) \tag{2.13}$$

and

$$(1-\tau'(t)\rho)z_\rho(x,\rho,t) = -\tau(t)(1-\tau'(t)\rho)u_{tt}(x,t-\tau(t)\rho) \tag{2.14}$$

adding (2.13) and (2.14), we obtain

$$\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0. \tag{2.15}$$

From (2.10) and (2.15) the problem (2.1) becomes

$$\begin{aligned} &u_{tt}(x,t) - au_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)z(x,1,t) = 0 \text{ in } \Omega\times]0,\infty[,\\ &v_{tt}(x,t) - bv_{xx}(x,t) = 0 \text{ in }]L_1,L_2[\times]0,\infty[\\ &\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 \text{ in } \Omega\times]0,1[\times]0,\infty[, \end{aligned} \tag{2.16}$$

under the transmission conditions

$$\begin{aligned} &u(L_i,t) = v(L_i,t), i = 1,2,\\ &au_x(L_i,t) = bv_x(L_i,t), i = 1,2, \end{aligned} \tag{2.17}$$

the boundary condition

$$u(0,t) = u(L_3,t) = 0 \tag{2.18}$$

and the initial conditions

$$\begin{aligned} &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ on } \Omega,\\ &v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) on \Omega,\\ &z(x,\rho,0) = u_t(x,-\tau(0)\rho = f_0(x,-\tau(0)\rho), (x,\rho) \text{ in } \Omega\times]0,1[. \end{aligned} \tag{2.19}$$

We define the energy for (2.16) by

$$\begin{aligned} E(t) =& \frac{1}{2}\int_{\Omega}(|u_t(x),t)|^2 + a|u_x(x,t)|^2)dx + \frac{1}{2}\int_{L_1}^{L_2}(|v_t(x,t)|^2 + b|v_x(x,t)^2)dx \\ &+ \frac{\xi(t)\tau(t)}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx \end{aligned} \tag{2.20}$$

where

$$\xi(t) = \bar{\xi}\mu_1(t) \tag{2.21}$$

is a non increasing function of class $C^1(\mathbb{R}_+)$ and $\bar{\xi}$ be a positive constant such that

$$\frac{\beta}{\sqrt{1-d}} < \bar{\xi} < 2 - \frac{\beta}{\sqrt{1-d}}. \tag{2.22}$$

Lemma 2.1. *For any regular solution* (u, v, z) *to the system* (2.16)-(2.19)*, the eneryy functional defined by* (2.20) *satisfies*

$$\begin{aligned} E'(t) \leq& -\mu_1(t)\left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right)\int_{\Omega} u_t^2 dx \\ &- \mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\beta(\sqrt{1-d})}{2}\right)\int_{\Omega} z_1^2(x,\rho,t) \\ \leq& \, 0 \end{aligned} \tag{2.23}$$

Proof. We multiply the first and second equations of (2.16) by u_t and v_t, then we integrate by parts on Ω and $]L_1, L_2[$ respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_t^2 + au_x^2)dx = -\mu_1(t)\int_{\Omega} u_t^2 dx - \mu_2(t)\int_{\Omega} z(x,1,t)u_t dx + a[u_x u_t]_{\partial\Omega}, \tag{2.24}$$

$$\frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2}(v_t^2 + bv_x^2)dx = b[v_x v_t]_{L_1}^{L_2} \tag{2.25}$$

Next, we multiply the third equation of (2.19) by $\xi(t)z(x,\rho,t)$ and we integate over $\Omega\times]0,1[$, we get

$$\tau(t)\xi(t)\int_{\Omega}\int_0^1 z_t(x,\rho,t)z(x,\rho,t)d\rho dx = -\frac{\xi(t)}{2}\int_{\Omega}\int_0^1(1-\tau'(t))\rho)\frac{\partial}{\partial\rho}(z(x,\rho,t))^2 d\rho dx.$$

Thus,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\xi(t)\tau(t)}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx\right)=&\frac{\xi(t)}{2}\int_{\Omega}(z^2(x,0,t)-z^2(x,1,t))dx\\&+\frac{\xi(t)\tau'(t)}{2}\int_{\Omega}\int_0^1 z^2(x,1,t)d\rho dx\\&+\frac{\xi'(t)\tau(t)}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx.\end{aligned}\tag{2.26}$$

Taking in account (2.20), (2.24), (2.25), (2.26) and the tow conditions (2.17) and (2.18), we have

$$\begin{aligned}E'(t)=&\frac{\xi(t)}{2}\int_{\Omega}u_t^2dx-\frac{\xi(t)}{2}\int_{\Omega}z^2(x,1,t)dx\\&+\frac{\xi(t)\tau'(t)}{2}\int_{\Omega}z^2(x,1,t)dx+\frac{\xi'(t)\tau(t)}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx\\&-\mu_1(t)\int_{\Omega}u_t^2dx-\mu_2(t)\int_{\Omega}z(x,1,t)u_tdx.\end{aligned}\tag{2.27}$$

Applying Young's inequality, we get

$$\mu_2(t)\int_{\Omega}z(x,1,t)u_tdx\leq\frac{|\mu_2(t)|}{2\sqrt{1-d}}\int_{\Omega}u_t^2dx+\frac{|\mu_2(t)|\sqrt{1-d}}{2}\int_{\Omega}z^2(x,1,t)dx\tag{2.28}$$

Inserting (2.28) into (2.27), we have

$$\begin{aligned}E'(t)\leq&-\left(\mu_1(t)-\frac{\xi(t)}{2}-\frac{|\mu_2(t)|}{2\sqrt{1-d}}\right)\int_{\Omega}u_t^2dx\\&-\left(\frac{\xi(t)}{2}-\frac{\xi(t)\tau'(t)}{2}-\frac{|\mu_2(t)|\sqrt{1-d}}{2}\right)\int_{\Omega}z^2(x,1,t)dx\\&+\frac{\xi'(t)\tau(t)}{2}\int_{\Omega}\int_0^1 z^2(x,1,t)d\rho dx\\\leq&-\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_{\Omega}u_t^2dx\\&-\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_{\Omega}z^2(x,1,t)dx\\\leq&0.\end{aligned}$$

Which complete the proof. □

2.2 Global solution

In this section, we prove existence and uniqueness of solutions for the system (2.16) -(2.19). Let $U = (u, v, \varphi, \psi, z)^T$ vector function, where $\varphi = u_t$ and $\psi = v_t$, then we write the system (2.16) -(2.19) as

$$\begin{cases} U_t - A(t)U = 0, \\ U(0) = U_0 = (u_0, v_0, u_1, v_1, f_0(., -, \tau(0)))^T \end{cases} \tag{2.29}$$

where the operator $\mathcal{A}(t)$ is defind by

$$\mathcal{A}(t)U = \begin{pmatrix} \varphi(x,t) \\ \psi(x,t) \\ au_{xx}(x,t) - \mu_1(t)\varphi(x,t) - \mu_2(t)z(x,1,t) \\ bv_{xx} \\ -\frac{1-\tau'(t)\rho}{\tau(t)} z_\rho \end{pmatrix} \tag{2.30}$$

where $\mathcal{A}(t)$ is given by

$$\mathcal{A}(t) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a\partial_{xx} & 0 & -\mu_1(t) & 0 & -\mu_2(t) \\ 0 & b\partial_{xx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho \end{pmatrix}$$

Now, we intreduce the set

$$X_* = (u, v) \in H^1(\Omega) \times H^1(]L_1, L_2[)/u(0) = u(L_3) = 0, u(L_i) = v(L_i), au_x(L_i) = bv_x(L_i), i =$$

and the phase space

$$\mathcal{H} = X_* \times L^2(\Omega) \times L^2(]L_1, L_2[) \times L^2(\Omega)\times]0, 1[)$$

equipped with the inner product

$$\langle U, \hat{U}\rangle_{\mathcal{H}} = \int_{\Omega}(\varphi\hat{\varphi} + au_x\hat{u_x})dx + \int_{L_1}^{L_2}(\psi\hat{\psi} + bv_x\hat{v_x})dx + \xi(t)\tau(t)\int_{\Omega}\int_0^1 z\hat{z}d\rho dx, \quad (2.31)$$

xhere $U = (u, v, \varphi, \psi, z)^T$and $\hat{U} = (\hat{u}, \hat{v}, \hat{\varphi}, \hat{\psi}, \hat{z})^T$.

We define the domain $D(\mathcal{A}(t))$ of $\mathcal{A}(t)$ by

$$D(\mathcal{A}(t)) = \{(u, v, \varphi, \psi, z)^t \in \mathcal{H}/(u, v) \in (H^2(\Omega) \times H^2(]L_1, L2[)) \cap X_*,$$
$$\varphi \in H^1(\Omega), \psi \in H^1(]L_1, L_2), z \in L^2(]0, L); H_0^1(]0, 1[)), \varphi = (., 0)\}. \quad (2.32)$$

Notice that

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \forall t > 0. \quad (2.33)$$

Now, we give the following important theorem (kato).

Theorem 2.1. *(see [10] and [11]) Assume that*

1. *$Y = D(\mathcal{A}(0))$ is dense supect of $\mathcal{H}$,*
2. *(2.33) holds,*
3. *for all $t \in [0, t], \mathcal{A}(t)$ genertes a stronyly continuous somegroup on $\mathcal{H}$ and the family $\mathcal{A}(t) = \mathcal{A}(t)/t \in [0, T]$ is stable with stability constants C and m independent of t(i.e.,the semigroup $(S_t(s))_{s\leq 0}$generated by $\mathcal{A}(t)$satisfies $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$,for all $u \in \mathcal{H}$ and $s \leq 0$)*
4. *$\partial_t\mathcal{A}(t)$ belongs to $L_*^\infty([0, T], B(Y, \mathcal{H}))$, wish is the space of equivalent classes of essentially bounded ,strongly measurable functions from [0,T] into the set $B(Y, \mathcal{H})$ of bounded linear operators from Y into $\mathcal{H}$.*

Then, problem (2.29) has a unique solution $U \in C([0, T], Y) \cap C^1([0, T], H)$ for any initial datum in Y.

The following theorem gives an existence and uniqueness result of solution to the problem.

Theorem 2.2. *For* $U_0 \in \mathcal{H}$ *there exists a unique solution* U *satifying* $U \in C([0,+\infty[,\mathcal{H})$ *for problem* (2.29). *Moreover, if* $U_0 \in D(\mathcal{A}(0))$*,then* $U \in C([0,+\infty[, D(\mathcal{A}(0))) \cap C^1([0,+\infty[,\mathcal{H})$.

Proof. First, we show that $D(\mathcal{A}(0))$ is dense in $\mathcal{H}$. Let $\hat{U} = (\hat{u}, \hat{v}, \hat{\varphi}, \hat{\psi}, \hat{z})^t \in \mathcal{H}$ be orthogonal to all elements of $D(\mathcal{A}(0))$,

$$0 = \langle U, \hat{U}\rangle_{\mathcal{H}} = \int_\Omega (\varphi\hat{\varphi} + au_x\hat{u}_x)dx + \int_{L_1}^{L_2} (\psi\hat{\psi} + bv_x\hat{v}_x)dx + \xi(t)\tau(t)\int_\Omega\int_0^1 z\hat{z}d\rho dx, \tag{2.34}$$

for $U = (u, v, \varphi, \psi, z)^T \in D(\mathcal{A}(0))$.

Take $u = v = \varphi = \psi = 0$ and $z \in D(\Omega\times]0,1[)$. As $U = (0,0,0,0,z)^T \in D(\mathcal{A}(0))$ and therefore, from (2.34), we conclude that

$$\int_\Omega\int_0^1 z\hat{z}d\rho dx = 0.$$

Since $C_0^\infty\Omega\times]0,1[)$ is deness in $L^2(\Omega\times]0,1[)$, then $\hat{z} = 0$. Next, let $\varphi \in D(\Omega)$,then $U = (0,0,\varphi,0,0)^T \in D(\mathcal{A}(0))$, which implies from (2.34) that

$$\int_\Omega \varphi\hat{\varphi}dx = 0$$

then it follows that $\hat{\varphi} = 0$. Similarly, take $\psi \in D(]L_1, L_2[)$, from (2.34) we have

$$\int_{L_1}^{L2} \psi\hat{\psi} = 0$$

by density of $D(]L_1, L_2[)$ in $L^2(]L_1, L_2[)$,we get $\hat{\psi} = 0$.

Now, for $(u, v) \in C_0^\infty(\Omega\times]L_1, L_2[)$ then $(u_x, v_x) \in D(\Omega\times]L_1, L_2[))$ we have

$$a\int_\Omega u_x\hat{u}_x dx + b\int_{L_1}^{L_2} v_x\hat{v}_x dx = 0.$$

Since $D(\Omega\times]L_1, L_2[)$ is dense in $L^2(\Omega\times]L_1, L_2[)$, we have $(\hat{u}_x\hat{v}_x) = (0,0)$ because $(\hat{u}, \hat{v}) \in X_*$.

Consequently,

$$D(\mathcal{A}(0)) \text{ is dense in } \mathcal{H} \tag{2.35}$$

To prove that the operator $\mathcal{A}(t)$ generates for a fixed t a $C_0 - semigroup$ in $\mathcal{H}$. Let $U = (u, v, \varphi, \psi, z)^T \in D(\mathcal{A}(t))$. Then

$$\langle \mathcal{A}(t)U, U\rangle_t = -\mu_1(t)\int_\Omega \varphi^2 dx - \mu_2(t)\int_\Omega z(x,1)\varphi dx - \frac{\xi(t)}{2}\int_\Omega\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)d\rho dx$$

using,

$$(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho) = \frac{\partial}{rho}((1-\tau'(t)\rho)z^2(x,\rho)) + \tau'(t)z^2(x,\rho),$$

we obtain

$$\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)d\rho = (1-\tau'(t))z^2(x,1) - z^2(x,0) + \tau'(t)\int_0^1 z^2(x,\rho)d\rho$$

thus

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t =& -\mu_1(t)\int_\Omega \varphi^2 dx - \mu_2(t)\int_\Omega z(x,1)\varphi dx + \frac{\xi(t)}{2}\int_\Omega \varphi^2 dx \\ &- \frac{\xi(t)(1-\tau'(t))}{2}\int_\Omega z^2(x,1)dx - \frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho)d\rho dx.\end{aligned}$$

from (2.28), we conclude

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t \leq& -\mu_1(t)\left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega \varphi^2 dx \\ &- \mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1)dx \\ &+ \frac{\xi(t)|\tau'(t)|}{2\tau(t)}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho)d\rho dx\end{aligned}$$

So

$$\begin{aligned}\langle \mathcal{A}(t)U,U\rangle_t \leq & -\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega \varphi^2 dx \\ & -\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1)dx \\ & +k(t)\langle U,U\rangle_t,\end{aligned}$$

where

$$k(t)=\frac{\sqrt{1-\tau'(t)^2}}{2\tau(t)}$$

Take in account (2.23) we get

$$\langle A(t)U,U\rangle_t - k(t)\langle U,U\rangle_t \leq 0, \tag{2.36}$$

then, the operator $\bar{A}(t)=A(t)-k(t)I$ is dissipative.

To prove surjectivity of the operator $\lambda I - A(t)$ for fixed $t>0$ and $\lambda>0$, take $F=(f_1,f_2,f_3,f_4,f_5)^T \in H$. we seek $U=(u,v,\varphi,\psi,z)^T \in D(A(t))$ solution of

$$(\lambda I - A(t))U = F,$$

which means that U satisfy the system

$$\lambda u - \varphi_1 = f_1, \tag{2.37}$$

$$\lambda v - \psi = f_2, \tag{2.38}$$

$$\lambda\varphi - a u_{xx} + \mu_1(t)\varphi + \mu_2(t)z(x,1) = f_3, \tag{2.39}$$

$$\lambda\psi - b v_{xx} = f_4, \tag{2.40}$$

$$\lambda z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = f_5. \tag{2.41}$$

For u and v with the appropriated regularity, (2.37) and (2.38) give

$$\varphi = \lambda u - f_1, \tag{2.42}$$

$$\psi = \lambda v - f_2 \tag{2.43}$$

Notice that $\varphi \in H^1(\Omega)$ and $\psi \in H^1(]L_1, L_2)$. Moreover, if $\tau'(t) = 0$, we get

$$z(x,\rho) = \varphi(x)e^{\sigma(\rho,t)} + \tau(t)e^{\sigma(\rho,t)} \int_0^\rho \frac{f_5(x,t)}{1 - s\tau'(s)} e^{-\sigma(s,t)} ds,$$

where

$$\sigma(\rho,t) = \lambda \frac{\tau(t)}{\tau'(t)} \ln(1 - \rho\tau'(t)),$$

is a solution of (2.41) such that

$$z(x,0) = \varphi(x). \tag{2.44}$$

On the other hand,

$$z(x,\rho) = \varphi e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho} \int_0^\rho f_5(x,s)e^{\lambda\tau(t)s} ds$$

is solution to (2.41) that satisfy (2.44). Now, we will consider $\tau'(t) \neq 0$. From (2.42) we have

$$\begin{aligned} z(x,1) &= \varphi e^{\sigma(1,t)} + \tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1 - s\tau'(s)} e^{-\sigma(s,t)} ds \\ &= (\lambda u - f_1)e^{\sigma(1,t)} + \tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1 - s\tau'(s)} e^{-\sigma(x,s)} ds \\ &= \lambda u e^{\sigma(1,t)} - f_1 e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1 - s\tau'(s)} e^{-\sigma(s,t)} ds. \end{aligned} \tag{2.45}$$

Inserting (2.42) and (2.45) in (2.39), and (2.43) in (2.40), we have

$$\begin{cases} \eta u - a u_{xx} = g_1 \\ \lambda^2 v - b v_{xx} = g_2 \end{cases} \tag{2.46}$$

where

$$\eta := \lambda^2 + \lambda\mu_1(t) + \lambda\mu_2(t)e^{\sigma(1,t)},$$

$$g_1 := f_3 + \lambda f_1 + \mu_1(t)f_1 + \mu_2(t)f_1 e^{\sigma(1,t)}$$

$$-\mu_2(t)\tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1 - s\tau'(s)} e^{-\sigma(s,t)} ds$$

$$g_2 := f_4 + \lambda f_2.$$

We consider the variational problem

$$\Phi((u,v),(\tilde{u},\tilde{v})) = l(\tilde{u},\tilde{v}), \tag{2.47}$$

where the bilinear from

$$\Phi : X_* \times X_* \to \mathbb{R}$$

and the linear from

$$l : X_* \to \mathbb{R}$$

are defined by

$$\Phi((u,v),(\tilde{u},\tilde{v})) = \eta \int_\Omega u\tilde{u}dx + a\int_\Omega u_x\tilde{u}_x dx + \lambda^2 \int_{L_1}^{L_2} v\tilde{v}dx + b\int_{L_1}^{L_2} v_x\tilde{v}_x - a[u_x\tilde{u}]_{\theta\Omega} - b[v_x\tilde{v}]_{L_1}^{L_2}$$

and

$$l(\tilde{u},\tilde{v}) = \int_\Omega g_1\tilde{u}dx + \int_{L_1}^{L_2} g_2\tilde{u}dx$$

We can easily verify that Ψ is continuous and coercive, and l is continuous, so we obtain by applying the Lax-Milgram theorem, a solution for $(u,v) \in X_+$ for (2.46). From (2.39) and (2.40) we conclude that $(u,v) \in H^2(]L_1,L_2[)$ and so $(u,v,\varphi,\psi,z) \in D(A(t))$.

Which implies that the operator $\lambda I - A(t)$ is surjective for any $\lambda > 0$. Since $k(t) > 0$, then

$$\lambda I - \tilde{A}(t) = (\lambda + k(t))I - A(t) \text{ is surjective,} \tag{2.48}$$

for any$\lambda > 0$ and $t > 0$.

Next we show that

$$\frac{\|\Psi\|_t}{\|\Psi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, t,s \in [0,T], \tag{2.49}$$

where $\Psi = (u, v, \varphi, \psi, z)^T$, $c > 0$. For all $s, t \in [0, T]$, we have

$$\|\Psi\|_t^2 - \|\Psi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\left[\int_\Omega (\varphi^2 + au_x^2)dx + \int_{L_1}^{L_2} (\psi^2 + bv_x^2)dx\right.$$
$$\left. + \left(\xi(t)\tau(t) - \xi(s)\tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_\Omega \int_0^1 z^2(x,\rho)d\rho dx\right].$$

It is easy to see that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. Now we will prove $\xi(t)\tau(t) - \xi(s)\tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. First notice that

$$\tau(t) = \tau(s) + \tau'(r)(t - s),$$

for $r \in (s, t)$. Since ξ is a non increasing function and $\xi > 0$, we obtain

$$\xi(t)\tau(t) \leq \xi(s)\tau(s) + \xi(s)\tau'(r)(t - s),$$

thus

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)} \leq 1 + \frac{|\tau'(t)|}{\tau(s)}|t - s|.$$

By (2.5) and the fact that τ' is bounded, we conclude

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)} \leq 1 + \frac{c}{\tau_0}|t - s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

which gives the proof of (2.49) and therefore (iii) follows

Furthermore , since $k'(t) = \frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+\tau'(t)^2}} - \frac{\tau'(t)\sqrt{1+\tau'(t)^2}}{2\tau(t)^2}$ is bounded on $[0, T]$ for all $T > 0$ (by (2.5) and (2.22))

we have

$$\frac{d}{dt}A(t)U = \begin{pmatrix} 0 \\ 0 \\ -\mu_1'(t)\varphi - \mu_2'(t)z(.,1) \\ 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with $\frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ bounded on $[0,T]$ by (2.5) and (2.22). Thus

$$\frac{d}{dt}\tilde{A}(t) \in L_*^\infty([0,T]), B(D(A(0)),H)), \tag{2.50}$$

where $L_*^\infty([0,T], B(D(A(0)),H))$ is the space of equivalence classes of essentlly bounded, strongy measurable functions from $[0,T]$ into $B(D(A(0)),H)$, and $B(D(A(0)),H)$ is the set of bounded linear operators from $D(A(0))$ into H.

Then, from (2.36) , (2.48) and (2.49),the family $\tilde{A} = \left\{\tilde{A}(t) : t \in [0,T]\right.$ is a stable family of generators in H. By (2.33), (2.35), (2.36), (2.48), (2.49) and (2.50) the assumptions (i)-(iv) of Theorem 2.1 are verifted. Thus, the problem

$$\begin{cases} \bar{U}_t = \tilde{A}(t)\tilde{U}, \\ \tilde{U}(0) = U_0 \end{cases} \tag{2.51}$$

has a unique solution $\tilde{U} \in C([0,+\infty[, D(A(0))) \cap C^1([0,+\infty[, H)$ for $U_0 \in D(A(0))$. Then the solution of (2.29) is given by

$$U(t) = e^{\int_0^t k(s)ds}\tilde{U}(t)$$

because

$$\begin{aligned} U_t(t) &= k(t)e^{\int_0^t k(s)ds}\tilde{U}(t) + e^{\int_0^t k(s)ds}\tilde{U}(t) \\ &= e^{\int_0^t k(s)ds}\Big(k(t) + \bar{A}(t)\Big)\tilde{U}(t) \\ &= A(t)e^{\int_0^t k(s)ds}\tilde{U}(t) \\ &= A(t)U(t) \end{aligned}$$

which ends the proof. □

Chapter 3

Exponential stability

In this chapter, we study the stability of solutions, more precisely, we need to understand the asymptotic behavior of our system and show the exponential stability of the solutions of problem. Here we follow the energy methode, which consists in constructing a suitable Lyapunov functional. In the following, sevral lemmas are presented.

Let

$$\mathfrak{F}_1(t) = \int_\Omega u u_t dx + \int_{L_1}^{L_2} v v_t dx, \tag{3.1}$$

where (u, v, z) be a solution of (2.16)-(2.19), then

Lemma 3.1. *For any $\varepsilon, c_1 > 0$ we have estimate*

$$\begin{aligned} \frac{d}{dt}\mathfrak{F}_1(t) \leq -(a - \mu_1^2(0)c_1^2\varepsilon_1) \int_\Omega u_x^2 dx - b \int_{L_1}^{L_2} v_x^2 dx \\ + \left(1 + \frac{1}{2\varepsilon_1}\right) \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 + \frac{\beta^2}{2\varepsilon_1} \int_\Omega z^2(x, 1, t) dx, \end{aligned} \tag{3.2}$$

Proof. Differentiating $\mathfrak{F}_1(t)$ and taking (2.16) in account, we get

$$\begin{aligned} \frac{d}{dt}\mathfrak{F}_1(t) = \int_\Omega u_t^2 dx - a \int_\Omega u_x^2 dx - \mu_1(t) \int_\Omega u u_t dx \\ - \mu_2(t) \int_\Omega u z(x, 1, t) dx + \int_{L_1}^{L_2} v_t^2 - b \int_{L_1}^{L_2} v_x^2 dx, \end{aligned}$$

then,

$$\frac{d}{dt}\mathfrak{F}_1(t) \leq \int_\Omega u_t^2 dx - a\int_\Omega u_x^2 dx + \left|\mu_1(t)\int_\Omega uu_t dx\right| + \left|\mu_2(t)\int_\Omega uz(x,1,t)dx\right| + \int_{L_1}^{L_2} v_t^2 dx - b\int_{L_1}^{L_2} v_x^2 dx$$

Take in account hypothesis (H_1) and (H_2), we get

$$\frac{d}{dt}\mathfrak{F}_1(t) \leq \int_\Omega u_t^2 - a\int_\Omega u_x^2 dx + \mu_1(0)\left|\int_\Omega uu_t dx\right| + \beta\mu_1(0)\left|\int_\Omega uz(x,1,t)dx\right| + \int_{L_1}^{L_2} v_t^2 dx - b\int_{L_1}^{L_2} v_x^2 dx \quad (3.3)$$

From (2.17) and (2.18), we get

$$u^2(x,t) = \left(\int_0^x u_x(s,t)ds\right)^2 \leq L_1\int_0^{L_1} u_x^2(x,t)dx, \mathrm{x} \in [0, L_1],$$

and

$$u^2(x,t) \leq (L_3 - L_2)\int_{L_2}^{L_3} u_x^2(x,t)dx, \mathrm{x} \in [L_2, L_3],$$

which imply that

$$\int_\Omega u^2(x,t)dx \leq c_1^2\int_\Omega u_x^2 dx, \mathrm{x} \in \Omega, \quad (3.4)$$

where $c_1 \max\{L_1, L_3 - L_2\}$ is the Pioncaré's constant. Using Yong's inuquality and (3.4), we have

$$\mu_1(0)\left|\int_\Omega uu_t dx\right| \leq \frac{\varepsilon_1\mu_1^2(0)c_1^2}{2}\int_\Omega u_x^2 dx + \frac{1}{2\varepsilon_1}\int_\Omega u_t^2 dx, \quad (3.5)$$

and

$$\beta\mu_1(0)\left|\int_\Omega uz(x,1,t)dx\right| \leq \frac{\varepsilon_1\mu_1^2(0)c_1^2}{2}\int_\Omega u_x^2 dx + \frac{\beta^2}{2\varepsilon_1}\int_\Omega z^2(x,1,t)dx. \quad (3.6)$$

Inserting (3.5 and (3.6) into (3.3) we get 3.2.

We introduce the functional

$$q(t) = \begin{cases} x - \frac{L_1}{2}, \text{x} \in [0, L_2] \\ \frac{L_2 - L_3 - L_1}{2(L_2 - L_1)}(x - L_1) + \frac{L_1}{2}, \text{x} \in [L_1, L_2]; \\ x - \frac{L_2 + L_3}{2}, \text{x} \in [L_2, L_3], \end{cases} \tag{3.7}$$

Notice that $|q(x)| \leq M$, where

$$M = \max\left\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\right\}.$$

We have the following result. □

Let

$$\mathfrak{F}_2(t) = -\int_\Omega q(x)u_x u_t dx \text{ and } \mathfrak{F}_3(t) = -\int_{L_1}^{L_2} q(x)v_x v_t dx. \tag{3.8}$$

where (u, v, z) be solution of (2.16)-(2.19), then

Lemma 3.2. *For any $\varepsilon_2 > 0$, the following estimaes holds true*

$$\frac{d}{dt}\mathfrak{F}_2(t) \leq \left(\frac{1}{2} + \frac{1}{2\varepsilon_2}\right)\int_\Omega u_t^2 dx + \left(\frac{a}{2} + M^2\mu_1^2(0)\varepsilon_2\right)\int_\Omega u_x^2 dx + \frac{\beta^2}{2\epsilon_2}\int_\Omega z^2(x,1,t)$$
$$-\frac{1}{4}[L_1 u_1^2(L_1,t) + (L_3 - L_2)u_t^2(L_2,t)] - \frac{a}{4}[L_1 u_t^2(L_1,t) + (L_3 - L_2)u_x^2(L_2,t)] \tag{3.9}$$

and

$$\frac{d}{dt}\mathfrak{F}_3(t) = \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\left(\int_{L_1}^{L_2} v_t^2 dx + b\int_{L_1}^{L_2} v_x^2 dx\right) + \frac{1}{4}[L_1 v_t^2(L_1,t) + (L_3 - L_2)v_t^2(L_2,t)]$$
$$+\frac{b}{4}[L_1 v_x^2(L_1,t) + (L_3 - L_2)v_x^2(L_2,t)], \tag{3.10}$$

Proof. Differentiating $\mathfrak{F}_2(t)$ and by (2.16), we have

$$\frac{d}{dt}\mathfrak{F}_2(t) = -\int_\Omega q(x)u_{xt}u_t dx - a\int_\Omega q(x)u_{xx}u_x dx$$
$$+\mu_1(t)\int_\Omega q(x)u_x u_t dx + \mu_2(t)\int_\Omega q(x)u_x z(x,1,t)dx$$

Integration by parts and hupothesis (H_1) and (H_2), we have

$$\begin{aligned}\frac{d}{dt}\mathfrak{F}_2(t) \leq \frac{1}{2}\int_\Omega q'(x)u_t^2dx - \frac{1}{2}[q(x)u_t^2]_{\partial\Omega} + \frac{a}{2}\int_\Omega q'(x)u_x^2dx - \frac{a}{2}[q(x)u_x^2]_{\partial\Omega}\\ +\mu_1(0)|\int_\Omega q(x)u_xu_tdx| + \beta\mu_1(0)|\int_\Omega q(x)u_xz(x,1,t)dx|\\ \leq \frac{1}{2}\int_\Omega u_t^2dx - \frac{1}{2}[q(x)u_t^2]_{\partial\Omega} + \frac{a}{2}\int_\Omega u_x^2dx - \frac{a}{2}[q(x)u_x^2]_{\partial\Omega}\\ +\mu_1(0)M|\int_\Omega u_xu_tdx| + \beta\mu_1(0)M|\int_\Omega u_xz(x,1,t)dx|\end{aligned} \tag{3.11}$$

By using the boundary conditions (2.18), we get

$$\frac{1}{2}[q(x)u_t^2]_{\partial\Omega} = \frac{1}{4}[L_1u_t^2(L_1,t) + (L_3-L_2)u_t^2(L_2,t)],$$
$$-\frac{a}{2}[q(x)u_x^2]_{\partial\Omega} \leq -\frac{a}{4}[L_1u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)$$

by using Young's inequality, we conclude that (3.11) gives (3.9).

Similarly, taking the derivative of $\mathfrak{F}_2(t)$, we obtian

$$\begin{aligned}\frac{d}{dt}\mathfrak{F}_2(t) = \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2dx - \frac{1}{2}[q(x)v_t^2]_{L_1}^{L_2} - \frac{b}{2}\int_{L_1}^{L_2} q'(x)v_x^2dx - \frac{b}{2}[q(x)v_x^2]_{L_1}^{L_2}\\ = \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2} v_t^2dx + b\int_{L_1}^{L_2} v_x^2dx\right) + \frac{1}{4}[L_1v_t^2(L_1,t) + (L_3-L_2)v_t^2(L_2,t)]\\ +\frac{b}{4}[L_1v_x^2(L_1,t) + (L_3-L_2)v_x^2(L_2,t)],\end{aligned}$$

Hence, the proof is complete. □

Now, we introduce the functional

$$\mathfrak{I}(t) = \bar{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2(x,1,t)d\rho dx \tag{3.12}$$

For this functional we have the following estimate.

Lemma 3.3. *Let (u,v,z) be a solution of* (2.16)-(2.19). *Then the functional* $\mathfrak{I}(t)$ *satisfies*

$$\frac{d}{dt}\mathfrak{I}(t) \leq -2\mathfrak{I}(t) + \bar{\xi}\int_\Omega u_t^2dx \tag{3.13}$$

Proof. Next, we write $z = z(x,1,t)$ for simplicity. Take the derivative of $\mathfrak{I}(t)$ over t we have

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(t) =& \bar{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 d\rho dx \\ &+ \bar{\xi}\tau(t)\int_\Omega\int_0^1 \left[-2\tau'(t)\rho e^{-2\tau(t)\rho}z^2 + 2e^{-2\tau(t)\rho}zz_t\right] d\rho dx\end{aligned}$$

From (2.15) we get

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(t) =& \bar{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 d\rho dx \\ &- 2\bar{\xi}\tau(t)\tau'(t)\int_\Omega\int_0^1 \rho e^{-2\tau(t)\rho}z^2 d\rho dx \\ &- \bar{\xi}\int_\Omega\int_0^1 2(1-\tau'(t)\rho)e^{-2\tau(t)\rho}zz_\rho d\rho dx\end{aligned} \tag{3.14}$$

Notice that

$$\frac{\partial}{\partial\rho}\left(e^{-2\tau(t)\rho}z^2\right) = -2\tau e^{-2\tau(t)\rho}z^2 + 2e^{-2\tau(t)\rho}zz_\rho \tag{3.15}$$

Inserting (3.15) in (3.14) we get

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(t) =& \bar{\xi}\tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2 d\rho dx \\ &- 2\mathfrak{I}(t) \\ &- \bar{\xi}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau(t)\rho}z^2\right) d\rho dx \\ &- \bar{\xi}\tau'(t)\int_\Omega\int_0^1 \rho\frac{\partial}{\partial\rho}\left(e^{-2\tau(t)\rho}z^2\right) d\rho dx\end{aligned} \tag{3.16}$$

by integration by parts we cancel the first and the last terms in (3.16), we have

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(t) =& -2\mathfrak{I}(t) \\ &- \bar{\xi}\int_\Omega\left(e^{-2\tau(t)}z^2(x,1,t) - z^2(x,0,t)\right) dx\end{aligned}$$

Since $z(x,0,t) = u_t(x,t-\tau(t)\dot{0}) = u_t(x,t)$ in Ω, we get (3.13). □

Now we state our main result of stability.

Theorem 3.1. *Let* $U(t) = (u(t), v(t), \varphi(t), \psi(t), z(t))$ *be the solution of* (2.16)-(2.19) *with initial data* $U_0 \in D(A(0))$ *and* $E(t)$ *the energy of* U*. Assume that the hypothesis* (2.5)*,* (2.6)*,* $(H1)$*,* $(H2)$ *and*

$$\max\{1, \frac{a}{b}\} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)} \tag{3.17}$$

hold. Then there exist ositive constants c *and* α *such that*

$$E(t) \leq ce^{-\alpha t}, \forall t \geq 0 \tag{3.18}$$

Proof. Define the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{3} N_i \mathfrak{F}_i(t) + \mathfrak{I}(t) \tag{3.19}$$

where N, $N_i, i = 1, 2, 3$ are positive real numbers.

Differentiating $\mathcal{L}(t)$ with respect to t, we have

$$\frac{d}{dt}\mathcal{L}(t) = N\frac{d}{dt}E(t) + \sum_{i=1}^{3} N_i \frac{d}{dt}\mathfrak{F}_i(t) + \frac{d}{dt}\mathfrak{I}(t) \tag{3.20}$$

Now, using lemma 2.1, we have the estimate

$$\frac{d}{dt}E(t) \leq -K\left[\int_\Omega u_t^2 dx + \int_\Omega z^2(x, 1, t)dx\right] \tag{3.21}$$

where K is a positive constante . From (2.17), we know that

$$a^2 u_x^2(L_i, t) = b^2 v_x^2(L_i, t), i = 1, 2. \tag{3.22}$$

Using (3.2), (3.9), (3.10), (3.13), (3.21) and (3.22) in (3.20), we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq &- \left[KN - \left(1+\frac{1}{2\varepsilon_1}\right)N_1 - \left(\frac{1}{2}+\frac{1}{2\varepsilon_2}\right)N_2 - \bar{\xi}\right]\int_\Omega u_t^2 dx \\
&- \left(KN - \frac{\beta^2}{2\varepsilon_1}N_1 - \frac{\beta^2}{2\varepsilon_1}N_2\right)\int_\Omega z^2(x,1,t)dx \\
&- \left[(a-\mu_1^2(0)c_1^2\varepsilon_1)N_1 - \left(\frac{a}{2}+M^2\mu_1^2(0)\varepsilon_2\right)N_2\right]\int_\Omega u_x^2 dx \\
&+ \left[N_1 + \frac{L_2-L_3-L_1}{4(L_2-L_1)}N_3\right]\int_{L_1}^{L_2} v_t^2 dx \\
&- \left[N_1 + \frac{L_2-L_3-L_1}{4(L_2-L_1)}N_3\right] b\int_{L_1}^{L_2} v_x^2 dx \\
&- (N_2 - \frac{a}{b}N_3)\frac{a}{4}\left[\frac{L_1}{4}u_t^2(L_1,t) + \frac{L_3-L_2}{4}u_t^2(L_2,t)\right] - 2\mathcal{J}(t).
\end{aligned} \tag{3.23}$$

Now we can find real constants N_1,N_2 and N_3 such that

$$N_1 + \frac{L_2-L_3-L_1}{4(L_2-L_1)}N_3 < 0, \quad N_2 > \max\left\{1, \frac{a}{b}N_3, N_1 > \frac{N_2}{2}.\right\}$$

Next, we take tow positive canstants ε_1 and ε_2 such that

$$\mu_1^2(0)c_1^2\varepsilon_1 N_1 + M^2\mu_1^2(0)\varepsilon_2 N_2 < a\left(N_1 - \frac{N_2}{2}\right)$$

Finally, we chose N large enough such that (3.23) satisfies

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq -\eta_1\int_\Omega (u_t^2+u_x^2)dx - \eta_1\int_{L_1}^{L_2}(v_t^2+v_x^2)dx - \eta_1\int_\Omega z^2(x,1,t)dx - \eta_1\int_\Omega z^2(x,1,t)dx \\
&\leq -\eta_1\int_\Omega (u_t^2+u_x^2)dx - \eta_1\int_{L_1}^{L_2}(v_t^2+v_x^2)dx - \eta_1\int_\Omega z^2(x,1,t)dx
\end{aligned}$$

where $\eta_1 > 0$.

From (2.20), there exists $\eta_2 > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_2 E(t), \forall t \leq 0 \tag{3.24}$$

Also, there exists two positive constants γ_1 and γ_2 such that

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t). \forall t \geq 0 \tag{3.25}$$

combining (3.24) and (3.25), we get

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha\mathcal{L}(t), \ \forall t \geq 0.$$

which gives

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\alpha t}, \ \forall t \geq 0. \tag{3.26}$$

By using estimates (3.25) and (3.26), we get

$$\gamma_1 E(t) \leq \mathcal{L}(0)e^{-\alpha t}, \ \forall t \geq 0$$

so

$$E(t) \leq \frac{\mathcal{L}(0)}{\gamma_1}e^{-\alpha t}, \ \forall t \geq 0$$

and the proof of Theorem 3.1 follows. □

Conclusion

Through this study, we knew the impact that time-varying delay has on the propagation of waves, even if those effects are applied to a small part of the medium in which the wave propagates.

Mathematically, we have seen that dealing with the time delay related to time requires some new techniques and additional studies with those used in the fixed time delay.

It is important, both from the theoretical and practical sides, to study other problems and to know the impact of these changes (delay) in different situations.

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