وزارة التعليم العالي والبحث العلمي

Université 20 Aout 1955 de Skikda

Faculté des Sciences

Département de Mathématiques

جامعة 20 أوت 1955 ، سكيكدة

كلـــــــية العلـــوم

قسم الرياضيات

N0 : U.S/F.S/D.M/2021/2022.

Faculté des Sciences
Département de Mathématiques

Mémoire

Présenté en vue de l'obtention du diplôme de
Master en Mathématiques

LA RESOLUTION DES PROBLEME SPECTRAUX INTEGRO-DIFFERENTIELS.

Option : Analyse Numérique des équation aux dérivées partielles

Par :

. TIFOUTI Chafiai

Encadré par : KHELLAF Ammar MCB ENPC

Devant le jury :

Président : BOUCENNA Ahcene MCB U. SKIKDA

Examinateur : BEDRANI Yassine MAB U. SKIKDA

Année : 2021/2022

ACKNOWLEDGEMENTS

I would like to thank, first of all, My God who gave me the strength to write this modest work.

I would like to thank the members of the jury, who agreed to evaluate my memory work.

I would like to thank KHELLAF Ammar, director of my memory, for his availability and her judicious advice throughout this work.

I would like to thank all the professors of the Mathematics Department, without forgetting also my colleagues and friends, as well as all others who have participated from afar or closely in the elaboration of this memory.

Thank you!

Cifouti Chafiai ✍

Dedication

I dedicate this memory to my parents :

To my dearest mother, who has not ceased to encourage me and to pray for me, she lights my way with her love, her support and all the sacrifices, receive through this work, as modest as it is, the expression of my feelings and of my eternal gratitude.

To my dearest father, who can be proud and find here the result of long years of sacrifices and hardships to help me advance in life. I dedicate this work to you as a testimony of my deep love. May God, the Almighty, preserve you and grant you health, long life and happiness

To my dear brother Ibrahim, and my dear sisters who have always been for me examples of perseverance, courage and generosity.

To my close friends: Zaki, Ilyes, Bilel, Abdou, Lamine, Rami, Mohamed Amine, Bachir.

Chafiai

CONTENTS

Notations **5**

Abstract **6**

Introduction **7**

1 General notions and concepts for bounded operators **8**

1.1 Notions on bounded linear operator 8

1.2 Properties of compact operators 10

1.3 Spectrum and resolvent 11

1.4 Application on integral operators 16

1.5 Conclusion and comments 17

2 Spectral approximation **18**

2.1 Convergence of operators 18

2.2 Property U 19

2.3 Kantorovich projection method 22

2.4 Conclusion and comments 25

3 Integral-differential spectral problem **26**

3.1 Position of the problem 26

3.2 Numerical application 28

3.3 Conclusion and comments 30

Bibliography **31**

NOTATIONS

X	A Banach space over $\mathbb{C}$
T	Compact operator
$BL(X)$	The set of all bounded linear operators on X
$\|T\|$	The subordinated operator norm of $T \in BL(X)$
$re(T)$	The resolvent set of $T \in BL(X)$
$R(z, T)$	The resolvent operator of $T \in BL(X)$ at $z \in re(T)$
$sp(T)$	The spectrum of $T \in BL(X)$
$\rho(T)$	The spectral radius of $T \in BL(X)$
$T_n \xrightarrow{p} T$	The pointwise convergence of T_n to T
$T_n \xrightarrow{n} T$	The norm convergence of T to T_n
λ_n	An eigenvalue of T_n

ABSTRACT

Abstract

In this memoir, we study spectral problems of integro-differential type. We will demonstrate the property U is obtained under a convergence mode. We will construct the matrix formulation of the problem based on the Kantorovich projection method.

Key words : Spectral problem, Kantorovich method, Compact operators, integro-differential operator.

Résumé

Dans ce mémoire, nous étudions les problèmes spectraux de type intégro-différentiel. Nous allons montrer que la propriété U est obtenue sous un mode de convergence. Nous allons construire la formulation matricielle du problème étudié en se basant sur la méthode de projection de Kantorovich.

Mot-clés : Problème spectral, Méthode de Kantorovich, Opérateurs compact, Opérateur intégro-différentiel.

ملخص

في هذه المذكرة قمنا بدراسة المشاكل الطيفية من نوع تكامل تفاضلي، و برهنا الخاصية التحتية باستعمال التقارب المنتظم، بالإضافة إلى تشكيل مصفوفة لحل هذه المشكلة بالاعتماد على طريقة الاسقاط لكونتوروفتش .

الكلمات المفتاحية : مشكل طيفي، طريقة كونتوروفيتش، المؤثر المتراص، المؤثر من نوع تكامل تفاضلي

INTRODUCTION

In this memoir, we will study an integro-differential spectral problem. This type of problem appears in several fields of applied mathematics. Our study is focused on the analytical and numerical way. Analytically, we will see that the property U is obtained by replacing the exact operator by a sequence of operators converging in norm. Numerically, we will put the matrix formulation of the spectral integral-differential problem using the Kantorovich projection method.

In the first chapter, we will provide the basic definitions around the theory of bounded operators, then we will see precisely, the notions of compact operators.

In the second chapter, we will study analytically the spectral problems approximated under a mode of convergence ”convergence in norm”. We will show that the property U is obtained.

In the third chapter, we will put the matrix formulation of the approximated spectral problem of integro-differential type.

CHAPTER 1

GENERAL NOTIONS AND CONCEPTS FOR BOUNDED OPERATORS

In this chapter, we will begin to review the notions and concepts of bounded operators, so we will recall the definition of compact operators on Banach space.

1.1 Notions on bounded linear operator

New definitions are thus needed, and things become more interesting.
In the following, X will denote a complex Banach space and $BL(X)$ the Banach algebra of all bounded linear operators from X into itself.
The identity operator will be denote by I. Let $\|\cdot\|_X$ denote the norm on X.
The so-called subordinated operator norm is defined as follows :

$$\|T\| = \sup\{\|Tx\|_X \;:\; x \in X, \;\; \|x\|_X < 1\} \quad \text{for} \;\; T \in BL(X).$$

Proposition 1.1.1. *The following properties are equivalent :*

1. *T is continuous.*

2. *T is continuous on X.*

3. There is a constant c such that.

$$\forall x \in X : \quad \|Tx\| \leqslant c\,\|x\|\,.$$

Proof : Its clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2) is obvious.

We will show the implication (2) $\Rightarrow$ (3), we suppose that T is continuous in 0,

so, $\forall\,\epsilon > 0\;\; \exists\,\delta > 0\;\; \forall x \in X$

$$||x|| \leq \delta \Rightarrow \|Tx\| < \epsilon. \tag{1.1}$$

Let $\epsilon > 0$ and $x \in X,$ such that $x \neq 0$, we put $x' = \delta\dfrac{x}{2\|x\|}$, thus, $\|x'\| = \dfrac{\delta}{2}$,

from where $\|x'\| < \delta$ and therefore and according to (1.1), $\|Tx'\| < \epsilon$. Then, follow that

$$\|Tx\| < 2\epsilon\frac{\|x\|}{\delta}.$$

Then there exists c satisfying the assertion (3). This is inequality is also true for $x = 0$ and (3) checked with

$$c = \frac{2\epsilon}{\delta}.$$

We will show the implication (3) $\Rightarrow$ (1). We suppose that (3) is verified so for all $x, y \in X$ we have

$$\|T(x-y)\|_X \leqslant c\|x-y\|_X.$$

That is to say,

$$\|Tx - Ty\|_X \leqslant c\|x-y\|_X,$$

from where, the application T is lipschitzian, so it is continuous.

■

Definition 1.1.1. *[1]*

An operator $T \in BL(X)$ is said to be compact, if it transforms any bounded subset of X into *arelatively compact set.* $\forall r > 0$ *the set* $T(B_r)$ *is compact such that*

$$B_r = \{x \in X \ : \ \|x\|_X < r\}.$$

Theorem 1.1.1. ***(Banach–Steinhaus)*** *[2]*

$T \in BL(X)$ *is compact, if only if for any bounded sequence* $(x_n)_{n\geq 1}$ *of* X *then we can extract a subsequence from* $(Tx_n)_{n\geq 1}$ *convergent.*

Remark 1.1.1. *The two definitions are equivalent because we work on Banach space.*

Theorem 1.1.2. *[2]*

Let X *and* Y *two normed vector spaces. Assume that* X *is a Banach space, where* $(T_i)_{i\in I}$ *is a collection of* $BL(X)$. *If for any* $x \in X$, *we have*

$$\sup_{i\in I} \|T_i(x)\| < \infty,$$

then $\sup_{i\in I} \|T_i\| < \infty.$

1.2 Properties of compact operators

In the following, we give properties concerning the operators compact.

Proposition 1.2.1. *[1]*

1. *Any* T *compact operator is a bounded operator, the converse is false.*

2. *Any* T *linear combination of compact operators is an operator compact.*

3. *Let* $T, S \in BL(X)$, *if* S *is a compact then* TS *and* ST *are two compact operators.*

4. *Let* $(T_n) \subset BL(X)$ *be sequence of compact operators, if*

$$\lim_{n\to\infty} \|T_n - T\| = 0,$$

   then T *is compact,*

5. *Let* $(T) \in BL(X)$ *with finite dimensional image* $\dim(\mathrm{Im}(T)) < +\infty$ *then* T *is compact, whene* $\mathrm{Im}(T) = \{y \in X : \ \exists\, x \in \ X, \ y = Tx\}.$

1.3 Spectrum and resolvent

Definition 1.3.1. *[3]*

The resolvent set of T is defined by

$$re\,(T) = \{z \in \mathbb{C} \ : \ T - zI \ \text{ is bijective}\}\,.$$

for $z \in re(T)$,

$$R(z\,,T) = (T - zI)^{-1}.$$

is called the resolvent operator of T at z.We shall write $R(\,z)$ for $R\,(z\,,T)$ when there is no ambiguity. The spectrum of T is the set

$$sp\,(T) = \mathbb{C} \backslash \; re\,(T)\,.$$

An element of $sp\,(T)$ will be called a spectral value of T. If X is finite dimensional, then $sp\,(T)$ consists of eigenvalue of T. The spectral radius of T is defined by

$$\rho\,(T) = \sup\{|\lambda| \quad : \quad \lambda \in sp\,(T)\}\,.$$

Proposition 1.3.1. *[4]*

Let $T \in BL(X)$, such that $z - T$ is bijective, then

$$(z - T)^{-1} \in BL(X).$$

Theorem 1.3.1. *[3]*

Let $T \in BL(X)$, for $z \in \mathbb{C}$ such that $|z| > \rho(T)$, we have $z \in \; re(T)$ and $R(z)$ has the Laurent Expansion

$$R(z) = -\sum_{k=0}^{\infty} T^k z^{-k-1}.$$

If in fact $|z| > \|T\|$, then

$$\|R(z)\| \leqslant \frac{1}{|z| - \|T\|}.$$

Proof : Let $t = \limsup\limits_{k\to\infty} \left\|T^k\right\|^{1/k}$. Then $t \leqslant \|T\|$. For $z \in \mathbb{C}$ such that $|z| > t$, it follows that the power series $\sum_{k=0}^{n} T^k z^{-k-1}$, converges in $BL(X)$. Settings

$$B_n(z) = -\sum_{k=0}^{n} T^k z^{-k-1},$$

we have

$$B_n(z)(T - zI) = (T - zI)B_n(z) = I - \left(\frac{T}{z}\right)^{n+1},$$

which tends to I as $n \to \infty$ provided $|z| > t$, and then the sum of the series is the operators $R(z)$. If $|z| > \rho(T)$, then $z \notin sp(T)$. Thus $\{z \in \mathbb{C} \ : \ |z| > \rho(T)\}$ is containded in $re(T)$ and the function $R(\cdot)$ is analytic in this set. Hence the Laurent Expansion

$$R(z) = -\sum_{k=0}^{\infty} T^k z^{-k-1}.$$

is valid for all z such that $|z| > \rho(T)$. For $|z| > \|T\|$, We have

$$\|R(z)\| = \left\| -\sum_{k=0}^{\infty} T^k z^{-k-1} \right\| \leqslant \frac{1}{|z|} \sum_{k=0}^{\infty} \left(\frac{\|T\|}{|z|}\right)^k \leqslant \frac{1}{|z| - \|T\|},$$

as desired.

■

Example 1.3.1. *In this example, some spectral values are not eigenvalues : Let* $X = \ell^2$. *Let* (e_k) *be the standard basis for* X. *Consider the left shift operator on* X *:*

$$Tx = \sum_{k=1}^{\infty} x(k+1)e_k, \quad x = \sum_{k=1}^{\infty} x(k)e_k \in \ell^2.$$

Let us first compute the eigenvalues of T. *The equation* $Tx = \lambda x$ *leads to* $x(k+1) = \lambda x(k)$ *for each positive integer* k. *Fixing* $x(1) = 1$, *we get*

$$x = \sum_{k=1}^{\infty} \lambda^{k-1} e_k.$$

Now $x \in X$ *if and only if* $|\lambda| < 1$. *Hence any complex number* λ *such that* $|\lambda| < 1$, *is an eigenvalue of* T. *Since* $sp(T)$ *is closed, this implies that* $sp(T)$ *contains* $\{\lambda \in : |\lambda| \leqslant 1\}$. *It can* *be easily seen that for any integer $k \geqslant 0$, $\left\|T^k\right\| = 1$, so that $\rho(T) = \lim_{k\to\infty} \left\|T^k\right\|^{1/k} = 1$ and hence*

$$sp(T) = \{\lambda \in : \ |\lambda| \leqslant 1\}.$$

However, no point λ on the boundary of the spectrum is an eigenvalue, because then $\sum_{k=1}^{\infty} \lambda^{k-1} e_k \notin \ell^2$.

Lemma 1.3.1. *[4]*

1. *If $T \in BL(X)$ with $\|T\| < 1$, then $(I-T)^{-1} \in BL(X)$ and*

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

2. *If $z \in \mathbb{C}$, with $\|T\| < |z|$, then $(z \pm T)^{-1} \in BL(X)$.*

3. *If $S^{-1} \in BL(X)$ and $T \in BL(X)$, then $(S+T)^{-1} \in BL(X)$ if and only if $(I + S^{-1}T)^{-1} \in BL(X)$.*

4. *Let $I(X)$ the space of invertible bounded operators in X. $I(X)$ is an open subset of BL(X).*

Proof : (1) Consider the series

$$\sum_{n=0}^{\infty} T^n. \tag{1.2}$$

In the Banach space $BL(X)$, Since $\|T^n\| \leq \|T\|^n$ and $\|T\| < 1$, the series $\sum_{n=0}^{\infty} \|T^n\|$ converges and therefore the series in (1.2) converges in $BL(X)$. Now observe that

$$(I-T)\sum_{n=0}^{\infty} T^n = \sum_{n=0}^{\infty} T^n - \sum_{n=0}^{\infty} T^{n+1} = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = I,$$

and, analogously

$$\left(\sum_{n=0}^{\infty} T^n\right)(I-T) = I.$$

The previous identities ensure that $I-T$ is invertible and

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

(2) Since $\|z^{-1}T\| < 1$, by property (1) we get that $(I \pm z^{-1}T)^{-1} \in BL(X)$ and so $(z \pm T)^{-1} = z(I \pm z^{-1}T)^{-1} \in BL(X)$.

(3) Since S is invertible, it follows that $S+T=S\left(I+S^{-1}T\right)$.

Hence, $(S+T)^{-1}\in BL\left(X\right)$. if and only if $\left(I+S^{-1}T\right)^{-1}\in BL\left(X\right)$

(4) Let $I(X)$ the space of invertible bounded operators in X. Then $I(X)$ is an open subset of $BL(X)$.

$$\left\|S^{-1}T\right\|\leq\left\|T^{-1}\right\|\|S\|<1.$$

Combining properties (1) and (3), we get that $\left(I+S^{-1}T\right)^{-1}\in BL\left(X\right)$, and therefore $(S+T)^{-1}\in BL\left(X\right)$, this means that the open ball with center S and radius $\|S^{-1}\|^{-1}$ is contained in $BL\left(X\right)$. By the arbitrariness of $S\in BL\left(X\right)$. It follows that $I(X)$ is an open subset of $BL\left(X\right)$.

Proposition 1.3.2. *[4]*

The following properties hold

1. *Let* $z_0\in re(T)$ *If* $z\in\mathbb{C}$ *and* $|z-z_0|<\|R(z_0,T)\|^{-1}$, *then* $z\in re(T)$ *and*

$$R(z,T)=\sum_{n=0}^{\infty}(z_0-z)^nR(z_0,T)^{n+1}, \tag{1.3}$$

where the series in (1.3) *Converges in the operator norm.*

2. $re(T)$ *is open.*

3. *If* $re(T)\neq\emptyset$, *then the maping* $R(\cdot,T):re(T)\to BL(X)$ *is analytic.*

4. *Resolvent identity for every* $z,\mu\in re(T)$

$$R(z,T)-R(\mu,T)=(\mu-z)R(z,T)R(z,T).$$

In particular, $R(z,T)$ *and* $R(\mu,T)$ *commute.*

5. *Let* $(z_n)_{n\in\mathbb{N}}\subset re\left(T\right)$ *be sequence converging to some* $z_0\in\mathbb{C}$.*Then* $z_0\in re(T)$ *if and only if* $\lim_{n\to\infty}\|R(z_n,T)\|=\infty$.

Proof : (1) Observe that

$$z-T=z-z_0+z_0-T=\left[I+(z-z_0)\,R\left(z_0,T\right)\right](z_0-T)\,,\quad z\in\mathbb{C} \tag{1.4}$$

If $|z - z_0| < \|R(z_0, T)\|^{-1}$, namely if $\|(z - z_0) R(z_0, T)\| < 1$, then, By applying, (**Lemma 1.3.1 (1)**) we conclude that

$$I + (z - z_0) R(z_0, T) \in BL(X).$$

Combining this fact with the identity (1.4) yields that operator $z - T$ is bijective with bounded inverse operator given by

$$\begin{aligned} R(z, T) &= R(z_0, T) \left[I - (z_0 - z) R(z_0, T)\right]^{-1} \\ &= R(z_0, T) \sum_{n=0}^{\infty} (z_0 - \lambda)^n R(z_0, T)^n \\ &= \sum_{n=0}^{\infty} (z_0 - z)^n R(z_0, T)^{n+1}. \end{aligned}$$

Properties (2) and (3) follow by property (1). In particular, the series representation of the resolvent (1.3) gives that $R(\cdot, T)$ is analytic on the open set $re(T) \neq \emptyset$. (4) Fix $z, \mu \in re(T)$. Then,

$$\begin{aligned} R(z, T) = R(z, T)(\mu - T) &= R(z, T)\left[(\mu - z) + (z - T)\right] R(\mu, T) \\ &= (\mu - z) R(z, T) R(\mu, T) + R(\mu, T). \end{aligned}$$

Thus

$$R(z, T) - R(\mu, T) = (\mu - z) R(z, T) R(\mu, T).$$

(5) Assume that $z_0 \in \rho(T)$. For every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for every $n > n_0$, it holds $|z_n - z_0| < \epsilon$. By property (1),

$$\epsilon > |z_n - z_0| \geq \frac{1}{\|R(z_n, T)\|}.$$

For every $n > n_0$. Hence $\lim_{n\to\infty} \|R(z_n, T)\| = \infty$. Vice versa, assume that $z_0 \in re(T)$ Then the function $R(\cdot, T)$ is clearly bounded on the compact set $\{z_n \; n \in \cup \{0\}\} \subseteq re(T)$, contracting the assumption that

$$\lim_{n\to\infty} \|R(z_n, T)\| = \infty.$$

■

Remark 1.3.1. *[3]*

1. *By (**Proposition 1.3.2 (3)**), $z \in re(T) \to \|R(z)\| \in \mathbb{R}$ is a continuous function which takes only positive values. Hence it attains a maximum value and a positive minimum value on each compact subset of $\mathbb{C}$ contained in $re(T)$.*

2. *If $X \neq \{0\}$, then $sp(T)$ is nonempty and the spectral radius of T is related to the powers of T in a peculiar manner.*

1.4 Application on integral operators

Theorem 1.4.1. *[5]*

The integral operator defined on

$$(Tu)(x) = \int_a^b k(x,y)u(y)dy.$$

is compact on the space $(C([a,b])\ ,\ \|.\|_\infty)$, where

$$\|u\|_\infty = \max_{x\in[a\,,b]} |u(x)|\,.$$

Proof : Let B the unit ball of $C([a\,,b])$, to show that our operator is compact, we are inspired by Ascoli's theorem [5], it suffices to establish that $H = T(B)$ is equicontinuous. For all $x \in [a;b]$ the set $H_x = \{u(x)\ :\ u \in H\}$. We first notice that K is uniformily continuous on $[a,b]\times[a,b]$ for everything u of B and everything x, x' of $[a,b]$ we write

$$\begin{aligned}
|Tu(x) - Tu(x')| &= \left|\int_a^b (k(x,y) - k(x',y))\, u(y)dy\right| \\
&\leq \left|\int_a^b k(x,y) - k(x',y)\right| |u(y)|\, dy \\
&\leq \|u\|_\infty \left|\int_a^b k(x,y) - k(x',y)dy\right| \\
&\leq \left|\int_a^b k(x,y) - k(x',y)dy\right|.
\end{aligned}$$

The uniform continuity of K on $[a,b]\times[a,b]$ allows associate to any real $\epsilon>0$ and other real $\alpha>0$ so that

$$|x-x'|\leq\alpha\Rightarrow|K(x,y)-K(x',y)|\leq\frac{\epsilon}{b-a}.$$

So

$$|x-x'|\leq\alpha\Rightarrow|Tu(x)-T(ux')|\leq\epsilon,\quad\forall u\in B.$$

which means that is equicontinuous. Let is move on to the second condition. so that the set $H_x=\{g(x)=Tu(x)\ :\ u\in B\}$ is relatively compact, it is enough that it is bounded let us calculate for this purpose

$$|Tu(x)|\ =|g(x)|=\int_a^b k(x,y)u(x)\leq\|u\|_\infty\int_a^b\sup_{x,y\in[a,b]}|k(x,y)|\,dy\leq(b-a)M.$$

$$M=\sup_{x,y\in[a,b]}|k(x,y)|\,.$$

Thus, It appears that H_x is bounded, which completes the demonstration.

■

1.5 Conclusion and comments

In this first chapter, we have approached some fundamental notions and concepts about bounded linear operators and in particularly compact operators. We showed the basic definitions of the spectral theory such as the resolvent, the spectrum set, the spectral radius. These notions will play major stone, for the next chapter. For more details about the spectral theory see [3].

CHAPTER 2

SPECTRAL APPROXIMATION

Our main interest lies in determining the spectral values of finite type of a given bounded linear operator T. Since exact computations are almost always impossible, we attempt to obtain numerical approximations. Usually these approximations are the exact results of spectral computations on a sequence of bounded operators $(T_n)_{n>1}$ which converges to T pointwise, or with the norm converges to T.

2.1 Convergence of operators

In this chapter, T and (T_n) denote bounded linear operators on a Banach space X. If (T_n) converges to T in some sense, the following questions arise naturally :

If $\lambda_n \in sp(T_n)$ and $\lambda_n \to \lambda$, does $\lambda \in sp(T)$?

We shall say that under a given mode of convergence, denoted by :

Property U holds if, whenever $T_n \to T$, λ_n belongs to $sp(T_n)$ and (λ_n) converges to λ, we have $\lambda \in sp(T)$.

Let us consider two well-known modes of convergence.

The pointwise convergence, denoted by $T_n \xrightarrow{p} T$:

$$\|T_n x - Tx\|_X \to 0 \text{ for every } x \in X.$$

The norm convergence, denoted by $T_n \xrightarrow{n} T$:

$$\|T_n - T\| \to 0.$$

If $T_n \xrightarrow{n} T$, then clearly $T_n \xrightarrow{p} T$ But the converse is not true.

Example 2.1.1. *In this example, $T_n \xrightarrow{p} I$ but $T_n \xrightarrow{n} I$ does not hold :*
Consider $X = \ell^p,\ 1 \leq p < \infty$. For $n = 1, 2...$ and $x = \sum_{k=1}^{\infty} x(k)e_k$ in X, let

$$T_n x = \sum_{k=1}^{\infty} x(k)e_k.$$

Then each T_n is a bounded finite rank operator on X and $T_n \xrightarrow{p} I$. But $T_n \xrightarrow{n} I$ does not hold : since $\|T_n - I\| = 1$ for each n. [4]

2.2 Property U

If $T_n \xrightarrow{p} T$, then we may have $\lambda_n \in\ sp\,(T_n)$ with $\lambda_n \to \lambda$, but $\lambda \notin sp\,(T_n)$.

Example 2.2.1. *Property U does not hold under pointwise convergence :*
Consider $X = \ell^p$. For $x = \sum_{k=1}^{\infty} x(k)e_k \in X$, let $Tx = x(1)e_1$ and for each integer $n \geq 2$,

$$T_n x = x(1)e_1 - x(n)e_n.$$

Since $\|T_n x - Tx\|_2 = |x(n)| \to 0$ for every $x \in X$, we see that $T_n \xrightarrow{p} I$:
Now

$$sp\,(T) = \ \{0, 1\} \ \text{ and } \ sp\,(T_n) = \{-1, 0, 1\}\,.$$

Since $\lambda_n = -1 \in sp\,(T_n)$ for each n, but $-1 \notin sp\,(T_n)$, we see that Property U does not hold.
For T and $\widetilde{T}$ in $BL\,(X)$, we shall denote $R\,(\cdot\,, T)$ and $R(\cdot\,, \widetilde{T})$ by $R\,(\cdot)$ and $\widetilde{R}\,(\cdot)$, respectively.

Proposition 2.2.1. *[4]*
Let T and $\widetilde{T}$ be in $BL\,(X)$.

1. *Second Resolvent Identity : Let* $z \in re(T) \cap re\left(\widetilde{T}\right)$*.Then*

$$\widetilde{R}(z) - R(z) = \widetilde{R}(z)\left(T - \widetilde{T}\right)R(z) = R(z)\left(T - \widetilde{T}\right)\widetilde{R}(z).$$

2. *Second Neumann Expansion : Let* $z \in re(T)$ *be such that* $\rho\left(\left(T - \widetilde{T}\right)R(z)\right) < 1$*. Then* $z \in re\left(\widetilde{T}\right)$ *and*

$$\widetilde{R}(z) = R(z)\sum_{k=0}^{\infty}\left[\left(T - \widetilde{T}\right)R(z)\right]^{k}.$$

If in fact $\left\|\left(T - \widetilde{T}\right)R(z)\right\| < 1$*, then*

$$\left\|\widetilde{R}(z)\right\| \leq \frac{\|R(z)\|}{1 - \left\|\left(T - \widetilde{T}\right)R(z)\right\|}$$

$$\left\|\widetilde{R}(z) - R(z)\right\| \leqslant \frac{\|R(z)\|\left\|\left(T - \widetilde{T}\right)R(z)\right\|}{1 - \left\|\left(T - \widetilde{T}\right)R(z)\right\|}.$$

Proof :

1. For $z \in re(T) \cap re\left(\widetilde{T}\right)$, we have

$$\widetilde{R}(z)\left(T - \widetilde{T}\right)R(z) = \widetilde{R}(z)\left[(T - zI) - \left(\widetilde{T} - zI\right)\right]R(z) = \widetilde{R}(z) - R(z).$$

Interchanging T and $\widetilde{T}$, we obtain the other equality.

2. For $z \in re(T)$, consider the identity

$$\widetilde{T} - zI = T - zI - \left(T - \widetilde{T}\right) = \left[I - \left(T - \widetilde{T}\right)R(z)\right](T - zI).$$

Since $\rho\left(\left(T - \widetilde{T}\right)R(z)\right) < 1$, the operator $I - \left(T - \widetilde{T}\right)R(z)$ is invertible.
The identity stated above shows that $z \in re\left(\widetilde{T}\right)$ and by (**Theorem 1.3.1**)

$$\widetilde{R}(z) = (T - zI)^{-1}\left[I - \left(T - \widetilde{T}\right)R(z)\right]^{-1}$$

$$= R(z)\sum_{k=0}^{\infty}\left[\left(T - \widetilde{T}\right)R(z)\right]^{k}.$$

Let $\left\|\left(T-\widetilde{T}\right)R(z)\right\|<1$. Then

$$\rho\left(\left(T-\widetilde{T}\right)R(z)\right)\leq\left\|\left(T-\widetilde{T}\right)R(z)\right\|<1.$$

and we have

$$\left\|\widetilde{R}(z)\right\|\leq\|R(z)\|\sum_{k=0}^{\infty}\left\|\left(T-\widetilde{T}\right)R(z)\right\|^{k}=\frac{\|R(z)\|}{1-\left\|\left(T-\widetilde{T}\right)R(z)\right\|}.$$

For T_n in $BL(X)$,we denote $R(\cdot\,,T_n)$ by $R_n(\cdot)$.

■

Theorem 2.2.1. *[4]*

Let $T\in BL(X)$ and E be a nonempty closed subset of $re(T)$. Then

$$\alpha_1(E)=\sup\{\|R(z)\|\ :\ z\in E\}<\infty.$$

If $T_n\xrightarrow{n}T$, then there is a positive integer n_0 such that $E\subset re(T_n)$ for all $n\geq n_0$ and

$$\alpha_2(E)=\sup\{\|R_n(z)\|\ :\ z\in E,\ n\geq n_0\}<\infty.$$

Proof : If $|z|>\|T\|$, then by (**Theorem 1.3.1**)

$$\|R(z)\|\leq\frac{1}{|z|-\|T\|}.$$

Hence $\|R(z)\|\to0$ as $|z|\to\infty$ and there is some $\alpha>0$ such that $\|R(z)\|\leq1$ for all $z\in\mathbb{C}$ with $|z|>\alpha$. Now $E_0=\{z\in E\ :\ |z|\leq\alpha\}$ is a compact subset of $re(T)$ and the function $z\mapsto\|R(z)\|$ is continuous on E_0 by (**Remark 1.3.1**) There is, therefore, some $\beta>0$ such that $\|R(z)\|\leq\beta$ for all $z\in E_0$.Thus

$$\alpha_1(E):=\sup\{\|R(z)\|\ :\ z\in E\}\leq\max\{1,\beta\}<\infty.$$

Let $T_n\xrightarrow{n}T$.

Case(i) : $0\in E$

Since $E \subset re(T)$, we see that $0 \notin sp(T)$. Find n_0 such that $\|T_n - T\| < \dfrac{1}{(2\alpha_1(E))}$, for all $n \geq n_0$. Then for $z \in E$ and $n \geq n_0$,

$$\|(T_n - T)R(z)\| \leq \|T_n - T\| \alpha_1(E) \leq \frac{1}{2}.$$

and by (**Proposition 2.2.1 (2)**) $z \in re(T_n)$ with

$$\|R_n(z)\| \leq \frac{\|R(z)\|}{1 - \|(T_n - T)R(z)\|} \leq 2\alpha_1(E).$$

This shows that

$$\alpha_2(E) = \sup\{\|R_n(z)\| \ : \ z \in E, \ \ n \geq n_0\} \leq 2\alpha_1(E) < \infty.$$

■

Theorem 2.2.2. *[4]*
Property U hold under norm convergence, that is, if $T_n \xrightarrow{n} T$, $\lambda_n \in sp(T_n)$ *and* $\lambda \to \lambda_n$, *then* $\lambda \in sp(T)$.

Proof : Suppose for a moment that $\lambda \in re(T)$. Since the set $re(T)$ is open in $\mathbb{C}$ by (**Theorem 1.3.1**) there is some $r > 0$ such that

$$E = \{z \in \mathbb{C} : \ |z - \lambda| \leq r\} \subset re(T).$$

By (**Theorem 2.2.1**), $E \subset re(T_n)$ for all large n Since and $\lambda_n \to \lambda$, we see that $\lambda_n \in E \subset re(T_n)$, for all large n, which is contradictory to the hypothesis that $\lambda_n \in sp(T_n)$ for each n. Hence λ must belong to $sp(T)$.

■

2.3 Kantorovich projection method

Consider the space $X = \mathcal{C}([0,1])$, provided by the uniform standard :

for $u \in X$,

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

For $n \in \mathbb{N}^*$ such as $n \geq 0$, we define the subdivision $(t_i)_{1\leq i\leq n}$ as following, $t_{i,n} = (1-i)h$ and $h = \frac{1}{n}$ for $i = 1, 2, \ldots, n+1$.

Thus, we define the hat functions $(e_{i,n})_{1\leq i\leq n}$ by the following relation:

$$e_{1,n}(t) = \begin{cases} \frac{(t_{2,n}-t)}{h} & \text{if} \quad t \in [0, t_{2,n}], \\ 0 \quad \text{else.} \end{cases}$$

$$e_{i,n} = \begin{cases} 1 + \frac{t-t_{i,n}}{h} & \text{if} \quad t \in [t_{i-1,n}, t], \\ 1 + \frac{t_{i,n}-t}{h} & \text{if} \quad t[t_{i,n}, t], \\ 0 \quad \text{else.} \end{cases}$$

$$e_{n,n} = \begin{cases} \frac{t-t_{n-1,n}}{h} & \text{if} \quad t \in [t_{n-1,n}, 1], \\ 0 \quad \text{else.} \end{cases}$$

Definition 2.3.1. *We say that an operator P is a projection if and only if :*

1. $P \in BL(X)$,
2. $PP = P$.

Let $n \geq 2$, then we define the projection operator π_n, by the following formula : for $u \in X$,

$$\pi_n u(t) = \sum_{i=1}^{n} e_{i,n}(t)u(t_i). \tag{2.1}$$

It's clear that,

$$(\pi_n^2) = \pi_n\pi_n = \pi_n \quad \textit{and} \quad \pi_n \in BL(X).$$

Definition 2.3.2. *We say that the approximation sequence $(K_n)_{n\geq 1} \subset BLX$ of $K \in BL(X)$ is from the kantorivich projection method if :*

$$K_n = \pi_n K.$$

Theorem 2.3.1. *: Let $u \in X$ et $(\pi_n)_{n\geq 1}$ the projection operator defined by* (2.1) *so,*

$$\lim_{n\to+\infty} \|(\pi_n - I)u\|_X = 0.$$

Proof : It is clear that for all $t \in [0,1]$,

$$\sum_{i=1}^{n} |e_{i,n}(t)| = 1,$$

so,

$$\|(I-\pi_n)u\|_X = \max_{0\leq t\leq \tau} |\sum_{i=1}^{n}(u(t)-u(t_{i,n}))e_{i,n}(t)| \leq w_\infty(u,h).$$

or

$$w_\infty(u,h) = \sup\{|u(t)-u(s)| \quad , t,s \in [0,1], \quad |t,s| < h\}.$$

There by, $w_\infty(u,h) \to 0$ when $h \to 0$, that implies :

$$\lim_{n\to+\infty} \|(I-\pi_n)u\| = 0.$$

when $h \to 0$

■

Let us now show that Kantorovich's projection method establishes a uniform approximation.

Theorem 2.3.2. *Let $n \in \mathbb{N}^*$, suppose K is a compact operator and that $K_n = \pi_n K$, so*

$$\lim_{n\to+\infty} \|K_n - K\| = 0.$$

Proof : Let $n \in \mathbb{N}^*$, we have :

$$K_n - K = \pi_n K - K = (\pi_n - I)K.$$

Like the projection operator $(\pi_n)_{n\geq 1}$ converges for punctually to the identity operator I, so, according to the Banach-Steinhaus (**Theorem 1.1.1**), we find that :

$$\lim_{n\to+\infty} \|(\pi_n - I)K\| = 0.$$

■

2.4 Conclusion and comments

In this chapter, we have shown the property U under convergence in norm. This property plays an important role in numerical approximation for the validation of adapted algorithms. For more details, we refer to the book of M.Ahues. [3]

CHAPTER 3

INTEGRAL-DIFFERENTIAL SPECTRAL PROBLEM

In this chapter, we study the main problem of this memory. We investigate the spectral problem for an integro-differential operator.

3.1 Position of the problem

We consider the following problem : find the pair $(\lambda, u) \in \mathbb{C} \times \mathcal{C}([0,1])$ such as

$$Ku(t) = \lambda u(t), \quad \text{for} \quad t \in [0,1]\,,$$

where the operator K, is a bounded integro-differential operator :

$$Ku(t) = \int_0^1 k(t,s)\,(u(s) + u'(s))\,ds, \quad t \in [0,1]\,.$$

the function $k(\cdot\,,\cdot)$ is given as :

$$(H)\ :\ \left\| \begin{array}{l} k(\cdot\,,\cdot) \in \mathcal{C}^1\left([0,1]\times[0,1]\right). \\ \dfrac{\partial k}{\partial t}(\cdot\,,\cdot) \in \mathcal{C}\left([0,1]\times[0,1]\right). \end{array} \right.$$

These conditions explain the regularity of the operator K.

Note that the operator K is a linear operator. Now, we going to show that the spectral problem associated with the integro-differential operator is equivalent to an operator system for a spectral problem.

Consider again the integro-differential spectral problem.

Find $(\lambda, u) \in \mathbb{C} \times \mathcal{C}^1([0,1])$.

$$\int_0^1 k(t,s)\,(u(s)+u'(s))\,ds = \lambda u(t), \quad t \in [0,1]\,.$$

By describing this last equation, we obtain the following system :

$$\begin{cases} \displaystyle\int_0^1 k(t,s)\,(u(s)+u'(s))\,ds = \lambda u(t), & t \in [0,1]\,, \\ \displaystyle\int_0^1 \frac{\partial k}{\partial t}\,(t,s)\,(u(s)+u'(s))\,ds = \lambda u'(t), & t \in [0,1]. \end{cases}$$

That implies

$$\begin{cases} \displaystyle\int_0^1 k(t,s)u(s)ds + \int_0^1 k(t,s)u'(s)ds = \lambda u(t), & t \in [0,1]\,, \\ \displaystyle\int_0^1 \frac{\partial k}{\partial t}\,(t,s)\,u(s)ds + \int_0^1 \frac{\partial k}{\partial t}\,(t,s)\,u'(s)ds = \lambda u'(t), & t \in [0,1]. \end{cases}$$

We make the following change of variables:

$$\begin{cases} u = u \in \mathcal{C}([0,1]), \\ u' = v \in \mathcal{C}([0,1]). \end{cases}$$

Moreover, we use the following notation

$$\begin{cases} K_1h(t) = \displaystyle\int_0^1 k(t,s)h(s)ds & \text{for} \quad h \in \mathcal{C}([0,1]), \\ K_2h(t) = \displaystyle\int_0^1 \frac{\partial k}{\partial t}\,(t,s)\,h(s)ds & \text{for} \quad h \in \mathcal{C}([0,1]), \end{cases}$$

and for all $(u,v) \in \mathcal{C}([0,1]) \times \mathcal{C}([0,1])$, we have

$$K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} K_1 & K_1 \\ K_2 & K_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Therefore, our differential integro spectral problem is written as :

To find $(u,v) \in \mathcal{C}([0,1]) \times \mathcal{C}([0,1])$, $\lambda \in \mathbb{C}$:

$$\begin{bmatrix} K_1 & K_1 \\ K_2 & K_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

3.2 Numerical application

First we define a uniform subdivision L_n on the interval $[0,1]$ by :

$$L_n = \left\{ t_i \ : \quad t_i = ih \ , \ \ h = \frac{1}{n} \ \ et \ \ n \in \mathbb{N}^* \right\}.$$

The operators K_1 and K_2 will be approached by the operators $K_{1,n}$ and $K_{2,n}$ respectively, using Kantorovich's method.

$$\begin{cases} K_{1,n}u(t) = \pi_n K_1 u(t) = \sum_{i=1}^{n} K_1 u(t_i) e_i(t), \\ K_{1,n}u(t) = \pi_n K_2 u(t) = \sum_{i=1}^{n} K_2 u(t_i) e_i(t). \end{cases}$$

where $u \in \mathcal{C}([0,1])$ and π_n is the projection operator defined previously in chapter 2. That implies

$$\begin{cases} K_{1,n}u(t) = \sum_{i=1}^{n} \int_0^1 k(t_i, s) u(s) ds e_i(t), \\ K_{2,n}u(t) = \sum_{i=1}^{n} \int_0^1 \frac{\partial k}{\partial t}(t_i, s) u(s) ds e_i(t). \end{cases}$$

Then, by substituting the approximate operators in the spectral problem :

To find $(\lambda_n \, , \, u_n \, , \, v_n) \in \mathbb{C} \times \mathcal{C}([0,1]) \times \mathcal{C}([0,1])$ such as :

$$\sum_{i=1}^{n} \left[\int_0^1 k(t_i, s) u_n(s) ds e_i(t) + \int_0^1 k(t_i, s) v_n(s) ds e_i(t) \right] = \lambda_n u_n(t), \ \ t \in [0,1]. \tag{3.1}$$

$$\sum_{i=1}^{n} \left[\int_0^1 \frac{\partial k}{\partial t}(t_i, s) u_n(s) ds e_i(t) + \int_0^1 \frac{\partial k}{\partial t}(t_i, s) v_n(s) ds e_i(t) \right] = \lambda_n v_n(t), \ \ t \in [0,1]. \tag{3.2}$$

Let now $j = 1, 2, ...n$. By multiplying the equation (3.1) by $k(t_j, t)$ then by integration over the interval $[0,1]$ on the interval $[0,1]$, we find

$$\sum_{i=1}^{n} \left(\int_0^1 k(t_i, s) u_n(s) ds \right) \int_0^1 k(t_j, t) e_i(t) dt$$

$$+\left(\int_0^1 k(t_i,s)v_n(s)ds\right)\int_0^1 k(t_j,t)e_i(t)dt=\lambda_n\int_0^1 k(t_j,t)u_n(t)dt.$$

We put :

$$x_i=\int_0^1 k(t_i,s)u_n(s)ds \quad , \quad y_n=\int_0^1 k(t_i,s)v_n(s)ds \quad i=1,...,n.$$

$$a_{ij}=\int_0^1 k(t_j,t)e_i(t)dt \quad , \quad i,j=1,2,...,n.$$

That implies

$$\sum_{i=1}^{n} a_{ij}(x_i+y_i)=\lambda_n x_j, \quad j=1,2,...,n.$$

Now, let $j=1,2,...,n$. By multiplying the equation (3.2) by $\frac{\partial k}{\partial t}(t_j,s)$, then by integration over the interval $[0,1]$, we find :

$$\sum_{i=1}^{n}\left[\left(\int_0^1 \frac{\partial k}{\partial t}(t_i,s)\,u_n(s)ds\int_0^1 \frac{\partial k}{\partial t}(t_j,t)\,e_i(t)dt\right)\right.$$

$$\left.+\left(\int_0^1 \frac{\partial k}{\partial t}(t_i,s)\,v_n(s)ds\int_0^1 \frac{\partial k}{\partial t}(t_j,t)\,e_i(t)dt\right)\right]=\lambda_n\int_0^1 \frac{\partial k}{\partial t}(t_i,t)\,v_n(t)dt.$$

we put :

$$\bar{x}_i=\lambda_n\int_0^1 \frac{\partial k}{\partial t}(t_i,s)\,u_n(s)ds \quad , \quad \bar{y}_n=\int_0^1 \frac{\partial k}{\partial t}(t_i,s)\,v_n(s)ds.$$

$$b_{ij}=\int_0^1 \frac{\partial k}{\partial t}(t_j,t)\,e_i(t)dt.$$

That implies

$$\sum_{i=1}^{n} b_{ij}(\bar{x}_i+\bar{y}_i)=\bar{y}_j \quad j=1,...,n.$$

Now let's take the equation (3.1) and multiply it by $\frac{\partial k}{\partial t}(t_j,t)$ then by integration over the interval $[0,1]$ We find :

$$\sum_{i=1}^{n}\int_0^1 k\,(t_i,s)u_n(s)\int_0^1 \frac{\partial k}{\partial t}(t_j,t)e_i(t)dt+\int_0^1 \left(k\,(t_i,s)\,v_n(s)\right)ds$$

$$\times\int_0^1 \frac{\partial k}{\partial t}(t_j,t)\,e_i(t)dt=\lambda_n\int_0^1 \frac{\partial k}{\partial t}(t_j,t)\,u_n(t)dt.$$

That implies

$$\sum_{i=1}^{n} b_{ij}(x_i+y_i)=\lambda_n\bar{x}_j \;\; pour \;\; j=1,...,n.$$

Finally we take the equation (3.2) and multiply it by $k(t_j, t)$ then by integration over the interval $[0, 1]$ We find :

$$\sum_{i=1}^{n} a_{ij}(\bar{x}_i + \bar{y}_i) = \lambda_n \bar{y}_j \ \ for \ \ j = 1, ..., n.$$

So, we can reformulate these equations, to get the following system :

$$\begin{bmatrix} A & 0 & A & 0 \\ B & 0 & A & 0 \\ 0 & A & 0 & B \\ 0 & B & 0 & B \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \\ y \\ \bar{y} \end{bmatrix} = \lambda_n \begin{bmatrix} x \\ \bar{x} \\ y \\ \bar{y} \end{bmatrix}$$

This system is finite dimensional. Finaly, we use the function ”eig” in Matlab to compute the eigenvalues of the problem

$$\begin{bmatrix} A & 0 & A & 0 \\ B & 0 & A & 0 \\ 0 & A & 0 & B \\ 0 & B & 0 & B \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \\ y \\ \bar{y} \end{bmatrix} = \lambda_n \begin{bmatrix} x \\ \bar{x} \\ y \\ \bar{y} \end{bmatrix}$$

3.3 Conclusion and comments

In this chapter, we have treated a spectral problem for an integro-differential operator. We have shown the finite dimensional system implemented on machine (Matlab) to solve this spectral integro-differential problem.

BIBLIOGRAPHY

[1] N.Brania, Mémoire Présenté en vue de l'obtention du diplôme de Master, Sur les équations intégrales linéaires définies sur un large intervalle, Université 20 Aout 1955 Skikda, Année 2020-2021.

[2] Master premiére année : Analyse fonctionnelle, Université Claude Bernard Lyon I, Année 2009-2010.

[3] M.Ahues, A.Largillier and B.V.Limaye, Spectral Computations for Bounded Operators, Chapman and Hall/CRC, New York, 2001.

[4] N.Dunford, J.Schwarts, Linear Operators, Vol.I, Interscience, New York, 1958.

[5] Z.Titraoui, Mémoire Présenté en vue de l'obtention du diplôme de Master, Opérateur Intégral de Fredholm, Université de M.SILA, Année 2018/2019.

[6] K.E. Atkinson. Elementary numerical analysis. Joho Wiley and sons singapore, second edition 1998.