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Qualitative analysis for some evolution equations of exponential form

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Presented by:
Faham Mohammed El Amin

Supervisor:
Dr: Lallouche Abdallah

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Jury Committee :

Latreche Abdelkrim	**M.C.A**	**Skikda University**	**Chair**
Lallouche Abdallah	**M.C.A**	**Skikda University**	**Supervisor**
Cheribet Fatima Zohra	**M.C.B**	**Skikda University**	**Examiner**

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Dedication

To:

My parents

my wife and my children, Khalil and Tasnim

All my familly

My teachers

My friends

I dedicate this modest work

Acknowledgments

Above all, I am truly grateful to Allah, the Almighty, whose guidance, strength, and patience have enabled me to complete this work.

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Abstract:

This work investigates the qualitative behavior of several classes of evolution nonlinear difference equation and systems; we focus on their stability properties, boundedness, and periodic nature. We also illustrate our theoretical findings with numerical examples .

Keywords:

Nonlinear difference equation ; Global and local asymptotic stability ; Linearized Stability Theorem ; Boundedness of solutions ; Periodicity and oscillation

الملخص:

يتناول هذا العمل دراسة السلوك النوعي لعدة فئات من المعادلات التطورية الفرقيه غير الخطية وأنظمتها، مع التركيز على خصائص الاستقرار، والتقييد، والطبيعة الدورية. كما نقوم بتوضيح النتائج النظرية بأمثلة عددية.

الكلمات المفتاحية:

معادلة فرقيه غير خطية؛ الاستقرار التقاربي الكلي والمحلي؛ نظرية الاستقرار الخطية؛ تقييد الحلول؛ الدورية و التذبذب.

Résumé:

Ce travail aborde le comportement qualitatif de certaines classes d'équations et de systèmes d'évolution de différences non linéaires ; nous nous concentrons sur leurs propriétés de stabilité, leur bornitude et leur nature périodique. Nous illustrons également nos résultats théoriques par des exemples numériques.

Mots clés:

Équation aux différences non linéaire ; Stabilité asymptotique globale et locale ; Théorème de stabilité linéarisée ; Bornitude des solutions ; Périodicité et oscillation

Contents

Introduction

In recent decades, difference equations have become a powerful tool for modeling various real-world phenomena in disciplines such as biology, economics, engineering, and population dynamics, These equations, which describe the evolution of a variable over discrete time steps, offer deep insights into the qualitative behavior of dynamic systems.

Among the various types of difference equations, nonlinear forms particularly those involving exponential and rational terms have attracted growing attention due to their ability to capture complex behaviors such as oscillations, chaos, and bifurcations. Despite their apparent simplicity, these equations often exhibit intricate dynamics that require sophisticated mathematical techniques for analysis

This work is devoted to the study of the stability properties, boundedness, and periodic nature of certain classes of nonlinear difference equations and systems.

Chapter 1, present the essential definitions and preliminary concepts used in the study of nonlinear difference equations, including stability types, periodicity, boundedness, and equilibrium analysis. It sets the theoretical groundwork for analyzing the stability, convergence, and oscillatory behavior of solutions to scalar and system-level difference equations explored in later chapters, [1,2,3,4,6,9,12] .

In chapter 2, we extend our analysis to the nonlinear equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}},$$

where the parameters α β and γ the initial conditions are arbitrary positive numbers. We establishe criteria for boundedness of all positive solutions, as well as conditions guaranteeing local and global asymptotic stability, [1,2,3,4,7,8,13].

Finally, **in chapter 3** addresses for couples system of two difference equation

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1 + \beta_1 e^{-x_n} + \gamma_1 e^{-x_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}}, \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 e^{-y_n} + \gamma_2 e^{-y_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}}, \end{cases}$$

where the parameters $\alpha_i, \beta_i, \gamma_i, a_i, b_i$, and c_i for $i \in \{1,2\}$ and initial conditions x_0, x_{-1}, y_0, and y_{-1} are positive real numbers. We prove boundedness, persistence, and global attractivity of the unique positive equilibrium under natural parameter constraints. We also illustrate our theoretical findings with numerical examples, [2,3,5,10,11,13].

Chapter 1

Definitions and preliminaries

This chapter present the essential definitions and preliminary concepts used in the study of nonlinear difference equations, including stability types, periodicity, boundedness, and equilibrium analysis. It sets the theoretical groundwork for analyzing the stability, convergence, and oscillatory behavior of solutions to scalar and system-level difference equations explored in later chapters,

1.1 Definitions

Definition 1.1.1. *Let $I \subset \mathbb{R}$ and let $f : \mathbb{R} \to I$ be a continuous function. Consider the difference equation:*

$$y_{n+1} = f(y_n, y_{n-1}), \quad n = 0, 1, 2, \dots \tag{1.1.1}$$

with initial conditions $y_{-1}, y_0 \in I$. We say that $\bar{y}$ is an ***equilibrium*** *of Eq (1.1.1) if:*

$$\bar{y} = f(\bar{y}, \bar{y}) \tag{1.1.2}$$

Definition 1.1.2.

(i) The equilibrium $\bar{y}$ of Eq (1.1.1) is called ***locally stable*** *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial values $y_{-1}, y_0 \in I$ satisfying*

$$|y_0 - \bar{y}| + |y_{-1} - \bar{y}| < \delta,$$

then:

$$|y_n - \bar{y}| < \varepsilon \quad \textit{for all } n \geq -1.$$

(ii) The equilibrium $\bar{y}$ is called ***locally asymptotically stable*** *if it is locally stable, and if there*

exists $\gamma > 0$ *such that :* $y_{-1}, y_0 \in I$ *with :*

$$|y_0 - \bar{y}| + |y_{-1} - \bar{y}| < \gamma \quad \Rightarrow \quad \lim_{n\to\infty} y_n = \bar{y}.$$

(iii) The equilibrium $\bar{y}$ *is called a* ***global attractor*** *if for every* $y_{-1}, y_0 \in I$*,*

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(iv) The equilibrium $\bar{y}$ *is called* ***globally asymptotically stable*** *if it is locally stable and a global attractor.*

(v) The equilibrium $\bar{y}$ *is called* ***unstable*** *if it is not stable.*

Definition 1.1.3. *The difference equation (1.1.1) is said to be permanent, if there exist constants P and Q with* $0 < P \le Q < \infty$*, such that for any initial conditions* y_{-1}, y_0*, there exists a positive integer N depending on the initial conditions such that:*

$$P \le y_n \le Q \quad \textit{for all } n \ge N$$

1.2 Linearized equation

Definition 1.2.1. *Let:*

$$s = \frac{\partial f}{\partial u}(\bar{y}, \bar{y}) \quad \textit{and} \quad t = \frac{\partial f}{\partial v}(\bar{y}, \bar{y})$$

denote the partial derivatives of $f(u,v)$ *evaluated at an equilibrium point* $\bar{y}$ *of Eq. (1.1.2). Then, the equation is:*

$$x_{n+1} = s x_n + t x_{n-1}, \quad n = 0, 1, 2, \dots \tag{1.2.1}$$

.

is called the linearized equation associated with Eq (1.1.1) about the equilibrium point $\bar{y}$.

Definition 1.2.2. *The sequence* $\{y_n\}$ *is said to be periodic with period p if:*

$$y_{n+p} = y_n \quad \textit{for } n = 0, 1, 2, \dots \tag{1.2.2}$$

Lemma 1.2.1. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, 2, \dots \tag{1.2.3}$$

where $k \in \{1,2,\dots\}$. Let $I = [a,b]$ be some interval of real numbers, and assume that

$$f : [a,b][a,b] \to [a,b]$$

is a continuous function satisfying the following properties:

(i) $f(u,v)$ is a non-increasing function both in u and v;

(ii) If $(m,M) \in [a,b][a,b]$ is a solution of the system

$$\begin{cases} M = f(m,m), \\ m = f(M,M), \end{cases}$$

then $m = M$

Then Eq. (1.2.1) has a unique positive equilibrium point $\bar{y}$, and every solution of Eq. (1.2.1) converges to $\bar{y}$

Theorem 1.2.1. *The equation:*

$$x_{n+1} = sx_n + tx_{n-1}, \quad n = 0,1,2,\dots \tag{1.2.4}$$

is the linearized equation associated with the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), \quad n = 0,1,2,\dots \tag{1.2.5}$$

about the equilibrium point $\bar{y}$. The characteristic equation associated with Eq. (1.2.5) is

$$\lambda^2 - s\lambda - t = 0. \tag{1.2.6}$$

*(i) If both roots of the quadratic equation (1.2.6) lie in the unit disk $|\lambda| < 1$, then the equilibrium $\bar{y}$ of Eq. (1.2.5) is **locally asymptotically stable**.*

*(ii) If at least one of the roots of Eq. (1.2.6) has absolute value greater than one, then the equilibrium $\bar{y}$ of Eq. (1.2.5) is **unstable**.*

(iii) A necessary and sufficient condition for both roots of Eq. (1.2.6) to lie in the open unit disk $|\lambda| < 1$ is

$$|s| < 1 - t < 2. \tag{1.2.7}$$

*In this case, the equilibrium $\bar{y}$ is called a **sink**.*

(iv) *A necessary and sufficient condition for both roots of Eq. (1.2.6) to have absolute value greater than one is*

$$|t| > 1 \quad \textit{and} \quad |s| < |1-t|. \tag{1.2.8}$$

In this case, $\bar{y}$ is called a ***repeller****.*

(v) *A necessary and sufficient condition for one root of Eq. (1.2.6) to have absolute value greater than one and the other less than one is:*

$$s^2 + 4t > 0 \quad \textit{and} \quad |s| > |1-t|. \tag{1.2.9}$$

In this case, the unstable equilibrium $\bar{y}$ is called a ***saddle point****.*

1.3 System of equations

Definition 1.3.1. *the system of the form:*

$$\begin{cases} x_{n+1} &= f(x_n, x_{n-1}, y_n, y_{n-1}) \\ y_{n+1} &= g(x_n, x_{n-1}, y_n, y_{n-1}), \end{cases} \quad n = 0, 1, \ldots \tag{1.3.1}$$

where $f : I^2J^2 \to I$ and $g : I^2J^2 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_i, y_i)\}_{i=-1}^{\infty}$ of system (1.3.1) is uniquely determined by initial conditions $(x_i, y_i) \in IJ$. Along with system (1.3.1), we consider the corresponding vector map $F = (f, f_1, g, g_1)$. An equilibrium point of (1.3.1) is a point $(\bar{x}, \bar{x}, \bar{y}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \bar{y}, \bar{y}) \\ \bar{y} &= g(\bar{x}, \bar{x}, \bar{y}, \bar{y}). \end{aligned} \tag{1.3.2}$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1.3.2. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (1.3.1)*

(i) *An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition $(x_{-1}, x_0, y_{-1}, y_0) \in I^2J^2$ with $\|(x_{-1}, x_0) - (\bar{x}, \bar{x})\| + \|(y_{-1}, y_0) - (\bar{y}, \bar{y})\| < \delta$, then $\|(x_n, x_{n-1}) - (\bar{x}, \bar{x})\| + \|(y_n, y_{n-1}) - (\bar{y}, \bar{y})\| < \varepsilon$ for all $n \geq 0$, where $\|\cdot\|$ is usual Euclidian norm in $\mathbb{R}^2$.*

(ii) *An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.*

(iii) *An equilibrium point $(\bar{x},\bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\|(x_{-1},x_0)-(\bar{x},\bar{x})\|+\|(y_{-1},y_0)-(\bar{y},\bar{y})\| < \eta$ implies $(x_n,y_n)\to(\bar{x},\bar{y})$ as $n\to\infty$.*

(iv) *An equilibrium point $(\bar{x},\bar{y})$ is called global attractor if $(x_n,y_n)\to(\bar{x},\bar{y})$ as $n\to\infty$.*

(v) *An equilibrium point $(\bar{x},\bar{y})$ is called asymptotic stable if it is a global attractor and stable.*

1.4 Linearized system

Definition 1.4.1. *Let $(\bar{x},\bar{y})$ be an equilibrium point of a map $F=(f,f_1,g,g_1)$, where f and g are continuously differentiable functions at $(\bar{x},\bar{x},\bar{y},\bar{y})$. The linearized of system (1.3.1) about the equilibrium point $(\bar{x},\bar{y})$ is*

$$X_{n+1}=F(x_n)=F_Jx_n \tag{1.4.1}$$

where $X_n=\begin{pmatrix}x_n\\ y_n\\ x_{n-1}\\ y_{n-1}\end{pmatrix}$ *and F_J is Jacobian matrix of system (1.3.1) about the equilibrium point $(\bar{x},\bar{y})$.*

To construct corresponding linearized system of system we consider the following transformation:

$$(x_n,y_n,x_{n-1},y_{n-1})\mapsto(f,g,f_1,g_1) \tag{1.4.2}$$

where $f=x_{n+1}, g=y_{n+1}, f_1=x_n$ and $g_1=y_n$. The linearized system about $(\bar{x},\bar{y})$ is given by

$$Z_{n+1}=F(\bar{x},\bar{y})Z_n \tag{1.4.3}$$

where $Z_n=\begin{pmatrix}x_n\\ y_n\\ x_{n-1}\\ y_{n-1}\end{pmatrix}$ *and $J_F(\bar{x},\bar{y})$the Jacobian matrix about the fixed point $(\bar{x},\bar{y})$ under transformation (1.4.2)*

Lemma 1.4.1. *Assume that $X_{n+1}=F(X_n)$, $n=0,1,\dots$ is a system of difference equations such that $\overline{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\overline{X}$ lie inside the open unit disk $|\lambda|<1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

Definition 1.4.2. *By [2], a positive semi-cycle of a solution* $\{x(n)\}_{n=-1}^{\infty}$ *of* $x(n+1) = f(x(n), x(n-1))$ *consists of a "string" of terms* $\{x(k), x(k+1), ..., x(m)\}$*, all greater than or equal to the equilibrium* $\bar{x^*}$*, with* $k \geq -1$ *and* $m \leq \infty$ *and such that either* $k = -1$ *or* $k > -1$ *and* $x(k-1) \geq \bar{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x(m+1) \geq \bar{x}$*.*

A negative semi-cycle of a solution $\{x(n)\}_{n=-1}^{\infty}$ *of* $x(n+1) = f(x(n), x(n-1))$ *consists of a "string" of terms* $\{x(k), x(k+1), ..., x(m)\}$*, all less than the equilibrium* $\bar{x^*}$*, with* $k \geq -1$ *and* $m \leq \infty$ *and such that either* $k = -1$ *or* $k > -1$ *and* $x(k-1) \geq \bar{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x(m+1) \geq \bar{x}$*.*

Chapter 2

Stability analysis of a nonlinear difference equation

In this chapter, the local asymptotic stability of the equilibrium point of Equation (2.1.4) was investigated by using the linearized stability theorem. Furthermore, the characterization of the stability was examined that depends on the conditions of the coefficients . In Section 3, the semi-cycle of positive solutions was analyzed. All this results will be shown theoretical and by simulations at the end of this chapter ,

2.1 Introduction

H. El-Metwally et al. [3] investigated the global asymptotic stability of the following difference equation:

$$x_{n+1} = \alpha + \beta e^{-x_n}, \quad n = 0, 1, \ldots \tag{2.1.1}$$

n this case, the parameters α and β are considered as positive constants, while the initial values are chosen as arbitrary non-negative real numbers. The equation serves as a model for a biological system, where α denotes the immigration rate and β stands for the population growth rate. The global asymptotic stability of the corresponding difference equation was examined in [5] as :

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}} \tag{2.1.2}$$

The parameters α, β, and γ are assumed to be positive real constants, and the initial conditions are taken as arbitrary non-negative values. In [6], the analysis focused on the boundedness and global asymptotic behavior of the following difference equation:

$$y_{n+1} = \frac{\alpha.e^{-(ny_n+(n-k)y_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}} \tag{2.1.3}$$

It was demonstrated that the parameters α and β are positive real numbers, and that $K \in 1,2,\ldots$, while the initial conditions $y_{-k},\ldots,y_{-1},y_0$ are arbitrarily chosen. Similar investigations can be found in [4,7]. The aim of this chapter is to examine the local and global behavior of the positive solutions of the difference equation:

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}} \tag{2.1.4}$$

2.2 Local and global asymptotic stability analysis

In this section, we analyze both the local and global asymptotic stability of the unique positive equilibrium point associated with Equation (2.1.4). The equilibrium points are determined by solving the following equation:

$$\bar{y} = \frac{(\alpha+\beta)e^{-\bar{y}}}{\gamma + (\alpha+\beta)\bar{y}} \tag{2.2.1}$$

Set

$$f(y) = \frac{(\alpha+\beta)e^{-y}}{\gamma + (\alpha+\beta)y} - y \tag{2.2.2}$$

For $y = 0$ and $y \to \infty$ we obtain respectively,

$$f(0) = \frac{\alpha+\beta}{\gamma} > 0, \quad \lim_{y\to\infty} f(y) = -\infty. \tag{2.2.3}$$

and

$$f'(y) = \frac{-(\alpha+\beta)e^{-y}[\gamma + (\alpha+\beta)(y+1)]}{[\gamma + (\alpha+\beta)y]^2} - 1 \tag{2.2.4}$$

As a result, Equation (2.2.1) admits a unique solution $\bar{y}$ which implies that Equation (2.1.4) possesses a single positive equilibrium $\bar{y}$ The linearized form of Equation (2.1.4) around this equilibrium, along with its corresponding characteristic equation, is given by:

$$x_{n+1} + \frac{\alpha(e^{-\bar{y}} + \bar{y})}{\gamma + (\alpha+\beta)\bar{y}} x_n - \frac{\beta(e^{-\bar{y}} + \bar{y})}{\gamma + (\alpha+\beta)\bar{y}} x_{n-1} = 0, \quad n = 1, 2, \ldots \tag{2.2.5}$$

and

$$\lambda^2 + \frac{\alpha(e^{-\bar{y}} + \bar{y})}{\gamma + (\alpha+\beta)\bar{y}} \lambda + \frac{\beta(e^{-\bar{y}} + \bar{y})}{\gamma + (\alpha+\beta)\bar{y}} = 0 \tag{2.2.6}$$

respectively.

Theorem 2.2.1. *The following statements are true:*

1. *Every solution of Equation (2.1.4) is bounded if* $0 < y_n$.
2. *The equilibrium point of Equation (2.1.4) is bounded if* $0 < \bar{y}$.

Proof. 1. Suppose that $0 < y_n$. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Equation (2.1.4). We have

$$0 < y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}} < \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma}$$
$$< \frac{(\alpha+\beta)e^0}{\gamma} = \frac{\alpha+\beta}{\gamma}$$

This implies that all positive solutions of Equation (2.1.4) remain bounded. Therefore, statement 1) holds true.

2. Assume that $0 < \bar{y}$. Then

$$0 < \bar{y} = \frac{\alpha e^{-\bar{y}} + \beta e^{-\bar{y}}}{\gamma + \alpha\bar{y} + \beta\bar{y}} < \frac{\alpha e^{-\bar{y}} + \beta e^{-\bar{y}}}{\gamma}$$
$$< \frac{(\alpha+\beta)e^0}{\gamma} = \frac{\alpha+\beta}{\gamma}$$

This leads to the conclusion that statement 2) is likewise valid.

□

Theorem 2.2.2. *Let* $\alpha > \beta$ *If :*

$$(\alpha+\beta) < \gamma e^{\frac{-\gamma + \sqrt{\gamma^2 + 4(\alpha+\beta)\gamma}}{2(\alpha+\beta)}} \tag{2.2.7}$$

then the positive equilibrium point of Equation (2.1.4) is locally asymptotically stable.

Proof. From the Linearized Stability Theorem, we can write

$$\left|-\frac{\alpha(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}}\right| < 1+\frac{\beta(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}} < 2 \tag{2.2.8}$$

The inequality (2.2.8) can be shown under two cases;

1. $\left|-\frac{\alpha e^{-\bar{y}}}{\gamma+(\alpha+\beta)\bar{y}}\right| < 1+\frac{\beta(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}}$
2. $1+\frac{\beta(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}} < 2$

From 2), we get

$$\bar{y} > \frac{\beta e^{-\bar{y}}-\gamma}{\alpha} \tag{2.2.9}$$

By 1), we will have:

$$0 < (\alpha+\beta)e^{-\bar{y}}+2(\alpha+\beta)\bar{y}+\gamma \tag{2.2.10}$$

which always holds and since $\alpha > \beta$, we can also obtain

$$\bar{y} > \frac{\beta e^{-\bar{y}}-\gamma}{\alpha}. \tag{2.2.11}$$

then we have

$$(\alpha+\beta)e^{-\bar{y}} < \gamma \tag{2.2.12}$$

Rewriting (2.2.12), we get

$$(\alpha+\beta)\bar{y}^2+\gamma\bar{y}-\gamma < 0 \tag{2.2.13}$$

In view of (2.2.12) and (2.2.13), we obtain

$$(\alpha+\beta) < \gamma e^{\frac{-\gamma+\sqrt{\gamma^2+4(\alpha+\beta)\gamma}}{2(\alpha+\beta)}}$$

□

Theorem 2.2.3. *Suppose that the assumptions of Theorem (2.2.2) are satisfied, and let $\bar{y_1}$ and $\bar{y_2}$ denote the equilibrium points of Equation (2.1.4), corresponding to parameters that fulfill the following conditions:*
$\gamma_2 < \gamma_1 < \frac{(\alpha+\beta)\beta}{\alpha}$ *then if the parameter γ decreases, then the local stability of the positive equilibrium point*

$$\bar{y} = \frac{-(\alpha\gamma-(\alpha+\beta)\beta)+\sqrt{(\alpha\gamma-(\alpha+\beta)\beta)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)} \tag{2.2.14}$$

decreases also.

Proof. By the linearized stability theorem, we have

$$\left|-\frac{\alpha(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}}\right| < 1+\frac{\beta(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}} < 2 \tag{2.2.15}$$

let

$$\left|-\frac{\alpha(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}}\right| = 1 \tag{2.2.16}$$

From (2.2.16), computations will give us

$$\alpha(\alpha+\beta)\bar{y}^2-((\alpha+\beta)\beta-\alpha\gamma)\bar{y}-(\alpha+\beta)\gamma=0 \tag{2.2.17}$$

where we get the positive equilibrium point.

$$\bar{y}=\frac{(\alpha+\beta)\beta-\alpha\gamma+\sqrt{((\alpha+\beta)\beta-\alpha\gamma)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)}$$

$$\bar{y}_1=\frac{(\alpha+\beta)\beta-\alpha\gamma_1+\sqrt{((\alpha+\beta)\beta-\alpha\gamma_1)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)} \tag{2.2.18}$$

$$\bar{y}_2=\frac{(\alpha+\beta)\beta-\alpha\gamma_2+\sqrt{((\alpha+\beta)\beta-\alpha\gamma_2)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)} \tag{2.2.19}$$

where $\gamma_2 < \gamma_1 < \frac{(\alpha+\beta)\beta}{\alpha}$

Considering both (2.2.18) and (2.2.19), we get

$$\bar{y}_1 < \bar{y}_2 \tag{2.2.20}$$

$$\frac{\beta(e^{-\bar{y}}+\bar{y})}{\gamma+(\alpha+\beta)\bar{y}} < 1 \tag{2.2.21}$$

from (2.2.21) assume that :

$$\frac{\beta(e^{-\bar{y}_1}+\bar{y}_1)}{\gamma+(\alpha+\beta)\bar{y}_1} < \frac{\beta(e^{-\bar{y}_2}+\bar{y}_2)}{\gamma+(\alpha+\beta)\bar{y}_2} < 1 \tag{2.2.22}$$

Given the conditions outlined in Theorem (2.2.2), and assuming that inequality (2.2.22) also holds, it follows that the stability of $\bar{y}_2$ is less robust than that of $\bar{y}_1$. By evaluating expression (2.2.22), we obtain:

$$\gamma(e^{-\bar{y}_2}-e^{-\bar{y}_1})+(\alpha+\beta)(\bar{y}_1e^{-\bar{y}_2}-\bar{y}_2e^{-\bar{y}_1})+\gamma(\bar{y}_2-\bar{y}_1) > 0 \tag{2.2.23}$$

The inequality can alternatively be expressed in the following form:

$$\gamma(\bar{y}_2^2-\bar{y}_1^2)+\frac{\gamma^2}{\alpha+\beta}(\bar{y}_2-\bar{y}_1)+\gamma(\bar{y}_2-\bar{y}_1)+(\alpha+\beta)\bar{y}_1\bar{y}_2(\bar{y}_2-\bar{y}_1)>0 \tag{2.2.24}$$

From (2.2.20) we get:

$$\gamma(\bar{y}_2+\bar{y}_1)+\frac{\gamma^2}{\alpha+\beta}+\gamma+(\alpha+\beta)\bar{y}_1\bar{y}_2>0 \tag{2.2.25}$$

This condition is satisfied for all valid parameter values, thereby concluding the proof. □

Theorem 2.2.4. *Assume that the sequence $\{y_n\}_{-1}^{\infty}$ is a monotonically decreasing solution to Equation (2.1.4), and that the assumptions of Theorem (2.2.2) are satisfied. If*

$$y_n > 2\bar{y} \tag{2.2.26}$$

then the positive equilibrium of Equation (2.1.4) is globally asymptotically stable.

Proof. Let us define a Lyapunov function $V(n)$ as follows:

$$V(n)=(y_n-\bar{y})^2, n=0,1,2,... \tag{2.2.27}$$

The change along the solutions of Equation (2.1.4) is

$$\begin{aligned}\Delta V(n) &= V(n+1)-V(n)\\ &= (y_{n+1}-y_n)(y_{n+1}+y_n-2\bar{y})\end{aligned} \tag{2.2.28}$$

From (2.2.28) we can write

$$y_{n+1}-\bar{y}=\frac{\alpha\ e^{-y_n}+\beta e^{-y_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}-y_n$$

and

$$y_{n+1}+y_n-2\bar{y}=\frac{\alpha e^{-y_n}+\beta e^{-y_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}+y_n-2\bar{y}.$$

Using the given assumptions, it can be shown that $\Delta V(n)<0$, which confirms the global asymptotic stability of the positive equilibrium point of Equation (2.1.4). □

2.3 The semi-cycle and oscillation

This section is devoted to the analysis of the semi-cyclic behavior and oscillatory nature of the positive solutions of Equation (2.1.4).

Theorem 2.3.1. *Assume that $f \in C[[0,\infty)[0,\infty),[0,\infty)]$ and that $f(x,y)$ is decreasing in both arguments. Let $\bar{y}$ be a positive equilibrium of $y(n+1) = f(y(n),y(n-1))$. Then every oscillatory solution of the difference equation $y(n+1) = f(y(n),y(n-1))$ has semi-cycle of length at most two.*

Theorem 2.3.2. *Let $f(x,y) = \frac{\alpha e^{-x}+\beta e^{-y}}{\gamma+\alpha x+\beta y}$ be a function such that $f \in C[[0,\infty)[0,\infty),[0,\infty)]$. Then every oscillatory solution of Equation (2.1.4) has semi-cycle of length at most two.*

Proof. According to Theorem (2.3.1), Equation (2.1.4) can be expressed in the following form:

$$f(x,y) = \frac{\alpha e^{-x} + \beta e^{-y}}{\gamma + \alpha x + \beta y} \tag{2.3.1}$$

The partial derivatives of Equation (2.3.1) with respect to x and y are given by:

$$\frac{\partial f}{\partial x} = \frac{-\alpha e^{-x}(\gamma + \alpha x + \beta y) - \alpha(\alpha e^{-x} + \beta e^{-y})}{(\gamma + \alpha x + \beta y)^2} \tag{2.3.2}$$

and

$$\frac{\partial f}{\partial y} = \frac{-\beta e^{-x}(\gamma + \alpha x + \beta y) - \beta(\alpha e^{-x} + \beta e^{-y})}{(\gamma + \alpha x + \beta y)^2} \tag{2.3.3}$$

Each of these derivatives is negative provided that the following conditions are satisfied:

$$-\alpha e^{-x}(\gamma + \alpha x + \beta y) - \alpha(\alpha e^{-x} + \beta e^{-y}) < 0 \tag{2.3.4}$$

and

$$-\beta e^{-x}(\gamma + \alpha x + \beta y) - \beta(\alpha e^{-x} + \beta e^{-y}) < 0 \tag{2.3.5}$$

Since the parameters are positive and the variables x and y lie within a positive interval, inequalities (2.3.4) and (2.3.5) are always satisfied. Hence, the proof is complete. □

2.4 Examples

Example In this Example, we use MATLAB for plotting, Figure 1(a) show the local stability of Equation (2.1.4) for the parameters $\alpha = 1.3$, $\beta = 0.9$, $\gamma = 1.5$, $y(-1) = 0.83$ and $y(0) = 0.96$ by using the conditions in Theorem (2.2.2) The parameters $\alpha = 3$, $\beta = 12$, $\gamma = 2$ and the initial conditions $y(-1) = 0.83$ and $y(0) = 0.96$ are selected to show in Figure 1(b) the unstable behavior of the solutions of Equation (2.1.4). In Figure 1(c), we can show that by decreasing of the parameter γ the local stability get be weaker.

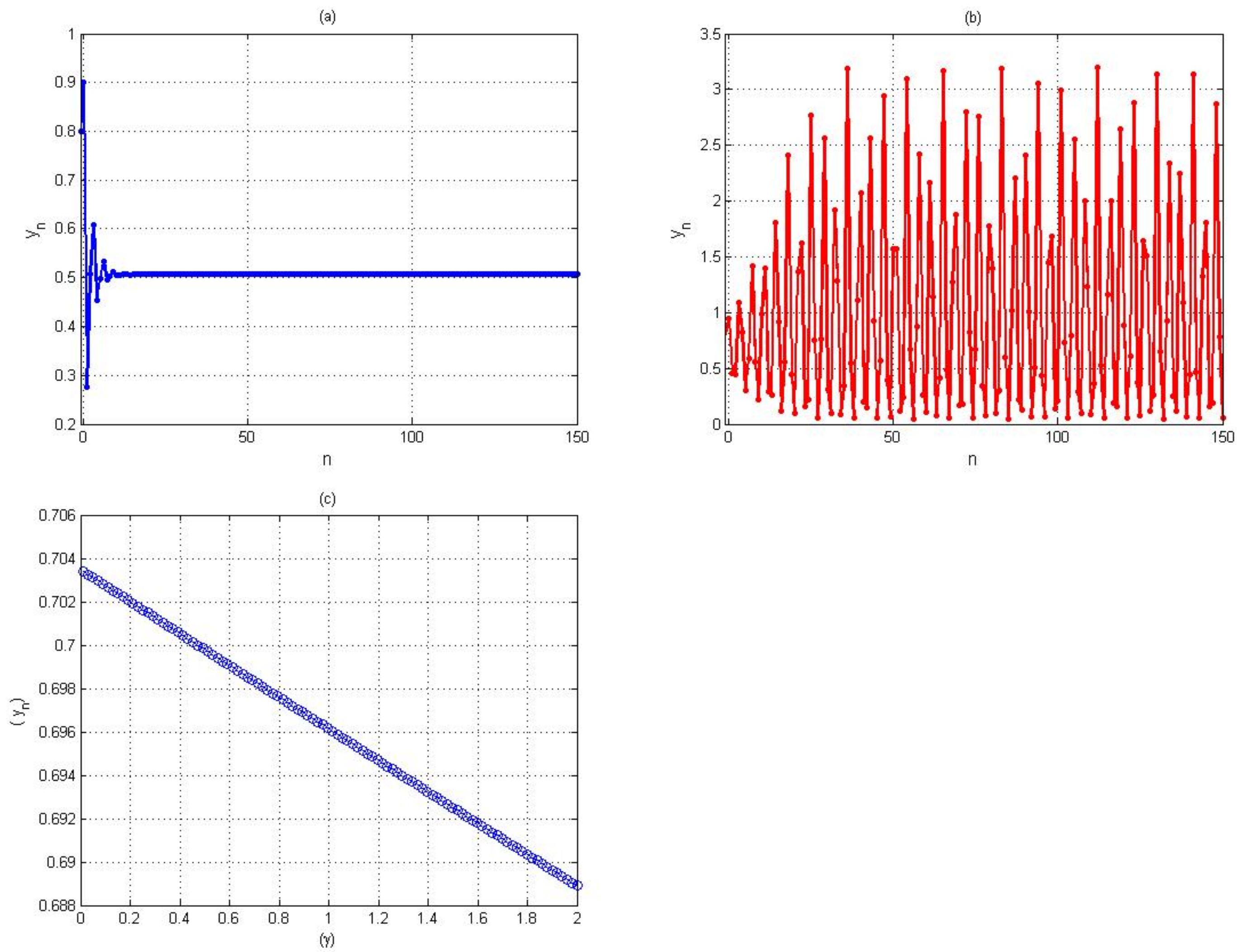

figure 1:
(a) Local stable behavior of Equation (2.1.4) for $\alpha = 1.3$, $\beta = 0.9$, $\gamma = 1.5$, $y(-1) = 0.83$ and $y(0) = 0.96$
(b) Unstable behavior of Equation (2.1.4) for $\alpha = 3$, $\beta = 12$, $\gamma = 2$ and the initial conditions $y(-1) = 0.83$ and $y(0) = 0.96$
(c) Stability analysis for $\alpha = 1.3$, $\beta = 0.9$, $\gamma \in [0.01, 2]$, $y(-1) = 0.83$ and $y(0) = 0.96$;

Chapter 3

Stability analysis of a system of difference equation

In this chapter, we explore the qualitative dynamics of a system of nonlinear difference equations. Our focus is on understanding the long term behavior of positive solutions through a rigorous mathematical framework, [1,3,4,5,8]

3.1 Introduction

Exponential difference equations play a central role in modeling population dynamics, as highlighted in [1–4] and related works. Higher-order nonlinear difference systems are particularly significant due to their wide range of applications. These equations often arise as discrete models or numerical approximations of differential and delay differential systems used to describe complex processes in biology, ecology, physiology, physics, engineering, and economics. Studying the behavior of their solutions—especially the local asymptotic stability of equilibrium points—remains a key aspect of understanding such systems.

El-Metwally et al. [3] explored the boundedness, long-term behavior, periodic nature of positive solutions, and the stability of the equilibrium point for the following population model.

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n} \tag{3.1.1}$$

Papachristodoulos and colleagues [2,3] examined the boundedness, persistence, and asymptotic dynamics of positive solutions in the context of the following two-species interactive and invasive population models.

$$x_{n+1} = a + bx_{n-1}e^{-y_n}, \quad y_{n+1} = c + dy_{n-1}e^{-x_n} \tag{3.1.2}$$

Papachinopoulos et al. [2,3] analyzed the long-term behavior of solutions for the following three systems of exponential-type difference equations.

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \quad y_{n+1} = \frac{\delta + \varepsilon e^{-x_n}}{\zeta + x_{n-1}}, \\ x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \quad y_{n+1} = \frac{\delta + \varepsilon e^{-x_n}}{\zeta + y_{n-1}}, \\ x_{n+1} &= \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \quad y_{n+1} = \frac{\delta + \varepsilon e^{-y_n}}{\zeta + x_{n-1}} \end{aligned} \tag{3.1.3}$$

Recently, Papachinopoulos and Schinas [2] explored the long-term dynamics of positive solutions for a system consisting of two difference equations.

$$x_{n+1} = a + by_{n-1}e^{-y_n}, \quad y_{n+1} = c + dx_{n-1}e^{-x_n}, \tag{3.1.4}$$

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \quad y_{n+1} = c + dx_{n-1}e^{-y_n}$$

In this chapter, we investigate the qualitative behavior of positive solutions of the following system of exponential difference equations:

$$\begin{cases} x_{n+1} &= \frac{\alpha_1 + \beta_1 e^{-x_n} + \gamma_1 e^{-x_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 e^{-y_n} + \gamma_2 e^{-y_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}} \end{cases} \tag{3.1.5}$$

where the parameters $\alpha_i, \beta_i, \gamma_i, a_i, b_i$, and c_i for $i \in \{1,2\}$ and initial conditions x_0, x_{-1}, y_0, and y_{-1} are positive real numbers.

More specifically, we study the boundedness and persistence of solutions, examine the existence and uniqueness of a positive steady state, and analyze both the local stability and global dynamics of the unique positive equilibrium point.

3.2 Boundedness and persistence

The next theorem demonstrates that all solutions of system (3.1.5) remain bounded and exhibit persistence.

Theorem 3.2.1. *Every positive solution* $\{(x_n, y_n)\}$ *of system (3.1.5) is bounded and persists.*

Proof. For any positive solution $\{(x_n, y_n)\}$ of system (3.1.5), one has

$$\begin{aligned} x_{n+1} &\leq \frac{\alpha_1+\beta_1+\gamma_1}{a_1} = U_1, \\ y_{n+1} &\leq \frac{\alpha_2+\beta_2+\gamma_2}{a_2} = U_2, \\ n &= 0,1,2,\ldots \end{aligned} \tag{3.2.1}$$

Moreover, by combining systems (3.1.5) and (3.2.1), we deduce the following

$$\begin{aligned} x_{n+1} &\geq \frac{\alpha_1+\beta_1 e^{-U_1}+\gamma_1 e^{-U_1}}{a_1+b_1U_2+c_1U_2} = L_1, \\ y_{n+1} &\geq \frac{\alpha_2+\beta_2 e^{-U_2}+\gamma_2 e^{-U_2}}{a_2+b_2U_1+c_2U_1} = L_2, \\ n &= 2,3,\ldots \end{aligned} \tag{3.2.2}$$

From (3.2.1) and (3.2.2), it follows that

$$L_1 \leq x_n \leq U_1, \quad L_2 \leq y_n \leq U_2, \quad n = 3,4,\ldots \tag{3.2.3}$$

Hence, the theorem is proved. □

Lemma 3.2.1. *Let* $(\{x_n, y_n\})$ *be a positive solution of system (3.1.5). Then,* $[L_1, U_1][L_2, U_2]$ *is invariant set for system (1.1.2).*

Theorem 3.2.2. *The system described by equation (3.1.5) admits a unique positive equilibrium* $(\bar{x}, \bar{y})$ *in* $[L_1, U_1][L_2, U_2]$*, provided that the following condition holds:*

$$a_2 + L_1(b_2+c_2) < \frac{a_2+(\beta_2+\gamma_2)e^{-K}}{K} \tag{3.2.4}$$

where

$$K = \frac{\alpha_1 + e^{-L_1}(\beta_1+\gamma_1) - a_1L_1}{L_1(b_1+c_1)} \tag{3.2.5}$$

Proof. Consider the following system of equations:

$$x = \frac{\alpha_1+(\beta_1+\gamma_1)e^{-x}}{a_1+(b_1+c_1)y}, \quad y = \frac{\alpha_2+(\beta_2+\gamma_2)e^{-y}}{a_2+(b_2+c_2)x} \tag{3.2.6}$$

From (3.2.6), it follows that :

$$y = \frac{\alpha_1 + (\beta_1 + \gamma_1)e^{-x} - a_1 x}{(b_1 + c_1)x}$$
$$x = \frac{\alpha_2 + (\beta_2 + \gamma_2)e^{-y} - a_2 y}{(b_2 + c_2)y} \tag{3.2.7}$$

Set :

$$F(x) = \frac{\alpha_2 + (\beta_2 + \gamma_2)e^{-f(x)} - a_2 f(x)}{(b_2 + c_2)f(x)} - x \tag{3.2.8}$$

where $f(x) = (\alpha_1 + (\beta_1 + \gamma_1)e^{-x} - a_1 x)/((b_1 + c_1)x)$ and $x \in [L_1, U_1]$. Then, it follows that :

$$F(L_1) = \frac{\alpha_2 + (\beta_2 + \gamma_2)e^{-(\alpha_1 + (\beta_1 + \gamma_1)e^{-L_1} - a_1 L_1)/(L_1(b_1 + c_1))}}{(b_2 + c_2)\frac{\alpha_1 + (\beta_1 + \gamma_1)e^{-L_1} - a_1 L_1}{L_1(b_1 + c_1)}} - L_1$$
$$= \left(\alpha_2 + (\beta_2 + \gamma_2)e^{-\frac{\alpha_1 + e^{-L_1}(\beta_1 + \gamma_1) - a_1 L_1}{L_1(b_1 + c_1)}}\right)\left(\frac{L_1(b_1 + c_1)}{(b_2 + c_2)(\alpha_1 + e^{-L_1}(\beta_1 + \gamma_1) - a_1 L_1)}\right) - L_1$$

Furthermore, it is easy to see that

$$(a_1 + (b_1 + c_1)U_2)(\alpha_1 + e^{-L_1}(\beta_1 + \gamma_1)) > a_1(\alpha_1 + (\beta_1 + \gamma_1)e^{-U_1}). \tag{3.2.9}$$

From (3.2.9) it follows that

$$\alpha_1 + e^{-L_1}(\beta_1 + \gamma_1) - a_1 L_1 > 0 \tag{3.2.10}$$

Let $K = (\alpha_1 + e^{-L_1}(\beta_1 + \gamma_1) - a_1 L_1)/(L_1(b_1 + c_1)) > 0$.
Then $F(L_1)$ can be expressed as:

$$F(L_1) = \frac{\alpha_2 + (\beta_2 + \gamma_2)e^{-K} - a_2 K}{(b_2 + c_2)K} - L_1 \tag{3.2.11}$$

Suppose that : $a_2 + L_1(b_2 + c_2) < \dfrac{\alpha_2 + (\beta_2 + \gamma_2 e^{-K})}{k}$;
then, it follows that $F(L_1) > 0$. Furthermore, we have

$$F(U_1) = U_1(b_1 + c_1)\Big((\beta_2 + \gamma_2)e^{(\beta_1 + \gamma_1)(1 - e^{-U_1})/U_1(b_1 + c_1)}$$
$$+ \alpha_2 + \frac{a_2(\beta_1 + \gamma_1)(1 - e^{-U_1})}{U_1(b_1 + c_1)}\Big)$$
$$\big((b_2 + c_2)(\alpha_1 + e^{-U_1}(\beta_1 + \gamma_1) - a_1 U_1)\big)^{-1} - U_1 \tag{3.2.12}$$

Then it is easy to see that :

$$\alpha_1 + e^{-U_1}(\beta_1+\gamma_1) - a_1U_1 = (\beta_1+\gamma_1)(e^{-U_1}-1) < 0 \tag{3.2.13}$$

which gives that $F(U_1) < 0$. Hence, $F(x) = 0$ has at least one positive solution in $[L_1, U_1]$. Moreover, we obtain that :

$$F'(x) = -1 - \frac{f'(x)}{(b_2+c_2)f(x)}[a_2 + (b_2+c_2)x + (\beta_2+\gamma_2)e^{-f(x)}] \tag{3.2.14}$$

where

$$f'(x) = -\frac{\alpha_1 + (\beta_1+\gamma_1)(1+x)e^{-x}}{(b_1+c_1)x^2} \tag{3.2.15}$$

Then, from (3.2.15) it follows that $f(U_1) < f(x) < f(L_1)$ and using (3.2.13) we obtain

$$\begin{aligned} F'(x) &= -1 + \frac{\alpha_1 + (\beta_1+\gamma_1)(1+x)e^{-x}}{(b_1+c_1)(b_2+c_2)x^2 f(x)}[a_2 + (b_2+c_2)x + (\beta_2+\gamma_2)e^{-f(x)}] \\ &\le -1 + \frac{\alpha_1 + (\beta_1+\gamma_1)(1+x)e^{-x}}{(b_1+c_1)(b_2+c_2)x^2 f(U_1)}[a_2 + (b_2+c_2)x + (\beta_2+\gamma_2)e^{-f(x)}] \quad < 0 \end{aligned} \tag{3.2.16}$$

Hence, $F(x) = 0$ has a unique positive solution in $[L_1, U_1]$. The proof is therefore completed. □

Theorem 3.2.3. *The unique positive equilibrium point of system (3.1.5) is locally asymptotically stable under the following condition:*

$$\begin{aligned} &\frac{\beta_1 e^{-L_1}}{a_1 + (b_1+c_1)L_2} + \left(1 + \frac{\beta_1 e^{-L_1}}{a_1+(b_1+c_1)L_2}\right)\left(\frac{c_2(\alpha_2 + (\beta_2+\gamma_2)e^{-L_2})}{(a_2+(b_2+c_2)L_1)^2}\right) \\ &+ \frac{b_2(\alpha_2+(\beta_2+\gamma_2)e^{-L_2})}{(a_2+(b_2+c_2)L_1)^2}\frac{(b_1+c_1)(\alpha_1+(\beta_1+\gamma_1)e^{-L_1})}{(a_1+(b_1+c_1)L_2)^2} + \frac{\gamma_1 e^{-L_1}}{a_1+(b_1+c_1)L_2} < 1 \end{aligned} \tag{3.2.17}$$

Proof. The characteristic polynomial of Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(\bar{x}, \bar{y})$ is given by:

$$P(\lambda) = \lambda^4 - (A_1+B_2)\lambda^3 - (A_2B_1 - A_1B_2 + B_3 + B_4)\lambda^2 - ((A_3+A_4)B_1 - (B_3+B_4)A_1)\lambda. \tag{3.2.18}$$

wher

$$J_F(\bar{x}, \bar{y}) = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$\begin{aligned}
A_1 &= -\frac{\beta_1 e^{-\bar{x}}}{a_1+(b_1+c_1)\bar{y}},\\
A_2 &= -\frac{b_1(\alpha_1+(\beta_1+\gamma_1)e^{-\bar{x}})}{(a_1+(b_1+c_1)\bar{y})^2},\\
A_3 &= -\frac{\gamma_1 e^{-\bar{x}}}{a_1+(b_1+c_1)\bar{y}},\\
A_4 &= -\frac{c_1(\alpha_1+(\beta_1+\gamma_1)e^{-\bar{x}})}{(a_1+(b_1+c_1)\bar{y})^2},\\
B_1 &= -\frac{b_2(\alpha_2+(\beta_2+\gamma_2)e^{-\bar{y}})}{(a_2+(b_2+c_2)\bar{x})^2},\\
B_2 &= -\frac{\beta_2 e^{-\bar{y}}}{a_2+(b_2+c_2)\bar{x}},\\
B_3 &= -\frac{c_2(\alpha_2+(\beta_2+\gamma_2)e^{-\bar{y}})}{(a_2+(b_2+c_2)\bar{x})^2},\\
B_4 &= -\frac{\gamma_2 e^{-\bar{y}}}{a_2+(b_2+c_2)\bar{x}}.
\end{aligned}$$

Clearly, one root of $P(\lambda)$ is 0. To check the behavior of the other three roots of $P(\lambda)$, we let $\Phi(\lambda) = \lambda^3$ and :
$\Psi(\lambda) = (A_1+B_2)\lambda^2 + (A_2B_1 - A_1B_2 + B_3 + B_4)\lambda + (A_3+A_4)B_1 - (B_3+B_4)A_1.$
Assume that (3.2.17) holds and $|\lambda| = 1$; then, one has:

$$\begin{aligned}
|\Psi(\lambda)| \le\ & |A_1+B_2| + |A_2B_1 - A_1B_2 + B_3 + B_4| + |(A_3+A_4)B_1 - (B_3+B_4)A_1|\\
<& \frac{\beta_1 e^{-\bar{x}}}{a_1+(b_1+c_1)\bar{y}} + \left(1+\frac{\beta_1 e^{-\bar{x}}}{a_1+(b_1+c_1)\bar{y}}\right)\left(\frac{c_2(\alpha_2+(\beta_2+\gamma_2)e^{-\bar{y}})}{(a_2+(b_2+c_2)\bar{x})^2} + \frac{(\beta_2+\gamma_2)e^{-\bar{y}}}{a_2+(b_2+c_2)\bar{x}}\right)\\
&+\frac{b_2(\alpha_2+(\beta_2+\gamma_2)e^{-\bar{y}})}{(a_2+(b_2+c_2)\bar{x})^2}\\
&\left(\frac{(b_1+c_1)(\alpha_1+(\beta_1+\gamma_1)e^{-\bar{x}})}{(a_1+(b_1+c_1)\bar{y})^2} + \frac{\gamma_1 e^{-\bar{x}}}{a_1+(b_1+c_1)\bar{y}}\right)\\
<& \frac{\beta_1 e^{-L_1}}{a_1+(b_1+c_1)L_2} + \left(1+\frac{\beta_1 e^{-L_1}}{a_1+(b_1+c_1)L_2}\right)\\
&\left(\frac{c_2(\alpha_2+(\beta_2+\gamma_2)e^{-L_2})}{(a_2+(b_2+c_2)L_1)^2} + \frac{(\beta_2+\gamma_2)e^{-L_2}}{a_2+(b_2+c_2)L_1}\right)\\
&+\frac{b_2(\alpha_2+(\beta_2+\gamma_2)e^{-L_2})}{(a_2+(b_2+c_2)L_1)^2}\\
&\frac{(b_1+c_1)(\alpha_1+(\beta_1+\gamma_1)e^{-L_1})}{(a_1+(b_1+c_1)L_2)^2}\\
&+\frac{\gamma_1 e^{-L_1}}{a_1+(b_1+c_1)L_2} < 1.
\end{aligned} \tag{3.2.19}$$

Then, by Rouché's Theorem, $\Phi(\lambda)$ and $\Phi(\lambda)-\Psi(\lambda)$ have the same number of zeros in an open unit disk $|\lambda|<1$. Hence, all the roots of (3.2.18) satisfy $|\lambda|<1$, and it follows from Lemma (3.2.2) that the unique positive equilibrium point $(\bar{x},\bar{y})$ of the system (3.1.5) is locally asymptotically stable.

□

Theorem 3.2.4. *The unique positive equilibrium point $(\bar{x},\bar{y})$ of system (3.1.5) is globally asymptotically stable, if the following condition is satisfied:*

$$\begin{aligned}\alpha_1+(\beta_1+\gamma_1)e^{-L_1} &< \bar{x}(a_1+(b_1+c_1)L_2),\\ \alpha_2+(\beta_2+\gamma_2)e^{-L_2} &< \bar{y}(a_2+(b_2+c_2)L_1).\end{aligned} \tag{3.2.20}$$

Proof. We consider the following discrete time analogue of Lyapunov function:

$$V_n=\bar{x}\left(\frac{x_n}{\bar{x}}-1-\ln\frac{x_n}{\bar{x}}\right)+\bar{y}\left(\frac{y_n}{\bar{y}}-1-\ln\frac{y_n}{\bar{y}}\right).$$

Then nonnegativity of V_n follows from the following inequality:

$$x-1-\ln x\geq 0 \quad \forall x>0.$$

Furthermore, we have

$$\begin{aligned}-\ln\left(\frac{x_{n+1}}{x_n}\right)&=\ln\left(1-\left(1-\frac{x_n}{x_{n+1}}\right)\right)\leq-\frac{x_{n+1}-x_n}{x_{n+1}},\\ -\ln\left(\frac{y_{n+1}}{y_n}\right)&=\ln\left(1-\left(1-\frac{y_n}{y_{n+1}}\right)\right)\leq-\frac{y_{n+1}-y_n}{y_{n+1}}.\end{aligned}$$

Assume that (3.2.20) holds true; then, it follows that

$$\begin{aligned}V_{n+1}-V_n&=\bar{x}\left(\frac{x_{n+1}}{\bar{x}}-1-\ln\frac{x_{n+1}}{\bar{x}}\right)+\bar{y}\left(\frac{y_{n+1}}{\bar{y}}-1-\ln\frac{y_{n+1}}{\bar{y}}\right)\\ &\quad-\bar{x}\left(\frac{x_n}{\bar{x}}-1-\ln\frac{x_n}{\bar{x}}\right)-\bar{y}\left(\frac{y_n}{\bar{y}}-1-\ln\frac{y_n}{\bar{y}}\right)\\ &\leq(x_{n+1}-x_n)+(y_{n+1}-y_n)-\frac{\bar{x}}{x_{n+1}}(x_{n+1}-x_n)-\frac{\bar{y}}{y_{n+1}}(y_{n+1}-y_n)\\ &=(x_{n+1}-x_n)\left(1-\frac{\bar{x}}{x_{n+1}}\right)+(y_{n+1}-y_n)\left(1-\frac{\bar{y}}{y_{n+1}}\right)\\ &\leq(U_1-L_1)\frac{\alpha_1+(\beta_1+\gamma_1)e^{-L_1}-\bar{x}(a_1+(b_1+c_1)L_2)}{\alpha_1+(\beta_1+\gamma_1)e^{-L_1}}\\ &\quad+(U_2-L_2)\frac{\alpha_2+(\beta_2+\gamma_2)e^{-L_2}-\bar{y}(a_2+(b_2+c_2)L_1)}{\alpha_2+(\beta_2+\gamma_2)e^{-L_2}}\leq 0\end{aligned}$$

Then it follows that $\lim_{n\to\infty} x_{n+1} = \bar{x}$ and $\lim_{n\to\infty} y_{n+1} = \bar{y}$. Furthermore, $V_n \leq V_0$ for all $n \geq 0$, which gives that $(\bar{x}, \bar{y}) \in [L_1, U_1][L_2, U_2]$ is uniformly stable. Hence, the unique positive equilibrium point $(\bar{x}, \bar{y}) \in [L_1, U_1][L_2, U_2]$ of system (3.1.5) is globally asymptotically stable

□

3.3 Examples

To validate our theoretical findings and reinforce the analytical discussion, we present several illustrative numerical examples in this section. These examples highlight various qualitative behaviors of solutions to the nonlinear difference system (3.1.5). Specifically, the first and last examples reveal the instability of the positive equilibrium point under certain parameter settings, while the remaining examples clearly demonstrate that the unique positive equilibrium is globally asymptotically stable for other choices of parameters. (we use MATLAB for plotting)

Example 1 Let $\alpha_1 = 5.6$, $\beta_1 = 9.1$, $\gamma_1 = 8.3$, $a_1 = 1.7$, $b_1 = 0.05$, $c_1 = 6.158$, $\alpha_2 = 2.5$, $\beta_2 = 1.04$, $\gamma_2 = 3.3$, $a_2 = 0.2$, $b_2 = 2.3$, and $c_2 = 0.04$. Then, system (3.1.5) can be written as:

$$\begin{cases} x_{n+1} &= \frac{5.6+9.1e^{-x_n}+8.3e^{-x_{n-1}}}{1.7+0.05y_n+6.158y_{n-1}}, \\ \\ y_{n+1} &= \frac{2.5+1.04e^{-y_n}+3.3e^{-y_{n-1}}}{0.2+2.3x_n+0.04x_{n-1}} \end{cases} \tag{3.4.1}$$

with initial conditions $x_{-1} = 1.1$, $x_0 = 1.3$, $y_{-1} = 2.2$, and $y_0 = 2.3$.

In this case the positive equilibrium point of the system (3.4.1) is unstable. Moreover, in Figure 1 the plot of x_n is shown in Figure 1(a), the plot of y_n is shown in Figure 1(b),

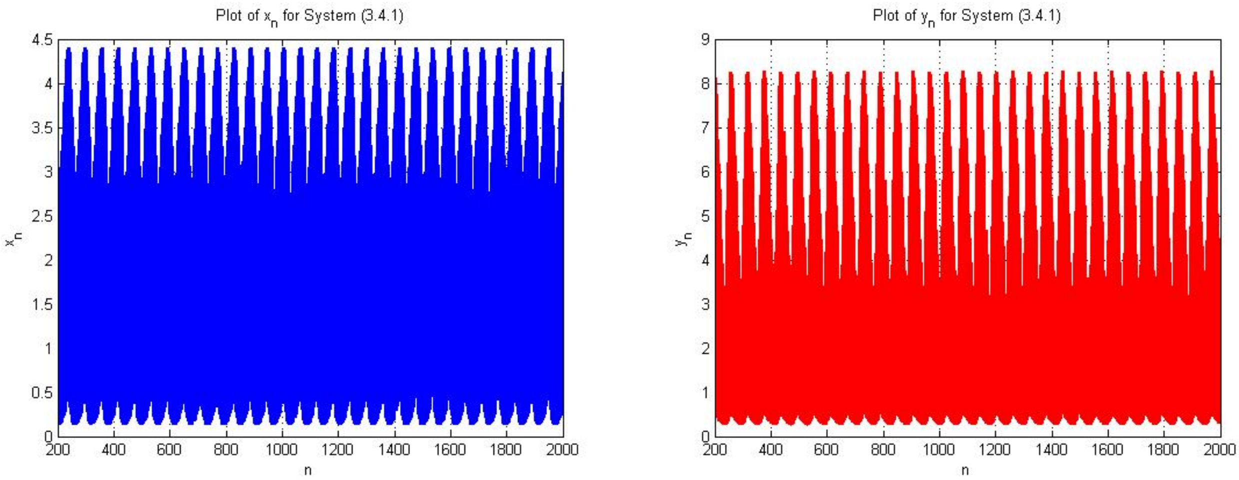

Figure 1 :plot for the systeme (3.4.1) where: $\alpha_1 = 5.6$, $\beta_1 = 9.1$, $\gamma_1 = 8.3$, $a_1 = 1.7$, $b_1 = 0.05$, $c_1 = 6.158$, $\alpha_2 = 2.5$, $\beta_2 = 1.04$, $\gamma_2 = 3.3$, $a_2 = 0.2$, $b_2 = 2.3$, and $c_2 = 0.04$.

Example 2 Let $\alpha_1 = 7.89$, $\beta_1 = 64$, $\gamma_1 = 99$, $a_1 = 19$, $b_1 = 80$, $c_1 = 4.9$, $\alpha_2 = 39$, $\beta_2 = 72.8$, $\gamma_2 = 17$, $a_2 = 7$, $b_2 = 3.1$, and $c_2 = 29$. Then, system (3.1.5) can be written as:

$$\begin{cases} x_{n+1} &= \frac{7.89+64e^{-x_n}+99e^{-x_{n-1}}}{19+80y_n+4.9y_{n-1}} \\ y_{n+1} &= \frac{39+72.8e^{-y_n}+17e^{-y_{n-1}}}{7+3.1x_n+29x_{n-1}} \end{cases} \tag{3.4.2}$$

with initial conditions $x_{-1} = 0.633$, $x_0 = 0.64$, $y_{-1} = 1.98$, and $y_0 = 1.9$

In this case the unique positive equilibrium point of the system (3.4.3) is given by $(\bar{x}, \bar{y}) = (0.5291954, 2.0894531)$. Moreover, in Figure 2 the plot of x_n is shown in Figure 2(a), the plot of y_n is shown in Figure 2(b)

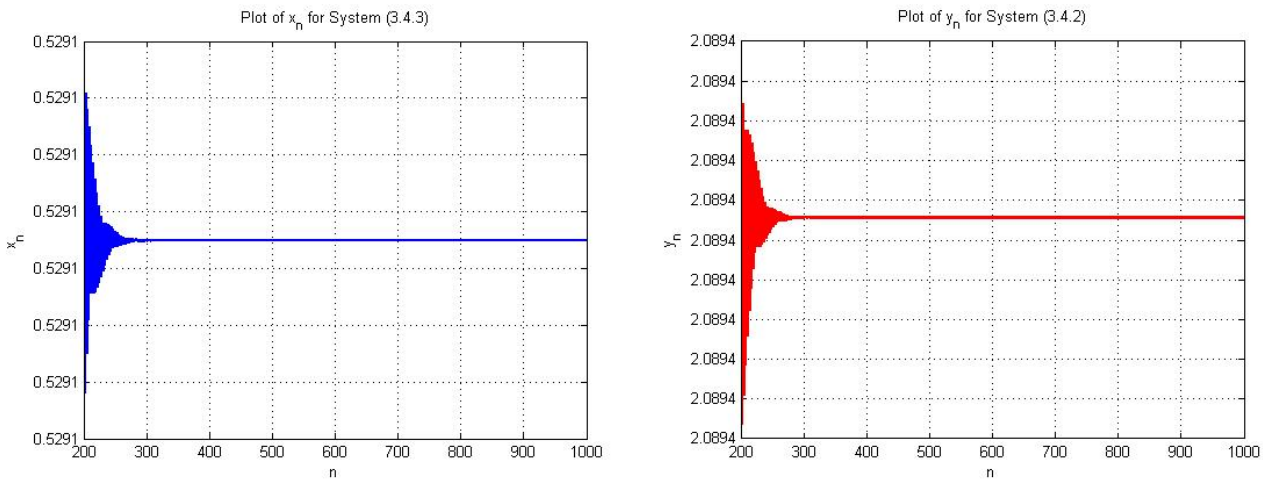

Figure 2 :plot for the systeme (3.4.2) where : $\alpha_1 = 8.98$, $\beta_1 = 75$, $\gamma_1 = 135$, $a_1 = 21$, $b_1 = 79$, $c_1 = 5$, $\alpha_2 = 41$, $\beta_2 = 71.9$, $\gamma_2 = 16$, $a_2 = 6$, $b_2 = 2$, and $c_2 = 30$.

Example 3 Let: $\alpha_1 = 32.9$, $\beta_1 = 299$, $\gamma_1 = 55$, $a_1 = 41$, $b_1 = 14$, $c_1 = 99$, $\alpha_2 = 61$, $\beta_2 = 29$, $\gamma_2 = 90$, $a_2 = 1.9$, $b_2 = 79$, and $c_2 = 0.9$. Then, system (3.1.5) can be written as:

$$\begin{cases} x_{n+1} & = \frac{32.9+299e^{-x_n}+55e^{-x_{n-1}}}{41+14y_n+99y_{n-1}} \\ y_{n+1} & = \frac{61+29e^{-y_n}+90e^{-y_{n-1}}}{1.6+79x_n+0.9x_{n-1}} \end{cases} \tag{3.4.3}$$

with initial conditions $x_{-1} = 0.81$, $x_0 = 0.83$, $y_{-1} = 1.3$, and $y_0 = 1.4$.

In this case the unique positive equilibrium point of the system (3.4.3) is given by $(\bar{x}, \bar{y}) = (0.944795, 1.23698)$. Moreover, in Figure 3 the plot of x_n is shown in Figure 3(a), the plot of y_n is shown in Figure 3(b)

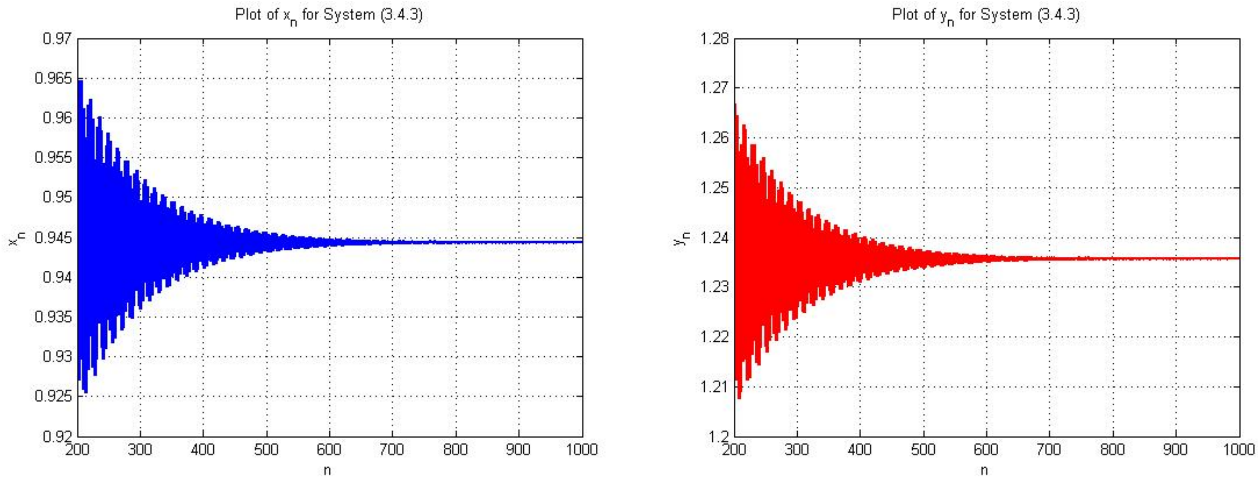

Figure 3:plot for the systeme (3.4.3) where : $\alpha_1 = 32.9$, $\beta_1 = 299$, $\gamma_1 = 55$, $a_1 = 41$, $b_1 = 14$, $c_1 = 99$, $\alpha_2 = 61$, $\beta_2 = 29$, $\gamma_2 = 90$, $a_2 = 1.9$, $b_2 = 79$, and $c_2 = 0.9$.

Example 4 Let : $\alpha_1 = 9$, $\beta_1 = 4$, $\gamma_1 = 0.02$, $a_1 = 0.03$, $b_1 = 3$, $c_1 = 9$, $\alpha_2 = 11$, $\beta_2 = 19$, $\gamma_2 = 320$, $a_2 = 1.02$, $b_2 = 10$, and $c_2 = 0.1$. Then, system (3.1.5) can be written as:

$$\begin{cases} x_{n+1} &= \frac{9+4e^{-x_n}+0.02e^{-x_{n-1}}}{0.03+3y_n+9y_{n-1}}, \\ y_{n+1} &= \frac{11+19e^{-y_n}+320e^{-y_{n-1}}}{1.02+10x_n+0.1x_{n-1}} \end{cases} \tag{3.4.4}$$

with initial conditions $x_{-1} = 0.4$, $x_0 = 0.3$, $y_{-1} = 1.3$, and $y_0 = 1.4$.

In this case the positive equilibrium point of the system (3.4.4) is unstable. Moreover, in Figure 4 the plot of x_n is shown in Figure 4(a), the plot of y_n is shown in Figure 4(b)

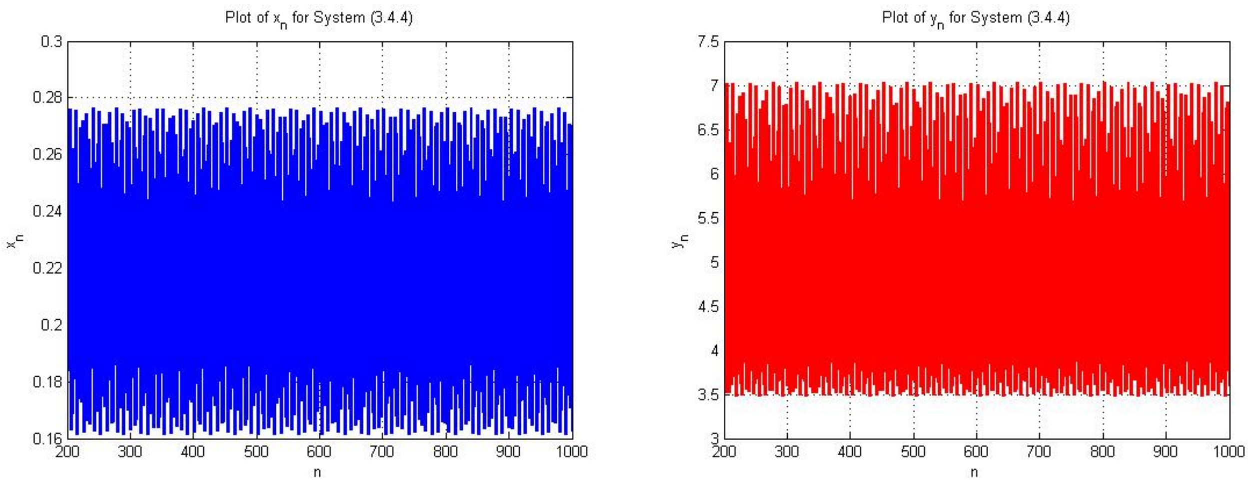

figure 4:plot for the systeme (3.4.4) where : $\alpha_1 = 9$, $\beta_1 = 4$, $\gamma_1 = 0.02$, $a_1 = 0.03$, $b_1 = 3$, $c_1 = 9$, $\alpha_2 = 11$, $\beta_2 = 19$, $\gamma_2 = 320$, $a_2 = 1.02$, $b_2 = 10$, and $c_2 = 0.1$

For system (3.1.5), we established the boundedness and persistence of positive solutions under specific parameter conditions. We also proved that its unique positive equilibrium is both locally and globally asymptotically stable. The core aim of dynamical systems theory is to predict long-term behavior based on current states. This involves identifying possible global dynamics and the parameters that govern them.

Conclusion

This work discusses a global dynamics of positive solutions for class of difference equation and systems. These equations constitute a foundational element of mathematical modelling exerting significat influence across a spectrum of scientific domains and practical applications,

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