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Faculté des Sciences
Département de Mathématiques

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Mémoire

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Stability and Bifurcation Analysis of a Fractional-Order Delayed SIRS Epidemic Model with Logistic Growth

Option : AFA

Par :

HANNACHI Choumaissa

Encadré par : BOULFOUL Bilal MCB U. SKIKDA

Devant le jury :

Président : *BEDRANI Yassine* MCB U. SKIKDA
Examinateur: HAMDI Zakaria MCB U. SKIKDA

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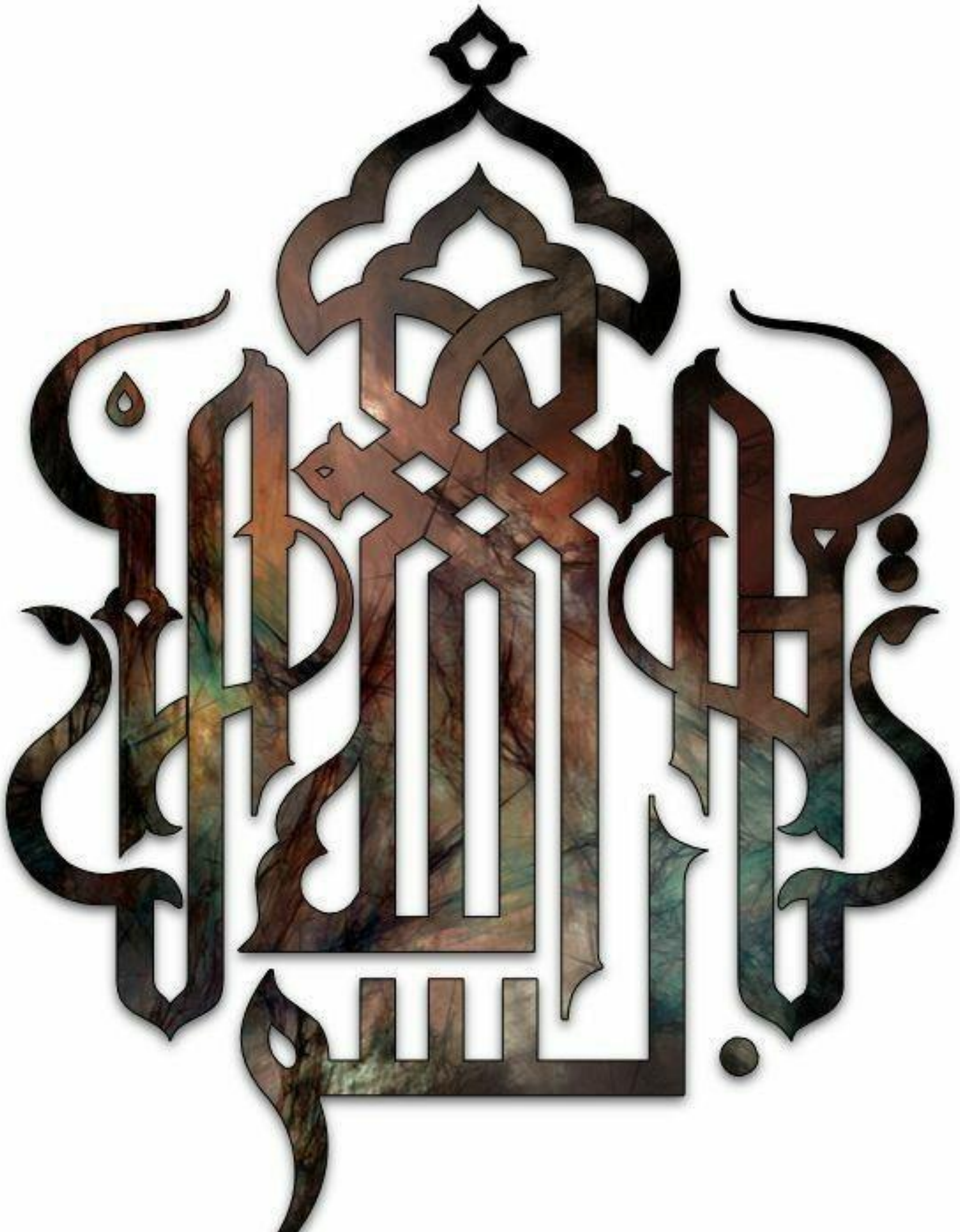

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It wasn't that easy but I did it.

TO EVERYONE I LOVE

Abstract

Mathematical modeling plays a vital role in the epidemiology of infectious diseases. Policy makers can provide the effective interventions by the relevant results of the epidemic models. In this work, we study a fractional-order SIRS epidemic model with time delay and logistic growth (see [54]), and we discuss the dynamical behavior of the model, such as the local stability of the equilibria and the existence of Hopf bifurcation around the endemic equilibrium. We present the numerical simulations to verify the theoretical analysis.

Keywords: fractional calculus, epidemic model, stability, Hopf bifurcation, delay differential equation.

Résumé

La modélisation mathématique joue un rôle essentiel dans l'épidémiologie des maladies infectieuses. Les décideurs politiques peuvent fournir des interventions efficaces grâce aux résultats pertinents des modèles épidémiques. Dans ce mémoire, nous étudions un modèle épidémique SIRS d'ordre fractionnaire avec retard temporel et croissance logistique (voir [54]), et nous discutons du comportement dynamique du modèle, tel que la stabilité locale des équilibres et l'existence d'une bifurcation de Hopf autour de l'équilibre endémique. Nous présentons les simulations numériques pour vérifier l'analyse théorique.

Mots-Clés: calcul fractionnaire, modèle épidémique, stabilité, bifurcation de Hopf, équations différentielles à retard.

ملخص:

تلعب النمذجة الرياضية دورا حيويا في وبائيات الأمراض المعدية، يمكن لواضعي السياسات توفير التدخلات الفعالة من خلال النتائج ذات الصلة للنماذج الوبائية.

في هذا العمل ندرس نموذجا لوباء SIRS الجزئي مع تأخير زمني ونمو لوجستي، ونناقش السلوك الديناميكي للنموذج، مثل الاستقرار المحلي للتوازن ووجود تشعب هوبف حول التوازن المتوطن، تقدم المحاكاة العددية للتحقق من التحليل النظري.

الكلمات المفتاحية: حساب التفاضل والتكامل الجزئي، النموذج الوبائي، الاستقرار، تشعب هوبف، معادلة تفاضلية ،التأخير.

Contents

Introduction

Recently, delay fractional differential equations (DFDEs) have received considerable attentions, because they provide mathematical models of real-world problems in which the fractional rate of change depends on the influence of their hereditary effects, see e .g., [6], [46] and the references therein. One of the simplest forms of DFDEs is:

$$\begin{cases} {}^{c}_{0+}D^{\alpha}x(t) = f(t, x(t), x(t-\tau)), \quad t \in [0, T], \\ x(t) = \phi(t), \quad \forall t \in [-\tau, 0], \end{cases} \tag{1}$$

where $\alpha > 0$ is the order of the Caputo fractional derivative ${}^{c}_{0+}D^{\alpha}$, and the initial condition ϕ is a continuous function on the interval $[-\tau, 0], \tau, T > 0$ are fixed real parameters. For this equation, the first basic and important problem is to show the existence and uniqueness of solutions under some reasonable conditions.

Mathematical models of infectious disease are widely used by many researchers. The first epidemic model was presented by Bernoulli in 1760 [17]. Epidemic models have become a valuable tool for the analysis of dynamics of infectious disease in recent years. Many deterministic or stochastic epidemic models were presented and analyzed by previous researches [23], [45], [21], [25].

In [28], the authors proposed an SIRS model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) + \delta R(t) - \mu S(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\gamma + \mu + \alpha)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (\delta + \mu)R(t), \end{cases}$$

where $S(t), I(t), R(t)$ denote the susceptible number, the infected number and the recovered number of individuals at time t, respectively. Parameters $A, \beta, \delta, \mu, \gamma$ and α are nonnegative constants. A is the recruitment rate of the population, β is the disease transmission coefficient, μ is the death rate of the population, γ is the rate constant for recovery, α is the death rate due to the disease, and δ is the rate constant for loss of immunity. Mena-Lorca and Hetheote discussed the dynamical behavior of this model.

In [35], Ranjith Kumar et al. considered an epidemic model with time delay and logistic growth of the susceptibles as follows:

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t-\tau)I(t-\tau) + \chi S(t), \\ \frac{dI(t)}{dt} = \beta S(t-\tau)I(t-\tau) - dI(t), \end{cases} \tag{2}$$

where $S(t), I(t)$ represents the number of susceptible and infected population, respectively. r represents intrinsic birth rate constant, K represents carrying capacity of susceptible, β represents the force of infection or the rate of transmission, χ represents immigration coefficient of $S(t), d$ represents death coefficient of $I(t)$, and τ is the latent period of the disease. The locally asymptotical stability of the disease-free equilibrium and endemic equilibrium of system (2) were studied. Hopf bifurcation around the endemic equilibrium was addressed.

Taking into account the latent period of the disease (time delay) and logistic growth of the susceptibles, we can present the following SIRS epidemic model:

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t-\tau)I(t-\tau) - \mu I(t), \\ \frac{dI(t)}{dt} = \beta S(t-\tau)I(t-\tau) - (\mu + \gamma + \varepsilon)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - \mu R(t), \end{cases}$$

The meanings of parameters r, K, β, τ are same as in system (2). μ is the death rate of the population, ε is the death rate due to the disease, γ is the rate of disease recovery.

In recent years, many scholars have proposed the idea of using fractional-order model to study infectious disease model [19], [20], [27], [24], [10], [22], [55], [58]. Fractional-order model is an extension of integer-order model, and fractional-order model has certain advantages in describing processes with memory and heritability [18], [29], [38].

In [55], the author studied a class of SIR infectious disease model with Caputo fractional-order derivative:

$$\begin{cases} D^{\alpha}S(t) = A - \beta S(t)I(t) - \mu I(t), \\ D^{\alpha}I(t) = \beta S(t)I(t) - (\mu + \gamma + \varepsilon)I(t), \\ D^{\alpha}R(t) = \gamma I(t) - \mu R(t), \end{cases}$$

However, the above fractional-order system does not take into account the influence of the latent period of the disease, i.e., time delay. In fact, it takes a certain time for an infected person to show symptoms from being infected, so it is of great significance to discuss the influence of time-delay factors.

This memory is organized as folow:

In the first chapter, We submitted reminder of some basic notions, in section 1, we define the fractional-order Caputo derivative, properties and laplace transform of the caputo fractional-order derivative, in section 2, we present the fractional-order differential equation and we study the existence, uniqueness, positivity and stability of solutions .
In the second chapter, we will study a SIRS model with Caputo fractional-order derivative and time delay, And we'll get some basic results such as the existence and uniqueness, nonnegativity, positive invariance of the solutions and we are presented the local asymptotic stability and bifurcation results for fractional-order epidemic model are derived and numerical examples .
In the third chapter, verify the obtained theoretical results.
Finally, the last chapter contains conclusion.

Chapter 1

Reminder of Some Basic Notions

1.1 Fractional-order Caputo Derivative

In this section, we present the definition of Caputo fractional-order derivative, and some useful lemmas are recalled for next analysis.

1.1.1 Definitions and properties

Definition 1.1.1 ([34]). The Riemann-Liouville fractional integral of order α for a function $f(t)$ is defined as:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where $t \geq t_0$, $\alpha > 0$, $\Gamma(.)$ is the gamma function, $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx$. Furthermore, $I^0 f(t) = f(t)$.

Definition 1.1.2 ([34]). The Caputo fractional derivative of order α for the function $f(t) \in C^n([t_0, \infty[, \mathbb{R})$ is defined by:

$${}_{t_0}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

where $t \geq t_0$, and n is a positive integer such that $n-1 \leq \alpha < n$. Furthermore, when $0 < \alpha < 1$:

$${}_{t_0}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f^{'}(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

Proposition 1.1.1 ([16]). *Let $f(t), g(t) : [t_0, \infty] \longrightarrow \mathbb{R}$ be such that ${}_{t_0}^{c}D_t^{\alpha} f(t)$ and ${}_{t_0}^{c}D_t^{\alpha} g(t)$ exist almost everywhere and let $c_1, c_2 \in \mathbb{R}$. Then, ${}_{t_0}^{c}D_t^{\alpha}(c_1 f(t) + c_2 g(t))$ exists almost everywhere, and ${}_{t_0}^{c}D_t^{\alpha}(c_1 f(t) + c_2 g(t)) = c_1 \, {}_{t_0}^{c}D_t^{\alpha} f(t) + c_2 \, {}_{t_0}^{c}D_t^{\alpha} g(t)$.*

Proposition 1.1.2 ([34]). *The fractional derivative for a constant function $f(t) = c$ is zero, that is,* ${}_{t_0}^{c}D_t^{\alpha}c = 0$.

In the following lemma we define the mean value theorem and another property [59].

Lemma 1.1.1 ([59]). *Suppose that $f \in C[a,b]$ and $D^{\alpha}f \in C(a,b]$ for $0 < \alpha \leq 1$, then we have*

1. *For each $t \in (a,b]$ there exists $\xi \in [a,t]$ suth that $f(t) = f(a) + \frac{1}{\alpha}D^{\alpha}f(\xi)(t-a)$.*
2. *If $D^{\alpha}f(t) \geq 0$ (resp: $D^{\alpha}f(t) \leq 0$) $\forall t \in (a,b)$, then f is non-decreasing (resp: is non-increasing) for each $t \in [a,b]$.*

Definition 1.1.3 ([34]). The Mittag-Leffler function is defined by

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)},$$

where $\alpha > 0,\ \gamma > 0,\ z \in \mathbb{C}$. In particular, when $\gamma = 1$, the Mittag-Leffler function of one parameter is defined as $E_{\alpha}(z) := E_{\alpha,1}(z)$. Furthermore, $E_{1,1}(z) = e^z$.

1.1.2 Laplace Transform of the Caputo Fractional Derivative

Definition 1.1.4 ([34]). The function $F(s)$ of the complex variable s denoted by:

$$F(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)dt,$$

is called the Laplace transform of the function $f(t)$, which is called the original.
For the existence of the integral, the function $f(t)$ must be exponential order a, which means that there exist $M > 0$ and $T > 0$ such that $e^{-at}|f(t)| \leq M$, $\forall t > T$.

Definition 1.1.5 ([34, 36]). The Laplace transform of the Caputo fractional derivative of order $n-1 < \alpha \leq n$ for a function $f \in C^n([t_0, \infty[, \mathbb{R})$ is:

$$L\{{}_{t_0}^{c}D_t^{\alpha}f(t); s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(t_0),$$

where $F(s)$ is the Laplace transform of $f(t)$, and $f^{(k)}(t_0)$ $(k = 0, ..., n-1)$ are the initial conditions. Obviously $L\{{}_{t_0}^{c}D_t^{\alpha}f(t); s\} = s^{\alpha}F(s)$, if $f^{(k)}(t_0) = 0$ for $k = 0, ..., n-1$.

1.2 Fractional-order Differential Equation

1.2.1 Existence, Uniqueness of Solutions

Consider the following initial condition problem on the infinite time interval $[-\tau, \infty[$ as well with the obvious change from finite T to ∞.
Let τ be an arbitrary positive constant, and $\phi \in C([-\tau, 0]; \mathbb{R}^n)$ be a given continuous function, we study the delay Caputo fractional differential equations:

$$^cD^{\alpha}_{0+}x(t) = f(t, x(t), x(t-\tau)), \quad t \in [0, T]. \tag{1.1}$$

with the initial condition:

$$x(t) = \phi(t), \quad \forall t \in [-\tau, 0]. \tag{1.2}$$

where $x \in \mathbb{R}^n), T > 0$, and $f : [0, T] \times \mathbb{R}^n) \times \mathbb{R}^n) \to \mathbb{R}^n)$ is continuous.

Let $\mathbb{R}_{\geq 0}$ be the set of all non-negative real numbers and $\mathbb{R}^n$ be the d-dimensional Euclidean space endowed with a norm $||.||$. For any $[-\tau, 0] \subset [-\tau, \infty[$, let $C([-\tau, 0]; \mathbb{R}^n)$ be the space of continuous functions $\xi : [-\tau, 0] \longrightarrow \mathbb{R}^n$ with the sup norm $||.||_\infty$, that is:

$$||\xi||_\infty = \sup_{-\tau \leqslant t \leqslant 0} ||\xi(t)||, \quad \forall \xi \in C([-\tau, 0]; \mathbb{R}^n).$$

A function $\varphi(, \phi) \in C([-\tau, T]; \mathbb{R}^n)$ is called a solution of the initial condition problem (1.1)-(1.2) over the interval $[-\tau, T]$ if:

$$\begin{cases} ^cD^{\alpha}_{0+}\varphi(t) = f(t, \varphi(t), \varphi(t-\tau)), \quad t \in [0, T], \\ \varphi(t) = \phi(t), \quad \forall t \in [-\tau, 0]. \end{cases}$$

Theorem 1.2.1 ([49]). *Assume that $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and satisfies the following Lipschitz condition, that is there exists $K > 0$ such that:*

$$||f(t, x(t), x(t-\tau)) - f(t, y(t), y(t-\tau))|| \leq K\left(||x(t) - y(t)|| + ||x(t-\tau) - y(t-\tau)||\right), \tag{1.3}$$

for all $t \in [0, T]$. Then, the initial condition problem (1.1)-(1.2) has a unique solution $\varphi(., \phi)$ on the interval $[-\tau, T]$, provided that the Lipschitz condition (1.3) is satisfied and

$$\bar{M} = \frac{2KT^\alpha}{\Gamma(\alpha + 1)} < 1.$$

Proof. We can apply a fractional integral operator to the differential equation (1.1) and incorporate the initial condition (1.2), thus we obtain the equivalent equation:

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(s-\tau)) ds, \tag{1.4}$$

which also is a Volterra equation of the second kind. Define the operator $L : C([-\tau, T], \mathbb{R}^n) \rightarrow C([-\tau, T], \mathbb{R}^n)$, such that:

$$Lx(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(s-\tau)) ds. \tag{1.5}$$

Then, we have

$$\begin{aligned}
||Lx(t) - Ly(t)|| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||f(s, x(s), x(s-\tau)) - f(s, y(s), y(s-\tau))|| ds, \\
&\leqslant \frac{K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (||x(s) - y(s)|| + ||x(s-\tau) - y(s-\tau)||) ds, \\
&\leqslant \frac{K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\sup_{s\in[-\tau,T]} ||x(s) - y(s)|| + \sup_{s\in[-\tau,0]} ||x(s) - y(s)|| \\
&+ \sup_{s\in[-\tau,T]} ||x(s) - y(s)||), \\
&\leqslant \frac{2K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{s\in[-\tau,T]} ||x(s) - y(s)|| ds, \\
&\leqslant \frac{2K}{\Gamma(\alpha+1)} ||x - y|| T^{\alpha}.
\end{aligned}$$

So, we obtain

$$||Lx(t) - Ly(t)|| \leqslant \bar{M} ||x - y||.$$

By Banach contraction principle [48], we can deduce that L has a unique fixed point which implies that our problem has a unique solution. □

1.2.2 Comparison principle and positivity

Lemma 1.2.1 ([36]). *The solution of the Cauchy problem:*

$$\begin{cases} {}_a^c D_t^{\alpha} x(t) = \lambda x(t) + f(t), \\ x(a) = b \quad (b \in \mathbb{R}), \end{cases}$$

with $0 < \alpha < 1$ *and* $\lambda \in \mathbb{R}$ *has the form*

$$x(t) = b E_{\alpha}[\lambda(t-a)^{\alpha}] + \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[\lambda(t-s)^{\alpha}] f(s) ds,$$

while the solution to the problem

$$\begin{cases} {}_{a}^{c}D_t^{\alpha}x(t) = \lambda x(t), \\ x(a) = b \quad (b \in \mathbb{R}), \end{cases}$$

is given by

$$x(t) = bE_{\alpha}[\lambda(t-a)^{\alpha}].$$

We need the following lemma, which generalizes Lemma 2.1.2 given in [3] on the integer-order system to fractional-order system.

Lemma 1.2.2 ([32]). *Let $u(t)$ be a continuous function on $[t_0, +\infty[$ and satisfying:*

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha}u(t) \leq -\lambda u(t) + \mu, \\ x(t_0) = u_{t_0}, \end{cases} \tag{1.6}$$

where $0 < \alpha < 1$, $(\lambda, \mu) \in \mathbb{R}^2$ and $\lambda \neq 0$, and $t_0 \geq 0$ is the initial time. Then

$$u(t) \leq \left(u_{t_0} - \frac{\mu}{\lambda}\right) E_{\alpha}[-\lambda(t - t_0^{\alpha})] + \frac{\mu}{\lambda}. \tag{1.7}$$

1.2.3 Stability Analysis

Lemma 1.2.3 ([14]). *Consider the following fractional-order differential system Caputo derivative:*

$$D^{\alpha}X(t) = AX(t), \quad X(0) = X_0, \tag{1.8}$$

where $\alpha \in [0,1]$, $X(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. The characteristic equation of system (1.8) is $|s^{\alpha}E - A| = 0$. If all of the roots of the characteristic equation have negative real parts, then the zero solution of the system is asymptotically stable.

Lemma 1.2.4 ([51]). *Consider the following fractional-order delay differential system with Caputo derivative:*

$$D^{\alpha}X(t) = AX(t) + BX(t-\tau), \quad X(t) = \phi(t), \quad t \in [-\tau, 0], \tag{1.9}$$

where $\alpha \in [0,1]$, $X(t) \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$, $\tau > 0$, then the characteristic equation of system (1.9) is $|s^{\alpha}E - A - Be^{-s\tau}| = 0$. If all of the roots of the characteristic equation have negative real parts, then the zero solution of the system is asymptotically stable.

Lemma 1.2.5 (Hopf Bifurcation, [52]). *Consider the following n-dimensional fractional-*

order system with delay:

$$ {}_{t_0}^{c}D_t^{\alpha} x_j(t) = f_j(x_1(t), ..., x_n(t); \tau), \quad j = 1, 2, ..., n, \tag{1.10}$$

where $0 < \alpha < 1$ *and the time delay* $\tau \geq 0$. *System (1.10) undergoes a Hopf bifurcation at the equilibrium* $x^* = (x_1^*, ..., x_n^*)$ *when* $\tau = \tau_0$ *if the following conditions are satisfied:*

1. *All the eigenvalues* $\lambda_j (j = 1, 2, ..., n)$ *of the coefficient matrix* A *of the linearized system of (1.10) with* $\tau = 0$ *satisfy* $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$.

2. *The characteristic equation of the linearized system of (1.10) has a pair of purely imaginary roots* $\pm i\omega_0$ *when* $\tau = \tau_0$.

3. *Re* $\left[\frac{ds(\tau)}{d\tau}\right] |_{(\tau=\tau_0,\omega=\omega_0)} > 0$, *where Re*[.] *denotes the real part of the complex number.*

Definition 1.2.1 ([40], [41], [57]). The constant x^* is an equilibrium point of Caputo fractional dynamic system (1.1), if and only if $f(t, x^*, x^*) = 0$.

Remark 1.2.1 ([13]). When $\alpha \in [0, 1]$, the Caputo fractional order system (1.1) has the same equilibrium points as the integer-order system $dx(t)/dt = f(t, x)$.

Chapter 2

Dynamic Analysis of a Fractional-order SIRS Model with Time Delay

Based on the above analysis, the following SIRS model with Caputo fractional-order derivative and time delay will be studied in this work:

$$\begin{cases} D^{\alpha}S(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t-\tau)I(t-\tau) + \rho R(t), \\ D^{\alpha}I(t) = \beta S(t-\tau)I(t-\tau) - (\mu+\delta+\sigma)I(t), \\ D^{\alpha}R(t) = \sigma I(t) - \mu R(t) - \rho R(t), \end{cases} \tag{2.1}$$

where $0 < \alpha \leq 1.S(t), I(t), R(t)$ represent the number of susceptible, infected and removed persons at time t, respectively; β denotes the infection coefficient; μ represents the natural mortality rate; δ denotes the death rate of the disease; K is the carrying capacity of susceptible population; ρ is the state transition rate from the recovered to the susceptible one; σ is the state transition rate from the infected to the recovered one;τ denotes the latent period of the disease.

The initial value condition of model (2.1) is:

$$S(\theta) = \varphi_1(\theta) > 0, \quad I(\theta) = \varphi_2(\theta) > 0, \quad R(0) = R_0 > 0, \quad \theta \in [-\tau, 0]. \tag{2.2}$$

In this chapter, stability and bifurcation problems system of (2.1) will be studied by using the theory of fractional-order stability and delay differential equation.

2.1 Existence, Uniqueness and positivity solution

In this section, we will discuss the existence and uniqueness of the solution for system (2.1). Furthermore, the solutions of system (2.1) with initial condition (2.2) are nonnegative and positively invariant.

Theorem 2.1.1 ([54]). *If* $C \in C([-\tau, 0], \mathbb{R}^3_+)$ *is the continuous function of Banach space and* $x_0(t) \in C$ *in an initial condition, then system (2.1) has a unique solution* $x(t) \in \Theta$, *where* $x(t) = (S(t), I(t), R(t))$, $x(t) = \{(S(t), I(t), R(t)) \in \mathbb{R}^3_+ : \max\{|S(t)|, |I(t)|, |R(t)|\} \leq L\}$ *and* $\hat{x}(t) = (\hat{S}(t), \hat{I}(t), \hat{R}(t))$, $y(t) = (S(t-\tau), I(t-\tau), R(t-\tau))$.

Proof. Consider the mapping $H(x) = (S(t), I(t), R(t))$, where:

$$\begin{aligned}
H_1(x(t), y(t)) &= rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t-\tau)I(t-\tau) + \rho R(t), \\
H_2(x(t), y(t)) &= \beta S(t-\tau)I(t-\tau) - (\mu + \delta + \sigma)I(t), \\
H_3(x(t), y(t)) &= \sigma I(t) - \mu R(t) - \rho R(t).
\end{aligned}$$

For any $t_1, t_2 \in \Theta$:

$$\begin{aligned}
||H(z(t_1)) - H(z(t_2))|| &\leq ||H_1(z(t_1)) - H_1(z(t_2))|| + ||H_2(z(t_1)) - H_2(z(t_2)))|| \\
&+ ||H_3(z(t_1)) - H_3(z(t_2)))||, \\
&= ||rS(t_1)\left(1 - \frac{S(t_1)}{K}\right) - \beta S(t_1 - \tau)I(t_1 - \tau) \\
&+ \rho R(t_1) - rS(t_2)\left(1 - \frac{S(t_2)}{K}\right) \\
&+ \beta S(t_2 - \tau)I(t_2 - \tau)n - \rho R(t_2)|| \\
&+ ||\beta S(t_1 - \tau)I(t_1 - \tau) - (\mu + \delta + \sigma)I(t_1) \\
&- \beta S(t_2 - \tau)I(t_2 - \tau) + (\mu + \delta + \sigma)I(t_2)|| \\
&+ ||\sigma I(t_1) - (\mu + \rho)R(t_1) - \sigma I(t_2) + (\mu + \rho)R(t_2)||, \\
&= ||r\left(1 - \frac{1}{K}\right)[S(t_1)S(t_1) + S(t_1)S(t_2) - S(t_2)S(t_1) - S(t_2)S(t_2)] \\
&+ \beta[S(t_1 - \tau)I(t_1 - \tau) + S(t_1 - \tau)I(t_2 - \tau) \\
&- S(t_2 - \tau)I(t_1 - \tau) - S(t_2 - \tau)I(t_2 - \tau)] + \rho R(t_1) - \rho R(t_2)|| \\
&+ ||\beta[S(t_1 - \tau)I(t_1 - \tau) + S(t_1 - \tau)I(t_2 - \tau) \\
&- S(t_2 - \tau)I(t_1 - \tau) - S(t_2 - \tau)I(t_2 - \tau)] - (\mu + \delta + \sigma)I(t_1) \\
&- \beta S(t_2 - \tau)I(t_2 - \tau) + (\mu + \delta + \sigma)I(t_2)|| \\
&+ ||\sigma I(t_1) - (\mu + \rho)R(t_1) - \sigma I(t_2) + (\mu + \rho)R(t_2)||, \\
&\leq ||\left(r + \frac{2rL}{K}\right)[S(t_1) - S(t_2)] + \rho[R(t_1) - R(t_2)]\,|| \\
&+ \beta L[||S(t_1 - \tau) - S(t_2 - \tau)|| + ||I(t_1 - \tau) - I(t_2 - \tau)||] \\
&+ \beta L[||S(t_1 - \tau) - S(t_2 - \tau)|| + ||I(t_1 - \tau) - I(t_2 - \tau)||] \\
&+ ||(\mu + \delta + \sigma)[I(t_1) - I(t_2)]||
\end{aligned}$$

$$\begin{aligned}
&\leq ||\left(r + \frac{2rL}{K}\right)[S(t_1) - S(t_2)] + \rho[R(t_1) - R(t_2)]\,|| \\
&+ 2\beta L[||S(t_1 - \tau) - S(t_2 - \tau)|| + ||I(t_1 - \tau) - I(t_2 - \tau)||] \\
&+ ||(\mu + \delta + \sigma)[I(t_1) - I(t_2)]|| \\
&+ ||\sigma[I(t_1) - I(t_2)] + (\mu + \rho)[R(t_1) - R(t_2)]||, \\
&\leq M||z(t_1) - z(t_2)||.
\end{aligned}$$

where $M = \max\{r + 2rL/K, 2\beta, \mu + \delta + \sigma\}$. Hence, $H(x(t), y(t)$ satisfies Lipschitz condition. From Lemma 5 in [33] we can obtain that system (2.1) has a unique solution $x(t)$. □

Theorem 2.1.2 ([54]). *The solutions of system (2.1) with initial condition (2.2) are nonnegative.*

Proof. Assume that $R^3_+ = \{(S,I,R) \in \mathbb{R} : S \geq 0, I \geq 0, R \geq 0\}$ is positively invariant. System (2.1) can be written in the vector form:

$$D^\alpha X(t) = H(z(t)).$$

Here $z(t) = (S(t), I(t), R(t))^\top$, and:

$$H(z(t)) = \begin{bmatrix} rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t-\tau)I(t-\tau) + \rho R(t) \\ \beta S(t-\tau)I(t-\tau) - (\mu+\delta+\sigma)I(t) \\ \sigma I(t) - \mu R(t) - \rho R(t) \end{bmatrix}.$$

$z_0 = (S(\theta), I(\theta), R(0))^\top \in R^3_+$. For that, we investigate the direction of the vector field $H(z(t))$ on each coordinate space and see whether the vector field points to the interior of $\mathbb{R}^3_+$. From (2.1) we have:

$$D^\alpha S(t)|_{S=0} = \rho R(t) \geq 0, \quad D^\alpha I(t)|_{I=0} = 0, \quad D^\alpha R(t)|_{R=0} = \sigma I(t) \geq 0. \qquad (2.3)$$

From Theorem 1 in [59], Lemma 1.1.1 and Eq. (2.3) the vector field $H(z(t))$ is interior of $\mathbb{R}^3_+$. The solution of (2.1) with initial condition $z_0 \in \mathbb{R}^3_+$; say $z(t) = z(t, X_0)$, in such a way, $z(t) \in \mathbb{R}^3_+$. □

Theorem 2.1.3 ([54]). *The set $\Omega = \{(S,I,R) \in R^3_+\} : S + I + R \leq rK/4$ is positively invariant with respect to system (2.1).*

Proof. Let $(S(t), I(t), R(t))$ be the solution of system (2.1) with initial condition (2.2). Set $N(t) = S(t) + I(t) + R(t)$. From system (2.1) we can obtain:

$$\begin{aligned} D^\alpha N(t) &= rS(t)\left(1 - \frac{S(t)}{K}\right) - \mu I(t) - \mu R(t) - \delta I(t), \\ &\leq rS(t)\left(1 - \frac{S(t)}{K}\right) - (\mu+\delta)N(t), \\ &\leq \frac{rK}{4} - (\mu+\delta)N(t). \end{aligned}$$

Hence

$$N(t) \leq \left(-\frac{rK}{4(\mu+\delta)} + N(0)\right) E_\alpha(-(\mu+\delta)t^\alpha) + \frac{rK}{4(\mu+\delta)}.$$

Obviously, $E_\alpha(-(\mu+\delta)t^\alpha) \geq 0$. Hence, $N(t) = S(t) + I(t) + R(t) \leq rK/4$ when $S(0) + I(0) + R(0) \leq rK/4$, and $\Omega = \{(S,I,R) \in R^3_+ : S + I + R \leq rK/4\}$ is positively invariant with respect to system (2.1). □

2.2 Analysis of Stability and Hopf Bifurcation

The equilibria of system (2.1) are the points of intersections at which $D^{\alpha}S(t) = 0, D^{\alpha}I(t) = 0$ and $D^{\alpha}R(t) = 0$. It is straightforward to see that for system (2.1), there always exists a trivial equilibrium $E_1(0,0,0)$ and a disease-free equilibrium $E_2(K,0,0)$.

The basic reproduction number is defined as the average number of secondary infections produced when one infected individual is introduced into a host population, where everyone is susceptible [44]. Now, we use next-generation matrix method in [44] to obtain the basic reproduction number R_0 of system (2.1).

If $x = (I, S, R)^{\top}$, then when $\tau = 0$, the original system can be expressed as:

$$\frac{dx}{dt} = F(x) - V(x),$$

where

$$F(x) = \begin{pmatrix} \beta SI \\ 0 \\ 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} (\mu + \delta + \sigma)I \\ \beta SI - rS(1 - \frac{S}{K}) - \rho R \\ \rho R + \mu R - \sigma I \end{pmatrix}.$$

We can get

$$F = \begin{pmatrix} \beta K & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu + \delta + \sigma & 0 \\ \beta K & r \end{pmatrix}.$$

The next-generation matrix for model (2.1) is

$$FV^{-1} = \begin{pmatrix} \frac{\beta K}{\mu+\delta+\sigma} & 0 \\ 0 & 0 \end{pmatrix}.$$

The spectral radius $\rho(FV^{-1}) = \beta K/(\mu + \delta + \sigma)$. According to Theorem 2 in [44], the basic reproduction number of system (2.1) is:

$$R_0 = \frac{\beta K}{\mu + \delta + \sigma}.$$

The basic reproduction number is affected by several factors including: the disease transmission coefficient β, the carrying capacity of susceptible population K, the natural death rate of the population μ, the death rate of the disease δ, the state transition rate from the infected to the recovered one σ.

Theorem 2.2.1 ([54]). *If $R_0 > 1$, system (2.1) has a unique endemic equilibrium E^*. If $R_0 < 1$, there is no endemic equilibrium of system (2.1).*

Proof. To obtain the endemic equilibrium E^* of system (2.1), we need to impose the right side of system (2.1) to be equal to 0. In other words, the equilibrium $E^*(S^*, I^*, R^*)$

should satisfy the following equations:

$$\begin{aligned} rS^*\left(1-SK^*\right)-\beta S^*I^*+\rho R^* &= 0,\\ \beta S^*I^*-(\mu+\delta+\sigma)I^* &= 0,\\ \sigma I^*-\mu R^*-\rho R^* &= 0. \end{aligned}$$

From above we can obtain

$$S^*=\frac{\mu+\delta+\sigma}{\beta},\quad R^*=\frac{\sigma}{\mu+\rho}I^*,\quad I^*=\frac{r(\mu+\rho)(\mu+\delta+\sigma)^2(R_0-1)}{K\beta^2[\mu^2+\delta(\rho+\mu)+\mu(\rho+\sigma)]}.$$

It is obvious that $S^*>0$. When $R_0>1$, $I^*>0$, system (2.1) has a unique endemic equilibrium, and when $R_0<1$, $I^*<0$, there is no endemic equilibrium of system (2.1).

In the following, we will discuss the locally asymptotical stability of the trivial equilibrium E_1, the disease-free equilibrium E_2, the endemic equilibrium for system (2.1) and the existence of Hopf bifurcation around the endemic equilibrium E^*.

To discuss the locally asymptotical stability of system (2.1), we have to linearize it. Let us consider the following coordinate transformation:

$$x(t)=S(t)-\bar{S},\quad y(t)=I(t)-\bar{I},\quad z(t)=R(t)-\bar{R},$$

where $(\bar{S},\bar{I},\bar{R})$ denotes any equilibrium of system (2.1). So we can obtain that the corresponding linearized system is of the form:

$$\begin{cases} D^\alpha x(t)=\left(r-\frac{2r\bar{S}}{K}\right)x(t)-\beta\bar{I}x(t-\tau)-\beta\bar{S}y(t-\tau)+\rho z(t),\\ D^\alpha y(t)=\beta\bar{I}y(t-\tau)+\beta\bar{S}x(t-\tau)-(\mu+\delta+\sigma)y(t),\\ D^\alpha z(t)=\sigma y(t)-\mu z(t)-\rho z(t). \end{cases} \tag{2.4}$$

Taking Laplace transform on both sides of (2.4), we get:

$$\begin{aligned} s^\alpha S(s)-s^{\alpha-1}x(0) &= \left(r-\frac{2r\bar{S}}{K}\right)S(s)-\beta\bar{I}e^{-s\tau}\left[S(s)+\int_{-\tau}^{0}e^{-st}\varphi_1(t)dt\right]\\ &- \beta\bar{S}e^{-s\tau}\left[I(s)+\int_{-\tau}^{0}e^{-st}\varphi_2(t)dt\right]+\rho R(s),\\ s^\alpha I(s)-s^{\alpha-1}y(0) &= \beta\bar{S}e^{-s\tau}\left[I(s)+\int_{-\tau}^{0}e^{-st}\varphi_2(t)dt\right]\\ &+ \beta\bar{I}e^{-s\tau}\left[S(s)+\int_{-\tau}^{0}e^{-st}\varphi_1(t)dt\right]-(\mu+\delta+\sigma)I(s), \end{aligned} \tag{2.5}$$

$$s^\alpha R(s)-s^{\alpha-1}z(0)=\sigma I(s)-(\mu+\rho)R(s).$$

Here $S(s), I(s), R(s)$ are the Laplace transform of $x(t), y(t), z(t)$, respectively. The above system (2.5) can be written as follows:

$$\Delta(s)\cdot\begin{pmatrix} S(s)\\ I(s)\\ R(s)\end{pmatrix}=\begin{pmatrix} b_1(s)\\ b_2(s)\\ b_3(s)\end{pmatrix},$$

where

$$\Delta(s)=\begin{pmatrix} s^\alpha - r + \frac{2r\bar{S}}{K} + \beta\bar{I}e^{-\lambda\tau} & \beta\bar{S}e^{-\lambda\tau} & -\rho\\ -\beta\bar{I}e^{-\lambda\tau} & s^\alpha - \beta\bar{S}e^{-\lambda\tau} + (\mu+\sigma+\delta) & 0\\ 0 & -\sigma & s^\alpha + (\mu+\rho)\end{pmatrix},$$

and

$$\begin{aligned} b_1(s) &= s^{\alpha-1}x(0) - \beta\overline{I}e^{-s\tau}\int_{-\tau}^{0} e^{-st}\varphi_1(t)dt - \beta\overline{S}e^{-s\tau}\int_{-\tau}^{0} e^{-st}\varphi_2(t)dt,\\ b_2(s) &= s^{\alpha-1}y(0) + \beta\overline{I}e^{-s\tau}\int_{-\tau}^{0} e^{-st}\varphi_1(t)dt + \beta\overline{S}e^{-s\tau}\int_{-\tau}^{0} e^{-st}\varphi_2(t)dt,\\ b_3(s) &= s^{\alpha-1}z(0). \end{aligned}$$

□

Theorem 2.2.2 ([54]). *The trivial equilibrium $E_1(0,0,0)$ is always unstable.*

Proof. The characteristic matrix at $E_1(0,0,0)$ is:

$$\Delta_1(s)=\begin{pmatrix} s^\alpha - r & 0 & \rho\\ 0 & s^\alpha + (\mu+\sigma+\delta) & 0\\ 0 & \sigma & s^\alpha + (\mu+\rho)\end{pmatrix}.$$

The characteristic equation at the trivial equilibrium $E_1(0,0,0)$ reduces to:

$$(s^\alpha - r)(s^\alpha + \mu + \sigma + \delta)(s^\alpha + \mu + \rho) = 0. \tag{2.6}$$

Obviously, Eq. (2.6) has a positive root $s^\alpha = r (0 < \alpha \geq 1)$. Then the trivial equilibrium $E_1(0,0,0)$ of system (2.1) is always unstable. □

Theorem 2.2.3 ([54]). *If $R_0 < 1$, then the disease-free equilibrium E_2 of system (2.1) is locally asymptotically stable for all $\tau \geq 0$.*

Proof. The characteristic matrix at $E_2(K,0,0)$ is:

$$\Delta_2(s) = \begin{pmatrix} s^\alpha + r & \beta K e^{-s\tau} & -\rho \\ 0 & s^\alpha - \beta K e^{-s\tau} + (\mu+\sigma+\delta) & 0 \\ 0 & -\sigma & s^\alpha + (\mu+\rho) \end{pmatrix}.$$

Then the characteristic equation at the disease-free equilibrium $E_2(K,0,0)$ is:

$$(s^\alpha + r)(s^\alpha + \mu + \rho)[s^\alpha + (\mu+\delta+\sigma) - \beta K e^{-s\tau}] = 0.$$

When $\tau = 0$, the characteristic equation can be translated into:

$$(s^\alpha + r)(s^\alpha + \mu + \rho)[s^\alpha + (\mu+\delta+\sigma) - \beta K] = 0. \tag{2.7}$$

Let $s^\alpha = \lambda$, Eq. (2.7) can be rewritten as:

$$(\lambda + r)(\lambda + \mu + \rho)[\lambda + (\mu+\delta+\sigma) - \beta K] = 0.$$

It's characteristic roots are:

$$\lambda_1 = -r, \quad \lambda_2 = -\mu - \rho, \quad \lambda_3 = \beta K - (\mu+\delta+\sigma) = (R_0 - 1)(\mu+\delta+\sigma).$$

Obviously, $\lambda_1 < 0, \lambda_2 < 0$. Hence, $|arg(\lambda_1)| = |\arg(\lambda_2)| = \pi > \alpha\pi/2. |\arg(\lambda_3)| = \pi > \alpha\pi/2$ when the basic reproduction number $R_0 < 1$. Hence, all the eigenvalues λ_i of $\Delta_2(s)$ satisfy $|\arg(\lambda_i)| + \pi > \alpha\pi/2 (i = 1,2,3)$ when $R_0 < 1$. According to Lemma 2, the disease-free equilibrium E_2 is locally asymptotically stable when $R_0 < 1$.

When $\tau \neq 0$, since the first two factors of the left side of Eq. (2.7) do not contain time delay τ, we only need to consider the third factor:

$$s^\alpha - \beta K e^{-s\tau} + (\mu+\delta+\sigma) = 0. \tag{2.8}$$

Assume $s = i\omega = \omega(\cos(\pi/2) + i\sin(\pi/2))(\omega > 0)$, then s is substituted in (2.8), we get:

$$(i\omega)^\alpha - \beta K e^{-i\omega\tau} + (\mu+\delta+\sigma) = 0.$$

Separating the imaginary parts and real parts leads to:

$$\omega^\alpha \cos\frac{\alpha\pi}{2} + (\mu+\delta+\sigma) = \beta K \cos\omega\tau, \quad \omega^\alpha \sin\frac{\alpha\pi}{2} = -\beta K \sin\omega\tau.$$

Squaring and adding both sides of this equation, we can obtain:

$$\omega^{2\alpha} + 2(\mu+\delta+\sigma)cos\frac{\alpha\pi}{2}\omega^\alpha + (\mu+\delta+\sigma+\beta K)(\mu+\delta+\sigma)(1 - R_0) = 0. \tag{2.9}$$

Obviously, $2(\mu+\delta+\sigma)\cos(\alpha\pi/2) > 0$, then by our assumption that $R_0 < 1$, Eq. (2.9) has no positive roots, which ensures that Eq. (2.7) has no purely imaginary roots if $R_0 < 1$. According to Lemma 1.2.4, the equilibrium E_2 is locally asymptotically stable for any delay $\tau \geq 0$ if $R_0 < 1$. The proof is completed. □

Next, we discuss the local stability and bifurcation results at the endemic equilibrium point E^*. When $R_0 > 1$, the endemic equilibrium point E^* exists. The characteristic matrix at E^* is:

$$\Delta_3(s) = \begin{pmatrix} s^\alpha - r + \frac{2rS^*}{K} + \beta I^* e^{-s\tau} & \beta S^* e^{-s\tau} & -\rho \\ -\beta I^* e^{s\tau} & s^\alpha - \beta S^* e^{-s\tau} + (\mu+\sigma+\delta) & 0 \\ 0 & -\sigma & s^\alpha + (\mu+\rho) \end{pmatrix}.$$

The associated characteristic equation of system (2.1) at E^* can be described as:

$$B_1(s) + B_2(s)e^{-s\tau} + B_3 e^{-s\tau} = 0, \tag{2.10}$$

where

$$B_1(s) = (s^\alpha)^3 + p_1(s^\alpha)^2 + p_2 s^\alpha + p_3, \quad B_2(s) = q_1(s^\alpha)^2 + q_2 s^\alpha, \quad B_3 = a_2a_4a_5 - a_1a_3a_5 - a_2\rho\sigma,$$

and

$$p_1 = a_1 + a_4 + a_5, \quad p_2 = a_1a_5 + a_4a_5 + a_1a_4, \quad p_3 = a_1a_4a_5,$$

$$q_1 = a_2 - a_3, \quad q_2 = a_2a_5 + a_2a_4 - a_1a_3 - a_3a_5,$$

$$a_1 = r - \frac{2rS^*}{K}, \quad a_2 = \beta I^*, a_3 = \beta S^*, \quad a_4 = \mu+\sigma+\delta, \quad a_5 = \mu+\rho.$$

Case 1. When $\tau = 0$, Eq. (2.10) becomes:

$$(s^\alpha)^3 + (p_1+q_1)(s^\alpha)^2 + (p_2+q_2)s^\alpha + (p_3+B_3) = 0.$$

On the basis of Routh–Hurwitz theorem, the endemic equilibrium point E^* is locally asymptotically stable if:

$$p_1+q_1 > 0, \quad p_3+B_3 > 0, \quad (p_1+q_1)(p_2+q_2) > p_3+B_3.$$

Case 2. When $\tau > 0$, let $s = i\omega = \omega(cos(\pi/2) + isin(\pi/2))(\omega > 0)$ be a root of Eq. (2.10). Substituting s in (2.10), we obtain:

$$(c_1 + id_1) + (c_2 + id_2)e^{-i\omega\tau} + B_3 e^{-i\omega\tau} = 0, \tag{2.11}$$

where

$$\begin{aligned}
c_1 &= \omega^{3\alpha}\cos\frac{3\alpha\pi}{2} + p_1\omega^{2\alpha}\cos(\alpha\pi) + p_2\omega^{\alpha}cos\frac{\alpha\pi}{2} + p_3,\\
d_1 &= \omega^{3\alpha}\sin\frac{3\alpha\pi}{2} + p_1\omega^{2\alpha}\sin(\alpha\pi) + p_2\omega^{\alpha}\sin\frac{\alpha\pi}{2},\\
c_2 &= q_1\omega^{2\alpha}\cos(\alpha\pi) + q_2\omega^{\alpha}\cos\frac{\alpha\pi}{2},\\
d_2 &= q_1\omega^{2\alpha}\sin(\alpha\pi) + q_2\omega^{\alpha}\sin\frac{\alpha\pi}{2}.
\end{aligned}$$

Separating the real and imaginary parts of (2.11) yields:

$$\begin{aligned}
(c_2 + B_3)\cos(\omega\tau) + d_2\sin(\omega\tau) + c_1 &= 0,\\
d_2\cos(\omega\tau) - (c_2 + B_3)\sin(\omega\tau) + d_1 &= 0.
\end{aligned} \tag{2.12}$$

From Eq. (2.12) we have:

$$\cos(\omega\tau) = -\frac{(c_2 + B_3)c_1 + d_1 d_2}{(c_2 + B_3)^2 + d_2^2}, \quad \sin(\omega\tau) = -\frac{(c_2 + B_3)d_1 + c_1 d_2}{(c_2 + B_3)^2 + d_2^2}.$$

It is obvious that $\cos^2(\omega\tau) + \sin^2(\omega\tau) = 1$, and:

$$\omega^{6\alpha} + v_1\omega^{5\alpha} + v_2\omega^{4\alpha} + v_3\omega^{3\alpha} + v_4\omega^{2\alpha} + v_5\omega^{1\alpha} + v_6\omega = 0,$$

where

$$\begin{aligned}
v_1 &= 2p_1\cos\frac{\alpha\pi}{2},\\
v_2 &= p_1^2 - q_1^2 + 2p_2\cos(\alpha\pi),\\
v_3 &= 2\left[(p_1p_2 - q_1q_2)\cos\frac{\alpha\pi}{2} - B_3\cos\frac{3\alpha\pi}{2}\right],\\
v_4 &= p_2^2 - q_2^3 - 2(p_1p_3 + q_1B_3)cos(\alpha\pi),\\
v_5 &= 2(p_2p_3 - q_2B_3)\cos\frac{\alpha\pi}{2},\\
v_6 &= p_3^2 - B_3^2.
\end{aligned}$$

Let

$$f(\omega) = \omega^{6\alpha} + v_1\omega^{5\alpha} + v_2\omega^{4\alpha} + v_3\omega^{3\alpha} + v_4\omega^{2\alpha} + v_5\omega^{1\alpha} + v_6\omega.$$

Then let us discuss the distribution of roots of Eq. (2.10). It is imperative that the following lemma is useful and needed.

Lemma 2.2.1 ([54]). *For Eq. (2.10), the following results hold:*

1. *If $v_1 > 0$, $v_2 > 0$, $v_3 > 0$, $v_4 > 0$, $v_5 > 0$, $v_6 > 0$ and $p_3^2 - B_3^2 \neq 0$, then Eq. (2.10)*

has no root with zero real parts for all $\tau \geq 0$.

2. *If* $v_1 < 0$, $v_2 < 0$, $v_3 < 0$, $v_4 < 0$, $v_5 < 0$ *and* $v_6 > 0$, *then Eq. (2.10) has a pair of purely imaginary roots* $\pm i\omega_+$ *when* $\tau = \tau_j$, $j = 1, 2, 3, ...,$ *where:*

$$\tau_j = \frac{1}{\omega_+}\left[\arccos\left(-\frac{(c_2 + B_3)c_1 + d_1 d_2}{(c_2 + B_3)^2 + d_2^2}\right) + 2j\pi\right], \quad j = 1, 2, 3, \ldots.$$

Let $s(\tau) = \varrho(\tau) + i\omega(\tau)$ *be the root of Eq. (2.10) such that when* $\tau = \tau_j$ *satisfies* $\varrho(\tau) = 0, \omega(\tau) = \omega_+$. *Taking the derivative of Eq. (2.10) with respect to* τ:

$$\frac{ds}{dt} = \frac{e^{-s\tau}[sB_2(s) + sB_3]}{B_1'(s) + B_2'(s)e^{-s\tau} - \tau B_2(s)e^{-s\tau} - B_3\tau e^{-s\tau}}. \tag{2.13}$$

In (2.13), consider the numerator and denominator terms described as:

$$e^{-s\tau}\left[sB_2 + sB_3\right] = \varsigma_1 + i\varsigma_2,$$
$$B_1'(s) + B_2'(s)e^{-s\tau} - \tau B_2(s)e^{-s\tau} - B_3\tau e^{-s\tau} = \varsigma_3 + i\varsigma_4,$$

where

$$
\begin{aligned}
\varsigma_1 &= q_1\omega^{2\alpha+1}\left[\cos\frac{2\alpha\pi}{2}\sin(\omega\pi)-\sin\frac{2\alpha\pi}{2}\cos(\omega\pi)\right]\\
&+ q_2\omega^{\alpha+1}\left[\cos\frac{\alpha\pi}{2}\sin(\omega\pi)-\sin\frac{\alpha\pi}{2}\cos(\omega\pi)\right]+B_3\omega\sin(\omega\pi),\\
\varsigma_2 &= q_1\omega^{2\alpha+1}\left[\cos\frac{2\alpha\pi}{2}\sin(\omega\pi)+\sin\frac{2\alpha\pi}{2}\cos(\omega\pi)\right]\\
&+ q_2\omega^{\alpha+1}\left[\cos\frac{\alpha\pi}{2}\sin(\omega\pi)+\sin\frac{\alpha\pi}{2}\cos(\omega\pi)\right]+B_3\omega\sin(\omega\pi),\\
\varsigma_3 &= 3\alpha\omega^{3\alpha-1}\cos\frac{(3\alpha-1)\pi}{2}+2\alpha\omega^{2\alpha-1}p_1\cos\frac{(2\alpha-1)\pi}{2}+\alpha\omega^{\alpha-1}p_2\cos\frac{(\alpha-1)\pi}{2}\\
&+ 2\alpha\omega^{2\alpha-1}p_1\left[\cos\frac{(2\alpha-1)\pi}{2}\cos(\omega\tau)+\sin\frac{(2\alpha-1)\pi}{2}\sin(\omega\tau)\right]\\
&+ \alpha\omega^{\alpha-1}p_2\left[\cos\left(\frac{(\alpha-1)\pi}{2}\right)\cos(\omega\tau)+\sin\left(\frac{(\alpha-1)\pi}{2}\right)\sin(\omega\tau)\right]\\
&- \tau\omega^{2\alpha}p_1\left[\cos\frac{2\alpha\pi}{2}\cos(\omega\tau)+\sin\frac{2\alpha\pi}{2}\sin(\omega\tau)\right]\\
&- \tau\omega^{\alpha}p_2\left[\cos\frac{\alpha\pi}{2}\cos(\omega\tau)+\sin\frac{\alpha\pi}{2}\sin(\omega\tau)\right],\\
\varsigma_4 &= \alpha\omega^{3\alpha-1}\sin\frac{(3\alpha-1)\pi}{2}+2\alpha\omega^{2\alpha-1}p_1\sin\frac{(2\alpha-1)\pi}{2}+\alpha\omega^{\alpha-1}p_2\sin\frac{(\alpha-1)\pi}{2}\\
&+ 2\alpha\omega^{2\alpha-1}p_1\left[\sin\frac{(2\alpha-1)\pi}{2}\cos(\omega\tau)-\cos\frac{(\alpha-1)\pi}{2}\sin(\omega\tau)\right]\\
&+ \alpha\omega^{\alpha-1}p_2\left[\sin\frac{(\alpha-1)\pi}{2}\cos(\omega\tau)-\cos\frac{(\alpha-1)\pi}{2}\sin(\omega\tau)\right]\\
&- \tau\omega^{2\alpha}p_1\left[\sin\frac{2\alpha\pi}{2}\cos(\omega\tau)-\cos\frac{2\alpha\pi}{2}\sin(\omega\tau)\right]\\
&- \tau\omega^{\alpha}p_2\left[\sin\frac{\alpha\pi}{2}\cos(\omega\tau)-\cos\frac{\alpha\pi}{2}\sin(\omega\tau)\right].
\end{aligned}
\tag{2.14}
$$

Then from Eq. (2.13) we have:

$$
\begin{aligned}
\frac{ds}{d\tau}\Big|_{\tau=\tau_j,\omega=\omega_j} &= \frac{\varsigma_1+i\varsigma_2}{\varsigma_3+i\varsigma_4}=\frac{(\varsigma_1\varsigma_3+\varsigma_2\varsigma_4)+i(\varsigma_2\varsigma_3+\varsigma_2\varsigma_4)}{\varsigma_3^2+\varsigma_4^2},\\
Re\frac{ds}{d\tau}\Big|_{\tau=\tau_j,\omega=\omega_j} &= \frac{\varsigma_1\varsigma_3+\varsigma_2\varsigma_4}{\varsigma_3^2+\varsigma_4^2}\neq 0.
\end{aligned}
$$

Define $\tau^*=\min\{\tau_j\}$*, and based on the bifurcation theorem for functional differential equations [30], we have the following theorem.*

Theorem 2.2.4 ([54]). *Assume* $R_0>1$*. For system (2.1), the following results hold:*

1. *If* $v_i>0\,(i=1,2,3,4,5,6)$*, then the endemic equilibrium* E^* *is locally asymptotically stable for* $\tau\geq 0$*;*

2. *If* $v_i < 0\,(i = 1, 2, 3, 4, 5)$ *and* $v_6 > 0$*, then the endemic equilibrium* E^* *is locally asymptotically stable for* $\tau \in [0, \tau_0]$*;and*

3. *System (2.1) undergoes a Hopf bifurcation at the endemic equilibrium* E^* *when* $\tau = \tau_j\,(j = 1, 2, 3, ...)$.

Chapter 3

Numerical Simulations

In this section, several illustrative numerical examples are presented to confirm the theoretical results and to examine the dynamical behavior of system (2.1)(Look chapter 2). All the figures are plotted by using Matlab 2018a. From chapter 2 section 2 we can find that delay τ and fractional order α are the important factors, which affect the convergence speed of solutions. We select parameters as follows: $r = 0.15$, $K = 90$, $\beta = 0.007$, $\rho = 0.004$, $\mu = 0.035$, $\delta = 0.1$, $\sigma = 0.2$ with initial conditions $S_0 = 50$, $I_0 = 5$, $R_0 = 50$. We can calculate $R_0 = 1.880597015 > 1$. System (2.1) have three equilibria $E_1(0, 0, 0), E_2(90, 0, 0)$ and $E^*(47.85714286, 10.68849472, 54.81279344)$. We only discuss the stability of E^*.

1. $\tau = 0.8$ and $\alpha = 0.98$. We can calculate $\tau^* = 3.827693574$ from (2.12). Obviously,$\tau < \tau^* = 3.827693574$. From Theorem 2.2.1 we can obtain that E^* is locally asymptotically stable (see Fig. 1).

2. $\tau = 12$ and $\alpha = 0.98$. It is to see that $\tau > \tau^* = 3.827693574$. From Theorem 2.2.1 we can find that E^* is unstable and Hopf bifurcation occurs (see Fig. 2).

3. $\tau = 12$ and $\alpha = 0.90$. We can calculate $\tau^* = 12.32545463$ and $\tau < \tau^*$. Then E^* islocally asymptotically stable (see Fig. 3).

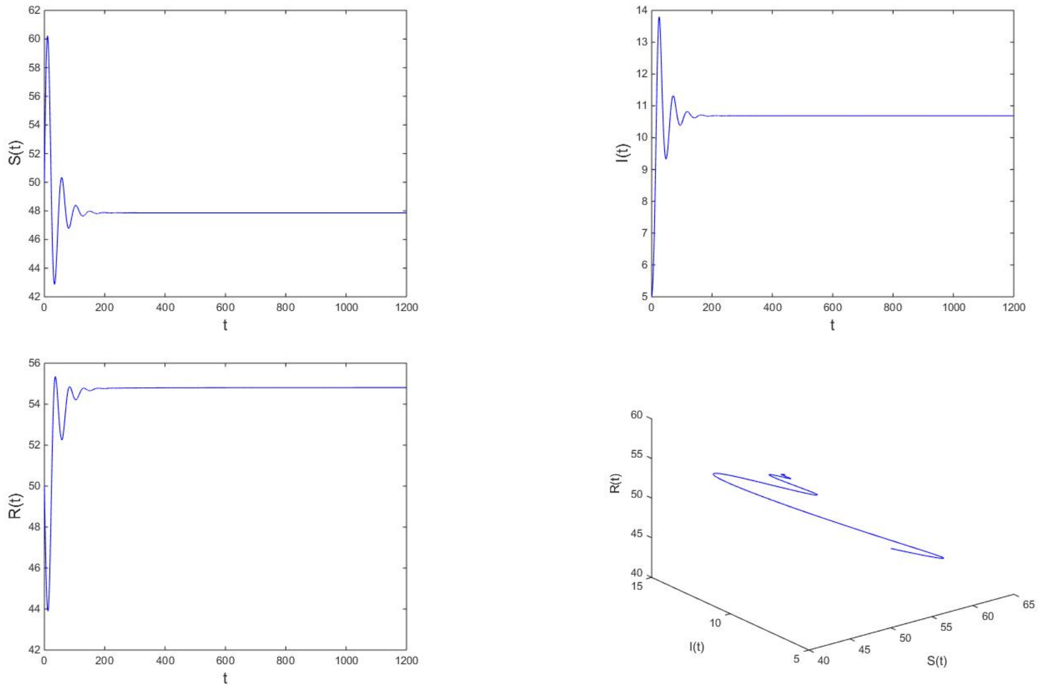

Figure 3.1: For the following parameter values $r = 0.15, K = 90, \beta = 0.007, \rho = 0.004, \mu = 0.035, \delta = 0.1, \sigma = 0.2, \tau = 0.8$ and $\alpha = 0.98$, the endemic equilibrium E^* of system (2.1) is locally asymptotically stable.

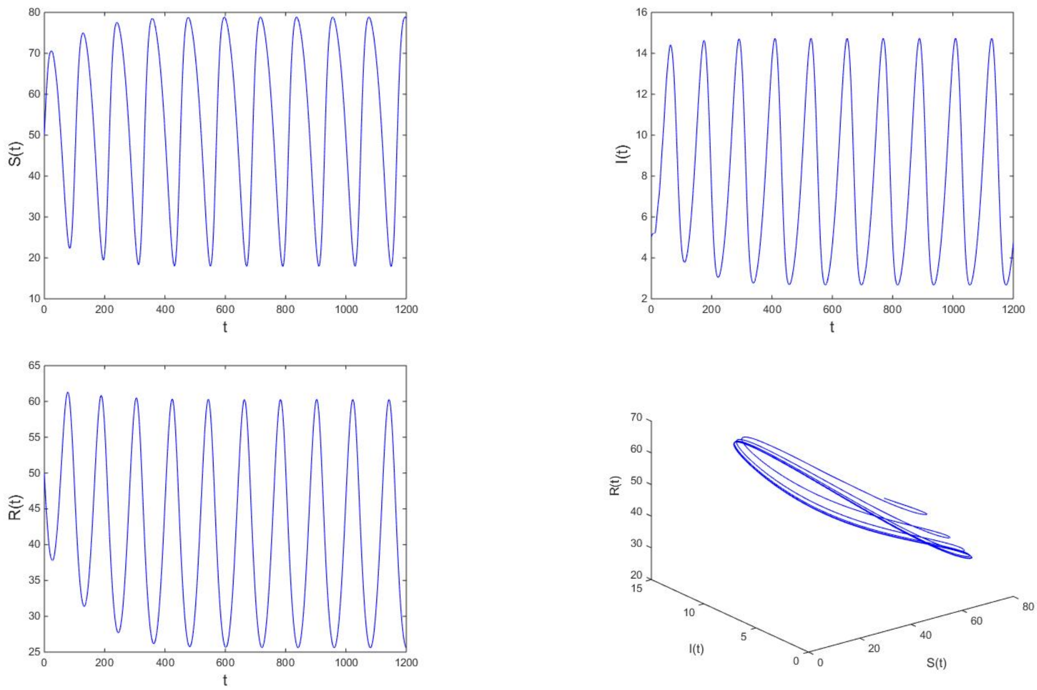

Figure 3.2: For the following parameter values $r = 0.15, K = 90, \beta = 0.007, \rho = 0.004, \mu = 0.035, \delta = 0.1, \sigma = 0.2, \tau = 12$ and $\alpha = 0.98$, the endemic equilibrium E^* of system (2.1) is unstable.

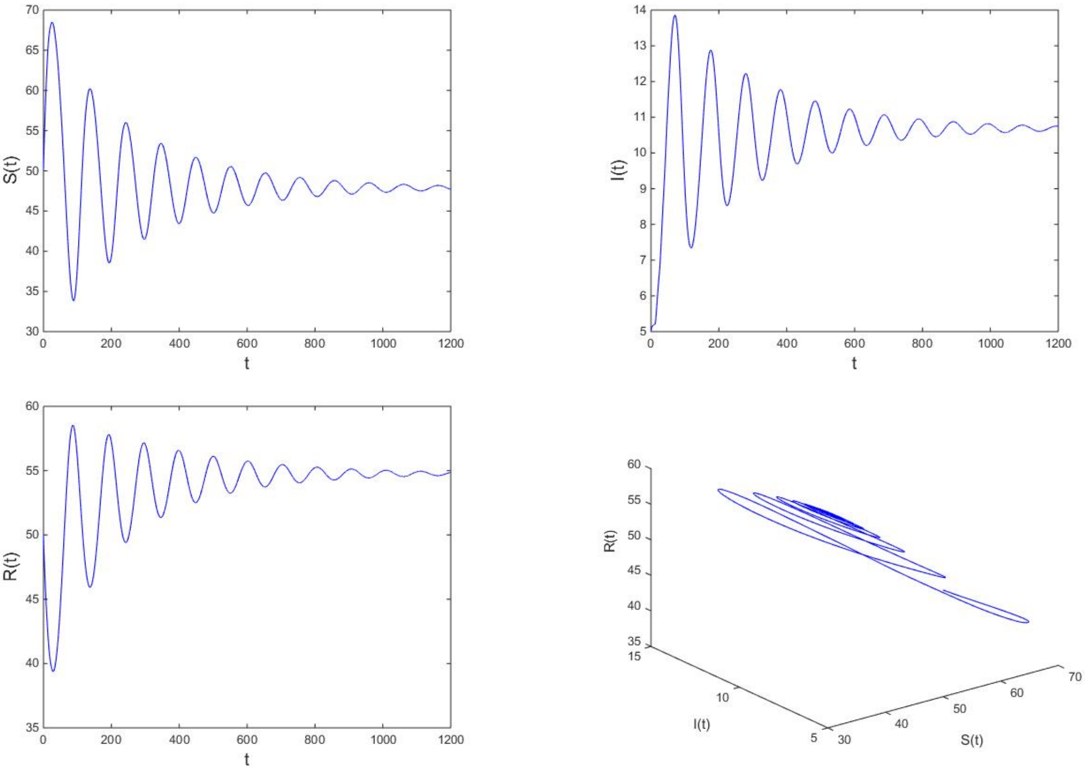

Figure 3.3: For the following parameter values $r = 0.15, K = 90, \beta = 0.007, \rho = 0.004, \mu = 0.035, \delta = 0.1, \sigma = 0.2, \tau = 12$ and $\alpha = 0.90$, the endemic equilibrium E^* of system (2.1) is locally asymptotically stable.

Conclusion

In this work, we studied a fractional-order SIRS epidemic model with delay and logistic growth of the susceptibles. The dynamical behavior of system (2.1) is studied. Local stability of the equilibria for system (2.1) and Hopf bifurcation are analyzed. The trivial equilibrium $E_1(0,0,0)$ of system (2.1) is always unstable. The disease-free equilibrium E_2 of system (2.1) is locally asymptotically stable for all $\tau \geq 0$ when $R_0 < 1$. When $R_0 > 1$ and $\tau = 0$, the endemic equilibrium is locally asymptotically stable. According to Theorem 2.2.1, when $R_0 > 1$ and the last two conditions of Theorem 2.2.4 satisfied, the stability of the endemic equilibrium changes at **Hopf** bifurcation point τ^*. Our findings illustrate that using the time delay τ as bifurcation parameter, one can conclude that the positive equilibrium loses its stability, and Hopf bifurcation occurs when time delay increases. The numerical simulations shown in Figs. 1 verified the effectiveness of the obtained theoretical results. From Fig. 2 we can speculate that the positive equilibrium loses its stability and Hopf bifurcation occurs when α is used as bifurcation parameter. It will be considered in future work.

Modeling of epidemic diseases by delayed fractional-order differential equations has more advantages and consistency rather than classical integer-order mathematical modeling. Our model takes into account several factors (time delay, Logistic growth, fractional order, etc.), which is more realistic. The model is thought to contribute valuable insight for public health, which is useful for some the prediction and control measures for some diseases.

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