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Two-dimensional wavelets transform: Basic properties and applications

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Dedication

To the Ever-Living, the Self-Sustaining, Whom neither slumber nor sleep overtakes, to the beloved Chosen One, may the best prayers and peace be upon him.

To the one who surrounded me with her tenderness and affection and made the path of knowledge easy for me, to the source of sacrifice and giving, the joy of my eyes "my mother", my God protect her.

To the one to whom I owe my success because he encouraged me to embark on the paths of knowledge "my father", my God prolong his life.

To my brothers, may God keep them in my life forever.

to my loyal friends.

To everyone who found a place in my

heart that words could never fully express

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Acknowledgements

Abstract

This graduation note focuses on the two-dimensional continuous wavelet transform (2D CWT) by using two different scale parameters (a_1, a_2), we are able to stretch or compress the wavelet independently along the horizontal and vertical directions. This anisotropic scaling offers more flexibility than isotropic scaling, which uses a single scale factor. In fact, it is a natural generalization of the idea of wavelet transform. This idea makes the 2D wavelet transform very natural in the study. So, mathematical and applied results have demonstrated the power of the wavelet transform in handling convolution operations, strengthening its role as an effective tool for accurate analysis in multiple fields.

Keywords:

- Wavelet transform

- Fourier transform

- Convolution product

- Mother wavelet.

Résumé

Ce mémoire se concentre sur la transformée en ondelettes continue bidimensionnelle (CWT 2D). Grâce à deux paramètres d'échelle différents (a_1, a_2), nous pouvons étirer ou compresser l'ondelette indépendamment dans les directions horizontale et verticale. Cette mise à l'échelle anisotrope offre plus de flexibilité que la mise à l'échelle isotrope, qui utilise un facteur d'échelle unique. Il s'agit en fait d'une généralisation naturelle du concept de transformée en ondelettes. Cette idée rend la transformée en ondelettes 2D très naturelle dans l'étude. Ainsi, les résultats mathématiques et appliqués ont démontré la puissance de la transformée en ondelettes dans la gestion des opérations de convolution, renforçant ainsi son rôle d'outil efficace pour des analyses précises dans de nombreux domaines.

Mots clés:

- Transformée en ondelettes
- Transformée de Fourier
- Produit de convolution
- Ondelette mère.

Abstract in arabic

تُركز هذه المذكرة على تحويل الموجات المستمر ثنائي الأبعاد (2 D CWT)، باستخدام معاملي مقياس مختلفين (a_2،a_1)، مما يُمكّننا من تمديد أو ضغط الموجات بشكل مستقل على طول الاتجاهين الأفقي والرأسي. يوفر هذا التحجيم متباين الخواص مرونة أكبر من التحجيم متساوي الخواص، الذي يستخدم عامل مقياس واحد. في الواقع، يُعد هذا تعميمًا طبيعيًا لفكرة تحويل الموجات. هذه الفكرة تجعل تحويل الموجات ثنائي الأبعاد أمرًا طبيعيًا للغاية في الدراسة. لذا، أظهرت النتائج الرياضية والتطبيقية قوة تحويل الموجات في معالجة عمليات الالتفاف، مما يعزز دوره كأداة فعالة للتحليل الدقيق في مجالات متعددة.

الكلمات المفتاحية:

- تحويل الموجات

- تحويل فورييه

- حاصل الالتفاف

- الموجة الأم.

List of Symbols

$L^2(R^2)$: The space of square integrable functions in $\mathbb{R}^2$

$\mathcal{F}[f(t_1, t_2)]$ or $\widehat{f}(\omega_1, \omega_2)$: The 2D Fourier transform of function $f(t_1, t_2)$

$\mathcal{F}^{-1}(\mathcal{F}[f(x, y)])$: The inversion formula of Fourier transform

$\overline{f(t_1, t_2)}$: the conjugate of $f(t_1, t_2)$.

$f * g$: The convolution of the function $f(t_1, t_2)$ and $g(t_1, t_2)$.

$W_f(a_1, a_2, b_1, b_2)$: The 2D continuous wavelet transform of function f.

$\psi(\frac{t_1-b_1}{a_1}, \frac{t_2-b_2}{a_2})$: The 2D mother wavelet.

General introduction

Joseph Fourier in 1822 published first work about Fourier transform, that is a mathematical process used to convert a mathematical function from the time domain to the frequency domain. Fourier transform measures the frequency component of a given function. This new function is often referred to as the frequency domain representation of the original function. The concept is similar to expressing a musical chord in terms of the notes that constitute it. In the original form of the Fourier transform, both the domain of the original function and the domain of the resulting function are continuous and unbounded. However, the Fourier transform has several drawbacks. Its main issue lies in its inefficiency with time-variant or non-stationary signals, as it does not provide information about frequency content over time. Additionally, it fails to preserve some important features of such signals, such as direction, skewness, and others, this limitation led to emergence of the wavelet transform.

The wavelet transform is one of the most powerful tools for analyzing time-varying signals because it provides an alternative to classical linear time–frequency representations with better time and frequency localization properties [8] by adapting the time-frequency window according to the signal's characteristics, It uses short windows at high frequencies and long windows at low frequencies, allowing for precise localization of both time and frequency components. This flexibility makes it particularly effective for analyzing non-stationary signals, where frequency components change over time.

The aim of the work is to enrich the theory of the wavelet transform and to provide new properties by studying several convolutions between two functions in the two-dimensional space.

The work is organized as:

The first chapter is dedicated to a brief study of the Fourier transform, covering its mathematical foundations. Its limitations are also addressed, justifying the transition to wavelet-based methods.

The second chapter introduces the two-dimensional wavelet transform, providing a clear mathematical definition and explaining its core structure. The mathematical properties of the transform in two dimensions are then discussed and rigorously proven.

The third chapter focuses on practical aspects by studying several convolutions between two functions in the two-dimensional space, using the convolution expression specific to the wavelet transform. Precise mathematical proofs are provided to support the effectiveness of this approach in analyzing two-dimensional signals.

CHAPTER 1

2-D Fourier transform

Contents

The Fourier transform (FT) introduced by Joseph Fourier in 1807, is one of the most valuable and widely-used integral transforms that converts a signal from time versus amplitude (and to frequency versus amplitude). Thus FT can be considered as the time-frequency representation tool in signal processing and analysis. It has evolved into a widely recognized discipline of harmonic analysis and has been successfully applied in diverse scientific and engineering pursuits.

Figure 1.1: Joseph Fourier (1768-1830).

1.1 Definition

The two-dimensional Fourier transform (2D Fourier transform) is a fundamental tool in signal processing, image processing, and various other fields of science and engineering. It extends the concept of the one-dimensional Fourier transform to two-dimensional signals or functions.

Definition 1.1 The 2−dimensional Fourier transform (2D FT), denoted by $\mathcal{F}\left[f(t_1,t_2)\right]$ of the function $f(t_1,t_2)\in L^2\left(\mathbb{R}^2\right)$ is defined as

$$\mathcal{F}\left[f(t_1,t_2)\right](\omega_1,\omega_2)=\widehat{f}(\omega_1,\omega_2)=\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(t_1,t_2)e^{-i(\omega_1 t_1+\omega_2 t_2)}dt_1dt_2. \tag{1.1}$$

where

- $\mathcal{F}\left[f(t_1,t_2)\right](\omega_1,\omega_2)$ is the transformed function in frequency domain,
- $f(t_1,t_2)$ is the original function in spatial domain,
- ω_1 and ω_2 are the frequency variables, representing spatial frequencies in the t_1- and t_2- directions, respectively,

and corresponding inversion formula is given by

$$\mathcal{F}^{-1}(f(\omega_1,\omega_2))(t_1,t_2) = \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathcal{F}(f(t_1,t_2))e^{i(\omega_1 t_1+\omega_2 t_2)}d\omega_1 d\omega_2. \tag{1.2}$$

1.2 Properties

The $2-D$ Fourier transform has the following properties (see [11]):

1. **Linearity:** Let $f(t_1,t_2)$ and $g(t_1,t_2)$ in $L^2(\mathbb{R}^2)$, then

$$\mathcal{F}[Af(t_1,t_2)+Bg(t_1,t_2)](\omega_1,\omega_2) = A\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2)+B\mathcal{F}[g(t_1,t_2)](\omega_1,\omega_2). \tag{1.3}$$

2. **Translation:** The $2-D$ Fourier transform of any function $f(x_1-k_1,x_2-k_2)$ is given by

$$\mathcal{F}[f(t_1-k_1,t_2-k_2)](\omega_1,\omega_2) = e^{-i(\omega_1 k_1+\omega_2 k_2)}\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2). \tag{1.4}$$

3. **Modulation:** The $2-D$ Fourier transform of any function $e^{-i(\nu_1^0 t_1+\nu_2^0 t_2)}f(t_1,t_2)$ is given by

$$\mathcal{F}[e^{-i(\nu_1^0 t_1+\nu_2^0 t_2)}f(t_1,t_2)](\omega_1,\omega_2) = \mathcal{F}[f(t_1,t_2)](\omega_1-\nu_1^0,\omega_2-\nu_2^0). \tag{1.5}$$

4. **Orthogonality relation:** The $2-D$ Fourier transform of the functions $f(t_1,t_2)$ and $g(u_1,u_2)$ in $L^2(\mathbb{R}^2)$ satisfies the following orthogonality relation

$$\langle\mathcal{F}[f(t_1,t_2)],\mathcal{F}[g(u_1,u_2]\rangle = \langle f(t_1,t_2),g(u_1,u_2)\rangle. \tag{1.6}$$

and we also have

$$\langle\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2),g(\omega_1,\omega_2)\rangle = \langle f(t_1,t_2),\mathcal{F}^{-1}[g](t_1,t_2)\rangle. \tag{1.7}$$

5. **Convolution:** Let $f(t_1,t_2)$ and $g(t_1,t_2)$ in $L^2\left(\mathbb{R}^2\right)$, then

$$\mathcal{F}[(f*g)](\omega_1,\omega_2) = \left(\sqrt{2\pi}\right)^2\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2)\mathcal{F}[g(t_1,t_2)](\omega_1,\omega_2), \tag{1.8}$$

where $f*g$ denotes the convolution of the function $f(t_1,t_2)$ and $g(t_1,t_2)$ and is given by

$$(f*g)(t_1,t_2) = \int_{\mathbb{R}^2} f(t_1,t_2)g(u_1-t_1,u_2-t_2)dt_1 dt_2,$$

6. **Multiplication:** Let $f(t_1,t_2)$ and $g(t_1,t_2)$ in $L^2(\mathbb{R}^2)$, then

$$\mathcal{F}[f(t_1,t_2)g(t_1,t_2)](\omega_1,\omega_2) = \frac{1}{\left(\sqrt{2\pi}\right)^2}\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2) * \mathcal{F}[g(t_1,t_2)](\omega_1,\omega_2). \tag{1.9}$$

7. **Differentiation:** Let $f(t_1,t_2) \in L^2(\mathbb{R}^2)$, then

$$\mathcal{F}\left[\nabla_{\mathbf{x}} f(t_1,t_2)\right](\omega_1,\omega_2) = i\,(\omega_1,\omega_2)\,\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2). \tag{1.10}$$

8. **Differentiation in frequency:** Let $f(t_1,t_2) \in L^2(\mathbb{R}^2)$, then

$$\mathcal{F}\left[(t_1,t_2)\,f(t_1,t_2)\right](\omega_1,\omega_2) = i\nabla_{\xi}\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2). \tag{1.11}$$

9. **Complex conjugation:** Let $f(t_1,t_2) \in L^2(\mathbb{R}^2)$, then

$$\mathcal{F}\left[\overline{f(t_1,t_2)}\right](\omega_1,\omega_2) = \overline{\mathcal{F}\left[f(t_1,t_2)\right](-\omega_1,-\omega_2)}. \tag{1.12}$$

10. **Duality:** Let $f(t_1,t_2) \in L^2(\mathbb{R}^2)$, then

$$\widehat{f}(t_1,t_2) = f(-\omega_1,-\omega_2). \tag{1.13}$$

11. **Parseval relation:** Let $f(t_1,t_2)$ and $g(t_1,t_2)$ in $L^2(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} f(t_1,t_2)\overline{g(t_1,t_2)}dt_1dt_2 = \int_{\mathbb{R}^2}(\mathcal{F}[f(t_1,t_2)](\omega_1,\omega_2))\left(\overline{\mathcal{F}[g(t_1,t_2)](\omega_1,\omega_2)}\right)d\omega_1 d\omega_2. \tag{1.14}$$

12. **Real and imaginary parts:**

$$\begin{aligned}(\mathcal{F}[\operatorname{Re} f(t_1,t_2)](\omega_1,\omega_2)) &= \frac{1}{2}(\widehat{f}(\omega_1,\omega_2) + \left(\overline{\widehat{f}(-\omega_1,-\omega_2)}\right). \qquad (1.15)\\ (\mathcal{F}[\operatorname{Im} f(t_1,t_2)](\omega_1,\omega_2)) &= \frac{1}{2i}(\widehat{f}(\omega_1,\omega_2) + \left(\overline{\widehat{f}(-\omega_1,-\omega_2)}\right).\end{aligned}$$

1.3 Fourier transform limitation

The Fourier transform, a cornerstone of signal processing, has revolutionized the analysis of signals in both time and frequency domains. However, like any tool, it comes with its own set of limitations. In addition to the inability to check continuity, Fourier transform suffers from fixed resolution, poor time-frequency localization, and limited time-frequency resolution trade-off. These limitations can hinder its effectiveness in analyzing signals with non-stationary or transient behavior. In the following, we'll explore these disadvantages of Fourier transform and highlight how Wavelet transform overcomes them.

- **Inability to check continuity**: One of the major disadvantages of Fourier transform is its inability to check the continuity of a signal. Fourier transform treats the entire signal as a whole and doesn't provide information about the local variations or discontinuities with in the signal. This limitation can be problematic in applications where identifying discontinuities or abrupt changes in audio processing, medical imaging, and fault detection.

- **Fixed resolution:** The Fourier transform provides frequency information with fixed resolution across the entire signal. This means that it may not capture localized changes in frequency content well, especially in signals with non-stationary or transient behavior.

- **Poor time-frequency localization:** Fourier analysis cannot provide simultaneous time and frequency localization. While it can accurately represent frequency content over time, it frequencies occur.

- **Limited time-frequency resolution Trade-off**: The trade-off between time and frequency resolution is fixed in the Fourier transform. Increasing time resolution reduces frequency resolution, and vice versa, limiting its ability to analyze signals with varying frequency content over time.

To overcome the lack of time information in the Fourier transform, a new analysis method is therefore introduced:

- Wavelet transform is a powerful mathematical tool that addresses many of the shortcomings of Fourier transform. Unlike Fourier transform, which decomposes a signal into sinusoidal components of different frequencies, Wavelet transform decomposes a signal into localized, transient components known as wavelets.

- Wavelet transform excels at analyzing localized features and capturing discontinuities in a signal. By using wavelets with different scales and positions, Wavelet transform can identify and characterize abrupt changes, edges, and other local variations within a signal.

- Wavelet transform provides variable resolution in both time and frequency domains. It achieves this by using wavelets, which are localized functions that can capture transient features and localize them in both time and frequency.

- Wavelet transform offers better time-frequency localization compared to Fourier transform. It can capture both transient and oscillatory behavior in signals with high precision, making it suitable for analyzing non-stationary signals.

- Wavelet transform allows for multiresolution analysis, meaning it can analyze signals at different scales or resolutions. This enables the detection of features at different levels of detail, making it useful for tasks such as denoising, feature extraction, and compression.

This chapter concludes by clarifying the need to search for new analysis tools that overcome these limitations, paving the way for presenting the wavelet transform as an effective alternative in the following chapters.

CHAPTER 2

2-D Continuous wavelet transform

Contents

2.1 General concepts

The concept of "wavelets" or "ondelettes" only began to appear in the literature in the early 1980's. This emerging concept represents a synthesis of various ideas originating from multiple disciplines, including:

- Mathematics (e.g., Calderón-Zygmund operators and Littlewood-Paley theory).

- Physics (e.g., coherent state formalism in quantum mechanics and the renormalization group).

- Engineering (e.g., quadratic mirror filters, sideband coding in signal processing, and pyramid algorithms in image processing).

In 1982, Jean Morlet, a French geophysical engineer, introduced the idea of the wavelet transform as a new mathematical tool for seismic signal analysis. His work was soon joined by Alex Grossmann, a French theoretical physicist, who quickly recognized the significance of the Morlet wavelet transform. Grossmann noted its similarity to coherent states in quantum mechanics and developed an exact inversion formula for the wavelet transform.

Figure. 2.1: Jean Morlet
(1931-2007).

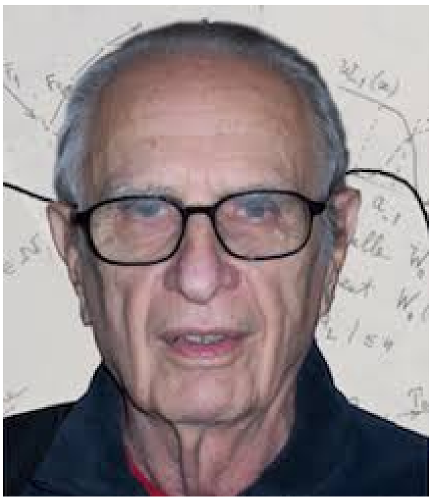

Figure. 2.2: Alex Grossmann
(1930-2019).

The wavelet transform is one of the most powerful tools used in signal and image processing.

Definition 2.1 The wavelet transform is a mathematical technique used to decompose a signal into scaled and translated versions of a simple, oscillating wave-like function called a wavelet. It is not only a local analysis, but also has variable temporal resolution.

We mainly distinguish two types of wavelet transforms are:

1. Continuous wavelet transform.
2. Discrete wavelet transform.

We will be interested in studying the continuous wavelet transform.

2.2 Mother wavelet

2.2.1 Definitions

Definition 2.2 A wavelet is an oscillating function (which explains the word "wave") with zero mean, possessing a certain degree of regularity and whose support is finite.

Definition 2.3 In mathematics, a wavelet ψ is a square summable function on Euclidean space $\mathbb{R} \times \mathbb{R}^+$, most often oscillating and with zero mean, chosen as a tool for multi-scale analysis and reconstruction.

Remark 2.1 Wavelets are a class of a functions used to localize a given function in both space and scaling.

A family of wavelets can be constructed from a function $\psi(t)$, sometimes know as a "mother wavelet", which is confined in a finite interval. We define a family $\psi_{a,b}$ of wavelets from the mother wavelet as

$$\forall t \in \mathbb{R}, \psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), (a,b) \in \mathbb{R}_*^+ \times \mathbb{R}, \tag{2.1}$$

where b is any real number and defines the shift and a is positive and defines the scale. The wavelet $\psi_{a,b}$ is simply the mother wavelet translated by b and dilated by a.

Consequently, when the scale a increase, the resolution increase. This means that the support of the non-zero part of the mother wavelet increase. In addition, it is usually assumed that following condition is satisfied

$$C_{\psi} = \int_{0}^{+\infty} \frac{\left\| \widehat{\psi}\left(v\right) \right\|^{2}}{v} dv < +\infty, \tag{2.2}$$

where $\widehat{\psi}$ is a Fourier transform of ψ. This admissibility condition impose for the functions of $L^2\left(\mathbb{R}\right)$ that $\psi\left(t\right)$ has zero mean.

2.2.2 Mother wavelet properties

- Vanishing Moments:

This property indicates how well the wavelet captures information about different frequency components. A wavelet with more vanishing moments can better distinguish between different frequencies in a signal

$$M_k = \int_{-\infty}^{+\infty} t^k \psi\left(t\right) dt = 0, 0 \leq k < N. \tag{2.3}$$

- Support Size:

Refers to the length of the wavelet function. A shorter support (compact support) provides better time resolution, while a longer support can offer better frequency resolution.

- Regularity:

A measure of the smoothness of the wavelet function. More regular wavelets (higher degree of differentiability) are often preferred for certain applications.

- Symmetry:

Symmetry in the wavelet function can be desirable for certain applications, especially in image processing, as it can lead to better reconstruction.

- Zero Mean:

A fundamental property of a mother wavelet is that its integral over all time is zero, meaning it has a zero average value.

- **Energy:** The total energy of the wavelet is usually normalized to one.

Remark 2.2

1. The mother wavelet is transformed (translated and scaled) to create a family of daughter wavelets. These transformations allow the wavelet to analyze signals at different locations and scales.
2. The choice of mother wavelet depends on the specific application and the characteristics of the signal being analyzed. Factors like the frequency content, time-domain behavior, and desired resolution need to be considered.

2.2.3 Examples of mother wavelets

1. **Haar wavelet:** A simple wavelet with a rectangular shape, known for its computational simplicity but limited smoothness.
2. **Daubechies wavelets:** A family of wavelets with varying degrees of regularity and compact support.
3. **Symlets wavelets:** Modified versions of Daubechies wavelets with improved symmetry properties.
4. **Coiflets wavelets:** Wavelets designed for good approximation properties and orthogonality.
5. **Morlet wavelets**: A complex wavelet often used in time-frequency analysis due to its band-pass characteristics.

2.3 Definition of 2-D Continuous wavelet transform

The two-dimensional continuous wavelet transform (2D CWT) is a natural extension of the one-dimensional CWT, with the translation parameter being a vector in $x - y$ plane. Dallard and Spedding [3] introduced the new 2D wavelet functions Halo and Arc and tried to generalize the notion of the Hardy functions for the isotropic wavelet in two-dimensional space. Antoine and Murenzi [1] introduced the 2D directional CWT with a rotation parameter to enhance the ability to detect data singularity along a particular direction. Farge et al. [5] first extend the Morlet wavelet to the two-dimensional space to analyze the homogeneous turbulence flows (see [10]).

The $2-D$ wavelet transform is a mathematical technique used to analyze and process $2-D$ signals, especially images, by breaking them down into components at different scales and orientations. It's commonly used in image compression, denoising, and feature extraction. It can be generally defined as follows:

$$W_f(\mathbf{a}, \mathbf{b}, \theta) = \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\psi}_{\mathbf{a},\mathbf{b},\theta}(\mathbf{t})\, d\mathbf{t}, \tag{2.4}$$

where

$$\overline{\psi}_{\mathbf{a},\mathbf{b},\theta}(\mathbf{t}) = \left(\sqrt{a_1 a_2}\right)^{-1} \psi\left(R_\theta^{-1}\left(\frac{\mathbf{t}-\mathbf{b}}{\mathbf{a}}\right)\right), \tag{2.5}$$

and

- $f(\mathbf{t}) \in L^2(\mathbb{R}^2)$ is a two-dimensional signal function,

- ψ is the wavelet mother function,

- $\mathbf{a} = (a_1, a_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ is the scale parameter vector (anisotropic scaling)or for ;

- $\mathbf{b} \in \mathbb{R}^2$ is the position vector of translation,

- $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta \in [0, 2\pi]$, is the standard rotation matrix of rotating angle θ.

Remark 2.3 The two-dimensional continuous wavelet transform (2D CWT) that incorporates both rotation through the angle θ and anisotropic scaling with two separate scale parameters (a_1, a_2). This formulation enables the analysis of directional features in two-dimensional signals or images with greater precision, particularly when the structures are oriented or non-isotropic. Furthermore, introducing a rotation parameter θ allows the wavelet to be oriented in various directions, making it especially useful for detecting and analyzing edges or patterns that are not aligned with the coordinate axes.

In this work, we focus on the two-dimensional continuous wavelet transform (2D CWT) by using two different scale parameters (a_1, a_2), we are able to stretch or compress the wavelet independently along the horizontal and vertical directions. This anisotropic scaling offers more flexibility than isotropic scaling, which uses a single scale factor. We have

Definition 2.4 The 2D continuous wavelet transform (2D CWT) of function $f(t_1, t_2)$ at scale $(a_1, a_2) \in \mathbb{R}^* \times \mathbb{R}^*$ and translational value $(a_1, a_2) \in \mathbb{R} \times \mathbb{R}$ is given by

$$W_f(a_1, a_2, b_1, b_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t_1, t_2)\overline{\psi}_{a_1,a_2,b_1,b_2}(t_1, t_2) dt_1 dt_2, \tag{2.6}$$

where

$$\psi_{a_1,a_2,b_1,b_2}(t_1, t_2) = \frac{1}{\sqrt{a_1 a_2}} \psi\left(\frac{t_1 - b_1}{a_1}, \frac{t_2 - b_2}{a_2}\right), \tag{2.7}$$

or for simply

$$W_f(a_1, a_2, b_1, b_2) = \frac{1}{\sqrt{a_1 a_2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t_1, t_2)\overline{\psi}\left(\frac{t_1 - b_1}{a_1}, \frac{t_2 - b_2}{a_2}\right) dt_1 dt_2, \tag{2.8}$$

where $\psi(.,.)$ is a $2-D$ continuous function in both the space domain and the frequency domain called the $2-D$ mother wavelet (for more detail see [10], [4]) and the inverse transform is given by the relation

$$f(t_1, t_2) = \frac{1}{C_\psi} \int\int_{\mathbb{R}_+^2} \int\int_{\mathbb{R}^2} W_f(a_1, a_2, b_1, b_2)\overline{\psi}\left(\frac{t_1 - b_1}{a_1}, \frac{t_2 - b_2}{a_2}\right) \frac{1}{a_1^2 a_2^2} da_1 da_2 db_1 db_2, \tag{2.9}$$

where C_ψ is a constant defined by

$$C_\psi = \int_{\mathbb{R}^2} \frac{\left|\widehat{\psi}(\omega_1, \omega_2)\right|^2}{\omega_1 \omega_2} d\omega_1 d\omega_2 < +\infty. \tag{2.10}$$

This inequality is often referred to as admissibility condition, which also implies that $\widehat{\psi}(\upsilon_1, \upsilon_2) = 0$ and $\int_{\mathbb{R}^2} \psi(\omega_1, \omega_2) d\omega_1 d\omega_2 = 0$.

2.4 Properties of 2-D continuous wavelet transform

In this section, the main properties of the wavelet transform are reviewed, along with mathematical proofs proving its effectiveness, confirming its position as an important analytical tool in two-dimensional signal processing. The properties of the 2D continous wavelet transform (2D WCT) are identical to those in one dimension, including energy conservation.

Theorem 2.1 (Linearity) *Let $f(t_1,t_2), g(t_1,t_2) \in L(\mathbb{R}^2)$, and $\alpha, \beta \in \mathbb{C}^2$. For $(a_1,a_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $(b_1,b_2) \in \mathbb{R}^2$, we have*

$$W_{\alpha f+\beta g}(a_1,a_2,b_1,b_2) = \alpha W_f(a_1,a_2,b_1,b_2) + \beta W_g(a_1,a_2,b_1,b_2). \tag{2.11}$$

Proof. We have

$$\begin{aligned}
W_{\alpha f_1+\beta f_2}(a_1,a_2,b_1,b_2) &= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\alpha f(t_1,t_2)+\beta g(t_1,t_2)] \\
&\quad \times\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)dt_1dt_2 \\
&= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\alpha f(t_1,t_2)\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)\right. \\
&\quad \left.+\,\beta g(t_1,t_2)\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)\right]dt_1dt_2 \\
&= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\alpha f(t_1,t_2)\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)dt_1dt_2 \\
&\quad +\frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\beta g(t_1,t_2)\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)dt_1dt_2 \\
&= \alpha[\frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t_1,t_2)\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)dt_1dt_2] \\
&\quad +\beta[\frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(t_1,t_2)\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)dt_1dt_2] \\
&= \alpha W_f(a_1,a_2,b_1,b_2)+\beta W_g(a_1,a_2,b_1,b_2).
\end{aligned}$$

■

Theorem 2.2 (Parseval) *Let ψ an analyzing wavelet and f be a function of $L^2(\mathbb{R}^2)$ then*

$$\int\int_{\mathbb{R}^2}|f(t_1,t_2)|^2\,dt_1dt_2 = \frac{1}{C_\psi}\int\int_{\mathbb{R}^2}\int\int_{\mathbb{R}_+^2}|W_f(a_1,a_2,b_1,b_2)|^2\,\frac{1}{a_1^2a_2^2}da_1da_2db_1db_2, \tag{2.12}$$

where $W_f(a_1,a_2,b_1,b_2)$ is defined by (2.6) and C_ψ by (2.10).

Proof. From Parseval theory, we get

$$W_f(a_1,a_2,b_1,b_2)=\int\int_{\mathbb{R}^2}\widehat{f}(\omega_1,\omega_2)\overline{\widehat{\psi}_{a_1,a_2,b_1,b_2}(\omega_1,\omega_2)}e^{i(b_1\omega_1+b_2\omega_2)}d\omega_1 d\omega_2,$$

where

$$\widehat{\psi}_{a_1,a_2,b_1,b_2}(\omega_1,\omega_2)=\sqrt{a_1a_2}\psi(a_1\omega_1,a_2\omega_2),$$

thus

$$W_f(a_1,a_2,b_1,b_2)=\int\int_{\mathbb{R}^2}\widehat{f}(\omega_1,\omega_2)\sqrt{a_1a_2}\psi(a_1\omega_1,a_2\omega_2)e^{i(b_1\omega_1+b_2\omega_2)}d\omega_1 d\omega_2,$$

and we have

$$E=\int\int_{\mathbb{R}^2}\int\int_{\mathbb{R}^2_+}|W_f(a_1,a_2,b_1,b_2)|^2\frac{1}{a_1^2a_2^2}da_1da_2db_1db_2.$$

For fixed a_1, a_2, integrate over b_1, b_2

$$\int\int_{\mathbb{R}^2}|W_f(a_1,a_2,b_1,b_2)|^2 db_1db_2=\int\int_{\mathbb{R}^2}\left|\widehat{f}(\omega_1,\omega_2)\right|^2 a_1a_2\left|\widehat{\psi}(a_1\omega_1,a_2\omega_2)\right|^2 d\omega_1d\omega_2.$$

Now integrate over scaling we get

$$\begin{aligned}E &= \int\int_{\mathbb{R}^2_+}\frac{1}{a_1^2a_2^2}\left(\int\int_{\mathbb{R}^2}\left|\widehat{f}(\omega_1,\omega_2)\right|^2 a_1a_2\left|\widehat{\psi}(a_1\omega_1,a_2\omega_2)\right|^2 d\omega_1d\omega_2\right)da_1da_2\\ &= \int\int_{\mathbb{R}^2}\left|\widehat{f}(\omega_1,\omega_2)\right|^2\left(\int\int_{\mathbb{R}^2_+}\int\int_{\mathbb{R}^2}\frac{\left|\widehat{\psi}(a_1\omega_1,a_2\omega_2)\right|^2}{a_1a_2}da_1da_2\right)d\omega_1d\omega_2,\end{aligned}$$

we change the variable: $u=a_1\omega_1, v=a_2\omega_2 \Longrightarrow da_1=\frac{du}{\omega_1}, da_2=\frac{dv}{\omega_2}$

$$\begin{aligned}E &= \int\int_{\mathbb{R}^2}\left|\widehat{f}(\omega_1,\omega_2)\right|^2\left(\int\int_{\mathbb{R}^2}\frac{\left|\widehat{\psi}(u,v)\right|^2}{|uv|}dudv\right)d\omega_1d\omega_2\\ &= C_\psi.\left\|\widehat{f}\right\|^2\\ &= C_\psi.\left\|f\right\|^2.\end{aligned}$$

We have

$$\|f\|^2=\int\int_{\mathbb{R}^2}|f(t_1,t_2)|^2dt_1dt_2,$$

thus

$$\int\int_{\mathbb{R}^2}|f(t_1,t_2)|^2dt_1dt_2=\frac{1}{C_\psi}\int\int_{\mathbb{R}^2}\int\int_{\mathbb{R}^2_+}|W_f(a_1,a_2,b_1,b_2)|^2\frac{1}{a_1^2a_2^2}da_1da_2db_1db_2.$$

■

Theorem 2.3 (Translation) *Let* $f(t_1,t_2) \in L(\mathbb{R}^2)$ *and* $(a_1,a_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *and* $(b_1,b_2) \in \mathbb{R}^2$. *The wavelet transform is invariant under translation* (x_0,y_0) *and we have*

$$W_{f(t_1-x_0,t_2-y_0)}(a_1,a_2,b_1,b_2) = W_f(a_1,a_2,b_1-x_0,b_2-y_0). \tag{2.13}$$

Proof. We have

$$\begin{aligned} W_{f(t_1-x_0,t_2-y_0)}(a_1,a_2,b_1,b_2) &= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_2(t_1-x_0,t_2-y_0) \\ &\quad \times\overline{\psi}\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right)dt_1dt_2. \end{aligned}$$

To simplify the integration, we change the variables, to set

$$\begin{aligned} u &= t_1-x_0 \Longrightarrow t_1 = u+x_0 \Longrightarrow dt_1 = du \\ v &= t_2-y_0 \Longrightarrow t_2 = v+y_0 \Longrightarrow dt_2 = dv \end{aligned}$$

thus

$$\begin{aligned} W_{f(t_1-x_0,t_2-y_0)}(a_1,a_2,b_1,b_2) &= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u,v) \\ &\quad \times\overline{\psi}\left(\frac{(u+x_0)-b_1}{a_1},\frac{(v+y_0)-b_2}{a_2}\right)dudv \\ &= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u,v) \\ &\quad \times\overline{\psi}\left(\frac{u-(b_1-x_0)}{a_1},\frac{v-(b_2-y_0)}{a_2}\right)dudv \\ &= W_f(a_1,a_2,b_1-x_0,b_2-y_0). \end{aligned}$$

■

Theorem 2.4 (Dilatation) *Let* $f(t_1,t_2) \in L(\mathbb{R}^2)$ *and* $(a_1,a_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *and* $(b_1,b_2) \in \mathbb{R}^2$. *If the function* f *is scaled in space with* $(\delta_{\mathbf{k}}f)(t_1,t_2) = \frac{1}{\sqrt{k_1k_2}}f\left(\frac{t_1}{k_1},\frac{t_2}{k_2}\right)$ *where* $\mathbf{k} = (k_1,k_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. *Then*

$$W(\delta_{\mathbf{k}}f)(a_1,a_2,b_1,b_2) = W_f\left(\frac{a_1}{k_1},\frac{a_2}{k_2},\frac{b_1}{k_1},\frac{b_2}{k_2}\right). \tag{2.14}$$

Proof. We have

$$(\delta_{\mathbf{k}}f)(t_1,t_2) = \frac{1}{\sqrt{k_1k_2}}f\left(\frac{t_1}{k_1},\frac{t_2}{k_2}\right),$$

and we want to find the wavelet transform scaling

$$\begin{aligned} W(\delta_{\mathbf{k}} f)(a_1,a_2,b_1,b_2) &= W_{\delta_{\mathbf{k}} f}(a_1,a_2,b_1,b_2) \\ &= \frac{1}{\sqrt{a_1 a_2}} \int\int \frac{1}{\sqrt{k_1 k_2}} f\left(\frac{t_1}{k_1},\frac{t_2}{k_2}\right) \overline{\psi}\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right) dt_1 dt_2 \\ &= \frac{1}{\sqrt{k_1 k_2}} \frac{1}{\sqrt{a_1 a_2}} \int\int f\left(\frac{t_1}{k_1},\frac{t_2}{k_2}\right) \overline{\psi}\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2}\right) dt_1 dt_2, \end{aligned}$$

we change the variable, we put

$$\begin{aligned} u &= \frac{t_1}{k_1} \longrightarrow t_1 = u k_1 \longrightarrow dt_1 = k_1 du \\ v &= \frac{t_2}{k_2} \longrightarrow t_2 = v k_2 \longrightarrow dt_2 = k_2 du \end{aligned}$$

thus

$$\begin{aligned} W(\delta_{\mathbf{k}} f)(a_1,a_2,b_1,b_2) &= \frac{1}{\sqrt{k_1 k_2}} \frac{1}{\sqrt{a_1 a_2}} \int\int f(u,v)\overline{\psi}\left(\frac{uk_1-b_1}{a_1},\frac{vk_1-b_2}{a_2}\right) k_1 k_2 du dv \\ &= \sqrt{k_1 k_2} \frac{1}{\sqrt{a_1 a_2}} \int\int f(u,v)\overline{\psi}\left(\frac{uk_1-b_1}{a_1},\frac{vk_1-b_2}{a_2}\right) du dv. \end{aligned}$$

Using the definition of scaled wavelet

$$\psi\left(\frac{uk_1-b_1}{a_1},\frac{vk_1-b_2}{a_2}\right) = \psi\left(\frac{u-\frac{b_1}{k_1}}{\frac{a_1}{k_1}},\frac{v-\frac{b_2}{k_2}}{\frac{a_2}{k_2}}\right),$$

so

$$\begin{aligned} W(\delta_{\mathbf{k}} f)(a_1,a_2,b_1,b_2) &= \frac{\sqrt{k_1 k_2}}{\sqrt{a_1 a_2}} \int\int f(u,v)\overline{\psi}\left(\frac{u-\frac{b_1}{k_1}}{\frac{a_1}{k_1}},\frac{v-\frac{b_2}{k_2}}{\frac{a_2}{k_2}}\right) du dv \\ &= \frac{1}{\sqrt{\frac{a_1 a_2}{k_1 k_2}}} \int\int f(u,v)\overline{\psi}\left(\frac{u-\frac{b_1}{k_1}}{\frac{a_1}{k_1}},\frac{v-\frac{b_2}{k_2}}{\frac{a_2}{k_2}}\right) du dv \\ &= W_f\left(\frac{a_1}{k_1},\frac{a_2}{k_2},\frac{b_1}{k_1},\frac{b_2}{k_2}\right). \end{aligned}$$

■

CHAPTER 3

2-D Wavelet transform and convolution

Contents

In this chapter we study one of the most practical applications of the wavelet transform by using convolution operations between functions in two dimensions.

3.1 The 2−dimensional generalized translation and convolution

Consider 2−dimensional general integral transform

$$\widehat{f}(x_1,x_2) = (I_\psi f)(x_1,x_2) = \int_a^b\int_a^b f(t_1,t_2)\psi(x_1,x_2,t_1,t_2)dt_1dt_2, (x_1,x_2)\in\mathbb{C}^2, \quad (3.1)$$

where a may take value 0 or $-\infty$ and b may be $+\infty$, we assume that it is possible to recover $f(y_1,y_2)$ from its transform $(I\psi f)(x_1,x_2)$ by means of the inversion formula

$$f(t_1,t_2) = I_\psi^{-1}(I_\psi f)(t_1,t_2). \quad (3.2)$$

We know that the integral transformation converts the convolution between two functions into the product of their transforms. Hence, we define $2D$ convolution of two functions f and g by means of the relation

$$(f * g)(z_1, z_2) = I_\psi^{-1}[(I_\psi f)(I_\psi g)](z_1, z_2), \tag{3.3}$$

from (3.2), (3.3) can expressed as

$$\begin{aligned}(f * g)(z_1, z_2) &= I_\psi^{-1}\Big[\int_a^b\int_a^b f(x_1,x_2)\psi(t_1,t_2,x_1,x_2)dx_1dx_2 \\ &\quad \int_a^b\int_a^b g(y_1,y_2)\psi(t_1,t_2,y_1,y_2)dy_1dy_2\Big](z_1,z_2) \\ &= I_\psi^{-1}\Big[\int_a^b\int_a^b f(x_1,x_2)\Big(\int_a^b\int_a^b g(y_1,y_2) \\ &\quad \psi(t_1,t_2,x_1,x_2)\psi(t_1,t_2,y_1,y_2)dy_1dy_2\Big)dx_1dx_2\Big](z_1,z_2).\end{aligned}$$

Under different condition of f, g and ψ, we can change the order of integration, so, for example $f, g \in L^1(a,b)^2$ and $\psi \in L^\infty(a,b)^2$.

Let assume that there exists a basic $2-$dimensional generalized function $D(x_1,x_2,y_1,y_2,z_1,z_2)$ such that

$$\psi(t_1,t_2,x_1,x_2)\psi(t_1,t_2,y_1,y_2) = \int_a^b\int_a^b D(x_1,x_2,y_1,y_2,z_1,z_2)\psi(t_1,t_2,z_1,z_2)dz_1dz_2. \tag{3.4}$$

In some cases $D(x_1,x_2,y_1,y_2,z_1,z_2)$ is a $2D$ measure and in other cases it turns out to be a $2D$ distribution. The integral exists if $\psi \in L^\infty(a,b)^2$ and

$$\int_a^b\int_a^b |D(x_1,x_2,y_1,y_2,z_1,z_2)|\, dz_1dz_2 \leq C. \tag{3.5}$$

Then

$$\begin{aligned}(f * g)(z_1, z_2) &= I_\psi^{-1}\Big[\int_a^b\int_a^b f(x_1,x_2)\Big(\int_a^b\int_a^b g(y_1,y_2)dy_1dy_2 \\ &\quad \int_a^b\int_a^b D(x_1,x_2,y_1,y_2,u_1,u_2)\psi(t_1,t_2,u_1,u_2)dx_1dx_2\Big)\Big](z_1,z_2),\end{aligned}$$

we can change the order of integration if $f, g \in L^1(a,b)^2$and $\psi \in L^\infty(a,b)^2$ and if (3.5) holds

$$\begin{aligned}(f * g)(z_1, z_2) &= I_\psi^{-1}\Big[\int_a^b\int_a^b \psi(t_1,t_2,u_1,u_2)\Big(\int_a^b\int_a^b\Big(\int_a^b\int_a^b D(x_1,x_2,y_1,y_2,u_1,u_2) \\ &\quad f(x_1,x_2)dx_1dx_2\Big)g(y_1,y_2)dy_1dy_2\Big)du_1du_2\Big](z_1,z_2).\end{aligned} \tag{3.6}$$

Now define the 2−dimension generalized translation τ_{z_1,z_2} by

$$\tau_{z_1,z_2}f(y_1,y_2) := f^*(z_1,z_2,y_1,y_2) := \int_a^b\int_a^b D(x_1,x_2,y_1,y_2,z_1,z_2)f(x_1,x_2)dx_1dx_2. \tag{3.7}$$

Then, by (3.5), we have

$$\int_a^b\int_a^b |\tau_{z_1,z_2}f(y_1,y_2)|\, dz_1dz_2 \leq C\,\|f\|_{L^1}\,, \tag{3.8}$$

and

$$\begin{aligned}(f*g)(z_1,z_2) &= I_\psi^{-1}\Big[\int_a^b\int_a^b \psi(t_1,t_2,u_1,u_2)(\int_a^b\int_a^b(\tau_{z_1,z_2}f)(y_1,y_2)\\ &\quad g(y_1,y_2)dy_1dy_2)du_1du_2](z_1,z_2)\\ &= I_\psi^{-1}[I_\psi(\tau_{z_1,z_2}f)(y_1,y_2)g(y_1,y_2)dy_1dy_2)\,(t_1,t_2)](z_1,z_2).\end{aligned}$$

This yields the 2−dimensional generalized convolution

$$(f*g)(z_1,z_2) := \int_a^b\int_a^b(\tau_{z_1,z_2}f)(y_1,y_2)g(y_1,y_2)dy_1yd_2. \tag{3.9}$$

This definition makes sense under different conditions on f and g.

Remark 3.1 1. If $\psi\in L^\infty(a,b)^2$, (3.5) holds, $f\in L^1(a,b)^2$ and $g\in L^\infty(a,b)^2$, then $f*g\in L^\infty(a,b)^2$.

2. If $\psi\in L^\infty(a,b)^2$, (3.5) holds and $f,g\in L^1(a,b)^2$, then $f*g\in L^1(a,b)$.

Properties

From (3.4) it follows that $D(x_1,x_2,y_1,y_2,z_1,z_2)$ is given by

$$D(x_1,x_2,y_1,y_2,z_1,z_2) = I_\psi^{-1}[\psi(t_1,t_2,x_1,x_2)\psi(t_1,t_2,y_1,y_2)](z_1,z_2), \tag{3.10}$$

where $x_i,y_i,z_i\in[a,b]\,,i=1,2$. Clearly, $D(x_1,x_2,y_1,y_2,z_1,z_2)$ is symmetric in (x_1,x_2) and (y_1,y_2), but in general, not symmetric in all the variables $(x_1,x_2)\,,(y_1,y_2)\,,(z_1,z_2)$. Setting $(t_1,t_2)=(t_{0_1},t_{0_2})$ a constant vector, (3.4) yields

$$\int_a^b\int_a^b D'(x_1,x_2,y_1,y_2,z_1,z_2)dz_1dz_2 = 1, \tag{3.11}$$

where

$$D'(x_1,x_2,y_1,y_2,z_1,z_2) = \frac{D(x_1,x_2,y_1,y_2,z_1,z_2)\psi(t_{0_1},t_{0_2},z_1,z_2)}{\psi(t_{0_1},t_{0_2},x_1,x_2)\psi(t_{0_1},t_{0_2},y_1,y_2)}, \tag{3.12}$$

provided $\psi(t_{0_1}, t_{0_2}, x_1, x_2)\psi(t_{0_1}, t_{0_2}, y_1, y_2) \neq 0$. In some of the cases specific forms of $D(x_1, x_2, y_1, y_2, z_1, z_2)$ are available in the literature and in some cases $D(x_1, x_2, y_1, y_2, z_1, z_2)$ is know to be symmetric in all the variables.

Now, an important property of the convolution $*$ is obtained from (3.9), which also follows from (3.3). Indeed, from (3.9) for $f, g \in L^1(a, b)$, we have

$$\begin{aligned} I_\psi(f * g)(t_1, t_2) &= \int_a^b \int_a^b \psi(t_1, t_2, z_1, z_2) \left(\int_a^b \int_a^b (\tau_{z_1, z_2} f)(y_1, y_2) g(y_1, y_2) dy_1 dy_2 \right) dz_1 dz_2 \\ &= \int_a^b \int_a^b \psi(t_1, t_2, z_1, z_2) \left(\int_a^b \int_a^b \left(\int_a^b \int_a^b D(x_1, x_2, y_1, y_2, z_1, z_2) \right.\right. \\ &\quad f(x_1, x_2) dx_1 dx_2 \big) \, g(y_1, y_2) dy_1 dy_2 \big) dz_1 dz_2 \\ &= \int_a^b \int_a^b f(x_1, x_2) \left(\int_a^b \int_a^b g(y_1, y_2) dy_1 dy_2 \right. \\ &\quad \left. \int_a^b \int_a^b \psi(t_1, t_2, z_1, z_2) D(x_1, x_2, y_1, y_2, z_1, z_2) dz_1 dz_2 \right) dx_1 dx_2 \\ &= \int_a^b \int_a^b f(x_1, x_2) \left(\int_a^b \int_a^b g(y_1, y_2) dy_1 dy_2 \right. \\ &\quad \psi(t_1, t_2, x_1, x_2) \psi(t_1, t_2, y_1, y_2) \big) dx_1 dx_2 \\ &= \int_a^b \int_a^b f(x_1, x_2) \psi(t_1, t_2, x_1, x_2) dx_1 dx_2 \\ &\quad \int_a^b \int_a^b g(y_1, y_2) \psi(t_1, t_2, y_1, y_2) dy_1 dy_2 \\ &= (I_\psi f)(t_1, t_2)(I_\psi g)(t_1, t_2). \end{aligned} \tag{3.13}$$

3.2 Special cases

In this section we discuss various known and unknown special cases of (3.1), (3.7), (3.9) and (3.10). For each case we give appropriate $D(x_1, x_2, y_1, y_2, z_1, z_2)$ and then convolution can be defined using (3.7) and (3.9).

3.2.1 Fourier convolution

The Fourier transform of $f \in L^1(-\infty, +\infty)^2$ is defined, for $\psi(x_1, x_2, t_1, t_2) = e^{-i(x_1 t_1 + x_2 t_2)}$, $(x_1, x_2), (t_1, t_2) \in \mathbb{R}^2$ by

$$\mathcal{F}(f)(x_1, x_2) = \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t_1, t_2) e^{-i(x_1 t_1 + x_2 t_2)} dt_1 dt_2,$$

and inverse Fourier transform is given by

$$f(t_1, t_2) = \mathcal{F}^{-1}(\mathcal{F}f)(t_1, t_2).$$

Similarly by formula (7.3.3) in Pathak [7], we have

$$\begin{aligned} D(x_1, x_2, y_1, y_2, z_1, z_2) &= \mathcal{F}^{-1}[e^{-i(x_1t_1+x_2t_2)}e^{-i(y_1t_1+y_2t_2)}](z_1, z_2) \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(x_1t_1+x_2t_2)-i(y_1t_1+y_2t_2)}e^{i(z_1t_1+z_2t_2)}dt_1dt_2 \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-ix_1t_1-ix_2t_2-iy_1t_1-iy_2t_2}e^{i(z_1t_1+z_2t_2)}dt_1dt_2 \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(x_1+y_1)t_1-i(x_2+y_2)t_2}e^{i(z_1t_1+z_2t_2)}dt_1dt_2 \\ &= \delta(z_1 - y_1 - x_1, z_2 - y_2 - x_2). \end{aligned}$$

Hence,

$$\begin{aligned} (\tau_{z_1,z_2}f)(y_1, y_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(z_1 - y_1 - x_1, z_2 - y_2 - x_2)f(x_1, x_2)dx_1dx_2 \\ &= f(z_1 - y_1, z_2 - y_2). \end{aligned}$$

Therefore,

$$(f * g)(z_1, z_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(z_1 - y_1, z_2 - y_2)g(y_1, y_2)dy_1dy_2.$$

Hence

$$\mathcal{F}(f * g)(t_1, t_2) = \mathcal{F}(f)(t_1, t_2)\mathcal{F}(g)(t_1, t_2).$$

3.2.2 Laplace convolution

The Laplace transform of f on $\mathbb{R}_+^2$ is defined by

$$L[f](s_1, s_2) = \int_0^{+\infty}\int_0^{+\infty} f(t_1, t_2)e^{-(s_1t_1+s_2t_2)}dt_1dt_2, (s_1, s_2) \in \mathbb{C}^2,$$

provided the integral is convergent. In this case

$$\psi(s_1, s_2, t_1, t_2) = e^{-(s_1t_1+s_2t_2)},$$

$$\begin{aligned} D(x_1, x_2, y_1, y_2, z_1, z_2) &= L^{-1}[e^{-s_1(x_1+y_1)-s_2(x_2+y_2)}](z_1, z_2) \\ &= \delta(z_1 - y_1 - x_1, z_2 - y_2 - x_2). \end{aligned}$$

Therefore,

$$\begin{aligned}(\tau_{z_1,z_2}f)(y_1,y_2) &= \int_0^{+\infty}\int_0^{+\infty}\delta(z_1-x_1-y_1,z_2-x_2-y_2)f(y_1,y_2)dy_1dy_2\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(z_1-x_1-y_1,z_2-x_2-y_2)H(y_1,y_2)f(y_1,y_2)dy_1dy_2\\ &= H(z_1-x_1,z_2-x_2)f(z_1-x_1,z_2-x_2),\end{aligned}$$

where $H(y_1, y_2)$ is the 2−dimensional Heaviside unit function. Hence

$$\begin{aligned}(f*g)(z_1,z_2) &= \int_0^{+\infty}\int_0^{+\infty}(\tau_{z_1,z_2}f)(x_1,x_2)g(x_1,x_2)dx_1dx_2\\ &= \int_0^{+\infty}\int_0^{+\infty}H(z_1-x_1,z_2-x_2)f(z_1-x_1,z_2-x_2)g(x_1,x_2)dx_1dx_2\\ &= \int_0^{z}\int_0^{z}f(z_1-x_1,z_2-x_2)g(x_1,x_2)dx_1dx_2.\end{aligned}$$

Clearly,

$$L(f*g)(s_1,s_2) = L(f)(s_1,s_2)L(g)(s_1,s_2).$$

3.2.3 Mellin convolution

The Mellin transform of f defined on $(0,+\infty)^2$ is given by

$$(Mf)(s_1,s_2) = \int_0^{+\infty}\int_0^{+\infty}t_1^{s_1-1}t_2^{s_2-1}f(t_1,t_2)dt_1dt_2, (s_1,s_2)\in\mathbb{C}^2,$$

provided the integral exists. Here

$$\psi(s_1,s_2,t_1,t_2) = t_1^{s_1-1}t_2^{s_2-1},$$

and we take

$$\begin{aligned}D(x_1,x_2,y_1,y_2,z_1,z_2) &= M^{-1}[x_1^{s_1-1}x_2^{s_1-1}y_1^{-s_1}y_2^{-s_2}](z_1,z_2)\\ &= M^{-1}[(x_1/y_1)^{s_1-1}y_1^{-1}(x_2/y_2)^{s_2-1}y_2^{-1}](z_1,z_2)\\ &= \delta(z_1-x_1/y_1,z_2-x_2/y_2)(y_1^{-1}y_2^{-1}).\end{aligned}$$

Then

$$\begin{aligned}(\tau_{z_1,z_2}f)(y_1,y_2) &= \int_0^{+\infty}\int_0^{+\infty}f(z_1,z_2)\delta(z_1-x_1/y_1,z_1-x_1/y_1)(y_1^{-1}y_2^{-1})dz_1dz_2\\ &= f(x_1/y_1,x_2/y_2)(y_1^{-1}y_2^{-1}),\end{aligned}$$

so that

$$(f * g)(x_1, x_2) = \int_0^{+\infty} \int_0^{+\infty} f(\frac{x_1}{y_1}, \frac{x_2}{y_2}) g(y_1, y_2) y_1^{-1} y_2^{-1} dy_1 dy_2.$$

Hence

$$M[f * g](s_1, s_2) = M[f](s_1, s_2) M[g](s_1, s_2).$$

3.2.4 Hankel convolution

Define the 2−dimensional Hankel transform as one dimensional Hankel transform (see [7])

$$F(x_1, x_2) = (h_{(\gamma_1,\gamma_2)} f)(x_1, x_2) = \int_0^{+\infty} \int_0^{+\infty} j(x_1 t_1, x_2 t_2) f(t_1, t_2) d\sigma(t_1, t_2),$$

where

$$j(x_1, x_2) = 2^{\left(\gamma_1 - \frac{1}{2}\right)\left(\gamma_2 - \frac{1}{2}\right)} \Gamma(\gamma_1 + \frac{1}{2}, \gamma_2 + \frac{1}{2}) x_1^{\frac{1}{2} - \gamma_1} x_2^{\frac{1}{2} - \gamma_2} J_{\left(\gamma_1 - \frac{1}{2}, \gamma_2 - \frac{1}{2}\right)}(x_1, x_2),$$

and $J_{\left(\gamma_1 - \frac{1}{2}, \gamma_2 - \frac{1}{2}\right)}(x_1, x_2)$ is the ordinary Bessel function of order $\left(\gamma_1 - \frac{1}{2}\right)\left(\gamma_2 - \frac{1}{2}\right)$, and

$$d\sigma(t_1, t_2) = \frac{t_1^{2\gamma_1} t_2^{2\gamma_2}}{2^{\gamma_1 - \frac{1}{2}} 2^{\gamma_2 - \frac{1}{2}} \Gamma(\gamma_1 + \frac{1}{2}, \gamma_2 + \frac{1}{2})} dt_1 dt_2.$$

The inverse 2−dimensional Hankel transform is also given by

$$(h^{-1}_{(\gamma_1,\gamma_2)} F)(t_1, t_2) = \int_0^{+\infty} \int_0^{+\infty} j(x_1 t_1, x_2 t_2) F(x_1, x_2) d\sigma(x_1, x_2).$$

In this case

$$\begin{aligned} D(x_1, x_2, y_1, y_2, z_1, z_2) &= \int_0^{+\infty} \int_0^{+\infty} j(x_1 t_1, x_2 t_2) j(y_1 t_1, y_2 t_2) j(z_1 t_1, z_2 t_2) d\sigma(t_1, t_2) \\ &= 2^{3\gamma_1 - \frac{5}{2}} 2^{3\gamma_2 - \frac{5}{2}} (\pi)^{-1} \Gamma(\gamma_1 + \frac{1}{2}, \gamma_2 + \frac{1}{2})]^2 \\ &\quad (\Gamma(\gamma_1, \gamma_2))^{-1} (x_1 y_1 z_1)^{-2\gamma_1 + 1} (x_2 y_2 z_2)^{-2\gamma_2 + 1} \\ &\quad [\triangle(x_1, x_2, y_1, y_2, z_1, z_2)]^{(2\gamma_1 - 2)(2\gamma_2 - 2)}, \end{aligned}$$

for $(\gamma_1, \gamma_2) > (0, 0)$. We note that $D(x_1, x_2, y_1, y_2, z_1, z_2) \geq 0$ and that $D(x_1, x_2, y_1, y_2, z_1, z_2)$ is symmetric in $x_1, x_2, y_1, y_2, z_1, z_2$ and we have the formula

$$\int_0^{+\infty} \int_0^{+\infty} j(z_1 t_1, z_2 t_2) D(x_1, x_2, y_1, y_2, z_1, z_2) d\sigma(z_1, z_2) = j(x_1 t_1, x_2 t_2) j(y_1 t_1, y_2 t_2).$$

Moreover, setting $(t_1, t_2) = (0, 0)$, we get

$$\int_0^{+\infty}\int_0^{+\infty} D(x_1, x_2, y_1, y_2, z_1, z_2)d\sigma_1(z_1, z_2)d\sigma_2(z_1, z_2) = 1.$$

In this case we have

$$(\tau_{(x_1,x_2)}f)(y_1, y_2) = \int_0^{+\infty}\int_0^{+\infty} D(x_1, x_2, y_1, y_2, z_1, z_2)f(z_1, z_2)d\sigma(z_1, z_2),$$

$$(f * g)(x_1, x_2) = \int_0^{+\infty}\int_0^{+\infty} (\tau_{(x_1,x_2)}f)(y_1, y_2)g(y_1, y_2)d\sigma(y_1, y_2),$$

and

$$h_{(\gamma_1,\gamma_2)}(f * g)(y_1, y_2) = (h_{(\gamma_1,\gamma_2)}f)(y_1, y_2)(h_{(\gamma_1,\gamma_2)}g)(y_1, y_2).$$

3.2.5 Legendre convolution

The 2−dimensional Legendre transformation of $f \in L^1(-1, 1)^2$ or $C(-1, 1)^2$ in 2D is defined by

$$P[f](n_1, n_2) = F(n_1, n_2) = \int_{-1}^{1}\int_{-1}^{1} P_{n_1,n_2}(t_1, t_2)f(t_1, t_2)dt_1dt_2,$$

with the inverse formula

$$f(t_1, t_2) = P^{-1}[f(n_1, n_2)](t_1, t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} F(n_1, n_2)P_{n_1,n_2}(t_1, t_2).$$

Define

$$\begin{aligned} D(x_1, x_2, y_1, y_2, z_1, z_2) &= P^{-1}[P_{n_1,n_2}(x_1, x_2)P_{n_1,n_2}(y_1, y_2)](z_1, z_2) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P_{n_1,n_2}(x_1, x_2)P_{n_1,n_2}(y_1, y_2)P_{n_1,n_2}(z_1, z_2), \end{aligned}$$

so that

$$\int_{-1}^{1}\int_{-1}^{1} D(x_1, x_2, y_1, y_2, z_1, z_2)P_{n_1,n_2}(z_1, z_2)dz_1dz_2 = P_{n_1,n_2}(x_1, x_2)P_{n_1,n_2}(y_1, y_2).$$

Setting $(n_1, n_2) = (0, 0)$, and using the fact that $P_0(x_1, x_2) = 1$, we get

$$\int_{-1}^{1}\int_{-1}^{1} D(x_1, x_2, y_1, y_2, z_1, z_2)dz_1dz_2 = 1.$$

Moreover,

$$(\tau_{(x_1,x_2)}f)(y_1, y_2) = \int_{-1}^{1}\int_{-1}^{1} D(x_1, x_2, y_1, y_2, z_1, z_2)f(z_1, z_2)dz_1dz_2.$$

Then

$$(f*g)(x_1,x_2)=\int_{-1}^{1}\int_{-1}^{1}(\tau_{(x_1,x_2)}f)(y_1,y_2)g(y_1,y_2)dy_1dy_2,$$

$$\begin{aligned}
P[f*g](n_1,n_2) &= \int_{-1}^{1}\int_{-1}^{1}P_{n_1,n_2}(x_1,x_2)\left(\int_{-1}^{1}\int_{-1}^{1}(\tau_{(x_1,x_2)}f)(y_1,y_2)\right. \\
&\quad g(y_1,y_2)dy_1dy_2{}^{\circ}dx_1dx_2 \\
&= \int_{-1}^{1}\int_{-1}^{1}g(y_1,y_2)dy_1dy_2\int_{-1}^{1}\int_{-1}^{1}P_{n_1,n_2}(x_1,x_2)(\tau_{(x_1,x_2)}f)(y_1,y_2)dx_1dx_2 \\
&= \int_{-1}^{1}\int_{-1}^{1}g(y_1,y_2)dy_1dy_2\int_{-1}^{1}\int_{-1}^{1}P_{n_1,n_2}(x_1,x_2) \\
&\quad \left(\int_{-1}^{1}\int_{-1}^{1}D(x_1,x_2,y_1,y_2,z_1,z_2)f(z_1,z_2)dz_1dz_2\right))dx_1dx_2 \\
&= \int_{-1}^{1}\int_{-1}^{1}g(y_1,y_2)dy_1dy_2\int_{-1}^{1}\int_{-1}^{1}f(z_1,z_2)dz_1dz_2 \\
&\quad \int_{-1}^{1}\int_{-1}^{1}P_{n_1,n_2}(x_1,x_2)D(x_1,x_2,y_1,y_2,z_1,z_2)dx_1dx_2 \\
&= \int_{-1}^{1}\int_{-1}^{1}g(y_1,y_2)dy_1dy_2\int_{-1}^{1}\int_{-1}^{1}f(z_1,z_2)P_{n_1,n_2}(y_1,y_2) \\
&\quad P_{n_1,n_2}(z_1,z_2)dz_1dz_2 \\
&= \int_{-1}^{1}\int_{-1}^{1}f(z_1,z_2)P_{n_1,n_2}(z_1,z_2)dz_1dz_2\int_{-1}^{1}\int_{-1}^{1}g(y_1,y_2) \\
&\quad P_{n_1,n_2}(y_1,y_2)dy_1dy_2 \\
&= (Pf)(n_1,n_2)(Pg)(n_1,n_2).
\end{aligned}$$

3.2.6 Dual Poisson-Laguerre convolution

The 2−dimensional Dual Poisson-Laguerre transform of $f\in L(0,+\infty)^2$ is defined by

$$PL(f)(n_1,n_2)=F(n_1,n_2)=\int_{0}^{\infty}\int_{0}^{\infty}f(x_1,x_2)L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)d\wedge(x_1,x_2),$$

with the inversion formula

$$f(x_1,x_2)=(PL)^{-1}[F(n_1,n_2)]=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}F(n_1,n_2)L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)\sigma(n_1,n_2),$$

where

$$\begin{aligned} L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)(x_1,x_2) &= \Gamma(\rho_1,\rho_2)\Gamma(\alpha_1+1,\alpha_2+1)L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x_1,x_2),\\ \rho_1(n_1,n_2) &= \frac{n_1!}{\Gamma(n_1+\alpha_1+1)}, \rho_2(n_1,n_2) = \frac{n_2!}{\Gamma(n_2+\alpha_2+1)}\\ d\wedge(x_1,x_2) &= \frac{e^{-x_1}x_1^{\alpha_1}e^{-x2}x_2^{\alpha_2}}{\Gamma(\alpha_1+1,\alpha_2+1)}dx_1dx_2\\ \sigma(n_1,n_2) &= [\Gamma(\alpha_1+1,\alpha_2+1)(\rho_1(n_1,n_2),\rho_2(n_1,n_2))]^{-1}. \end{aligned}$$

We set

$$\begin{aligned} D(x_1,x_2,y_1,y_2,z_1,z_2) &= (PL)^{-1}[L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(y_1,y_2)](z_1,z_2)\\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(y_1,y_2)L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(z_1,z_2), \end{aligned}$$

$$(\tau_{(x_1,x_2)}f)(y_1,y_2) = \int_0^\infty\int_0^\infty D(x_1,x_2,y_1,y_2,z_1,z_2)f(z_1,z_2)d\wedge(z_1z_2),$$

and

$$(f*g)(x_1,x_2) = \int_0^\infty\int_0^\infty (\tau_{(x_1,x_2)}f)(y_1,y_2)g(y_1,y_2)d\wedge(y_1y_2).$$

As in the previous case we get

$$PL(f*g)(n_1,n_2) = PL(f)(n_1,n_2)PL(g)(n_1,n_2).$$

3.2.7 Chebyshev convolution

2−dimensional Chebyshev transform of $f \in L_w^p(-1,1)^2$ or $C[-1,1]^2$ is defined by

$$\mathcal{F}[f](k_1,k_2)\widehat{f}(k_1,k_2) = \frac{1}{\pi^2}\int_{-1}^1\int_{-1}^1 f(u_1,u_2)T_{k_1,k_2}(u_1,u_2)w(u_1,u_2)du_1du_2,$$

where

$$T_{k_1,k_2}(x_1,x_2) = \left(\cos(k_1\cos^{-1}x_1),\cos(k_2\cos^{-1}x_2)\right),(x_1,x_2)\in[-1,1]^2,$$

is the kth degree Chebyshev polynomial where $k = k_1k_2$ and $w(x_1,x_2) = \sqrt{(1-x_1^2)(1-x_2^2)}$. The 2−dimensional inverse Chebyshev transform is given by

$$\mathcal{F}^{-1}[\widehat{f}](x_1,x_2) = \frac{1}{\pi^2}\widehat{f}(0,0) + \frac{4}{\pi^2}\sum_{k=1}^{+\infty}\sum_{k=1}^{+\infty}\widehat{f}(k_1,k_2)T_k(x_1,x_2).$$

Set

$$D(x_1, x_2, y_1, y_2, z_1, z_2) = \mathcal{F}^{-1}[\frac{T_{k_1,k_2}(x_1, x_2)w(x_1, x_2)}{\pi^2}\frac{T_{k_1,k_2}(y_1, y_2)w(y_1, y_2)}{\pi^2}](z_1, z_2).$$

Then

$$(\tau_{(x_1,x_2)}f)(y_1, y_2) = \int_{-1}^{1}\int_{-1}^{1} D(x_1, x_2, y_1, y_2, z_1, z_2)f(z_1, z_2)dz_1dz_2,$$

$$(f * g)(x_1, x_2) = \int_{-1}^{1}\int_{-1}^{1}(\tau_{(x_1,x_2)}f)(y_1, y_2)g(y_1, y_2)dy_1dy_2,$$

and

$$\mathcal{F}(f * g)(x_1, x_2) = \mathcal{F}(f)(x_1, x_2)\mathcal{F}(g)(x_1, x_2).$$

3.2.8 Sturm-Liouville convolution

Let I denote any open interval $a < x < b$ where a may be $-\infty$ and b may take the value ∞. Let $\{\psi_{(n_1,n_2)}(x_1, x_2)\}_{(n_1,n_2)=(0,0)}^{\infty}$ be the orthonormal system of a 2−dimensional Sturm-Liouville boundary value problem. Assume that the 2−dimensional Sturm-Liouville transform of f is defined by

$$S\{f\}(n_1, n_2) = \mathbf{F}(n_1, n_2) = \int_a^b\int_a^b f(x_1, x_2)\overline{\psi_{(n_1,n_2)}(x_1, x_2)}dx_1dx_2.$$

Then (under certain conditions) 2−dimensional inversion formula is

$$f(x_1, x_2) = S^{-1}(\mathbf{F})(x_1, x_2) = \sum_0^{\infty}\sum_0^{\infty}\mathbf{F}(n_1, n_2)\psi_{(n_1,n_2)}(x_1, x_2).$$

For conditions of validity of the formula we may refer [8] and refrences therein.

In this case we define

$$D(x_1, x_2, y_1, y_2, z_1, z_2) = S^{-1}[\overline{\psi}_{(n_1,n_2)}(x_1, x_2)\overline{\psi}_{(n_1,n_2)}(y_1, y_2)](z_1, z_2),$$

$$(\tau_{(x_1,x_2)}f)(y_1, y_2) = \int_{-1}^{1}\int_{-1}^{1} D(x_1, x_2, y_1, y_2, z_1, z_2)f(z_1, z_2)dz_1dz_2,$$

and

$$(f * g)\,(x_1, x_2) = \int_a^b\int_a^b(\tau_{(x_1,x_2)}f)(y_1, y_2)f(y_1, y_2)dy_1dy_2,$$

$$\mathbf{F}(f * g)(x_1, x_2) = \mathbf{F}(f)(x_1, x_2)\mathbf{F}(g)(x_1, x_2).$$

3.3 Convolution for 2-D wavelet transform

The above approach can be used in the development of a theory of convolution associated with the wavelet transform. For fixed and arbitrary $\rho \geq 0$, and for any real number $\alpha \neq 0$, the 2−dimensional continuous wavelet transform (2D-CWT) of a function $f(t_1,t_2) \in L^2(\mathbb{R}^2)$ with respect to the wavelet $\psi \in L^2(\mathbb{R})$ is defined by

$$W_f(a_1, a_2, b_1, b_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t_1,t_2)\overline{\psi_{a_1,a_2,b_1,b_2}}(t_1, t_2)dt_1dt_2, \tag{3.14}$$

where

$$\psi_{a_1,a_2,b_1,b_2}(t_1, t_2) = (|a_1a_2|^{-\rho}\, \psi\left(\frac{t_1 - b_1}{a_1}\frac{t_2 - b_2}{a_2}\right). \tag{3.15}$$

A reconstruction formula for (3.14) with respect to the wavelet ψ is given by

$$\begin{aligned} f(t_1,t_2) &= W_f^{-1}[W_f(a_1, a_2, b_1, b_2)](t_1, t_2) \\ &= C_\psi^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_f(a_1, a_2, b_1, b_2)\psi_{a_1,a_2,b_1,b_2}(t_1, t_2) \\ &\quad \times |a_1a_2|^{2\rho}\, db_1db_2da_1da_2, \end{aligned} \tag{3.16}$$

where

$$0 < C_\phi = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\left|\widehat{\phi}(\omega_1, \omega_2)\right|}{|\omega_1\omega_2|} d\omega_1d\omega_2 < \infty. \tag{3.17}$$

We defined the basic function for the present case. Assume that $D(x_1, x_2, y_1, y_2, z_1, z_2)$ is such that

$$\begin{aligned} & W_f[D(x_1, x_2, y_1, y_2, z_1, z_2)](a_1, a_2, b_1, b_2) \\ = & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D(x_1, x_2, y_1, y_2, z_1, z_2)\overline{\psi_{a_1,a_2,b_1,b_2}(x_1, x_2)}dx_1dx_2 \\ = & \overline{\phi_{a_1,a_2,b_1,b_2}}(z_1, z_2)\overline{\theta_{a_1,a_2,b_1,b_2}}(y_1, y_2), \end{aligned}$$

so that formally by (3.16),

$$\begin{aligned} & D(x_1, x_2, y_1, y_2, z_1, z_2) \\ = & C_\psi^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{\phi_{a_1,a_2,b_1,b_2}}(z_1, z_2)\overline{\theta_{ba_1,a_2,b_1,b_2}}(y_1, y_2) \\ & \psi_{a_1,a_2,b_1,b_2}(x_1, x_2)\, |a_1|^{2\rho_1-3}\, |a_2|^{2\rho_2-3}\, db_1db_2da_1da_2. \end{aligned} \tag{3.18}$$

Clearly, $D(x_1, x_2, y_1, y_2, z_1, z_2)$ is symmetric in all the three variables whene $\theta = \phi = \psi$ and ψ is real valued.

The translation τ_{x_1,x_2} is defined by

$$(\tau_{(x_1,x_2)}f)(y_1,y_2) = f^*(x_1,x_2,y_1,y_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D(x_1,x_2,y_1,y_2,z_1,z_2)f(z_1,z_2)dz_1dz_2$$

$$= C_\phi^{-1}\int_{\mathbb{R}^6}\overline{\phi_{a_1,a_2,b_1,b_2}}(z_1,z_2)\overline{\theta_{a_1,a_2,b_1,b_2}}(y_1,y_2)\psi_{a_1,a_2,b_1,b_2}(x_1,x_2)f(z_1,z_2) \quad (3.19)$$
$$|a_1a_2|^{2\rho-3}\,db_1db_2da_1da_2dz_1dz_2.$$

Using (3.16) this can be expressed as

$$(\tau_{(x_1,x_2)}f)(y_1,y_2) = W_f^{-1}[\overline{\theta_{a_1,a_2,b_1,b_2}}(y_1,y_2)(W_f)(b_1,b_2,a_1,a_2)](x_1,x_2). \quad (3.20)$$

The associated convolution is defined by

$$\begin{aligned}(f*g)(x_1,x_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^*(x_1,x_2,y_1,y_2)g(y_1,y_2)dy_1dy_2 \quad (3.21)\\ &= \int_{\mathbb{R}^4} D(x_1,x_2,y_1,y_2,z_1,z_2)f(z_1,z_2)g(y_1,y_2)dz_1dz_2dy_1dy_2\end{aligned}$$

$$= C_\phi^{-1}\int_{\mathbb{R}^6}\overline{\phi_{a_1,a_2,b_1,b_2}}(z_1,z_2)\overline{\theta_{a_1,a_2,b_1,b_2}}(y_1,y_2)\psi_{a_1,a_2,b_1,b_2}(x_1,x_2)$$
$$f(z_1,z_2)g(y_1,y_2)\,|a_1a_2|^{2\rho-3}\,db_1db_2da_1da_2dz_1dz_2dy_1dy_2.$$

Using definition (3.14) we can write it as

$$\begin{aligned}(f*g)(x_1,x_2) &= C_\phi^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(W_f)(a_1,a_2,b_1,b_2)(W_g)(a_1,a_2,b_1,b_2)\\ &\quad \psi_{a_1,a_2,b_1,b_2}(x_1,x_2)\,|a_1a_2|^{2\rho-3}\,db_1db_2da_1da_2\\ &= W_f^{-1}[(W_f)(a_1,a_2,b_1,b_2)(W_g)(a_1,a_2,b_1,b_2)](x_1,x_2),\end{aligned}$$

so that

$$W_{(f*g)}(a_1,a_2,b_1,b_2) = (W_f)(a_1,a_2,b_1,b_2)(W_g)(a_1,a_2,b_1,b_2). \quad (3.22)$$

From representation (3.21), we conclude that if $\theta = \phi = \psi$, then the following commutative and associative properties hold:

$$f*g = g*f,$$

$$(f*g)*h = f*(g*h).$$

Remark 3.2 1. $D(x_1, x_2, y_1, y_2, z_1, z_2)$, $\tau_{(x_1,x_2)}f(y_1, y_2)$ and $f * g$ can be defined by other expression similar to (3.18), (3.19) and (3.21) respectively. Then result of type (3.22) will stiil hold.

2. The following differentiability results can also be established. For $f, g \in \ell(\mathbb{R})$, the Schwartz space of rapidly decreasing functions, we have for $i = 1, 2$

$$\begin{aligned} D^k_{x_i}(f * g)(x_1, x_2) &= (D^k f * g)(x_1, x_2) + (f * D^k g)(x_1, x_2), k \in \mathbb{N}_0, \\ \Delta^k_{x_i}(f * g)(x_1, x_2) &= (\Delta^k f * g)(x_1, x_2) + (f * \Delta^k g)(x_1, x_2), k \in \mathbb{N}_0, \end{aligned}$$

where $\Delta_{x_i} f(x_1, x_2) = x_i(d/dx_i)f(x_1, x_2)$.

CHAPTER 4

Conclusion

Through this Master's thesis, we have explored mathematical transforms applied in signal processing, with a particular focus on the Two-Dimensional Wavelet Transform, due to its significant ability to represent and analyze signals that contain localized details which traditional transforms, such as the Fourier transform, fail to capture accurately. First, we began by presenting the fundamentals of the Fourier transform, highlighting its limitations that motivated the development of more efficient alternatives. After, we proved on to study the two-dimensional wavelet transform, providing its mathematical definition and discussing its important properties supported by rigorous proofs, which demonstrate its capability to represent complex signals. Then, we focused on the practical aspect by analyzing several convolution operations between functions in two-dimensional space using the wavelet transform, which revealed its high effectiveness in analyzing complex signals. With this, we have laid a foundational understanding of the two-dimensional wavelet transform and opened the door for future research and applications that can leverage this powerful tool in various fields such as image compression, video analysis, and pattern recognition.

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