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Mémoire

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STUDY OF LOCAL AND NONLOCAL ELLIPTIC AND PARABOLIC PROBLEMS

Option : Analyse fonctionnelle appliquée(AFA)

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Dedication

I dedicate this work

To

My pride and my guide, my beacon when I am lost, who never stopped encouraging me over the days and paved the way to success for me ,

my father: ***ABDELHALIM****.*

The flower of my life and the queen of my heart, the moon of my home, who never stopped giving me what I needed for my goals and filled my heart with love and tenderness,

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KESMOUNE IMANE

ABSTRACT

In this work, we focus on partial differential equations of the elliptic and parabolic types. The aim of this thesis is to find weak solutions for these equations, Which includes both the classical and fractional Laplace operator.

We first present the basic concepts and general results necessary for the study. We then address the existence and uniqueness of solutions for elliptic equations with homogeneous Dirichlet boundary conditions, using methods from convex analysis and critical point theory.

The study is then extended to fractional Laplace operator with the same boundary condition. After , we study the existence of non-trivial solutions for equations driven by a non-local integrodifferential operator with semi-linear term and homogeneous Dirichlet boundary conditions. employing variational methods : the Mountain Pass theorem and the Linking theorem.

Finally, we discuss a model problem of parabolic partial differential equations, in both linear and nonlinear cases. For linear case, we apply the Faedo-Galerkin method, while for nonlinear case, we use the Schauder Fixed Point theorem.

Key words : Weak solution ,classical Laplace operator ,fractional Laplace operator,non-local operator,linear,nonlinear,semi-linear, Mountain pass theorem,Linking theorem,Faedo-Galerkin methode,Schauder Fixed point theorem .

RÉSUMÉ

Dans ce travail, nous nous concentrons sur les équations aux dérivées partielles de type elliptique et parabolique. L'objectif de cette mémoire est de trouver des solutions faibles pour ces équations, incluant à la fois l'opérateur de Laplace classique et l'opérateur de Laplace fractionnaire. Nous présentons d'abord les concepts de base et les résultats généraux nécessaires à l'étude. Ensuite, nous abordons l'existence et l'unicité des solutions pour les équations elliptiques avec des conditions aux limites de Dirichlet homogènes, en utilisant des méthodes d'analyse convexe et de théorie des points critiques. L'étude est ensuite étendue à l'opérateur de Laplace fractionnaire avec la même condition aux limite. Ensuite, nous étudions l'existence de solutions non triviales pour des équations conduites par un opérateur intégral-différentiel non local avec un terme semi-linéaire et des conditions aux limites de Dirichlet homogènes, en utilisant des méthodes variationnelles :le théorème du pass mountain et le théorème de liaison. Enfin, nous discutons d'un problème modèle d'équations aux dérivées partielles paraboliques, dans les cas linéaire et non linéaire. Pour le cas linéaire, nous appliquons la méthode de Faedo-Galerkin, tandis que pour le cas non linéaire, nous utilisons le théorème du point fixe de Schauder.

Mots clés : Solution faible, opérateur de Laplace classique, opérateur de Laplace fractionnaire, opérateur nonlocal, linéaire, non linéaire, semi-linéaire, théorème du pass montagne, théorème de liaison, méthode de Faedo-Galerkin, théorème du point fixe de Schauder.

ملخّص

في هذا العمل ، نركز على المعادلات التفاضلية الجزئية من صنف القطع الناقص والقطع المكافئ . الهدف من هذه المذكرة هو إيجاد حلول ضعيفة لهذه المعادلات ، والتي تشتمل على كل من مؤثر لابلاس الكلاسيكي والمؤثر الكسري.

نقدم أولا المفاهيم الأساسية والنتائج العامة الضرورية للدراسة . ثم نتطرق إلى وجود و وحدانية الحلول لمعادلات من صنف القطع الناقص مع شروط حدية متجانسة من نوع ديركلي ،باستخدام طرق من التحليل المحدب ونظرية النقاط الحرجة .

تُوسَّع الدراسة بعد ذلك لتشمل مؤثر لابلاس الكسري من نفس الشرط الحدي .ثم ندرس وجود حلول غير تافهة لمعادلات تعتمد على مؤثرات تكاملية-تفاضلية غير محلية مع حد شبه خطي وشروط حدية متجانسة من نوع ديركلي ، وذلك باستخدام طرق تغايراتية : نظرية الممر الجبلي ونظرية الربط

وأخيراً ، نناقش نوع خاص من المعادلات التفاضلية الجزئية من صنف القطع المكافئ والتي تتمثل في معادلة الحرارة . في كلا الحالتين الخطية والغير الخطية.

للحالة الخطية ، نطبق طريقة فادو-غاليركين بينما للحالة الغيرِ خطية ، نستخدم نظرية النقطة الثابتة لشودرِ.

الكلمات المفتاحية: حل ضعيف ، مؤثر لابلاس الكلاسيكي ، مؤثر لابلاس الكسري ،مؤثر غير محلي ، خطي، غير خطي، شبه خطي، نظرية الممر الجبلي ، نظرية الربط ،طريقة فادو-غاليركين ، نظرية النقطة الثابتة لشودر.

Contents

Introduction

A partial differential equation (PDE) establishes a relationship between a function whose behavior is not fully known and its partial derivatives. PDEs are prevalent across all fields of physics and engineering . Recently, there has been a significant surge in their application in disciplines like biology, chemistry, computer science (especially in image processing and graphics), and economics (finance) ,(see [12], [2], [13], [1]). In each of these fields, where interactions among multiple independent variables occur, efforts are made to define functions involving these variables. This enables the modeling of various processes through the formulation of equations. In our work, we focus on certain elliptic and parabolic problems. There is no general theory known regarding the solvability of all partial differential equations. The existence of such a theory is highly unlikely, given the rich variety of physical, geometric, and probabilistic phenomena that can be modeled by PDEs.

We will focus on two important operators that are involved in many PDEs:

The classical Laplacian, defined by $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, is commonly used in elliptic and parabolic equations. The fractional Laplacian, $(-\Delta)^s$ with $0 < s < 1$, is a generalization of the classical Laplacian operator .

Well-posed Problems and Classical Solutions

The informal concept of a well-posed problem includes many desirable characteristics of solving a PDE. A problem for a PDE is considered well-posed if:

1. The solution exists;
2. The solution is unique;
3. The solution depends continuously on the problem's given data.

The third condition is especially important for problems derived from physical applications, where we prefer that our unique solution changes minimally when the problem's conditions change slightly. For many problems, however, uniqueness might not be expected. In these cases, the main mathematical tasks are to classify and characterize the solutions.

Clearly, it would be ideal to solve PDEs such that conditions $(1) - (3)$ are met. However, we still need to carefully define what we mean by a "solution". Should a solution, for instance, be real analytic or at least infinitely differentiable? While this might be desirable, it could be too demanding. It might be more reasonable to require a solution of a PDE of order k to be at least k times continuously differentiable. This ensures that all derivatives appearing in the PDE are continuous, even if some higher derivatives do not exist. We can informally call a solution with this level of smoothness a classical solution of the PDE, which is the most straightforward notion of a solution.

Thus, solving a partial differential equation in the classical sense means, if possible, to write down a formula for a classical solution that meets the criteria $(1)-(3)$ above, or at least to demonstrate that such a solution exists and to deduce various properties of it.

Limitations of Classical Methods

Traditionally, solutions to PDEs are sought in the form of strong solutions, meaning functions that are sufficiently smooth to satisfy the given equations pointwise. However, in many cases, the conditions required for the existence and uniqueness of strong solutions are too strict and are not always achievable for complex practical problems.

The Weak Solution

Faced with the limitations of strong solutions, weak solutions offer a more general and flexible approach. A weak solution is a function that satisfies the equation in an integral sense rather than pointwise, allowing for consideration of less regular functions.

This concept appeared as a result of the emergence of distribution theory. This theory was formalized by the French mathematician **Laurent Schwartz**, earning him the Fields Medal in 1950.

The idea is that we usually evaluate a function by calculating its value at a point. However, this method heavily emphasizes the irregularities (such as discontinuities) of the function. The underlying idea of distribution theory is that there exists a better evaluation process: calculating an average of the function's values over a domain that narrows progressively around the point of interest. By considering weighted averages, we are thus led to examine expressions of the form

$$I_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)\,dx$$

where φ is a test function.

Objectives and Motivation of the Study

The objective of this thesis is finding weak solutions to elliptic and parabolic equations involving the classical Laplacian and the fractional Laplacian. The main motivation is to overcome the limitations of strong solutions and provide effective tools for addressing complex problems where classical methods fail.

Structure of the Thesis

This thesis is organized as follows:

The first chapter is a review of some concepts and general results on standard functional tools.

The second chapter discusses the existence and uniqueness of solutions for elliptic problems involving the Laplacian operator under homogeneous Dirichlet boundary conditions.
The first problem is defined as follows:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then we generalized this problem to:

$$\begin{cases} -\sum_{ij} \partial_i (A_{ij}\partial_j u) = f & \text{in } \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

When $A = (A_{ij})_{ij} \in C^1(\overline{\Omega}, \mathbb{R}^{N\times N})$.
The third chapter extends the previous study to include the fractional Laplacian operator, considering homogeneous Dirichlet boundary condition.

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

Both chapters utilize a method derived from **convex analysis and critical points**.

In the fourth chapter,we study the existence of non-trivial solutions for equation driven by a non-local integrodifferential operator $\mathcal{L}_K$ with semi-linear term , and homogeneous Dirichlet boundary condition More precisely, we consider the problem

$$\begin{cases} -\mathcal{L}_k u - \lambda u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

As a particular case, we derive an existence theorem for the following equation driven by the fractional Laplacian

$$\begin{cases} (-\Delta)^s u - \lambda u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

Employing two distinct variational methods: **the Mountain Pass theorem** and **the Linking theorem**.

In the last chapter, we discuss a model problem of parabolic PDEs dependent on the Laplacian operator called the heat equation ,including both linear and nonlinear cases.
The first one defined as:

$$\begin{cases} \partial_t u - \Delta u = f \quad \text{in} \quad \Omega \times]0,T[, \\ u = 0 \quad \text{on} \quad \partial\Omega \times]0,T[, \\ u(\cdot,0) = u_0 \end{cases}$$

by using **The Faedo- Galerkin method**

And the second one defined as :

$$\begin{cases} \partial_t u = \operatorname{div}\left(A(u)\nabla u\right) + f, & \in \Omega \times]0,T[\\ u = 0, & \partial\Omega \times]0,T[\\ u(.,0) = u_0, \end{cases}$$

Where we prove the existence of the solution using **Schauder's fixed point theorem** ,and we presente a condition in order to obtain the unicity of the solution.

Chapter 1

Preliminaries

In this chapter, we will establish and review the essential mathematical tools, fundamental definitions, and results required for the subsequent chapters. This tools will be presented without proofs.

Contents

1.1 *Element of functional Analysis*

***Definition* 1.1.1.** [3].*Let Ω be a set , $p \in \mathbb{R}$ with $1 \leq p < \infty$, we set*

$$Lp(\Omega) = \{f : \Omega \to \mathbb{R}; f \text{ is measurable and } \int_\Omega |f|^p < +\infty\}$$

with

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

when $p = +\infty$

$$L^{\infty}(\Omega) = \{f : \Omega \to \mathbb{R}\ f \text{ is measurable and } \exists\, C > 0 \text{ such that } |f(x)| \leq C\, a.e.\ \text{ in } \Omega\}$$

with

$$\|f\|_{L^{\infty}(\Omega)} = \|f\|_{\infty} = \{\ inf\,\{C; |f(x)| \leq C \ a.e.\ \ in\,\Omega\}\}$$

.

***Proposition* 1.1.1.** [3]

(i) For $1 \leq p \leq \infty$*, the space* $(L^p(\Omega), \|\cdot\|_p)$ *is a Banach space.*

(ii) For $1 \leq p < \infty$*, the space* $(L^p(\Omega), \|\cdot\|_p)$ *is a separable space.*

(iii) For $1 < p < \infty$*, the space* $(L^p(\Omega), \|\cdot\|_p)$ *is a reflexive space.*

Theorem* 1.1.1.** [3] ***(Hölder's inequality)*. Assume that* $f \in L^p$ *and* $g \in L^q$ *with* $1 \leq p \leq \infty$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$
then $fg \in L^1$ *and :*

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

In the particular case $p = q = 2$ *we get the* ***Cauchy-Schwartz*** *inequality:*

$$\|fg\|_{L^1} \leq \|f\|_2\|g\|_2$$

Lemma* 1.1.1.** [3](Young inequality)****. Let* $a,b \in \mathbb{R}_+$ *and* $p,q \in]1, +\infty[$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$ *,then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem* 1.1.2.** [10] ***(Dominated convergence theorem, Lebesgue) *.Let* Ω *be a set ,* $(f_n)_n$ *be a sequence of functions in* L^p *and* $1 \leq p < \infty$ *that satisfy:*
(a) $f_n(x) \to f(x)$ *a.e. on* Ω
(b) there is a function $g \in L^p$ *such that* $\forall\ n$*,* $|f_n(x)| \leq g(x)$ *a.e. on* Ω *.*
Then $f \in L^p$ *and*

$$f_n \underset{n\to+\infty}{\longrightarrow} f \ \ in\ L^P i.e. \|f_n - f\|_p \underset{n\to+\infty}{\longrightarrow} 0$$

Theorem* 1.1.3.** [10] ***(Inverse of dominated convergence theorem)*. Let* Ω *be a set ,* (f_n) *be a sequence in* L^p *and let* $f \in L^p$ $1 \leq p \leq \infty$ *, be such that* $\|fn - f\|_p \underset{n\to+\infty}{\longrightarrow} 0$
Then, there exist a subsequence (f_{n_k}) *and a function* $g \in L^p$ *such that*
(a) $f_{n_k}(x) \to f(x)$ *a.e. on* Ω
(b) $|f_{n_k}(x)| \leq h(x) \forall k$*, a.e. on* Ω *.*

Theorem* 1.1.4.** [5] ***(Fatou's lemma)*. If* (f_n) *is a sequence of nonnegative measurable functions, then*

$$\int \left(\liminf_{n\to+\infty} f_n\right) \leq \liminf_{n\to+\infty} \int f_n$$

Theorem* 1.1.5.** [3] ***(Fibini). *Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two measure spaces that are σ-finite.*
Assume that $F \in L^1(\Omega_1 \times \Omega_2)$. Then for a.e. $x \in \Omega_1, F(x,y) \in L^1_y(\Omega_2)$ and $\int_{\Omega_2} F(x,y)d\mu_1 \in L^1_x(\Omega_1)$. Similarly, for a.e. $y \in \Omega_2; F(x,y) \in L^1_x(\Omega_1)$ and $\int_{\Omega_1} F(x,y)d\mu_1 \in L^1_y(\Omega_2)$.
Moreover, one has

$$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} F(x,y)d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} F(x,y)d\mu_1 = \iint_{\Omega_1\times\Omega_2} F(x,y)d\mu_1 d\mu_2$$

Proposition* 1.1.2.** [3] ***(Dual of** L^p **)

- *For $p \in [1, +\infty[$, and $\frac{1}{p} + \frac{1}{q} = 1$ the dual space of L^p is L^q ,and we note that : $(L^p)^* = L^q$.*
- *For $p = +\infty$, We have $(L^\infty)^*$ is strictly bigger than L^1 ,so :*

$$(L^\infty)^* \neq L^1$$

***Corollaire* 1.1.1.** [3] *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then $C_c^\infty(\Omega)$is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$.*

***Corollaire* 1.1.2.** [3] *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^p(\Omega)$ be such that*

$$\int_\Omega uf = 0 \qquad \forall f \in C_c^\infty(\Omega)$$

Then $u = 0$ a.e. on Ω .

Theorem* 1.1.6.** *[3](Ascoli-Arzelà)***) Let K be a compact metric space and let H be a bounded subset of $C(K)$. Assume that H is uniformly equicontinuous, that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$ for all $f \in H$. Then the closure of H in $C(K)$ is compact.*

1.2 ***Sobolev space***

Let $\Omega \subset \mathbb{R}^N$,be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

***Definition* 1.2.1.** [3] *The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ \begin{array}{l} u \in L^p(\Omega) \\ \exists g_1, g_2, \dots, g_N \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega),\ \forall i = 1, 2, \dots, N \end{array} \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \in W^{1,p}(\Omega)$ *we define* $\dfrac{\partial u}{\partial x_i} = g_i$, *and we write*

$$\nabla u = grad\, u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N}\right).$$

The space $W^{1,p}(\Omega)$ *is equipped with the norm*

$$\|u\|_{W^{1,p}} = \|u\|_p + \left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_p\right)^{1/p}$$

or sometimes with the equivalent norm $(\|u\|_p^p + \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_p^p)^{1/p}$ *(if* $1 \leq p < \infty$*).*

The space $H^1(\Omega)$ *is equipped with the scalar product*

$$\langle u, v\rangle_{H^1} = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.$$

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{L^2}$$

$$= \int_\Omega uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$$

The associated norm is defined by

$$\|u\|_{H^1} = \left(\|u\|_2^2 + \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_2^2\right)^{1/2}.$$

***Proposition* 1.2.1.** [3] $W^{1,p}(\Omega)$ *is a Banach space for every* $1 \leq p \leq \infty$. $W^{1,p}(\Omega)$ *is reflexive for* $1 < p < \infty$, *and it is separable for* $1 \leq p < \infty$. $H^1(\Omega)$ *is a separable Hilbert space.*

***Definition* 1.2.2.** [3] *Let* $1 \leq p < \infty$; $W_0^{1,p}(\Omega)$ *denotes the closure of* $C_c^1(\Omega)$ *in* $W^{1,p}(\Omega)$.
Set

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}$, *equipped with the* $W^{1,p}$ *norm, is a separable Banach space; it is reflexive if* $1 < p < \infty$. H_0^1, *equipped with the* H^1 *scalar product, is a Hilbert space.*

***Proposition* 1.2.2.** [3] $C_c^\infty(\Omega)$ *is dense in* $W_0^{1,p}(\Omega)$.

Notation. [3] Given $x \in \mathbb{R}^N$, write

$$x = (x', x_N) \text{ with } x' \in \mathbb{R}^{N-1}, \quad x' = (x_1, x_2, \dots, x_{N-1}),$$

and set

$$|x'| = \left(\sum_{i=1}^{N-1} x_i^2 \right)^{1/2}.$$

We define

$$\mathbb{R}^N_+ = \{x = (x', x_N); x_N > 0\},$$

$$Q = \{x = (x', x_N); |x'| < 1 \text{ and } |x_N| < 1\},$$

$$Q_+ = Q \cap \mathbb{R}^N_+,$$

$$Q_0 = \{x = (x', 0); |x'| < 1\}.$$

***Definition* 1.2.3.** [3] *We say that an open set Ω is of class C^1 if for every $x \in \partial\Omega = \Gamma$ there exist a neighborhood U of x in $\mathbb{R}^N$ and a bijective map $H : Q \to U$ such that*

$$H \in C^1(\overline{Q}), \quad H^{-1} \in C^1(U),$$

$$H(Q_+) = U \cap \Omega, \quad \textit{and} \quad H(Q_0) = U \cap \Gamma.$$

The map H is called a local chart.

Corollaire* 1.2.1.** [3] ***(Poincaré's inequality). *Suppose that $1 \leq p < \infty$ and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

In particular, the expression $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$, and it is equivalent to the norm $\|u\|_{W^{1,p}}$. On $H_0^1(\Omega)$, the expression

$$\left(\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right)^{1/2}$$

is a scalar product that induces the norm $\|\nabla u\|_{L^2}$ and is equivalent to the norm $\|u\|_{H^1}$.

***Proposition* 1.2.3.** [3] *We denote by $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$, $1 \leq p < \infty$, and by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$. The dual of $L^2(\Omega)$ is identified with $L^2(\Omega)$ We have the inclusions*

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

where these injections are continuous and dense.

If Ω is bounded then

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega) \quad \textit{if} \quad \frac{2N}{N+2} \leq p < \infty,$$

with continuous and dense injections. If Ω is not bounded, the same holds, but only for the range $\frac{2N}{N+2} \leq p \leq 2$.

Proposition 1.2.4. *[18]**(Gauss-Green Formula for the Laplacian).*** *Let Ω be an open set of class $C^{0,1}$. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then:*

$$-\int_\Omega \Delta u(x)v(x)\,dx = \int_\Omega \nabla u(x)\nabla v(x)\,dx - \int_{\partial\Omega} v(x)\frac{\partial u}{\partial n}(x)\,d\Gamma(x)$$

where $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$. *Moreover, if $v \in H^2(\Omega)$, then:*

$$\int_\Omega \{u(x)\Delta v(x) - \Delta u(x)v(x)\}\,dx = \int_{\partial\Omega} \left\{u(x)\frac{\partial v}{\partial n}(x) - v(x)\frac{\partial u}{\partial n}(x)\right\} d\Gamma(x).$$

1.3 *Fractional Sobolev spaces*

Definition 1.3.1. [7] *Let $\Omega \subseteq \mathbb{R}^N$. For $0 < s < 1$ and any $1 \leq p < +\infty$, we define $W^{s,p}(\Omega)$ as follows*

$$W^{s,p}(\Omega) : = \left\{ u \in L^p(\Omega); \quad \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

i.e., an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} : = \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

where the term

$$[u]_{W^{s,p}(\Omega)} : = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

is the Gagliardo semi-norm of u.

Proposition 1.3.1. [7] *Let Ω be an open subset of $\mathbb{R}^N$, $0 < s < 1$, then we have*

(a) *For $1 \leq p < \infty$, $W^{s,p}(\Omega)$ is a Banach and separable space.*

(b) *For $1 < p < \infty$, $W^{s,p}(\Omega)$ is a reflexive space.*

Theorem 1.3.1. [7] *For any $s > 0$, the space $C_c^\infty(\mathbb{R}^N)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^N)$.*

Definition 1.3.2. [7] *We note $W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|.\|_{W^{s,p}(\Omega)}}$, according to above Theorem (1.3.1), we have*

$$W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N).$$

In fact, when Ω is a subset of $\mathbb{R}^N$, $W_0^{s,p}(\Omega) \neq W^{s,p}(\Omega)$.

1.3.1 *Fractional Sobolev inequalities*

***Theorem* 1.3.2.** [7] *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < n$. Then there exists a positive constant $C = C(n,\ p,\ s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$,we have :*

$$\|u\|_{L^{p*}}^p \leq C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy$$

where $p = p*(n,s)$ is the so-called "fractional critical exponent" and it is equal to $\dfrac{Np}{N-sp}$. Consequently, the space $W^{s,p}(\mathbb{R}^N)$is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p,p*]$*

***Theorem* 1.3.3.** [7] *Let $s \in (0,1)$ and $p \in [1,+\infty[$ be such that $sp < n$. Let$\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}$ .Then there exists a positive constant $C = C(n,p,s,\Omega)$such that, for any $u \in W^{s,p}$, we have :*

$$\|u\|_{L^q(\Omega)} \leq c\|u\|_{W^{s,p}(\Omega)}$$

for any $q \in [p,p*]$; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,p*]$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$for any $q \in [1,p*]$.

***Remark* 1.3.1.** [7] *In the critical case $q = p*$the constant C in Theorem (1.3.3) does not depend on Ω.*

***Theorem* 1.3.4.** [7] *Let $s \in (0,1), p \in [1,+\infty), q \in [1,p], \Omega \subset \mathbb{R}^N$be a bounded extension domain for $W^{s,p}$ and $\mathscr{T}$be a bounded subset of $L^p(\Omega)$. Suppose that :*

$$\sup_{u\in\mathscr{T}} \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < +\infty.$$

Then $\mathscr{T}$ pre-compact in $L^q(\Omega)$.

1.3.2 H^s *Space*

Now, we looks at the case (p=2) in the space $W^{s,p}(\Omega)$ this is a very special cases when the Fractional Sobolev spaces $W^{s,2}(\Omega)$ and $W_0^{s,2}(\Omega)$ which are Hilbert spaces, they are noted $H^s(\Omega)$ and $H_0^s(\Omega)$ respectively, for $s \in (0,1)$.

✍ We define the space $H^s(\Omega)$ as follows

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\Omega\times\Omega) \right\},$$

with the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.$$

The space $H^s(\Omega)$ equipped with the inner product $\langle .,. \rangle_{H^s(\Omega)}$ is a Hilbert space .
For more informations see [7].

1.4 *Review of Convex Analysis*

In this section, we recall results on convexity . In the following, X denotes a Banach space. X^* denotes its topological dual and we denote $\langle .,. \rangle$ the duality pairing between X and X^*. In this section, we assume the functions to be valued in $\overline{\mathbb{R}} = \mathbb{R}^N \cup \{+\infty\} \cup \{-\infty\}$.

1.4.1 *Convex sets,Lower semi-continuous functions*

***Definition* 1.4.1.** [6] *A subset C of X is said to be convex if it is stable by convex combination, i.e.:*

$$\forall (x, y) \in C^2 \ \forall \ \lambda \in [0, 1] \ \ \lambda x + (1 - \lambda)y \in C$$

***Definition* 1.4.2.** [6] *Let J be a function defined on X and taking values in $\overline{\mathbb{R}}$. It is said to be lower semi-continuous at x, if for any sequence x_n such that x_n converges to x,we have:*

$$J(x) \leq \underline{lim}_{n \to +\infty} J(x_n)$$

***Definition* 1.4.3.** [6] *A functional J from X to $\overline{\mathbb{R}}$ is said to be proper if it is not identically equal to $+\infty$ and does not take the value $-\infty$ In particular, its domain $dom(J) = \{x \in X | J(x) \in \mathbb{R}\}$ is non-empty.*

1.4.2 *Fréchet and Gâteaux differentiability.*

***Definition* 1.4.4.** [6] *Let Ω be a subset of a Banach space X and $F : \Omega \to \mathbb{R}$. If $u \in \Omega$, we say that F is Gâteaux-differentiable (or G-differentiable) at u if there exists $l \in X^*$ such that for any direction $v \in X$ where F(u + tv) exists for t > 0 sufficiently small, he directional derivative $F'_v(u)$ exists ,and we have :*

$$\lim_{l \to 0^+} \frac{F(u + tv) - F(u)}{t} = \langle l, v \rangle.$$

We will denote $F'(u) := l$.

It should be noted that a Gâteaux-differentiable function is not necessarily continuous. Finally, we introduce the classical derivative (or Fréchet derivative). (***The Landau notation o(x) is used to denote a function of x such that*** $\lim_{\|x\| \to 0} o(x) \setminus \|x\| = 0$ *)*

***Definition* 1.4.5.** [6] *Let X be a Banach space, Ω an open set in X, and F: $\to \mathbb{R}$ a function. If $u \in \Omega$, we say that F is differentiable (or derivable) at u (in the Fréchet sense) if there exists*

$l \in X^{*}$ such that :

$$\forall v \in \Omega \quad F(u+v) - F(v) = \langle l, v \rangle + o(v)$$

If F is differentiable, the derivative is unique and we denote it by $DF(u) := l$. The set of differentiable functions from Ω to $\mathbb{R}$ will be denoted by $C^{1}(\Omega, \mathbb{R})$.

***Proposition* 1.4.1.** [6] *Let F be a continuous function from Ω to $\mathbb{R}$ and G-differentiable in a neighborhood of $u \in \Omega$. Let F'(v) denote the G-derivative of F at v, and suppose that the application $v \to F'(v)$ is continuous in a neighborhood of u. Then*

$$F(u+v) - F(v) = \langle F'(u), V \rangle + o(v)$$

i.e.F is Fréchet differentiable and its (classical) derivative DF(u) coincides with F'(u).

1.5 *Propertie of the Weak Topology* $\sigma(E, E^{*})$

***Proposition* 1.5.1.** [3] *Let E be a Banach space and let $f \in E^{*}$,and let (x_n) be a sequence in E. Then*

(i) $[x_n \to x$ *weakly in* $\sigma(E, E^{\star})] \Leftrightarrow [\langle f, x_n \rangle \to \langle f, x \rangle \, \forall f \in E^{\star}]$.

(ii) If $x_n \to x$ *strongly, then* $x_n \rightharpoonup x$ *weakly in* $\sigma(E, E^{\star})$.

(iii) If $x_n \rightharpoonup x$ *weakly in* $\sigma(E, E^{\star})$, *then* $(\|x_n\|)$ *is bounded and* $\|x\| \leq \liminf \|x_n\|$.

(iv) If $x_n \to x$ *weakly in* $\sigma(E, E^{\star})$ *and if* $f_n \to f$ *strongly in* $E^{\star}$ *(i.e.,* $\|f_n - f\|_{E^{\star}} \to 0$*), then* $\langle f_n, x_n \rangle \to \langle f, x \rangle$.

Chapter 2

Solving Dirichlet problem of linear elliptic PDEs

In this chapter, we present a method for solving certain elliptic partial differential equations (PDEs), specifically those of the form $DJ(u) = 0$, where DJ denotes the differential (in a weak sense) of a functional J, which is typically convex. The characteristics of convex functions enable us to find a solution to the PDE by minimizing a functional, assuming this functional tends towards $+\infty$ at infinity. For this reason, we will first present some theoretical concepts that ensure the existence of a minimum for J and how this leads to the existence of the weak solution to the PDEs. Afterwards, we will illustrate several classic examples of boundary problems governed by linear elliptic PDEs, which can be resolved using the variational method. For details on the regularity of the solutions to these problems, please refer to the book [6].

Contents

2.1 *Minimization of a Convex Functional*

In the following, X denotes a Banach space. X^* denotes its topological dual, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In this section, we consider functions taking values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

***Definition* 2.1.1.** *A functional J defined on a separable Banach space X is said to be* *coercive if*

$$\lim_{\|x\|_X \to +\infty} J(x) = +\infty$$

We are concerned with the problem of minimizing J over a closed convex set in X. Two results are useful:

Proposition 2.1.1. *If there exists a solution $u \in X$ to the problem $\inf_{u \in X} J(u)$ and if J is Gâteaux-differentiable at this point u, then the subdifferential at u, denoted $\partial J(u)$, is reduced to $\{J'(u)\}$. Conversely, if J is convex and Gâteaux-differentiable at u with $J'(u) = 0$, then u is a minimum.*

Theorem 2.1.1. *Let X be a separable and reflexive Banach space, U a closed convex subset of X, and J a convex, proper, coercive, and lower semi-continuous functional. Then there exists a solution to the problem:*

$$\inf_{u \in U} J(u)$$

.

2.2 *Solving linear elliptic PDEs with Dirichlet boundary conditions*

2.2.1 *Introduction*

Consider the physical problem of studying the equilibrium position of an elastic membrane in the plane. The membrane projects onto the plane in an open set Ω , its edge remaining coincident with the boundary curve $\partial\Omega$ of this open set.At each point $x = (x_1, x_2)$, the membrane is subjected to a tension, a vertical force defined by a function $x \mapsto f(x)$. The displacement of this point x is identified with the elevation of the membrane $z = u(x)$ at this point. The physical equations lead to the equation satisfied by u, namely $-\Delta u = f$, with, in addition, the boundary condition u = 0 on $\partial\Omega$. This is one of the physical models of the Dirichlet problem.

2.2.2 *Dirichlet problem associated to the Laplacian*

Let Ω be a bounded domain of class $C^1(\mathbb{R}^N)$. Let f $\in L^2(\Omega)$,we are looking for u a solution to the problem :

$$(P_1) : \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{2.1}$$

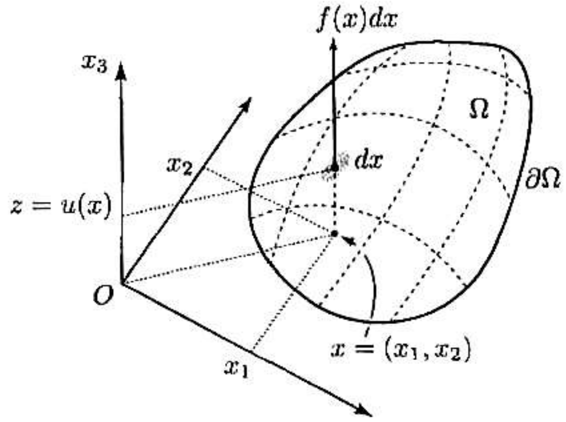


Figure 2.1:

Let u $\in H^2(\Omega) \cap H_0^1(\Omega)$ a strong solution of (P_1).
Since $\Delta u \in L^2(\Omega)$,and f $\in L^2(\Omega)$, and by using the fact that $L^2(\Omega) \subset D'(\Omega)$, then the equation of (P_1) has a sense in $D'(\Omega)$.

Existence of a weak solution of (P_1)

1) ***The variational formulation :*** We have

$$\int_\Omega -\Delta u \; vdx = \int_\Omega f \; vdx \quad \forall v \in D(\Omega)$$

With Green formula we obtain :

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in D(\Omega)$$

$D(\Omega)$ dense in $H_0^1(\Omega)$, then

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

Let $a(u,v) = \int_\Omega \nabla u \nabla v dx$; $l(v) = \int_\Omega f v dx$

2) ***The variational problem :*** Let $J : H_0^1(\Omega) \mapsto \mathbb{R}$ defined by :

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx$$

The variational problem associate to (2.1)is :

$$\begin{cases} Find \; u \in H_0^1(\Omega) \; such \; that \\ J(u) = \underset{v \in H_0^1(\Omega)}{inf} J(v) \end{cases} \tag{2.2}$$

3) ***The existence of the minimum of J in*** $H_0^1(\Omega)$

For that ,using Theorem (2.1.1) .

Note that $H_0^1(\Omega)$ is a closed convex subset of $H^1(\Omega)$

i) We show that J is convex in $H_0^1(\Omega)$ *:*

Let $u_1, u_2 \in H_0^1(\Omega)$,and $\lambda \in \mathbb{R}$

$$\begin{aligned} J(\lambda u_1 + (1-\lambda)u_2) &= \frac{1}{2}a(\lambda u_1 + (1-\lambda)u_2, \lambda u_1 + (1-\lambda)u_2) - l(\lambda u_1 + (1-\lambda)u_2) \\ &= \frac{1}{2}\lambda^2 a(u_1, u_1) + 2\lambda(1-\lambda)a(u_1, u_2) + (1-\lambda)^2 a(u_2, u_2) - \lambda l(u_1) \end{aligned}$$
$$-(1-\lambda)l(u_2)$$

On the other hand we have :

$a(u_1 - u_2, u_1 - u_2) \geq 0 \Rightarrow a(u_1, u_2) \leq \dfrac{a(u_1, u_1) + a(u_2, u_2)}{2}$

then

$$J(\lambda u_1 + (1-\lambda)u_2) \leq \lambda\left(\frac{1}{2}a(u_1, u_1) - l(u_1)\right) + (1-\lambda)\left(\frac{1}{2}a(u_2, u_2) - l(u_2)\right)$$

So

$$J(\lambda u_1 + (1-\lambda)u_2) \leq \lambda J(u_1) + (1-\lambda)J(u_2)$$

Then J is convex in $H_0^1(\Omega)$.

ii) We show that J is coercive in $H_0^1(\Omega)$: let v $\in H_0^1(\Omega)$

$$\begin{aligned} J(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx \\ &\geq \frac{1}{2}\|v\|^2_{H_0^1(\Omega)} - \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \quad \text{(Cauchy Schwartz inequality)} \end{aligned}$$

Also we have : $\|v\|_{L^2(\Omega)} \leq Cp\|\nabla v\|_{L^2(\Omega)}$ (Poincaré's inequality)

Then

$$\|v\|_{L^2(\Omega)} \leq Cp\|v\|_{H_0^1(\Omega)}$$

We deduce that :

$$\begin{aligned} J(v) &\geq \frac{1}{2}\|v\|^2_{H_0^1(\Omega)} - Cp\|f\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)} \\ &\geq \|v\|_{H_0^1(\Omega)}\left(\frac{1}{2}\|v\|_{H_0^1(\Omega)} - Cp\|f\|_{L^2(\Omega)}\right) \end{aligned}$$

Finally

$$\lim_{\|v\|_{H_0^1(\Omega)} \to +\infty} J(v) = +\infty$$

Then J is coercive in $H_0^1(\Omega)$.

iii) *We show that J is lower semi-continuous functional :*

***Remark* 2.2.1.** *We can prove that J is continuous,and thus we will get that J is lower semi-continuous.*

Indeed, let $(v_n)_{n\in\mathbb{N}} \underset{n\to+\infty}{\to} v$ in $H_0^1(\Omega)$ *then* $(v_n)_{n\in\mathbb{N}} \underset{n\to+\infty}{\to} v$ in $L^2(\Omega)$ *and* , $\nabla v_n \underset{n\to\infty}{\to} \nabla v$ *in* $L^2(\Omega)$, *and we show that*
$J(v_n) \underset{n\to\infty}{\to} J(v)$ in $\mathbb{R}$.

$$\begin{aligned}|J(v_n) - J(v)| &= \left|\frac{1}{2}\int_\Omega (|\nabla v_n|^2 - |\nabla v|^2)dx - \int_\Omega f(v_n - v)dx\right| \\ &\leq \frac{1}{2}|\int_\Omega (\nabla v_n - \nabla v)(\nabla v_n + \nabla v)dx| + \int_\Omega |f||v_n - v|dx\end{aligned}$$

Then with Cauchy Schwartz inequality we obtain :

$$|J(v_n) - J(v)| \leq \frac{1}{2}\|\nabla v_n - \nabla v\|_{L^2(\Omega)}\|\nabla v_n + \nabla v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|v_n - v\|_{L^2(\Omega)} \underset{n\to+\infty}{\to} 0$$

Then $J(v_n) \underset{n\to+\infty}{\to} J(v)$ in $\mathbb{R}$,so J is coninuous in $H_0^1(\Omega)$, then J is lower semi-continuous in $H_0^1(\Omega)$. Finally, according to the Theorem (2.1.1) ,the problem (2.2) has a solution in $H_0^1(\Omega)$.

4) ***The solution of the variational problem(2.2) is a weak solution of the problem(2.1).***

i) *The minimum of J is a solution of the variatioal formulation :* Indeed, we will use the proposition (2.1.1).So we have to prove that J is Gâteau differentiable at u where u is the minimum of J . Let u,v $\in H_0^1(\Omega)$

$$J(u+tv) - J(u) = \frac{1}{2}a(u+tv, u+tv) - l(u+tv) - \frac{1}{2}a(u,u) + l(u)$$

as $a(.,.)$is bilinear and symmetric ,and also l(v) is linear ,we obtain :

$$J(u+tv) - J(u) = ta(u,v) + \frac{t^2}{2}a(v,v) - t\,l(v)$$

then

$$\frac{J(u+tv) - J(u)}{t} = a(u,v) + \frac{t}{2}a(v,v) - l(v)$$

So

$$\lim_{t\to 0}\frac{J(u+tv) - J(u)}{t} = a(u,v) - l(v)$$

i.e.

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

After that ,we prove that $J'(u) \in H^{-1}$ is continous . Indeed ,let $u_n \underset{n\to+\infty}{\to} u$ in $H_0^1(\Omega)$ then $\nabla u_n \underset{n\to+\infty}{\to} \nabla u$ in $L^2(\Omega)$.

$$\begin{aligned}\forall v \in H_0^1(\Omega) \quad |\langle J'(u_n), v\rangle - \langle J'(u), v\rangle| &= \int_\Omega (\nabla u_n) - \nabla u)\nabla v dx \\ &\leq \|\nabla u_n) - \nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \underset{n\to+\infty}{\to} 0\end{aligned}$$

Then we conclude that :

$$\forall v \in H_0^1(\Omega) \quad \langle J'(u_n), v\rangle \underset{n\to+\infty}{\to} \langle J'(u), v\rangle$$

i.e. $J'(u)$is continuous , then J is Gâteaux differentiable ,(moreover J is Fréchet differentiable then $J'(u) = DJ(u)$).

Since u minimizes J , then according to the proposition (2.1.1), DJ(u)=0
i.e.

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

ii) *u is a solution of the Dirichlet problem:* Indeed ,as $u \in H_0^1(\Omega)$, then $u = 0$ a.e. in $\partial\Omega$,and since $D(\Omega) \subset H_0^1(\Omega)$ then :

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in D(\Omega)$$

with Green formula we obtain :

$$\int_\Omega -\Delta u \; v dx = \int_\Omega f v \; dx \quad \forall v \in D(\Omega)$$

then

$$\int_\Omega (-\Delta u - f) \; v \; dx = 0 \quad \forall v \in D(\Omega)$$

then

$$-\Delta u = f \quad a.e. \; in \; \Omega$$

So the minimum of J is the solution of Dirichlet problem (2.1) .

Uniqueness of the solution

Suppose that u,v $\in H_0^1(\Omega)$ are two solutions of the problem (2.1) .by difference ,we obtain :

$$\Delta(u - v) = 0$$

Multiplying by $u-v$,integrate over Ω ,using Green's formula we obtain :

$$\int_{\Omega} |\nabla(u-v)|^2(x)dx = 0$$

Then $u-v = cte$
since $u-v=0$ on $\partial\Omega$,we conclude that $u-v=0$ in Ω ,then $u=v$ a.e.in Ω

2.2.3 *Dirichlet problem for another operator A*

Let f $\in L^2(\Omega)$,and let $A=(A_{ij})_{ij} \in C^1(\overline{\Omega}, \mathbb{R}^{N\times N})$ such that :

1. $\forall i,j \in [1,N] A_{ij} = A_{ji}$
2. $\exists \alpha > 0, \forall x \in \mathbb{R}^N : \sum_{ij} A_{ij}x_ix_j \geq \alpha |x|^2$ (**(uniform ellepticity of A**)

Looking for a solution of the problem :

$$(p_2) : \begin{cases} -\sum_{ij} \partial_i(A_{ij}\partial_j u) = f & \text{in } \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{2.3}$$

Remark 2.2.2. *1. The problem (2.3)can be written as $-div(A(x)\nabla u) = f$, thus as a PDE in divergence form ,this formulation gives us an advantage among several other advantage is to obtain a variational formulation to our problem .*

2. This problem is a generalization of problem (p_1),sence it is enough to choose $A = \delta_{ij}$to obtain it.

Then ,we we are searching of u ,a solution of:

$$(\tilde{p}_2) : \begin{cases} -div(A(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

We will do the same way ,we did in the first problem .

Existence of a weak solution of $(\tilde{P}_2)$

1) ***Variational formulation :***

$$\int_{\Omega} -div(A(x)\nabla u)v dx = \int_{\Omega} fv dx \quad \forall v \in D(\Omega)$$

Using Green formula and density of $D(\Omega) in H_0^1(\Omega)$,we obtain :
$\int_{\Omega} A(x)\nabla u.\nabla v dx = \int_{\Omega} fv dx \quad \forall v \in H_0^1(\Omega)$

2) ***Variational problem***

The functional J associated for this problem is :

$$J(v) = \frac{1}{2}\int_\Omega A(x)\nabla v.\nabla v - \int_\Omega fvdx$$

Or $A(x).X.Y$ denotes the scalar $A_{ij}X_iY_j$.

The variational problem of (p_2) is :

$$\begin{cases} \text{Find u } \in H_0^1(\Omega) \\ J(u) = \inf\limits_{v\in H^1 0(\Omega)} J(v) \end{cases} \tag{2.5}$$

3) ***Existence of solution for the variational problem***

i) ***The continuity of J :***

Let $v_n \to v$ in $H_0^1(\Omega)$, we have:

$$\|v_n - v\|_{H_0^1(\Omega)} \to 0$$

This implies both:

$$\|v_n - v\|_{L^2(\Omega)} \to 0$$

and

$$\|\nabla v_n - \nabla v\|_{L^2(\Omega)} \to 0$$

$$\begin{aligned} J(v_n) - J(v) &= \frac{1}{2}\int_\Omega A(x)\nabla v_n.\nabla v_n dx - \frac{1}{2}\int_\Omega A(x)\nabla v.\nabla v dx - \int_\Omega f(v_n - v)dx \\ &= \frac{1}{2}\int_\Omega A(x)\nabla v_n.(\nabla v_n - \nabla v)dx + \frac{1}{2}\int_\Omega A(x)(\nabla v_n - \nabla v).\nabla v dx - \int_\Omega f(v_n - v)dx \end{aligned}$$

$A(x) \in C^1(\overline{\Omega}, \mathbb{R}^{N\times N})$, then each component $A_{ij}(x)$ is continuous, on the compact set $\overline{\Omega}$. This implies that each $A_{ij}(x)$ is bounded on $\overline{\Omega}$.
then

$$\sum_{j=1}^{N} |A_{ij}(x)| \leq M_i \quad \text{for all } x \in \overline{\Omega}$$

such that $M_i = \underset{1\leq j\leq N}{max} A_{ij}$ and we have :

$$\|A(x)\|_{C(\overline{\Omega})} = \max_{1\leq i\leq N} \sum_{j=1}^{N} |A_{ij}(x)|$$

then

$$\|A(x)\|_{C(\overline{\Omega})} \leq \max_{1\leq i\leq N} Mi = M$$

using the Cauchy-Schwarz inequality and we get :

$$\begin{aligned}|J(v_n) - J(v)| &\leq \frac{1}{2}\|A(x)\|_{C(\overline{\Omega})}\|\nabla v_n - \nabla v\|_{L^2(\Omega)}\|\nabla v_n\|_{L^2(\Omega)}\\ &+ \frac{1}{2}\|A(x)\|_{C(\overline{\Omega})}\|\nabla v_n - \nabla v\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|v_n - v\|_{L^2(\Omega)}\\ &\leq \frac{1}{2}M\|\nabla v_n - \nabla v\|_{L^2(\Omega)}\|\nabla v_n\|_{L^2(\Omega)}\\ &+ \frac{1}{2}M\|\nabla v_n - \nabla v\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|v_n - v\|_{L^2(\Omega)} \underset{n\to+\infty}{\to} 0\end{aligned}$$

Then $J(v_n) \underset{n\to\infty}{\to} J(v)$ in $\mathbb{R}$,so J is coninuous in $H_0^1(\Omega)$,wich leads to that J is lower semi-continuous .

2) ***The convexity***

Remark 2.2.3. *we can use the same way in the previous problem ,because* $a(v,v) = \int_\Omega A(x)\nabla v(x).\nabla v(x)dx \geq 0$ *,wich is positive as a result of the uniform ellepticity of* A *.*

3) ***Coercivity***

Let $v \in H_0^1(\Omega)$we have :

$$\begin{aligned}J(v) &= \frac{1}{2}\int_\Omega A(x)\nabla v(x).\nabla v dx - \int_\Omega f v dx\\ &\geq \frac{1}{2}\alpha\|\nabla v\|_2^2 - \|f\|_2\|v\|_2 (\textit{using Cauchy Schwartz inequality and uniforme ellepticity of } A)\\ &\geq \frac{1}{2}\alpha\|v\|^2_{H^10(\Omega)} - C_p\|f\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} (\textit{using poincaré's inequality})\\ &\geq \|v\|_{H_0^1(\Omega)}\left(\frac{\alpha}{2}\|v\|_{H_0^1(\Omega)} - C_p\|f\|_2\right) \underset{\|v\|_{H_0^1(\Omega)}\to+\infty}{\to +\infty}\end{aligned}$$

4) ***The solution of the variational problem(2.5) is a weak solution of the problem(2.4).***

The functional J is Gâteaux-differentiable, with its derivative defined by:

$$(J'(u), v) = \int_\Omega A(x)\nabla u.\nabla v\, dx - \int_\Omega f v\, dx.$$

Using Green's formula, we prove that the minimum of J on $H_0^1(\Omega)$ is indeed the solution to the problem (2.4). Finally ,we deduce that the minimum of J is a weak solution of Dirichlet problem (2.4).

Remark 2.2.4. *Note also that* $f \in L^2(\Omega)$ *can be replaced by* $f \in H^{-1}(\Omega)$*. In this case, we* *replace the integral* $\int_\Omega fudx$ *by the duality pairing* $\langle f,u\rangle$.

Uniqueness of the solution

Suppose that u,v $\in H_0^1(\Omega)$ are two solutions of the problem (2.2) .by difference ,we obtain :

$$div(A(x)\nabla(u-v)) = 0$$

Multiplying by $(u-v)$,integrate on Ω ,using Green's formula we obtain :

$$\int_\Omega A(x)\nabla(u-v).\nabla(u-v)(x)dx = 0$$

In the other hand we have :

$$\alpha\|\nabla(u-v)\|^2_{L^2(\Omega)} \le \int_\Omega A(x)\nabla(u-v).\nabla(u-v)dx \le M\|\nabla(u-v)\|^2_{L^2(\Omega)} \Rightarrow \alpha\|\nabla(u-v)\|_2^2 = 0$$

But

$$\alpha > 0 \Rightarrow \|\nabla(u-v)\|^2_{L^2(\Omega)} = 0$$

Then $u-v = cte$,but $u-v=0$ on $\partial\Omega$,so $u-v=0$ in Ω ,then $u=v$ in Ω

Chapter 3

Solving Dirichlet problem depends on the fractional Laplacian operator of elliptic type

In this chapter we are insterested in studying the nonlocal version of the linear problem

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where, from now on, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $f : \mathbb{R} \to \mathbb{R}$ is some suitable function. Namely,

$$\begin{cases} (-\Delta)^s u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.2}$$

We'll begin by establishing a weak formulation for the equation $(-\Delta)^s u = f$. This will allow us to prove the existence and uniqueness of a solution in the weak sense .To lay the groundwork, we'll first establish some essential concepts and notations that will be used throughout the chapter .

3.1 *Functional setting*

For $u \in \mathscr{S}(\mathbb{R}^N)$ [1]

the fractional Laplace operator can be expressed in Fourier frequency , as stated in the following Definition.

[1]The Schwartz space is defined as $\mathscr{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty \right\}$.

***Definition* 3.1.1.** *[4] For $u \in \mathscr{S}(\mathbb{R}^N)$, we have that*

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}((2\pi|\xi|)^{2s}\hat{u}(\xi)) \tag{3.3}$$

Roughly speaking, formula (3.3)characterizes the fractional Laplace operator in the Fourier space, by taking the s-power of the multiplier associated to the classical Laplacian operator. Indeed, by using the inverse Fourier transform, one has that

$$\Delta u(x) = \Delta\left(F^{-1}(u(\xi))\right)(x) = \Delta\int_{\mathbb{R}^n} u(\xi)e^{2\pi ix\cdot\xi}\,d\xi = \int_{\mathbb{R}^n}(2\pi|\xi|)^2 u(\xi)e^{2\pi ix\cdot\xi}\,d\xi = F^{-1}\left((2\pi|\xi|)^2 u(\xi)\right),$$

From this and Lemma (3.1.1)it also follows that the classical Laplacian is the limit case of the fractional one, namely for any u $\in \mathscr{S}(\mathbb{R}^N)$

(a) $\lim_{s\to 0^+}(-\Delta)^s u(x) = u(x)$,

(b) $\lim_{s\to 1^-}(-\Delta)^s u(x) = -\Delta u(x)$.

Consider the function $g(x) = (2\pi|x|)^{2s}\hat{u}(x), x \in \mathbb{R}^N$ .In general, the function g does not belong to $\mathscr{S}(\mathbb{R}^N, \mathbb{C})$,since $|x|^{2s}$or its derivatives eventually become singular at $x = 0$. However, g belongs to $L^1(\mathbb{R}^N, \mathbb{C})$. So the inverse Fourier transform of g is well defined.
Hence, we can define the fractional Laplacian $(-\Delta)^s u$ as follows

$$(-\Delta)^s u(x) := (2\pi)^{2s}\int_{\mathbb{R}^N} e^{2\pi ix.y}|y|^{2s}\hat{u}(y)dy$$

.

Furthermore,we can define the fractional laplacien as in the following definition.

***Definition* 3.1.2.** *[7] The fractional Laplacian operator is given by the Cauchy principal value integral in the real space, for all $s \in (0,1)$ and every $u \in \mathscr{S}(\mathbb{R}^N)$*

$$(-\Delta)^s u(x) = C(N,s)P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy \quad \forall x \in \mathbb{R}^N,$$

where

$$C(N,s) = \left(\int_{\mathbb{R}^N}\frac{1-\cos y}{|y|^{N+2s}}dy\right)^{-1}$$

is a normalization constant.

In other side, P.V. denotes the Cauchy principal value defined by

$$P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy, = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(x,\epsilon)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy.$$

***Theorem* 3.1.1.** *[15]*

Let $0 < s < 1$ and $\phi \in \mathscr{S}(\mathbb{R}^N)$. Then for $x \in \mathbb{R}^N$,

$$(-\Delta)^s\phi(x) = -\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\frac{\phi(x+y)-2\phi(x)+\phi(x-y)}{|y|^{N+2s}}\,dy \tag{3.4}$$

Why we do call the fractional laplacien as the nonlocal operator ?

Observe that the classical Laplacian, denoted as $-\Delta$, is a local operator because the value of $-\Delta u$ at a point x is determined only by the value of u in a small neighborhood of x. On the other hand, the fractional Laplacian is a nonlocal operator; as described in equation (3.4), the value of $(-\Delta)^s$ at a point x depends on the values of u far away from x.

Definition of $(-\Delta)^s$ ***in the space of tempered distribution*** $\mathscr{S}'$

Lemma 3.1.1. *[16] Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded open set. Then there exists a constant $C = C(n,s,\Omega) > 0$ such that $\forall \varphi \in C_c^\infty(\Omega)$*

$$\left|\int_{|x-y|>\varepsilon}\frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}dy\right| \le C\frac{\|\varphi\|_{C^2(\Omega)}}{1+|x|^{N+2s}} \quad \forall x \in \mathbb{R}^{\mathbb{N}}, \varepsilon \le 1.$$

We will introduce another functional class relevant to the nonlocal operator $(-\Delta)^s$. Previously, to define $(-\Delta)^s u(x)$ pointwise as in definition (3.1.2), we considered functions u in $\mathscr{S}(\mathbb{R}^N)$. However, there are larger spaces where it is still possible to define the nonlocal Laplacian either pointwise or as a tempered distribution.

Definition **3.1.3.** *[11]*

Let $0 < s < 1$. We denote by $\mathscr{L}^s(\mathbb{R}^N)$ the space of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ for which the norm

$$\|u\|_{\mathscr{L}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+2s}}\,dx < \infty.$$

Notice that we trivially have $u \in \mathscr{L}^s(\mathbb{R}^N) \implies u \in L^1_{loc}(\mathbb{R}^N)$. This inclusion implies in particular that

$$\mathscr{S}(\mathbb{R}^N) \subset \mathscr{L}^s(\mathbb{R}^N) \subset \mathscr{S}'(\mathbb{R}^N).$$

We also note that for any $1 \le p \le +\infty$, we have $L^p(\mathbb{R}^N) \subset \mathscr{L}^s(\mathbb{R}^n)$

Corollaire **3.1.1.** *By the previous result, we have that for any $u \in \mathscr{L}_s(\mathbb{R}^N)$,then $(-\Delta u)^s \in \mathscr{S}'(\mathbb{R}^{\mathbb{N}})$. Then we have*

$$\langle(-\Delta)^s u, \varphi\rangle = \int_{\mathbb{R}^N} u(-\Delta)^s\varphi dx \quad \forall v \in D(\mathbb{R}^N)$$

3.2 *Existence and Uniqueness of the Weak Solution*

Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ be a bounded open set with Lipschitz boundary.

Definition 3.2.1. *[19] A natural functional space to find the weak solutions of the problem (3.5) is given by following non−local version of $H_0^1(\Omega)$*

$$H_\Omega^s(\mathbb{R}^N) := \{u \in H^s(\mathbb{R}^N) : u = 0 \; in \; \mathbb{R}^N \setminus \Omega\}$$

Remark 3.2.1. *$H_\Omega^s(\mathbb{R}^N)$ preserves some important density properties of the classical Sobolev spaces such as*

- *If Ω is Lipschitz then $H_\Omega^s(\mathbb{R}^N)$ is the closure of $C_c^\infty(\Omega)$ in $H^s(\mathbb{R}^N)$.*
- *If $\Omega$is Lipschitz and $s \neq \frac{1}{2}$ then $H_\Omega^s(\mathbb{R}^N)$ is the completion of $C_c^\infty(\Omega)$ with the norme of $H^s(\Omega)$*

Another important feature of this space is that the functions contained in it satisfy a nonlocal version of the Poincaré inequality.

Theorem 3.2.1. *(poincaré inequality) .If Ω is Lipschitz then there exist a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C[u]_{H^s(\mathbb{R}^N)} \forall u \in H_\Omega^s(\mathbb{R}^N)$$

Remark 3.2.2. *Theorem (3.2.1) implies that, in $H_\Omega^s(\mathbb{R}^N)$, the Gagliardo seminorm is actually a norm equivalent to the H^s norm. Hence, in the sequel, given u,v $\in H_\Omega^s(\mathbb{R}^N)$ we will use the following conventions*

$$\|u\|_{H_\Omega^s(\mathbb{R}^N)} := [u]_{H^s(\mathbb{R}^N)}$$

$$\langle u, v \rangle_{H_\Omega^s(\mathbb{R}^N)} := \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy.$$

Now , we consider our problem as follows

$$\begin{cases} (-\Delta)^s u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.5}$$

Proposition 3.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $0 < s < 1$, $f \in L^2(\Omega)$. A function $u \in H_\Omega^s(\mathbb{R}^N)$ is a weak solution to the Dirichlet problem (3.5) if*

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_\Omega f v \, dx = 0 \quad \forall v \in H_\Omega^s(\mathbb{R}^N)$$

Remark 3.2.3. *[15] Let $0 < s < 1$ and $u \in \mathscr{S}(\mathbb{R}^N)$, then $u \in H^s(\mathbb{R}^N)$*

Proof. ***Step 1:*** Weak formulation

By definition (3.1.3) , $u \in \mathscr{S}(\mathbb{R}^N)$ then $u \in \mathscr{L}^s(\mathbb{R}^N)$, then by Corollary (3.1.1) we get:

$$\langle (-\Delta)^s u, v\rangle = \int_\Omega u(-\Delta)^s v dx \quad \forall v \in D(\Omega)$$

Using the fact that $v = 0$ in $\mathbb{R}^N \setminus \Omega$

$$\int_\Omega v(x) \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dydx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx = I$$

then

$I = \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx + \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx$

Using Fubini theorem ,we get

$$\begin{aligned} I &= \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx + \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(y,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx - \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(y,\varepsilon)} \frac{u(y)-u(x)}{|x-y|^{N+2s}} v(x) dxdy \end{aligned}$$

taking $x = X$,and $y = Y$,then

$I = \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx - \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(Y,\varepsilon)} \frac{u(Y)-u(X)}{|X-Y|^{N+2s}} v(X) dXdY$

Choosing $X = y$,and $Y = x$ then

$$\begin{aligned} I &= \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x) dydx - \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(y) dydx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx \end{aligned}$$

Then,we get

$$C_{N,s}\int_\Omega v(x) \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dydx = C_{N,s}\frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx \tag{3.6}$$

The next step is to prove that :

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx = \frac{1}{2}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx$$

for that, will use the dominted convergence theorem taking

$$I_\varepsilon(x) = I_{\frac{1}{n}}(x) = \int_{\mathbb{R}^N \setminus B(x,\frac{1}{n})} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dy$$

$i)$ $\underset{n\to+\infty}{lim} I_{\frac{1}{n}}(x) = \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dy$

$ii)$ $|I_{\frac{1}{n}}(x)| \leq \int_{\mathbb{R}^N} \frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{N+2s}} dy$

by Young inequality ,we get :

$$|I_{\frac{1}{n}}(x)| \leq \frac{1}{2}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy + \frac{1}{2}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dy$$

Now,taking

$$g(x) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy + \frac{1}{2}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dy$$

$$\int_{\mathbb{R}^N} |g(x)| dx \leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dydx$$

using the fact that $D(\mathbb{R}^N) \subset \mathscr{S}(\mathbb{R}^N)$,and by By remark (3.2.3), we obtain $u \in H^s(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx < +\infty$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dydx < +\infty$$

So g $\in L^1(\mathbb{R}^N)$ Finally ,by the dominated convergence theorem ,we get

$$\underset{\varepsilon\to0}{lim} C_{N,s}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx = C_{N,s}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx \tag{3.7}$$

Now,we will prove that

$$\underset{\varepsilon\to0}{lim}\int_\Omega v(x)\int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dydx = \int_\Omega v(x)(-\Delta)^s u(x)dx$$

again , using dominated convergence theorem, taking

$$I'_\varepsilon(x) = \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x)dy$$

$i)$

$$\underset{\varepsilon\to0}{lim} I'_\varepsilon(x) = \underset{n\to+\infty}{lim} I'_{\frac{1}{n}}(x) = \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy.v(x)$$

ii) By lemma (3.1.1) we get

$$|I'_\varepsilon(x)| \leq C\|u\|_{C^2(\Omega)} \frac{|v(x)|}{1+|x|^{N+2s}}$$

$$h(x) = C\|u\|_{C^2(\Omega)} \frac{|v(x)|}{1+|x|^{N+2s}} \text{ ,then}$$

$$\int_{\mathbb{R}^N} |h(x)|dx = C\|u\|_{C^2(\Omega)} \int_{\mathbb{R}^N} \frac{|v(x)|}{1+|x|^{N+2s}}dx$$

as $D(\mathbb{R}^N) \subset \mathscr{S}(\mathbb{R}^N) \subset \mathscr{L}_s^1(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} \frac{|v(x)|}{1+|x|^{N+2s}}dx < \infty$$

then $h \in L^1(\mathbb{R}^N)$ So ,by dominated convergence theorem ,we get

$$\lim_{\varepsilon\to 0} C_{N,s} \int_\Omega v(x) \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dydx = \int_\Omega v(x)(-\Delta)^s u(x)dx \tag{3.8}$$

then by (3.6), (3.7), and (3.8) we get

$$\int_\Omega v(x)(-\Delta)^s u(x)dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx \quad \forall v \in D(\mathbb{R}^N)$$

we deduce that

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx = \int_\Omega fvdx \quad \forall v \in D(\Omega)$$

by remark (3.2.1),we have $H^s_\Omega(\mathbb{R}^N) = \overline{D(\Omega)}^{H^s(\mathbb{R}^N)}$,then $\forall v \in H^s_\Omega(\mathbb{R}^N), \exists (\varphi_n)_{n\in\mathbb{N}}$ such that $\varphi_n \overset{\|\,\|_{H^s(\mathbb{R}^N)}}{\longrightarrow} v$,then

$$\tilde{I} = \left| \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}} dydx - \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dydx \right| \leq$$
$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)||(\varphi_n-v)(x)-(\varphi_n-v)(y)|}{|x-y|^{N+2s}} dydx$$

Applying Cauchy Schwartz inequality we get :

$$|\tilde{I}| \leq \frac{C_{N,s}}{2} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dy\, dx \right)^{1/2} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\varphi_n-v)(x)-(\varphi_n-v)(y)|^2}{|x-y|^{N+2s}}\, dy\, dx \right)^{1/2}.$$

Since $\varphi_n \overset{\|\,\|_{H^s(\mathbb{R}^N)}}{\longrightarrow} v$, we deduce that $\varphi_n \overset{\|\,\|_{L^2(\mathbb{R}^N)}}{\longrightarrow} v$ and $\varphi_n \overset{[\cdot]_{H^s(\mathbb{R}^N)}}{\longrightarrow} v$

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\varphi_n-v)(x)-(\varphi_n-v)(y)|^2}{|x-y|^{N+2s}}\,dy\,dx\right)^{1/2}\underset{n\to+\infty}{\longrightarrow}0$$

then

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}dydx\underset{n\to+\infty}{\longrightarrow}\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx$$

On the other hand

$$\begin{aligned}|\int_\Omega f\varphi_n dx-\int_\Omega fvdx| &\le \int_\Omega |f||\varphi_n-v|dx\\ &\le \|f\|_{L^2(\Omega)}\|\varphi_n-v\|_{L^2(\Omega)}\underset{n\to+\infty}{\longrightarrow}0\end{aligned}$$

Finally we get

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx=\int_\Omega fvdx\quad\forall v\in H^s_\Omega(\mathbb{R}^N)$$

by remark (3.2.2),we obtain

$$\frac{C_{N,s}}{2}\langle u,v\rangle_{H^s_\Omega}(\mathbb{R}^N)=\int_\Omega fvdx\quad\forall v\in H^s_\Omega(\mathbb{R}^N)$$

Step 2:Existence of minimizers

***Proposition* 3.2.2.** *Consider the following variational problem*

$$(P_s):\begin{cases}Find\ u\in H^s_\Omega(\mathbb{R}^N)\quad such\ that\\ J(u,u)=\underset{v\in H^s_\Omega(\mathbb{R}^N)}{min}J(v,v)\end{cases}$$

Where

$$J(v,v)=\frac{C_{N,s}}{4}\langle v,v\rangle_{H^s_\Omega(\mathbb{R}^N)}-\int_\Omega fvdx$$

Then (P_s) *admits a unique solution.*

We need to prove some results before the proof of the existence

***Lemma* 3.2.1.** *Let* J *be such that*

$$J(u,v)=\frac{C_{N,s}}{4}\langle u,v\rangle_{H^s_\Omega(\mathbb{R}^N)}-\int_\Omega fvdx\quad\forall v\in H^s_\Omega(\mathbb{R}^N)$$

Then J *is continuous.*

Proof. We can put $J(u,v)=a(u,v)-L(v)$ such that :
$a(u,v)=\frac{C_{N,s}}{4}\langle u,v\rangle_{H^s_\Omega(\mathbb{R}^N)}$

$L(v) = \int_{\Omega} fvdx$
For all $v \in H^s_{\Omega}(\mathbb{R}^N)$ and by using Cauchy-Schwartz inequality we have :

$$\begin{aligned} |a(u,v)| &= \frac{C_{N,s}}{4} |\langle u, v\rangle_{H^s_{\Omega}(\mathbb{R}^N)}| \\ &\leq \frac{C_{N,s}}{4} \|u\|_{H^s_{\Omega}(\mathbb{R}^N)} \|v\|_{H^s_{\Omega}(\mathbb{R}^N)} \end{aligned}$$

Then a(.,.)is continuous in $H^s_{\Omega}(\mathbb{R}^N) \times H^s_{\Omega}(\mathbb{R}^N)$

$$\begin{aligned} |L(v)| &= |\int_{\Omega} fvdx| \\ &\leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq C\|f\|_{L^2(\Omega)} \|u\|_{H^s_{\Omega}(\mathbb{R}^N)} \quad (\text{ Poincaré inequality}) \end{aligned}$$

Then , L(.) is continuous in $H^s_{\Omega}(\mathbb{R}^N)$.
Finelly , J(.,.) is continous in $H^s_{\Omega}(\mathbb{R}^N) \times H^s_{\Omega}(\mathbb{R}^N)$. □

***Lemma* 3.2.2.** *Let* $(u_n)_{n\geq 1} \subset H^s_{\Omega}(\mathbb{R}^N)$ *be a minimizing sequence, i.e.*

$$lim_{n\to+\infty} J(u_n, u_n) = \underset{v\in H^s_{\Omega}(\mathbb{R}^N)}{inf} J(v,v) = m$$

Then, $(u_n)_{n\geq 1}$ *is bounded in* $H^s_{\Omega}(\mathbb{R}^N)$.

***Proof*.** According to the definition of the inf of function we have :

$$m \leq J(\varphi,\varphi) \leq m + \frac{1}{n} \quad n \geq 1, \varphi \in D(\Omega) \tag{3.9}$$

With Cauchy-Schwartz inequality and (3.9) we get that:

$$\begin{aligned} m \leq J(\varphi,\varphi) &= \frac{C_{N,s}}{4} \|\varphi\|_{H^s_{\Omega}(\mathbb{R}^N)} - \int_{\Omega} f\varphi dx \\ &\leq \frac{C_{N,s}}{4} \|\varphi\|^2_{H^s_{\Omega}(\mathbb{R}^N)} + \|g\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\ m &< \infty \end{aligned}$$

on the other hand

$$J(u_n,u_n) = \frac{C_{N,s}}{4} \|u_n\|^2_{H^s_{\Omega}(\mathbb{R}^N)} - \int_{\Omega} fu_n dx \tag{3.10}$$

Using Cauchy-Schwartz inequality and Young's inequality we obtain :

$$\begin{aligned}\int_\Omega f u_n dx \le \|f\|_{L^2(\Omega)} &\le C\|f\|_{L^2(\Omega)}\|u_n\|_{H^s_\Omega(\mathbb{R}^N)}\\ &\le \frac{\sqrt{C_{N,s}}}{2}\|u_n\|_{H^s_\Omega(\mathbb{R}^N)}\,\frac{2C}{\sqrt{C_{N,s}}}\|f\|_{L^2(\Omega)}\\ &\le \frac{C_{N,s}}{8}\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} + \frac{2C^2}{C_{N,s}}\|f\|^2_{L^2(\Omega)}\end{aligned}$$

then we get :

$$J(u_n,u_n) \ge \frac{C_{N,s}}{4}\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{C_{N,s}}{8}\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{2C^2}{C_{N,s}}\|f\|^2_{L^2(\Omega)}$$

hence

$$\frac{C_{N,s}}{8}\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} \le J(u_n,u_n) + \frac{2C^2}{C_{N,s}}\|f\|^2_{L^2(\Omega)}$$

Then

$$\frac{C_{N,s}}{8}\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} < m + \frac{1}{n} + \frac{2C^2}{C_{N,s}}\|f\|^2_{L^2(\Omega)}$$

So

$$\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} \le \frac{8}{C_{N,s}}\left(m + \frac{1}{k} + \frac{1}{2}\|f\|^2_{L^2(\Omega)}\right) \to R \in \mathbb{R}, k \to +\infty$$

Then $(u_n)_{n\ge1}$ is bounded in $H^s_\Omega(\mathbb{R}^N)$ □

Proof*. (of* ***Proposition*** (3.2.2) *)* With the regularity assumptions of Ω , the compact embedding Theorem with the fractional H^snorm tells us that $H^s_\Omega(\mathbb{R}^N)$ is compactly embedded (pre-compact) in $L^2(\Omega)$

Since $(u_n)_n \ge 1$ is bounded in $H^s_\Omega(\mathbb{R}^N)$ by lemma (3.2.2) , then there exists a subsequence $(u_{nk})_{k\ge1}$ of $(u_n)_n \ge 1$ such that

$$u_{nk} \rightharpoonup u \text{ in } H^s_\Omega(\mathbb{R}^N)(\textit{ weak convergence})$$

and also :

$$u_{nk} \to u \text{ in } L^2(\Omega)\text{by compact embedding}$$

that implies a pointwise convergence in Ω and a weak one i.e.

$$u_{nk} \rightharpoonup u \text{ in } L^2(\Omega) \text{ as } l \to +\infty$$

. Notice that we have here three different convergences with different subsequence, but we keep the same notation. With the pointwise convergence, we get the same limit by unicity of the limit.

Therefore, we get that:

$$J(u_{n_k}, u_{n_k}) = \frac{C_{N,s}}{4}\int_{\mathbb{R}^{2N}} \frac{|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}dxdy - \int_\Omega f u_{n_k} dx < m + \frac{1}{k}, \forall k > 0$$

So

$$\frac{C_{N,s}}{4}\int_{\mathbb{R}^{2N}} \frac{|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}dxdy < \int_\Omega f u_{n_k} dx + m + \frac{1}{k}, \forall k > 0$$

then

$$\frac{C_{N,s}}{4} \lim\inf_{k\to+\infty} \int_{\mathbb{R}^{2N}} \frac{|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}\, dy\, dx \le \liminf_{k\to+\infty} \int_\Omega f u_{n_k} dx + m$$

Since $u_{n_k} \rightharpoonup u$ in $H^s_\Omega(\mathbb{R}^N)$, ,then by proposition (1.5.1),we get:

$$\frac{C_{N,s}}{4}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dy\, dx \le \frac{C_{N,s}}{4} \liminf_{l\to+\infty} \int_{\mathbb{R}^{2N}} \frac{|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{N+2s}}\, dy\, dx.$$

Then

$$\frac{C_{N,s}}{4}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx \le \liminf_{k\to+\infty} \int_\Omega f u_{n_k} dx + m$$

Since $u_{n_k} \rightharpoonup u$ in $L^2(\Omega)$, it implies that for all $v \in L^2(\Omega)$,

$$\int_\Omega u_{n_k} v\, dx \to \int_\Omega uv\, dx.$$

Specifically, choosing $v = f$, we get

$$\int_\Omega u_{n_k} f\, dx \to \int_\Omega uf\, dx.$$

we deduce that

$$\liminf_{k\to+\infty} \int_\Omega f u_{n_k} dx = \lim_{k\to+\infty} \int_\Omega f u_{n_k} dx = \int_\Omega f u dx$$

. Then

$$\frac{C_{N,s}}{4}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx \le \int_\Omega f u dx + m$$

Therefore

$$J(u,u) = \frac{C_{N,s}}{4}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx - \int_\Omega f u dx \le m$$

So

$$J(u,u) \le m \Rightarrow J(u,u) = m$$

Finally J has a minimizer $u \in H^s_\Omega(\mathbb{R}^N)$.

Step 3: Weak Euler–Lagrange equation Given $w \in H^s_\Omega(\mathbb{R}^N)$ and $t \in \mathbb{R}$ we have that $u + tw \in H^s_\Omega(\mathbb{R}^N)$, so $\psi(t) := \mathcal{F}(u+tw) \ge \mathcal{F}(u) = \psi(0)$. Hence, the function ψ has a minimum at $t = 0$. If ψ is differentiable at $t = 0$, then $\psi'(0) = 0$. But

$$(u(x)+tw(x)-u(y)-tw(y))^2 = (u(x)-u(y))^2 + 2t(u(x)-u(y))(w(x)-w(y)) + t^2(w(x)-w(y))^2,$$

which implies

$$\psi(t)-\psi(0)=\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy+\frac{tC_{N,s}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,dxdy$$
$$-\int_\Omega fw\,dx.$$

$$\to\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy-\int_\Omega fw\,dx.$$

Thus, we have shown that

$$0=\psi'(0)=\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy-\int_\Omega fw\,dx.$$

This proves that u is the weak solution to our problem.

□

□

Uniqueness of minimizers

let u_1 and $u_2\in H^s_\Omega(\mathbb{R}^N)$ be solutions of (3.5) , by difference ,end by put $w=u_1-u_2$ we obtain:

$$\begin{cases}(-\Delta)^s w=0 & \text{in }\Omega\\ w=0 & \text{in }\mathbb{R}^N\setminus\Omega\end{cases}\tag{3.11}$$

The weak formulation of the equation of (3.11),with chossing the test fuction $\varphi=w$,we get :

$$\int_{\mathbb{R}^{2N}}\frac{|w(x)-w(y)|^2}{|x-y|^2}dydx=0,w\in H^s_\Omega(\mathbb{R}^{\mathbb{N}})$$

On the other hand by Poincaré's inequlity we deduce:

$$\|w\|^2_{L^2(\Omega)}\le c\|w\|^2_{H^s_\Omega(\mathbb{R}^N)}=0\Rightarrow\|w\|^2_{L^2(\Omega)}=0\Rightarrow w=0\text{ in }\Omega\Rightarrow u_1=u_2\text{ in }\Omega$$

Finally $u_1=u_2$ in $\mathbb{R}^N$.

Stability

Proposition 3.2.3. *The equation 3.5 has a unique solution that depends continuously on the data $f\in L^2(\Omega)$*

Proof. To show that the solution depends continuously on the data f, let $u_1,u_2\in H^s_\Omega(\mathbb{R}^N)$ with $u_1\neq u_2$,be solutions of the same type of equation as 3.5 with data $f_1,f_2\in L^2(\Omega)$ respectively. *Note that*

$$\langle u,v\rangle_{H^s_\Omega(\mathbb{R}^N)}=\int_\Omega fvdx\tag{3.12}$$

$$\|u_1-u_2\|_{H^s_\Omega(\mathbb{R}^N)}=\frac{\langle u_1-u_2,u_1-u_2\rangle_{H^s_\Omega(\mathbb{R}^N)}}{\|u_1-u_2\|_{H^s_\Omega(\mathbb{R}^N)}}$$

According to (3.12) *we get :*

$$\begin{aligned}\|u_1 - u_2\|_{H^s_\Omega(\mathbb{R}^N)} &\leq \frac{\|u_1 - u_2\|_{L^2(\Omega)}\|f_1 - f_2\|_{L^2(\Omega)}}{\|u_1 - u_2\|_{H^s_\Omega(\mathbb{R}^N)}} \\ &\leq C\frac{\|u_1 - u_2\|_{H^s_\Omega(\mathbb{R}^N)}\|f_1 - f_2\|_{L^2(\Omega)}}{\|u_1 - u_2\|_{H^s_\Omega(\mathbb{R}^N)}} \\ &\leq c\|f_1 - f_2\|_{L^2(\Omega)}\end{aligned}$$

we are done. □

Chapter 4

Eigen values problem of nonlocal elleptic operator with semilinear term

Contents

This chapter investigates the existence of non zero solutions to equations governed by the nonlocal operator $\mathcal{L}_k$ with homogeneous Dirichlet boundary conditions .Specifically we consider the problem:

$$(p_\lambda) : \begin{cases} -\mathcal{L}_k u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

Where the set $\Omega \subset \mathbb{R}^N, N > 2s$, is open, bounded and with Lipschitz boundary. $\lambda \in \mathbb{R}$,and f is the non linear term.

4.1 *Introduction*

We are insterested in studying the problem :

$$\begin{cases} (-\Delta)^s u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \partial\Omega \end{cases} \tag{4.1}$$

for $s \in (0,1)$and $(-\Delta)^s$ is the fractional laplace operator which (up to normalization factors) may be defined as :

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} dy, x \in \mathbb{R}^N \tag{4.2}$$

To be precise ,the aim of this work is studying the following equation

$$(p_\lambda) : \begin{cases} -\mathcal{L}_K u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{4.3}$$

where $\mathcal{L}_K$ is the nonlocal operator defined as follows :

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^N} \left(u(x+y)+u(x-y)-2u(x)\right) K(y) dy, x \in \mathbb{R}^N \tag{4.4}$$

Here , $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a function such that :

i)

$$mK \in L^1(\mathbb{R}^N) : m(x) = min\{|x|^2, 1\} \tag{4.5}$$

ii)

$$\exists \theta > 0 : K(x) \geq \frac{\theta}{|x|^{N+2s}}, \forall x \in \mathbb{R}^N \setminus \{0\} \tag{4.6}$$

iii)

$$K(x) = K(-x), \forall x \in \mathbb{R}^N \setminus \{0\} \tag{4.7}$$

A typical example for K is given by $K(x) = \dfrac{1}{|x|^{N+2s}}$,then $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$ define in (4.2) The weak formulation of (4.3) is given by the following problem:

$$\begin{cases} u \in X_0 \text{ such that :} \\ \int_{\mathbb{R}^N} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \lambda \int_\Omega u(x)\varphi(x)dx = \int_\Omega f(x,u(x))\varphi(x)dx \quad \forall \varphi \in X_0 \end{cases} \tag{4.8}$$

Here the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map :

$(x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}$is in $L^2(\mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega, dxdy)$,where $\mathcal{C}\Omega =: \mathbb{R}^N \setminus \Omega$. Moreover

$$X_0 = \{g \in X : g = 0 \text{a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

We note that

$$C_0^2(\Omega) \subseteq X_0 \tag{4.9}$$

see, e.g., [[23], Lemma 11] (for this we need condition (4.5), and so X and X_0 are non-empty.

Finally, we suppose that the nonlinear term in equation (4.3)is a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verifying the following conditions:

1)

$$\text{f is continuous in } \overline{\Omega} \times \mathbb{R} \tag{4.10}$$

2) $\exists \alpha_1, \alpha_2 > 0$,and $q \in (2, 2^*), 2^* = \dfrac{2N}{N-2s}$ (the element 2^* is called the critical Sobolev exponent) such that :

$$|f(x,t)| \leq \alpha_1 + \alpha_2 |t|^{q-1}, \forall x \in \Omega, t \in \mathbb{R} \tag{4.11}$$

3)

$$\lim_{t \to 0} \frac{f(x,t)}{t} = 0 \tag{4.12}$$

uniformely in $x \in \Omega$ (f is **super -linear**)

4)

$$tf(x,t) \geq 0 \; \forall x \in \Omega, t \in \mathbb{R} \tag{4.13}$$

5) $\exists \mu > 2$, and $r > 0$ such that $\forall x \in \overline{\Omega}, t \in \mathbb{R} \; |t| \geq r$

$$0 < \mu F(x,t) \leq tf(x,t) \tag{4.14}$$

Where the function F is the primitive of f with respect to its second variable ,that is

$$F(x,t) = \int_0^t f(x,\tau)d\tau \tag{4.15}$$

(f is **super -quadratic**) ,this hypothesis is also known in the literature as the " **Ambrosetti-Rabinowitz hypothesis**".

▷ we remark that $f(x, 0) = 0$ thanks to (4.12) $\Rightarrow u = 0$ is a trivial solution of (4.3).

Our scope

We will constrate non-trivial solutions for (4.3). Using two different variational tricks .
When $\lambda < \lambda_1$(where$\lambda_1$is the first eigenvalue of $-\mathcal{L}_K$,see Section 3),we can find a non -trivial solution using "**The Mountain Pass theorem** " .
When $\lambda \geq \lambda_1$ we can achieve our goal using "**the Linking theorem**".

The main result of the present chapter is an existence theorem for equation driven by general integrodifferential operators of non-local fractional type ,as stated here below .

***Theorem* 4.1.1.** *Let $s \in (0,1), N > 2s$and Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary. Let $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$be a function satisfying conditions (4.5) to (4.6) and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verify (4.10) to (4.14). Then, for any $\lambda \in \mathbb{R}$ problem (4.8) admits a solution $u \in X_0$which is not identically zero.*

In the nonlocal framework, the simplest example we can deal with is given by the fractional Laplacian $(-\Delta)^s$, according to the following result:

***Theorem* 4.1.2.** *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Consider the following equation*

$$\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dx\, dy - \lambda \int_{\Omega} u(x)\varphi(x)\, dx = \int_{\Omega} f(x,u(x))\varphi(x)\, dx \tag{4.16}$$

for any $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.
If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function verifying (4.10) – (4.14), then, for any $\lambda \in \mathbb{R}$, problem (4.16) admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

4.2 *Some preliminary results*

This section establishes preliminary results essential for later sections.

4.2.1 *The variational methods*

Palais-Smale Condition

To express the compactness of minimizing sequences, or more generally of sequences that converge to a point which one hopes to show is a critical point, one often resorts to the Palais-Smale condition.

***Definition* 4.2.1.** *[14] Let E be a Banach space, and $J : E \to \mathbb{R}$ a function of class C^1. If $c \in \mathbb{R}$, we say that J satisfies the Palais-Smale condition (at level c) if every sequence $(u_n)_n$ in*

E such that

$$J(u_n) \to c \text{ in } \mathbb{R}, \text{ and } J'(u_n) \to 0 \text{ in } E'$$

contains a convergent subsequence $(u_{n_k})_k$.

Mountain Pass Theorem

Theorem 4.2.1. *[14] Let E be a Banach space, $J \in C^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. We assume that $J(0) = 0$ and that:*

1. *There exist $R > 0$ and $a > 0$ such that if $\|u\| = R$, then $J(u) \geq a$;*

2. *There exists $u_0 \in E$ such that $\|u_0\| > R$ and $J(u_0) < a$.*

Then J has a critical value c such that $c \geq a$. More precisely, if we define

$$B := \{\varphi([0,1]); \varphi \in C([0,1], E), \varphi(0) = 0, \varphi(1) = u_0\},$$

and:

$$c := \inf_{A \in B} \max_{v \in A} J(v),$$

then c is a critical value of J, and $c \geq a$.

Linking Theorem

Theorem 4.2.2. *[17]*

Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $J \in C^1(E, \mathbb{R})$ satisfies (PS), and

(i) there are constants $\rho, a > 0$ such that $\|J_{\partial B_\rho \cap X}\| \geq a$, and

(ii) there is an $u_0 \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\overline{B_R} \cap V) \oplus \{re \mid 0 < r < R\}$, then $I|_{\partial Q} \leq 0$.

Then J possesses a critical value $c \geq a$ which can be characterized as

$$c \equiv \inf_{h \in \Gamma} \max_{u \in Q} J(h(u)),$$

where

$$\Gamma = \{h \in C(\overline{Q}, E) \mid h = id \text{ on } \partial Q\}.$$

∂Q refers to the boundary of Q relative to $V \oplus spane\{u_0\}$

4.2.2 *Estimates on the nonlinearity and its primitive*

We gather some basic results that will be crucial for the main estimates later. Leveraging the growth conditions on f, to deduce some bounds from above and below for the nonlinear term and its primitive.

***Lemma* 4.2.1.** *Assume $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying conditions (4.10)–(4.12). Then, for any $\varepsilon > 0, \exists \delta = \delta(\varepsilon)$such that $\forall x \in \Omega$ and $t \in \mathbb{R}$*

$$|f(x,t)| \leq 2\varepsilon|t| + q\delta(\varepsilon)|t|^{q-1} \tag{4.17}$$

as a consequence

$$|F(x,t)| \leq \varepsilon|t|^2 + \delta(\varepsilon)|t|^q \tag{4.18}$$

Where F is defined as in (4.15).

For the proof of Lemma (4.2.1) see ([22], Lemma 3)(similar estimates are also in ([17], [24])

***Lemma* 4.2.2.** *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions (4.10) and (4.14). Then, there exist two positive constants α_3 and α_4 such that $\forall x \in \Omega$and $t \in \mathbb{R}$*

$$F(x,t) \geq \alpha_3|t|^\mu - \alpha_4 \tag{4.19}$$

***Proof*.** Let $r > 0$, be as in (4.14) then, $\forall x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geq r > 0$:

$$\frac{tf(x,t)}{F(x,t)} \geq \mu$$

***case 1*:** suppose $t > r$

Dividing by t and integrating both terms in $[r,t]$ we obtain

$$\int_r^t \frac{f(x,\tau)}{F(x,\tau)} d\tau \geq \int_r^t \mu \frac{1}{\tau} d\tau$$

then

$$\ln F(x,\tau)|_r^t \geq \mu \ln \tau|_r^t$$

So

$$F(x,t) \geq \frac{F(x,r)}{r^\mu}\, t^\mu, \;\; \forall x \in \overline{\Omega}, \;\; t \in \mathbb{R}, \;\; t > r \tag{4.20}$$

since $x \mapsto F(x,r)$ is continuous in $\overline{\Omega}$, by the Weierstrass Theorem there exists $\min_{x\in\overline{\Omega}} F(x,r)$. Hence we get:

$$F(x,t) \geq \min_{x\in\overline{\Omega}} F(x,r)\; r^{-\mu} t^\mu, \;\; \forall x \in \overline{\Omega}, \;\; t \in \mathbb{R}, \;\; t > r \tag{4.21}$$

***case 2*:** $t < -r < 0$

with the same argument we get: [we dividing by $t < 0$]

$$\int_t^{-r} \frac{f(x,\tau)}{F(x,\tau)} d\tau \le \int_t^{-r} \frac{\mu}{\tau} d\tau.$$

we put $T = -\tau$ we get:

$$\begin{aligned}
&\int_{-t}^{r} \frac{-f(x,-T)}{F(x,-T)} dT \le \mu \int_{-t}^{r} \frac{dT}{T} \\
&\ln \frac{F(x,-r)}{F(x,t)} - \ln\left(\frac{r^\mu}{(-t)^\mu}\right) \le 0 \\
&\ln\left(\frac{\frac{F(x,-r)}{F(x,t)}}{\frac{r^\mu}{|t|^\mu}}\right) \le 0 \\
\Rightarrow &0 \le \frac{F(x,-r)}{F(x,t)} \frac{|t|^\mu}{r^\mu} \le 1 \\
\Rightarrow &F(x,t) \ge \frac{F(x,-r)}{r^\mu} |t|^\mu
\end{aligned}$$

by Weierstrass Theorem we obtain:

$$F(x,t) \ge \min_{x\in\overline{\Omega}} F(x,-r)\, r^{-\mu}\, t^\mu, \ \forall x \in \overline{\Omega}, \ \ t \in \mathbb{R}, \ \ t < -r$$

so, $\forall x \in \overline{\Omega}$, and $t \in \mathbb{R}$ with $|t| \ge r$ we get:

$$F(x,t) \ge a_3 |t|^\mu \tag{4.22}$$

where $a_3 = r^{-\mu} \min\left\{\min_{x\in\overline{\Omega}} F(x,r)_{x\in\overline{\Omega}}, \ \min_{x\in\overline{\Omega}} F(x,-r)\right\} > 0.$
Note that $a_3 > 0$ being $F(x,t) > 0$, $\forall x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \ge r$,(see assumption (4.14)). Since the function F is continuous in $\overline{\Omega} \times \mathbb{R}$, by the Weierstrass Theorem, it is bounded for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \le r$, say:

$$|F(x,t)| \le \tilde{a_4} \text{ in } \overline{\Omega} \times \{|t| \le r\} \tag{4.23}$$

fore some positive constant $\tilde{a_4}$. Then we obtain:

$$F(x,t) \ge -\tilde{a_4} \quad \text{in } \overline{\Omega} \times \{|t| \le r\} \tag{4.24}$$

on the other hand: for $|t| \leq r$ we have: $a_3|t|^\mu \leq a_3 r^\mu$ then

$$a_3|t|^\mu - a_3 r^\mu \leq 0$$
$$a_3|t|^\mu - a_3 r^\mu - \tilde{a_4} \leq -\tilde{a_4}$$

by (4.24):

$$F(x,t) \geq a_3|t|^\mu - a_3 r^\mu - \tilde{a_4}$$

Finally: $F(x,t) \geq a_3|t|^\mu - a_4$ with: $a_4 = a_3 r^\mu + \tilde{a_4} > 0$ □

4.2.3 *The functional sitting*

Now we recall some basic results on the space X and X_0. In the sequal we set $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$ where

$$\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2N} \tag{4.25}$$

and $\mathcal{C}\Omega = \mathbb{R}^{\mathbb{N}} \setminus \Omega$. The space X is endowed with the norm defined as:

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \left(\int\limits_Q |g(x) - g(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}} \tag{4.26}$$

***Proposition* 4.2.1.** *$\|.\|_X$ is a norm on X.*

***Proof*.** We only show that if $||g||_X = 0 \Rightarrow$ then $g = 0$ a.e in $\mathbb{R}^N$ Let $g \in X$: such that $||g||_X = 0$, then:

1. $$\|g\|_{L^2(\Omega)} = 0 \Rightarrow g = 0 \ \text{ a.e } in \ \Omega \tag{4.27}$$

2. $$\int\limits_Q |g(x) - g(y)|^2 K(x-y) dxdy = 0 \Rightarrow g(x) = g(y) \text{ a.e. } (x,y) \in Q$$

   (because of $K(x-y) > 0 \forall x, y \in \mathbb{R}^N \setminus \{0\}$)

Then: $g = c$ a.e in $\mathbb{R}^N$.

By (4.27) we conclude that: $c = 0$ so that: $g = 0$ a.e in $\mathbb{R}^N$. □

Why didn't we use $H^s(\Omega)$ instead of X ?

***Remark* 4.2.1.** *We denote by $H^s(\Omega)$ the usual fractional sobolev space endowed with the norm:*

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left(\int\limits_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}. \tag{4.28}$$

In the model case which $K(x) = \dfrac{1}{|x|^{N+2s}}$*, the norms in* (4.26) *and* (4.28) *is not the same, because* $\Omega\times\Omega$ *is strictly contained in* Q *that makes the classical fractional sobolev approach not sufficient for studying the problem.*

For further details on the fractional Sobolev spaces we refer to [7] and to the references therein.

***Lemma* 4.2.3.** *(The relation between* X *and* X_0 *with* H^s*) Let* $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions* (4.5)-(4.7)*. Then the following assertions true:*

a) *if* $v \in X$ *then* $v \in H^s(\Omega)$*. Moreover:*

$$\|v\|_{H^s(\Omega)} \leq c(\theta).\|v\|_X$$

b) *if* $v \in X_0$ *then* $v \in H^s(\mathbb{R}^N)$*. Moreover:*

$$\|v\|_{H^s(\Omega)} \leq ||v||_{H^s(\mathbb{R}^N)} \leq c(\theta).\|v\|_X$$

$c(\theta) = \max\left\{1, \dfrac{1}{\sqrt{\theta}}\right\}$, *where* θ *is given in* (4.6).

Proof. Let us prove part a.
By (4.4) we get:

$$\begin{aligned}\int\limits_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy &\leq \frac{1}{\theta}\int\limits_{\Omega\times\Omega} |v(x)-v(y)|^2K(x-y)dydy\\ &\leq \frac{1}{\theta}\int\limits_{Q} |v(x)-v(y)|^2K(x-y)dxdy\end{aligned}$$

$$\|v\|_{L^2(\Omega)} + \left(\int\limits_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}} \leq \|v\|_{L^2(\Omega)} + \frac{1}{\sqrt{\theta}}\left(\int\limits_{Q} |v(x)-v(y)|^2K(x-y)dxdy\right)^{\frac{1}{2}}$$

$$\|v\|_{H^s(\Omega)} \leq \max\left\{1, \frac{1}{\sqrt{\theta}}\right\}\left(\int\limits_{Q} |v(x)-v(y)|^2K(x-y)dxdy\right)^{\frac{1}{2}}$$

since $v \in X$ we obtain:

$$\|v\|_{H^2(\Omega)} \leq c(\theta)\|v\|_X$$

$$c(\theta) = \max\left\{1, \frac{1}{\sqrt{\theta}}\right\}$$

. *part b*

since $v \in X$ then $v = 0$ a.e in $\mathcal{C}\Omega$ then $\|v\|_{L^2(\mathbb{R}^N)} = \|v\|_{L^2(\Omega)}$

So:

$$\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dxdy = \int_Q \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dxdy \leq \frac{1}{\theta} \int_Q |v(x)-v(y)|^2 K(x-y) dxdy < \infty$$

$$\|v\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}} \leq \min\left\{1, \frac{1}{\sqrt{\theta}}\right\} \left(||v||_{L^2(\Omega)} + \int_Q |v(x)-v(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}}$$

then

$$||v||_{H^s(\mathbb{R}^N)} \leq c(\theta)||v||_X$$

so

$$\|v\|_{H^s(\Omega)} \leq \|v\|_{H^s(\mathbb{R}^N)} \leq c(\theta)\|v\|_X$$

. □

Now we give a sort of Poincaré–Sobolev inequality for functions inX_0

***Lemma* 4.2.4.** *Let $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ satisfy assumptions (4.5) − (4.7). Then:*

a) $\exists\ c = c(n,s) > 0$ *such that:*

$$\forall v \in X_0 : \quad \|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^N)} \leq c \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dxdy$$

where 2^* *is given in* (4.11).

b) $\exists C > 1,\ C = C(n,s,\theta,\Omega)$ *such that:*

$$\forall v \in X_0 : \quad \int_Q |v(x)-v(y)|^2 k(x-y) dxdy \leq ||v||^2_X \leq C \int_Q |v(x)-v(y)|^2 k(x-y) dxdy$$

that is:

$$||v||_{X_0} = \left(\int_Q |v(x)-v(y)|^2 k(x-y) dxdy \right)^{\frac{1}{2}} \tag{4.29}$$

is a norm on X_0 equivalent to the usual one define in (4.26)

Proof. *Part a)*:

Let $v \in X_0$: by lemma (4.2.3) , we have $v \in H^s(\mathbb{R}^N)$ by theorem (1.3.2) , with $p = 2$, we get

$$\|v\|^2_{L^{2^*}(\mathbb{R}^N)} \leq c \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dxdy$$

and $c = c(N, s)$

Since $v = 0$ a.e in $\mathbb{R}^N \setminus \Omega$ then $\|v\|_{L^{2^*}(\mathbb{R}^N)} = \|v\|_{L^{2^*}(\Omega)}$

then we obtain:

$$\|v\|^2_{L^{2^*}(\Omega)} \leq c \int\limits_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dxdy$$

Part b): by (4.26) it follows that:

$$||v||^2_X \geq \int\limits_Q |v(x) - v(y)|^2 k(x - y) dxdy$$

$$||v||^2_X = \left(||v||_{L^2(\Omega)} + \left(\int\limits_Q |v(x) - v(y)|^2 k(x - y) dxdy \right)^{\frac{1}{2}} \right)^2$$
$$\leq 2\|v\|^2_{L^2(\Omega)} + 2 \int\limits_Q |v(x) - v(y)|^2 k(x - y) dxdy.$$

Now, using that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously [1]

$$||v||^2_X \leq 2|\Omega|^{\frac{2^*-2}{2^*}} ||v||^2_{L^{2^*}(\Omega)} + 2 \int\limits_Q |v(x) - v(y)|^2 k(x - y) dxdy.$$

By lemma (4.2.4) part a) we have:

$$\leq 2c|\Omega|^{\frac{2^*-2}{2^*}} \int\limits_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dxdy + 2 \int\limits_Q |v(x) - v(y)|^2 k(x - y) dxdy$$

by assumption (4.6) we deduce:

$$||v||^2_X \leq 2 \left(\frac{c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta} + 1 \right) \int\limits_Q |v(x) - v(y)|^2 k(x - y) dxdy$$

we take $C = 2 \left(\frac{c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta} + 1 \right) > 1.$

Now we show that (4.29) is a norm in X_0 .we just prove that $\|g\|_{X_0} = 0 \Rightarrow g = 0$ in a.e. R^N.

[1] $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously (being Ω bounded and $.2 < 2^* = \frac{2N}{N - 2s}$) we have :

$$\|v\|^2_{L^2(\Omega)} \leq |\Omega|^{\frac{2^*-2}{2^*}} \|v\|^2_{L^{2^*}(\Omega)}$$

Let $g \in X_0$:

$$||g||_{X_0} = 0 \Rightarrow \int_Q |g(x) - g(y)|^2 k(x-y)dxdy = 0$$

then $g(x) - g(y) = 0$ a.e $(x,y) \in Q \Rightarrow g = c \in \mathbb{R}$ a.e in $\mathbb{R}^N$
since $g = 0$ a.e $\mathbb{R}^N \setminus \Omega \Rightarrow c = 0 \Rightarrow g = 0$ a.e in $\mathbb{R}^N$. □

***Lemma* 4.2.5.** *$(X_0, ||.||_{X_0})$ is a Hilbert space with scalar product*

$$\langle u, v \rangle_{X_0} = \int_Q (u(x) - u(y))(v(x) - v(y))k(x-y)dxdy \tag{4.30}$$

***Remark* 4.2.2.** *Note that in (4.30) and (4.29) the integrals come be extended to all $\mathbb{R}^{2N}$, since $v \in X_0$ (and so $v = 0$ a.e in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$)*

Proof. *Part a)* proving that $\langle u, v \rangle_{X_0}$ is scalar product in X_0:
let $u, v, w \in X_0$ and $\lambda \in \mathbb{R}$:

1. $\langle u, u \rangle_{X_0} = \int_Q |u(x) - u(y))|^2 k(x-y)dxdy \geq 0 \quad (k(x-y) > 0)$

2. $\langle u, u \rangle_{X_0} = 0 \Rightarrow \int_Q |u(x) - u(y))|^2 k(x-y)dxdy = 0 \Rightarrow u = 0$ a.e in $\mathbb{R}^{\mathbb{N}}$ (It has been proven before)

3. $$\begin{aligned} \langle u+v, w \rangle_{X_0} &= \int_Q ((u+v)(x) - (u+v)(y))\,(w(x) - w(y))k(x-y)dxdy \\ &= \int_Q (u(x) - u(y))(w(x) - w(y))k(x-y)dxdy \\ &+ \int_Q (v(x) - v(y))(w(x) - w(y))k(x-y)dxdy \\ &= \langle u, w \rangle_{X_0} + \langle v, w \rangle_{X_0} \end{aligned}$$

4. $\langle \lambda u, v \rangle_{X_0} = \lambda \int_Q (u(x) - u(y))(v(x) - v(y))k(x-y)dxdy = \lambda \langle u, v \rangle_{X_0}$

5. $\langle u, v \rangle_{X_0} = \langle v, u \rangle_{X_0}$ (*Multiplication is commutative*).

Part b):

we show that X_0 is a Hilbert space, i.e X_0 is complete with respect to the norm $||.||_{X_0}$. (see [22] Lemma **7** for the prove) □

For the following lemma we need thatΩ has a Lipschitz boundary .

***Lemma* 4.2.6.** *(see [22] Lemma **8**) let $k : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (4.5) – (4.7) and let v_j be a bounded sequence in X_0 then, there exists $v_\infty \in L^\eta(\mathbb{R}^N)$ such that up to a sequence $v_j \xrightarrow[n\to+\infty]{} v_\infty$ for any $\eta \in [1, 2^*)$.*

***Proposition* 4.2.2.** *(Relation between $\|.\|_{L^{2^*}(\Omega)}$ and $\|.\|_{X_0}$). Let $v \in X_0$,then*

$$\|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^N)} \leq \frac{c}{\theta}\|v\|^2_{X_0}$$

***Proof*.** By assumption (4.7) we get

$$\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy \leq \frac{1}{\theta}\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 K(x-y)dxdy$$

by lemma (4.2.4) part b ,we obtain :

$$\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy \leq \frac{1}{\theta}\|v\|^2_{X_0}$$

then by lemma (4.2.4) part a ,we deduce that :

$$\|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^N)} \leq \frac{c}{\theta}\|v\|^2_{X_0}$$

□

***Proposition* 4.2.3.** *(Relation between $\|.\|_{L^q(\Omega)}$and$\|.\|_{X_0}$) . Let $v \in X_0$ then*

$$\|v\|^q_{L^q(\Omega)} \leq |\Omega|^{\frac{2^*-q}{2^*}} \left(\frac{c}{\theta}\right)^{\frac{q}{2}} \|v\|^q_{X_0}$$

***Proof*.** Let $v \in X_0$
using the fact that : $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously [2] and by proposition (4.2.2) we get

$$\|v\|^q_{L^{2^*}(\Omega)} \leq \left(\frac{c}{\theta}\right)^{\frac{q}{2}} \|v\|^q_{X_0}$$

so that

$$\|v\|^q_{L^q(\Omega)} \leq |\Omega|^{\frac{2^*-q}{2^*}} \left(\frac{c}{\theta}\right)^{\frac{q}{2}} \|v\|^q_{X_0}$$

□

***Remark* 4.2.3.** *(Relation between $\|.\|_{L^2(\Omega)}$and $\|.\|_{X_0}$) by proposition (4.2.3) we get*

$$\|v\|^2_{L^2(\Omega)} \leq |\Omega|^{\frac{2^*-2}{2^*}} \frac{c}{\theta}\|v\|^2_{X_0}$$

[2] $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously being Ω bounded and $\max\{2, q\} = q < 2^*$). i.e

$$\|v\|^q_{L^q(\Omega)} \leq |\Omega|^{\frac{2^*-q}{2^*}} \|v\|^q_{L^{2^*}(\Omega)}$$

4.3 An eigenvalue problem

We consider the following eigenvalue problem

$$\begin{cases} -\mathcal{L}_k u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{4.31}$$

where $s \in (0,1), N > 2s$, Ω is on open bounded set of $\mathbb{R}^N$.
More precisely, we discuss the weak formulation of (4.31) which consists in the following eigenvalue problem

$$\begin{cases} \int_{\mathbb{R}^{2N}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy = \lambda \int_{\Omega} u(x)\varphi(x)\,dx \quad \forall \varphi \in X_0, \\ u \in X_0. \end{cases} \tag{4.32}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\mathcal{L}_k$ provided there exists a non-trivial solution $u \in X_0$ of problem (4.32) , in fact, of its weak formulation(4.32), a and in this case any solution with called an eigenfunction corresponding to the eigenvalue λ.

We will present result specific to this problem without proof. Which we need in studying our basic problem, but you will find all the results and proof in [23].

***Proposition* 4.3.1.** *(Eigenvalues and eigenfunctions of $-\mathcal{L}_K$). Let $s \in (0,1), N > 2s$, Ω be an open bounded set of $\mathbb{R}^N$ and let $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ be a function satisfying assumptions* (4.5) – (4.7)*. Then,*

a) problem (4.32) *admits an eigenvalue λ_1 which is positive and that can be characterized as follows*

$$\lambda_1 = \min_{\substack{u \in X_0 \\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy, \tag{4.33}$$

or, equivalently,

$$\lambda_1 = \min_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy}{\int_{\Omega} |u(x)|^2 dx}; \tag{4.34}$$

b) there exists a non-negative function $e_1 \in X_0$, which is an eigenfunction corresponding to λ_1, attaining the minimum in (4.33)*, that is $\|e_1\|_{L^2(\Omega)} = 1$ and*

$$\lambda_1 = \int_{\mathbb{R}^{2N}} |e_1(x)-e_1(y)|^2 K(x-y)dxdy \tag{4.35}$$

c) λ_1 is simple, that is if $u \in X_0$ is a solution of the following equation

$$\int_{\mathbb{R}^{2N}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy = \lambda_1 \int_{\Omega} u(x)\varphi(x)dx \quad \forall \varphi \in X_0, \tag{4.36}$$

then $u = \zeta e_1$, *with* $\zeta \in \mathbb{R}$;

d) the set of the eigenvalues of problem (4.32) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with* [3]

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots \tag{4.37}$$

and

$$\lambda_k \to +\infty \text{ as } k \to +\infty. \tag{4.38}$$

Moreover, for any $k \in \mathbb{N}$ *the eigenvalues can be characterized as follows:*

$$\lambda_{k+1} = \min_{\substack{u\in\mathbb{P}_{K+1} \\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy, \tag{4.39}$$

or, equivalently,

$$\lambda_{k+1} = \min_{u\in\mathbb{P}_{k+1}\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy}{\int_\Omega |u(x)|^2 dx}, \tag{4.40}$$

where

$$\mathbb{P}_{k+1} := \left\{u \in X_0 \ s.t. \ \left\langle u, e_j \right\rangle_{X_0} = 0 \quad \forall j = 1,\ldots,k\right\} \tag{4.41}$$

e) for any $k \in \mathbb{N}$ *there exists a function* $e_{k+1} \in \mathbb{P}_{k+1}$, *which is an eigenfunction corresponding to* λ_{k+1}, *attaining the minimum in* (4.39), *that is* $\|e_{k+1}\|_{L^2(\Omega)} = 1$ *and*

$$\lambda_{k+1} = \int_{\mathbb{R}^{2N}} \left|e_{k+1}(x) - e_{k+1}(y)\right|^2 K(x-y)dxdy \tag{4.42}$$

f) the sequence $\{e_k\}_{k\in\mathbb{N}}$ *of eigenfunctions corresponding to* λ_k *is an orthonormal basis of* $L^2(\Omega)$ *and an orthogonal basis of* X_0;

g) each eigenvalue λ_k *has finite multiplicity* [4] *, more precisely, if* λ_k *is such that*

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1} \tag{4.43}$$

for some $h \in \mathbb{N}_0$, *then the set of all the eigenfunctions corresponding to* λ_k *agrees with*

$$\operatorname{span}\{e_k,\ldots,e_{k+h}\}. \tag{4.44}$$

[3] As usual, here we call λ_1 the first eigenvalue of the operator $-\mathcal{L}_K$. This notation is justified by (4.37). Notice also that some of the eigenvalues in the sequence $\lambda_{k\,k\in\mathbb{N}}$ may repeat, i.e. the inequalities in (4.37) may be not always strict.

[4]We observe that we already know that the eigenfunctions corresponding toλ_1 are spane_1, thanks to c), so g) is interesting only when $k \geq 2$.

4.4 *Existence of solutions depends on* $\mathcal{L}_K$

We will prove theorem (4.1.1)

4.4.1 *A variational problem*

1) ***Another structure of*** $\mathcal{L}_K$

We have:

$$\begin{aligned}\mathcal{L}_K u(x) &= \int_{\mathbb{R}^N} [u(x+y)+u(x-y)-2u(x)]K(y)dy \quad x \in \mathbb{R}^N \\ &= \int_{\mathbb{R}^N} [u(x+y)-u(x)]K(y)dy + \int_{\mathbb{R}^N} [u(x-y)-u(x)]K(y)dy\end{aligned}$$

put : $y = -z$

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^N} [u(x-z)-u(x)]K(-z)dz + \int_{\mathbb{R}^N} [u(x-y)-u(x)]K(y)dy$$

K is symetric so:

$$L_K u(x) = \int_{\mathbb{R}^N} [u(x-z)-u(x)]K(z)dz + \int_{\mathbb{R}^N} [u(x-y)-u(x)]K(y)dy$$

put $y = z$

$$\begin{aligned}\mathcal{L}_K u(x) &= \int_{\mathbb{R}^N} [u(x-y)-u(x)]K(y)dy + \int_{\mathbb{R}^N} [u(x-y)-u(x)]K(y)dy \\ &= 2\int_{\mathbb{R}^N} [u(x-y)-u(x)]K(y)dy \\ &= -2\int_{\mathbb{R}^N} [u(x)-u(x-y)]K(y)dy\end{aligned}$$

put $\gamma = x - y$

$$= -2\int_{\mathbb{R}^N} [u(x)-u(\gamma)]K(x-\gamma)d\gamma$$

put $\gamma = y$

$$\mathcal{L}_K u(x) = -2\int_{\mathbb{R}^N} [u(x)-u(y)]K(x-y)dy$$

2) *The Weak formulation*

Let $u \in X_0$

$$\begin{aligned}I = \int_{\mathbb{R}^N} -\mathcal{L}_K u(x)\ \varphi(x)\ dx &= 2\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} [u(x)-u(y)]K(x-y)dy\right)\varphi(x)dx \\ &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))\varphi(x)\ K(x-y)\ dydx\end{aligned}$$

then

$$I = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y)\varphi(x)K(x-y)dydx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\varphi(x)K(x-y)dydx$$

Using changing of variable $x = y$ and Fubini theorem for the second term, we get:

$$\begin{aligned} I &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\varphi(x)K(x-y)dydx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\varphi(y)K(x-y)dydx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dydx \end{aligned}$$

then the weak formulation of 4.3 is given by the following problem

$$\begin{cases} u \in X_0 \quad \text{such that} \\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dydx - \lambda\int_\Omega u(x)\varphi(x)dx = \int_\Omega f(x,u(x))\varphi(x)dx \\ \forall \varphi \in X_0 \end{cases} \tag{4.45}$$

3) *The variational problem*

The problem (4.3),has a variational structure,it is the Euler-Lagrange equation of the functional $J_\lambda : X_0 \to \mathbb{R}$ defined by:

$$J_\lambda(x) = \frac{1}{2}\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^2K(x-y)dydx - \frac{\lambda}{2}\int_\Omega |u(x)|^2dx - \int_\Omega F(x,u(x))dx$$

J_λ is well defined, indeed

$$\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^2K(x-y)dydx = \|u\|^2_{X_0} < \infty$$

$$\int_\Omega |u(x)|^2dx = \|u\|^2_{L^2(\Omega)} \le \frac{c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta}\|u\|^2_{X_0} < \infty \quad \text{(by remark (4.2.3))}$$

$$\begin{aligned} |\int_\Omega F(x,u(x))dx| &\le \int_\Omega |F(x,u(x))|dx \\ &\le \int_\Omega \left(\varepsilon|u(x)|^2 + \delta(\varepsilon)|u(x)|^q\right)dx \quad (\text{ by } 4.18) \\ &\le \varepsilon\int_\Omega |u(x)|^2dx + \delta(\varepsilon)\int_\Omega |u(x)|^qdx \\ &\le \epsilon\|u\|^2_{L^2(\Omega)} + \delta(\varepsilon)\|u\|^q_{L^q(\Omega)} \\ &\le \varepsilon|\Omega|^{\frac{2^*-2}{2^*}}\frac{C}{\theta}\|u\|^2_{X_0} + \delta(\varepsilon)|\Omega|^{\frac{2^*-q}{2}}(\frac{C}{\theta})^{\frac{q}{2}}\|u\|^q_{X_0} \text{ by proposition(4.2.3)} \\ &\text{and remark (4.2.3)} \\ &< \infty \end{aligned}$$

Then J_λ is well defined in X_0

4) *J_λ is Fréchet differentiable in X_0 ($J_\lambda \in C^1(X_0)$)*

Indeed, let $u, \varphi \in X_0$ we pose $J_\lambda(u) = J_1(u) - J_2(u)$ such that :
$J_1(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dydx - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx$
and
$J_2(u) = \int_\Omega F(x, u(x))dx$
It is easy to check that J_1 is Fréchet differentiable and we get:

$$< J_1'(u), \varphi >= \int_{\mathbb{R}^{2N}} (u(x)-u(y))(\varphi(x)-\varphi(y))k(x-y)dydx - \lambda\int_\Omega u(x)\varphi(x)dx$$

Now, we will work on J_2
first remarking that :

$$\int_0^1 vf(x, u+t\varphi)dt = F(x, u+t\varphi)|_0^1 = F(x, u+\varphi) - F(x, u)$$

then we obtain

$$\begin{aligned} F(x, u+\varphi) - F(x,u) - f(x,u)\varphi &= \int_0^1 vf(x, u+t\varphi) - f(x,u)\varphi\int_0^1 dt \\ &= \int_0^1 vf(x, u+t\varphi)dt - \int_0^1 f(x,u)\varphi dt \\ &= \int_0^1 v(f(x, u+t\varphi) - f(x,u))dt \end{aligned}$$

$$J_2(u+\varphi) - J_2(u) - \int_\Omega f(x,u)\varphi dx = \int_\Omega\int_0^1 v(f(x,u+t\varphi) - f(x,u))dtdx$$

Using Fubini theorem ,Cauchy-Schwartz inequality,and ramark (4.2.3),we get:

$$\begin{aligned} |J_2(u+\varphi) - J_2(u) - \int_\Omega f(x,u)\varphi dx| &\leq \int_0^1\int_\Omega |\varphi||f(x,u+t\varphi) - f(x,u)|dxdt \\ &\leq \int_0^1 \|\varphi\|_{L^2(\Omega)}\|f(x,u+tv) - f(x,u)\|_{L^2(\Omega)}dt \\ &\leq \left(|\Omega|^{\frac{2^*-2}{2^*}}\frac{C}{\theta}\right)^{\frac{1}{2}}\|u\|_{X_0}\|f(x,u+t\varphi) - f(x,u)\|_{L^2(\Omega)} \\ \frac{|J_2(u+\varphi) - J_2(u) - \int_\Omega f(x,u)\varphi dx|}{\|u\|_{X_0}} &\leq (|\Omega|^{\frac{2^*-2}{2^*}}\frac{C}{\theta})^{\frac{1}{2}}\|f(x,u+t\varphi) - f(x,u)\|_{L^2(\Omega)} \underset{\varphi\to 0}{\to} 0 \end{aligned}$$

then

$$< J_2'(u), \varphi >= \int_\Omega f(x,u)\varphi dx$$

Finally we obtain :

$$< J'_\lambda(u),\varphi >=< J'_1(u),\varphi > - < J'_2(u),\varphi >= \int_{\mathbb{R}^{2N}}(u(x)-u(y))(\varphi(x)-\varphi(y))k(x-y)dydx$$
$$-\lambda\int_\Omega u(x)\varphi(x)dx - \int_\Omega f(x,u)\varphi(x)dx$$

***Proposition* 4.4.1.** *The critical points of J_λ are solutions to problem (4.8).*

In order to find these critical points, we will apply two classical variational results. the first one is "the mountain pass theorem" and the second one "is the linking theorem " respectively in the case when $\lambda < \lambda_1$ and $\lambda \geq \lambda_1$, where λ_1 is the first eigen value of the non-local operator - $\mathcal{L}_K$ (as we introduced in proposition 4.3.1)
Both these minimax theorems require that the functional J_λ

1. has a suitable geometric structure (for the Mountain Pass Theorem in conditions $(1),(2)$ of (4.2.1) and for the Linking Theorem conditions $(i),(ii)$ of (4.2.2).

2. satisfies the Palais–Smale compactness condition at any level $c \in \mathbb{R}$ (see, for instance, definition (4.2.1) , that is for any $c \in \mathbb{R}$ any sequence u_n in X_0 such that

$$J_\lambda(u_n) \to c \quad \text{and} \quad \sup\Big\{|\langle J'_\lambda(u_n),\varphi\rangle| : \varphi \in X_0, \|\varphi\|_{X_0} = 1\Big\} \to 0 \quad \text{as} \quad n \to +\infty,$$

admits a subsequence strongly convergent in X_0.

4.4.2 *Solutions of the variational problem with the Mountain Pass theorem ($\lambda < \lambda_1$)*

Here we assume that the non-linearity f satisfies conditions $(4.10) - 4.12)$ and (4.14).
We will verify that the functional J_λ satisfies the assumptions of the Mountain pass theorem for $(\lambda < \lambda_1)$ In this subsection, in order to verify that the functional J_λ satisfies the assumptions of the Mountain Pass Theorem, we need the following lemma.

***Lemma* 4.4.1.** *Let $k : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ satisfy assumptions (4.5) - (4.7) and let $\lambda \in \mathbb{R}$, then there existe two positive constant m^1_λ and M^1_λ. depending only on λ,such that for any $v \in X_0$*

$$m^1_\lambda\|v\|^2_{X_0} \leq \int_{\mathbb{R}^{2N}}|v(x)-v(y)|^2k(x-y)dxdy - \lambda\int_\Omega|v(x)|^2dx \leq M^1_\lambda\|v\|^2_{X_0} \tag{4.46}$$

that is

$$\|v\|_{X_0,\lambda} = (\int_{\mathbb{R}^{2N}}|v(x)-v(y)|^2k(x-y)dxdy - \lambda\int_\Omega|v(x)|^2dx)^{\frac{1}{2}} \tag{4.47}$$

is a norm on X_0 *equivalent to the ones in* 4.26 *and* 4.29 *.*
The constant m^1_λ *and* M^1_λ *are given by*

$$m^1_\lambda = \min\left\{1, 1-\frac{\lambda}{\lambda_1}\right\} \text{ and } M^1_\lambda = \max\left\{1, 1-\frac{\lambda}{\lambda_1}\right\}$$

Proof. 1. *If* $v=0$ *the* 4.46 *is trivially verified. so we take* $v \in X_0 \setminus \{0\}$ *.*
To make the computation clearer, let's examine two possibilities which $(0 \le \lambda < \lambda_1$ *or* $(\lambda < 0)$*. (Remark that* $\lambda_1 > 0$ *by proposition* 4.3.1*)*

i) we assume that $0 \le \lambda < \lambda_1$:
In this case we have

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \le \int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy$$

then

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \le \|u\|^2_{X_0} \quad ....(1)$$

Now, using the variational characterization of λ_1 in 4.33 we get

$$\lambda_1 < \frac{\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy}{\int_\Omega |v(x)|^2 dx}$$

then

$$\int_\Omega |v(x)|^2 dx < \frac{1}{\lambda_1}\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy$$

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \ge \int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \frac{1}{\lambda_1}\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy$$

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \ge (1-\frac{\lambda}{\lambda_1})\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy \quad ...(2)$$

from (1) and (2) ,we get

$$\left(1-\frac{1}{\lambda_1}\right)\|v\|^2_{X_0} \le \int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \le \|v\|^2_{X_0} \quad ...(3)$$

ii) We assume that $\lambda < 0$*:*

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \geq \int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy$$

then

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \geq \|v\|^2_{X_0} \quad ..(4)$$

According to 4.33 we get

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \leq \left(1-\frac{\lambda}{\lambda_1}\right)\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy$$

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \leq (1-\frac{\lambda}{\lambda_1})\|v\|^2_{X_0} \quad ..(5)$$

by (4) and (5), we deduce that

$$\|v\|^2_{X_0} \leq \int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \leq (1-\frac{\lambda}{\lambda_1})\|v\|^2_{X_0} \quad ...(6)$$

from (3) and (6) we get

$$\min\left\{1, 1-\frac{\lambda}{\lambda_1}\right\} \leq \int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |v(x)|^2 dx \leq \max\left\{1, 1-\frac{\lambda}{\lambda_1}\right\}$$

which $1-\dfrac{\lambda}{\lambda_1} > 0$ because $\lambda_1 > 0$.

2. Now we have to show that formula 4.48 defines a norm on X_0. For this we claim that

$$< u, v >_{X_0,\lambda} = \int_{\mathbb{R}^{2N}} (u(x)-u(y))(v(x)-v(y))k(x-y)dxdy - \lambda \int_\Omega u(x)v(x)dx \qquad (4.48)$$

is a scalar product on X_0.
for that ,let $u, v, w \in X_0, \alpha, \beta \in \mathbb{R}$

i) $< u, u >_{X_0,\lambda} = \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 k(x-y)dxdy - \lambda \int_\Omega |u(x)|^2 dx \geq m^1_\lambda \|u\|^2_{X_0} > 0$ (by 4.46)

ii) $< u, u >_{X_0,\lambda} = 0 \Rightarrow m^1_\lambda \|u\|^2_{X_0} = 0 \Rightarrow \|u\|^2_{X_0} = 0 \Rightarrow u = 0 \; in \; X_0$
(ramrk that $m^1_\lambda \|u\|^2_{X_0} \leq 0 \leq M^1_\lambda \|u\|^2_{X_0} \Rightarrow m^1_\lambda \|u\|^2_{X_0} = 0$)
$u = 0 \Rightarrow < u, u >_{X_0,\lambda} = 0$ (trivially)

3i)

$$\begin{aligned} < \alpha u + \beta v, w > &= \int_{\mathbb{R}^{2N}} \left(\alpha u + \beta v(x) - \alpha u + \beta v(y)\right)\left(w(x) - w(y)\right) k(x-y) dx dy \\ &- \lambda \int_{\Omega} \left(\alpha u + \beta v\right)(x) w(x) dx \\ &= \alpha \left(\int_{\mathbb{R}^{2N}} (u(x) - u(y))(w(x) - w(y)) k(x-y) dx dy - \lambda \int_{\Omega} u(x) w(x) dx \right) \\ &+ \beta \left(\int_{\mathbb{R}^{2N}} (v(x) - v(y))(w(x) - w(y)) k(x-y) dx dy - \lambda \int_{\Omega} v(x) w(x) dx \right) \\ &= \alpha < u, v >_{X_0,\lambda} + \beta < v, w > \end{aligned}$$

4i) $< u, v >_{X_0,\lambda} = < v, u >_{X_0,\lambda}$
Hence,the claim is proved. since

$$\|v\|_{X_0,\lambda} = \sqrt{< v, v >_{X_0,\lambda}}$$

the formula 4.48 defines a norm on X_0

3. the equivalence of the norms
with lemma 4.2.4 part b ,we have :

$$\forall x \in X_0 : \quad \|v\|^2_{X_0} \leq \|v\|^2_X \leq C \, \|v\|^2_{X_0} \quad ...(1)$$

and now, we can write 4.46 as the following:

$$m^1_\lambda \|v\|^2_{X_0} \leq \|v\|^2_{X_0,\lambda} \leq M^1_\lambda \|v\|^2_{X_0}$$

then

$$m^1_\lambda \leq \frac{\|v\|^2_X}{\|v\|^2_{X_0}} \leq M^1_\lambda$$

$$\frac{1}{M_\lambda} \leq \frac{\|v\|^2_{X_0}}{\|v\|^2_X} \leq \frac{1}{m_\lambda}$$

$$\frac{1}{M^1_\lambda} \|v\|^2_X \leq \|v\|^2_{X_0} \leq \frac{1}{m^1_\lambda} \|v\|^2_X \quad ...(2)$$

by (1) and (2), we get

$$\frac{1}{M^1_\lambda} \|v\|^2_X \leq \|v\|^2_{X_0} \leq \|v\|^2_X \leq C \|v\|^2_{X_0} \leq \frac{C}{m^1_\lambda} \|v\|^2_X$$

then $\|.\|_X, \|.\|_{X_0}, \|.\|_{X_0,\lambda}$ are equivalent in X_0.

□

Now we can prove that the functional J_λ has the geometric features required by the Mountain Pass Theorem.

J_λ **has the geometric features required by Mountain Pass theorem**

***Proposition* 4.4.2.** *Let* $(\lambda < \lambda_1)$ *and let* f *be a function satisfying conditions* (4.10)- (4.12), *then there exist* $\rho > 0$ *and* $\beta > 0$ *such that for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$ *it results that* $J_\lambda(u) \geq \beta$.

Proof. Let $u \in X_0$

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 k(x-y)dxdy - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx - \int_\Omega F(x, u(x))dx.$$

from(4.18) we get that for any $\varepsilon > 0$

$$\begin{aligned} J_\lambda(u) \geq & \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 k(x-y)dxdy - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx \\ & - \epsilon \int_\Omega |u(x)|^2 dx - \delta(\varepsilon)\int_w |u(x)|^q dx. \\ & \geq \frac{1}{2}\|u\|_{X_0,\lambda} - \varepsilon\|u\|^2_{L^2(\Omega)} - \delta(\varepsilon)\|u\|^2_{L^q(\Omega)} \end{aligned}$$

By lemma (4.4.1) and proposition (4.2.3) and remark (4.2.3) , we get

$$\begin{aligned} J_\lambda(u) &\geq \frac{m^1_\lambda}{2}\|u\|^2_{X_0} - \varepsilon|\Omega|^{\frac{2^*-2}{2^*}}\left(\frac{c}{\theta}\right)\|u\|^2_{X_0} - \delta(\varepsilon)|\Omega|^{\frac{2^*-q}{2^*}}\left(\frac{c}{\theta}\right)^{\frac{q}{2}}\|u\|^q_{X_0} \\ &\geq \left(\frac{m^1_\lambda}{2} - \frac{\varepsilon c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta}\right)\|u\|^2_{X_0} - \left(\frac{\delta(\varepsilon)|\Omega|^{\frac{2^*-q}{2^*}}c^{\frac{q}{2}}}{\theta^{\frac{q}{2}}}\right)\|u\|^q_{X_0} \\ &\geq \left(\frac{m^1_\lambda}{2} - \frac{\varepsilon c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta}\right)\|u\|^2_{X_0} - \left(\frac{\delta(\varepsilon)|\Omega|^{\frac{2^*-q}{2^*}}c^{\frac{q}{2}}}{\theta^{\frac{q}{2}}}\right)\|u\|^{q-2}_{X_0}\|u\|^2_{X_0}, \quad \Big(q \in (2, 2^*) \Rightarrow q > 2 \Rightarrow q-2 > 0\Big), \end{aligned}$$

choosing $\varepsilon > 0$ such that $\left(\frac{m^1_\lambda}{2} - \frac{\epsilon c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta}\right) > 0 \Rightarrow m^1_\lambda > 2\varepsilon c|\Omega|^{\frac{2^*-2}{2^*}}$, then we pose

$$\alpha = \left(\frac{m^1_\lambda}{2} - \frac{\varepsilon c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta}\right)$$

we get

$$J_\lambda(u) \geq \alpha\|u\|^2_{X_0}\left(1 - k\|u\|^{q-2}_{X_0}\right) \text{ where } k = \left(\frac{\delta(\varepsilon)|\Omega|^{\frac{2^*-q}{2^*}}c^{\frac{q}{2}}}{\theta^{\frac{q}{2}}\alpha}\right).$$

Now let $\|u\|^2_{X_0} = \rho > 0$

$$J_\lambda(u) \geq \alpha\rho^2(1 - k\rho^{q-2})$$

since q >2 ,then we can choose ρ such that $(1 - k\rho^{q-2}) > 0 i.e. 0 < \rho^{q-2} < \frac{1}{k}$,so that

$$J_\lambda(u) \geq \alpha\rho^2(1 - k\rho^{q-2}) =: \beta > 0,$$

Hence, the proposition is proved. □

***Proposition* 4.4.3.** *Let $\lambda < \lambda_1$ and let f be a function satisfying (4.10) to (4.12) and (4.14), then there existe $e \in X_0$ such that $e \geq 0$ a.e in $\mathbb{R}^N$, $\|e\|_{X_0} > \rho$ and $J_\lambda(e) < \beta$, where ρ and β are given in proposition (4.4.2).*

***Proof*.** Let $u \in X_0$ such that $\|u\|_{X_0} = 1$ and $u \geq 0$ $a.e$ in $\mathbb{R}^N$, now let $\zeta > 0$

$$\begin{aligned} J_\lambda(\zeta u) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} |\zeta u(x) - \zeta u(y)|^2 k(x-y)dxdy - \frac{\lambda}{2}\int_\Omega \zeta^2|u(x)|^2dx - \int_\Omega F(x, \zeta u(x))dx \\ &= \frac{1}{2}\|\zeta u\|^2_{X_0,\lambda} - \int_\Omega F(x,\zeta u(x))dx \\ &= \frac{1}{2}\zeta^2\|u\|^2_{X_0,\lambda} - \int_\Omega F(x,\zeta u(x))dx, \end{aligned}$$

by lemma (4.2.2) and (4.4.1) using the choice we made in the first, we get

$$\begin{aligned} J_\lambda(\zeta u) &\leq \frac{M^1_\lambda}{2}\zeta^2 - \alpha_3\zeta^\mu \int_\Omega |u(x)|^\mu dx + \alpha_4|\Omega| \\ &\leq \zeta^2\left(\frac{M^1_\lambda}{2} - \alpha_3\zeta^{\mu-2}\int_\Omega |u(x)|^\mu dx\right) + \alpha_4|\Omega|. \end{aligned} \tag{4.49}$$

Since $\mu > 2$, passing to the limit as $\zeta \longrightarrow +\infty$ we get $J_\lambda(\zeta u) \longrightarrow -\infty$, so that we can take $e = \zeta u$ with ζ sufficiently large. □

The palais-Smale condition

This will be accomplished with forthcoming propositions (4.4.4) and (5.28)

***Proposition* 4.4.4.** *Let $\lambda < \lambda_1$ and let f be a function satisfying conditions (4.10) to (4.12) and (4.14).Let $c \in \mathbb{R}$ and let u_n be a sequence in X_0 such that*

$$J_\lambda(u_n) \longrightarrow c \tag{4.50}$$

and

$$\sup_{\|\varphi\|_{X_0}=1} \{|\langle J'_\lambda(u_n), \varphi\rangle|, \varphi \in X_0\} \longrightarrow 0 \quad as \quad n \longrightarrow +\infty, \tag{4.51}$$

then u_n is bounded in X_0.

***Proof*.** Let $n \in \mathbb{N}$ by (4.50) and (4.51) it follows that there exists $k > 0$ such that

$$J_\lambda(u_n) \leq k, \tag{4.52}$$

and

$$\left|\langle J'_\lambda(u_n), \frac{u_n}{\|u_n\|_{\mathcal{X}_0}}\rangle\right| \leq k. \tag{4.53}$$

Indeed, we have

$$\Big(J_\lambda(u_n) \longrightarrow c \ \ in \ \ \mathbb{R}\Big) \Leftrightarrow \Big(\forall \varepsilon > 0, \exists\, \sigma > 0 : \forall n > \sigma \ \ |J_\lambda(u_n) - c| \leq \varepsilon\Big),$$

on the other hand we have

$$|J_\lambda(u_n)| = |J_\lambda(u_n) - c + c| \leq |J_\lambda(u_n) - c| + |c| \leq \varepsilon + |c|,$$

choosing $\varepsilon = 1$, we get

$$|J_\lambda(u_n)| \leq 1 + |c|.$$

Then taking $k = 1 + |c| > 0$ we obtain $|J_\lambda(u_n)| \leq k$
also we have

$$\Big(\sup_{\|\varphi\|_{X_0}=1} \{|\langle J_\lambda^{'}(u_n), \varphi\rangle|, \varphi \in \lambda_0\} \longrightarrow 0 \ \ as \ \ n \longrightarrow +\infty\Big)$$

that implies for any $\varphi \in X_0$,with $\|\varphi\|_{X_0} = 1$

$$\langle J_\lambda^{'}(u_n), \varphi\rangle \longrightarrow 0 \ \ as \ \ n \longrightarrow +\infty$$

then this implies that

$$\forall \varepsilon^{'} > 0, \exists\, \sigma^{'} > 0 : \forall n > \sigma^{'} \ \ |\langle J_\lambda^{'}(u_n), \varphi\rangle| \leq \varepsilon^{'},$$

Since $\varphi \in X_0$ and $\|\varphi\|_{X_0} = 1$,we take

$$\varphi = \frac{u_n}{\|u_n\|_{X_0}} \Rightarrow \|\varphi\|_{X_0} = \frac{\|u_n\|_{X_0}}{\|u_n\|_{X_0}} = 1,$$

then we get

$$\langle J_\lambda^{'}(u_n), \frac{u_n}{\|u_n\|_{\mathcal{X}_0}}\rangle \leq \varepsilon^{'},$$

Choosing $\epsilon^{'} = 1 + |c| = k$ getting

$$\langle J_\lambda^{'}(u_n), \frac{u_n}{\|u_n\|_{\mathcal{X}_0}}\rangle \leq k.$$

Moreover, by lemma (4.2.2) applied with $\varepsilon = 1$ we have

$$|f(x, u_n(x))| \leq 2|u_n(x)| + q\delta(1)|u_n(x)|^{q-1}$$

and

$$|F(x, u_n(x))| \leq |u_n(x)|^2 + \delta(1)|u_n(x)|^q \ \ with \ \ |u_n(x)| \leq r,$$

then we obtain

$$|f(x, u_n(x))| \leq 2r + q\delta(1)r^{q-1}$$

and

$$|F(x,u_n(x))| \leq r^2 + \delta(1) r^q,$$

$$\begin{aligned}
|\int_{\Omega\cap\{|u_n|\leq r\}} F(x,u_n(x)) - \frac{1}{\mu} f(x,u_n(x))u_n(x)dx| &\leq \int_\Omega \left(|F(x,u_n(x))| + \frac{1}{\mu}|f(x,u_n(x))u_n(x)| \right) dx \\
&\leq \int_\Omega \left(2r + q\delta(1)r^q + \frac{2}{\mu} r + \frac{q\delta(1)}{\mu} r^{q-1} \right) dx \\
&\leq \left(2r + q\delta(1)r^q + \frac{2}{\mu} r + \frac{q\delta(1)}{\mu} r^{q-1} \right) |\Omega| := \tilde{k}
\end{aligned}$$

then

$$\int_{\Omega\cap\{|u_n|\leq r\}} \left(F(x,u_n(x)) - \frac{1}{\mu} f(x,u_n(x))u_n(x)dx \right) dx \leq \tilde{k}, \tag{4.54}$$

such that $\tilde{k} > 0$ since $r > 0$.

Also by lemma (4.4.1) and (4.50) ,(4.51)

$$\begin{aligned}
J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle &= (\frac{1}{2} - \frac{1}{\mu})\|u_n\|^2_{X_0} - \lambda(\frac{1}{2} - \frac{1}{\mu})\|u_n\|^2_{L^2(\Omega)} - \int_\Omega F(x,u_n(x))dx \\
&+ \frac{1}{\mu}\int_\Omega f(x,u_n(x))u_n(x)dx \\
&= (\frac{1}{2} - \frac{1}{\mu}) \left(\|u_n\|^2_{X_0} - \lambda(\frac{1}{2} - \frac{1}{\mu})\|u_n\|^2_{L^2(\Omega)} \right) - \int_\Omega F(x,u_n(x))dx \\
&+ \frac{1}{\mu}\int_\Omega f(x,u_n(x))u_n(x)dx \\
&\geq (\frac{1}{2} - \frac{1}{\mu}) m^1_\lambda \|u_n\|^2_{X_0} - \int_{\Omega\cap\{|u_j|\leq r\}} \left(F(x,u_n(x)) - \frac{1}{\mu} f(x,u_n(x))u_n(x) \right) dx.
\end{aligned}$$

By (4.54), we get

$$J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle \geq (\frac{1}{2} - \frac{1}{\mu}) m^1_\lambda \|u_n\|^2_{X_0} - \tilde{k}. \tag{4.55}$$

As a consequence of (4.50) and (4.51) we also have $J_\lambda(u_n) \leq k$

and

$$\begin{aligned}
|\langle J_\lambda'(u_n), \frac{u_n}{\|u_n\|_{X_0}}\rangle| &\Rightarrow |\langle J_\lambda'(u_n), u_n\rangle| \leq k\|u_n\|_{X_0} \\
&\Rightarrow \langle J_\lambda'(u_n), u_n\rangle \geq -k\|u_n\|_{X_0},
\end{aligned}$$

then

$$J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle \leq k - \frac{k}{\mu}\|u_n\|_{X_0}.$$

So

$$J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda^{'}(u_n), u_n\rangle \leq k\left(1 + \frac{1}{\mu}\|u_n\|_{X_0}\right).$$

Since $\mu > 2 \Rightarrow 0 < \frac{1}{\mu} < \frac{1}{2} \Rightarrow \frac{1}{\mu}\|u_n\|_{X_0} < \frac{1}{2}\|u_n\|_{X_0} \leq \|u_n\|_{X_0}$, we get

$$J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda^{'}(u_n), u_n\rangle \leq k(1 + \|u_n\|_{X_0}).$$

This and (4.55) implies that, for any $n \in \mathbb{N}$

$$(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1\|u_n\|_{X_0}^2 - \tilde{k} \leq k(1 + \|u_n\|_{X_0})$$

$$(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1\|u_n\|_{X_0}^2 - k\|u_n\|_{X_0} - (k + \tilde{k}) \leq 0.$$

Taking $Z = \|u_n\|_{X_0}$, we get

$$(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1 Z^2 - kZ - (k + \tilde{k}) \leq 0.$$

$\Delta = k^2 + 4(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1(k + \tilde{k}) > 0$ $(\tilde{k} > 0, k > 1, m_\lambda^1 > 0, \mu > 2)$,then

$$Z_1 = \frac{k + \sqrt{k^2 + 4(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1(k + \tilde{k})}}{2(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1} > 0,$$

and

$$Z_2 = \frac{k - \sqrt{k^2 + 4(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1(k + \tilde{k})}}{2(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1},$$

$X_2 < 0$, indeed posing that $X_2 \geq 0 \Rightarrow k^2 \geq k^2 + 4(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1(k + \tilde{k})$, which implies that

$$4(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1(k + \tilde{k}) \leq 0,$$

and this is impossible because $m_\lambda^1 > 0, \mu > 2, k > 1, \tilde{k} > 0$, then $X_2 < 0$.
Since we are interested to $Z = \|u_n\|_{X_0}$, we get

$$(\frac{1}{2} - \frac{1}{\mu})m_\lambda^1 Z^2 - kZ - (k + \tilde{k}) \leq 0.$$

hold when $0 < Z < Z_1$. □

$$\|u_n\|_{X_0} < \frac{k + \sqrt{k^2 + 4\left(\frac{1}{2} - \frac{1}{\mu}\right) m_\lambda^1 (k + \tilde{k})}}{2\left(\frac{1}{2} - \frac{1}{\mu}\right) m_\lambda^1}$$

Hence $(u_n)_{n\in\mathbb{N}}$ is bounded in X_0.

***Proposition* 4.4.5.** *Let f be a function satisfying condition (4.10) − (4.12) and (4.14). Let (u_n) be a sequence in X_0, such that u_n is bounded in X_0 and (4.51)holds true. Then there exists $u_\infty \in X_0$ such that up to a subsequence $\|u_n - u_\infty\| \xrightarrow[n\to+\infty]{} 0$.*

***Proof*.** Since $(u_n)_{n\in\mathbb{N}}$ is bounded in X_0 and $X_0$is a reflexif space (X_0 is a Hilbert space by lemma (4.2.5)), then there exists a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ such that u_n converges weekly to some limit noted $u_\infty \in X_0$

i.e

$$\langle u_n, \varphi\rangle \to \langle u_\infty, \varphi\rangle, \quad \forall \varphi \in X_0 \tag{4.56}$$

Then $\forall \varphi \in X_0$

$$\int_{\mathbb{R}^{2n}} (u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) K(x-y)dxdy \to \int_{\mathbb{R}^{2n}} (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y)) K(x-y)dxdy \forall \varphi \in X_0$$

on other hand we have:

$$\begin{aligned}\lim_{n\to\infty} \|u_n - u_\infty\| &= \lim_{n\to\infty} \int_{\mathbb{R}^{2n}} |(u_n - u_\infty)(x) - (u_n - u_\infty)(y)|^2 K(x-y)dxdy \\ &= \lim_{n\to\infty} \left(\|u_n\|^2 + \|u_\infty\|^2 - 2\langle u_n, u_\infty\rangle_{X_0}\right).\end{aligned}$$

So we will prove that

$$\begin{cases} \lim\limits_{n\to\infty} \|u_n\|_{X_0} = \|u_\infty\|_{X_0} \\ \lim\limits_{n\to\infty} \langle u_n, u_\infty\rangle_{X_0} = \langle u_\infty, u_\infty\rangle_{X_0} \end{cases}$$

First, by lemma (4.64), up to a subsequence

$$u_n \to u_\infty \text{in } L^2(\Omega) \tag{4.57}$$

$$u_n \to u_\infty \text{in } L^q(\Omega), \quad q \in (2, 2^*) \tag{4.58}$$

$$u_n \to u_\infty \quad \text{in} \quad L^q(\Omega) \overset{\text{by Dominated.C.T}}{\Longrightarrow} \begin{cases} \exists u_{n_j} \quad \text{and} \quad g \in L^q(\Omega) \quad \text{Such that} \\ 1) u_{n_j} \to u \quad a.e \quad \text{on} \quad \Omega. \\ 2) |u_{n_j}(x)| \le g(x), \quad \forall j, \quad a.e \quad \text{on} \quad \Omega. \end{cases} \tag{4.59}$$

Next, we have $\int\limits_{\Omega} f(x,u_n(x))u_n(x)dx \longrightarrow \int\limits_{\Omega} f(x,u_\infty(x))u_\infty(x)dx.$,indeed, by Dominated convergence theorem, and de fact that the map $t \to f(.,t)$ is continuous in $t \in \mathbb{R}$, we will obtain our convergence as follows:

Taking $h_n(x) = f(x,u_n(x))u_n(x)$

$$\begin{aligned}\lim_{n\to+\infty} h_n(x) &= \lim_{n\to+\infty} f(x,u_n(x))u_n(x)\\ &= f(x,\lim_{n\to+\infty} u_n(x))\lim_{n\to+\infty} u_n(x)(\text{usinge the fact that } t\to f(.,t) \text{ is continous})\end{aligned}$$

Then by (4.59) $\underset{n\to+\infty}{lim} h_n(x) = f(x,u_\infty(x))u_\infty(x)$

and we have

$$\begin{aligned}|h_n(x)| &= |f(x,u_n(x))u_n(x)|\\ &= |f(x,u_n(x))||u_n(x)|\end{aligned}$$

by (4.11) ,we get

$$\begin{aligned}|h_n(x)| &\leq \left(\alpha_1 + \alpha_2|u_n(x)|^{q-1}\right)|u_n(x)|\\ &\leq \alpha_1|u_n(x)| + \alpha_2|u_n(x)|^q\end{aligned}$$

again by (4.59) we deduce

$$|h_n(x)| \leq \alpha_1 g(x) + \alpha_2(g(x))^q$$

taking

$$R(x) = \alpha_1 g(x) + \alpha_2(g(x))^q$$

$$\begin{aligned}\int\limits_{\Omega}|R(x)|dx &\leq \alpha_1\int\limits_{\Omega}|g(x)|dx + \alpha_2\int\limits_{\Omega}|g(x)|^q dx\\ &\leq \alpha_1\|g(x)\|_{L^1(\Omega)} + \alpha_2\|g(x)\|^q_{L^q(\Omega)}\end{aligned}$$

because of $g \in L^q(\Omega)$ and $q \in (2,2^*)$ we obtain $L^q(\Omega) \hookrightarrow L^1(\Omega)$

so

$$\int\limits_{\Omega}|R(x)|dx \leq \alpha_1 c\|g(x)\|_{L^q(\Omega)} + \alpha_2\|g(x)\|^q_{L^q(\Omega)} < \infty$$

then $R \in L^1(\Omega)$.

Hence, by the dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_{\Omega} h_n(x)dx = \int_{\Omega}\lim_{n\to+\infty} h_n(x)dx$$

i.e

$$\lim_{n\to+\infty}\int_{\Omega} f(x,u_n(x))u_n(x)dx = \int_{\Omega} f(x,u_\infty(x))u_\infty(x)dx$$

and here we have proven what we wanted.
Also we can conclude that

$$\int_{\Omega} f(x,u_n(x))u_\infty(x)dx \xrightarrow[n\to+\infty]{} \int_{\Omega} f(x,u_\infty(x))u_\infty(x)dx$$

because $u_\infty \in X_0$.
Moreover, by (4.50) we have that

$$\left\langle J'_\lambda(u_n), \frac{u_n}{\|u_n\|_{X_0}}\right\rangle \xrightarrow[n\to+\infty]{} 0 \Rightarrow \left\langle J'_\lambda(u_n), u_n\right\rangle \xrightarrow[n\to+\infty]{} 0$$

i.e

$$\int_{\mathbb{R}^{2n}} |u_n(x)-u_n(y)|^2K(x-y)dxdy - \lambda\int_{\Omega}|u_n(x)|^2dx - \int_{\Omega} f(x,u_n(x))u_n(x)dx \xrightarrow[n\to+\infty]{} 0.$$

Then,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{2n}} |u_n(x)-u_n(y)|^2K(x-y)dxdy = \lim_{n\to+\infty}\left[\lambda\int_{\Omega}|u_n(x)|^2dx - \int_{\Omega} f(x,u_n(x))u_n(x)dx\right]$$

by (4.57) we have $u_n \longrightarrow u_\infty$ in $L^2(\Omega)$ i.e $\|u_n-u_\infty\|_{L^2(\Omega)} \xrightarrow[n\to+\infty]{} 0$
$\big|\|u_n\|_{L^2(\Omega)} - \|u_\infty\|_{L^2(\Omega)}\big| \le \|u_n-u_\infty\|_{L^2(\Omega)} \xrightarrow[n\to+\infty]{} 0$ then $\|u_n\|_{L^2(\Omega)} \longrightarrow \|u_\infty\|_{L^2(\Omega)}$.
Now, we have:

$$\begin{cases} \|u_n\|_{L^2(\Omega)} \xrightarrow[n\to+\infty]{} \|u_\infty\|_{L^2(\Omega)} \\ \int_{\Omega} f(x,u_n(x))u_n(x)dx \xrightarrow[n\to+\infty]{} \int_{\Omega} f(x,u_\infty(x))u_\infty(x)dx \end{cases}$$

then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{2n}} |u_n(x)-u_n(y)|^2K(x-y)dxdy = \lambda\|u_\infty(x)\|^2_{L^2(\Omega)} + \int_{\Omega} f(x,u_\infty(x))u_\infty(x)dx \qquad (4.60)$$

Furthermore, using again(4.51)

$$\left\langle J_\lambda'(u_n), u_\infty \right\rangle \xrightarrow[n\to+\infty]{} 0$$

i.e

$$\begin{aligned}\lim_{n\to\infty} \int\limits_{\mathbb{R}^{2n}} & (u_n(x) - u_n(y)) - (u_\infty(x) - u_\infty(y))\, K(x-y)dxdy \\ & - \lambda \int\limits_{\Omega} u_n(x)u_\infty(x)dx - \int\limits_{\Omega} f(x, u_n(x))u_\infty(x)dx = 0\end{aligned}$$

then

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^{2n}} (u_n(x) - u_n(y))\,(u_\infty(x) - u_\infty(y))\, K(x-y)dxdy$$

$$= \lim_{n\to\infty} \left[\lambda \int\limits_{\Omega} u_n(x)u_\infty(x)dx + \int\limits_{\Omega} f(x, u_n(x))u_\infty(x)dx\right] \tag{4.61}$$

$$\tag{4.62}$$

by (4.56) we obtain

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^{2n}} (u_n(x) - u_n(y))\,(u_\infty(x) - u_\infty(y))\, K(x-y)dxdy = \|u_\infty\|^2_{X_0}$$

and we have $u_n \longrightarrow u_\infty$ in $L^2(\Omega) \Rightarrow u_n \rightharpoonup u_\infty$ in $L^2(\Omega)$

i.e $\langle u_n, \varphi\rangle \longrightarrow \langle u_\infty, \varphi\rangle, \quad \forall \varphi \in L^2(\Omega)$,so we get form (4.61):

$$\|u_\infty\|^2_{X_0} = \lambda \|u_\infty\|^2_{L^2(\Omega)} + \int\limits_{\Omega} f(x, u_\infty(x))u_\infty(x)dx. \tag{4.63}$$

Finally, by (4.63) and (4.60) we get

$$\|u_n\|^2_{X_0} \xrightarrow[n\to+\infty]{} \|u_\infty\|^2_{X_0}.$$

Now we have $\begin{cases} \|u_n\|^2_{X_0} \xrightarrow[n\to+\infty]{} \|u_\infty\|^2_{X_0} \\ \langle u_n, u_\infty\rangle_{X_0} \underset{n\to+\infty}{\rightarrow} \|u_\infty\|^2_{X_0} \quad \text{(by (4.56))} \end{cases}$

so

$$\begin{aligned}\lim_{n\to\infty}\|u_n-u_\infty\|^2_{X_0} &= \lim_{n\to\infty}\|u_n\|^2_{X_0}+\|u_\infty\|^2_{X_0}-2\lim_{n\to\infty}\langle u_n,u_\infty\rangle_{X_0}\\ &=\|u_\infty\|^2_{X_0}+\|u_\infty\|^2_{X_0}-2\|u_\infty\|^2_{X_0}\\ &=0\end{aligned}$$

then the subsequence u_n converges to u_∞ strangely in X_0.
Then, the assertion of proposition is proved. □

Hence, the Palais-Smale conditions is verified by proposition (4.4.4) and (4.4.5).

***Remark* 4.4.1.** *Note that Proposition* (4.4.5), *holds true for any value of the parameter* λ, *so we can use such result also for* $\lambda \geq \lambda_1$

conclusion: When $\lambda < \lambda_1$, the geometry of the Mountain Pass theorem of J_λ and the Palais-Smale condition is verified. Then we conclude that there exists a Critical point $u \in X_0$ of J_λ such that

$$J_\lambda(u) \geq \beta > 0 = J_\lambda(0)$$

so that $u \neq 0$

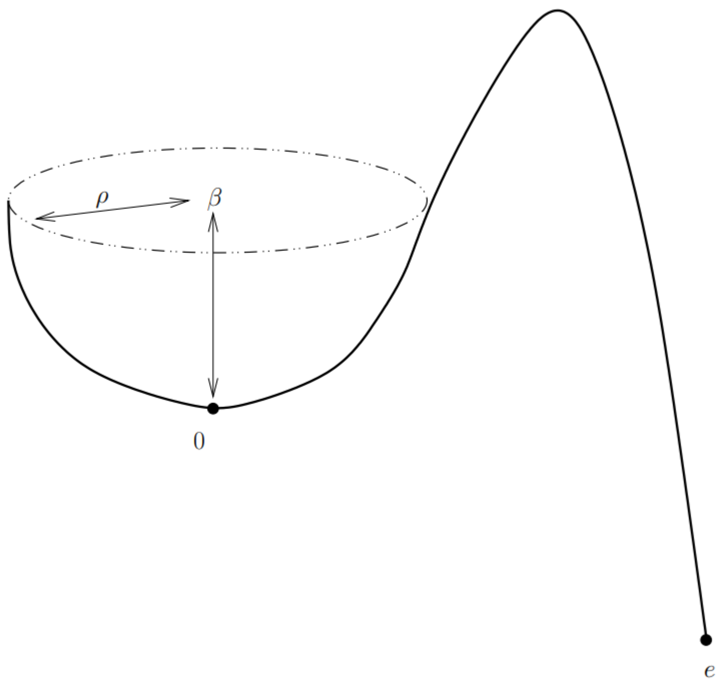


Figure 4.1: The Mountain Pass type geometry of J_λ when $\lambda < \lambda_1$.

4.4.3 Solutions of the variational problem with the linking theorem ($\lambda \geq \lambda_1$)

Since $\lambda \geq \lambda_1$, we can suppose that $\lambda \in [\lambda_k, \lambda_k + 1), k \in \mathbb{N}$ where λ_k is the $k-$th eigenvalue of the operator $-\mathcal{L}_K$.
We recall that, in what follows, e_k will be the $k-$th eigenfunctions corresponding to the eigenvalue λ_k of $\mathcal{L}_K$, and

$$\mathbb{P}_{k+1} := \{u \in X_0, \quad \langle u, e_n\rangle_{X_0} = 0 \quad \forall n = 1, .., k\}$$

as defined in Proposition (4.3.1), while span $\{e_1, ..., e_k\}$ will denote the linear subspace generated by the first k eigenfunctions of $-\mathcal{L}_K$ for any $k \in \mathbb{N}$.
First of all, we need a preliminary lemma.

***Lemma* 4.4.2.** *Let k, $\mathbb{R}^N\backslash\{0\} \longrightarrow (0, +\infty)$ satisfy assumptions* (4.5)- (4.7) *and let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Then for any $v \in \mathbb{P}_{k+1}$*

$$\int\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y)dxdy - \lambda \int\limits_{\Omega} |v(x)|^2 dx \geq m^{\lambda}_{k+1} \|v\|^2_{X_0} \tag{4.64}$$

where

$$m^{\lambda}_{k+1} = 1 - \frac{\lambda}{\lambda_{k+1}} > 0 \tag{4.65}$$

Proof. First of all not that $\lambda \geq \lambda_k \geq \lambda_1 > 0$, (thanks to proposition (4.3.1)).
Let $v \in \mathbb{P}_{k+1}$.
If $v \equiv 0$, then (4.64) is trivially verified.
Now, assume $v \not\equiv 0$.
The variational characterization of λ_{k+1} (formula 4.40) gives as

$$\lambda_{k+1} \leq \frac{\int\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y)dxdy}{\int\limits_{\Omega} |v(x)|^2 dx}$$

then

$$\int\limits_{\Omega} |v(x)|^2 dx \leq \frac{1}{\lambda_{k+1}} \int\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y)dxdy$$

then

$$-\lambda \int\limits_{\Omega} |v(x)|^2 dx \geq \frac{-\lambda}{\lambda_{k+1}} \int\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y)dxdy$$

$$\int\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y)dxdy - \lambda \int\limits_{\Omega} |v(x)|^2 dx$$
$$\geq \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \int\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y)dxdy.$$

So

$$\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 K(x-y)dxdy - \lambda \int_{\Omega} |v(x)|^2 dx \geq \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|v\|^2_{X_0}.$$

Then the lemma is proved. □

No we prove that the functional J_λ has the geometric structure required by the linking theorem.

The geometric features required by Linking theorem

***Proposition* 4.4.6.** *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let f be a function satisfying conditions* (4.10)-(4.12).
Then there exists $\rho > 0$ and $\beta > 0$ such that for any $u \in \mathbb{P}_{k+1}$ with $\|u\|_{X_0} = \rho$ it results that $J_\lambda(u) \geq \beta$.

Proof. This proof is quite similar to that of proposition (4.4.2). The key difference is that lemma (4.4.1)is not applicable here, so we must use the previous lemma (4.4.2) instead. This substitution changes m_1^λ from proposition (4.4.1) to m_{k+1}^λ.
While the rest of the proof remains largely unchanged. We provide a detailed explanation for the reader's convenience.
Let $u \in \mathbb{P}_{k+1}$. By (4.18) we get that for any $\varepsilon > 0$

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{2} \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy - \frac{\lambda}{2} \int_{\Omega} |u(x)|^2 dx - \varepsilon \int_{\Omega} |u(x)|^2 dx - \delta(\varepsilon) \int_{\Omega} |u(x)|^q dx. \\
&\geq \frac{m_{k+1}^\lambda}{2} \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy - \varepsilon \|u\|^2_{L^2(\Omega)} - \delta(\varepsilon) \|u\|^q_{L^q(\Omega)} \\
&\geq \frac{m_{k+1}^\lambda}{2} \|u\|^2_{X_0} - \varepsilon |\Omega|^{\frac{2^*-2}{2^*}} \frac{c}{\theta} \|u\|^2_{X_0} - \delta(\varepsilon) |\Omega|^{\frac{2^*-q}{2^*}} \|u\|^q_{X_0} \\
&\geq \left(\frac{m_{k+1}^\lambda}{2} - \varepsilon \frac{c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta} \right) \|u\|^2_{X_0} - \delta(\varepsilon) |\Omega|^{\frac{2^*-q}{2^*}} \left(\frac{c}{\theta}\right)^{\frac{q}{2}} \|u\|^q_{X_0}
\end{aligned}$$

choosing $\varepsilon > 0$ such that $\left(\frac{m_{k+1}^\lambda}{2} - \varepsilon \frac{c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta} \right) > 0 \Rightarrow m_{k+1}^\lambda \theta > 2\varepsilon c |\Omega|^{\frac{2^*-2}{2^*}}$

then pose $\alpha = \frac{m_{k+1}^\lambda}{2} - \varepsilon \frac{c|\Omega|^{\frac{2^*-2}{2^*}}}{\theta}$ we get:

$$J_\lambda(u) \geq \alpha \|u\|^2_{X_0} \left(1 - k\|u\|^{q-2}_{X_0}\right)$$

with $k = \dfrac{\delta(\varepsilon)|\Omega|^{\frac{2^*-q}{2^*}} c^{\frac{q}{2}}}{\alpha \theta^{\frac{q}{2}}}$

Now, let $u \in \mathbb{P}_{k+1}$ be such that $\|u\|_{X_0} = \rho > 0$, since $q > 2$ we can choose ρ such that

$$1 - k\rho^{q-2} > 0 \Rightarrow 0 < \rho^{q-2} < \frac{1}{k}$$

so that

$$\inf_{\substack{u \in \mathbb{P}_{k+1} \\ \|u\|_{X_0} = \rho}} J_\lambda(u) \geq \alpha\rho^2(1 - k\rho^{q-2}) := \beta > 0$$

□

***Proposition* 4.4.7.** *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let f be a function satisfying conditions* (4.10),(4.11) *and* (4.13)*. Then $J_\lambda(u) \leq 0$ for any $u \in Span\{e_1, ..., e_k\}$.*

Proof. Let $u \in Span\{e_1, ..., e_k\}$. Then $u(x) = \sum_{i=1}^{k} u_i e_i(x)$ with $u_i \in \mathbb{R}$, $\quad i = 1, ..., k$. Since $\{e_1, ..., e_k, ...\}$ is an orthonormal basis of $L^2(\Omega)$ and orthogonal one of X_0 (by proposition 4.3.1 -f). We have:

$$\begin{cases} \left\langle e_i, e_j \right\rangle_{L^2(\Omega)} = 0, & \text{for } i \neq j, \\ \|e_i\|^2_{L^2(\Omega)} = 1, & \text{for all } i, \\ \left\langle e_i, e_j \right\rangle_{X_0} = 0, & \text{for } i \neq j, \quad \forall i, j \in \{1, \dots, k\}. \end{cases}$$

then

$$\begin{aligned} \int_\Omega |u(x)|^2 dx &= \langle u, u \rangle_{L^2(\Omega)} \\ &= \left\langle \sum_{i=1}^{k} u_i e_i(x), u \right\rangle_{L^2(\Omega)} \\ &= \sum_{i=1}^{k} u_i \langle e_i(x), u(x) \rangle_{L^2(\Omega)} \\ &= \sum_{i=1}^{k} u_i \langle e_i(x), \sum_{j=1}^{K} u_j e_j(x) \rangle_{L^2(\Omega)} \\ &= \sum_{i=1}^{k} u_i \sum_{j=1}^{K} u_j \langle e_i(x), e_j(x) \rangle_{L^2(\Omega)} \end{aligned}$$

when $i = j$ we get

$$\int_\Omega |u(x)|^2 dx = \sum_{i=1}^{k} |u_i|^2$$

and

$$\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x - y) dx dy = \|u\|^2_{X_0} = \langle u, u \rangle_{X_0}$$

$$\begin{aligned}\langle u,u\rangle_{X_0} &= \left\langle \sum_{i=1}^{k} u_i e_i(x), u \right\rangle_{X_0}\\ &= \sum_{i=1}^{k} u_i \langle e_i(x), \sum_{j=1}^{k} u_j e_j(x)\rangle_{X_0}\\ &= \sum_{i=1}^{k} u_i \sum_{j=1}^{k} u_j \langle e_i(x), e_j(x)\rangle_{X_0}\end{aligned}$$

when $i=j$

$$\langle u,u\rangle_{X_0} = \sum_{i=1}^{k} |u_i|^2 \|e_i\|^2_{X_0}$$

Morever, we have $F(x,t)\geq 0,\ \forall x\in\Omega,\ t\in\mathbb{R}$.

Indeed, we have $\begin{cases} F(x,t)=\displaystyle\int_0^t f(x,\tau)d\tau \\ \qquad \text{and} \\ tf(x,t)\geq 0,\ \forall x\in\Omega,\ t\in\mathbb{R}\end{cases}$

If $t>0 \Rightarrow f(x,t)\geq 0,\ \forall x\in\Omega \Rightarrow f(x,\tau)\geq 0,\ \tau\in[0,t]$

There fore, the integral of a positive function on a positive interval is positive.

then $F(x,t)\geq 0$

If $t<0 \Rightarrow f(x,t)\leq 0,\ \forall x\in\Omega \Rightarrow F(x,t) = -\displaystyle\int_t^0 f(x,\tau)d\tau$

$$f(x,\tau)\leq 0,\ \forall\tau\in[t,0] \Rightarrow F(x,t)\geq 0.$$

If $t=0 \Rightarrow F(x,t)=0$

then $F(x,t)\geq 0,\ \forall x\in\Omega,\ t\in\mathbb{R}$.

We get

$$\begin{aligned}J_\lambda(u) &= \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega F(x,u(x))dx\\ &= \frac{1}{2}\sum_{i=1}^{k} u_i^2\|e_i\|^2_{X_0} - \frac{\lambda}{2}\sum_{i=1}^{k} u_i^2 - \int_\Omega F(x,u(x))dx\\ &= \frac{1}{2}\sum_{i=1}^{k} u_i^2\left(\|e_i\|^2_{X_0}-\lambda\right) - \int_\Omega F(x,u(x))dx\end{aligned}$$

because of $F(x,u(x))\geq 0,\ \forall x\in\Omega \Rightarrow -\displaystyle\int_\Omega F(x,u(x))\leq 0$

then

$$J_\lambda(u)\leq \frac{1}{2}\sum_{i=1}^{k} u_i^2\left(\|e_i\|^2_{X_0}-\lambda\right)$$

and by proposition 4.3.1 , we have

$$\begin{cases} \lambda_1 = \int\limits_{\mathbb{R}^{2N}} |e_1(x) - e_1(y)|^2 K(x-y)dxdy, \ e_i \in X_0 \\ \lambda_{k+1} = \int\limits_{\mathbb{R}^{2N}} |e_{k+1}(x) - e_{k+1}(y)|^2 K(x-y)dxdy, \ e_{k+1} \in \mathbb{P}_{k+1} \end{cases}$$

Then $\lambda_i = \|e_i\|^2_{X_0}, \ \forall i \in \overline{1,k}$

so

$$J_\lambda(u) \leq \frac{1}{2}\sum_{i=1}^{k}(\lambda_i - \lambda)$$

since $\lambda_i \leq \lambda_K \leq \lambda, \ \forall i = 1, ..., k \Rightarrow (\lambda_i - \lambda) \leq 0$

then $J_\lambda(u) \leq 0$.

□

***Proposition* 4.4.8.** *Let $\lambda \geq 0$ and let f be a function satisfying (4.10)- (4.1)and (4.14).*

Moreover,let $\mathbb{F}$ be finite dimensional subspace of X_0. Then, there exists $R > \rho$ such that $J_\lambda(u) \leq 0$ for any $u \in \mathbb{F}$ with $\|u\|_{X_0} \geq R$. Where ρ is given in the previous proposition 4.4.6. Let $u \in \mathbb{F}$, by lemma 4.2.2, we get:

$$J_\lambda(u) \leq \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - a_3\int_\Omega |u(x)|^u dx + a_4|\Omega|.$$

$\lambda \geq 0$ then $J_\lambda(u) \leq \frac{1}{2}\|u\|^2_{X_0} - a_3\|u(x)\|^\mu_{L^\mu(\Omega)} + a_4|\Omega|$

Thanks to the fact that in any finite dimensional Space all the norms are equivalent we deduce that

$$\exists c > 0, \ \|u\|_{L^\mu(\Omega)} \leq c\|u\|_{X_0}$$

Then $J_\lambda(u) \leq \frac{1}{2}\|u\|^2_{X_0} - \tilde{a_3}\|u(x)\|^\mu_{X_0} + a_4|\Omega|$

for some positive constant $\tilde{a_3}$.Then

$$J_\lambda(u) \leq \|u\|^2_{X_0}\left(\frac{1}{2} - \tilde{a_3}\|u(x)\|^{\mu-2}_{X_0}\right) \ \textit{(since } \mu > 2\textit{)}$$

Hence, if $\|u\|_{X_0} \longrightarrow +\infty$, then $J_\lambda(u) \longrightarrow -\infty$,and then the assertion of proposition (4.4.8)is proved.

Proposition (4.4.6)-(4.4.8) give [5] *that J_λ has the geometric structure required by the Linking theorem.*

[5] In particular, we use Proposition (4.4.8) with $\lambda \in [\lambda_k, \lambda_{k+1})$ and

$$F := \text{span}\{e_1, \ldots, e_{k+1}\} = \text{span}\{e_1, \ldots, e_k\} \oplus \text{span}\{e_{k+1}\},$$

while [17, Remark 5.5-(iii)] is used here with $V := \text{span}\{e_1, \ldots, e_k\}$ and $e := e_{k+1}$. With this choice, $F = V \oplus \text{span}\{e\}$.

The Palais-Smale condition:

To prove the Palais-Smale compactness condition we essentially follow the approach used for the Mountain pass theorem. However some significant technical differences arise, particularly regarding the boundedness of the Palais-Smale sequence. There fore, we provide a detailed explanation for the reader's convenience.

***Proposition* 4.4.9.** *Let* $\lambda \geq \lambda_1$ *and let of be a function satisfying conditions* (4.10)-(4.15)*. Let* $c \in \mathbb{R}$ *and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* X_0 *such that:*

$$J_\lambda(u_n) \longrightarrow c \tag{4.66}$$

and

$$\sup_{\|\varphi\|_{X_0}=1} \left\{ \left| \left\langle J'_\lambda(u_n), \varphi \right\rangle \right|, \ \varphi \in X_0 \right\} \xrightarrow[n\to+\infty]{} 0 \tag{4.67}$$

then $(u_n)_{n\in\mathbb{N}}$ *is bounded in* X_0*.*

Proof. However, lemma (4.4.1) cannot be applied in this situation leading to some technical challenges that necessitate the introduction of an additional parameter γ.
there are the details of the proof.
For any $n \in \mathbb{N}$, by (4.66)and (4.67) it follows that $\exists k > 0$ such that

$$\begin{aligned} &|J_\lambda(u_n)| \leq k \\ &\left| \left\langle J'_\lambda(u_n), \frac{u_n}{\|u_n\|_{X_0}} \right\rangle \right| \leq k \end{aligned}$$

Let us fix $\gamma \in (2, \mu)$ where $\mu > 2$, by lemma (4.2.1) applied with $\varepsilon = 1$ we have that

$$\begin{aligned} \Bigg| \int_{\Omega\cap\{|u_n|<r\}} & \left(F(x, u_n(x)) - \frac{1}{\gamma} f(x, u_n(x))u_n(x) \right) dx \ | \\ & \leq \left(r^2 - \delta(1)r^q + \frac{2}{\gamma} r + \frac{q}{\gamma} \delta(1) r^{q-1} \right) |\Omega| =: \tilde{K} \end{aligned}$$

So that, using also (4.14) and lemma (4.2.2).

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\gamma} \left\langle J'_\lambda(u_n), u_n \right\rangle = & \left(\frac{1}{2} - \frac{1}{\gamma} \right) \left(\|u_n\|^2_{X_0} - \lambda \|u_n\|^2_{L^2(\Omega)} \right) \\ & - \int_\Omega \left(F(x, u_n(x)) - \frac{1}{\gamma} f(x, u_n(x))u_n(x) \right) dx \end{aligned}$$

on the other hand

$$\int_\Omega \left(F(x,u_n(x)) - \frac{1}{\gamma} f(x,u_n(x))u_n(x) \right) dx = \int_{\Omega\cap\{|u_n|\geq r\}} \left(F(x,u_n(x)) - \frac{1}{\gamma} f(x,u_n(x))u_n(x) \right) dx + \int_{\Omega\cap\{|u_n|\leq r\}} \left(F(x,u_n(x)) - \frac{1}{\gamma} f(x,u_n(x))u_n(x) \right) dx$$

by (4.14) we have:

$$\mu F(x,u_n(x)) \leq f(x,u_n(x))u_n(x), \ \forall x \in \mathbb{R}, \ |u_n(x)| \geq r.$$

Then

$$\int_{\Omega\cap\{|u_n|\geq r\}} \left(F(x,u_n(x)) - \frac{1}{\gamma} f(x,u_n(x))u_n(x) \right) dx \leq \int_{\Omega\cap\{|u_n|\geq r\}} \left(1 - \frac{\mu}{\gamma} \right) F(x,u_n(x))dx$$

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\gamma}\left\langle J'_\lambda(u_n), u_n \right\rangle &\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\left(\|u_n\|^2_{X_0} - \lambda\|u_n\|^2_{L^2(\Omega)}\right) \\ &+ \left(\frac{\mu}{\gamma} - 1\right) \int_{\Omega\cap\{|u_n|\geq r\}} F(x,u_n(x))dx \\ &- \int_{\Omega\cap\{|u_n|\leq r\}} \left(F(x,u_n(x)) - \frac{1}{\gamma} f(x,u_n(x))u_n(x) \right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\left(\|u_n\|^2_{X_0} - \lambda\|u_n\|^2_{L^2(\Omega)}\right) \\ &+ \left(\frac{\mu}{\gamma} - 1\right) \int_{\Omega\cap\{|u_n|\geq r\}} F(x,u_n(x))dx - \tilde{K} \end{aligned}$$

by lemma (4.2.2) and the fact that $\left(\frac{\mu}{\gamma} - 1\right) > 0$ since $\gamma \in (2,\mu)$ we get

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\gamma}\left\langle J'_\lambda(u_n), u_n \right\rangle &\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\left(\|u_n\|^2_{X_0} - \lambda\|u_n\|^2_{L^2(\Omega)}\right) \\ &+ a_3\left(\frac{\mu}{\gamma} - 1\right)\|u_n\|^\mu_{L^\mu(\Omega)} - a_4 \int_{\Omega\cap\{|u_n|\geq r\}} dx - \tilde{K} \end{aligned}$$

$\int\limits_{\Omega\cap\{|u_n|\geq r\}} dx \leq \int\limits_{\Omega} dx = |\Omega|$ then

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\gamma}\left\langle J'_\lambda(u_n), u_n\right\rangle &\geq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\left(\|u_n\|^2_{X_0} - \lambda\|u_n\|^2_{L^2(\Omega)}\right) \\ &\quad + a_3\left(\frac{\mu}{\gamma}-1\right)\|u_n\|^{\mu}_{L^\mu(\Omega)} - a_4\left(\frac{\mu}{\gamma}-1\right)|\Omega| - \tilde{K}. \end{aligned}$$

Moreover, for any $\varepsilon > 0$

$$\begin{aligned} \|u_n\|^2_{L^2(\Omega)} &\leq |\Omega|^{\frac{\mu-2}{\mu}}\|u_n\|^2_{L^\mu(\Omega)} \\ &\leq \varepsilon^{\frac{2}{\mu}}\|u_n\|^2_{L^\mu(\Omega)}\frac{1}{\varepsilon^{\frac{2}{\mu}}}|\Omega|^{\frac{\mu-2}{\mu}} \end{aligned}$$

Using Young inequality with conjugate exponent $p = \dfrac{\mu}{2} > 1,\ q = \dfrac{\mu}{\mu-2}$ such that $\dfrac{1}{p}+\dfrac{1}{q} = 1$, we get

$$\|u_n\|^2_{L^2(\Omega)} \leq \frac{2}{\mu}\varepsilon\|u_n\|^{\mu}_{L^\mu(\Omega)} + \frac{\mu-2}{\mu}\varepsilon^{\frac{-2}{\mu-2}}|\Omega|.$$

Hence, we deduce

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\gamma}\left\langle J'_\lambda(u_n), u_n\right\rangle &\geq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_n\|^2_{X_0} - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2}{\mu}\varepsilon\|u_n\|^{\mu}_{L^\mu(\Omega)} \\ &\quad - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{\mu-2}{\mu}\varepsilon^{\frac{-2}{\mu-2}}|\Omega| \\ &\quad + a_3\left(\frac{\mu}{\gamma}-1\right)\|u_n\|^{\mu}_{L^\mu(\Omega)} - a_4\left(\frac{\mu}{\gamma}-1\right)|\Omega| - \tilde{K}. \\ &\geq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_n\|^2_{X_0} + \left[a_3\left(\frac{\mu}{\gamma}-1\right) - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2}{\mu\varepsilon}\right]\|u_n\|^{\mu}_{L^\mu(\Omega)} - c_\varepsilon \end{aligned}$$

$$c_3 = \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{\mu-2}{\mu}\varepsilon^{\frac{-2}{\mu-2}}|\Omega| + a_4\left(\frac{\mu}{\gamma}-1\right)|\Omega| + \tilde{K}$$

such that $c_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} +\infty$ ($2 < \gamma < \mu$ and $\lambda > \lambda_1 > 0$).

Now, choosing ε such that:

$$a_3\left(\frac{\mu}{\gamma}-1\right) - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2}{\mu\varepsilon} > 0$$

then, we get

$$J_\lambda(u_n) - \frac{1}{\gamma}\left\langle J'_\lambda(u_n), u_n\right\rangle \geq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_n\|^2_{X_0} - c_\varepsilon. \tag{4.68}$$

As a consequence of (4.66) and (4.67)we also have

$$J_\lambda(u_n) - \frac{1}{\mu}\left\langle J'_\lambda(u_n), u_n\right\rangle \leq k(1 + \|u_n\|_{X_0})$$

so that by (4.68), we obtain for any $n \in \mathbb{N}$

$$\left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u_n\|^2_{X_0} - c_\varepsilon \leq k(1 + \|u_n\|_{X_0})$$

then we have:

$$\left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u_n\|^2_{X_0} - k\|u_n\|_{X_0} - (c_\varepsilon + k) \leq 0$$

pose that $Z = \|u_n\|_{X_0}$ then

$$\Delta = k^2 + 4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k) > 0$$

then

$$Z_1 = \frac{k + \sqrt{k^2 + 4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k)}}{2\left(\frac{1}{2} - \frac{1}{\gamma}\right)} > 0$$

$$Z_2 = \frac{k - \sqrt{k^2 + 4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k)}}{2\left(\frac{1}{2} - \frac{1}{\gamma}\right)} < 0$$

$Z_2 < 0$, Indeed let taking $Z_2 > 0$ so

$$k - \sqrt{k^2 + 4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k)} \geq 0$$

$$k^2 \geq k^2 + 4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k)$$

$$4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k) \leq 0$$

but

$$\gamma > 2 \Rightarrow \left(\frac{1}{2} - \frac{1}{\gamma}\right) > 0$$

and

$$c_\varepsilon + k > 0$$

So, contruduction.

Finally, we deduce the solution of our equation is:

$$\|u_n\|_{X_0} = \frac{k + \sqrt{k^2 + 4\left(\frac{1}{2} - \frac{1}{\gamma}\right)(c_\varepsilon + k)}}{2\left(\frac{1}{2} - \frac{1}{\gamma}\right)} < \infty.$$

then $(u_n)_{n\in\mathbb{N}}$ is bounded in X_0.

Hence, the assertion of proposition is provided.

By proposition 20 and remark 15 we deduce the validity of the Palais-Smale condition for the Functional J_λ, $\lambda \geq \lambda_1$. □

Conclusion :

The geometry of the linking theorem is assured by proposition (4.4.6)- (4.4.8) and the Palais-Smale condition is given by proposition 4.4.9and remark(4.4.1).

So we deduce that there exists a critical point $u \in X_0$ of J_λ.

Furthermore

$$J_\lambda(u) \geq \beta > 0 = J_\lambda(0)$$

and so $u \not\equiv 0$, this ends the proof of theorem 1.

4.5 ***Existence of solutions depends on*** $(-\Delta)^s$

4.5.1 ***Proof of theorem*** (4.1.2)

It is a consequence of theorem (4.1.1)by choosing $K(x) = \dfrac{1}{|x|^{N+2s}}$ and by recalling that $X_0 \subseteq H^s(\mathbb{R}^N)$, due to lemma $(4.2.3) - b)$.

Chapter 5

Some of parabolic problems

Contents

5.1 *The heat equation*

5.1.1 *Introduction*

In this chapter, we'll consider a typical model of a parabolic problem called the heat equation, or diffusion equation. Let Ω be a bounded open set of $\mathbb{R}^N$, $T > 0$, and u_0 a function from Ω to $\mathbb{R}$, and let f be a function from $\Omega\times]0,T[$ to $\mathbb{R}$. We consider the heat equation:

$$\begin{cases} \partial_t u - \Delta u = f \quad \text{in} \quad \Omega\times]0,T[, \\ u = 0 \quad \text{on} \quad \partial\Omega\times]0,T[, \\ u(\cdot,0) = u_0 \end{cases}$$

We can model the problem by a heat flow within a body (see [25])such that:

1) The term $\partial_t u$ describes the evolution of the heat distribution over time. Specifically, we expect to be able to define the value of the solution at any time $T > 0$ by knowing the distribution at time 0.

2) The term Δu corresponds to a variation of u related to its local average. A point x where $\Delta u(x) < 0$ is colder than its immediate surroundings, and its temperature will increase, and vice versa. This term thus corresponds to averaging phenomena and tends to regularize the solutions of the equation.

3) The term u_0 corresponds to the heat distribution at the initial instant.

4) When we put $u(x,t) = g(x,t)$ on $\partial\Omega\times]0,T[$, the term g corresponds to a thermostat located at the boundary of the domain, imposing its heat on the system's boundary. Therefore, in our case where $g(x,t) = 0$ on $\partial\Omega\times]0,T[$, it means that there is no heat imposed on the boundary of the system by a thermostat.

5) The term f represents an external source or sink of heat.

Let's go back to the mathematical concepts: for $u_0 \in L^2(\Omega)$ and f in a suitably chosen space, we seek a weak solution to this problem. For that, we will use Faedo's Galerkin method.

The Faedo-Galerkin method is a very general and robust method. The idea of the method is as follows:

1) Starting from a problem posed in an infinite-dimensional space, we first proceed with an approximation in an increasing sequence of finite-dimensional subspaces.

2) Then we solve the approximated problem, which is generally easier than solving directly in infinite dimensions.

3) Finally, we pass in one way or another to the limit as the dimension of the approximation spaces tends to infinity to construct a solution to the weak formulation of the problem.

We previously mentioned that we will search for the weak solution in the space $L^2(]0,T[,H_0^1(\Omega))$, which we note is characterized by the presence of time.
To investigate evolving partial differential equations, we require specialized spaces that incorporate both spatial and temporal dimensions, underscoring the significance of the forthcoming subsection.(See [21]).

5.1.2 *Preliminaries*

Functions with values in Banach spaces

The natural functional setting for evolution problems requires spaces which involve time. Given a function $u = u(x,t)$, it is often convenient to separate the roles of space and time adopting the following point of view. Assume that $t \in [0,T]$ and that for every t, or at least for a.e. t, the function $u(\cdot,t)$ belongs to a Banach space E . Then, we may consider u as a function of the real variable t with values in E:

$$u : [0, T] \to E.$$

When we adopt this convention, we write $u(t)$ and $u'(t)$ instead of $u(x,t)$ and $u_t(x,t)$.

We can extend to these types of functions to the notions of measurability and integral . First, we introduce the set of functions $s : [0,T] \to V$ which assume only a finite number of values. These functions are called simple and are of the form :

$$s(t) = \sum_{j=1}^{N} \chi_{V_j}(t) u_j \qquad 0 \le t \le T$$

Where $u_1, \ldots, u_N \in E$ and $V_1, \ldots, V_N$ are Lebesgue measurable, mutually disjoint subsets of [0, T].

We say that a function $f : [0,T] \to E$ is measurable if there exists a sequence of simple functions $S_k : [0,T] \to E$ such that, as $k \to \infty$,

$$\|S_k(t) - f(t)\|_E \to 0, \quad \text{a.e. in } [0,T]. \tag{5.1}$$

It is not difficult to prove that, if f is measurable and $v \in E$, the (real) function $t \mapsto (f(t), v)_E$ is Lebesgue measurable in $[0,T]$.

The notion of integral is defined first for simple functions. If s is given by (5.1) we define

$$\int_0^T s(t)\, dt = \sum_{j=1}^{N} |V_j| u_j$$

Then

***Definition* 5.1.1.** *[21] We say that $f : [0,T] \to E$ is summable in $[0,T]$ if there exists a sequence $S_k : [0,T] \to E$ of simple functions such that*

$$\int_0^T \|S_k(t) - f(t)\|_E\, dt \to 0 \quad as \quad k \to +\infty. \tag{5.2}$$

If f is summable in $[0,T]$, we define the integral of f as follows:

$$\int_0^T f(t)\, dt = \lim_{k \to +\infty} \int_0^T S_k(t)\, dt \quad as \quad k \to +\infty. \tag{5.3}$$

Since (check it)

$$\| \int_0^T [S_h(t) - S_k(t)]\, dt \|_E \le \int_0^T \|S_h(t) - S_k\|_E\, dt,$$

$$\le \int_0^T \|S_k(t) - f(t)\|_E\, dt + \le \int_0^T \|S_k(t) - f(t)\|_E\, dt,$$

it follows from (5.2) that the real sequence

$$\left\{\int_0^T S_k(t)\,dt\right\}$$

is a Cauchy sequence, so that the limit (5.1.1) is well defined and does not depend on the choice of the approximating sequence $\{S_k\}$.

Moreover, the following important theorem holds:

***Theorem* 5.1.1.** *(**Bochner**)[25] A measurable function* $f : [0,T] \to E$ *is summable in* $[0,T]$ *if and only if the real function* $t \mapsto \|f(t)\|_E$ *is summable in* $[0,T]$*. Moreover,*

$$\left\|\int_0^T f(t)\,dt\right\|_E \leq \int_0^T \|f(t)\|_E\,dt, \tag{5.4}$$

and for any $u \in E$,

$$\langle u, \int_0^T f(t)\,dt\rangle_V = \int_0^T \langle u, f(t)\rangle_V\,dt. \tag{5.5}$$

The inequality (5.4) is well known in the case of real or complex functions.
By Riesz's Representation Theorem,(5.1.1) shows that the action of any element of V^* *commutes with the integrals.*

L^P *Spaces involving Time*

Once the definition of integral has been given, we can introduce the spaces $C([0,T],E)$ and $L^P(]0,T[,E)$, $1 \leq p \leq \infty$.

***Definition* 5.1.2.** *[21]*

The symbol $C([0,T],E)$ *denotes the set of continuous functions* $u : [0,T] \to E$*, endowed with the norm:*

$$\|u\|_{L^\infty([0,T],E)} = \max_{t\in[0,T]} \|u(t)\|_E.$$

$C([0,T],E)$ *is a Banach space.*

We define

$$L^p([0,T],E)$$

as the set of measurable fuctions $u : [0,T] \to E$ *such that :*

if $1 \leq p < \infty$

$$\|u\|_{L^P([0,T],E)} = \left(\int_0^T \|u(t)\|_E^P\,dt\right)^{\frac{1}{P}} < \infty$$

while if $p = \infty$

$$\|u\|_{L^\infty([0,T],E)} = ess\ sup_{t\in[0,T]}\|u(t)\|_E$$

***Proposition* 5.1.1.** *[9]*

Let $1 \leq p \leq \infty$, E is a Banach space ,then :

1. *$L^P(]0,T[,E)$ (with its usual norm) is a Banach space.*
2. *If $p < +\infty$ and E is separable, $L^P(]0,T[,E)$ is separable.*
3. *If $p = 2$ and E is a Hilbert space, then $L^2(]0,T[,E)$ is a Hilbert space whose scalar product is defined by:*

$$\langle u, v\rangle_{L^2(]0,T[,E)} = \int_0^T \langle u(t), v(t)\rangle_E \, dt$$

Consequently, for any bounded sequence in $L^2(]0,T[,E)$, we can extract a weakly convergent subsequence in $L^2(]0,T[,E)$.

Proposition* 5.1.2.** *[9]* ***Dominated Convergence Theorem

Let $1 \leq p < +\infty$ and $(U_n)_{n\in\mathbb{N}}$ is a subsequence of $L^P(]0,T[,E)$ .If :

a) $U_n \to U$ a.e.

b) $\|U_n\|_E \leq G$ a.e. $\forall n \in \mathbb{N}$ with $G \in L^2(]0,T[,\mathbb{R})$

Then $U_n \to U$ in $L^P(]0,T[,E)$.

Different Notions of Derivative for a Vector-Valued Function

We will properly define the derivative with respect to time, $\partial_t u = \dfrac{du}{dt}$, for a function defined on the interval $]0,T[$ in V, which is not differentiable in the classical sense (meaning the existence of the limit in E of the usual differential quotient).

Definition* 5.1.3.** *[9]* ***(Derivative by Transposition)

Let E be a Banach space, $1 \leq p < +\infty$, and $u \in L^p(]0,T[,E)$. We denote $\mathcal{D}$ as the space $C_c^\infty(]0,T[,\mathbb{R})$ and $\mathcal{D}_E^*$ as the set of linear mappings from $\mathcal{D}$ to E.

We define $\partial_t u \in \mathcal{D}_E^*$ by :

$$\langle \partial_t u, \phi\rangle_{\mathcal{D}_E^*,\mathcal{D}} = -\int_0^T u(t)\phi'(t)\, dt \in E$$

Definition* 5.1.4.** *[9]*(Weak Derivative)***

Let E and F be two Banach spaces and $1 \leq p, q \leq +\infty$. Assume there exists a vector space G such that $E \subset G$ and $F \subset G$. Let $u \in L^p(]0,T[,E)$ (thus, $\partial_t u \in \mathcal{D}_E^*$).We say that $\partial_t u \in L_F^q(]0,T[,F)$ if there exists a function $v \in L_F^q(]0,T[,F)$ such that

$$\langle \partial_t u, \phi\rangle_{\mathcal{D}_E^*,\mathcal{D}} = -\int_0^T u(t)\phi'(t)\, dt = \int_0^T v(t)\phi(t)\, dt$$

this equality only makes sense if there exists a vector space G such that $E \subset G$ and $F \subset G$. In this case ,we equate $\partial_t u \in \mathcal{D}_E^*$ and $v \in L^p(]0,T[,F)$.

Examples

1. For $1 \leq p < +\infty$, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and $E = W_0^{1,p}(\Omega),\ F = W^{-1,q}(\Omega)$,we have $E, F \subset G = \mathcal{D}^*(\Omega)$.

2. We can conclude from Example 1 that for $E = H_0^1(\Omega),\ F = H^{-1}(\Omega)$,we have $E, F \subset G = \mathcal{D}^*(\Omega)$.

In summary, we have three notions of the derivative of a function $u :]0, T[\to E$ (a Banach space):

1. Classical derivative: $u' : [0, T] \to E$ (rarely exists).

2. Derivative by transposition: $\partial_t u \in \mathcal{D}_E^*$ (exists as soon as $u \in L^p(]0, T[, E)$).

3. Weak derivative: $\partial_t u \in L^q(]0, T[, F)$ where F is a Banach space such that $E, F \subset G$ with G being a vector space.

Remark* 5.1.1.** *[9]* ***(The comparison between the spaces $L^p(]0, T[, L^p(\Omega))$ ***and*** $L^p(\Omega\times]0, T[)$***)***

Given $T > 0$, Ω an open subset of $\mathbb{R}^N$, and $1 \leq p < +\infty$, let $u \in L^p(]0, T[, L^p(\Omega))$. Then, there exists $v \in L^p(\Omega\times]0, T[)$ such that $u(t) = v(., t)$ a.e. in Ω, and for almost every $t \in]0, T[$. (Note that this equality holds regardless of the representatives chosen for u and v).

Conversely, if $v \in L^p(\Omega\times]0, T[)$, there exists $u \in L^p(]0, T[, L^p(\Omega))$ such that $u(t) = v(., t)$ a.e. in Ω and for almost every $t \in]0, T[$.

Using these notations, we can then compare $\dfrac{du}{dt}$ (which applies to an element of $C_c^\infty(]0, T[, \mathbb{R})$) and $\dfrac{\partial v}{\partial t}$ (which applies to an element of $C_c^\infty(\Omega\times]0, T[, \mathbb{R})$). The answer is that for every $\phi \in C_c^\infty(]0, T[, \mathbb{R})$ and every $\psi \in C_c^\infty(\Omega, \mathbb{R})$, we have:

$$\begin{aligned}\int_\Omega \langle \frac{du}{dt}, \phi \rangle_{\mathcal{D}'_E, \mathcal{D}}(x)\psi(x)\,dx &= -\int_\Omega \left(\int_0^T u(t)\phi'(t)\,dt \right)(x)\psi(x)\,dx \\ &= -\int_\Omega \int_0^T v(x,t)\phi'(t)\,dt\psi(x)\,dxdt \\ &= \langle \frac{\partial v}{\partial t}, \phi\psi \rangle_{\mathcal{D}^*\left(]0,T[,(\Omega)\right)), \mathcal{D}\left(]0,T[,(\Omega)\right))}\end{aligned}$$

***Lemma* 5.1.1.** *[9]**(Continuity in time)** Let E be a Banach space, $1 \leq p \leq +\infty$, and let $u \in L^p(]0, T[, E)$. Suppose that $\partial_t u \in L^p(]0, T[, E)$ (thus we write $u \in W^{1,p}(]0, T[, E)$).*

Then $u \in C([0, T], E)$ and even $u \in C^{0, 1-\frac{1}{p}}([0, T], E)$.

More precisely, there exists $a \in E$ such that $u(t) = a + \int_0^t \partial_t u ds$ for almost every $t \in]0, T[$ and u is then identified with the function (continuous on $]0, T[$) $t \mapsto a + \int_0^t \partial_t u(s) ds$.

In particular, this gives for any $0 \leq t_1 < t_2 \leq T$:

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \partial_t u(s) ds.$$

***Lemma* 5.1.2.** *[9]**(Continuity condition)** Let E be a Banach space and F a Hilbert space such that $E \subset F$, with a continuous injection, and E is dense in F. We identify F with F' (So that $E \subset F = E' \subset F'$)*

Let $u \in L^2(]0, T[, E)$ and suppose that $\partial_t u \in L^2(]0, T[, E')$, then $u \in C([0, T], F)$, and for any $0 \leq t_1 < t_2 \leq T$, we have :

$$\|u(t_2)\|_F^2 - \|u(t_1)\|_F^2 = 2 \int_{t_1}^{t_2} \langle \partial_t u, u \rangle_{E',E} dt$$

.

5.1.3 *Study of the heat equation by Faedo-Galerkine methode*

Let Ω be a bounded open set of $\mathbb{R}^n$, $T > 0$, $f : \Omega \times]0, T[\to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$.

We seek for u such that:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times]0, T[\\ u = 0 & \text{on } \partial\Omega \times]0, T[\\ u(\cdot, 0) = u_0 \end{cases} \tag{5.6}$$

The weak formulation of the heat equation

We multiply the diffusion equation by a test function $\varphi \in C_c^\infty(\Omega)$ and integrate over Ω. We find:

$$- \int_\Omega \partial_t u(t, x) \varphi(x) \, dx - \int_\Omega \Delta u(t, x) \varphi(x) \, dx = \int_\Omega f(t, x) \varphi(x) \, dx \tag{5.7}$$

We can consider $u = u(x, t)$ as a function of t with values into a suitable Hilbert space E:

$$u : [0, T] \to E$$

When we adopt this convention, we write $u(t)$ instead of $u(t, x)$, and u' instead of $\partial_t u$. Accordingly, we write $f(t)$ instead of $f(x, t)$. With this notations, (5.7) becomes

$$\int_\Omega u'(t) \phi(x) dx + \int_\Omega \nabla u(t) \nabla \phi(x) dx = \int_\Omega f(t) \phi(x) dx. \quad \forall \phi \in C_c^\infty(\Omega) \tag{5.8}$$

In order for the integrals to be well defined, it is enough to choose that:

$$u'(t) \in L^2(\Omega)$$

$$\nabla u(t) \in (L^2(\Omega))^N$$

$$f(t) \in L^2(\Omega)$$

From $\nabla u(t) \in (L^2(\Omega))^N$ and the homogeneous Dirichlet condition $u(t) = 0$ on $\partial\Omega$ for $t \in]0, T[$, it suggests that the natural space for $u(t)$ is $V = H_0^1(\Omega)$ for $t \in]0, T[$.

After multiplying equation (5.8) by a test function $\psi \in C_c^\infty([0, T[)$ and integrating over $[0, T[$, we obtain:

$$\begin{cases} \int_0^T (\int_\Omega u'(t)\phi(x)dx)\psi(t)dt - \int_0^T (\int_\Omega \nabla u(t)\nabla\phi(x)dx)\psi(t)dt = \int_0^T (\int_\Omega f(t)\phi(x)dx)\psi(t)dt \\ \forall\phi \in C_c^\infty(\Omega), \forall\psi \in C_c^\infty([0, T[) \end{cases} \tag{5.9}$$

$$\begin{aligned} |\int_0^T \int_\Omega u'(t)\phi(x)\psi(t)dxdt| &\leq \int_0^T \int_\Omega |u'(t)||\phi(x)||\psi(t)|dxdt \\ &\leq \sup_{x\in\Omega} |\varphi(x)| \sup_{t\in[0,T[} |\psi(t)| \int_0^T \left(\int_\Omega |u'(t)|\, dx\right) dt \\ &\leq \sup_{x\in\Omega} |\varphi(x)| \sup_{t\in[0,T[} |\psi(t)| \int_0^T \|u'(t)\|_{L^2(\Omega)} |\Omega|^{1/2}\, dt \\ &\leq M \int_0^T \|u'(t)\|_{L^2(\Omega)} dt \end{aligned}$$

Then if

$$u_t \in L^2\left(]0, T[, L^2(\Omega)\right)$$

we get

$$|\int_0^T \int_\Omega u'(t)\varphi(x)\psi(t)\, dx\, dt| < \infty$$

Then we can apply Fubini's Theorem

$$\int_0^T \left(\int_\Omega u'(t)\varphi(x)\, dx\right) \psi(t)\, dt = \int_\Omega \left(\int_0^T u'(t)\psi(t)\, dt\right) \varphi(x)\, dx$$

$$\int_0^T u'(t)\psi(t)\, dt = [u(t)\psi(t)]_0^T - \int_0^T u(t)\psi'(t)\, dt = u(T)\psi(T) - u(0)\psi(0) - \int_0^T u(t)\psi'(t)\, dt$$

$$\psi \in C_c^\infty([0, T[) \implies \psi(T) = 0$$

$$\int_0^T u'(t)\psi(t)\, dt = -u_0(x)\psi(0) - \int_0^T u(t)\psi'(t)\, dt$$

Then (5.9) becomes:

$$\begin{cases} -\int_\Omega u_0(x)\psi(0)\cdot\phi(x)\,dx - \int_\Omega \left(\int_0^T u(t)\psi'(t)\,dt\right)\phi(x)\,dx \\ -\int_0^T \left(\int_\Omega \nabla u(t)\nabla\phi(x)\,dx\right)\psi(t)\,dt = \int_0^T \left(\int_\Omega f(t)\phi(x)\,dx\right)\psi(t)\,dt \quad \forall\phi\in C_c^\infty(\Omega), \quad \forall\psi\in C_c^\infty([0,T[) \end{cases} \tag{5.10}$$

$$\begin{aligned}\left|\int_\Omega u_0(x)\psi(0)\phi(x)\,dx\right| &\le \int_\Omega |u_0(x)||\psi(0)||\phi(x)|\,dx \\ &\le \sup_{t\in[0,T[}|\psi(t)|\cdot\sup_{x\in\Omega}|\phi(x)|\int_\Omega |u_0(x)|\,dx\end{aligned}$$

Then if $u_0 \in L^2(\Omega)$ we can deduce by Cauchy-Schwarz inequality that

$$\left|\int_\Omega u_0(x)\psi(0)\phi(x)\,dx\right| < \infty$$

on the other hand

$$\begin{aligned}\left|\int_0^T\int_\Omega u(t)\psi'(t)\phi(x)\,dx\,dt\right| &\le \int_0^T\int_\Omega |u(t)||\psi'(t)||\phi(x)|\,dx\,dt \\ &\le \sup_{t\in[0,T[}|\psi'(t)|\cdot\sup_{x\in\Omega}|\phi(x)|\int_0^T\int_\Omega |u(t)|\,dx \\ &\le M'\int_0^T \|u(t)\|_{L^2}|\Omega|^{1/2}\,dt\end{aligned}$$

Since

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ then:}$$

then

$$\left|\int_0^T\int_\Omega u(t)\psi'(t)\phi(x)\,dx\,dt\right| \le \int_0^T \|u(t)\|_{H_0^1(\Omega)},dt$$

Then if $u \in L^2(]0,T[, H_0^1(\Omega))$, we can apply again Fubini's theorem and get:

$$\int_\Omega\left(\int_0^T u(t)\psi'(t)\,dt\right)\phi(x)\,dx = \int_0^T\left(\int_\Omega u(t)\psi'(t)\phi(x)\,dx\right)dt$$

With the same method, we conclude that the second integral is well defined for $\psi \in C_c^\infty([0,T[)$.

$$\left|\int_0^T\int_\Omega f(t)\phi(x)\psi(t)\,dx\,dt\right| \le M\int_0^T \|f(t)\|_{L^2(\Omega)}\,dt$$

Then we suggest that $f \in L^2(]0,T[, L^2(\Omega))$.

Finally, by taking $\varphi(x,t) = \phi(x)\psi(t)$,we obtain

For $f \in L^2(]0,T[, L^2(\Omega))$,"We thus obtain for the heat problem (5.6) that u satisfies:

$$\begin{cases} u \in L^2(]0,T[, H_0^1(\Omega)), \text{ such that} \\ \displaystyle\int_0^T \int_\Omega u\varphi_t \, dx \, dt + \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi \, dx \, dt - \int_\Omega u_0(x)\varphi(x,0) \, dx = \int_0^T \int_\Omega f\varphi \, dx \, dt, \quad \forall \varphi \in C_c^\infty([0,T[\times\Omega) \end{cases} \tag{5.11}$$

This formulation is equivalent to the another weak formulation given by the following lemma

***Lemma* 5.1.3.** *Let Ω be a bounded open set of $\mathbb{R}^N$, $T > 0$, $u_0 \in L^2(\Omega)$ and $f \in L^2(]0,T[, L^2(\Omega))$. Then u is a solution of (5.11) if and only if u is a solution of*

$$\begin{cases} u \in L^2(]0,T[, H_0^1(\Omega)), \quad \partial_t u \in L^2(]0,T[, H^{-1}(\Omega)), \quad u \in C([0,T]; L^2(\Omega)) \\ u(0) = u_0 \quad a.e., \\ \displaystyle\int_0^T \langle \partial_t u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt + \int_0^T \left(\int_\Omega \nabla u \cdot \nabla v \, dx \right) dt \\ \quad = \displaystyle\int_0^T \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt, \quad \forall v \in L^2(]0,T[, H_0^1(\Omega)) \\ f \in L^2(]0,T[, H^{-1}(\Omega)) \end{cases} \tag{5.12}$$

The prove of this equivalent is in [9].

Faedo-Galerkin method

As we say in the introduction, we will use the Faedo-Galerkin method for the existence of a weak solution. We can construct by approximations a sequence of problems whose solution exists and we show the convergence of the solutions of the approximate problems towards a function which satisfies a weak formulation of (5.6).

Since $H_0^1(\Omega)$ is separable, there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of spaces included in $H_0^1(\Omega)$, of finite dimension, for example:

$$\dim E_n = n, \text{ such that } E_n \subset E_{n+1} \text{ and}$$

$$\overline{\bigcup_{n\in\mathbb{N}} E_n} = H_0^1(\Omega).$$

Let $\{e_1, \ldots, e_n\}$ be a basis of E_n. We seek for $u_n(t)$ as

$u_n(t) = \sum_{i=1}^{n} \alpha_i(t) e_i,$

where the coefficients $\alpha_i(t)$ are chosen such that:

$$u_n(0) = \sum_{i=1}^{n} \alpha_i(0) e_i \to u_0 \text{ in } L^2(\Omega) \text{ as } n \to \infty.$$

Solving the problem using the Faedo-Galerkin method.

In this subsection , we will give for this linear problem an existence result by the Faedo-Galerkin method and uniqueness of weak solution .

Theorem 5.1.2. *Let Ω be an open bounded set in $\mathbb{R}^N$ $(T > 0)$, and $u_0 \in L^2(\Omega)$. We identify $L^2(\Omega)$ with its dual and we assume that $f \in L^2(]0, T[; H^{-1}(\Omega))$. Then there exists a unique u such that:*

$$\begin{cases} u \in L^2(]0, T[, H_0^1(\Omega)), \quad \partial_t u \in L^2(]0, T[; H^{-1}(\Omega)), \\ \displaystyle\int_0^T \langle \partial_t u(s), v(s) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} \, ds + \int_0^T \int_\Omega \nabla u \cdot \nabla v \, dx \, ds, = \int_0^T \int_\Omega \langle f v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, ds, \\ \qquad \forall v \in L^2(]0, T[; H_0^1(\Omega)), \\ u(0) = u_0 \ a.e. \end{cases} \tag{5.13}$$

We recall that $u(s)$ (resp $v(s)$)denotes the function $x \mapsto u(x, t)$(resp $v(x,t)$)
We also have the following estimates on u and $\partial_t u$:

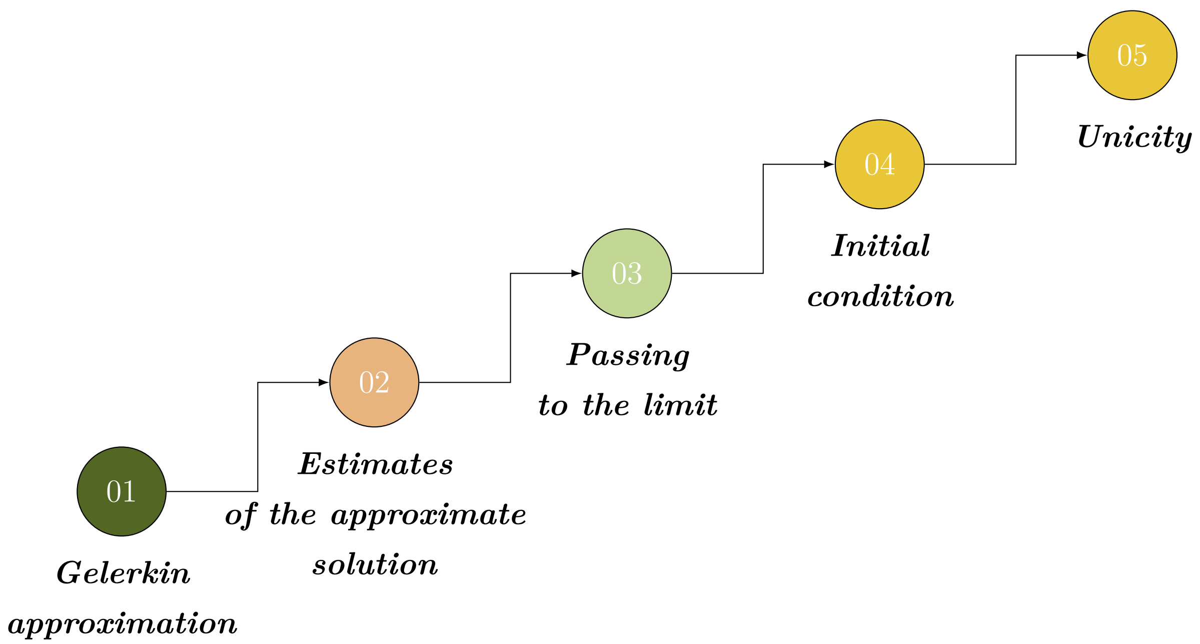
$$\|u\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

$$\|\partial_t u\|_{L^2(]0,T[,H^{-1}(\Omega))} \leq \|u_0\|_2 + 2\|f\|_{L^2(]0,T[;H^{-1}(\Omega))}$$

for all $t \in [0, T[$:

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} + \|\partial_t u\|^2_{L^2(]0,T[,H^{-1}(\Omega))} + \|u\|^2_{L^2(]0,T[,H_0^1(\Omega))}$$

Proof. we will present the demonstation in several steps :



A) Galerkim approximation

The idea is to construct the approximate solutions using the Faedo-Galerkin method .This will be done in three steps :

Step 1: Construction of the problem in finite dimension

The idea to prove this theorem is to first solve the problem in finite dimensional spaces. One could do this, for example, using finite element spaces. But it is simple to use a Hilbert basis formed by eigenfunctions of the Laplacian, that is a Hilbert basis of $L^2(\Omega)$, denoted $\{e_n\}_{n\in\mathbb{N}^*}$ such that:

for any n e_n is a weak solution of :

$$\begin{cases} -\Delta e_n = \lambda_n e_n & \text{in } \Omega \\ e_n = 0 & \text{on } \partial\Omega \end{cases}$$

with $\lambda_n \in \mathbb{R}$.

The family $\{e_n\}_{n\in\mathbb{N}^*}$ forms a Hilbert basis for $L^2(\Omega)$ and satisfies:

$$\begin{cases} e_n \in H_0^1(\Omega) \\ \int_\Omega \nabla e_n \cdot \nabla v \, dx = \lambda_n \int_\Omega e_n v \, dx, \quad \forall v \in H_0^1(\Omega) \end{cases}$$

with $\lambda_n > 0, \forall n \in \mathbb{N}^*$, and $\lambda_n \to +\infty$ as $n \to +\infty$

.

Since $\{e_n\}_{n\in\mathbb{N}^*}$ is a Hilbert basis of $L^2(\Omega)$, we have:

$$\forall w \in L^2(\Omega) : \quad w = \sum_{n\in\mathbb{N}^*} \langle w, e_n \rangle e_n$$

We will show that the family $\left(\frac{e_n}{\sqrt{n}}\right)_{n\in\mathbb{N}^*}$ is a Hilbert basis for $H_0^1(\Omega)$.

i.e.

$$\begin{cases} 1. \left(\frac{e_n}{\sqrt{\lambda_n}}\right)_{n\in\mathbb{N}^*} \quad \text{is an orthonormal basis:} \\ \left\langle \frac{e_n}{\sqrt{\lambda_n}}, \frac{e_m}{\sqrt{\lambda_m}} \right\rangle_{H_0^1(\Omega)} = 0, \quad n \neq m, \ n, m \in \mathbb{N}^* \\ \left\| \frac{e_n}{\sqrt{\lambda_n}} \right\|^2_{H_0^1(\Omega)} = 1 \\ \\ 2. \text{ The vector space generated by } \left(\frac{e_n}{\sqrt{\lambda_n}}\right)_{n\in\mathbb{N}^*} \text{ is dense in } H_0^1(\Omega). \end{cases}$$

We show (1) :

we recall that :

$$H_0^1(\Omega) \text{ is a Hilbert space with } \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$$

and the inner product:

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v\, dx, \quad \forall u, v \in H_0^1(\Omega)$$

First, note that for all $n, m \in \mathbb{N}^*$ we have:

$$\int_\Omega \nabla e_n \cdot \nabla e_m\, dx = \lambda_n \int_\Omega e_n e_m\, dx, \quad \forall e_n, e_m \in H_0^1(\Omega)$$

i.e.,

$$\langle \nabla e_n, \nabla e_m\rangle_{H_0^1(\Omega)} = \lambda_n \langle e_n, e_m\rangle_{L^2(\Omega)}$$

Since $\lambda_n \neq 0$ for all $n \in \mathbb{N}^*$, then

$$\langle e_n, e_m\rangle_{L^2(\Omega)} = 0 \quad \text{if} \quad n \neq m$$

Then

$$\left\langle \frac{e_n}{\sqrt{\lambda_n}}, \frac{e_m}{\sqrt{\lambda_m}} \right\rangle = 0 \quad \text{if} \quad n \neq m \quad \cdots (1)$$

and

$$\left\| \frac{e_n}{\lambda_n} \right\|^2_{H_0^1(\Omega)} = \int_\Omega \nabla \left(\frac{e_n}{\sqrt{\lambda_n}} \right) \cdot \nabla \left(\frac{e_n}{\sqrt{\lambda_n}} \right) dx = \frac{1}{\lambda_n} \lambda_n \int_\Omega e_n e_n\, dx = 1 \quad \cdots (2)$$

From (1) and (2), $\left\{ \frac{e_n}{\sqrt{\lambda_n}} \right\}_{n \in \mathbb{N}}$ is an orthonormal basis in $H_0^1(\Omega)$.

2. We show that the vector space generated by the family $(R)_{n \in \mathbb{N}^*}$, denoted by $\left\{ \frac{e_n}{\sqrt{n}}, n \in \mathbb{N}^* \right\}$, is dense in $H_0^1(\Omega)$.

***Remark* 5.1.2.** *If H is a Hilbert space, then an orthonormal system is a Hilbert basis if and only if*

$$\langle x, e_i\rangle = 0,\ \forall i \in I \Rightarrow x = 0$$

Let $v \in H_0^1(\Omega)$ such that $\langle v, R_n\rangle_{H_0^1(\Omega)} = 0$, $\forall n \in \mathbb{N}^*$.
Therefore, $\forall n \in \mathbb{N}^*$:

$$\langle v, R_n\rangle_{H_0^1(\Omega)} = \int_\Omega \nabla v \cdot \nabla (R_n)\, dx = \lambda_n \int_\Omega v \cdot (R_n)\, dx = \lambda_n \langle v, R_n\rangle_{L^2(\Omega)}$$

Then

$$\lambda_n \langle v, R_n\rangle_{L^2(\Omega)} = 0 \implies \lambda_n \langle v, \frac{e_n}{\sqrt{\lambda_n}}\rangle_{L^2(\Omega)} = 0$$

$$\implies \sqrt{\lambda_n} \langle v, e_n\rangle_{L^2(\Omega)} = 0$$

Since $\{e_n\}_{n\in\mathbb{N}}$ is a Hilbert basis in $L^2(\Omega)$ and $\lambda_n \neq 0 \forall n \in \mathbb{N}^*$, then $v = 0$ a.e. in Ω.

This shows that the orthogonal in $H_0^1(\Omega)$ of the set $\left\{\dfrac{e_n}{\sqrt{n}}\right\}_{n\in\mathbb{N}^*}$ is reduced to $\{0\}$. Therefore, that vect $\{e_n\}_{n\in\mathbb{N}^*}$ is dense in $H_0^1(\Omega)$. Finally, we obtain that the family $\left\{\dfrac{e_n}{\sqrt{n}}\right\}_{n\in\mathbb{N}^*}$ is a Hilbert basis of $H_0^1(\Omega)$.

Step 2: Approximate Solution

Now, we denoteR_n by e_n.

Let $n \in \mathbb{N}^*$, we define $E_n = \text{vect}\{e_p, p = 1, \ldots, n\}$. We seek an approximate solution in the form:

$$u_n(t) = \sum_{i=1}^{n} \alpha_i(t) e_i \quad \text{with} \quad \alpha_i \in C([0,T];\mathbb{R})$$

Assuming that α_i are differentiable for all t (which is not true in general). So we have:

$$u_n'(t) = \sum_{i=1}^{n} \alpha_i'(t) e_i$$

Therefore, for any $\psi \in H_0^1(\Omega)$ and $t \in]0,T[$, we have:

$$\langle u_n'(t), \psi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \sum_{i=1}^{n} \alpha_i'(t) \langle e_i, \psi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

Taking into account the countinuous embedding of $L^2(\Omega)$ into $H^{-1}(\Omega)$,
we have:

$$\langle e_i, \psi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega e_i \psi \, dx$$

So,

$$\langle u_n'(t), \psi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \sum_{i=1}^{n} \alpha_i'(t) \int_\Omega e_i \psi \, dx$$

Moreover, $\forall t \in]0,T[$, we have:

$$\Delta u_n(t) = \sum_{i=1}^{n} \alpha_i(t) \Delta e_i$$

But we have $\Delta e_i = \lambda_i e_i$,

$$\Delta u_n(t) = \sum_{i=1}^{n} \alpha_i(t) \lambda_i e_i \quad \text{in } \mathcal{D}^*(\Omega) \text{ and in } H^{-1}(\Omega)$$

i.e. $\forall \varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
\langle -\Delta u_n(t), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \int_\Omega \nabla u_n(t) \nabla \varphi \, dx \\
&= \int_\Omega \nabla (\sum_{i=1}^n \alpha_i(t) e_i, \nabla \varphi) \, dx \\
&= \sum_{i=1}^n \alpha_i(t) \int_\Omega \nabla e_i \nabla \varphi \, dx
\end{aligned}$$

but we have:

$$\int_\Omega \nabla e_i \varphi \, dx = \lambda_i \int_\Omega e_i \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega)$$

Then,

$$\langle -\Delta u_n(t), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \sum_{i=1}^n \alpha_i(t) \lambda_i \int_\Omega e_i \varphi \, dx$$

Finally, since $f \in L^2(]0, T[, H^{-1}(\Omega))$, we have, $\forall \varphi \in H_0^1(\Omega)$,

$$\langle f(t), \varphi \rangle_{H^{-1}(\Omega), H_0^{-1}(\Omega)} \in L^1(]0, T[)$$

Hence, the quantity $\langle f(.), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ is defined for almost every t.

For almost every t and for any $\varphi \in H_0^1(\Omega)$, we obtain:

$$\langle u_n'(t) - \nabla u_n(t) - f(t), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \sum_{i=1}^n \Big(\alpha_i'(t) + \lambda_i \alpha_i(t) \Big) \int e_i \varphi dx - \langle f(t), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

To obtain u_n, the idea is to choose α_i such that

$$\left\langle \frac{du_n}{dt}(t) - \Delta u_n(t) - f(t), \varphi \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$$

For $\varphi \in E_n$, let $f_i(t) = \langle f(t), e_i \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$.

$$\sum_{i=1}^n \Big(\alpha_i'(t) + \lambda_i \alpha_i(t) \Big) - f_i(t) = 0$$

$$\sum_{i=1}^n \Big(\alpha_i'(t) + \lambda_i \alpha_i(t) \Big) = f_i(t) \quad \forall i = 1, \ldots, n$$

$$\alpha_i'(t) + \lambda_i \alpha_i(t) = f_i(t) \quad \forall i = 1, \ldots, n$$

$$\alpha_i(0) = \langle u_0, e_i \rangle_2 = \alpha_i^{(0)}$$

Then we have:

$$\begin{cases} \alpha_i'(t) + \lambda_i \alpha_i(t) = f_i(t) \\ \alpha_i(0) = \alpha_i^{(0)} \end{cases} \tag{5.14}$$

First, we need to solve the homogeneous equation:

$$\alpha_i'(t) + \lambda_i \alpha_i(t) = 0$$

we get:

$$\alpha_i(t)_h = ce^{-\lambda_i t}$$

Now, we find a particular solution for the equation using the method of variation of the constant:

$$\text{put } \alpha_i(t)_p = c(t)e^{-\lambda_i t}$$

then we substitute it in (5.14) , we find for $t \in [0, T]$:

$$c'(t) = e^{\lambda_i t} f_i(t) \Rightarrow c(t) = \int_0^t e^{\lambda_i s} f_i(s)\, ds + c_1$$

Then

$$\alpha_i(t)_p = c_1 e^{-\lambda_i t} + \int_0^t e^{\lambda_i (s-t)} f_i(s)\, ds$$

Then

$$\alpha_i(t) = \alpha_i{}^{(0)} e^{-\lambda_i t} + \int_0^t e^{-\lambda_i (t-s)} f_i(s)\, ds$$

The functions α_i thus defined belong to $C([0, T], \mathbb{R})$, and consequently $u_n \in C([0, T], E_n) \subset C([0, T], H_0^1(\Omega))$. With

$$u_n(t) = \sum_{i=1}^{n} \alpha_i(t) e_i$$

Step 3: Accuracy of the time derivative

Let $n \in \mathbb{N}$ and u_n be the approximation solution given by the previous step. The functions α_i are not necessarily differentiable. We will specify the value of the derivative and denote this derivative by $(u_n)_t$.

A tronsposition derivative $(u_n)_t$ is an element of D_E^* , where $E = H_0^1(\Omega))$,Let $\varphi \in C_c^\infty(\Omega)$, we have :

$$\langle u_n(t), \varphi \rangle_{D_E^*, D} = -\int_0^T u_n(t) \varphi'(t)\, dt \in E_n \subset H_0^1(\Omega)$$

Since

$$u_n(t) = \sum_{i=1}^{n} \alpha_i(t) e_i$$

then we have

$$\begin{aligned}\langle u_n(t), \varphi \rangle &= -\int_0^T \sum_{i=1}^{n} \alpha_i(t) e_i \varphi'(t)\, dt \\ &= -\sum_{i=1}^{n} e_i \left(\int_0^T \alpha_i(t) \varphi'(t)\, dt \right)\end{aligned}$$

$$\begin{aligned}I = \int_0^T \alpha_i(t)\varphi'(t)\, dt &= \int_0^T \alpha_i^{(0)} e^{-\lambda_i t} \varphi(t)\, dt + \int_0^T \int_0^t e^{-\lambda_i (t-s)} f_i(s)\, ds\, \varphi'(t)\, dt \\ &= \int_0^T \alpha_i^{(0)} e^{-\lambda_i t} \varphi(t)\, dt + \int_0^T \int_0^t e^{-\lambda_i (t-s)} f_i(s)\, ds\, \varphi'(t)\, dt \\ &= T_i + S_i\end{aligned}$$

using integration by part we obtain

$$T_i = \int_0^T \alpha_i^{(0)} e^{\lambda_i t} \varphi'(t) = \left[\alpha_i^{(0)} e^{-\lambda_i t} \varphi \right]_0^T + \int_0^T \lambda_i \alpha_i e^{-\lambda_i t} \varphi(t)\, dt$$

Since $\varphi \in C_0^\infty\left(]0, T[, \mathbb{R})\right)$ then $\varphi(0) = \varphi(T) = 0$

$$\text{then} \quad T_i = \int_0^T \alpha_i^{(0)} \lambda_i e^{-\lambda_i t} \varphi(t)\, dt$$

$$S_i = \int_0^T \left(\int_0^t e^{-\lambda_i (t-s)} f_i(s)\, ds \right) \varphi'(t)\, dt$$

$$= \int_0^T \left(\int_0^T \chi_{[0,t]}(s) e^{-\lambda_i (t-s)} f_i(s)\, ds \right) \varphi'(t)\, dt$$

By Fubini's Theorem, we get:

$$S_i = \int_0^T \left(\int_s^T \chi_{[s,T]}(t) e^{-\lambda_i (t-s)} \varphi'(t)\, dt \right) f_i(s)\, ds$$

$$= \int_0^T \left(\int_s^T e^{-\lambda_i (t-s)} \varphi'(t)\, dt \right) f_i(s)\, ds$$

On the other hand,

$$\int_s^T e^{-\lambda_i (t-s)} \varphi'(t)\, dt = \left[e^{-\lambda_i (t-s)} \varphi(t) \right]_s^T + \lambda_i \int_s^T e^{-\lambda_i (t-s)} \varphi(t)\, dt$$

$$= -\varphi(s) + \int_s^T \lambda_i e^{-\lambda_i (t-s)} \varphi(t)\, dt$$

Then,

$$S_i = \int_0^T \left(\int_s^T \lambda_i e^{-\lambda_i(t-s)} \varphi(t)\,dt \right) f_i(s)\,ds - \int_0^T \varphi(s) f_i(s)\,ds$$

$$= \int_0^T \left(\int_s^T \chi_{[s,T]}(t) \lambda_i e^{-\lambda_i(t-s)} \varphi(t)\,dt \right) f_i(s)\,ds - \int_0^T f_i(t)\varphi(t)\,dt$$

Using again Fubini's Theorem, we get:

$$S_i = \int_0^T \left(\int_0^t \chi_{[s,T]}(t) \lambda_i e^{-\lambda_i(t-s)} f_i(s)\,ds \right) \varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt$$

$$= \int_0^T \left(\int_0^t \lambda_i e^{-\lambda_i(t-s)} f_i(s)\,ds \right) \varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt$$

Since $\alpha_i(t) = \alpha_i^{(0)} e^{-\lambda_i t} + \int_0^t e^{-\lambda_i(t-s)} f_i(s)\,ds$,then

$$\int_0^T e^{-\lambda_i(t-s)} f_i(s)\,ds = \alpha_i(t) - \alpha_i^{(0)} e^{-\lambda_i t}$$

$$S_i = \int_0^T \left(\lambda_i \alpha_i(t) - \lambda_i \alpha_i^{(0)} e^{-\lambda_i t} \right) \varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt$$

$$= \lambda_i \int_0^T \alpha_i(t)\varphi(t)\,dt - \lambda_i \int_0^T \alpha_i^{(0)} e^{-\lambda_i t} \varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt$$

$$T_i + S_i = \int_0^T \alpha_i^{(0)} \lambda_i e^{-\lambda_i t} \varphi(t)\,dt + \int_0^T \lambda_i \alpha_i(t)\varphi(t)\,dt$$

$$- \int_0^T \lambda_i \alpha_i^{(0)} e^{-\lambda_i t} \varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt$$

$$= \int_0^T \lambda_i \alpha_i(t)\varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt$$

then

$$\begin{aligned}
\langle (u_n)_t, \varphi \rangle &= -\sum_{i=1}^n \left(\int_0^T \lambda_i \alpha_i(t)\varphi(t)\,dt - \int_0^T f_i(t)\varphi(t)\,dt \right) e_i \\
&= -\sum_{i=1}^n \int_0^T \lambda_i e_i \alpha_i(t)\varphi(t)\,dt + \sum_{i=1}^n \int_0^T f_i(t) e_i \varphi_i(t)\,dt, \quad \forall \varphi \in C_0^\infty(]0,T[,\mathbb{R}) \\
&= \int_0^T \left(-\sum_{i=1}^n \lambda_i e_i \alpha_i(t) + \sum_{i=1}^n f_i(t) e_i \right) \varphi(t)\,dt
\end{aligned}$$

$$\text{Then} \quad (u_n)_t = -\sum_{i=1}^{n} \lambda_i e_i \alpha_i(t) + \sum_{i=1}^{n} f_i(t) e_i \quad \in L^2(]0,T[, E_n)$$

Put $f^{(n)} = \sum_{i=1}^{n} f_i e_i$, and since $\Delta u_n = -\sum_{i=1}^{n} \lambda_i \alpha_i e_i$,
we obtain

$$(u_n)_t = \Delta u_n + f^{(n)} \in L^2(]0,T[, E_n) \subset L^2(]0,T[, H_0^1(\Omega)) \subset L^2(]0,T[; H^{-1}(\Omega)) \tag{5.15}$$

and now, let $v \in L^2(]0,T[, H_0^1(\Omega))$, since $(u_n)_t \in L^2(]0,T[, H^{-1}(\Omega))$, we get:

$$\langle (u_n)_t, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \in L^1(]0,T[)$$

Indeed, by Cauchy-Schwarz inequality, getting:

$$\begin{aligned}
\int_0^T \left|\langle (u_n)_t, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\right| dt &= \int_0^T |\langle (u_n)_t, v\rangle_2|\ dt \\
&= \int_0^T \left|\int_\Omega (u_n)_t v \, dx\right| dt \\
&\leq \int_0^T \|(u_n)_t\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} < \infty \\
&\leq \|(u_n)_t\|_{L^2(]0,T[, L^2(\Omega))} \|v\|_{L^2(]0,T[, L^2(\Omega))}
\end{aligned}$$

Then, we deduce from (5.15) :

$$\int_0^T \langle (u_n)_t, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt = -\int_0^T \int_\Omega \nabla u_n \cdot \nabla v \, dx \, dt$$

$$+ \sum_{i=1}^{n} \int_0^T \int_\Omega f_i e_i v \, dx \, dt$$

This gives, returning to the definition of f_i,

$$\begin{aligned}
\int_0^T \langle (u_n)_t, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt + \int_0^T \int_\Omega \nabla u_n \cdot \nabla v \, dx \, dt &= \sum_{i=1}^{n} \int_0^T \langle f(t), e_i\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \left(\int_\Omega e_i v \, dx\right) dt \\
&= \int_0^T \langle f(t), \sum_{i=1}^{n} e_i\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \left(\int_\Omega e_i v \, dx\right) dt \\
&= \int_0^T \langle f(t), \sum_{i=1}^{n} e_i\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \langle e_i, v\rangle_{L^2(\Omega)} dt \\
&= \int_0^T \langle f(t), \sum_{i=1}^{n} \langle e_i, v\rangle_{L^2(\Omega)} e_i\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt
\end{aligned}$$

Next, we denote P_n as the orthogonal projection operator such that:

$$P_n : L^2(\Omega) \to E_n, \ \text{therefore, the operator } P_n$$

can be seen as an operator from $L^2(\Omega)$ to $H_0^1(\Omega)$

$$\Big(\text{since } E_n \subset H_0^1(\Omega)\Big)$$

we then denote P_n^t as the transposed operator from $H^{-1}(\Omega)$ into $(L^2(\Omega))'$, which is identified with $L^2(\Omega)$ and is also a subspace of $H^{-1}(\Omega)$

We obtain them for all $v \in L^2(]0,T[, H_0^1(\Omega))$:

$$\int_0^T \langle \partial_t u_n, v\rangle_{H^{-1},H_0^1}\, dt + \int_0^T \int_\Omega \nabla u_n \cdot \nabla v\, dx\, dt = \int_0^T \langle f, P_n v\rangle_{H^{-1},H_0^1}\, dt = \int_0^T \langle P_n^t f, v\rangle_{H^{-1},H_0^1}\, dt. \tag{5.16}$$

also we have $u_n \in C([0,T], H_0^1(\Omega))$ and

$$u_n(0) = \sum_{i=1}^n \alpha_i(0) e_i = \sum_{i=1}^n \langle u_0, e_i\rangle_{L^2(\Omega)} e_i = P_n u_0$$

Thus we have arrived at the approximate problem with all its concepts.

B) Estimates of the approximate solution

Let $n \in \mathbb{N}^*$, we have $(u_n) \in C([0,T], H_0^1(\Omega)) \subset L^2(]0,T[, L^2, H_0^1(\Omega))$ and $(u_n)_t = \Delta u_n + f^{(n)} \in L^2(]0,T[, H^{-1}(\Omega))$. Then by lemma (5.1.2), we get:

$$\frac{1}{2}\|u_n(T)\|_2^2 - \frac{1}{2}\|u_n(0)\|_2^2 = \int_0^T \langle (u_n)_t, u_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt$$

taking $v = u_n$ in (5.16)we get:

$$\int_0^T \langle (u_n)_t, u_n\rangle_{H^{-1},H_0^1} dt + \int_0^T \int_\Omega |\nabla u_n|^2 dx dt = \int_0^T \langle f, P_n u_n\rangle_{H^{-1},H_0^1} dt$$

So

$$\frac{1}{2}\|u_n(T)\|_2^2 - \frac{1}{2}\|u_0\|_2^2 = \int_0^T \langle f, P_n u_n\rangle_{H^{-1},H_0^1} dt - \int_0^T \int_\Omega |\nabla u_n|^2 dx dt$$

then

$$\frac{1}{2}\|u_n(T)\|_2^2 - \frac{1}{2}\|u_0\|_2^2 = \int_0^T \langle f, P_n u_n\rangle_{H^{-1},H_0^1} dt - \|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega))}$$

then

$$\begin{aligned}\|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega)} &= \frac{1}{2}\|u_0\|_2^2 - \frac{1}{2}\|u_n(T)\|_2^2 + \int_0^T \langle f, P_n u_n\rangle_{H^{-1},H_0^1} dt \\ &\leq \frac{1}{2}\|u_0\|_2^2 + \int_0^T \langle f, P_n u_n\rangle_{H^{-1},H_0^1} dt\end{aligned}$$

Noting that $P_n u_n = u_n$, because $P_n : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ and $(u_n)_n \subset H_0^1(\Omega) \Rightarrow u \subset L^2(\Omega)$, then $P_n u_n(t) \in H_0^1(\Omega)$ but $u_n(t) \in H_0^1(\Omega)$ so, the orthogonal projection of an element from a space onto the same space is the same element, then we get:

$$\|u_n\|^2_{L^2(]0,T[,H^1_0(\Omega)} \leq \frac{1}{2}\|u_0\|^2_2 + \int_0^T \langle f, u_n\rangle_{H^{-1},H^1_0} dt$$

by Cauchy Shwartz, and Young inequalities, we obtain:

$$\|u_n\|^2_{L^2(]0,T[,H^1_0(\Omega))} \leq \frac{1}{2}\|u_0\|^2_2 + \frac{1}{2}\|f\|^2_{L^2(]0,T[,H^{-1}(\Omega))} + \frac{1}{2}\|u_n\|^2_{L^2(]0,T[,H^1_0(\Omega))}$$

So

$$\|u_n\|^2_{L^2(]0,T[,H^1_0(\Omega))} \leq \frac{1}{2}\|u_0\|^2_2 + \frac{1}{2}\|f\|^2_{L^2(]0,T[,H^{-1}(\Omega))}\|u_n\|^2_{L^2(]0,T[,H^1_0(\Omega))} \quad \leq \|u_0\|^2_2 + \|f\|^2_{L^2(]0,T[,H^{-1}(\Omega)}$$

That also means:

$$\|u_n\|_{L^2(]0,T[,H^1_0(\Omega)} \leq \frac{1}{\sqrt{2}}\left(\|u_0\|_2 + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))}\right)$$

(using that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$)

then

$$\|u_n\|_{L^2(]0,T[,H^1_0(\Omega))} \leq \|u_0\|_2 + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \tag{5.17}$$

so, we obtain the first estimate for $(u_n)_{n\in\mathbb{N}^*}$.
For obtain an estimation for $((u_n)_t)_{n\in\mathbb{N}^*}$ using that $(u_n)_t = \Delta u_n + P^t_n$ by (5.16) , then we get:

$$\begin{aligned}\|(u_n)_t\|_{L^2(]0,T[)H^{-1}(\Omega)} &= \|\Delta u_n + P^t_n f\|_{L^2(]0,T[)H^{-1}(\Omega)} \\ &\leq \|\Delta u_n\|_{L^2(]0,T[)H^{-1}(\Omega)} + \|P^t_n f\|_{L^2(]0,T[)H^{-1}(\Omega)}\end{aligned}$$

on the other hand , for all $v \in H^1_0(\Omega)$

$$\langle \Delta u_n, v\rangle_{H^{-1},H^1_0} = -\int_\Omega \nabla u_n \nabla v dx$$

so

$$\begin{aligned}|\langle \Delta u_n, v\rangle_{H^{-1},H^1_0}| &\leq \|\nabla u_n\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ &\leq \|\nabla u_n\|_{L^2(\Omega)}\|v\|_{H^1_0(\Omega)}\end{aligned}$$

then $\|\Delta u_n\|_{H^{-1}(\Omega)} = \sup\limits_{\|v\|_{H^1_0(\Omega)}\leq 1} |\langle \Delta u_n, v\rangle_{H^{-1},H^1_0}| \leq \|\nabla u_n\|_{L^2(\Omega)}$
so

$$\|\Delta u_n\|_{H^{-1}(\Omega)} \leq \|u_n\|_{H^1_0(\Omega)}$$

and we obtain that:

$$\|\Delta u_n\|_{L^2(]0,T[,H^{-1}(\Omega)} \leq \|u_n\|_{L^2(]0,T[,H^1_0(\Omega)} \tag{5.18}$$

also for all $v \in H_0^1(\Omega)$

$$\begin{aligned}|\langle P_n^t f, v\rangle_{H^{-1},H_0^1}| &= |\langle f, P_n v\rangle_{H-1,H_0^1}| \\ &\leq \|f\|_{H-1(\Omega)}\|P_n v\|_{H_0^1(\Omega)}\end{aligned}$$

ysing the fact that :

$$\|P_n v\|_{H_0^1(\Omega)} \leq \|v\|_{H_0^1(\Omega)} \quad \text{for all} \quad v \in H_0^1(\Omega),$$

then

$$|\langle P_n^t f, v\rangle_{H^{-1},H_0^1}| \leq \|f\|_{H-1(\Omega)}\|v\|_{H_0^1(\Omega)}$$

then

$$\|P_n^t f\|_{H^{-1}} = \sup_{\|v\|_{H_0^1(\Omega)}\leq 1} |\langle P_n^t f, v\rangle_{H^{-1},H_0^1}| \leq \|f\|_{H^{-1}(\Omega)}$$

we can now deduce that

$$\|P_n^t f\|_{L^2(]0,T[,H-1(\Omega)} \leq \|f\|_{L^2(]0,T[,H-1(\Omega)} \tag{5.19}$$

by (5.18) and (5.19) we get

$$\|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))} \leq \|u_n\|_{L^2(]0,T[,H_0^1(\Omega))} + \|f\|_{L^2(]0,T[,H-1(\Omega))}$$

Finally by (5.17)we get

$$\|u_n\|_{L^2(]0,T[,H^{-1}(\Omega))} \leq \|u_0\|_2 + 2\|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \tag{5.20}$$

with this two estimates 5.17 and (5.20), we conclude that $(u_n)_{n\in\mathbb{N}^*}$ is bounded in $L^2(]0,T[,H_0^1(\Omega))$ and $((u_n)_t)_{n\in\mathbb{N}^*}$ is bounded in $L^2(]0,T[,H^{-1}(\Omega))$.

C) Existence of a week solution by passing to the limit

By the previous step , and by using the fact that $L^2(]0,T[,H_0^1(\Omega))$ and $L^2(]0,T[,H^{-1}(\Omega))$ are reflexif spaces ,we can extract a subsequence still denoted (u_n) such that:

$$\begin{aligned}u_n &\longrightarrow u \text{ weakly in } L^2(]0,T[,H_0^1(\Omega)) \\ (u_n)_t &\longrightarrow w \text{ weakly in } L^2(]0,T[,H^{-1}(\Omega))\end{aligned}$$

Previously we proved that

$$\begin{aligned}\|u_n\|_{L^2(]0,T[,H_0^1(\Omega))} &\leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \\ \|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))} &\leq \|u_0\|_{L^2(\Omega)} + 2\|f\|_{L^2(]0,T[,H^{-1}(\Omega))}\end{aligned}$$

then we deduce:

$$\lim_{n\to+\infty}\inf \|u_n\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))}$$
$$\lim_{n+\to\infty}\inf \|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + 2\|f\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

and by (1.5.1) we deduce that:

$$\|u\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \lim_{n\to+\infty}\inf \|(u_n)\|_{L^2(]0,T[,H_0^1(\Omega))}$$
$$\|w\| \leq \lim_{n\to+\infty}\inf \|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

then

$$\begin{cases}\|u\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))}\\ \|w\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + 2\|f\|_{L^2(]0,T[,H^{-1}(\Omega))}\end{cases}$$

We will first show that $w = \partial tu$ after we will show that u ia a solution of $\partial t_u = \Delta u + f$ in the sense required by(5.1.2). We null show that $w = \partial_t u = \Delta u + f$ in the sence required by (5.1.2).

1)Showing that $w = \partial_t u$

By definition of $\partial_t u$,we have for all $\varphi \in C_c^\infty(]0,T[,\mathbb{R})$

$$\int_0^T \partial_t u(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt$$

To demonstrate that $\partial_t u = w$, it suffices to show that for all $\varphi \in C_c^\infty(]0,T[,\mathbb{R})$

$$\int_0^T w(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \tag{5.21}$$

We recall that the left term of the equality is in $H^{-1}(\Omega)$ while the term on the right is in $H_0^1(\Omega)$.

This equality therefore uses the fact that $H_0^1(\Omega) \subset H^{-1}(\Omega)$, this inclusion being due to the fact that we have identified $(L^2(\Omega))'$ with $L^2(\Omega)$.

Let therefore. $\varphi \in C_c^\infty(]0,T[,\mathbb{R})$. We will show (5.21). For $\psi \in H_0^1(\Omega)$, we consider the application.

$$S : L^2(]0,T[,H_0^1(\Omega)) \longrightarrow \mathbb{R}$$
$$v \longrightarrow S(v) = \int_\Omega \left(-\int_0^T v(t)\varphi'(t)dt\right)\psi(x)dx$$

the application S is linear, continous from $L^2(]0,T[,H_0^1(\Omega))$ to $\mathbb{R}$

$$\begin{aligned} S(u) &= \int_\Omega \left(-\int_0^T v(t)\varphi'(t)\,dt \right) \psi(x)\,dx \\ &= \left\langle -\int_0^T v(t)\varphi'(t)\,dt, \psi \right\rangle_{L^2(\Omega)} \\ &= \left\langle -\int_0^T v(t)\varphi'(t)\,dt, \psi \right\rangle_{H^{-1},H_0^1} \end{aligned}$$

since $u_n \to u$ weakly in $L^2(]0,T[,H_0^1(\Omega))$ then we have

$$S(u_n) \xrightarrow[n\to+\infty]{} S(u)$$

then we get

$$\left\langle -\int_0^T u_n(t)\varphi'(t)\,dt, \psi \right\rangle_{H^{-1},H_0^1} \xrightarrow[n\to+\infty]{} \left\langle -\int_0^T u(t)\varphi'(t)\,dt, \psi \right\rangle_{H^{-1},H_0^1}$$

We now use the fact that

$$-\int_0^T u_n(t)\varphi'(t)\,dt = \int_0^T (u_n)_t(t)\varphi(t)\,dt \quad \text{(by definition of}(u_n)_t)$$

Therefore, we have

$$\left\langle \int_0^T (u_n)_t(t)\varphi(t)\,dt, \psi \right\rangle_{H^{-1},H_0^1} \xrightarrow[n\to+\infty]{} -\left\langle \int_0^T u(t)\varphi'(t)\,dt, \psi \right\rangle_{H^{-1},H_0^1} \tag{5.22}$$

We now consider the application $\overline{S}$ from $L^2(]0,T[,H^{-1}(\Omega))$ to $\mathbb{R}$ defined by

$$\overline{S}(v) = \left\langle \int_0^T v(t)\phi(t)\,dt, \psi \right\rangle_{H^{-1},H_0^1}$$

for $v \in L^2(]0,T[,H^{-1}(\Omega))$.

The application $\overline{S}$ is a continuous linear map from $L^2(]0,T[,H^{-1}(\Omega))$ to $\mathbb{R}$. since $(u_n)_t \longrightarrow w$ weekly in $L^2(]0,T[,H^{-1}(\Omega))$ then we have $\overline{S}((u_n)_t) \xrightarrow[n\to+\infty]{} \overline{S}(w)$ i.e .

$$\langle \int_0^T (u_n)_t(t)\varphi(t)dt, \psi \rangle_{H^{-1},H_0^1} \xrightarrow[n\to+\infty]{} \langle \int_0^T w(t)\varphi(t)dt, \psi \rangle_{H^{-1},H_0^1} \tag{5.23}$$

Then we conclude from (5.23) and (5.25)

$$\begin{aligned}\forall \psi \in & H_0^1(\Omega) : \\ & -\langle \int_0^T u(t)\varphi'(t)dt, \psi\rangle_{H^{-1},H_0^1} = \langle \int_0^T w(t)\varphi(t)dt, \psi\rangle_{H^{-1},H_0^1}\end{aligned}$$

then

$$-\int_0^T u(t)\varphi'(t)dt = \int_0^T w(t)\varphi(t)dt, \forall \varphi \in C_c^\infty(]0,T[,\mathbb{R})$$

So $\partial_t u = w$.

we therefore know that:

$$\begin{cases} u_n \longrightarrow u \text{ weakly in } L^2(]0,T[,H_0^1(\Omega)) \\ \partial_t u_n \longrightarrow \partial_t u \text{ weakly in } L^2(]0,T[,H^{-1}(\Omega)) \end{cases} \tag{5.24}$$

2) showing that u is *a* solution of $\partial_t u = \Delta u + f$.

To show that u is *a* solution of $\partial_t u = \Delta u + f$ in the sense required by (5.13), it is now sufficient to pass the limite in (5.16)

Let $v \in L^2(]0,T[,H_0^1(\Omega))$, we have for $n \in \mathbb{N}^*$, according to (5.16) ,

$$\int_0^T \langle (u_n)_t, v\rangle_{H^{-1},H_0^1} dt + \int_0^T \langle u_n, v\rangle_{H_0^1(\Omega)} dt = \int_0^T \langle f, P_n v\rangle_{H^{-1},H_0^1} dt$$

Thanks to the convergence in 5.24 we obtain:

$$\int_0^T \langle (u_n)_t, v\rangle_{H^{-1},H_0^1} dt \xrightarrow[n\to+\infty]{} \int_0^T \langle \partial_t u, v\rangle_{H^{-1},H_0^1} dt$$

and we have

$$\int_0^T \langle u_n, v\rangle_{H_0^1(\Omega)} dt \xrightarrow[n\to+\infty]{} \int_0^T \langle u, v\rangle_{H_0^1(\Omega)} dt$$

For demonstrate that $\int_0^T \langle f, P_n v\rangle_{H^{-1},H_0^1} dt \xrightarrow[n\to+\infty]{} \int_0^T \langle f, v\rangle_{H^{-1},H_0^1} dt$ we will use the dominated convergence theorem.

we put $h_n(t) = \langle f, P_n v\rangle_{H^{-1},H_0^1}$

i) $P_n \longrightarrow I$ uniformely in $H_0^1(\Omega)$ then we get

$$\begin{aligned}|\langle f, P_n v\rangle_{H^{-1},H_0^1} - \langle f, v\rangle_{H^{-1},H_0^1} &= \left|\langle f, (P_n - I)v\rangle_{H^{-1},H_0^1}\right| \\ &\leq \|f\|_{H^{-1}} \|(P_n - I)v\|_{H_0^1} \\ &\leq \|f\|_{H^{-1}} \|P_n - I\|_{H_0^1} \|v\|_{H_0^1} \xrightarrow[n\to+\infty]{} 0\end{aligned}$$

then

$$\langle f, P_n v\rangle_{H^{-1},H_0^1} \xrightarrow[n\to+\infty]{} \langle f, v\rangle_{H^{-1},H_0^1} \quad \text{a.e.}$$

ii)

$$|h_n(t)| = |\langle f, P_n v\rangle_{H^{-1},H_0^1}| \leq \|f\|_{H^{-1}}\|P_n v\|_{H_0^1} \leq \|f\|_{H^{-1}}\|v\|_{H_0^1}$$

we pose $g(t) = \|f\|_{H^{-1}}\|v\|_{H_0^1}$, using Cauchy-Schwartz inequality, we get:

$$\int_0^T |g(t)|dt = \int_0^T \|f\|_{H^{-1}}\|v\|_{H_0^1}dt$$
$$\leq \|f\|_{L^2(]0,T[,H^{-1}(\Omega))}\|v\|_{L^2(]0,T[,H_0^1(\Omega))} < \infty$$

then $g \in L^1(]0,T[)$.

So, by Domited convergence theoreme, we obtain

$$\lim_{n\to+\infty}\int_0^T \langle f, P_n v\rangle dt = \int_0^T \lim_{n\to+\infty}\langle f, P_n v\rangle dt$$
$$= \int_0^T \langle f, v\rangle dt.$$

Finally we get:

$$\int_0^T \langle \partial_t u, u\rangle_{H^{-1},H_0^1}dt + \int_0^T \langle u, v\rangle_{H_0^1}dt = \int_0^T \langle f, v\rangle_{H^{-1},H_0^1}dt$$

and this is the desired sense in formulation (5.16).

D)Initial condition

Since $u \in L^2(]0,T[, H_0^1(\Omega))$ and $\partial_t u \in L^2(]0,T[, H^{-1}(\Omega))$. then $u \in C([0,T], L^2(\Omega))$. see (5.1.2).

To complete the demonstration of the fact that u is a solution to (5.16), it remains therefore only to show that $u(0) = u_0$ a.e. (that is, $u(0) = u_0$ in $L^2(\Omega)$).
We know that $u \in C([0,T], L^2(\Omega))$ then

$$\lim_{t\to 0} u(t) = u(0) \in L^2(\Omega)$$

We also know that $u_n(0) = \sum_{i=1}^{N} \alpha_i^{(0)} e_i \underset{n\to+\infty}{\to} u_0$ in $L^2(\Omega)$.
It suffices to show that $(u_n)_{n\in\mathbb{N}^*}$is relatively compact in $C([0,T], H^{-1}(\Omega))$ because ,with it , we can conclude that there exists a subsequence ,still denoted $(u_n)_{n\in\mathbb{N}^*}$,and a function
$\in C([0,T], H^{-1}(\Omega))$,such that

$$u_n(t) \to w(t) \text{ in } H^{-1}(\Omega)$$

uniformly with respect to $t \in [0,T]$ (and then also in $L^2(]0,T[, H^{-1}(\Omega))$........$(I)$

So,we have

$$\begin{cases} u_n(0) \underset{n\to+\infty}{\to} u_0 \in L^2(\Omega) \subset H^{-1}(\Omega) \text{ continously} \\ u_n(0) \underset{n\to+\infty}{\to} w(0) \in H^{-1}(\Omega) \end{cases}$$

Then $w(0) = u_0$ (from the uniqueness of the limite) but we know that $u_n \to n$ weakly in $L^2(]0,T[, H_0^1(\Omega))$, then also weakly in $L^2(]0,T[, H^{-1}(\Omega))......(II)$
then by (I and (II)) ,and by unicity of the limite we have

$$w = u \quad \text{a.e. in }]0,T[$$

but for all $t \in [0,T]$ u and w are countinous on $[0,T]$ then

$$u(t) = w(t), \quad \text{for all } t \in [0,T]$$

Finally, we get $w(0) = u(0) = u_0$.

It remains to show that $(u_n)_{n\in\mathbb{N}^*}$ is relatively compact in $C([0,T], H^{-1}(\Omega))$. By Ascoli-Arzelà theorem (1.1.6) , it suffices to show that:

a) $\forall t \in [0,T],\ (u_n)_{n\in\mathbb{N}^*}$ is relatively compact in $H^{-1}(\Omega)$

b) $\|u_n(t) - u_n(s)\|_{H^{-1}(\Omega)} \longrightarrow 0$ as $s \to t$. uniformly with respect to $n \in \mathbb{N}^*$ (and for all $t \in [0,T]$).

***Proof*.** **For a):** we have $u_n \in L^2(]0,T[, H_0^1(\Omega))$ and $(u_n)_t \in L^2(]0,T[, H^{-1}(\Omega))$ by lemma (5.1.2), we have for all $t, s \in [0,T]$

$$\|u_n(t)\|_2^2 = \|u_n(s)\|_2^2 + 2\int_s^t \langle (u_n)_t(\xi), u_n(\xi)\rangle_{H^{-1},H_0^1} d\xi$$

and then

$$\begin{aligned}\|u_n(t)\|_2^2 &\le \|u_n(s)\|_2^2 + 2\int_s^t \left|\langle (u_n)_t(\xi), u_n(\xi)\rangle_{H^{-1},H_0^1}\right| d\xi \\ &\le \|u_n(s)\|_2^2 + 2\int_0^T \left|\langle (u_n)_t(\xi), u_n(\xi)\rangle_{H^{-1},H_0^1}\right| d\xi\end{aligned}$$

Using Cauchy-Schwartz, we get

$$\|u_n(t)\|_2^2 \le \|u_n(s)\|_2^2 + 2\|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))}\|u_n\|_{L^2(]0,T[,H_0^1(\Omega))}$$

integrate this inequality. with respect to s in $[0,T]$, one can deduce;

$$T\|u_n(t)\|_2^2 \le \int_0^T \|u_n(s)\|_2^2 ds + 2T\|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega)}\|u_n\|_{L^2(]0,T[,H_0^1(\Omega)}$$

then

$$T\|u_n(t)\|_2^2 \leq \|u_n\|^2_{L^2(]0,T[,L^2(\Omega))} + 2T\|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))}\|u_n\|_{L^2(]0,T[,H_0^1(\Omega))}$$

This shows that the sequence $(u_n(t))_{n\in\mathbb{N}^*}$ is bounded in $L^2(\Omega)), \forall t \in [0,T]$ (and even uniformly with respect to t.

Now, we will demonstrate (b)
For b) :
using the facy that

$$(u_n)_t \in L^1([0,T], H^{-1}(\Omega))$$

for all $t_1, t_2 \in [0,T]$, $t_1 > t_2$, $\forall n \in \mathbb{N}^*$ we have (in $H^{-1}(\Omega)$):

$$u(t_1) - u(t_2) = \int_{t_2}^{t_1} (u_n)_t(s)\, ds$$

and then

$$\begin{aligned}\|u(t_1) - u(t_2)\|_{H^{-1}(\Omega)} &\leq \int_{t_2}^{t_1} \|(u_n)_t(s)\|_{H^{-1}(\Omega)}\, ds \\ &\leq \left(\int_{t_2}^{t_1} 1^2 ds\right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} \|(u_n)_t(s)\|^2_{H^{-1}(\Omega)}\, ds\right)^{\frac{1}{2}} \\ &\leq \sqrt{t_1 - t_2} \int_0^T \|(u_n)_t(s)\|^2_{H^{-1}(\Omega)}\, ds\end{aligned}$$

Since $((u_n)_t)_{n\in\mathbb{N}^*}$ is bounded in $L^2(]0,T[, H^{-1}(\Omega))$: (i.e $\|(u_n)_t\|_{L^2(]0,T[,H^{-1}(\Omega))} < \infty$) then $\|u_n(t_1) - \|u_n(t_2)\|_{H^{-1}(\Omega)} \xrightarrow[t_2\to t_1]{} 0$ uniformly, with respect to $n \in \mathbb{N}^*$ (and for all $t \in [0,T]$).
So, we conclude finally by Ascoli-Arzelà theorem that $(u_n)_{n\in\mathbb{N}^*}$ is relatively compact in $C([0,T], H^{-1}(\Omega))$.

□

E) Unicity

Now we demonstrate the "uniqueness" part of Theorem (5.1.2) Let u_1 and u_2 be two solutions of ((5.13)) . let $u = u_1 - u_2$. By taking the difference of the equations satisfied by u_1 and u_2 and choosing for $t \in [0,T], v = u\chi_{]0,T[}$, as a test function, we obtain:

$$\int_0^t \langle \partial_t u(s), u(s)\rangle_{H^{-1},H_0^1}\, ds + \int_0^t \int_\Omega \nabla u(s)\nabla u(s) dx ds = 0$$

Since $u \in L^2(]0,T[, H_0^1(\Omega))$ and $\partial_t u \in L^2(]0,T[, H^{-1}(\Omega))$ by lemma (5.1.2), we get:

$$\frac{1}{2}\left(\|u(t)\|_2^2 - \|u(0)\|_2^2\right) = \int_0^t \langle \partial_t u(s), u(s)\rangle_{H^{-1},H_0^1}\, ds$$

We deduce that, for all $t \in [0,T]$,

$$\left(\|u(t)\|_2^2 - \|u(0)\|_2^2\right) + 2\int_0^t \int_\Omega \nabla u(s)\cdot\nabla u(s)\,dx\,ds = 0$$

Finally, since $u(0) = 0$,then

$$\|u(t)\|_2^2 + 2\int_0^t \|u(s)\|_{H_0^1(\Omega)}\,ds = 0$$

$$\Rightarrow \begin{cases} \|u(t)\|_2^2 = 0 \Rightarrow u(t) = 0 \quad \text{a.e in } \Omega \\ \text{and} \\ \displaystyle\int_0^t \|u(s)\|_{H_0^1(\Omega)} = 0 \Rightarrow \|u(s)\|_{H_0^1(\Omega)} = 0 \Rightarrow u(s) = 0 \quad \text{a.e in } \Omega \end{cases}$$

then $u_1 = u_2 =$ a.e. in Ω.

Thus we have finished proving the existence and uniqueness of the weak solution.

Proof of the last estimation in Theorem (5.1.2)
Since $u \in L^2(]0,T[,H_0^1(\Omega))$ and $\partial_t u \in L^2(]0,T[,H^{-1}(\Omega))$ by Lemma (5.1.2) ,for all $t \in [0,T]$

$$\|u(t)\|_2^2 = \|u(0)\|_2^2 + 2\int_0^t \langle \partial_t u(s), u(s)\rangle_{H^{-1},H_0^1}\,ds$$

with Cauchy-Schwartz inequality, and young inequality we get:

$$\begin{aligned} \|u(t)\|_2^2 &\le \|u_0\|_2^2 + 2\int_0^t \|\partial_t u(s)\|_{H^{-1}(\Omega)}\|u(s)\|_{H_0^1(\Omega)}ds \\ &\le \|u_0\|_2^2 + 2\int_0^T \|\partial_t u(s)\|_{H^{-1}(\Omega)}\|u(s)\|_{H_0^1(\Omega)}ds \\ &\le \|u_0\|_2^2 + 2\|\partial_t u\|_{L^2(]0,T[,H^{-1}(\Omega))}\|u\|_{L^2(]0,T[,H_0^1(\Omega))}. \end{aligned}$$

Then we conclude

$$\|u(t)\|_2^2 \le \|u_0\|_2^2 + \|\partial_t u\|^2_{L^2(]0,T[,H^{-1}(\Omega))}\|u\|^2_{L^2(]0,T[,H_0^1(\Omega))}.$$

Proposition 5.1.3. Continuous dependence *Let Ω be an open bounded set in $\mathbb{R}^N, T > 0$. For $u_0 \in L^2(\Omega)$ and $f \in L^2(]0,T[,H_0^1(\Omega))$, we note $T(u_0,f)$. the solution of ((5.13)). T is continuous linear operator from $L^2(\Omega) \times L^2(]0,T[,H^{-1}(\Omega))$ to $L^2(]0,T[,H_0^1(\Omega))$ and to $C([0,T];L^2(\Omega))$.*

Proof.

$$\|\cdot\|_{L^2(\Omega)\times L^2(]0,T[,H^{-1}(\Omega))} = \|\cdot\|_{L^2(\Omega))} + \|\cdot\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

Then T est continuous from $L^2(\Omega) \times L^2(]0, T[, H^{-1}(\Omega))$ means that:

$\exists c > 0$, for $(u_0, f) \in L^2(\Omega) \times L^2(]0, T[, H^{-1}(\Omega))$:

$$\|T(u_0, f)\|_{L^2(]0,T[,H_0^1(\Omega))} \leq c \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \right)$$

i.e:

$$\|u\|_{L^2(]0,T[,H_0^1(\Omega))} \leq c \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \right)$$

and by the first estimation of u, we get:

$$\|u\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

Then We conclude that what is stated in the proposition is verified for $L^2(]0, T[, H_0^1(\Omega))$. We have the third estimate of $u(t)$:

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \|\partial_t u\|^2_{L^2(]0,T[,H^{-1}(\Omega))} \|u\|^2_{L^2(]0,T[,H_0^1(\Omega))} ds, \quad \forall t \in [0, T]$$

Then by the first and the second estimate of $\partial_t u$, we get:

$$\|u(t)\|_2^2 \leq 3\|u_0\|_2^2 + 3\|f\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

then

$$\sup_{t \in [0,T]} \|u(t)\|_2^2 \leq 3\|u_0\|_2^2 + 3\|f\|_{L^2(]0,T[,H^{-1}(\Omega))}$$

So

$$\|u\|^2_{C([0,T];L^2(\Omega))} \leq 3 \left(\|u_0\|_2^2 + \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \right)$$

then

$$\|T(u_0, f)\|_{(C([0,T];L^2(\Omega)))} \leq 3\|(u_0, f)\|_{L^2(\Omega) \times L^2(]0,T[,H^{-1}(\Omega))}$$

so we obtain what is stated in proposition for $C([0, T]; L^2(\Omega))$. □

□

5.2 ***Nonlinear Diffusion***

Let's first consider the following example.

Let Ω be a bounded open set in $\mathbb{R}^N$ and $A : \mathbb{R} \to M_N(\mathbb{R})$ (where $M_N(\mathbb{R})$ denotes the set of $N \times N$ matrices with real coefficients) such that:

$$\forall s \in \mathbb{R}, \quad A(s) = \left(a_{ij}(s) \right)_{i,j=1,\ldots,N} \quad \text{where} \quad a_{ij} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R} \times \mathbb{R}) \tag{5.25}$$

$$\exists \alpha > 0, \quad A(s)\xi \cdot \xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R} \tag{5.26}$$

We have the following problem:

$$\begin{cases} \partial_t u = \text{div}\,(A(u)\nabla u) = f, & x \in \Omega, t \in]0,T[\\ u(.,0) = 0, & \partial\Omega \times]0,T[\\ u(0) = u_0, & \end{cases}$$

Then we can show by Schauder's theorem that there exists a solution u of (Still with $L^2(\Omega)$ identified with $(L^2(\Omega)\,)$ '):

$$\begin{cases} u \in L^2\left(]0,T[, H_0^1(\Omega)\right), \quad \partial_t u \in L^2\left(]0,T[, H^-1(\Omega)\right) \quad \left(\text{and then} \quad u \in C\left([0,T], L^2(\Omega)\right)\right) \\ \displaystyle\int_0^T \langle \partial_t u, v \rangle_{H^{-1},H_0^1}\, dt + \int_0^T \int_\Omega A(u)\nabla u.\nabla v \, dx\, dt = \int_0^T \langle f, v \rangle_{H^{-1},H_0^1}\, dt \\ u(.,0) = u_0 \quad \text{and} \quad f \in L^2\left(]0,T[, H^{-1}(\Omega)\right), \quad u_0 \in L^2(\Omega). \end{cases} \tag{5.27}$$

To use Schauder's theorem, we use the resolution of linear problems:
Let $\bar{u} \in L^2(]0,T[, L^2(\Omega))$, we define the operator:

$$\begin{cases} T : L^2(]0,T[, L^2(\Omega)) & \to L^2(]0,T[, L^2(\Omega)) \\ & u \to T(\bar{u}) = u \end{cases}$$

where u is the solution of the problem (5.27) where we replaced $A(u)$ with $A(\bar{u})$,i.e.:

$$\begin{cases} u \in L^2(]0,T[, H_0^1(\Omega)), \quad \partial_t u \in L^2(]0,T[, H^{-1}(\Omega)) \\ \displaystyle\int_0^T \langle \partial_t u, v \rangle_{H^{-1},H_0^1}\, dt + \int_0^T \int_\Omega A(\bar{u})\nabla u.\nabla v \, dx\, dt = \int_0^T \langle f, v \rangle_{H^{-1},H_0^1}\, dt, \\ u(.,0) = u_0 \qquad\qquad \forall v \in L^2(]0,T[, H_0^1(\Omega)) \end{cases} \tag{5.28}$$

First, We need some preliminaries

5.2.1 *Preliminary result*

Theorem 5.2.1. *[9]**(Schauder Fixed Point Theorem)** .Let E be a Banach space. For any $R > 0$, define the closed ball B_R by*

$$B_R = \{x \in E \mid \|x\| \leq R\}.$$

Let f be a compact mapping from B_R to B_R, meaning that f is continuous and the set

$$\{f(x) \mid x \in B_R\}$$

is relatively compact in E. Then f has a fixed point, i.e., there exists $x \in B_R$ such that $f(x) = x$.

Theorem 5.2.2. *[9] **(Aubin-Simon)** .Let $1 \leq p < +\infty$, and let X, B, and Y be three Banach spaces such that*

(i) $X \subset B$ with compact injection,

(ii) $X \subset Y$, and if $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements in X such that the sequence $(\|f_n\|_X)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$, $f_n \to f$ in B, and $f_n \to 0$ in Y (as $n \to +\infty$), then $f = 0$.

Let $T > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements in $L^p((0,T), X)$ such that

1. *$(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T), X)$,*
2. *$\left(\dfrac{du_n}{dt}\right)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), Y)$.*

Then there exists $u \in L^p((0,T), B)$ such that, up to a subsequence, $u_n \to u$ in $L^p((0,T), B)$.

Lemma 5.2.1. *[23] **(Space-Time Compactness in the L^2 Framework)**.Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(]0,T[, L^2(\Omega))$. We identify $L^2(\Omega)_0$ with $L^2(\Omega)$. Suppose that:*

1. *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(]0,T[, H_0^1(\Omega))$,*
2. *The sequence $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(]0,T[, H^{-1}(\Omega))$.*

Then, the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(]0,T[, L^2(\Omega))$.

5.2.2 *The existence of Solutions by Schauder's fixed point theorem*

1- We show that T is continuous from $L^2(]0,T[, L^2(\Omega))$ to $L^2(]0,T[, L^2(\Omega))$.

Let $(\bar{u}_n)_{n\in\mathbb{N}}$ be a sequence of $L^2(]0,T[, L^2(\Omega))$, and $\bar{u} \in L^2(]0,T[, L^2(\Omega))$ such that:

$$\bar{u}_n \xrightarrow[n\to\infty]{} \bar{u} \text{ in } L^2(]0,T[, L^2(\Omega)).$$

We set $u_n = T(\bar{u}_n)$ and $u = T(\bar{u})$. To show that u_n converges to u in $L^2(]0,T[, L^2(\Omega))$ as $n \to \infty$, we proceed by contradiction.

If $u_n \nrightarrow u$ in $L^2(]0,T[, L^2(\Omega))$, then $\exists \varepsilon > 0$ and a subsequence still denoted (u_n) such that:

$$\|u_n - u\|_{L^2(]0,T[, L^2(\Omega))} \geq \varepsilon, \quad \forall n \in \mathbb{N}$$

u_n is the solution of (5.28) with $\bar{u}_n$ instead of $\bar{u}$. By taking $v = u_n$ as a test function, we get:

$$\int_0^T \langle \partial_t u_n, u_n \rangle_{H^{-1},H_0^1} \, dt + \int_0^T \int_\Omega A(\bar{u}_n) \nabla u_n . \nabla u_n \, dx \, dt = \int_0^T \langle f, u_n \rangle_{H^{-1},H_0^1} \, dt \tag{5.29}$$

$$I_1 = \int_0^T \langle \partial_t u_n, u_n \rangle_{H^{-1},H_0^1} \, dt = \frac{1}{2} \|u_n(T)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_n(0)\|^2_{L^2(\Omega)}$$

$$I_2 = \int_0^T \int_\Omega A(\bar{u}_n) \nabla u_n . \nabla u_n \, dx \, dt$$

by the assumption (5.26) we get:

$$I_2 \geq \alpha \int_0^T \int_\Omega |\nabla u_n|^2 dx \, dt = \alpha \|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega))}$$

$$I_3 = \int_0^T \langle f, u_n \rangle_{H^{-1},H_0^1} \, dt.$$

By Cauchy-Schwartz inequality, we get:

$$I_3 \leq \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \|u_n\|_{L^2(]0,T[,H_0^1(\Omega))}$$

By Young's inequality, we have:

$$\frac{1}{\sqrt{\alpha}} \|f\|_{L^2(]0,T[,H^{-1}(\Omega))} \cdot \sqrt{\alpha} \|u_n\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \frac{\alpha}{2} \|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega))} + \frac{1}{2\alpha} \|f\|^2_{L^2(]0,T[,H^{-1}(\Omega))}$$

then

$$I_3 \leq \frac{\alpha}{2} \|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega))} + \frac{1}{2\alpha} \|f\|^2_{L^2(]0,T[,H^{-1}(\Omega))}$$

after (5.29) becomes:

$$\alpha \|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_n(0)\|^2_{L^2(\Omega)} + \frac{1}{\alpha} \|f\|^2_{L^2(]0,T[,H^{-1}(\Omega))}$$

then $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^2(]0,T[, H_0^1(\Omega))$
on the other hand we have:

$$(u_n)_t = div(A(\bar{u}_n) \nabla u_n) + f$$

***Remark* 5.2.1.** $\bar{u}_n \in L^2(]0,T[, L^2(\Omega))$ *, since the* $a_{i,j}$ *are continuous and bounded then* $A(\bar{u}_n) \in L^\infty(]0,T[, L^\infty(\Omega))$ *,so* $A(\bar{u}_n) \nabla u_n \in L^2(]0,T[, L^2(\Omega))$
then $div(A(\bar{u}_n) \nabla u_n) \in L^2(]0,T[, H^{-1}(\Omega))$.

$div(A(\bar{u}_n) \nabla u_n) \in L^2(]0,T[, H^{-1}(\Omega))$; and since $f \in L^2(]0,T[, H^{-1}(\Omega))$
then

$$(u_n)_t \in L^2(]0,T[, H^{-1}(\Omega))$$

we deduce that $((u_n)_t)_{n\in\mathbb{N}}$ is thus bounded in $L^2(]0,T[,H^{-1}(\Omega))$. Now, we have:

$$\begin{cases}(u_n)_{n\in\mathbb{N}} \text{ bounded in } L^2(]0,T[,H_0^1(\Omega))\\ ((u_n)_t)_{n\in\mathbb{N}} \text{ bounded in } L^2(]0,T[,H^{-1}(\Omega))\end{cases}$$

then we can extract a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases}\exists w\in L^2(]0,T[,H_0^1(\Omega)) \text{ and } \xi\in L^2(]0,T[,H^{-1}(\Omega))\\ u_n \longrightarrow w \text{ weakly in } L^2(]0,T[,H_0^1(\Omega))\\ (u_n)_t \longrightarrow \xi \text{ weakly in } L^2(]0,T[,H^{-1}(\Omega)).\end{cases}$$

By theorem (5.2.2) we also conclude that the subsequence:

$$u_n \to w \text{ in } L^2(]0,T[,L^2(\Omega)). \tag{I}$$

On the other hand, as shown in the Proof of theorem (5.1.2), we necessarily have $\xi = w_t$. And as we have $\bar{u}_n \xrightarrow[n\to+\infty]{} \bar{u}$ in $L^2(]0,T[,L^2(\Omega))$ then

We can also assume, after extracting a subsequence that $\bar{u}_n \to \bar{u}$ a.e. on $]0,T[\times\Omega$

Then, we deduce that: $A(\overline{u}_n) \to A(u)$ a.e. on $]0,T[\times\Omega$ (thanks to the continuity of A) .

Let $v\in L^2(]0,T[,H_0^1(\Omega))$ .Now, take the limit as $n\to+\infty$ in (5.28).written with $\overline{u}_n$ and $\overline{u}$ instead of $\overline{u}$ and u.

$$\begin{aligned}\lim_{n\to+\infty}\int_0^T \langle \partial u_n, v\rangle_{H^{-1},H_0^1}\,dt + \lim_{n+\to\infty}\int_0^T\int_\Omega A(\bar{u}_n)\nabla u_n.\nabla v dxdt\\ = \lim_{n+\to\infty}\int_0^T \langle f, v\rangle_{H^{-1},H_0^1}\,dt.\end{aligned} \tag{5.30}$$

As $(u_n)_t \to w_t$ weakly in $L^2(]0,T[,H^{-1}(\Omega))$

then

$$\int_0^T \langle \partial_t u_n, v\rangle_{H^{-1},H_0^1(\Omega)}dt \xrightarrow[n\to+\infty]{} \int_0^T \langle w_t, v\rangle_{H^{-1},H_0^1(\Omega)}dt, \quad \forall v\in L^2(]0,T[,H_0^1(\Omega)).$$

Also, for demonstrating that:

$$\int_0^T\int_\Omega A(\bar{u}_n)\nabla u_n.\nabla v\,dx\,dt \xrightarrow[n\to+\infty]{} \int_0^T\int_\Omega A(\bar{u})\nabla w.\nabla v\,dx\,dt, \quad \forall v\in L^2(]0,T[,H_0^1(\Omega)).$$

We will use Fubini Theorem.

Taking $T_n = A(\bar{u}_n)\nabla u_n \in L^2(]0,T[,(L^2(\Omega))^N)$.

i) We have $A(\bar{u}_n)\to A(\bar{u})$ a.e. on $]0,T[\times\Omega$

and $\nabla u_n \to \nabla w$ a.e. on $]0,T[\times\Omega$ since $u_n\to w$ weakly in $L^2(]0,T[,H_0^1(\Omega))$

then

$$A(\bar{u}_n)\nabla u_n \to A(\bar{u})\nabla w \quad a.e.\text{on }]0,T[\times\Omega$$

ii)

$$\|T_n\|_{(L^2(\Omega))^N} = \left(\int_\Omega |A(\bar{u}_n)\nabla u_n|^2 dx\right)^{\frac{1}{2}} = \|A(\bar{u}_n)\nabla u_n\|_{(L^2(\Omega))^N}$$

Taking $G(t) = \|A(\bar{u}_n)\nabla u_n\|_{(L^2(\Omega))^N}$

$$\int_0^T |G(t)|^2\, dt = \|A(\bar{u}_n)\nabla u_n\|^2_{L^2(]0,T[,(L^2(\Omega))^N)} < \infty.$$

Then $G \in L^2(]0,T[,\mathbb{R})$.

By Dominated Convergence Theorem, we get that

$$T_n \xrightarrow[n\to+\infty]{} T \text{ in } L^2(]0,T[,(L^2(\Omega))^N).$$

Then

$$T_n \xrightarrow[n\to+\infty]{} T \quad \text{weakly in} \quad L^2(]0,T[,(L^2(\Omega))^N)$$

. So

$$\int_0^T \langle T_n, v\rangle_{(L^2(\Omega))^N} dt \xrightarrow[n\to+\infty]{} \int_0^T \langle T, v\rangle_{(L^2(\Omega))^N}\, dt \quad \forall v \in L^2(]0,T[,(L^2(\Omega))^N)$$

i.e.

$$\int_0^T \int_\Omega A(\bar{u}_n)\nabla u_n.v dx dt \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega A(\bar{u})\nabla w.\ v\ dxdt, \quad \forall v \in L^2(]0,T[,(L^2(\Omega))^N)$$

then

$$\int_0^T \int_\Omega A(\bar{u}_n)\nabla u_n \cdot \nabla v\, dx\, dt \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega A(\bar{u})\nabla w.\nabla v\, dx\, dt, \quad \forall v \in L^2(]0,T[,H_0^1(\Omega))$$

Then (5.30) becomes

$$\int_0^T \langle w_t, v\rangle_{H^{-1},H_0^1}\, dt + \int_0^T \int_\Omega A(\bar{u})\nabla w \cdot \nabla v\, dx\, dt = \int_0^T \langle f, v\rangle_{H^{-1},H_0^1}\, dt, \quad \forall v \in L^2(]0,T[,H_0^1(\Omega))$$

Then w is a solution of (5.28).

which means that $w = u = T(\bar{u})$.

Then we have $u_n \to u$ in $L^2(]0,T[,L^2(\Omega))$ (by (I)) , in contradiction with our hypothesis.

Then T is continuous. From $L^2(]0,T[,L^2(\Omega))$ to $L^2(]0,T[,L^2(\Omega))$.

2) Shown that T is compact from $L^2(]0,T[,L^2(\Omega))$ into $L^2(]0,T[,L^2(\Omega))$. The beginning of the reasoning from the previous question shows that $\text{Im}(T)$ is bounded in $L^2(]0,T[,H_0^1(\Omega))$ and that $\{\partial_t u \mid u \in \text{Im}(T)\}$ is a bounded subset of $L^2(]0,T[,H^{-1}(\Omega))$. By lemma (5.2.1) of compactness then implies that $\mathrm{Im}(T)$ is relatively compact in $L^2(]0,T[,L^2(\Omega))$, which establishes the compactness of T.

3) We show that there exists $R>0$***, such that*** $\|T(\bar{u})\|_{L^2(]0,T[,L^2(\Omega))} \leq R$***,*** for all $u \in L^2(]0,T[,L^2(\Omega))$.

We have $Im(T)$ is bounded in $L^2(]0,T[,H_0^1(\Omega))$, then by the continuous injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that

$Im(T)$ is bounded in $L^2(]0,T[,L^2(\Omega))$. Then

$\exists R>0$, such that $\|T(\bar{u})\|_{L^2(]0,T[,L^2(\Omega))} \leq R$

4) Show that there exists a solution u ***for*** (5.27) .

Let B_R denote the ball in $L^2(]0,T[,L^2(\Omega))$ centered at 0 with radius R, where R is given as in the preceding step. The operator T is continuous and compact from B_R to B_R.

The Schauder's fixed point theorem guarantees the existence of $u \in B_R$ (hence $u \in L^2(]0,T[,L^2(\Omega))$) such that $u = T(u)$, meaning u is a solution of (5.27).

5.2.3 ***Unicity of the solution if*** $a_{i,j}$ ***is lipschitz functions***

Let u_1, u_2 be two solutions of (5.27). Define $u = u_1 - u_2$, and we will show that $u = 0$ a.e. For $\varepsilon > 0$, we define the function

$$\begin{aligned} T_\varepsilon :& \mathbb{R} \to \mathbb{R} \\ & s \to T_\varepsilon(s) = \max\{-\varepsilon, \min\{s, \varepsilon\}\}. \end{aligned}$$

We can also denote φ_ε as the primitive of T_ε that vanishes at 0.

By taking $v = T_\varepsilon(u)$ in the weak formulation satisfied by u_1 and u_2, we obtain:

$$\begin{aligned} \int_0^T \langle \partial_t u, T_\varepsilon(u)\rangle_{H^{-1},H_0^1} dt + \int_0^T \int_\Omega \langle A(u_1)\nabla u . \nabla T_\varepsilon(u)\rangle dx dt \\ = \int_0^T \int_\Omega (A(u_2) - A(u_1))\nabla u_2 . \nabla T_\varepsilon(u) dx dt \end{aligned} \tag{5.31}$$

$$\begin{aligned} \int_0^T \langle \partial_t u, T_\varepsilon(u)\rangle_{H^{-1},H_0^1} dt &= \int_0^T \int_\Omega \partial_t u(x,t) T_\varepsilon u(x,t) dx dt \\ &= \int_0^T \int_\Omega \partial_t \phi_\varepsilon(u(x,t)) dx dt \\ &= \int_\Omega \int_0^T \partial_t \phi_\varepsilon(u(x,t)) dt dx \\ &= \int_\Omega \phi_\varepsilon(u(x,T)) dx - \int_\Omega \phi_\varepsilon(u(x,0)) dx \end{aligned}$$

Since $\nabla T_\epsilon(u) = \nabla u \chi_{0<|u|<\epsilon}$,then (5.31) becomes:

$$\int_\Omega \phi(u(x,T))dx - \int_\Omega \phi(u(x,0))dx + \alpha \int_0^T \int_\Omega \nabla u. \nabla u \underset{0<|u|<\varepsilon}{\chi} dxdt$$
$$\leq \int_0^T \int_\Omega |(A(u_1) - A(u_2))\nabla u_2||\nabla u| \underset{0<|u|<\varepsilon}{\chi} dxdt \tag{5.32}$$

Since the a_{ij} are Lipschitz functions, there exists an L such that, for all $i, j \in \{1, \ldots, N\}$ and $s_1, s_2 \in \mathbb{R}$

$$|a_{i,j}(s_1) - a_{i,j}(s_2)| \leq L|s_1 - s_2|.$$

We then use the fact that $u_0 = 0$ a.e. and $\phi_\varepsilon \geq 0$, to deduce from (5.32), with $A_\varepsilon = \{0 < |u| < \varepsilon\}$ and $y = (x, t)$:

$$\alpha \int_0^T \int_u |\nabla T_\varepsilon(u)|^2 dx\, dt \leq N^2 L\varepsilon \left(\int_{A_\varepsilon} |\nabla u_2|^2 dy\right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega |\nabla T_\varepsilon(u)|^2 dxdt\right)^{\frac{1}{2}}$$

So we have: $\alpha \||\nabla T_\varepsilon(u)|\|_{L^2(Q)} \leq a_\varepsilon \varepsilon$ with $Q =]0, T[\times\Omega$ and

$$a_\varepsilon = N^2 L \left(\int_{A_\varepsilon} |\nabla u_2|^2\, dy\right)^{\frac{1}{2}}$$

Since $\bigcap_{\varepsilon>0} A_\varepsilon = \phi$, then, The decreasing continuity of a measure implies that Lebesgue measure ($(N + 1)$-dimensional) of A_ε tends to 0 as $\varepsilon \to 0$, and therefore, since $\nabla u_2 \in L^2(Q)^N$. (Note that $L^2(Q)$ can be identified with $L^2(]0, T[, L^2(\Omega)))$

$$\lim_{\varepsilon\to 0} \int_{A_\varepsilon} |\nabla u_2|^2\, dy = 0$$

which gives $\lim_{\varepsilon\to 0} a_\varepsilon = 0$.

Now we can use, for example, the injection of $W_0^{1,1}(\Omega)$ in to $L^{1^*}(\Omega)$. This gives a, for $t \in]0, T[$:

$$||T_\varepsilon(u(t))||_{L^{1^*}(\Omega)} \leq ||\nabla T_\varepsilon(u(t))||_{L^1(\Omega)}. \tag{5.33}$$

We denote by mes the Lebesgue measure in $\mathbb{R}^N$. We now notice that for $t \in]0, T[$:

$$\varepsilon \operatorname{mes}\{|u(t)| \geq \varepsilon\}^{\frac{1}{1^*}} \leq \left(\int_\Omega |T_\varepsilon(u)|^{1^*}\, dx\right)^{\frac{1}{1^*}}$$

then we have with (5.33):

$$\varepsilon \operatorname{mes}\{|u(t)| \geq \varepsilon\}^{\frac{1}{1^*}} \leq \||\nabla T_\varepsilon(u(t))|\|_{L^1(\Omega)} = \int_\Omega |\nabla T_\varepsilon(u(x,t))|\, dx$$

and integrating with respect to t, such that $\dfrac{1}{1^*} = \dfrac{N-1}{N}$, and using Cauchy-Schwartz inequality:

$$\begin{aligned}\varepsilon \int_0^T \mathrm{mes}\{|u(t)| \geq \varepsilon\}^{\frac{N-1}{N}}\, dt &\leq \int_0^T \int_\Omega |\nabla T_\varepsilon(u(x,t))|\, dx\, dt \\ &\leq \ \||\nabla T_\varepsilon(u)|\|_{L^2(Q)}\, (T\,\mathrm{mes}(\Omega))^{\frac{1}{2}} \\ &\leq \frac{(T\,\mathrm{mes}(\Omega))^{\frac{1}{2}}}{\alpha}\, a_\varepsilon\, \varepsilon.\end{aligned}$$

then we have:

$$\int_0^T \mathrm{mes}\{|u(t)| \geq \varepsilon\}^{\frac{N-1}{N}}\, dt \leq \frac{(T\,\mathrm{mes}(\Omega))^{\frac{1}{2}}}{\alpha}\, a_\varepsilon.$$

As $\varepsilon \to 0$, by dominated convergence, we deduce:

(since $\lim_{\varepsilon\to 0} a_\varepsilon = 0$): $\int_0^T \mathrm{mes}\{|u(t)|\}^{\frac{N-1}{N}}\, dt \leq 0$

which gives:

$\mathrm{mes}\{|u(t)| > 0\} = 0$ a.e. in for $t \in]0, T[$, and therefore $u = 0$ a.e. which completes this proof of uniqueness.

Bibliography

[1] Ames, W. F., *Nonlinear Partial Differential Equations in Engineering*, volume **II**. Elsevier Science, (1972).

[2] Betounes, D., *Partial Differential Equations for Computational Science: With Maple and Vector Analysis*. Springer, (1999).

[3] Brezis, G., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext. Springer, New York, 2011.

[4] Bucur, C. and Valdinoci, E., *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana, 20. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[5] Carothers, L. N., *Real Analysis*, Cambridge University Press, (2000).

[6] Demengel, F. and Demengel, G., *Espaces Fonctionnels*, Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)]. EDP Sciences, Les Ulis, 2007.

[7] Di Nezza, E., Palatucci, G. and Valdinoci, E., *Hitchhiker's Guide to the Fractional Sobolev Spaces*, Bull. Sci. Math., 136 (2012), 521–573.

[8] Evans, L. C. and American Mathematical Society, *Partial Differential Equations*. Graduate Studies in Mathematics, American Mathematical Society, 1998.

[9] Gallouët, T. and Herbin, R., *Equations aux Dérivées Partielles*, Master 2 de Mathématiques, Université Aix Marseille, (2018).

[10] Gallouët, T. and Herbin, R., *Mesure, Intégration, Probabilités*, Ellipses Edition Marketing, In French, (2013).

[11] Garofalo, N., *Fractional Thoughts*, In New Developments in the Analysis of Nonlocal Operators, volume 723 of Contemp. Math., pages 1–135. Amer. Math. Soc., Providence, RI, 2019.

[12] Goodson, D. Z., *Mathematical Methods for Physical and Analytical Chemistry*. John Wiley & Sons, (2011).

[13] Karniadakis, G. E. and Raissi, M., *Hidden Physics Models: Machine Learning of Nonlinear Partial Differential Equations*, J. Comput. Phys., 357:125–141, (2018).

[14] Kavian, O., *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, Springer, Paris, (1993).

[15] Leoni, G., *A First Course in Fractional Sobolev Spaces*, Graduate Studies in Mathematics, (2023).

[16] Niang, A., *Fractional Elliptic Equations*, African Institute for Mathematical Sciences, Senegal, (2014).

[17] Rabinowitz, P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math., 65, American Mathematical Society, Providence, RI, (1986).

[18] Rakotoson J.E and Rakotoson J.M ., *Analyse fonctionnelle appliquée aux équations aux dérivée partielles*, Presses Universitaires de France, 1999

[19] Restrepo Montoya, D. E., *On the Fractional Laplacian and Nonlocal Operators*, Escuela de Matematicas, (2018).

[20] Roux, R., *Partie 1: Etude de l'équation de la Chaleur*, (2018).

[21] Salsa, S., *Partial Differential Equations in Action: From Modelling to Theory*, Springer–Verlag, New York, 2008.

[22] Servadei, R. and Valdinoci, E., *Mountain Pass Solutions for Non-Local Elliptic Operators*, J. Math. Anal. Appl., 389 (2012), 887–898.

[23] Servadei, R. and Valdinoci, E., *Variational Methods for Non-Local Operators of Elliptic Type*, Discrete Contin. Dyn. Syst., 33 (2013), no. 5, 2105–2137.

[24] Struwe, M., *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Springer Verlag, Berlin–Heidelberg, (1990).

[25] Roux, R., *Partie 1: Etude de l'équation de la Chaleur*, (2018).