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Thèse

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Option : ***Mathématique***

Comportement Asymptotique du Problème de Stokes dans un milieu poreux avec des Conditions de type Robin Non Homogènes

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Presented by:

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theme :

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MELL KAREK CHAFIA

Notation

Symbole	**Signification**
∇u	The gradient of u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)$
Δu	$= div(\nabla u)$
$\mathcal{D}(\Omega)$	The set of continuous functions whose support is a compact set of $\mathbb{R}^N$ contained in Ω
$L^p(\Omega)$	The Lebesgue space, $\begin{cases} \{u : \Omega \longmapsto \mathbb{R} \text{ mesurable };\int_\Omega \lvert u\rvert^p < +\infty\} & \text{if } 1 \leq p \leq +\infty, \\ \{u : \Omega \longmapsto \mathbb{R},\ \text{u mesurable and such that there exists } C \in \mathbb{R};\ \text{with } \lvert u\rvert \leq C\} & \text{if } p = +\infty \end{cases}$
$L^p_{loc}(\Omega)$	$= \{u \in L^p(\omega),\ \text{for any open bounded set } \omega \text{ with } \overline{\omega} \subset \Omega\}$
$L^{p'}(\Omega)$	The dual space of $L^p(\Omega)$
$L^p(\Omega; X)$	The set of measurable functions: $u : x \in \Omega \rightarrow u(x) \in X$ such that $\lVert u(x)\rVert_X \in L^{(}(\Omega)$.
$W^{1,p}(\Omega)$	The Sobolev spaces, $W^{1,p}(\Omega) = \{u \in L^p(\Omega); \lvert\nabla u\rvert \in L^p(\Omega)\}$
$H^1(\Omega)$	$\left\{u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \dots, N\right\}$
$H^1_0(\Omega)$	The space of function in $H^1(\Omega)$ will vanish on the boundary in the sens of the trace.
$H^{-1}(\Omega)$	the dual space of $H^1_0(\Omega)$

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Introduction

Various physical phenomena can be described in terms of fluid flow in porous medium. It occurs in the study of filtration in sandy soils or blood circulation in capillaries, see Bear [8] and Hornung [39] for more examples and motivation. In the study of such processes, one would like to find some averaged characteristics of the flow, e.g. permeability, macro-velocity, and macro-pressure. To obtain such quantities, there exist several mathematical approaches collectively referred to as homogenization theory. The homogenization theory is a branch of the mathematical analysis which treats the asymptotic behavior of differential operators with rapidly oscillating coefficients.

The aim of homogenization theory is to establish the macroscopic behavior of a system which is microscopically heterogeneous, in order to describe some characteristics of the heterogeneous medium (for instance, its thermal or electrical conductivity). This means that the heterogeneous material is replaced by a homogeneous fictitious one (the 'homogenized' material), whose global (or overall) characteristics are a good approximation of the initial ones. From the mathematical point of view, this signifies mainly that the solutions of a boundary value problem, depending on a small parameter, converge to the solution of a limit boundary value problem which is explicitly described.

In this thesis we study the asymptotic behavior of the Stokes system with two different kinds of boundary condition on the boundary of the holes, which are nonhomogeneous slip boundary conditions, we consider the flow of an incompressible viscous fluid through a porous medium, under the action of an exterior electrical field.

The problem of the homogenization of the Stokes equations in a perforated domain has been thoroughly investigated by Grégoire Allaire in 1990 [4], [5].

The classical approach is to model the perforated domain with holes periodically distributed with a period ε.

The periodicity assumption is neither realistic nor absolutely necessary, but greatly simplifies the analysis: namely, methods that rely on the construction of test functions in a given geometry do not lend themselves easily to the case of truly random media. Letting ε approach zero, one obtains one of the three possible homogeneous problems, depending on the ratio of the obstacle and cell sizes. The first of them, Darcy's law, was first formulated by Henry Darcy in his 1856 a report on the construction of the municipal water system of the town of Dijon, France [38]. Another extension to the traditional form of Darcy's low is the Brinkman term, which is used to account for transitional flow between the Stokes and Darcy equation. Finally, when the permeability is large enough and the obstacles asymptotically null, they cease to have any effect and the process converges to an ordinary Stokes problem.

On the other hand, we also discuss the study of a class of hyperbolic-parabolic problems which modelize many kinds of phenomena arising in electricity and magnetism, in the theory of elasticity, in vibrations theory and in hydrodynamics. Our objective is to homogenize the Stokes and hyperbolic-parabolic systems.

To study the homogenization of the Stokes and hyperbolic-parabolic problems we apply the periodic unfolding method. The periodic unfolding method was introduced in 2002 by D. Cioranescu, A. Damlamian, and G. Griso [15].This method is based on two ingredients: the unfolding operator and a macro-micro decomposition of functions which allows to separate the macroscopic and microscopic scales. The interest of the method comes from the fact that it only deals with functions and classical notions of convergence in L^p spaces. This renders the proof of homogenization results quite elementary. It also provides error estimates and corrector results.
This thesis is organized as flows:
In the first chapter, we present the the multiple-scale and energy method for the Stokes equation.
In the second chapter, we define first the periodic infolding for one and two parameters in perforated domains and some properties and we treat an elliptic model problem with a Robin boundary condition.
chapter 03 is dedicated to the first problem of Stokes equation in perforated domains with small holes. we study the asymptotic behavior of the velocity and pressure of the fluid as $\varepsilon \longrightarrow 0$.
In chapter 04 we present the homogenization results of the second Stokes problem in porous medium with Robin-type condition depending on a parameter γ.
Finally, in chapter 05 we are concerned with the asymptotic behavior of hyperbolic-parabolic problem in perforated domains with small holes.
In the end, we give a conclusion about the thèsis with some perspectives.

Chapter 1

The Multiple-Scale and Energy Method for The Stokes Equation

In this chapter we apply the multiple-scale and the energy methods to the study the asymptotic behavior of the Stokes equation [49].

1.1 The Multiple-Scale Method

Let a small positive parameter ε and the fluid domain defined by

$$\Omega_\varepsilon = \{x; x \in \varepsilon Y\} \tag{1.1}$$

where Ω is a given domain. We consider a formal expansion of the velocity and pressure out a neighborhood of $\partial\Omega$; consequently.
We consider the Stokes problem

$$\begin{cases} -grad \;\; p_\varepsilon + \Delta v_\varepsilon + f = 0 & in \; \Omega_\varepsilon \\ div \;\; v_\varepsilon = 0 & in \; \Omega_\varepsilon \\ v_\varepsilon|_{\partial\Omega} = 0 & on \; \partial\Omega_\varepsilon \end{cases} \tag{1.2}$$

Now, we postulate an asymptotic expansion

$$\begin{cases} v_\varepsilon(x) = \varepsilon^2 v_0(x,y) + \varepsilon^3 v_1(x;y) + ... \\ p_\varepsilon(x) = p_0(x) + \varepsilon p_1(x,y) + ... \end{cases} \tag{1.3}$$

with $y = \dfrac{x}{\varepsilon}, v_i, p_i$ Y-periodic in Y, for $x \in \Omega, y \in Y$. Note that we postulate that p_0 does not depend on Y, $gradp_\varepsilon$, depends on y from the first term because of the classical relation

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i} \tag{1.4}$$

The same relation (1.4) shows that the expansion of Δ is

$$\Delta = \frac{1}{\varepsilon^2}\Delta_{yy} + \frac{1}{\varepsilon}... \tag{1.5}$$

where Δ_{yy} denotes the laplacian with respect to the variables y_i (x_i being paramaters). The form of the relation (1.5) shows that the first significative term in (1.3) must be in the ε^2 term. As usual in homogenization problems, if we postulate an expansion beginning by ε_0 terms, the two first terms will be find to be zero.

To study the problem (1.2), we replace (1.3) into (1.2). Then, by taking the ε_0 term for the first equation of (1.2), the ε term for the second equation of (1.2) and the ε^2 term for the last equation of (1.2), we have

$$\begin{cases} -\dfrac{\partial p_1}{\partial y_i} + \Delta_y v_i^0 + \left(f_i - \dfrac{\partial p_0}{\partial x_i}\right) = 0 & in\ Y \\ div_y v_0 = 0 & in\ Y \\ v_0|_\Gamma = 0 & on\ \Gamma \end{cases} \tag{1.6}$$

with the supplementary conditions v_0, p_1 are Y-periodic in Y. This is the local problem, where x is a parameter and the term in parenthesis of the first equation of (1.6) plays the role of given force, v_0 *and* p_1 are the unknowns. Before , we consider the ε^2 term for the second equation of (1.2)

$$div_x v_0 + div_y v_1 = 0 \tag{1.7}$$

and we apply the classical mean value operator

$$\mathcal{M}_Y(.) = \tilde{.} = \frac{1}{|Y|}\int_Y .\ \ dy \tag{1.8}$$

Not that v_i as functions of y are defined on Y: it is natural to extend them to Y with value zero on ∂Y. Then, we have

$$(div_y\ v_1)^\sim = \frac{1}{|Y|}\int_Y \frac{\partial v_i^1}{\partial y_i} dy = \frac{1}{|Y|}\int_{\partial Y} n_i v_i^1 d\sigma = 0 \tag{1.9}$$

To see that the surface integral of (1.9) is zero it suffices to see that $n_i v_i^1$ is zero on Γ and that the integral on the parts of ∂Y lying on ∂Y annihilate by periodicity. On the other hand, the operator $\partial/\partial x_i$ commutes with $\sim$ as usual. Then, by applying $\sim$ to (1.7) we obtain

$$div_x \widetilde{v}^0 = 0 \tag{1.10}$$

which is the macroscopic equation.

Now we study the local problem (1.6). We define an appropriate space of Y-periodic functions

$$V = \{u; u \in H^1(Y); u|_\Gamma = 0, div_y u = 0;\ \ Y - periodic\} \tag{1.11}$$

$$(u, w)_V = \int_Y \frac{\partial u_i}{\partial y_k}\frac{\partial w_i}{\partial y_k} dy \tag{1.12}$$

which is a Hilbert space; the associated norm is equivalent to the $H^1(Y)$ norm.
To obtain a variational formulation of (1.6), we take a test function $w \in V$ and we multiply the first equation of (1.6) by w_i; by integrating over Y we have

$$-\int_Y \frac{\partial p_1}{\partial y_i} w_i dy = -\int_Y \frac{\partial}{\partial y_i}(p_1 w_i) dy = \int_{\partial Y} p_1 w_i n_i d\sigma = 0 \tag{1.13}$$

$$\begin{aligned} \int_Y \Delta_y v_i^0 w_i dy &= \int_Y \left[\frac{\partial}{\partial y_k}\left(\frac{\partial v_i^0}{\partial y_k} w_i\right) - \frac{\partial v_i^0}{\partial y_k}\frac{\partial w_i}{\partial y_k} \right] dy \\ \int_{\partial Y} n_k \frac{\partial v_i^0}{\partial y_k} w_i d\sigma - \int_Y \frac{\partial v_i^0}{\partial y_k}\frac{\partial w_i}{\partial y_k} dy &= -\int_Y \frac{\partial v_i^0}{\partial y_k}\frac{\partial w_i}{\partial y_k} dy \end{aligned} \tag{1.14}$$

Then, by using (1.12), we get

$$(v_0, w)_V - (f_i - \frac{\partial p_0}{\partial x_i}) \int_Y w_i dy = 0 \quad \forall w \in V \tag{1.15}$$

Conversely, if $v_0 \in V$ and satisfies (1.15), by integrating by parts it satisfies

$$\int_Y \left[\Delta_y v_i^0 + (f_i - \frac{\partial p_0}{\partial x_i}) \right] w_i dy = 0 \qquad \forall w \in V \tag{1.16}$$

Now, we show that the function p_1 just found is Y-periodic. In fact, $grad p_1$ is periodic because p_1 satisfies the first equation of (1.6). We multiply this equation by w_i and we integrate on Y. By comparing with (1.16) we have

$$-\int_Y \frac{\partial p_1}{\partial y_i} w_i dy = 0$$

and because $div w = 0$, we have

$$\int_Y \frac{\partial}{\partial y_i}(p^1 w_i) dy = \int_{\partial Y} p_1 w_i n_i d\sigma = 0 \tag{1.17}$$

Consequently, the local problem (1.6) is equivalent to the following variational problem:

$$Find \;\; v_0 \in V \; satisfying \;\; (1.15) \tag{1.18}$$

Moreover, by using the standard linearity property, we have

Proposition 1.1. *If we postulate an asymptotic expansion* (1.3) *the first term* $v_0(x, y)$ *is given by* $f_i(x)$ *and* $\partial p_0/\partial x_i(x)$ *as*

$$v_0 = (f_i - \frac{\partial p_0}{\partial x_i}) v_i \tag{1.19}$$

where $v_i(y)(i = 1, 2, 3)$ *are the solutions of*

$$\begin{cases} Find \;\; v_i \in V \;\; such \;\; that \\ (v_i, w)_V = \displaystyle\int_Y w_i dy \qquad\qquad \forall w \in V \end{cases} \tag{1.20}$$

The existence and uniqueness of the solutions of (1.18) or (1.20) are immediate consequences of the Lax-Milgram theorem on V.
Now, if we apply the mean operator $\sim$ (defined by (1.8)) to (1.19), we have

$$\widetilde{v}_j^0 = k_{ij}(f_i - \frac{\partial p_0}{\partial x_i}); \quad k_{ij} = \widetilde{v}_j^i \tag{1.21}$$

Relation (1.21) is the Darcy's law. The mean value value of the velocity of the fluid is equal to $f - gradp_0$ multiplied by a constant tensor with components k_{ij} which only depend on the geometry of the period Y. It is noticeable that (1.21) was obtained from the first equation of (1.2), i.e., for the viscosity coefficient ν equal to one. If we consider

$$-gradp_\varepsilon + \nu\Delta v_\varepsilon + f = 0 \tag{1.22}$$

instead of (1.2), we obtain

$$\widetilde{v}_j^0 = \frac{k_{ij}}{\nu}(f_i - \frac{\partial p_0}{\partial x_i}); \quad k_{ij} = \widetilde{v}_j^i \tag{1.23}$$

instead of (1.21). It is also necessary to introduce the coefficient ν^{-1} at the right hand side of (1.19)

Proposition 1.2. *The matrix k_{ij}, defined by* $(1.21)_2$ *will be called permeability tensor. It is a symmetric, positive definite matrix.*

Proposition 1.3. *The macroscopic equation* (1.10) *may be written in the form*

$$k_{ij}\frac{\partial^2 p_0}{\partial x_i \partial x_j} = k_{ij}\frac{\partial f_j}{\partial x_i} \tag{1.24}$$

which is an elliptic equation for the unknown $p_0(x)$. If $p_0(x)$ is obtained, the velocity field $v_0(x,y)$ is given by (1.19) *and the mean value of the velocity satisfies the Darcy's law* (1.21).

1.2 The Energy Method for The Stokes Problem

We consider the solution $u_\varepsilon \in H_0^1(\Omega_\varepsilon), p_\varepsilon \in L^2(\Omega_\varepsilon)$ of Stokes problem in Ω_ε

$$\begin{cases} -\dfrac{\partial p_\varepsilon}{\partial x_i} + \Delta u_i^\varepsilon + f_i = 0 & in\ \Omega_\varepsilon \\ divu_\varepsilon = 0 & in\ \Omega_\varepsilon \end{cases} \tag{1.25}$$

Moreover, we consider the local periodic problem in Y_ε : if e_k is the unitary vector in the direction of the axis k v_k and q_k are defined by

$$\begin{cases} \Delta_y v_k - grad_y q_k + e_k = 0 \\ div_y v_k = 0 : \quad v_k|_\Gamma = 0 \\ v_k, q_k \quad Y - periodic \end{cases} \tag{1.26}$$

We continue v_k with value zero to Y_S. (The notation is not changed after continuation). The mean value $\widetilde{v}_k$ is defined by

$$\widetilde{v}_k = \frac{1}{|Y|} \int_Y v_k dy$$

and the permeability tensor k_{ij} (which is symmetric and positive definite is defined by

$$k_{ij} = \widetilde{v}_j^i \tag{1.27}$$

Moreover, we define the asymptotic velocity u_0 and pressure p_0 by

$$\begin{cases} div u_0 = 0 & in\ \Omega \\ u_0^i = k_{ij}(f_i - \dfrac{\partial p_0}{\partial x_j}) & in\ \Omega \\ u_0.n = 0 & on\ \partial\Omega \end{cases} \tag{1.28}$$

Theoreme 1.1. *Let $u_\varepsilon, p_\varepsilon$ (resp u_0, p_0) be defined by* (1.25) *(resp* (1.28)*). There exists a continuation of u_ε and p_ε to Ω(in fact u_ε is continuated by zero) such that*

$$\frac{u_\varepsilon}{\varepsilon^2} \to u_0 \qquad in\ L^2(\Omega)\ weakly \tag{1.29}$$

$$p_\varepsilon \to p_0 \qquad in\ L^2(\Omega)\ weakly \tag{1.30}$$

The proof of this theorem will be given after several lemmas.

Lemma 1.1. *The constant of the Friedreich's inequality in Ω_ε is of the form $C.\varepsilon^2$, i.e:*

$$\int_{\Omega_\varepsilon} |u|^2 dx \leq C.\varepsilon^2 \int_{\Omega_\varepsilon} |grad_x u|^2 dx \tag{1.31}$$

Lemma 1.2. *We continue u_ε (initially defined on Ω_ε) to Ω with value zero out of Ω_ε. Then, after extraction of a subsequence (it will be proved to be the whole sequence by uniqueness of it limit):*

$$\frac{u_\varepsilon}{\varepsilon^2} \to u^* \quad in\ L^2(\Omega)\ \ weakly \tag{1.32}$$

where

$$div u^* = 0 \quad on\ \Omega \ \ ; \ \ u^*.n = 0\ \ on\ \ \partial\Omega \tag{1.33}$$

moreover

$$\|grad u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \ \ ; \ \ \|u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^2 \tag{1.34}$$

Lemma 1.3. *We consider a period Y. We consider a smooth surface γ strictly contained in Y, enclosing Y_S and we note Y_M the domain contained between γ and Γ (see fig 1). Then for given $u \in H^1(Y)$, there exist $v \in H^1(Y_M)/\mathbb{R}$ satisfying*

$$\Delta v = -\Delta u + grad q \quad in\ \ Y_M \tag{1.35}$$

$$divv = g \equiv divu + \frac{1}{Y_M}\int_{Y_S} divudy \quad in\ Y_M \tag{1.36}$$

$$v|_{\gamma} = u|_{\Gamma} \quad ; \quad v|_{\Gamma} = 0 \tag{1.37}$$

$$\|v\|_{H^1(Y_M)} \leq C\|u\|_{H^1(Y)} \tag{1.38}$$

Lemma 1.4. *There exists a (restriction) operator*

$$\begin{aligned} &R_\varepsilon : H_0^1(\Omega) \mapsto H_0^1(\Omega_\varepsilon) \quad such\ that \\ &w \in H_0^1(\Omega_\varepsilon) \Rightarrow R_\varepsilon(w) = w \end{aligned} \tag{1.39}$$

(elements of $H_0^1(\Omega_\varepsilon)$ are extended by continuity by 0 to Ω)

$$divw = 0 \Rightarrow \quad divR_\varepsilon w = 0 \tag{1.40}$$

$$\|R_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \leq C\|w\|_{L^2(\Omega)} + C\varepsilon\|gradw\|_{L^2(\Omega)} \tag{1.41}$$

$$\|gradR_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}\|w\|_{L^2(\Omega)} + C\|gradw\|_{L^2(\Omega)} \tag{1.42}$$

Proof. **of theorem 1.1** We see from (1.25) that $gradp_\varepsilon \in H^{-1}(\Omega_\varepsilon)$. We continue it to Ω in the following way. Let us define $F_\varepsilon \in H^{-1}(\Omega)$ by

$$\langle F_\varepsilon, w\rangle_\Omega = \langle gradp_\varepsilon, R_\varepsilon w\rangle_{\Omega_\varepsilon} \qquad \forall w \in H_0^1(\Omega) \tag{1.43}$$

where R_ε is defined in lemma 1.4. We calculate the right side of (1.43) by using (1.25) and we have

$$\langle F_\varepsilon, w\rangle_\Omega = -\int_{\Omega_\varepsilon} \frac{\partial u_i^\varepsilon}{\partial x_j}\frac{\partial (R_\varepsilon w)}{\partial x_j}dx + \int_{\Omega_\varepsilon} f_i(R_\varepsilon w)_i dx \tag{1.44}$$

and by using (1.41) and (1.42) for fixed ε we see that it is a bounded functional on $H_0^1(\Omega)$, and in fact $F_\varepsilon \in H^{-1}(\Omega)$. Moreover, if $w \in H_0^1(\Omega_\varepsilon)$ and we continue it by zero out of Ω_ε, we see from (1.43) and (1.39) that

$$F_\varepsilon|_{\Omega_\varepsilon} = gradp_\varepsilon \qquad (\ on\ \Omega_\varepsilon) \tag{1.45}$$

Moreover, if $divw = 0$, by (1.40), (1.43) and the classical orthogonality property, $\langle F_\varepsilon, w\rangle_\Omega = 0$ and this implies that F_ε is the gradient of a function of $L^2(\Omega)$.
This means that F_ε is a continuation of $gradp_\varepsilon$ to Ω, and that this continuation is a gradient. We also may say that p_ε has been extended by contituity to Ω and

$$F_\varepsilon \equiv gradp_\varepsilon \quad ; \quad p_\varepsilon \in L^2(\Omega)/\mathbb{R} \tag{1.46}$$

Let us estimate p_ε and its gradient. From (1.44), with (1.34), (1.41) and (1.42) we have

$$|\langle gradp_\varepsilon, w\rangle_\Omega| \leq C\left(\|w\|_{L^2(\Omega)} + \varepsilon\|gradw\|_{L^2(\Omega)}\right) \tag{1.47}$$

then, as $\varepsilon < 1$, we see that the right hand side of (1.47) is bounded by $C\|w\|_{H_0^1}$ and consequently

$$\|gradp_\varepsilon\|_{H^{-1}(\Omega)} \leq C \tag{1.48}$$

and from the inequality

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega)\|gradp\|_{H^{-1}(\Omega)} \tag{1.49}$$

we see that p_ε remains bounded in $L^2(\Omega)/\mathbb{R}$ and consequently, after extraction of a subsequence

$$\begin{aligned} p_\varepsilon \to p^* \quad & in\ L^2(\Omega)/\mathbb{R}\ \ weakly \\ gradp_\varepsilon \to gradp^* \quad & in\ H^{-1}(\Omega)\ \ weakly \end{aligned} \tag{1.50}$$

Moreover, let w_ε be a sequence of elements of H_0^1 such that

$$w_\varepsilon \to w \quad in \quad H_0^1(\Omega)\ \ weakly. \tag{1.51}$$

From (1.47) applied to $w_\varepsilon - w^*$ we have

$$\begin{aligned} & |\langle gradp_\varepsilon, w_\varepsilon\rangle - \langle gradp^*, w^*\rangle| \\ & \leq |\langle gradp_\varepsilon, w_\varepsilon - w^*\rangle| + |\langle gradp_\varepsilon - gradp^*, w^*\rangle| \\ & \leq C\left(\|w_\varepsilon - w^*\|_{L^2} + \varepsilon\|w_\varepsilon - w^*\|_{H_0^1}\right) + (term\ which \to 0) \end{aligned} \tag{1.52}$$

which tends to zero by virtue of (1.51) and the Rellich theorem. This implies

$$gradp_\varepsilon \to p^* \quad in\ H^{-1}(\Omega)\ \ strongly \tag{1.53}$$

And from (1.49) we have

$$p_\varepsilon \to p^*\ in\ L^2(\Omega)/\mathbb{R}\ \ strongly \tag{1.54}$$

From (1.32), (1.33) and (1.54) it only remains to prove that $u^*\ and\ p^*$ satisfy equation two of problem (1.28), then $u^*\ and\ p^*$ are the solution $u_0\ and\ p_0$ of (1.28).
We write the local problem (1.26) in terms of $x = \varepsilon y$:

$$\begin{cases} v_\varepsilon^k(x) \equiv v^k(x/\varepsilon) \quad ; \quad q_\varepsilon^k(x) \equiv q^k(x/\varepsilon) \\ \varepsilon^2\Delta_x v_\varepsilon^k - \varepsilon grad_x q_\varepsilon^k + e_k = 0 \\ div_x v_\varepsilon^k = 0 \end{cases} \tag{1.55}$$

and because $v^k(y),\ q^k(y)$ are independent of ε, we have

$$\|q_\varepsilon^k\|_{L^2(\Omega_\varepsilon)} \leq C \quad ; \quad \|v_\varepsilon^k\|_{L^2(\Omega)} \leq C \tag{1.56}$$

$$\|grad_x v_\varepsilon^k\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon} \tag{1.57}$$

Now, we apply the standard method to prove convergence in homogenization problems. We take $\phi \in \mathcal{D}(\Omega)$ and we multiply the second equation of problem (1.55) by ϕu_i^ε and integrate on Ω we get

$$\int_\Omega \frac{\partial v_{\varepsilon i}^k}{\partial x_j}\frac{\partial(\phi u_i^\varepsilon)}{\partial x_j}dx = \frac{1}{\varepsilon}\int_\Omega q_\varepsilon^k div(\phi u^\varepsilon)dx + \frac{1}{\varepsilon^2}\int_\Omega e_k \phi u^\varepsilon dx \tag{1.58}$$

then, by $(1.56)_1, (1.34)_2$ and second equation of problem (1.25):

$$\left|\frac{1}{\varepsilon}\int_\Omega q_\varepsilon^k div(\phi u^\varepsilon)dx\right| = \left|\frac{1}{\varepsilon}\int_\Omega q_\varepsilon^k \frac{\partial \phi}{\partial x_i}u_i^\varepsilon dx\right| \leq C\varepsilon \to 0$$

then, passing to the limit in the right hand side of (1.58) by (1.32) we have:

$$\int_\Omega \frac{\partial v_{\varepsilon i}^k}{\partial x_j}\frac{\partial(\phi u_i^\varepsilon)}{\partial x_j}dx \to \int_\Omega \phi u_k^* dx \qquad as \ \ \varepsilon \to 0 \tag{1.59}$$

On the other hand, we take the duality product of the first equation of problem (1.25) by ϕv_ε^k:

$$\begin{aligned}\int_\Omega \frac{\partial u_i^\varepsilon}{\partial x_j}\frac{\partial(\phi v_{\varepsilon i}^k)}{\partial x_j}dx &= \langle f, \phi v_\varepsilon^k\rangle_{\Omega_\varepsilon} + \langle p_\varepsilon, div(\phi v_\varepsilon^k)\rangle_{\Omega_\varepsilon} \\ &= \int_\Omega f_i \phi v_{\varepsilon i}^k dx + \int_\Omega p_\varepsilon \frac{\partial \phi}{\partial x_i} v_{\varepsilon i}^k dx\end{aligned} \tag{1.60}$$

(note that $v \ \ and \ u$) are zero out of Ω_ε, and we may write the integrals either on $\Omega_\varepsilon \ or \ \Omega$). Moreover, by the classical lemma on Y-periodic function

$$v_{\varepsilon i}^k \to \widetilde{v_i^k} = K_{ki} \qquad in \ \ L^2(\Omega) \ \ weakly \tag{1.61}$$

where K_{ki} are the components of the permeability tensor (1.27). We pass to the limit in (1.60) by using (1.53) and (1.61):

$$\int_\Omega \frac{\partial u_i^\varepsilon}{\partial x_j}\frac{\partial(\phi v_{\varepsilon i}^k)}{\partial x_j}dx \to K_{ki}\int_\Omega f_i \phi dx + K_{ik}\int_\Omega p^* \frac{\partial \phi}{\partial x_i}dx \tag{1.62}$$

(it is to be noticed that

$$\int_\Omega \frac{\partial \phi}{\partial x_i} v_{\varepsilon i}^k dx = \int_\Omega div(\phi v_\varepsilon^k)dx = \int_{\partial\Omega} \phi v_\varepsilon^k . dS = 0$$

and then p_ε is defined up to an additive constant in (1.60); Consequently the convergence in $L^2(\Omega)/\mathbb{R}$ suffices to pass to the limit). Now, we compare the left hand sides of (1.59) and (1.62). Their difference is

$$\left|\int_\Omega \frac{\partial v_{\varepsilon i}^k}{\partial x_j} u_i^\varepsilon \frac{\partial \phi}{\partial x_j}dx - \int_\Omega \frac{\partial u_i^\varepsilon}{\partial x_j} v_{\varepsilon i}^k \frac{\partial \phi}{\partial x_j}\right| \leq \left|\int\right| + \left|\int\right| \leq C\varepsilon \to 0$$

where (1.34), (1.56) and (1.57) have been used. Consequently, the right hand sides of (1.59) and (1.62) are equal and by writing them as distribution we have:

$$\langle u_k^*, \phi\rangle = K_{ki}\left\langle f_i - \frac{\partial p^*}{\partial x_i}, \phi\right\rangle \tag{1.63}$$

which is satisfied for any $\phi \in \mathcal{D}(\Omega)$

$$u_k^* = K_{ik}\left(f_i - \frac{\partial p^*}{\partial x_i}\right)$$

which is (1.28) for u^*, p^* and theorem 1.1 is proved □

Chapter 2

The Periodic Unfolding Method in Perforated Domains

In this chapter we recall the definition and some properties of the periodic unfolding operators in perforated domains $\mathcal{T}_\varepsilon$ for the classical homogenization and $\mathcal{T}_{\varepsilon\delta}$ depending on two parameters ε and δ and in the end we apply these method for a Robin model problem.

2.1 The periodic unfolding operator $\mathcal{T}_\varepsilon$

In this section, we introduce the periodic unfolding operator in the case of perforated domains introduced by Cioranescu et all [15] and [22].
In the following we denote:

- Ω an open set in $\mathbb{R}^N$

- $Y = \Pi_{i=1}^N [0; l_i[$ the reference cell $l_i > 0$ for all $1 \leq i \leq N$, or more generally a set having the paving property to a basis $(b_1, ..., b_N)$ defining the periods,

- T an open set included in Y such that ∂T does not contain the summits of Y. We can be, sometimes, transported to this situation by a simple change of period,

- $Y^* = Y \setminus \overline{T}$ a connected open set.

we define

$$T_\varepsilon = \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + T) \qquad \text{and} \qquad \Omega_\varepsilon \setminus T_\varepsilon.$$

We assume in the following that Ω_ε is a connected set and we take the regularity hypothesis

$$|\partial \Omega| = 0. \tag{2.1}$$

In the sequel, we will use the following notation:

- $\widetilde{\varphi}$ for the extension by 0 outside Ω_ε for any function $\varphi \in L^p(\Omega_\varepsilon)$,

- χ_ε for the characteristic function of Ω_ε,

- θ for the proportion of the material in the elementary cell, i.e. $\theta = \dfrac{|Y^*|}{|Y|}$

- $\rho(Y)$ for the diameter of the cell Y,

- T_ε^{int} for the set of holes that do not intersect the boundary $\partial\Omega$.

By analogy to the $1D$ notation, for $z \in \mathbb{R}^N$, $[z]_Y$ denotes the unique integer combination $\displaystyle\sum_{j=1}^{j=N} k_j b_j$, such that $z)\,[z]_Y$ belongs to Y. Set $\{z\}_Y = z - [z]_Y$. Then, for almost every $x \in \mathbb{R}^N$, there exists a unique element in $\mathbb{R}^N$, denoted by $\left[\dfrac{x}{y}\right]_Y$, such that

$$x - \varepsilon\left[\frac{x}{\varepsilon}\right]_Y = \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y,$$

where

$$\left\{\frac{x}{\varepsilon}\right\}_Y \in Y.$$

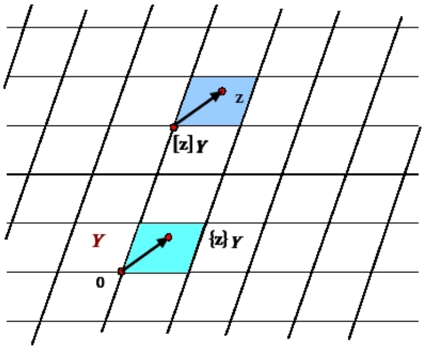

Fig. 1. The basic decomposition.

definition 2.1. *Let $\varphi \in L^p(\Omega_\varepsilon), p \in [1, +\infty]$. We define the function*

$$\mathcal{T}_\varepsilon(\varphi)(x,y) = \widetilde{\varphi}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right), \tag{2.2}$$

for every $x \in \mathbb{R}^N$ and $y \in Y^*$

Remark 2.1. *Notice that the oscillations due to perforations are shifted into the second variable y which belongs to the fixed domain Y^*, while the first variable x belongs to $\mathbb{R}^N$.*
One see immediately the interest of the unfolding operator. Indeed, when trying to pass to the limit in a sequence defined on Ω_ε one needs first, while using standard methods, to extend it to a fixed domain. With $\mathcal{T}_\varepsilon$, such extensions are more necessary.

The main properties given in [15] for fixed domains can easily be adapted for the perforated ones without any major difficulty in the proofs. These properties are

listed in the proposition below.
To do so, let us first define the following domain:

$$\widetilde{\Omega_\varepsilon} = int\left(\bigcup_{\xi\in\Lambda_\varepsilon}(\xi+Y)\right)$$

where

$$\Lambda_\varepsilon\left\{\xi\in\mathbb{Z}^N;\varepsilon(\xi+\overline{Y})\cap\Omega\neq\emptyset\right\}$$

The set $\widetilde{\Omega_\varepsilon}$ is the smallest finite union of εY cells containing Ω.

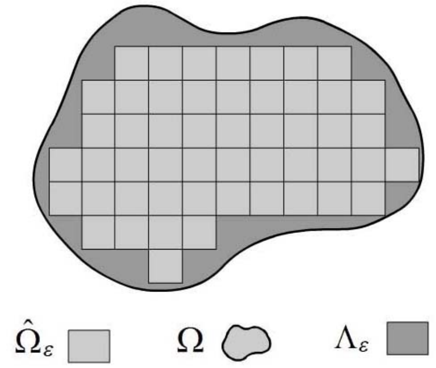

Fig. 2. The sets $\widehat{\Omega_\varepsilon}, \Omega$ and Λ_ε

Proposition 2.1. *[15] The unfolding operator $\mathcal{T}_\varepsilon$ has the following properties :*

1. *$\mathcal{T}_\varepsilon$ is a linear operator.*

2. $\mathcal{T}_\varepsilon(\varphi)\left(x,\left\{\frac{x}{\varepsilon}\right\}_Y\right)=\varphi(x)\quad\forall\varphi\in L^p(\Omega_\varepsilon)$ *and* $x\in\mathbb{R}^N$

3. $\mathcal{T}_\varepsilon(\varphi\psi)=\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi),\quad\forall\varphi,\psi\in L^p(\Omega_\varepsilon)$

4. *Let φ in $L^p(Y)$ or $L^p(Y^*)$ be a Y-periodic function. Set $\varphi_\varepsilon(x)=\varphi\left(\frac{x}{\varepsilon}\right)$. Then,*

   $$\mathcal{T}_\varepsilon(\varphi_\varepsilon)(x,y)=\varphi(y),\quad a.e.in\ \widetilde{\Omega_\varepsilon}$$

5. *One has the integration formula*

   $$\int_{\Omega_\varepsilon}\varphi dx=\frac{1}{|Y|}\int_{\widetilde{\Omega_\varepsilon}\times Y^*}\mathcal{T}_\varepsilon(\varphi)dxdy\quad\forall\varphi\in L^1(\Omega_\varepsilon)$$

6. *For every* $\varphi \in L^2(\Omega_\varepsilon), \mathcal{T}_\varepsilon(\varphi)$ *belongs to* $L^2(\mathbb{R}^N \times Y^*)$. *It also belongs to* $L^2(\widetilde{\Omega_\varepsilon} \times Y^*)$.

7. *For every* $\varphi \in L^2(\Omega_\varepsilon)$, *one has*
$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N \times Y^*)} = \sqrt{|Y|}\|\varphi\|_{L^2(\Omega_\varepsilon)}.$$

8. $\nabla_y \mathcal{T}_\varepsilon(\varphi)(x,y) = \varepsilon \mathcal{T}_\varepsilon\left(\nabla_x \varphi\right)(x,y) \qquad$ *for every* $(x,y) \in \Omega \times Y^*$.

9. *If* $\varphi \in H^1(\widetilde{\Omega_\varepsilon})$, *then* $\mathcal{T}_\varepsilon(\varphi)$ *is in* $L^2(\mathbb{R}^N, H^1(Y^*))$.

10. *One has the estimate*
$$\|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega \times Y^*))^N} = \varepsilon\sqrt{|Y|}\|\nabla_x \varphi\|_{(L^2(\Omega_\varepsilon))^N}.$$

Proposition 2.2. *[15] Let* $\varphi \in L^2(\Omega)$. *Then,*

1. $\mathcal{T}_\varepsilon(\varphi) \longrightarrow \widetilde{\varphi}$ *strongly in* $L^2(\mathbb{R}^N \times Y^*)$

2. $\varphi\chi_\varepsilon \rightharpoonup \theta\varphi$ *weakly in* $L^2(\Omega)$.

3. *Let* (φ_ε) *be in* $L^2(\Omega)$ *such that*
$$\varphi_\varepsilon \longrightarrow \varphi \qquad \text{strongly in } L^2(\Omega).$$
Then
$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \longrightarrow \widetilde{\varphi} \qquad \text{strongly in } L^2(\mathbb{R}^N \times Y^*)$$

Proposition 2.3. *[15] Let* φ_ε *in* $L^2(\Omega_\varepsilon)$ *and* $\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C$ *for every* ε, *such that*
$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi} \qquad \text{weakly in } L^2(\Omega \times Y^*).$$
Then,
$$\widetilde{\varphi_\varepsilon} \rightharpoonup \frac{1}{|Y|}\int_{Y^*} \widehat{\varphi}(.,y)dy \qquad \text{weakly in } L^2(\Omega)$$

Proposition 2.4. *[15] Let* φ_ε *be in* $L^2(\widetilde{\Omega}_\varepsilon)$ *for every* ε, *with*
$$\|\varphi_\varepsilon\|_{L^2(\widetilde{\Omega_\varepsilon})} \leq C,$$
$$\varepsilon\|\nabla_x \varphi_\varepsilon\|_{(L^2(\widetilde{\Omega_\varepsilon}))^N} \leq C$$
Then, there exists $\widehat{\varphi}$ *in* $L^2(\Omega; H^1(Y^*))$ *such that, up to subsequences*

1. $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^2(\Omega; H^1(Y^*))$.

2. $\varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi_\varepsilon) \rightharpoonup \nabla_y\widehat{\varphi}$ *weakly in* $L^2(\Omega \times Y)$,
   where
   $$y \longmapsto \widehat{\varphi} \in L^2(\Omega; H^1_{per}(Y^*))$$

2.1.1 Macro-Micro Decomposition

Following [15], we decompose any function φ in the from

$$\varphi = \mathcal{Q}_\varepsilon(\varphi) + \mathcal{R}_\varepsilon(\varphi),$$

where $\mathcal{R}_\varepsilon$ is designed in order to capture the oscillations.
As in the case of fixed domains, we start by defining $\mathcal{Q}_\varepsilon$ on the nodes $\varepsilon\xi_k$ of the εY-lattice.
Here, it is no longer possible to take the average on the entire cell Y as in [15], but it will be taken on a small ball B_ε centered on $\varepsilon\xi_k$ and not touching the holes. This is possible using the fact that ∂T does not contain the summits of Y. However, B_ε must be entirely contained in Ω_ε.
To guarantee that, we are let to define $\mathcal{Q}_\varepsilon(\varphi)$ on a subdomain of Ω_ε only. To do so, for every $\delta > 0$, let us set

$$\Omega^\varepsilon_\delta = \{x \in \Omega; d(x, \partial\Omega) > \delta\} \qquad \text{and} \qquad \widehat{\Omega^\varepsilon_\delta} = int\left(\bigcup_{\xi\in\Pi^\delta_\varepsilon} \varepsilon(\xi + \overline{Y})\right),$$

where

$$\Pi^\varepsilon_\delta = \left\{\xi \in \mathbb{Z}^N; \varepsilon(\xi + \overline{Y}) \subset \Omega^\varepsilon_\delta\right\}$$

- For every node $\varepsilon\xi_k$ in $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$ we define

$$\mathcal{Q}_\varepsilon(\varphi)(\varepsilon\xi_k) = \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} \varphi(\varepsilon\xi_k + \varepsilon z)dz.$$

Observe that by definition, any ball B_ε centered in a node of $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$ s entirely contained in Ω_ε, since actually they all belong to $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$.

- We define $\mathcal{Q}_\varepsilon(\varphi)$ on the whole $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, by taking a Q_1-interpolate, as in the finite element method, of the discrete function to $\mathcal{Q}_\varepsilon(\varphi)(\varepsilon\xi_k)$.

- On $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, $\mathcal{R}_\varepsilon$ will be defined as the remainder : $\mathcal{R}_\varepsilon(\varphi) = \varphi - \mathcal{Q}_\varepsilon(\varphi)$.

Proposition 2.5. *[15] For φ belonging to $H^1(\Omega_\varepsilon)$, one has the following properties*

1. $\|\mathcal{Q}_\varepsilon(\varphi)\|_{H^1(\widehat{\Omega}_{2\varepsilon\rho(Y)})} \leq \|\varphi\|_{H^1(\widehat{\Omega}_{2\varepsilon\rho(Y)})}$

2. $\|\mathcal{R}_\varepsilon(\varphi)\|_{L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)})} \leq C\varepsilon\|\nabla_x\varphi\|_{(L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)}))^N}$

3. $\|\nabla_x\mathcal{R}_\varepsilon(\varphi)\|_{(L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)}))^N} \leq C\|\nabla_x\varphi\|_{(L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)}))^N}$

We can now state the main result of this section.

Theoreme 2.1. *[15] Let φ_ε be in $H^1(\Omega_\varepsilon)$ for every ε, with $\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ bounded. There exists φ in $H^1(\Omega)$ and $\widehat{\varphi}$ in $L^2(\Omega; H^1_{per}(Y^*))$ such that, up to subsequences*

1. *$\mathcal{Q}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi$ weakly in $H^1_{loc}(\Omega)$*

2. *$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi$ weakly in $L^2_{loc}(\Omega; H^1(Y^*))$,*

3. *$\dfrac{1}{\varepsilon}\mathcal{T}_\varepsilon(\mathcal{R}_\varepsilon(\varphi_\varepsilon)) \rightharpoonup \widehat{\varphi}$ weakly in $L^2_{loc}(\Omega; H^1(Y^*))$,*

4. *$\mathcal{T}_\varepsilon(\nabla_x(\varphi_\varepsilon)) \rightharpoonup \nabla_x\varphi + \nabla_y\widehat{\varphi}$ weakly in $L^2_{loc}(\Omega; L^2(Y^*))$.*

2.1.2 The Averaging Operator $\mathcal{U}_\varepsilon$

definition 2.2. *For $\varphi \in L^2(\Omega \times Y^*)$, we set*

$$\mathcal{U}_\varepsilon(\varphi)(x) = \frac{1}{|Y^*|}\int_{Y^*} \widetilde{\varphi}\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right) dz, \qquad \text{for every } x \in \Omega$$

Remark 2.2. *For $V \in L^1(\Omega \times Y^*)$, the function $x \longmapsto V\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right)$ is generally not measurable (for example, we refer to [18]). Hence, it cannot be used as a test function. We replace it by the function $\mathcal{U}_\varepsilon(V)$.*

The next result extends the corresponding one given in [15].

Proposition 2.6. *[22] One has the following properties*

1. *The operator $\mathcal{U}_\varepsilon$ is linear and continous from $L^2(\Omega \times Y^*)$ into $L^2(\Omega)$, and one has for every $\varphi \in L^2(\Omega \times Y^*)$*

$$\|\mathcal{U}_\varepsilon\|_{L^2(\Omega)} \leq \|\varphi\|_{L^(\Omega\times Y^*)}$$

,

2. *$\mathcal{U}_\varepsilon$ is the left inverse of $\mathcal{T}_\varepsilon$ on Ω_ε, which means that $\mathcal{U}_\varepsilon \circ \mathcal{T}_\varepsilon = Id$ on Ω_ε,*

3. $\mathcal{T}_\varepsilon(\chi_\varepsilon \mathcal{U}_\varepsilon(\varphi))(x,y) = \dfrac{1}{|Y^*|}\displaystyle\int_{Y^*} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, y\right) dz, \quad \forall \varphi \in L^2(\Omega \times Y^*),$

4. *$\mathcal{U}_\varepsilon$ is the formal adjoint of $\mathcal{T}_\varepsilon$.*

Theoreme 2.2. *[22] Let φ_ε be in $L^2(\Omega_\varepsilon)$ for every ε, and let $\varphi \in L^2(\mathbb{R}^N \times Y^*)$. Then,*

1. *$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi$ strongly in $L^2(\mathbb{R}^N \times Y^*) \Longleftrightarrow \widetilde{\varphi_\varepsilon} - \mathcal{U}_\varepsilon(\varphi) \longrightarrow 0$ strongly in $L^2(\mathbb{R}^N)$*

2. *$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi$ strongly in $L^2_{loc}(\mathbb{R}^N; L^2(Y^*)) \Longleftrightarrow \widetilde{\varphi_\varepsilon} - \mathcal{U}_\varepsilon(\varphi) \longrightarrow 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$*

This result is essential for proving corrector results when studying homogenization problems.
Let us first state the following proposition.

Proposition 2.7. *[22] For every $\varphi \in H^1(\Omega_\varepsilon)$ one has*

$$\|\mathcal{R}_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)} = \|\varphi - \mathcal{Q}_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla\varphi\|_{(L^2(\Omega_\varepsilon))^N}.$$

Theoreme 2.3. *[22] Let v_ε be in $H^1(\Omega_\varepsilon)$ for every ε and $v \in H^1(\Omega)$ such that*

1. *$\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ is bounded*

2. *$\widetilde{v}_\varepsilon \rightharpoonup \theta v$ weakly in $L^2(\Omega)$.*
   Then,
   $$\mathcal{T}_\varepsilon(v_\varepsilon) \longrightarrow v \qquad \text{strongly in } L^2_{loc}(\Omega, L^2(Y^*))$$

2.1.3 The Boundary Unfolding Operator

We define here the unfolding operator on the boundary of the holes ∂T_ε, which is specific to the case of perforated domain. To do that, we need to suppose that T has Lipschitz boundary.

definition 2.3. *Suppose that T has a Lipschitz boundary, and let $\varphi \in L^p(\partial T_\varepsilon), p \in [1, +\infty]$.*
We define the function $\mathcal{T}^b_\varepsilon(\varphi) \in L^p(\mathbb{R}^N \times \partial T)$ by setting

$$\mathcal{T}^b_\varepsilon(\varphi)(x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right)$$

for every $x \in \mathbb{R}^N$ and $y \in \partial T$

Observe that if $\varphi \in W^{1,p}(\Omega_\varepsilon)$ and $\varphi = 0$ on $\partial\Omega_\varepsilon \setminus \partial T_\varepsilon^{int}$ is the set holes that do not intersect the boundary $\partial\Omega$,one has $\mathcal{T}_\varepsilon^b(\varphi) = \mathcal{T}_\varepsilon(\varphi)$ on ∂T. The next assertion reformulate those presented in proposition 2.1, when functions are defined on the boundary ∂T_ε.

Proposition 2.8. *[22] The boundary unfolding operator $\mathcal{T}_\varepsilon^b$ has the following properties :*

1. *$\mathcal{T}_\varepsilon^b$ is a linear operator.*
2. *$\mathcal{T}_\varepsilon^b(\varphi)\left(x, \left\{\dfrac{x}{\varepsilon}\right\}_Y\right) = \varphi(x) \quad \forall\varphi \in L^p(\partial T_\varepsilon)$ and $x \in \mathbb{R}^N$*
3. *$\mathcal{T}_\varepsilon^b(\varphi\psi) = \mathcal{T}_\varepsilon^b(\varphi)\mathcal{T}_\varepsilon^b(\psi), \quad \forall\varphi, \psi \in L^p(\partial T_\varepsilon)$*
4. *Let φ in $L^p(\partial T)$ be a Y-periodic function. Set $\varphi_\varepsilon(x) = \varphi\left(\dfrac{x}{\varepsilon}\right)$. Then,*
$$\mathcal{T}_\varepsilon^b(\varphi_\varepsilon)(x,y) = \varphi(y).$$
5. *For every $\varphi \in L^1(\partial T_\varepsilon)$ we have the integration formula*
$$\begin{aligned}\int_{\partial T_\varepsilon} \varphi(x)d\sigma(x) &= \frac{1}{\varepsilon|Y|}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}_\varepsilon^b(\varphi)dx\sigma(y)\\ &= \frac{1}{\varepsilon|Y|}\int_{\widetilde{\Omega_\varepsilon}\times\partial T} \mathcal{T}_\varepsilon^b(\varphi)dxd\sigma(y)\end{aligned}$$
6. *For every $\varphi \in L^2(\partial T_\varepsilon), \mathcal{T}_\varepsilon^b(\varphi)$ belongs to $L^2(\mathbb{R}^N \times \partial T)$. It also belongs to $L^2(\widetilde{\Omega_\varepsilon} \times \partial T)$.*
7. *For every $\varphi \in L^2(\partial T_\varepsilon)$, one has*
$$\|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^2(\mathbb{R}^N\times\partial T)} = \sqrt{\varepsilon|Y|}\|\varphi\|_{L^2(\partial T_\varepsilon)}.$$

Proposition 2.9. *[22] Let $g \in L^2(\partial T)$ and $\varphi \in H^1(\Omega)$ s.t. $\varphi = 0$ on $\partial\Omega_\varepsilon \setminus \partial T_\varepsilon^{int}$. One has the estimate*

$$\begin{aligned}\varepsilon|Y|\left|\int_{\partial T_\varepsilon} g_\varepsilon(x)\varphi(x)d\sigma(x)\right| &= \left|\int_{\mathbb{R}\times\partial T} g(y)\mathcal{T}_\varepsilon^b(\varphi)(x,y)dxd\sigma(y)\right|\\ &\leq C(|\mathcal{M}_{\partial T}(g)| + \varepsilon)\|\nabla\varphi\|_{(L^2(\Omega_\varepsilon))^N},\end{aligned}$$

where $\mathcal{M}_{\partial T}(g) = \dfrac{1}{|\partial T|}\displaystyle\int_{\partial T} g(y)d\sigma(y)$.

Proof. See [22] □

Proposition 2.10. *[22] Let $g \in L^2(\partial T)$ a Y-periodic function, and set $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$. On has the following convergence results as $\varepsilon \longrightarrow 0$*

1. *If $\mathcal{M}_{\partial T}(g) \neq 0$, then*

$$\varepsilon \int_{\partial T_\varepsilon} g_\varepsilon(x)\varphi(x)d\sigma(x) \longrightarrow \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega \varphi(x)dx \qquad \forall \varphi \in H^1(\Omega).$$

2. *If $\mathcal{M}_{\partial T}(g) = 0$, then*

$$\int_{\partial T_\varepsilon} g_\varepsilon(x)\varphi(x)d\sigma(x) \longrightarrow 0 \qquad \forall \varphi \in H^1(\Omega).$$

The next result is the equivalent of proposition 2.2 (1) and 2.3, to the case of functions defined on the boundaries of the holes.

Proposition 2.11. *[22]*

1. *Let $\varphi \in H^1(\Omega)$. Then, as $\varepsilon \to 0$, one has the convergence*

$$\int_{\mathbb{R}^N \times \partial T} \mathcal{T}^b_\varepsilon(\varphi)(x,y)dxd\sigma(y) \to \int_{\mathbb{R}^N \times \partial T} \widetilde{\varphi} dxd\sigma(y)$$

2. *Let $\varphi \in H^1(\Omega)$. Then*

$$\mathcal{T}^b_\varepsilon(\varphi) \to \widetilde{\varphi} \ \text{ strongly in } L^2(\mathbb{R}^N \times \partial T)$$

3. *Let φ_ε be in $L^2(\partial T_\varepsilon)$ for every ε, such that*

$$\mathcal{T}^b_\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi} \ \text{ weakly in } L^2(\mathbb{R}^N \times \partial T)$$

   Then,

$$\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \psi d\sigma(x) \to \frac{1}{|Y|}\int_{\mathbb{R}^N \times \partial T} \widehat{\varphi}(x,y)\psi(x)dxd\sigma(y) \quad \forall \psi \in H^1(\Omega).$$

2.2 The unfolding operator $\mathcal{T}_{\varepsilon\delta}$ depending on two parameters ε and δ

Following [19], the geometry of domains with small holes requires a specific unfolding operator depending on both parameters ε and . So, we will consider functions which vanish on the whole boundary of the perforated domain $\Omega_{\varepsilon\delta}$, namely belonging to the space $H^1_0(\Omega_{\varepsilon\delta})$. These functions are naturally extended by zero to the whole of Ω and these extensions belong to $H^1_0(\Omega)$. Consequently, from now on, we will not distinguish elements of $H^1_0(\Omega_{\varepsilon\delta})$ and their extensions in $H^1_0(\Omega)$.

definition 2.4. *For $\phi \in L^p(\Omega)$, $p \in [1, +\infty[$, the linear and continuous unfolding operator $\mathcal{T}_{\varepsilon\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^n)$ is defined by*

$$\mathcal{T}_{\varepsilon\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x,z) & a.e.\ for \quad (x,z) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \\ 0 & otherwise. \end{cases}$$

Proposition 2.12. *[19]*

1. *For any $u, v \in L^p(\Omega), \mathcal{T}_{\varepsilon\delta}(vw) = \mathcal{T}_{\varepsilon\delta}(v)\mathcal{T}_{\varepsilon\delta}(w)$.*
2. $\|\mathcal{T}_{\varepsilon\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leq \frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}$.
3. *Let $u \in H^1(\Omega)$. Then*

$$\mathcal{T}_{\varepsilon\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z\left(\mathcal{T}_{\varepsilon\delta}(u)\right) \qquad in \ \ \Omega \times \frac{1}{\delta}Y.$$

Concerning the integral formulas; we have the following results, similar to those of the previous section.

Proposition 2.13. *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying $\int_{\Lambda_\varepsilon} |w_\varepsilon|\, dx \to 0$, then*

$$\int_\Omega w_\varepsilon \, dx \overset{\mathcal{T}_{\varepsilon\delta}}{\simeq} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(w_\varepsilon)\, dxdz.$$

definition 2.5. *Let $p \in [1, \infty[$. The local average $M^\varepsilon_Y : L^p(\Omega) \longmapsto L^p(\Omega)$, is defined for $\phi \in L^p(\Omega)$ by*

$$M^\varepsilon_Y(\phi)(x) = \int_{\frac{1}{\delta}Y} \mathcal{T}_{\varepsilon\delta}(\phi)(x,z)dz.$$

Note that

$$\mathcal{T}_{\varepsilon\delta}(M^\varepsilon_Y(\phi)) = M^\varepsilon_Y(\phi) \quad on \quad \Omega \times \frac{1}{\delta}$$

Also, if $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$, such that $v_\varepsilon \longrightarrow v$ strongly in $L^p(\Omega)$, then $M^\varepsilon_Y(v_\varepsilon) \longrightarrow v$ strongly in $L^p(\Omega)$

Proposition 2.14. *[19] Suppose $N \geq 3$ and denote by 2^* the Sobolev exponent $\frac{2N}{N-2}$ associated to 2. Let ω be open and bounded in $\mathbb{R}^N$. Then the following estimates hold:*

$$\|\nabla_z\big(\mathcal{T}_{\varepsilon\delta}(u)\big)\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.3}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u - M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \leq \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.4}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u\big)\|^2_{L^2(\Omega\times\omega)} \leq \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{\frac{2}{N}}\|\nabla u\|^2_{L^2(\Omega)} + 2|\,\omega\,|\ \|u\|^2_{L^2(\Omega)}, \tag{2.5}$$

where C denotes the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$ and M^ε_Y is actually given by Definition 2.5.

2.2.1 The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$

We suppose in the sequel that the set B has Lipschitz boundary. We define a linear unfolding operator on the boundary of the holes $B_{\varepsilon\delta}$, specific to the case of domains with small holes.

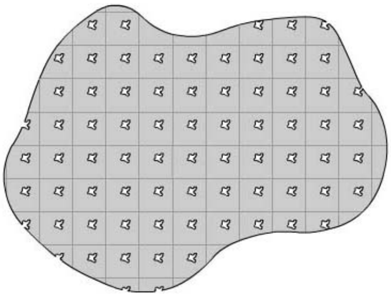

Fig. 3. The domain $\Omega_{\varepsilon\delta}$.

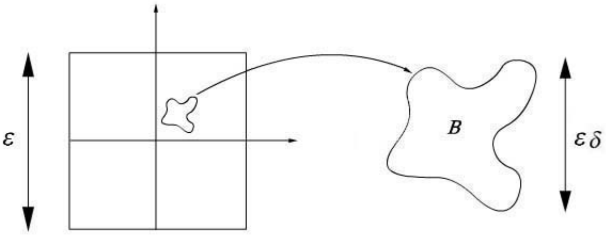

Fig. 4. The cell εY_δ and the set B.

definition 2.6. *For ϕ in $L^p(\partial B_{\varepsilon\delta}), p \in [1,\infty[$, the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta} : L^p(\partial B_{\varepsilon\delta}) \longmapsto L^p(\mathbb{R}^N \times \partial B)$ is defined by $\mathcal{T}^b_{\varepsilon\delta} : L^p(\partial B_{\varepsilon\delta}) \mapsto L^p(\mathbb{R}^N \times \partial B)$ is defined by*

$$\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\delta z\Big), \quad x \in \mathbb{R}^N,\ z \in \partial B. \tag{2.6}$$

Proposition 2.15. *[47] The linear and continuous boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$ has the following properties:*

1. $\mathcal{T}_{\varepsilon\delta}{}^b(\phi)\left(x, \frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}_Y\right) = \phi(x) \quad \forall\phi \in L^p(\partial B_{\varepsilon\delta}), x \in \mathbb{R}^N.$
2. *Let $\phi \in L^p(\partial B_{\varepsilon\delta})$ and set*
$$\phi_{\varepsilon\delta}(x) = \phi\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right) \qquad \text{then} \quad \mathcal{T}_{\varepsilon\delta}{}^b(\phi)(x,z) = \phi(z).$$
3. *For every $\phi \in L^1(\partial B_{\varepsilon\delta})$,*
$$\int_{\partial B_{\varepsilon\delta}} \phi(x)d\sigma(x) = \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}\times\partial B} \mathcal{T}_{\varepsilon\delta}{}^b(\phi)(x,z)dxd\sigma(z).$$
4. *Let $\phi \in L^2(\Omega)$, then*
$$\mathcal{T}_{\varepsilon\delta}{}^b(\phi) \longrightarrow \widetilde{\phi} \quad \text{strongly in} \quad L^2(\mathbb{R}^N \times \partial B).$$

5. *Let* $\{\phi_\varepsilon\}_\varepsilon$ *be a sequence in* $L^2(\partial B_{\varepsilon\delta})$*, such that*

$$\mathcal{T}_{\varepsilon\delta}{}^b(\phi) \rightharpoonup \phi \quad weakly \quad inL^2(\mathbb{R}^N \times \partial B.)$$

Then

$$\frac{\varepsilon}{\delta^{N-1}} \int_{\partial B_{\varepsilon\delta}} \phi_\varepsilon(x)\psi(x)d\sigma(x) \longrightarrow \int_{\mathbb{R}^N\times\partial B} \phi(x,z)\psi(x)dxd\sigma(z) \quad \forall\psi \in H^1(\Omega)$$

Lemma 2.1. *[27] Let* $g \in L^2(\partial B)$ *and* $\phi \in H^1(\Omega)$*. The following estimate holds:*

$$\left| \int_{\mathbb{R}^N\times\partial B} g(z)\mathcal{T}^b_{\varepsilon\delta}(\phi)dxd\sigma(z) \right| \leq C\left(|\mathcal{M}_{\partial B}(g)| + \varepsilon\delta\right) \|\nabla\phi\|_{L^2(\Omega)^N}.$$

Proposition 2.16. *[47] Let* $g \in L^2(\partial B)$ *and set*

$$g_\varepsilon(x) = g\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right) \qquad \forall x \in \mathbb{R}^N \setminus \bigcup_{\xi\in\mathbb{Z}} \varepsilon(\xi + \delta B). \tag{2.7}$$

Then for all $\phi \in H^1(\Omega)$*, one has the following estimate:*

$$\left| \int_{\partial B_{\varepsilon\delta}} g_\varepsilon(x)\phi(x)d\sigma(x) \right| \leq C\frac{\delta^{N-1}}{\varepsilon}\left(|\mathcal{M}_{\partial B}(g)| + \varepsilon\delta\right) \|\nabla\phi\|_{L^2(\Omega)^N}. \tag{2.8}$$

Proof. By using (2) and (3) of proposition 2.15, the result is a direct consequence of lemma 2.1 above. □

Proposition 2.17. *[47] Let* $g \in L^2(\partial B)$ *and* g_ε *be defined by* (2.7)*. Then, for all* $\phi \in H^1(\Omega)$*, one has the convergences:*

1. *If* $\mathcal{M}_{\partial B}(g) \neq 0 \to \frac{\varepsilon}{\delta^{N-1}} \int_{\partial B_{\varepsilon\delta}} g_\varepsilon(x)\phi(x)ds \to |\partial B|\mathcal{M}_{\partial B}(g) \int_\Omega \phi(x)dx$

2. *If* $\mathcal{M}_{\partial B}(g) = 0 \to \int_{\partial B_{\varepsilon\delta}} g_\varepsilon(x)\phi(x)ds \to 0$

Proof.

1. Let $\phi \in D(\mathbb{R}^N)$. One has, by unfolding with $\mathcal{T}^b_{\varepsilon\delta}$,

   $$\frac{\varepsilon}{\delta^{N-1}} \int_{\partial B_{\varepsilon\delta}} g_\varepsilon(x)\phi(x)d\sigma(x) = \int_{\mathbb{R}^N\times\partial B} g(z)\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)dxd\sigma(z),$$

   which, as ε tends to zero,converges to

   $$|\partial B|\mathcal{M}_{\partial B}(g) \int_\Omega \phi(x)dx,$$

   so, by density, the result is still true for every ϕ in $H^1(\Omega)$

2. The result is straightforward from proposition 2.16

□

Remark 2.3. *A simple consequence of proposition 2.17 is the convergence,*

$$\int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)dxd\sigma(z) \longrightarrow |\partial B| \int_{\mathbb{R}^N} \phi dxd\sigma(x)$$

2.2.2 Application: Homogenization of a Model Problem with Robin condition($\mathcal{P}_{\varepsilon\delta}$)

We consider the following problem ($\mathcal{P}_{\varepsilon\delta}$)

$$\begin{cases} -div(A^{\varepsilon}\nabla u_{\varepsilon\delta}) = f & in \;\; \Omega_{\varepsilon\delta} \\ A^{\varepsilon}\nabla u_{\varepsilon\delta}\eta_{\varepsilon\delta} = g_{\varepsilon} & on \;\; \partial T^{int}_{\varepsilon\delta} \\ u_{\varepsilon} = 0 & on \;\; \partial\Omega_{\varepsilon\delta} \setminus \partial T^{int}_{\varepsilon\delta} \end{cases} \tag{2.9}$$

where $\eta_{\varepsilon\delta}$ is the unit outward normal vector to $T_{\varepsilon\delta}$.
Introduce the functional space

$$V_{\varepsilon\delta} = \{\varphi \in H^1(\Omega_{\varepsilon\delta}) \mid \varphi = 0 \;\; on \;\; \partial\Omega_{\varepsilon\delta} \setminus \partial T^{int}_{\varepsilon\delta}\}$$

Then, the variational formulation of problem (2.9) is

$$\begin{cases} \text{Find } u_{\varepsilon} \in V_{\varepsilon\delta} \text{ satisfying} \\ \displaystyle\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}\nabla u_{\varepsilon\delta}\nabla\phi dx = \int_{\Omega_{\varepsilon\delta}} f\phi dx + \int_{\partial T_{\varepsilon\delta}} g_{\varepsilon}\phi d\sigma(x) \\ \text{for every } \phi \in V_{\varepsilon\delta} \end{cases} \tag{2.10}$$

Suppose that ε and δ are such that there exists a constant k satisfying

$$k = \lim_{\varepsilon\to 0} \frac{\delta^{N-1}}{\varepsilon}, \quad 0 \leq k < +\infty \tag{2.11}$$

We now state the unfolded formulation of the limit problem for (2.9) as ε goes to zero. At the limit we will observe the contribution of the periodic oscillations of the operator A^{ε}, as well as the contribution of small perforations.

Theoreme 2.4. *Assume that* (2.11) *is satisfied. Let* A^{ε} *belong to* $M(\alpha,\beta,\Omega)$, f *to* $L^2(\Omega)$ *and* g *to* $L^2(\partial B)$.
Suppose furthermore that; as tends to 0., there exists a matrix A *such that*

$$\mathcal{T}_{\varepsilon}(A^{\varepsilon})(x,y) \longrightarrow A(x,y) \quad a.e \quad in \;\; \Omega \times Y \tag{2.12}$$

Let $u_{\varepsilon\delta}$ *be the solution of the problem* (2.10). *Then there exist* u_0 *in* $H^1_0(\Omega)$ *and* $\widehat{u}$ *in* $L^2(\Omega; H^1_{per}(Y))$ *such that (up to a subsequence)*

$$\widetilde{u}_{\varepsilon\delta} \rightharpoonup u_0 \qquad weakly \;\; in \;\; L^2(\Omega) \tag{2.13}$$

with u_0 *and* $\widehat{u}$ *satisfying*

$$\int_Y A(x,y)(\nabla_x u_0(x) + \nabla_y \widehat{u}(x,y)\nabla_y\phi(y)dy = 0 \quad a.e \;\; x \in \Omega, \forall\phi \in H^1_{per}(Y) \tag{2.14}$$

and

$$\begin{cases} \displaystyle\int_{\Omega\times Y} A(x,y)(\nabla_x u_0(x) + \nabla_y \widehat{u}(x,y)\nabla\psi(y)dxdy = \int_{\Omega} f\psi dx + k|\partial B|\mathcal{M}_{\partial B}(g)\int_{\Omega}\psi dx, \\ \forall\psi \in H^1_0(\Omega) \end{cases} \tag{2.15}$$

Proof. Let $K > 0$. Observe first that Lax-Milgram theorem, there exists a unique solution $u_{\varepsilon\delta}$ of (2.9). Taking it as a test function in (2.10), one has

$$\alpha\|\nabla u_{\varepsilon\delta}\|^2_{(L^2(\Omega_{\varepsilon\delta}))^N} \leq \|f\|_{L^2(\Omega)}\|\nabla u_{\varepsilon\delta}\|_{(L^2(\Omega_{\varepsilon\delta}))^N} + \left|\int_{\partial B_{\varepsilon\delta}} g_\varepsilon u_{\varepsilon\delta} d\sigma(x)\right|$$

Then by proposition 2.16, we get

$$\alpha\|\nabla u_{\varepsilon\delta}\|^2_{(L^2(\Omega_{\varepsilon\delta}))^N} \leq C\left(1 + \frac{\delta^{N-1}}{\varepsilon}(|\mathcal{M}_{\partial B}(g)| + \varepsilon\delta)\right)$$

and so, the following a priori estimate holds:

$$\|u_{\varepsilon\delta}\|_{H^1(\Omega_{\varepsilon\delta})} \leq C.$$

This implies convergence (2.13) up to a subsequence. Next, by theorem 2.1, there exists $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$ such that

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \widehat{u} \quad weakly \;\; in \; (L^2(\Omega; L^2_{loc}(Y)))^N \tag{2.16}$$

With $\psi \in D(\Omega)$ as test function in (2.10), we have

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon \nabla u_{\varepsilon\delta} \nabla\psi dx = \int_{\Omega_{\varepsilon\delta}} f\psi dx + \int_{\partial B_{\varepsilon\delta}} g_\varepsilon \psi d\sigma(x). \tag{2.17}$$

By the unfolding criterion for integrals and proposition 2.15, we get

$$\begin{aligned}&\int_{\Omega\times Y_\delta} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathcal{T}_\varepsilon(\nabla\psi)dxdy \\ &\overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\psi)dxdy + \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial B} g(z)\mathcal{T}^b_\varepsilon(\psi)dxd\sigma(z).\end{aligned}$$

In view of (2.17), theorem 2.1 and proposition 2.15, at the limit we obtain equation (2.15) for all $\psi \in D(\Omega)$, and by density for all $\psi \in H^1_0(\Omega)$.

Use now $\Phi = \varepsilon(.)\phi(./\varepsilon)$ with $\psi \in D(\Omega)$ and $\phi \in C^1_{per}(Y)$, as test function in (2.10), to get

$$\begin{aligned}&\varepsilon\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x)\nabla u_{\varepsilon\delta}\psi(x)\nabla\phi\left(\frac{x}{\varepsilon}\right)dx + \int_{\Omega_{\varepsilon\delta}} A^\varepsilon\nabla u_{\varepsilon\delta}(x)\psi(x)\nabla\phi\left(\frac{x}{\varepsilon}\right)dx \\ &= \varepsilon\int_{\Omega_{\varepsilon\delta}} f(x)\psi(x)\phi\left(\frac{x}{\varepsilon}\right)dx + \varepsilon\int_{\partial B_{\varepsilon\delta}} g_\varepsilon(x)\psi(x)\phi\left(\frac{x}{\varepsilon}\right)d\sigma(x).\end{aligned}$$

The first integral, as well as the right-hand side of the above equality, converge to zero . the second integral above is unfolded $\mathcal{T}_\varepsilon$ to get

$$\begin{aligned}&\int_{\Omega_{\varepsilon\delta}} A^\varepsilon\nabla u_{\varepsilon\delta}(x)\psi(x)\nabla\phi\left(\frac{x}{\varepsilon}\right)dx \\ &\overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y_\delta} \mathcal{T}_\varepsilon(A^\varepsilon)(x,y)\mathcal{T}_\varepsilon(\nabla_x u_{\varepsilon\delta})(x,y)\nabla\phi(y)\mathcal{T}_\varepsilon(\psi)(x,y)dxdy,\end{aligned}$$

since the unfolding criterion of integrals (theorem 2.1) is satisfied due to the choice of test functions. Then, as above, we can pass to the limit with respect to ε and obtain (2.14), the first equation of the unfolded formulation for the limit problem. This equation describes the effect of the periodic oscillations of the coefficients in (2.15).

□

Strong formulation of the limit problem of $\mathcal{P}_{\varepsilon\delta}$

We now show that the unfolded problem is well posed and give the formulation in terms of the macroscopic solution u_0 alone.
First, let us introduce the classical correctors $\widehat{\chi}^j, (j = 1, ..., N)$ defined by the cell problems (see, for example, [9])

$$\begin{cases} \widehat{\chi}^j \in L^\infty(\Omega; H^1_{per}(Y)), \\ \displaystyle\int_Y A(x,y)\nabla(\widehat{\chi}^j - y_j)\nabla\phi dx = 0 \quad a.e.\ x \in \Omega. \\ \forall\phi \in H^1_{per}(Y). \end{cases}$$

Assuming u_0 is known and solving equation(2.14) for $\widehat{u}$ as a function u_0, gives

$$\widehat{u}(x,y) = \sum_{j=1}^N \frac{\partial u_0}{\partial x_j}(x)\widehat{\chi}^j(x,y),$$

Which used in equation (2.15) from theorem '2.4, yields

$$\int_\Omega A^{hom}\nabla u_0 \nabla\psi dx = \int_\Omega f\psi dx + k|\partial B|\mathcal{M}_{\partial B}(g)\int_\Omega \psi dx \quad \forall\psi \in H^1_0(\Omega) \tag{2.18}$$

where, for a.e. $x \in \Omega, A^{hom}(x)$ is the homogenized matrix.

$$a^{hom}_{ij}(x) = \int_Y \left(a_{ij}(x,y) - \sum_{k=1}^N a_{ik}(x,y)\frac{\partial \widehat{\chi}^j}{\partial y_k}(x,y) \right) dy.$$

Equation (2.18) is actually the variational formulation of the equation

$$\begin{cases} -div(A^{hom}\nabla u_0) = f + k\displaystyle\int_{\partial B} g d\sigma & in \ \ \Omega \\ u = 0 & on \ \ \partial\Omega. \end{cases}$$

Theoreme 2.5. *The limit function u_0 given by theorem 2.4, is the unique solution of the homogenized equation*

$$\begin{cases} u_0 \in H^1_0(\Omega) \\ \displaystyle\int_\Omega A^{hom}\nabla u_0\nabla\psi dx = \int_\Omega f\psi dx + k|\partial B|\mathcal{M}_{\partial B}(g)\int_\Omega \psi dx \\ \forall\psi \in H^1_0(\Omega) \end{cases}$$

Remark 2.4. *The contribution of the oscillations of the matrix A^ε in the homogenized problem is reflected by the presence of the homogenized operator $A^{hom}fl$ in the left-hand side. The contribution of the perforations is the constant $k\displaystyle\int_{\partial B} g d\sigma = k|\partial B|\mathcal{M}_{\partial B}$ in the right-hand side term.*

Remark 2.5. *If $k = 0$ the small holes have no influence at the limit.*

Remark 2.6. *If $\mathcal{M}_{\partial B}(g) = 0$, we obtain the result for domains without holes from [9]*

Chapter 3

Asymptotic Behavior of the Stokes Problem with Robin Condition in a Domain with small Holes

abstract We consider the Stokes problem in a domain $\Omega_{\varepsilon\delta_1\delta_2}$ of $\mathbb{R}^N, N \geq 3$ ε-periodically perforated by holes of size $r_1(\varepsilon\delta_1)$ and $r_2(\varepsilon\delta_2)(\varepsilon, \delta_1 = \delta_1(\varepsilon), \delta_2 = \delta_2(\varepsilon)$ being small parameters), with $r_1(\varepsilon\delta_1)/\varepsilon \longrightarrow 0$ and $r_2(\varepsilon\delta_2)/\varepsilon \to 0$ as $\varepsilon \to 0$. Our aim is to descrabe the asymptotic behavior of the velocity and the pressure of the fluid as $\varepsilon \to 0$ and give, if possible, a limit ("homogenized") problem. To do so, we use the periodic unfolding method introduced by Cioranescu, Damlamian and Griso(C.R. Acad. Sci. Paris, Ser. I **335**(2002)99-104; SIAM J. of Math. Anal. **40**(4)(2008)1585-1620). It allows us to consider ageneral geometric framework and so, to extend the result from Cioranescu, Donato and Ene (Math. Models and Methods in Appl. Sciences **19**(1996)857-881) and Capatina and Ene (European J. of Math. **22**(2011)333-345)

Keywords : Stokes problem, Robin condition, unfolding method.

3.1 Introduction

The aim of this work is to apply the unfolding method to perforated domains by holes size $\varepsilon\delta$, distributed periodically with period ε. We consider here the case where $\delta = \delta(\varepsilon)$ is such that $\delta \to 0$ as $\varepsilon \to 0$. Such kind of holes are known in literature as "small holes" (see e.g [25, 26]), or "tiny holes" (see e.g. [1, 41]).

Let Ω be a bounded open domain of $\mathbb{R}^N$, $N \geq 3$ with $|\partial\Omega| = 0$. The reference (periodicity) cell Y is given by

$$Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N.$$

Introduce now the compact subsets sets T and B of Y (the holes in Y), such that $T \cap B = \emptyset$. We assume that B and T have Lipschitz boundary. For two small positive parameters δ_1 and δ_2 with $\delta_2 \neq \delta_1$, the perforated (periodicity) cell $Y^*_{\delta_1\delta_2}$ is

$$Y^*_{\delta_1\delta_2} = Y \setminus (\delta_1\overline{B} \cup \delta_2\overline{T}). \tag{3.1}$$

then the holes in Ω are the following sets:

$$T_{\varepsilon\delta_1} = \bigcup_{\xi\in\Xi_\varepsilon} (\varepsilon\delta_1 T + \varepsilon\xi) \qquad B_{\varepsilon\delta_2} = \bigcup_{\xi\in\Xi_\varepsilon} (\varepsilon\delta_2 B + \varepsilon\xi), \tag{3.2}$$

where

$$\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N : \varepsilon(\xi + Y) \subset \Omega\}.$$

By construction, the size of the holes in $T_{\varepsilon\delta_1}$is of order $\varepsilon\delta_1$, and the size of those in $B_{\varepsilon\delta_2}$ is of order $\varepsilon\delta_2$.
Notice also that by construction, the boundary $\partial\Omega$ do notintersect that of the holes. Then, the perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ is defined as

$$\Omega_{\varepsilon\delta_1\delta_2} = \Omega \setminus (T_{\varepsilon\delta_1} \cup B_{\varepsilon\delta_2}).$$

In the following, we suppose that $\delta_1 = \delta_1(\varepsilon)$, and $\delta_2 = \delta_2(\varepsilon)$ depend on ε, and satisfy

$$\lim_{\varepsilon\to 0} \delta_1(\varepsilon) = \lim_{\varepsilon\to 0} \delta_2(\varepsilon) = 0. \tag{3.3}$$

This means that we are typically in the framework of domains with "small" holes mentioned at the beginning of this section.

We now consider the Stokes problem in $\Omega_{\varepsilon\delta_1\delta_2}$ with a Robin-type condition on the boundary the set of holes $T_{\varepsilon\delta_1}$, and a homogeneous Dirichlet one on the external boundary $\partial\Omega$ of the domain and on the boundary of the holes in $B_{\varepsilon\delta_2}$. The unknowns $u_{\varepsilon\delta_1\delta_2}$ the velocity field, and p_ε the (scalar) pressure, are characterized by the system,

$$\begin{cases} -\nu\Delta u_{\varepsilon\delta_1\delta_2} + \nabla p_\varepsilon &= f_\varepsilon \quad in \ \ \Omega_{\varepsilon\delta_1\delta_2}, \\ div u_{\varepsilon\delta_1\delta_2} &= 0 \quad in \ \ \Omega_{\varepsilon\delta_1\delta_2}, \\ u_{\varepsilon\delta_1\delta_2} &= 0 \quad on \ \ \partial\Omega \cup \partial B_{\varepsilon\delta_2}, \\ \nu\dfrac{\partial u_{\varepsilon\delta_1\delta_2,\tau}}{\partial n} + \alpha\varepsilon^\gamma u_{\varepsilon\delta_1\delta_2} &= g_\varepsilon \quad on \ \ \partial T_{\varepsilon\delta_1}, \end{cases} \tag{3.4}$$

where f_ε is the field of exterior body forces and g_ε is the field of exterior surface forces. The constants $\alpha \geq 0$ and γ are given, as well as $\nu > 0$ which is the viscosity of the fluid. The outward normal to $T_{\varepsilon\delta}$ n, while τ is the tangent vector to $T_{\varepsilon\delta}$, so that for a function $v, v_\tau = v - v_n.n$ where $v_n = V.n$
There is an extensive literature on the homogenization in perforated domains in

$\mathbb{R}^N$. For the case of "small" holes of size $\varepsilon^\alpha, \alpha > 0$, let us mention the works of Cioranescu and Murat [25, 26], concerning the homogeneous Dirichlet problem for the Poisson equation. They showed that the size $\varepsilon^{N/N-2}$ $(N > 2)$ is "critical" in the sense that the limit problem not only contains the laplacian but also an additional zero order term, called by the authors "strange term", it depends on the capacity of the set of holes at the limit.

The non homogeneous Neumann problem for the laplacian in the same geometrical framework, was studied by Conca and Donato [32]. In this case, the critical size of the holes is of order $\varepsilon^{N/N-1}$, and the contribution of the holes at the limit is an additional right-hand side integral term.

As concerning the Stokes problem, a pioneering work is that of Ene and Sanchez Palencia [49] who considered a periodic porous medium (with ε-size holes) with a Dirichlet condition on the boundary. They obtained at the limit the Darcy law, by applying the multiple scale method (introduced in [9]), together with sharp error estimates. This is the first mathematical justification of the experimental Darcy's law.

The Stokes problem with non homogeneous slip boundary condition depending on a parameter $\gamma \in \mathbb{R}^N$ (with still ε-size holes) was studied by Cioranescu, Donato and Ene in [21] by energy methods. They obtained at the limit, for different values of γ, either a Darcy-type law, or a Brinkman equation or a Stokes-type system.

In the framework of small holes, Allaire considered the Stokes system in [2] and the Navier-Stokes equation in [4, 3]. He obtained at the limit the same laws (Darcy, Brinkman or Stokes) but now it is the order of the size of holes with respect to ε who determines the type of the homogenized problem.

When working in perforated domains, the main difficulties are related to the fact that the equations and their solutions are defined on domains which strongly depend on ε. In order to speak about convergence when $\varepsilon \to 0$ (that is to "homogenize"), one needs to introduce extension operators to a fixed domain and to construct test functions, specific for each situation. To do that, several restrictions on the geometry of the holes and of the domain are necessary.

These difficulties are solved by the periodic unfolding method ([15] and [16]), as can be seen for example, in Cioranescu et al [20]. The advantage of the unfolding method, it is the fact that it separates the scales. So, one can treat in particular, holes at different scales in the same period. Such kind of problems, because of their complexity, cannot be solved by the classical methods in homogenization. For the Stokes problem, the method was applied by Capatina and Ene [12] and by Zaki [55] in the case of ε-size holes. This method was also extended to domains with small holes in Cioranescu et all [19] and used by Ould Hammouda in [46] and [47]. Our aim here is to apply it for the Stokes problem (3.4).

The paper is organized as follows. In Section 2 we give the variational formulation of the problem (3.4). In the Section 3, for the reader's convenience, we recall briefly the different definitions of the periodic unfolding operators and their properties. So, in Section 3.1, we list some notations. Section 3.2 is devoted to the periodic unfolding operator for fixed domains introduced in [15]. Section 3.3 recalls the definition and the properties of the unfolding operator for perforated domains with holes of size ε and finally Section 3.4 those of the unfolding operator corresponding to the case of volume-distributed small holes. In Section 3.5 we introduce the boundary operator related to the case of small holes, and prove some of its properties.

Finally, the homogenization result concerning problem (3.4) is stated and proved in Section 4.

3.2 The variational formulation

From now on, we make the following hypotheses on the data f_ε and g_ε:

$$f_\varepsilon \in (L^2(\Omega_{\varepsilon\delta_1\delta_2}))^N, \quad \varepsilon\widetilde{f_\varepsilon} \rightharpoonup f \quad \text{weakly in } (L^2(\Omega))^N. \tag{3.5}$$

and

$$g_\varepsilon(x) = g\Big(\frac{1}{\delta_1}\Big\{\frac{x}{\varepsilon}\Big\}\Big), \quad g \in (L^2(\partial T))^N,\ Y-\text{periodic function.} \tag{3.6}$$

Let use introduce the functional space

$$V_{\varepsilon\delta_1\delta_2} = \left\{\varphi \middle| \varphi \in (H^1(\Omega_{\varepsilon\delta_1\delta_2}))^N, \varphi = 0 \text{ on } \partial\Omega \cup \partial B_{\varepsilon\delta_2}, \varphi\cdot n = 0 \text{ on } \partial T_{\varepsilon\delta_1}\right\},$$

which is Hilbert for scalar product

$$\langle\varphi,\psi\rangle = \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla\varphi\, nabla\psi\, dx = \sum_{i,j=1}^{N}\int_{\Omega_{\varepsilon\delta_1\delta_2}} \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\psi_i}{\partial x_j}dx.$$

The variational formulation of system (3.4) is then the following:

$$\begin{cases} \text{Find } u_{\varepsilon\delta_1\delta_2} \in \mathrm{V}_{\varepsilon\delta_1\delta_2},\ \ p_\varepsilon \in L^2(\Omega_{\varepsilon\delta_1\delta_2}) \text{ satisfying} \\ \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\nabla\varphi\, dx + \alpha\varepsilon^\gamma \int_{\partial T_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}\,\varphi\, d\sigma(x) - \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \operatorname{div}\varphi\, dx, \\ \qquad\displaystyle = \int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon\,\varphi\, dx + \int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon\,\varphi\, d\sigma(x), \quad \forall\varphi \in V_{\varepsilon\delta_1\delta_2}, \\ \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} u_{\varepsilon\delta_1\delta_2}\,\nabla\varphi\, dx = 0, \quad \forall\varphi \in V_{\varepsilon\delta_1\delta_2}. \end{cases} \tag{3.7}$$

It is classical that this problem (see e.g [53]) is well-defined, it has a unique solution.

3.3 The periodic unfolding operators

3.3.1 Some notations

In the sequel, we will use the following notations:

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big)\Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$

$$Y_\delta = Y \setminus \ \delta\overline{T} \quad \text{for} \ \ \delta > 0,$$

$$\widetilde{\varphi} \ = \text{the extension by 0 to } \Omega, \ \forall \varphi \in L^p(\mathcal{O}), \ \mathcal{O} \subset \Omega.$$

For every z in $\mathbb{R}^N$, we denote by [z] the unique integer combination of periods such that $\{z\}_Y = z - [z]_Y$ belongs to Y. It follows that any $x \in \mathbb{R}^N$ can be written in the (unique) form

$$x = \varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big]_Y + \Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \quad \text{with } \Big\{\frac{x}{\varepsilon}\Big\}_Y \in Y.$$

For any subset K of Y, the average of φ over K and on ∂K, are given respectiely, by

$$\mathcal{M}_K(\varphi) = \frac{1}{|K|}\int_K \varphi \, dy, \quad \mathcal{M}_{\partial K}(\varphi) = \frac{1}{|\partial K|}\int_{\partial K} \varphi \, d\sigma.$$

3.3.2 The unfolding operator $\mathcal{T}_\varepsilon$ for fixed domains

We now recall from [46] and [27], the definition and some properties of a variant of the periodic unfolding operator introduced in [15] and [16].

definition 3.1. *For $\phi \in L^p(\Omega)$, $p \in [1, +\infty]$, the unfolding operator*

$$\mathcal{T}_\varepsilon : L^p(\Omega) \longrightarrow L^p(\Omega \times Y_\delta),$$

is defined as follows:

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & a.e. \ \ for \ (x,y) \in \widehat{\Omega}_\varepsilon \times Y_\delta, \\ 0 & a.e. \ \ for \ (x,y) \in \ \Lambda_\varepsilon \times Y_\delta. \end{cases}$$

Proposition 3.1. *If $\{w_\varepsilon\}_\varepsilon$ is a sequence in $L^1(\Omega)$ satisfying $\displaystyle\int_{\Lambda_\varepsilon} |w_\varepsilon| \, dx \to 0$, then*

$$\int_\Omega w_\varepsilon \, dx - \int_{\Omega \times Y_\delta} \mathcal{T}_\varepsilon(w_\varepsilon) \, dxdy \to 0.$$

Notice that if we replace Y_δ by Y, we retrieve the original unfolding operator $\mathcal{T}_\varepsilon$ as defined in [15, 16].

definition 3.2. *The local average $M_Y^\varepsilon : L^p(\Omega) \longrightarrow L^p(\Omega)$, is defined for any ϕ in $L^p(\Omega)$, $1 \leq p < \infty$, by*

$$M_Y^\varepsilon(\phi)(x) = \int_Y \mathcal{T}_\varepsilon(\phi)(x,y)\, dy.$$

It is known that if $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$, such that $v_\varepsilon \to v$ strongly in $L^p(\Omega)$, then

$$M_Y^\varepsilon(v_\varepsilon) \to v \quad \text{strongly in } L^p(\Omega). \tag{3.8}$$

3.3.3 The unfolding operator $\mathcal{T}_\varepsilon^*$ for perforated domains

Following [20], an unfolding operator $\mathcal{T}_\varepsilon^*$ for functions defined on perforated domains Ω_ε^* with holes of size ε. We will give here some of its main properties, needed later on. For more details and proofs, we refer the reader to [20]. Let first define the geometric settings for this special case.

With the notations from Section 3.1 (recall that T is a closed strict subset of Y), the part occupied by the material in the cell Y, supposed to be connected, is now denoted $Y^* = Y \setminus \overline{T}$. Then the perforated domain Ω_ε^* is obtained by removing from Ω the set of holes T_ε,

$$\Omega_\varepsilon^* = \Omega \setminus T_\varepsilon = \Big\{ x \in \Omega \Big| \; \Big\{\frac{x}{\varepsilon}\Big\} \in Y^* \Big\}. \tag{3.9}$$

With this definition, we also set

$$\begin{aligned} \widehat{\Omega}_\varepsilon^* &\doteq \widehat{\Omega}_\varepsilon \setminus \overline{T_\varepsilon}, \\ \Lambda_\varepsilon^* &\doteq \Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*. \end{aligned} \tag{3.10}$$

definition 3.3. *For ϕ Lebesgue-measurable on Ω_ε^*, the linear and continuous unfolding operator*

$$\mathcal{T}_\varepsilon^* : L^p(\Omega_\varepsilon^*) \longrightarrow L^p(\Omega \times Y^*),\ p \in [1, +\infty[,$$

is defined by

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & a.e. \text{ for } \ (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & a.e. \text{ for } \ (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

The main characteristic of this operator is that it maps functions defined on the oscillating domain Ω_ε^*, into functions defined on the fixed domain $\Omega \times Y^*$.

It is easily seen that for any function $w \in L^p(\Omega_\varepsilon^*)$, one has

$$\mathcal{T}_\varepsilon^*(w) = \mathcal{T}_\varepsilon(\widetilde{w})\big|_{\Omega \times Y^*}. \tag{3.11}$$

This implies that the main of properties of $\mathcal{T}_\varepsilon^*$ are consequences of those of $\mathcal{T}_\varepsilon$, as in particular, the integration formula written for every ϕ in $L^p(\Omega_\varepsilon^*)$

$$\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)\; dx\; dy = \int_{\Omega_\varepsilon^*} \phi(x)\; dx - \int_{\Lambda_\varepsilon^*} \phi(x)\; dx = \int_{\widehat{\Omega}_\varepsilon^*} \phi(x)\; dx.$$

An equivalent of Proposition 4.1 holds, allowing to pass from integrals over Ω_ε^* to integrals over the fixed domain $\Omega \times Y^*$.

Proposition 3.2. *If $\{\phi_\varepsilon\}$ is a sequence in $L^1(\Omega_\varepsilon^*)$ satisfying $\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon|\; dx \to 0$, then*

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon\, dx - \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(w_\varepsilon)\; dxdy \to 0.$$

Proposition 3.3. *Let $\{w_\varepsilon\}$ a sequence in $L^p(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$, then $\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w$ strongly in $L^p(\Omega \times Y^*)$.*

The main result concerning perforated domains is the next theorem.

Theoreme 3.1. *Let p be in $]1,+\infty]$. Let $\{w_\varepsilon\}$ be a sequence in $W^{1,p}(\Omega_\varepsilon^*)$ such that*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C.$$

Then, there exist a subsequence (still denoted ε), w in $W^{1,p}(\Omega)$ and $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(Y^*))$, such that

$$\begin{aligned}
&\widetilde{w_\varepsilon} \rightharpoonup |Y^*|w \quad \textit{weakly in } L^p(\Omega),\\
&\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w \quad \textit{strongly in } L^p_{loc}(\Omega; W^{1,p}(Y^*)),\\
&\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w \quad \textit{weakly in } L^p(\Omega; W^{1,p}(Y^*)),\\
&\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \textit{weakly in } L^p(\Omega \times Y^*)^N.
\end{aligned} \tag{3.12}$$

3.3.4 The unfolding operator $\mathcal{T}_{\varepsilon\delta}$ depending on two parameters ε and δ

We start by recalling the definition and the main properties of the periodic unfolding operator with two parameters δ and ε introduced in [19] (see also [46]).

Following [19], the geometry of domains with small holes requires a specific unfolding operator depending on both parameters ε and δ. When considering functions belonging to the space $H_0^1(\Omega_{\varepsilon\delta})$, one naturally can extend them by zero to the whole of Ω, and obviously, these extensions belong to $H_0^1(\Omega)$. It is why one may not distinguish the functions of $H_0^1(\Omega_{\varepsilon\delta})$ and their extensions in $H_0^1(\Omega)$.

definition 3.4. *For $\phi \in L^p(\Omega)$, $p \in [1,+\infty[$, the linear and continuous unfolding operator*

$$\mathcal{T}_{\varepsilon\delta} : L^p(\Omega) \longrightarrow L^p(\Omega \times \mathbb{R}^n),$$

is defined by

$$\mathcal{T}_{\varepsilon\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x,\delta z) & a.e.\ for \quad (x,z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y, \\ 0 & otherwise. \end{cases}$$

Proposition 3.4. *1. For any* $u \in L^2(\Omega)$

$$\|\mathcal{T}_{\varepsilon\delta}\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leq \frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}$$

2. If $\{w_\varepsilon\}$ *is a sequence in* $L^1(\Omega)$ *satisfying* $\displaystyle\int_{\Lambda_\varepsilon} |w_\varepsilon|\, dx \to 0$, *then*

$$\int_\Omega w_\varepsilon\, dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(w_\varepsilon)\, dxdz \to 0.$$

Proposition 3.5. *Suppose* $N \geq 3$ *and denote by* 2^* *the Sobolev exponent* $\dfrac{2N}{N-2}$ *associated to* 2. *Let* ω *be an open and bounded set in* $\mathbb{R}^N$. *Then the following estimates hold:*

$$\|\nabla_z\big(\mathcal{T}_{\varepsilon\delta}(u)\big)\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{3.13}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u - M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \leq \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{3.14}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u\big)\|^2_{L^2(\Omega\times\omega)} \leq \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{\frac{2}{N}}\|\nabla u\|^2_{L^2(\Omega)} + 2|\,\omega\,|\ \|u\|^2_{L^2(\Omega)}, \tag{3.15}$$

where C *denotes the Sobolev-Poincaré-Wirtinger constant for* $H^1(Y)$ *and* M^ε_Y *is actually given by Definition 3.2.*

We will also make use later on, of the following definition:

definition 3.5. *The local average* $M^{\varepsilon\delta}_Y : \quad L^P(\Omega) \longrightarrow L^P(\Omega)$ *is given for any* ϕ *in* $L^p(\Omega), \quad 1 \leq p < \infty$, *by*

$$M^{\varepsilon\delta}_Y(\phi)(x) = \delta^N \int_{\frac{1}{\delta}Y} \mathcal{T}_{\varepsilon\delta}(\phi)(x,y)dy.$$

3.3.5 The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$

definition 3.6. *For* ϕ *in* $L^p(\partial T_{\varepsilon\delta})$, $p \in [1,+\infty[$, *the boundary unfolding operator* $\mathcal{T}^b_{\varepsilon\delta} : L^p(\partial T_{\varepsilon\delta}) \mapsto L^p(\mathbb{R}^N \times \partial T)$ *is defined by*

$$\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\delta z\Big), \quad x \in \mathbb{R}^N,\ z \in \partial T. \tag{3.16}$$

Remark 3.1. *Observe that if* $\delta = 1$, $\mathcal{T}^b_{\varepsilon 1}$ *is nothing else than the boundary unfolding operator* $\mathcal{T}^b_\varepsilon$ *defined in [20].*

Proposition 3.6. *The linear and continuous boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$ has the following property:*

$$\int_{\partial T_{\varepsilon\delta}} \phi(x)\, d\sigma(x) = \frac{\delta^{N-1}}{\varepsilon} \int_{\mathbb{R}^N \times \partial T} \mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)\, dxd\sigma(z), \quad \forall \phi \in L^1(\partial T_{\varepsilon\delta}).$$

Proposition 3.7. *[47] Let φ_ε in $W^{1,p}(\Omega_{\varepsilon\delta})$ for every ε, and let $\varphi \in W^{1,p}(\frac{1}{\delta}Y))$ such that*

$$\mathcal{T}_{\varepsilon\delta}(\varphi_\varepsilon) \rightharpoonup \varphi \quad \text{weakly in } L^p(\Omega; W^{1,p}(\frac{1}{\delta}Y)).$$

Then the following convergence holds true:

$$\mathcal{T}^b_{\varepsilon\delta}(\varphi_\varepsilon) \rightharpoonup \varphi \quad \text{weakly in } L^p(\Omega; W^{1-\frac{1}{p},p}(\partial T)),$$

Proposition 3.8. *[47] Let $g \in L^2(\partial T)$ and set*

$$g_\varepsilon(x) = g\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}\Big), \quad \forall x \in \mathbb{R}^N \setminus \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + \delta T). \tag{3.17}$$

Then, for all $\phi \in H^1(\Omega)$, one has

$$\Big|\int_{\partial T_{\varepsilon\delta}} g_\varepsilon(x)\, \phi(x)\, dx\Big| \leq C \frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\Big(\varepsilon\delta + |\mathcal{M}_{\partial B}(g)|\Big)\|\nabla\phi\|_{L^2(\Omega)^N}. \tag{3.18}$$

Proposition 3.9. *[47] Let g be in $L^2(\partial T)$ and g_ε be defined by (3.17). Then, for all $\phi \in H^1(\Omega)$, one has the convergences*

$$\frac{\varepsilon}{\delta^{N-1}} \int_{\partial T_{\varepsilon\delta}} g_\varepsilon(x)\, \phi(x)\, ds \to |\partial T| \mathcal{M}_{\partial T}(g) \int_\Omega \phi(x). \tag{3.19}$$

Proposition 3.10. *For $g \in L^2(\partial T)$ and $v \in H^1(\Omega)$, we have the estimate,*

$$\Big|\int_{\partial T} g(x) v(x) d\sigma(x)\Big| \leq c \left(\delta^{\frac{N}{2}} \|\nabla v\|_{L^2(\Omega)} + \frac{\delta^{N-1}}{\varepsilon}\|v\|_{L^2(\Omega)}\right)$$

Proof. By using the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$, we have successively,

$$\begin{aligned}\Big|\int_{\partial T} g(x) v(x) d\sigma(x)\Big| &= \Big|\frac{\delta^{N-1}}{\varepsilon}\int_{\Omega\times\partial T} g(z) \mathcal{T}^b_{\varepsilon\delta}(v) dxd\sigma(z)\Big| \\ &\leq \frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial T} g(z)\left(\mathcal{T}^b_{\varepsilon\delta}(v) - M^{\varepsilon\delta}_Y(v)\right) dxd\sigma(z)\Big| \\ &\quad + \frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial T} g(z) M^{\varepsilon\delta}_Y(v) dxd\sigma(z)\Big|.\end{aligned}$$

Let now estimate the two integral terms on the right-hand side of this inequality. On the one hand, by using Cauchy-Schwarz inequality and recalling (3.14), we get

$$\frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial T} g(z)\left(\mathcal{T}^b_{\varepsilon\delta}(v) - M^{\varepsilon\delta}_Y(v)\right) dxd\sigma(z)\Big| \leq c\delta^{\frac{N}{2}}\|\nabla v\|_{L^2(\Omega)} \tag{3.20}$$

On the other hand we have

$$\frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial T} g(z)M_Y^{\varepsilon\delta}(v)dxd\sigma(z)\right| \leq c\frac{\delta^{N-1}}{\varepsilon}\left\|g\right\|_{L^2(\Omega)}\left\|M_Y^{\varepsilon\delta}(v)\right\|_{L^2(\Omega\times\partial T)}$$

$$\leq c\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^2(\partial T)}.\left(\int_{\Omega\times\partial T}\left|M_Y^{\varepsilon\delta}(v)\right|^2 dxd\sigma(z)\right)^{\frac{1}{2}}$$

Applying the Fubini theorem we immediately obtain

$$\frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial T} g(z)M_Y^{\varepsilon\delta}(v)dxd\sigma(z)\right| \leq c\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^2(\partial T)}|\partial T|^{\frac{1}{2}}\left(\int_{\Omega}\left|M_Y^{\varepsilon\delta}(v)\right|^2 dx\right)^{1/2}. \tag{3.21}$$

Let have a look to the last integral above. By Cauchy-Schwarz inequality and Proposition 3.4(1), it follows that

$$\begin{aligned}\left(\int_{\Omega}\left|M_Y^{\varepsilon\delta}(v)\right|^2 dx\right)^{1/2} &= \delta^N\left(\int_{\Omega}\left(\int_{\frac{1}{\delta}Y}\mathcal{T}_{\varepsilon\delta}(v)dz\right)^2 dx\right)^{1/2}\\ &\leq \delta^{\frac{N}{2}}\|\mathcal{T}_{\varepsilon\delta}(v)\|_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \|v\|_{L^2(\Omega)}.\end{aligned}$$

This inequality used in (3.21) yields

$$\frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial T} g(z)M_Y^{\varepsilon\delta}(v)dxd\sigma(z)\right| \leq c\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^2(\partial T)}.\|v\|_{L^2(\Omega)},$$

which, together with (3.20), gives the result. □

3.4 Main results

3.4.1 A priori estimates for $u_{\varepsilon\delta_1\delta_2}$ and p_ε

Proposition 3.11. *Let $\gamma\in\mathbb{R}$ and $(u_{\varepsilon\delta_1\delta_2}, p_\varepsilon)$ be the solution of problem (3.4). Then there exists a positive constant C independent of ε such that,*

$$\|u_{\varepsilon\delta_1\delta_2}\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})^N} \leq C, \tag{3.22}$$

$$\|\varepsilon u_{\varepsilon\delta_1\delta_2}\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})^N} \leq C, \tag{3.23}$$

and

$$\|\varepsilon^{-1}u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})^{N\times N}} \leq C. \tag{3.24}$$

Proof. Taking $u_{\varepsilon\delta_1\delta_2}$ as test function in (3.7) (which makes sense), we get

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial T_{\varepsilon\delta_1})}\leq\left|\int_{\Omega_{\varepsilon\delta_1\delta_2}}f_\varepsilon u_{\varepsilon\delta_1\delta_2}dx\right|+\left|\int_{\partial T_{\varepsilon\delta_1}}g_\varepsilon u_{\varepsilon\delta_1\delta_2}d\sigma(x)\right|.$$

We have successively, by using Cauchy-Schwarz and Poincaré inequalities as well as Proposition 3.10,

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial T_{\varepsilon\delta_1})}\leq C\bigg(\|f\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\delta_1^{\frac{N}{2}}+\frac{\delta_1^{N-1}}{\varepsilon}\bigg)\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}. \tag{3.25}$$

which implies that

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial T_{\varepsilon\delta_1})}\leq C\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

Using herein the Poincaré inequality yields

$$\|u_{\varepsilon\delta_1\delta_2}\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})}\leq C.$$

On the other hand, by using the Young inequality in (3.25), we have

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial T_{\varepsilon\delta_1})}\leq C\bigg(1+\delta_1^{\frac{N}{2}}+\eta\frac{1}{\varepsilon^2}+\frac{1}{\eta}(\delta_1^{n-1})^2\bigg)\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})},$$

which implies, in view of (3.22) and Poincaré's inequality,

$$\|\varepsilon u_{\varepsilon\delta_1\delta_2}\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})}\leq C. \tag{3.26}$$

Finally, by using estimate (3.22) and the fact that for $v\in H^1(\Omega_\varepsilon)$ (see for example, [3, 32]),

$$\|v\|_{(L^2(\Omega_\varepsilon))^N}\leq C\varepsilon\|\nabla v\|_{(L^2(\Omega_\varepsilon))^{N\times N}},$$

we get estimate (3.24).

□

To describe the limit behavior of system (3.7), we need to use some special test functions belonging to $V_{\varepsilon\delta_1\delta_2}$. For the construction of these functions, we follow the procedure from [12].

Consider the local periodic problem in Y^*:

$$(P_1)\begin{cases}-\Delta_y v^k+\nabla_y q^k &= e^k \ in\ Y^*\\ \mathrm{div}_y v^k &= 0\ in\ Y^*\\ v^k.n &= 0\ on\ \partial T\\ \left(\dfrac{\partial v^k}{\partial n}\right)_\tau &= 0\ on\ \partial T\\ (v^k;q^k)\ \ Y-periodic & \\ \mathcal{M}_{Y^*}(q^k) &= 0,\end{cases}$$

where $(e^k)_{k=1,\dots,N}$ is the canonical basis in $\mathbb{R}^n$.

Now, in order to write the local problem (P_1) in terms of $x = \varepsilon y$, we extend v^k and q^k by periodicity to $\Omega_{\varepsilon\delta_1\delta_2}$ and set

$$v^{k\varepsilon}(x) = v^k\Big(\frac{x}{\varepsilon}\Big), \quad q^{k\varepsilon} = q^k\Big(\frac{x}{\varepsilon}\Big) \text{ for } x \in \Omega_\varepsilon.$$

Consequently, problem (P_1) writes

$$(P_\varepsilon)\begin{cases} -\varepsilon^2\Delta_x v^{k\varepsilon} + \varepsilon\nabla_x q^{k\varepsilon} = & e^k \quad \textit{in } \Omega_\varepsilon, \\ \operatorname{div}_x v^{k\varepsilon} = & 0 \quad \textit{in } \Omega_\varepsilon, \end{cases}$$

and it is easily seen that the following estimates hold:

$$\|v^{k\varepsilon}\|_{(L^2(\Omega_{\varepsilon\delta_1\delta_2}))^N} \le C, \tag{3.27}$$

as well as

$$\|\nabla_x v^{k\varepsilon}\|_{(L^2(\Omega_{\varepsilon\delta_1\delta_2}))^{N\times N}} \le \frac{C}{\varepsilon} \tag{3.28}$$

Multiply now (P_ε) by $v \in V_{\varepsilon\delta_1\delta_2}$ and integrate on $\Omega_{\varepsilon\delta_1\delta_2}$, to obtain

$$\varepsilon\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla v^{k\varepsilon}\nabla v\, dx + \int_{\Omega_{\varepsilon\delta_1\delta_2}} q^{k\varepsilon}\operatorname{div} v\, dx = \frac{1}{\varepsilon}\int_{\Omega_{\varepsilon\delta_1\delta_2}} e^k\cdot v\, dx, \quad \forall v \in V_{\varepsilon\delta_1\delta_2}. \tag{3.29}$$

At this point, we recall the fact ([29, 55]) that, if $\phi \in L^2(\Omega_{\varepsilon\delta_1\delta_2})$, then there exists $\varphi \in V_{\varepsilon\delta_1\delta_2}$, such that

$$\operatorname{div}\varphi = \phi \quad \text{with} \quad \|\varphi\|_{V_{\varepsilon\delta_1\delta_2}} \le C\,\|\phi\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

Using this in (3.29), together with (3.27) and (3.28), gives to the estimate

$$\|q^{k\varepsilon}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \le C. \tag{3.30}$$

Other tools that will be needed below are some results from [19]. To formulate them, define the space K_B as

$$K_B = \{\Phi \in L^{2^*}(\mathbb{R}^N) \mid \nabla\Phi \in L^2(\mathbb{R}^N),\ \Phi \text{ constant on } B\}.$$

Lemma 3.1. *([19]) Let $N \ge 3$. Then, for every $\delta_0 > 0$, the set*

$$\bigcup_{0<\delta_2<\delta_0} \{\phi \in H^1_{per}(Y) \mid \phi = 0 \ \textit{on } \delta_2 B\},$$

is dense in $H^1_{per}(Y)$.

Lemma 3.2. *([19]) Let $v \in D(\mathbb{R}^N) \cap K_B$ and*

$$w_{\varepsilon\delta_2}(x) = v(B) - v\left(\frac{1}{\delta_2}\left\{\frac{x}{\varepsilon}\right\}\right) \quad \textit{for} \quad x \in \mathbb{R}^N.$$

Then

$$w_{\varepsilon\delta_2} \rightharpoonup v(B) \quad \textit{weakly in } H^1(\Omega). \tag{3.31}$$

Now, as in [19], let the function χ be the solution of the following cell problem:

$$\begin{cases} \chi \in L^\infty(\Omega; K_B), \\ \displaystyle\int_{\mathbb{R}^N \setminus \overline{T}} \nabla_z \chi(x,z)\, \nabla_z \Psi(z)\, dz = 0 \quad \text{for a.e.} x \in \Omega, \;\; \forall \Psi \in K_B \text{ with } \Psi(B) = 0, \end{cases} \tag{3.32}$$

and introduce the function $\varTheta$ given by

$$\varTheta(x) = \int_{\mathbb{R}^N \setminus \overline{T}} \nabla_z \chi(x,z)\, \nabla_z \chi(x,z)\, dz. \tag{3.33}$$

Notice that $\varTheta$ can be interpreted as a local capacity of the set B since it is positive by definition. Assume now that there exist k_1 with $0 \leq k_1 < +\infty$, and k_2 with $0 \leq k_2 < +\infty$, such that

$$k_1 = \lim_{\varepsilon \to 0} \frac{\delta_1(\varepsilon)^{N-1}}{\varepsilon} \quad \text{and} \quad k_2 = \lim_{\varepsilon \to 0} \frac{\delta_2(\varepsilon)^{\frac{N}{2}-1}}{\varepsilon}. \tag{3.34}$$

Remark 3.2. *It is easily seen that k_1 corresponds to the critical size of Dirichlet small holes from [25], that is $C\varepsilon^{N/N2-2}$. As concerning k_2, it corresponds to the critical size from [32], that is $C\varepsilon^{N-2}$. Observe also that, as $\varepsilon \to 0$, by* (3.34)*, the holes in $T_{\varepsilon\delta_1}$ approach quicker zero, than those from $B_{\varepsilon\delta_2}$. Consequently, for ε small enough, the holes from $T_{\varepsilon\delta_1}$ do not intersect those from $B_{\varepsilon\delta_2}$ for each ε. It is why that one could make the hypothesis that from now on, in the limiting process $\varepsilon \to 0$, the holes $T_{\varepsilon\delta_1}$ and $B_{\varepsilon\delta_2}$ do not intersect.*

We are now in position to give the homogenization result for the Stokes problem (3.4).

Theoreme 3.2 (Main Theorem)**.** *Assume that* (3.34) *hold true and that the data f_ε satisfies the the additional property*

$$\varepsilon \widetilde{f_\varepsilon} \rightharpoonup f \;\; \textit{weakly in } (L^2(\Omega))^N$$

and Let $(u_{\varepsilon\delta_1\delta_2}, p_\varepsilon)$ be the solution of (3.4)*. Then, there exist $u \in (L^2(\Omega))^n, p \in L^2(\Omega)$ and an extension $\widehat{p_\varepsilon}$ of the pressure p_ε such that*

$$\widetilde{u_{\varepsilon\delta_1\delta_2}} \rightharpoonup u \qquad \textit{weakly in } (L^2(\Omega))^N, \tag{3.35}$$

and

$$\varepsilon \widehat{p_\varepsilon} \rightharpoonup p \qquad \textit{weakly in } L^2(\Omega), \tag{3.36}$$

(i) *If $\gamma < -1$, then $u = 0$.*

(ii) *If $\gamma = -1$, then (u, p) $(u = (u_1, \ldots, u_N)$ is solution in Ω of the problem*

$$\begin{aligned} \nu\, u_k + \alpha\, k_1 |\partial T|\, \mathcal{M}_{\partial T}(v^k) u + k_2^2\, \mathcal{M}_{Y^*}(v^k)\, \varTheta u \\ = \mathcal{M}_{Y^*}(v^k)(f - \nabla p) + |\partial T| \mathcal{M}_{\partial T}(g v^k), \end{aligned} \tag{3.37}$$

for $k \in \{1, \dots, N\}$.

(iii) *If* $\gamma > -1$, *then* (u,p) $(u = (u_1, \dots, u_N)$ *is solution in* Ω *of the problem*

$$u_k + k_2^2\, \mathcal{M}_{Y^*}(v^k)\, \Theta u = \frac{1}{\nu}\mathcal{M}_{Y^*}(v^k)(f - \nabla p) + \frac{|\partial T|}{\nu}\mathcal{M}_{\partial T}(gv^k), \tag{3.38}$$

for $k \in \{1, \dots, N\}$.

Remark 3.3. *Observe that in both systems* (3.37) *and* (3.38), *we have the "strange term" involving the function* Θ, *it represents the contribution of the set of small holes* $B_{\varepsilon\delta_2}$ *with the homogeneous Dirichlet condition on their boundary. This makes the link with the results from [25], where this phenomenon was observed for the Poisson equation, as mentioned in the Introduction.*

In turn, the contribution of the holes of the set of small holes $T_{\varepsilon\delta_2}$ *with Neumann condition, is reflected by the additional term involving* g *in the right-hand side of the limit equation. This makes the link with the result from [32] mentioned in the Introduction.*

Proof of Theorem 3.2. Thanks to estimates (3.24) and (3.23), we can apply Proposition 3.5. Thus, there exists some U in $L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence,

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon\delta_2}(\varepsilon u_{\varepsilon\delta_1\delta_2}) \rightharpoonup U \text{ weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{3.39}$$

Using (3.8), one also has the convergence

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}^\varepsilon_Y(\varepsilon u_{\varepsilon\delta_1\delta_2}) \to k_2\, u \text{ strongly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{3.40}$$

On the other hand, due to Proposition 3.5, there exists W in $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Omega \times \mathbb{R}^N)$, such that

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon\delta_2}(\varepsilon u_{\varepsilon\delta_1\delta_2}) - \mathcal{M}^\varepsilon_Y(\varepsilon u_{\varepsilon\delta_1\delta_2})) \rightharpoonup W \text{ weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)). \tag{3.41}$$

From (3.39),(3.40) and (3.41), one conclude that

$$U = W + k_2 u \;\text{ and }\; \nabla_z U = \nabla_z W.$$

By Proposition 3.5 again, one also has convergence

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon\delta_2}(\varepsilon u_{\varepsilon\delta_1\delta_2}) = \delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla\varepsilon u_{\varepsilon\delta_1\delta_2}) \rightharpoonup \nabla_z U \text{ weakly in } L^2(\Omega \times \mathbb{R}^N). \tag{3.42}$$

Let go back to (3.7) where we would like to pass to the limit. To do so, we introduce the function $v_{\varepsilon\delta_2}$ defined by

$$v_{\varepsilon\delta_2}(x) = \varepsilon v^k\Big(\frac{x}{\varepsilon}\Big) w_{\varepsilon\delta_2}\, \varphi(x),$$

where $\varphi \in \mathcal{D}(\Omega)$, $v^k \in (H^1_{per}(Y^*))$ is solution of (P_1), and $w_{\varepsilon\delta_2}$ was defined in Lemma 3.2. We can choose $v_{\varepsilon\delta_2}$ as test function in (3.7). We get

$$\begin{aligned}\varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} &\nabla u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2} \nabla(v^{k\varepsilon}\varphi) dx + \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} \nabla w_{\varepsilon\delta_2} (v^{k\varepsilon}\varphi) dx \\ &+ \alpha\varepsilon^{\gamma+1} \int_{\partial T_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}\, v^{k\varepsilon}\, \varphi\, w_{\varepsilon\delta_2}\, d\sigma(x) - \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon p_\varepsilon \operatorname{div}(v^{k\varepsilon}\varphi w_{\varepsilon\delta_2})\, dx \\ &= \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon f_\varepsilon\, v^{k\varepsilon}\, w_{\varepsilon\delta_2} \varphi\, dx + \varepsilon \int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon\, v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \varphi\, d\sigma(x),\end{aligned} \tag{3.43}$$

where to simplify, we used the notation

$$v^{k\varepsilon}(x) = v^k\Big(\frac{x}{\varepsilon}\Big).$$

Now, we shall pass to the limit as $\varepsilon \to 0$ in (3.43).

Case 1. $\gamma \geq -1$. The first term in (3.43) is written as

$$\begin{aligned} I^1_\varepsilon &= \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\, w_{\varepsilon\delta_2} \nabla(\varphi v^{k\varepsilon}) dx \\ &= \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\, w_{\varepsilon\delta_2}\, v^{k\varepsilon}\, \nabla\varphi\, dx + \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\, w_{\varepsilon\delta_2}\, \varphi\, \nabla v^{k\varepsilon}\, dx.\end{aligned}$$

On the other hand, considering the integral

$$\begin{aligned} J_\varepsilon &= \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} w_{\varepsilon\delta_2}\, \nabla v^{k\varepsilon}\, \nabla(\varphi u_{\varepsilon\delta_1\delta_2})\, dx \\ &= \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \nabla\varphi u_{\varepsilon\delta_1\delta_2}\, dx + \varepsilon\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \nabla u_{\varepsilon\delta_1\delta_2}\, \varphi\, dx,\end{aligned} \tag{3.44}$$

and using estimates (3.22), (3.24) and (3.28), we obtain

$$\begin{aligned} |I^1_\varepsilon - J_\varepsilon| &\leq \varepsilon\nu \Big| \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\, w_{\varepsilon\delta_2}\, \nabla\varphi v^{k\varepsilon})\, dx \Big| + \varepsilon\nu \Big| \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \nabla\varphi u_{\varepsilon\delta_1\delta_2}\, dx \Big| \\ &\leq C\varepsilon \big(\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{(L^2(\Omega_{\varepsilon\delta_1\delta_2}))^{N\times N}} \|v^{k\varepsilon}\|_{(L^2(\Omega_{\varepsilon\delta_1\delta_2}))^N} \\ &\quad + \|\nabla v^{k\varepsilon}\|_{(L^2(\Omega_{\varepsilon\delta_1\delta_2}))^{N\times N}} \|u_{\varepsilon\delta_1\delta_2}\|_{(L^2(\Omega_{\varepsilon\delta_1\delta_2}))^N} \big) \leq C\,\varepsilon.\end{aligned}$$

Consequently, we have

$$\lim_{\varepsilon\to 0} I^1_\varepsilon = \lim_{\varepsilon\to 0} J_\varepsilon. \tag{3.45}$$

Now take $v = w_{\varepsilon\delta_2} u_{\varepsilon\delta_1\delta_2} \varphi$ as test function in (3.29), multiply by ν and use (3.44) to get

$$\begin{aligned} J_\varepsilon = \varepsilon\nu &\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla v^{k\varepsilon}\, \nabla(w_{\varepsilon\delta_2})\, u_{\varepsilon\delta_1\delta_2}\, \varphi\, dx + \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} q^{k\varepsilon}\, \nabla(w_{\varepsilon\delta_2})\, u_{\varepsilon\delta_1\delta_2}\, \varphi\, dx \\ &+ \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} q^{k\varepsilon}\, w_{\varepsilon\delta_2}\, u_{\varepsilon\delta_1\delta_2}\, \nabla\, \varphi\, dx + \frac{1}{\varepsilon}\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} e^k\, w_{\varepsilon\delta_2}\, u_{\varepsilon\delta_1\delta_2}\, \varphi\, dx.\end{aligned}$$

Unfolding with $\mathcal{T}_{\varepsilon\delta_2}$ the right-hand term, yields

$$\begin{aligned} J_\varepsilon = & -\nu\varepsilon\delta_2^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta_2}(\nabla v^{k\varepsilon})\, \mathcal{T}_{\varepsilon\delta_2}\nabla(w_{\varepsilon\delta_2})\, \mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}\varphi)\, dx \\ & + \nu\delta_2^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta_2}(q^{k\varepsilon})\, \mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2})\, \mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}\varphi)\, dx \\ & + \nu \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_\varepsilon(q^{k\varepsilon})\, \mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})\, \mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}\nabla\varphi)\, dx \\ & + \nu \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_\varepsilon(e^k)\, \mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})\, \mathcal{T}_\varepsilon\Big(\frac{u_{\varepsilon\delta_1\delta_2}}{\varepsilon}\Big)\, \mathcal{T}_\varepsilon(\varphi)\, dxdy. \end{aligned} \tag{3.46}$$

Since $\widetilde{u}_{\varepsilon\delta_1\delta_2} \to 0$ strongly in $(L^2(\Omega))^N$, using the boundedness of $q^{k\varepsilon}$, convergences (3.22) and (3.24) from (3.46), as well as (3.45), we deduce that

$$\lim_{\varepsilon\to 0} I_\varepsilon^1 = \nu\, v(B) \int_\Omega u_k\, \varphi\, dx. \tag{3.47}$$

Unfolding the second term of (3.43) by $\mathcal{T}_{\varepsilon\delta_2}$ and using lemma 3.2, we get

$$\begin{aligned} & \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon\nabla u_{\varepsilon\delta_1\delta_2}\, \nabla w_{\varepsilon\delta_2}\, (v^{k\varepsilon}\varphi)\, dx \\ & \qquad - \nu\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \int_{\Omega\times\mathbb{R}^N} \delta_2^{\frac{N}{2}}\, \mathcal{T}_{\varepsilon\delta_2}(\varepsilon\nabla u_{\varepsilon\delta_1\delta_2})\, (-\nabla_z v)\, v^k(y)\, \varphi(x)\, dxdz \to 0. \end{aligned}$$

Then, convergence (3.42) as well as hypothesis (3.34), imply that

$$\begin{aligned} & \lim_{\varepsilon\to 0} \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon\nabla u_{\varepsilon\delta_1\delta_2}\, \nabla w_{\varepsilon\delta_2}(v^{k\varepsilon}\varphi)\, dx \\ & \qquad = -\nu k_2 \int_{\Omega\times(\mathbb{R}^N\setminus B)} \nabla_z U(x,z)\, \nabla_z v(z)\, v^k(y)\, \varphi(x)\, dxdz. \end{aligned} \tag{3.48}$$

For the third term in (3.43), we have two cases.

1. if $\gamma > -1$, it is clear that this term tends to zero as $\varepsilon \to 0$,

2. if $\gamma = -1$, after unfolding with $\mathcal{T}^b_{\varepsilon\delta_1}$ we have,

$$\begin{aligned} & \alpha\varepsilon^{\gamma+1} \int_{\partial T_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}\, w_{\varepsilon\delta_2}\, v^{k\varepsilon}\varphi(x)\, d\sigma(x) \\ & \qquad = \alpha\varepsilon^{\gamma+1}\frac{\delta_1^{N-1}}{\varepsilon} \int_{\Omega\times\partial T} \mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta_1\delta_2})\, \mathcal{T}^b_{\varepsilon\delta_1}(w_{\varepsilon\delta_2})\, v^k(y)\, \varphi(x)\, dxd\sigma(z). \end{aligned}$$

Passing to limit and using hypothesis (3.34), we get

$$\begin{aligned} & \lim_{\varepsilon\to 0} \alpha \int_{\partial T_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}\, w_{\varepsilon\delta_2}\, v^{k\varepsilon}\, \varphi(x)\, d\sigma(x) \\ & \qquad = \alpha k_1\, v(B) \int_{\Omega\times\partial T} u\, v^k(z)\, \varphi(x)\, dxd\sigma(z). \end{aligned} \tag{3.49}$$

For the fourth term, we define the extension $\widehat{p_\varepsilon}$ of the pressure p_ε, by

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \operatorname{div}(v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \varphi)\, dx = \int_\Omega \widehat{p_\varepsilon}\, \mathcal{P}(v^{k\varepsilon})\, \nabla(w_{\varepsilon\delta_2}\varphi)\, dx, \qquad \forall\varphi \in \mathcal{D}(\Omega), \tag{3.50}$$

where $\mathcal{P}(v^{k\varepsilon})$ is an extension of v^k to Y . It is easy to verify that a possible of such an extension choice is the following one:

$$\mathcal{P}(v^{k\varepsilon}) = \begin{cases} v^k & \text{in } Y^*, \\ w^k & \text{in } T, \end{cases}$$

where w^k is the solution of

$$\begin{cases} -\Delta_y w^k + \nabla_y q^k &= 0 \quad \text{in } T \\ \operatorname{div}_y w^k &= 0 \quad \text{in } T, \\ w^k &= v^k \quad \text{on } \partial T, \\ \mathcal{M}_T(w^k) &= 0. \end{cases}$$

Arguing as in [21], it can be shown that up to a subsequence, convergence (3.36) holds. Therefore we have,

$$\begin{aligned} \varepsilon \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \operatorname{div}(v^{k\varepsilon} w_{\varepsilon\delta_2}\varphi)(x)\, dx &= \int_\Omega (\varepsilon\widehat{p_\varepsilon})\, \mathcal{P}(v^{k\varepsilon})\, \nabla(w_{\varepsilon\delta_2}\varphi)\, dx \\ &= \int_\Omega (\varepsilon\widehat{p_\varepsilon}) \mathcal{P}(v^{k\varepsilon})\, w_{\varepsilon\delta_2}\, \nabla\varphi\, dx + \int_\Omega (\varepsilon\widehat{p_\varepsilon})\, \mathcal{P}(v^{k\varepsilon})\, \nabla w_{\varepsilon\delta_2}\, \varphi\, dx. \end{aligned}$$

Unfolding by $\mathcal{T}_{\varepsilon\delta_2}$ the last integral gives

$$\begin{aligned} &\int_\Omega (\varepsilon\widehat{p_\varepsilon})\, \mathcal{P}(v^{k\varepsilon})\, \nabla(w_{\varepsilon\delta_2}\varphi)\, dx \\ &= \int_\Omega (\varepsilon\widehat{p_\varepsilon})\, \mathcal{P}(v^{k\varepsilon})\, w_{\varepsilon\delta_2}\, \nabla(\varphi)\, dx + \delta_2^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta_2}(\varepsilon\widehat{p_\varepsilon}) \mathcal{T}_{\varepsilon\delta_2}(\widetilde{v^{k\varepsilon}}) \mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2}) \mathcal{T}_{\varepsilon\delta_2}(\varphi) dx dz \\ &= \int_\Omega (\varepsilon\widehat{p_\varepsilon}) \widetilde{v^{k\varepsilon}} w_{\varepsilon\delta_2} \nabla(\varphi) dx + \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \int_{\Omega\times\mathbb{R}^N} \delta_2^{\frac{N}{2}} \mathcal{T}_{\varepsilon\delta_2}(\varepsilon\widehat{p_\varepsilon}) \mathcal{T}_{\varepsilon\delta_2}(\widetilde{v^{k\varepsilon}})(-\nabla_z v) \mathcal{T}_{\varepsilon\delta_2}(\varphi) dx dz. \end{aligned}$$

Clearly, the last integral goes to 0 as $\varepsilon \to 0$. Consequently,

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_\varepsilon \operatorname{div}(v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \varphi)\, dx = v(B) \int_\Omega p\, v^k\, \nabla\varphi\, dx. \tag{3.51}$$

Using $\mathcal{T}_\varepsilon$ in the fifth integral of (3.43), we get

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon f_\varepsilon\, v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \varphi\, dx = \int_{\mathbb{R}^n\times Y} \mathcal{T}_\varepsilon(\varepsilon f_\varepsilon)\, v^k(y) \mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})\, \varphi\, dx dy,$$

which implies that

$$\lim_{\varepsilon\to 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon f_\varepsilon\, v^{k\varepsilon}\, w_{\varepsilon\delta_2}\, \varphi\, dx = v(B) \int_{\mathbb{R}^n\times Y} f\, v^k(y)\, \varphi\, dx dy. \tag{3.52}$$

Finally, the sixth term after unfolding by $\mathcal{T}^b_\varepsilon$ becomes

$$\varepsilon \int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon \, w_{\varepsilon\delta_2} \, v^{k\varepsilon} \, \varphi \, d\sigma(x) = \int_{\mathbb{R}^n \times \partial T} g(y) \, \mathcal{T}^b_\varepsilon(w_{\varepsilon\delta_2}) \, v^k(y) \, \varphi(x) \, dx d\sigma(y),$$

so that, passing to the limit yields

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon \, w_{\varepsilon\delta_2} \, v^{k\varepsilon} \, \varphi(x) \, d\sigma(x) = v(B) \int_{\mathbb{R}^n \times \partial T} g(y) \, v^k(y) \, \varphi \, dx d\sigma(y). \tag{3.53}$$

Let now pass to the limit in (3.43). To do so, we use convergences (3.47)-(3.49), (3.51)-(3.53) and assumption on f_ε. We have to distinguish the three cases mentioned above.

Case $\gamma = -1$. We obtain the equation,

$$\begin{aligned} v(B) \int_\Omega u_k \, \varphi(x) \, dx - k_2 \int_{\Omega \times (\mathbb{R}^N \setminus B)} & \nabla_z U(x,z) \, \nabla_z v(z) \, v^k(y) \, varphi \, dxdz \\ + \alpha k_1 \, v(B) \int_{\Omega \times \partial T} u \, v^k(z) \, \varphi \, dx d\sigma(z) & + v(B) \int_\Omega p \, v^k(y) \, \mathrm{div}(\varphi) \, dx \\ = v(B) \int_{\Omega \times Y} f \, v^k(y) \, \varphi \, dxdy & + v(B) \int_{\Omega \times \partial T} g(y) \, v^k(y\,) \varphi \, dx d\sigma(y), \end{aligned} \tag{3.54}$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Integrating by parts in second term, it is easily seen that

$$\begin{aligned} k_2 v^k(y) \int_{\Omega \times (\mathbb{R}^N \setminus B)} & \nabla_z U(x,z) \, \nabla_z v(z) \, \varphi \, dxdz \\ & = k_2 v(B) v^k(y) \int_{\Omega \times \partial B} \nabla_z U \cdot \nu_B \, \varphi \, dx d\sigma(z) \\ & = k_2 v(B) \int_\Omega \left\{ \int_{\partial B} \nabla_z U \cdot \nu_B d\sigma(z) \right\} v^k(y) \, \varphi \, dx. \end{aligned} \tag{3.55}$$

On the other hand, from (3.32) by Green's formula, we have

$$\int_{\partial B} \nabla_z U \cdot \nu_B \, d\sigma = \int_{\partial B} \nabla_z (U - k_2 u) \cdot \nu_B \, d\sigma = -k_2 \, u \int_{\partial B} \nabla_z \chi \cdot \nu_B \, d\sigma. \tag{3.56}$$

Passing to the limit in (3.55), using (3.56) and unfolding with $\mathcal{T}_\varepsilon$, gives

$$\begin{aligned} k_2 \, v^k(y) \int_{\Omega \times (\mathbb{R}^N \setminus B)} & \nabla_z U(x,z) \, \nabla_z v(z) \, \varphi(x) \, dxdz \\ & = -k_2^2 \, v(B) |Y| \mathcal{M}_{Y^*}(v) \int_\Omega \mathit{\Theta}(x) \, u(x) \, \varphi(x) \, dx, \end{aligned} \tag{3.57}$$

where $\mathit{\Theta}$ is defined by (3.33). Then equation (3.37) is simply obtained by replacing (3.57) into (3.54).

Case $\gamma > -1$. We now have at the limit

$$\begin{aligned} \nu v(B) \int_\Omega u_k \varphi(x) dx + k_2 \int_{\Omega \times (\mathbb{R}^N \setminus B)} & \nabla_z U(x,z) \nabla_z v(z) v^k(y) \varphi(x) dxdz \\ & + v(B) \int_\Omega p v^k(y) \nabla \varphi dx \\ = v(B) \int_{\Omega \times Y} f v^k(y) \varphi dxdy + v(B) & \int_{\Omega \times \partial T} v^k(y) g(y) \varphi(x) dx d\sigma(y), \end{aligned}$$

from which (3.38) follows by using (3.55).

Case $\gamma < -1$. Multiply (3.43) by $\varepsilon^{-1-\gamma}$ to get

$$\begin{aligned}\varepsilon^{-1-\gamma} I^1_\varepsilon + \nu\varepsilon \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} \nabla w_{\varepsilon\delta_2}(v^{k\varepsilon}\varphi)dx + \alpha \int_{\partial T_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2} v^{k\varepsilon}\varphi w_{\varepsilon\delta_2} d\sigma(x) \\ - \varepsilon^{-1-\gamma} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon p_\varepsilon div(v^{k\varepsilon}\varphi w_{\varepsilon\delta_2})dx - \varepsilon^{-1-\gamma} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \varepsilon f_\varepsilon v^{k\varepsilon} w_{\varepsilon\delta_2}\varphi dx \\ - \varepsilon^{-1-\gamma}\varepsilon \int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon v^{k\varepsilon} w_{\varepsilon\delta_2}\varphi d\sigma(x) = 0, \quad \forall\varphi \in \mathcal{D}(\Omega).\end{aligned}$$

Passing to the limit we observe that all the integrals vanish, except for the second integral by using (3.49), which gives

$$\alpha k_1 v(B) \int_{\mathbb{R}^n \times \partial T} u v^k \varphi(x) dx d\sigma(z) = 0,$$

and so $u = 0$. This ends the proof of Theorem 3.2. ☐

Remark 3.4. *The case $\delta_1 = 1$ and $k_2 = 0$, corresponds to the classical homogenization. The results above in this case are those from Capatina and Ene [12].*

In the cases listed below, we have various variants of the Darcy law for $\gamma \geq -1$.

1. *If $k_1 = 0$ and $k_2 = 0$, then we retrieve the result from [55],*

$$\begin{aligned} &u = 0 && \text{if } \gamma < -1, \\ &u_k = \frac{1}{\nu}\mathcal{M}_{Y^*}(v^k)(f - \nabla p) + \frac{|\partial T|}{\nu}\mathcal{M}_{\partial T}(g v^k) && \text{if } \gamma \geq -1. \end{aligned}$$

2. *If $k_1 = 0$ and $k_2 \neq 0$,*

$$\begin{aligned} &u = 0 && \text{if } \gamma < -1, \\ &\nu u_k + k_2^2 \mathcal{M}_{Y^*}(v^k)\Theta u = \mathcal{M}_{Y^*}(v_k)(f - \nabla p) + |\partial T|\mathcal{M}_{\partial T}(g v^k) && \text{if } \gamma \geq -1. \end{aligned}$$

3. *If $k_1 \neq 0$ and $k_2 = 0$,*

$$\begin{aligned} &u = 0 && \text{if } \gamma < -1, \\ &\nu u_k + \alpha k_1 |\partial T| \mathcal{M}_{\partial T}(v^k) u = \mathcal{M}_{Y^*}(v_k)(f - \nabla p) + |\partial T|\mathcal{M}_{\partial T}(g v^k) && \text{if } \gamma = -1, \\ &u_k = \frac{1}{\nu}\mathcal{M}_{Y^*}(v^k)(f - \nabla p) + \frac{|\partial T|}{\nu}\mathcal{M}_{\partial T}(g v^k) && \text{if } \gamma > -1. \end{aligned}$$

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Chapter 4

Asymptotic Behavior of the Stokes Problem in a Domain with Small Holes under Non-Homogeneous Slip Boundary Conditions

Abstract We consider the Stokes problem in a domain of $\mathbb{R}^{\mathbb{N}}, N \geq 3$, ε-periodically perforated by holes of size $r(\varepsilon)$ with $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. A Robin-type condition depending on a parameter γ is prescribed on the boundary of some holes while on the boundary of the others as well as on the external boundary of the domain, a Dirichlet condition is imposed. The aim is to give the asymptotic behavior of the velocity and the pressure of the fluid as ε goes to zero. We use the periodic unfolding method introduced by Cioranescu, Damlamian and Griso in [16] which allows to consider a general geometric framework. and so, to generalize the results of R. Zaki [55]. According to the values of γ, we obtain at the limit a Darcy law, a Brinkmann or Stokes type equations.

4.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N,\ N > 2$ with $|\partial\Omega| = 0$. To define the perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$, we introduce the following sets: Let $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N$ and T and B are compacts strictly included in Y with Lipschitz boundary such that $\overline{T} \cap \overline{B} = \emptyset$. We assume that the holes, B and T have Lipschitz boundary.

We denote by ε, δ_1 and δ_2 three small parameters.

Define the following sets:

$$\begin{aligned} B_{\varepsilon\delta_1} &= \text{interior}\Big(\bigcup_{\xi\in\Pi_\varepsilon}(\varepsilon\delta_1\overline{B}+\varepsilon\xi)\Big), & T_{\varepsilon\delta_2} &= \bigcup_{\xi\in\Pi_\varepsilon}(\varepsilon\delta_2 T+\varepsilon\xi), \\ B^{int}_{\varepsilon\delta_1} &= \bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\delta_1 B+\varepsilon\xi), & T^{int}_{\varepsilon\delta_2} &= \bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\delta_2 T+\varepsilon\xi), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \Pi_\varepsilon &= \{\xi\in\mathbb{Z}^N \mid \varepsilon(\xi+Y)\cap\Omega\neq\emptyset\} \\ \Xi_\varepsilon &= \{\xi\in\mathbb{Z}^N \mid \varepsilon(\xi+Y)\subset\Omega\}, \qquad \widetilde{\Omega}_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Pi_\varepsilon}(\varepsilon\overline{Y}+\varepsilon\xi)\Big). \end{aligned}$$

such that $\forall\ (\xi_1,\xi_2)\in\Pi_\varepsilon^2,\ \xi_1\neq\xi_2,\Rightarrow(\xi_1+Y)\cap(\xi_2+Y)=\emptyset$.

Then, we set $\Omega_{\varepsilon\delta_1\delta_2}=\Omega\setminus(B_{\varepsilon\delta_1}\cup T_{\varepsilon\delta_2})$.

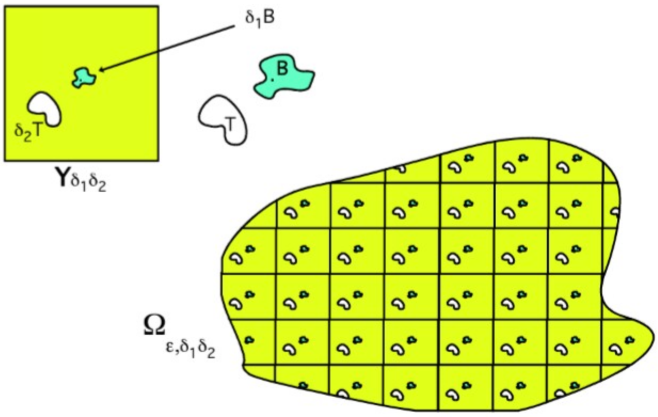

Fig. 5. the domain $\Omega_{\varepsilon\delta_1\delta_2}$

The main feature of the perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ is that the size $r(\varepsilon)=\varepsilon\delta_1,\varepsilon\delta_2,\ 0<\delta_1,\delta_2<1$ of the holes is not proportional to the size of the reference cell εY and $r(\varepsilon)/\varepsilon$ tends to zero as ε is taking values in a decreasing sequence of positive numbers which tends to zero. Let us consider the following problem:

$$(\mathcal{P}_{\varepsilon\delta_1\delta_2})\begin{cases} -\nu\Delta u_{\varepsilon\delta_1\delta_2}+\nabla p_{\varepsilon\delta_1\delta_2}=f_\varepsilon \text{ in } \Omega_{\varepsilon\delta_1\delta_2} \\ \text{div } u_{\varepsilon\delta_1\delta_2}=0 \text{ on } \Omega_{\varepsilon\delta_1\delta_2} \\ -p_{\varepsilon\delta_1\delta_2}.n+\nu\dfrac{\partial u_{\varepsilon\delta_1\delta_2}}{\partial n}+\alpha\varepsilon^\gamma u_{\varepsilon\delta_1\delta_2}=G_\varepsilon \text{ on } \partial B^{int}_{\varepsilon\delta_1} \\ u_{\varepsilon\delta_1\delta_2}=0 \text{ on } \partial\Omega_{\varepsilon\delta_1\delta_2}\cup\partial T^{int}_{\varepsilon\delta_2} \end{cases} \tag{4.2}$$

These problems modelize the flow of an incompressible viscous fluid through a porous medium under the action of an exterior electric field.

In [49] Ene and Sanchez-Palencia study the Stokes flow in a periodic porous medium with Dirichlet condition on the boundary of the holes. The limit law describing the homogenized medium is a Darcy's law. In [32], Cioranescu and Donato consider the Laplace equation with non-homogeneous Robin conditions on the boundary of the holes containing a term of type $\alpha\varepsilon^{\gamma}u_{\varepsilon\delta_1\delta_2}$. According to the values of γ, several a priori estimates are obtained. They lead to different limit laws.

In the first cas we present here (i.e the Stokes system with non-homogeneous Fourier boundary conditions) we obtain at the limit, according to the values of γ, a Darcy's law ($\gamma < 2$), a Brinkmann-type equation ($\gamma = 2$) or a Stokes-type equation ($\gamma > 2$). This phenomenon was already observed by Conca [27] when studying the two-dimensional Stokes equation with homogeneous Fourier boundary conditions. It was also noticed by Allaire [4] and [5], who considered the Stokes equation in a perforated domain with holes of size r_ε with $r_\varepsilon \ll \varepsilon$. The boundary conditions on the holes are of Dirichlet type in [4] and of slip type in [5]. In these situations it is the geometry of the domain, more precisely the size r_ε which determines the type of the limit law. Similar results have been obtained by Brillard [10] by using Γ-convergence methods.

Here, our aim is to study the asymptotic behaviour of problem 4.2 using the periodic unfolding method due to Cioranescu, Damlamian and Griso (see [16]) and Cioranescu et al. (see [19]). This model was studied in 1996 in the case of the classical homogenization ($\delta_1 = 1$ and $T_{\varepsilon\delta_2} = \phi$ by Cioranescu, Donato and Ene [21] by the energy method and again in 2012 using the periodic unfolding method by Zaki [55]. But in our case, the $B_{\varepsilon\delta_1}$ is formed by "small" holes and the set $T_{\varepsilon\delta_2}$ is non empty.

Let us now turn back to problem $(\mathcal{P}_{\varepsilon\delta_1\delta_2})$. Suppose now that ε, $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ are such that there exist two constants k_1 and k_2

$$k_1 = \lim_{\varepsilon\to 0}\frac{\delta_1^{N-1}}{\varepsilon} \in [0,+\infty[\quad \text{and } k_2 = \lim_{\varepsilon\to 0}\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \in [0,+\infty[. \tag{4.3}$$

Under this assumption, and following the values of γ we show that the solution of (4.2) converges in an approprite space to u which is characterized as the solution of a limit problem (see Theorems 4.2, 4.3 and 4.4). These theorems show the presence of four types of contributions in the limit systems.

The term $k_2^2\Theta$ from equations (4.51) and (4.52) corresponds to the "strange term" introduced for the Dirichlet problem in Cioranescu and Murat [25] and represents the contribution of the Dirichlet holes $T_{\varepsilon\delta_2}$.

Assuming this condition, we show that the solution of $(\mathcal{P}_{\varepsilon\delta_1\delta_2})$ converges in appropriate spaces to u characterized as solution of some limit problem as follows. The paper is organized as follows. In Section 2, we list some notations and define the different perforated domains. Afterwards, we recall the different definitions of the periodic unfolding operators and their properties. Section 3 is devoted to the periodic unfolding operator for fixed domains, introduced in [16]. And we recalls the definition and the properties of the unfolding operator for perforated domains with holes of size ε and Subection 3.4 those of the unfolding operator corresponding to the case of volume-distributed small holes. In Subsection 3.5 we introduce the boundary operator related to the case of small holes, and prove some of its properties.

Finally, the homogenization result concerning problem $(\mathcal{P}_{\varepsilon\delta_1\delta_2})$ is proved in Section 4.

4.2 Setting of the problem and variational formulation

As in [16], for every $z \in \mathbb{R}^N$, we denote by $[z]$ the unique integer combination of periods such that $\{z\} = z - [z]$ is in Y. Due to the periodicity,

$$x = \varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big] + \Big\{\frac{x}{\varepsilon}\Big\}\Big) \quad \text{for a.e. } x \in \mathbb{R}^N$$

Introduce now the following sets:

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon \tag{4.4}$$

By this definition, the set $\widehat{\Omega}_\varepsilon$ is the largest finite union of εY cells contained in Ω while Λ_ε is the subset of Ω containing parts of the εY cells intersecting the boundary $\partial\Omega$.

Consider the system (4.2), where, $u_{\varepsilon\delta_1\delta_2}$ is the velocity field, $p_{\varepsilon\delta_1\delta_2}$ is the pressure, f_ε is the field of exterior body force, G_ε is the field of exterior surface force. Recall that $\alpha \geq 0$ and γ are constant, and assume that the data f_ε and G_ε satisfy

(i) $f_\varepsilon \in L^2(\Omega_{\varepsilon\delta_1\delta_2})^N$

(ii) $G_\varepsilon = g_0 + g_\varepsilon$, $g_\varepsilon = g\Big(\dfrac{\cdot}{\varepsilon\delta_1}\Big)$, where $g_0 \in L^2(\Omega)^N$ and $g = (g_1, ..., g_N) \in H^{-1/2}(\partial B)^N$ is Y-periodic. If $g_0 = 0$, we assume that there exists $i \in \{1, \dots, N\}$ such that $\langle g_i, 1\rangle_{H^{-1/2}(\partial B), H^{1/2}(\partial B)} \neq 0$.

Let us introduce the following functional spaces:

$$\begin{aligned}
\mathbf{H}_{\varepsilon\delta_1\delta_2} &= \left\{v \mid v \in H^1(\Omega_{\varepsilon\delta_1\delta_2})^N, \;\; v = 0 \text{ on } \partial\Omega \cup \partial T_{\varepsilon\delta_2}\right\}, \\
\mathbf{V}_{\varepsilon\delta_1\delta_2} &= \{v \mid v \in \mathbf{H}_{\varepsilon\delta_1\delta_2}, \;\; \text{div} v = 0 \text{ in } \Omega_{\varepsilon\delta_1\delta_2}\}
\end{aligned}$$

Which is Hilbert for scalar product

$$\langle \varphi, \psi\rangle = \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla\varphi : \nabla\psi \, dx = \sum_{i,j=1}^{N} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \frac{\partial \varphi_i}{\partial x_j}\frac{\partial \psi_i}{\partial x_j} dx$$

and

$$K_T = \{\Phi \in L^{2^*}(\mathbb{R}^N) \mid \nabla\Phi \in L^2(\mathbb{R}^N), \Phi \text{ is constant on } T\}. \tag{4.5}$$

The variational formulation of system (4.2) is

$$\begin{cases} \text{Find } u_{\varepsilon\delta_1\delta_2} \in \mathbf{V}_{\varepsilon\delta_1\delta_2},\ p_{\varepsilon\delta_1\delta_2} \in L^2(\Omega_{\varepsilon\delta_1\delta_2}) \text{ satisfying} \\ \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} : \nabla\varphi\, dx + \alpha\varepsilon^{\gamma} \int_{\partial B_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2} \cdot \varphi\, d\sigma_{\varepsilon\delta_1}(x) - \int_{\Omega_{\varepsilon\delta_1\delta_2}} p_{\varepsilon\delta_1\delta_2} \operatorname{div}\varphi\, dx \\ \displaystyle = \int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon \cdot \varphi\, dx + \int_{\partial B_{\varepsilon\delta_1}} g_0 \cdot \varphi\, d\sigma_{\varepsilon\delta_1}(x) + \langle g_\varepsilon, \varphi\rangle_{H^{-1/2}(\partial B_{\varepsilon\delta_1})^N, H^{1/2}(\partial B_{\varepsilon\delta_1})^N} \\ \forall\varphi \in \mathbf{H}_{\varepsilon\delta_1\delta_2}. \end{cases} \tag{4.6}$$

Lax-Milligram theorem gives the existence of a unique solution of problem (4.6). Note that this solution is defined in the highly oscillating (with ε) domain $\Omega_{\varepsilon\delta_1\delta_2}$. Our aim is to give the asymptotic behavior of $(u_{\varepsilon\delta_1\delta_2}, p_{\varepsilon\delta_1\delta_2})$ as $\varepsilon \longrightarrow 0$. To do so, we apply the periodic unfolding method.

4.3 Some recalls on the periodic unfolding operators

4.3.1 Notations

$\widehat{\Omega}_\varepsilon = \text{interior}\big\{ \bigcup\limits_{\xi\in\Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big)\big\}$, and set $\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon$.

Denote $Y_\delta = Y \setminus \delta\overline{B}$. If δ is small enough this open set is connected.

In the sequel, $\widetilde{\varphi}$ denotes the extension by 0 outside $\Omega_{\varepsilon\delta}$ for any function φ in $L^p(\Omega_{\varepsilon\delta})$.

For every z in $\mathbb{R}^N$, we denote by $[z]$ the vector whose coordinates are the integer parts of the corresponding coordinates of z.

4.3.2 The unfolding operator $\mathcal{T}_\varepsilon$ for fixed domains

Below we recall the definition and some properties of a variante of the periodic unfolding operator (see [27, 46]).

definition 4.1. *For $\phi \in L^p(\Omega)$, $p \in [1, +\infty]$, the unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y_\delta)$ is defined as follows:*

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon y\Big) & a.e.\ \ for\ (x,y) \in \widehat{\Omega}_\varepsilon \times Y_\delta, \\ 0 & a.e.\ \ for\ (x,y) \in\ \Lambda_\varepsilon \times Y_\delta. \end{cases}$$

Proposition 4.1. *If* $\{w_\varepsilon\}$ *is a sequence in* $L^1(\Omega)$ *satisfying* $\displaystyle\int_{\Lambda_\varepsilon} |w_\varepsilon|\,dx \to 0$. *Then*

$$\int_{\Omega_\varepsilon} w_\varepsilon dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y_\delta} \mathcal{T}_\varepsilon(w_\varepsilon)\,dxdy.$$

Proposition 4.2. *Let* w_ε *in* $H^1(\Omega_\varepsilon)$ *for every* ε, *with* $\|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ *bounded. Then, there exist* w *in* $H^1(\Omega)$ *and* $\widehat{w} \in L^2(\Omega; H^1_{per}(Y))$ *such that, up to a subsequence*

$$\begin{aligned} &\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w \quad \text{weakly in } L^2(\Omega; H^1_{loc}(Y)),\\ &\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^2(\Omega; L^2_{loc}(Y))^N. \end{aligned} \tag{4.7}$$

We note that if we replace Y_δ by Y, we retrieve $\mathcal{T}_\varepsilon$ as defined in [16], and the related convergences in proposition4.2 become global with respect to the variable y.

definition 4.2. *The local average* $M^\varepsilon_Y : L^p(\Omega) \mapsto L^p(\Omega)$, *is defined for any* ϕ *in* $L^p(\Omega)$, $1 \leq p < \infty$, *by*

$$M^\varepsilon_Y(\phi) = \int_Y \mathcal{T}_\varepsilon(\phi)(\cdot, y)\,dy.$$

It is classical that, if $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$, such that $v_\varepsilon \to v$ strongly in $L^p(\Omega)$, then

$$M^\varepsilon_Y(v_\varepsilon) \to v \quad \text{strongly in } L^p(\Omega). \tag{4.8}$$

4.3.3 The unfolding operator $\mathcal{T}^*_\varepsilon$ for perforated domains

Following [20], an unfolding operator $\mathcal{T}^*_\varepsilon$ for functions defined on perforated domains Ω_ε with holes of size ε. Let first define the geometric settings for this specific case. With the notations of Section 2.1,

$$\begin{cases} \widehat{\Omega}_\varepsilon \doteq \widehat{\Omega}_\varepsilon \setminus \overline{B}_\varepsilon,\\ \Lambda_\varepsilon \doteq \Omega_\varepsilon \setminus \widehat{\Omega}_\varepsilon. \end{cases} \tag{4.9}$$

The part occupied by the material in the cell Y, supposed to be connected, is now denoted $Y^* = Y \setminus \overline{B}$. So, perforated domain Ω_ε is obtained by removing from Ω the set of holes B_ε,

$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon = \Big\{x \in \Omega \;\Big|\; \Big\{\frac{x}{\varepsilon}\Big\} \in Y^*\Big\}. \tag{4.10}$$

definition 4.3. *For* ϕ *Lebesgue-measurable on* Ω_ε, *the linear and continuous unfolding operator* $\mathcal{T}^*_\varepsilon : L^p(\Omega_\varepsilon) \mapsto L^p(\Omega \times Y^*)$, $p \in [1, +\infty[$ *is defined by*

$$\mathcal{T}^*_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon y\Big) & \text{for a.e. } \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*,\\ 0 & \text{for a.e. } \quad (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

The main characteristic of this operator is that it maps functions defined on the oscillating domain Ω_ε, into functions defined on the fixed domain $\Omega \times Y^*$.

It is easily seen that

$$\mathcal{T}_\varepsilon^*(w) = \mathcal{T}_\varepsilon(\widetilde{w})\big|_{\Omega\times Y^*}. \tag{4.11}$$

Consequently, the most of properties of $\mathcal{T}_\varepsilon^*$ follow from those of $\mathcal{T}_\varepsilon$. In particular, the integration formula written for every ϕ in $L^1(\Omega_\varepsilon)$

$$\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)\, dx\, dy = \int_{\Omega_\varepsilon} \phi(x)\, dx - \int_{\Lambda_\varepsilon} \phi(x)\, dx = \int_{\widehat{\Omega}_\varepsilon} \phi(x)\, dx.$$

There fore, an equivalent of Proposition 4.1 holds, allowing to pass from integrals over Ω_ε to integrals over the fixed domain $\Omega \times Y^*$.

Proposition 4.3. *If $\{\phi_\varepsilon\}$ is a sequence in $L^1(\Omega_\varepsilon)$ satisfying $\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon|\, dx \to 0$, then*

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon\, dx \overset{\mathcal{T}_\varepsilon^*}{\simeq} \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)\, dxdy$$

Proposition 4.4. *Let $\{w_\varepsilon\}$ a sequence in $L^p(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$, then $\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w$ strongly in $L^p(\Omega \times Y^*)$.*

noindent Now, we add an assumption on Y^*. For each $i \in \{1,\dots,N\}$, the interior of $\overline{Y^*} \cup \overline{(\mathbf{e}_i + Y^*)}$ is connected, where $(\mathbf{e}_1,\dots,\mathbf{e}_N)$ is the usual basis of $\mathbb{R}^N$.

Theoreme 4.1 (Theorem 2.12 in [20]). *Assume p in $(1,+\infty)$. Let $\{w_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega_\varepsilon^*)$ satisfying*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C.$$

Then, there exist a subsequence (still denoted ε), w in $W^{1,p}(\Omega)$ and $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(Y^*))$, such that

$$\begin{cases} \mathcal{T}_\varepsilon^*(w_\varepsilon) \to w \quad \text{strongly in } L^p(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*)^N. \end{cases} \tag{4.12}$$

4.3.4 The unfolding operator $\mathcal{T}_{\varepsilon\delta}$ depending on two parameters ε and δ

We start by recalling the definition and the main property of the periodic unfolding operator with two parameters δ and ε introduced in [19].

According to [19], the geometry of the domains with small holes requires a specific unfolding operator depending on both parameters ε and δ. So, we will consider functions which vanish on the whole boundary of the perforated domain $\Omega_{\varepsilon\delta}$, namely belonging to the space $H_0^1(\Omega_{\varepsilon\delta})$. These functions are naturally extended by zero to the whole of Ω and these extensions belong to $H_0^1(\Omega)$. Consequently, from now on, we will not distinguish elements of $H_0^1(\Omega_{\varepsilon\delta})$and their extensions in $H_0^1(\Omega)$.

definition 4.4. *For $\phi \in L^p(\Omega)$, $p \in [1,+\infty[$, the linear and continuous unfolding operator $\mathcal{T}_{\varepsilon\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ is defined by*

$$\mathcal{T}_{\varepsilon\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x,\delta z) & \text{for a.e.} \quad (x,z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases}$$

For $u \in L^2(\Omega)$, from definition 4.4 the estimates

$$\|\mathcal{T}_{\varepsilon\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leq \frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)} \tag{4.13}$$

Proposition 4.5. *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying $\displaystyle\int_{\Lambda_\varepsilon} |w_\varepsilon|\, dx \to 0$, then*

$$\int_\Omega w_\varepsilon\, dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(w_\varepsilon)\, dxdz \to 0.$$

Proposition 4.6. *Suppose $N \geq 3$ and denote by 2^* the Sobolev exponent $\dfrac{2N}{N-2}$ associated to 2. Let ω be open and bounded in $\mathbb{R}^N$. Then the following estimates hold:*

$$\|\nabla_z\big(\mathcal{T}_{\varepsilon\delta}(u)\big)\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{4.14}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u - M^\varepsilon_Y(u)\big)\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \leq \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{4.15}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u)\|^2_{L^2(\Omega\times\omega)} \leq \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{\frac{2}{N}}\|\nabla u\|^2_{L^2(\Omega)} + 2|\,\omega\,|\ \|u\|^2_{L^2(\Omega)}, \tag{4.16}$$

where C denotes the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$ and M^ε_Y is actually given by Definition 4.2.

From now on, we assume that $0 < \displaystyle\lim_{\varepsilon\to 0} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < \infty$,

definition 4.5. *The local averge $M^{\varepsilon\delta}_Y : \quad L^P(\Omega) \longrightarrow L^P(\Omega)$ is difined for any ϕ in $L^p(\Omega)$, $\quad 1 \leq p < \infty$ by*

$$M^{\varepsilon\delta}_Y(\phi)(x) = \delta^N \int_{\frac{1}{\delta}Y} \mathcal{T}_{\varepsilon\delta}(\phi)(x,z)dz$$

4.3.5 The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$

definition 4.6. *For ϕ in $L^p(\partial B_{\varepsilon\delta})$, $p \in [1,+\infty[$, the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta} : L^p(\partial B_{\varepsilon\delta}) \mapsto L^p(\mathbb{R}^N \times \partial B)$ is defined by*

$$\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\delta z\Big), \quad x \in \mathbb{R}^N,\ z \in \partial B. \tag{4.17}$$

The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$ is linear and continuous, it satisfies

$$\int_{\partial B_{\varepsilon\delta}} \phi(x)\, d\sigma(x) = \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)\, dxd\sigma(z), \qquad \forall\phi \in L^1(\partial B_{\varepsilon\delta}). \tag{4.18}$$

Proposition 4.7. *Let φ^ε in $W^{1,p}(\Omega_{\varepsilon\delta})$ for every ε and $\varphi \in W^{1,p}(Y))$ such that*

$$\mathcal{T}_{\varepsilon\delta}(\varphi_\varepsilon) \rightharpoonup \varphi \quad \textit{weakly in } L^p(\Omega; W^{1,p}(Y)),$$

then

$$\mathcal{T}^b_{\varepsilon\delta}(\varphi_\varepsilon) \rightharpoonup \varphi \quad \textit{weakly in } L^p(\Omega; W^{1-\frac{1}{p},p}(\partial T)),$$

For every domain K included in Y with Lipschitz boundary and for every $\varphi \in L^1(\partial K)$, we denote $\mathcal{M}_{\partial K}(\varphi) = \dfrac{1}{|\partial K|}\displaystyle\int_{\partial K} \varphi\, d\sigma_y$ the average of φ over ∂K.

Proposition 4.8 ([47]Proposition 18)**.** *Let g be in $L^2(\partial B)$. Set*

$$g_\varepsilon(x) = g\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}\Big) \quad \textit{for a.e. } x \in \mathbb{R}^N \setminus \bigcup_{\xi\in\mathbb{Z}^N} \varepsilon(\xi + \delta B). \tag{4.19}$$

Then, for every $\phi \in H^1(\Omega)$, one has

$$\Big|\int_{\partial B_{\varepsilon\delta}} g_\varepsilon(x)\,\phi(x)\; d\sigma_{\varepsilon\delta}(x)\Big| \le C\frac{\delta^{N-1}}{\varepsilon}\Big(\varepsilon\delta + |\mathcal{M}_{\partial B}(g)|\Big)\|\nabla\phi\|_{L^2(\Omega)^N}. \tag{4.20}$$

Proposition 4.9. *Let g be in $L^2(\partial B)$ and v be in $H^1(\Omega)$, one has*

$$\Big|\int_{\partial B_{\varepsilon\delta}} g(x)v(x)d\sigma(x)\Big| \le c\Big(\delta^{\frac{N}{2}}\|\nabla v\|_{L^2(\Omega)} + \frac{\delta^{N-1}}{\varepsilon}\|v\|_{L^2(\Omega)}\Big)$$

Proof. Transforming by unfolding with $\mathcal{T}^b_{\varepsilon\delta}$ gives

$$\left|\int_{\partial B} g(x)v(x)d\sigma(x)\right| = \left|\frac{\delta^{N-1}}{\varepsilon}\int_{\Omega\times\partial B} g(y)\mathcal{T}^b_{\varepsilon\delta}(v)dxd\sigma(z)\right|$$

$$\le \frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial B} g(z)\Big(\mathcal{T}^b_{\varepsilon\delta}(v) - M^{\varepsilon\delta}_Y(v)\Big)dxd\sigma(z)\right| + \frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial B} g(y)M^{\varepsilon\delta}_Y(v)dxd\sigma(z)\right|$$

We estimate now the two integral termes on the right-hand side of this inequality, indeed:

Using Proposition 4.6, the first integral in the right hand side becomes:

$$\frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial B} g(z)\Big(\mathcal{T}^b_{\varepsilon\delta}(v) - M^{\varepsilon\delta}_Y(v)\Big)dxd\sigma(z)\right| \le c\delta^{\frac{N}{2}}\|\nabla v\|_{L^2(\Omega)} \tag{4.21}$$

On the other hand we have

$$\frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial B} g(z)M^{\varepsilon\delta}_Y dxd\sigma(z)\right| \le c\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^2(\partial B)}\cdot\left(\int_{\Omega\times\partial B}\Big|M^{\varepsilon\delta}_Y(v)\Big|^2 dxd\sigma(z)\right)^{\frac{1}{2}}$$

and by Fibini theorem we obtain

$$\frac{\delta^{N-1}}{\varepsilon}\left|\int_{\Omega\times\partial B} g(z)M^{\varepsilon\delta}_Y dxd\sigma(z)\right| \le c\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^2(\partial B)}|\partial B|^{\frac{1}{2}}\left(\int_{\Omega}\Big|M^{\varepsilon\delta}_Y(v)\Big|^2 dx\right)^{\frac{1}{2}}$$

But, by Cauchy Schwartz inequality, using definition 4.5 and (4.13), one has:

$$\Big(\int_\Omega \Big|M_Y^{\varepsilon\delta}(v)\Big|^2 dx\Big)^{\frac{1}{2}} = \delta^N\Big(\int_\Omega .\Big(\int_{\frac{1}{\delta}Y} \mathcal{T}_{\varepsilon\delta}(v)dz\Big)^2 dx\Big)^{\frac{1}{2}},$$
$$\leq \delta^{\frac{N}{2}}\|\mathcal{T}_{\varepsilon\delta}(v)\|_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \|v\|_{L^2(\Omega)}.$$

Finally, we obtain:

$$\frac{\delta^{N-1}}{\varepsilon}\Big|\int_{\Omega\times\partial B} g(z)M_Y^{\varepsilon\delta}dxd\sigma(z)\Big| \leq c\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^2(\partial B)}.\|v\|_{L^2(\Omega)}$$

This estimate, together with (4.21), gives the result . □

4.4 Main results

Assume that the data $f_\varepsilon \in L^2(\Omega_{\varepsilon\delta_1\delta_2})^N$ and satisfies the additional property

$$\widetilde{f_\varepsilon} \rightharpoonup f \text{ weakly in } L^2(\Omega)^N \tag{4.22}$$

From now on, we assume that ε, δ_1 and δ_2 satisfy (4.3).

Theoreme 4.2 (Darcy type law). *Let* $\big(u_{\varepsilon\delta_1\delta_2}, p_{\varepsilon\delta_1\delta_2}\big)$ *be the solution of* (4.6). *For* $\gamma < 0$ *the sequence* $\big\{\varepsilon^\gamma\widetilde{u_{\varepsilon\delta_1\delta_2}}, \varepsilon^{-\gamma}\widetilde{p_{\varepsilon\delta_1\delta_2}}\big\}_\varepsilon$ *is uniformly bounded in* $L^2(\Omega)^N \times L^2(\Omega)$. *So, up to a subsequence, there exist* $(u,p) \in L^2(\Omega)^N \times L^2(\Omega)$ *such that:*

$$\varepsilon^\gamma\widetilde{u_{\varepsilon\delta_1\delta_2}} \rightharpoonup u \quad \textit{weakly in } \ L^2(\Omega)^N,$$
$$\varepsilon^{-\gamma}\widetilde{p_{\varepsilon\delta_1\delta_2}} \rightharpoonup p \quad \textit{weakly in } \ L^2(\Omega).$$

Note $U = \mathcal{M}_{\partial B}(u)$ *then*

$$U = \frac{-1}{\alpha k_1|\partial B|}\nabla p.$$

Theoreme 4.3 ($0 \leq \gamma < 2$). *The sequence* $\big\{\varepsilon^{1+\gamma}\widetilde{u_{\varepsilon\delta_1\delta_2}}, \varepsilon\widetilde{p_{\varepsilon\delta_1\delta_2}}\big\}_\varepsilon$ *is bounded in* $L^2(\Omega)^N \times L^2(\Omega)$. *So, up to a subsequence, there exist* $(u,p)\in L^2(\Omega)^N \times L^2(\Omega)$, *such that*

$$\varepsilon^{1+\gamma}\widetilde{u_{\varepsilon\delta_1\delta_2}} \rightharpoonup u \quad \textit{weakly in } \ L^2(\Omega)^N,$$
$$\varepsilon p_{\varepsilon\delta_1\delta_2} \rightharpoonup p \quad \textit{weakly in } \ L^2(\Omega).$$

It satisfies the following equation:

$$u = \frac{1}{\alpha k_1|\partial B|}\Big(-\nabla p + |\partial B|g_0 + |\partial B|\mathcal{M}_{\partial B}(g)\Big).$$

Theoreme 4.4 (Brinkman type law). *Assume that* $\varepsilon, \delta_1, \delta_2$ *satisfy* (4.3). *Let* $(u_{\varepsilon\delta_1\delta_2}, p_{\varepsilon\delta_1\delta_2})$ *be the solution of problem* (4.6).

If $\gamma = 2$, *the sequence* $\left\{\left(\varepsilon^2\widetilde{u_{\varepsilon\delta_1\delta_2}}, p_{\varepsilon\delta_1\delta_2}\right)\right\}_\varepsilon$ *is uniformly bounded in* $L^2(\Omega)^N \times L^2(\Omega)$. *So, up to a subsequence, there exists* $(u,p) \in L^2(\Omega)^N \times L^2(\Omega)$ *such that*

$$\begin{aligned} \varepsilon^2\widetilde{u_{\varepsilon\delta_1\delta_2}} &\rightharpoonup u \quad \textit{weakly in } \ L^2(\Omega)^N, \\ \widetilde{p_{\varepsilon\delta_1\delta_2}} &\rightharpoonup p \quad \textit{weakly in } \ L^2(\Omega). \end{aligned}$$

Moreover, there exist $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))^N$, $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ *with* $U - k_2 u \in L^2(\Omega; K_T)$, *such that* $(u, \widehat{u}, U, p)$ *solves the equations:*

$$\nu \int_Y (\nabla u(x) + \nabla_y \widehat{u}(x,y)) : \nabla_y \phi(y) dy = 0 \textit{ for a.e. } x \in \Omega, \ \ \forall \phi \in H^1_{per}(Y)^N \tag{4.23}$$

$$\int_{\mathbb{R}\setminus T} \nabla_z U(x,z) : \nabla_z v(z) dz = 0 \textit{ for a.e. } x \in \Omega, \ \ \forall v \in K_T, \ \ v(T) = 0 \tag{4.24}$$

$$\begin{aligned} \nu \int_{\Omega\times Y} (\nabla u + \nabla_y \widehat{u}) \nabla \psi dx dy - \nu k_2 \int_{\Omega\times\partial T} \nabla_z U n_T \psi dx d\sigma + \alpha k_1 |\partial B| \int_\Omega u\psi dx + \int_\Omega \nabla p \psi dx \\ = \int_\Omega f\psi dx + k_1|\partial B| \int_\Omega g_0 \psi dx + k_1 |\partial B| \ M_{\partial B} \int_\Omega \psi dx \end{aligned} \tag{4.25}$$

If $\gamma > 2$, *the sequence* $\left\{\left(\widetilde{u_{\varepsilon\delta_1\delta_2}}, \widetilde{p_{\varepsilon\delta_1\delta_2}}\right)\right\}_\varepsilon$ *is uniformly bounded in* $L^2(\Omega)^N \times L^2(\Omega)$. *Then, up to a subsequence, there exist* $(u,p) \in H^1_0(\Omega)^N \times L^2(\Omega)$ *such that*

$$\begin{aligned} &\widetilde{u_{\varepsilon\delta_1\delta_2}} \rightharpoonup u \quad \textit{weakly in } \ L^2(\Omega)^N, \\ &\|\widetilde{u}_{\varepsilon\delta_1\delta_2} - u\|_{L^2(\Omega)} \longrightarrow 0, \\ &\widetilde{p_{\varepsilon\delta_1\delta_2}} \rightharpoonup p \quad \textit{weakly in } \ L^2(\Omega). \end{aligned}$$

Moreover, there exist $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))^N$, $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))^N$ *with* $U - k_2 u \in L^2(\Omega; K_T)$, *such that* $(u, \widehat{u}, U, p)$ *solves the equations* (4.23), (4.24) *and*

$$\begin{aligned} \nu \int_{\Omega\times Y} (\nabla u + \nabla_y \widehat{u}) : \nabla \psi dx dy - \nu k_2 \int_{\Omega\times\partial T} \nabla_z U n_T \psi dx d\sigma + \int_\Omega \nabla p \psi dx = \\ \int_\Omega f \cdot \psi dx + k_1 |\partial B| \int_\Omega g_0 \cdot \psi \, dx + k_1 |\partial B| \ M_{\partial B} \int_\Omega \psi dx \ \ \forall \psi \in H^1_0(\Omega) \end{aligned} \tag{4.26}$$

Lemma 4.1. *There exists a positive constant* C *independent of* ε *and* δ *such that*

$$\|v_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} \le C\Big(\varepsilon\delta \|\nabla v_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} + (\varepsilon\delta)^{\frac{1}{2}} \|v_{\varepsilon\delta}\|_{L^2(\partial B_{\varepsilon\delta})}\Big) \ \forall v_{\varepsilon\delta} \in H_{\varepsilon\delta_1\delta_2}$$

Proof. We use the following inequality from [[29], Lemma 6.1]

$$\|v\|^2_{L^2(\Omega_{\varepsilon\delta})} \le C\Big(\|\nabla_y v\|^2_{L^2(\Omega_{\varepsilon\delta})} + \|v\|^2_{L^2(\partial B)}\Big) \ \forall v_{\varepsilon\delta} \in H_{\varepsilon\delta_1\delta_2}$$

and by the change $z = \dfrac{y}{\delta}$, we get the result. □

Lemma 4.2 ([29], Lemma 5.1). *For every* $\phi \in L^2(\Omega_{\varepsilon\delta_1\delta_2})$, *there exists* $\varphi \in V_{\varepsilon\delta_1\delta_2}$ *such that*

$$\begin{cases} div\,\varphi = \phi, \\ \|\varphi\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})} \leq C\|\phi\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \end{cases}$$

4.4.1 Apriori estimates for $u_{\varepsilon\delta_1\delta_2}$ and $p_{\varepsilon\delta_1\delta_2}$

Proposition 4.10. *Let* $(u_{\varepsilon\delta_1\delta_2}, p_{\varepsilon\delta_1\delta_2})$ *be the solution of problem* (4.6). *Then the following a priori estimates hold true:*
(i) For $\gamma < 0$

$$\begin{aligned} &\|\varepsilon^{\gamma} u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|\varepsilon^{-\gamma} p_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \end{aligned} \tag{4.27}$$

(ii) For $0 \leq \gamma < 2$

$$\begin{aligned} &\|\varepsilon^{1+\gamma} u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|\varepsilon^{\frac{\gamma}{2}} \nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|\varepsilon p_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \end{aligned} \tag{4.28}$$

(iii) For $\gamma = 2$

$$\begin{aligned} &\|\varepsilon^{2} u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|p_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \end{aligned} \tag{4.29}$$

(iv) For $\gamma > 2$

$$\begin{aligned} &\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \\ &\|p_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C \end{aligned} \tag{4.30}$$

Proof. To prove this proposition, we combine the estimates given in Proposition 4.9. Using $u_{\varepsilon\delta_1\delta_2}$ as test fonction in (4.6) we get

$$\begin{aligned} &\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} |\nabla u_{\varepsilon\delta_1\delta_2}|^2 dx + \alpha\varepsilon^{\gamma} \int_{\partial B_{\varepsilon\delta_1}} |u_{\varepsilon\delta_1\delta_2}|^2 d\sigma(x) \\ &= \int_{\Omega_{\varepsilon\delta_1\delta_2}} f_{\varepsilon} \cdot \cdot u_{\varepsilon\delta_1\delta_2} dx + \int_{\partial B_{\varepsilon\delta_1}} g_0 \cdot u_{\varepsilon\delta_1\delta_2} d\sigma(x) + \langle g_{\varepsilon}, u_{\varepsilon\delta_1\delta_2} \rangle_{(H^{-1/2}(\partial B_{\varepsilon\delta_1}))^N, (H^{1/2}(\partial B_{\varepsilon\delta_1}))^N} \cdot \end{aligned}$$

Then

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial B_{\varepsilon\delta_1})}\leq\Big|\int_{\Omega_{\varepsilon\delta_1\delta_2}}f_\varepsilon\cdot u_{\varepsilon\delta_1\delta_2}dx\Big|+\Big|\int_{\partial B_{\varepsilon\delta_1}}g_0\cdot u_{\varepsilon\delta_1\delta_2}d\sigma(x)\Big|$$
$$+\Big|\langle g_\varepsilon,u_{\varepsilon\delta_1\delta_2}\rangle_{H^{-1/2}(\partial B_{\varepsilon\delta_1})^N,H^{1/2}(\partial B_{\varepsilon\delta_1})^N}\Big|$$

By the Cauchy Schwartz and Poincaré inequalities , we have

$$\Big|\int_{\Omega_{\varepsilon\delta_1\delta_2}}f_\varepsilon\cdot u_{\varepsilon\delta_1\delta_2}dx\Big|\leq\|f_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\tag{4.31}$$

and we have successively, by using Proposition 4.9

$$\Big|\int_{\partial B_{\varepsilon\delta_1}}g_0\cdot u_{\varepsilon\delta_1\delta_2}d\sigma(x)\Big|\leq C\delta_1^{\frac{N}{2}}\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+C\frac{\delta_1^{N-1}}{\varepsilon}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}$$

and finally

$$\Big|\langle g_\varepsilon,u_{\varepsilon\delta_1\delta_2}\rangle\Big|_{H^{-1/2}(\partial B_{\varepsilon\delta_1})^N,H^{1/2}(\partial B_{\varepsilon\delta_1})^N}\leq C\delta_1^{\frac{N}{2}}\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+C\frac{\delta_1^{N-1}}{\varepsilon}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

Hence

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial B_{\varepsilon\delta_1})}\leq C(1+\delta_1^{\frac{N}{2}})\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+C\frac{\delta_1^{N-1}}{\varepsilon}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

From which, using the Poincaré inequality and due to assumption on k_1 in $(4.3)_1$, that gives

$$\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\leq C\Big(1+\frac{\delta_1^{N-1}}{\varepsilon}\Big)\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}.$$

Thus

$$\|u_{\varepsilon\delta_1\delta_2}\|_{H^1(\Omega_{\varepsilon\delta_1\delta_2})}\leq C.\tag{4.32}$$

This estimate can be refined following the different values of γ. To do so, observe that according to Lemma 4.1

$$\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\leq C\Big(\varepsilon\delta_1\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+(\varepsilon\delta_1)^{\frac{1}{2}}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\partial B_{\varepsilon\delta_1})}\Big)$$

then

$$\frac{\delta_1^{N-1}}{\varepsilon}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\leq C\Big(\delta_1^N\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\Big(\frac{\delta_1^{N-1}}{\varepsilon}\Big)^{\frac{1}{2}}\delta_1^{\frac{N}{2}}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\partial B_{\varepsilon\delta_1})}\Big)$$

Using Young's inequality, we get

$$\frac{\delta_1^{N-1}}{\varepsilon}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}\leq C\Big(\delta_1^N\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\frac{1}{\eta}\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-\gamma}+\eta\varepsilon^\gamma\delta_1^N\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial B_{\varepsilon\delta_1})}\Big).$$

Consequently

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+(\alpha-C\eta\delta_1^N)\varepsilon^\gamma\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial B_{\varepsilon\delta_1})}\leq C(1+\delta_1^N)\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\frac{C}{\eta}\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-\gamma}$$

Then for suitable η we finally obtain the following a priori estimate:

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+\alpha\varepsilon^{\gamma}\|u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\partial B_{\varepsilon\delta_1})} \leq C\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}+C\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-\gamma} \quad (4.33)$$

(i) Case $\gamma < 0$. From (4.33), one has

$$\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C$$

and then

$$\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\partial B_{\varepsilon\delta_1})} \leq C\varepsilon^{-\frac{\gamma}{2}}.$$

By lemma 4.1 and using the Young inequality, we get

$$\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C(\varepsilon\delta_1+\frac{\varepsilon\delta_1}{\eta}+\eta\varepsilon^{-\gamma}).$$

Hence

$$\|\varepsilon^{\gamma}u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C$$

which is the first estimate in (4.27)

(ii) Case $0 \leq \gamma < 2$. On one hand by (4.33), one has

$$\|\varepsilon^{\frac{\gamma}{2}}\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C.$$

On the other hand we have

$$\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C\Big(\frac{\delta_1^{N-1}}{\varepsilon}\Big)^{\frac{1}{2}}\varepsilon^{-\frac{\gamma}{2}} \implies \|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C\delta_1^{\frac{N-1}{2}}\varepsilon^{\frac{-1-\gamma}{2}}.$$

Then by the Poincaré inequality and again using the Young inequality, we get

$$\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq \frac{C}{\eta}\delta_1^{N-1}+\eta\varepsilon^{-1-\gamma}$$

Consequently,

$$\|\varepsilon^{1+\gamma}u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C.$$

(iii) Case $\gamma = 2$. From (4.33), one has

$$\nu\|\nabla u_{\varepsilon\delta_1\delta_2}\|^2_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C\frac{\delta_1^{N-1}}{\varepsilon}\varepsilon^{-2}$$

then, again successively by Poincaré and Young inequalities, we get

$$\|\varepsilon^{2}u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} \leq C.$$

The estimate of $\nabla u_{\varepsilon\delta_1\delta_2}$ follows easily from (4.32).

(iiii) Case $\gamma > 2$. The estimate of $\nabla u_{\varepsilon\delta_1\delta_2}$ and $u_{\varepsilon\delta_1\delta_2}$ in (4.30) follow easily from (4.32).

Now, we prove the upper estimates of the pressure.

In the end, we shall establish the a priori estimates of the pressure p_ε. Indeed, we choose $\phi \in L^2(\Omega_{\varepsilon\delta_1\delta_2})$ as a test function in the variational formulation (4.6) and by Proposition 4.9 and Lemma 4.2, we get

$$\Big|\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_{\varepsilon\delta_1\delta_2}\,\phi\,dx\Big| \leq C\Big[(1+\varepsilon^{\gamma}\delta_1^{\frac{N}{2}})\|\nabla u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \varepsilon^{\gamma}\frac{\delta_1^{N-1}}{\varepsilon}\|u_{\varepsilon\delta_1\delta_2}\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})} + \frac{\delta_1^{N-1}}{\varepsilon}\Big]\|\phi\|_{L^2(\Omega_{\varepsilon\delta_1\delta_2})}. \tag{4.34}$$

The a priori estimates for the pressure follow now from (4.34) and estimates the $u_{\varepsilon\delta_1\delta_2}$ and $\nabla u_{\varepsilon\delta_1\delta_2}$ for the different values of γ obtained above. □

4.4.2 Proof of theorem 4.2

Proof. Case $\gamma < 0$. The corresponding estimates (4.27) from proposition 4.10 as wel as the following ones:

$$\varepsilon^{\gamma}\widetilde{u_{\varepsilon\delta_1\delta_2}} \rightharpoonup u \quad \text{weakly in } \ L^2(\Omega)^N, \qquad \varepsilon^{-\gamma}\widetilde{p_{\varepsilon\delta_1\delta_2}} \rightharpoonup p \quad \text{weakly in } \ L^2(\Omega).$$

Let $\varphi \in D(\Omega)^N$ be a test fonction in (4.6). Multiplying (4.6) by $\varepsilon^{-\gamma}$, then unfolding. That gives

$$\begin{aligned}&\nu\varepsilon^{-\gamma}\int_{\Omega\times Y}\mathcal{T}_{\varepsilon}(\nabla u_{\varepsilon\delta_1\delta_2}) : \mathcal{T}_{\varepsilon}(\nabla\varphi)dxdy + \alpha\frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial B}\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta_1\delta_2})\cdot\mathcal{T}^b_{\varepsilon\delta_1}(\varphi)dxd\sigma(z)\\ &-\int_{\Omega_{\varepsilon\delta_1\delta_2}}\varepsilon^{-\gamma}p_{\varepsilon\delta_1\delta_2}\operatorname{div}\varphi\,dx = \varepsilon^{-\gamma}\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_{\varepsilon}\cdot\varphi dx + \varepsilon^{-\gamma}\int_{\partial B_{\varepsilon\delta_1}} g_0\cdot\varphi d\sigma_{\varepsilon\delta_1}(x)\\ &+\varepsilon^{-\gamma}\int_{\partial B_{\varepsilon\delta_1}} g_{\varepsilon}\cdot\varphi d\sigma_{\varepsilon\delta_1}(x)\end{aligned}$$

Now, we can pass to limit in all terms. Using again the fact that $\nabla u_{\varepsilon\delta_1\delta_2}$ is uniformly bounded in $L^2(\Omega_{\varepsilon\delta_1\delta_2})^{N\times N}$ and as $-\gamma > 0$, we get at the limit the following identity:

$$\alpha k_1\int_{\Omega\times\partial B} u\varphi dxd\sigma(z) - \int_{\Omega} p\operatorname{div}\varphi dx = 0 \tag{4.35}$$

one has Darcy law

$$u = \frac{-1}{\alpha k_1|\partial B|}\nabla p$$

Remark 4.1. *If* $k_1 = 0$ *we get:* $\displaystyle\int_{\Omega}\nabla p\varphi dx = 0$ *implies* $\nabla p = 0$*, i.e* $p = constant$

4.4.3 Proof of theorem 4.3

Proof. $0 \leq \gamma < 2$. From corresponding estimates (4.28) in Proposition 4.10,it follow that

$$\varepsilon^{1+\gamma}\widetilde{u_{\varepsilon\delta_1\delta_2}} \rightharpoonup u \quad \text{weakly in } \ L^2(\Omega), \qquad \varepsilon p_{\varepsilon\delta_1\delta_2} \rightharpoonup p \quad \text{weakly in } \ L^2(\Omega).$$

Let $\varphi \in (D(\Omega))^N$ be a test function in (4.6). Multiply equation (4.6) by ε and we use the unfolding operator $\mathcal{T}_\varepsilon$ we get

$$\nu\varepsilon^{1-\frac{\gamma}{2}}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\varepsilon^{\frac{\gamma}{2}}\nabla u_{\varepsilon\delta_1\delta_2}):\mathcal{T}_\varepsilon(\nabla\varphi)dxdy+\alpha\frac{\delta_1^{N-1}}{\varepsilon}\int_{\Omega\times\partial B}\mathcal{T}^b_{\varepsilon\delta_1}(\varepsilon^{1+\gamma}u_{\varepsilon\delta_1\delta_2})\cdot\mathcal{T}^b_{\varepsilon\delta_1}(\varphi)dxd\sigma(z)$$

$$-\int_{\Omega_{\varepsilon\delta_1\delta_2}}\varepsilon p_{\varepsilon\delta_1\delta_2}\mathrm{div}\,\varphi\,dx=\int_{\Omega_{\varepsilon\delta_1\delta_2}}\varepsilon f_\varepsilon\cdot\varphi dx+\int_{\Omega\times\partial B}\mathcal{T}^b_\varepsilon(g_0)\cdot\varphi dxd\sigma(y)+\int_{\Omega\times\partial B}\mathcal{T}^b_\varepsilon(g_\varepsilon)\varphi dxd\sigma(y)$$

We can now pass to limit in all the expression. By using the estimate (4.28) and the assumption for f_ε (4.22), we get

$$\alpha k_1\int_{\Omega\times\partial B}u\varphi dxd\sigma(z)-\int_\Omega p\mathrm{div}\,\varphi dx=\int_{\Omega\times\partial B}g_0\cdot\varphi dxd\sigma(y)+\int_{\Omega\times\partial B}g\varphi dxd\sigma(y).$$

Then

$$\int_\Omega\Big(\alpha k_1|\partial B|u+\nabla p-|\partial B|g_0-|\partial B|\mathcal{M}_{\partial B}(g)\Big)\varphi dx=0.$$

So that

$$u=\frac{1}{\alpha k_1|\partial B|}\Big(-\nabla p+|\partial B|g_0+|\partial B|\mathcal{M}_{\partial B}(g)\Big).$$

□

Remark 4.2. *If* $k_1=0$ *we obtain:*

$$\nabla p=|\partial B|g_0+|\partial B|M_{\partial B}(g)$$

4.4.4 Proof of theorem 4.4

The proof of theorem 4.4, makes use the next two elementary results.

Lemma 4.3 (see [19]). *Suppose* $N\geq 3$. *Then, there exists* $\delta_0>0$ *such that*

$$\bigcup_{0<\delta<\delta_0}\{\phi\in H^1_{per}(Y)\ \mid\ \phi=0\ \text{ on }\delta T\}$$

is dense in $H^1_{per}(Y)$.

Lemma 4.4 ([19], Lemma 3.3). *Let* v *be in* $\mathcal{D}(\mathbb{R}^N)\cap K_T$ *(i.e* $v=v(T)$ *on* T*). Set*

$$w_{\varepsilon\delta_2}(x)=v(T)-v\Big(\frac{1}{\delta_2}\Big\{\frac{x}{\varepsilon}\Big\}\Big)\ \text{ for a.e. } x\in\mathbb{R}^N.$$

Then

$$w_{\varepsilon\delta_2}\rightharpoonup v(T)\ \text{ weakly in } H^1(\Omega). \tag{4.36}$$

Proof of Theorem 4.4. Due to the estimates (4.29) in Proposition 4.10, the case $\gamma=2$ is the easiest.

From (4.32) we have $\|u_{\varepsilon\delta_1\delta_2}\|_{(H^1(\Omega_{\varepsilon\delta_1\delta_2}))^N} \leq C$. Applying Proposition 4.2 one has the convergences

$$\begin{aligned} \mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta_1\delta_2}) &\rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y)^{N\times N}, \\ \mathcal{T}_\varepsilon(u_{\varepsilon\delta_1\delta_2}) &\rightharpoonup u \quad \text{weakly in } \ L^2(\Omega; H^1(Y))^N \end{aligned}$$

where $u \in H_0^1(\Omega)^N$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))^N$.

By Proposition 4.6, there exists U in $L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence,

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}) \rightharpoonup U \text{ weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{4.37}$$

On one hand, by (4.8), one has

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}_Y^\varepsilon(u_{\varepsilon\delta_1\delta_2}) \longrightarrow k_2 u \text{ in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)). \tag{4.38}$$

On the other hand, by Proposition 4.6 there exists a W in $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Omega \times \mathbb{R}^N)$ such that

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\Big(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}) - \mathcal{M}_Y^\varepsilon(u_{\varepsilon\delta_1\delta_2})\Big) \rightharpoonup W \text{ weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)) \tag{4.39}$$

From (4.37),(4.38) and (4.39), it yields

$$U = W + k_2 u \ \text{ and } \ \nabla_z U = \nabla_z W$$

and again by Proposition 4.6

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\nabla_z \mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}) = \delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2}) \rightharpoonup \nabla_z U \text{ weakly in } L^2(\Omega \times \mathbb{R}^N). \tag{4.40}$$

Now, in (4.6) choose the test function $v_\varepsilon = \varepsilon \Phi \psi\Big(\frac{\cdot}{\varepsilon}\Big)$ with $\Phi \in \mathcal{D}(\Omega)^N$ and ψ in $H^1_{per}(Y)^N$. Since $\nabla v_\varepsilon = \varepsilon\psi\Big(\frac{x}{\varepsilon}\Big)\nabla\Phi + \Phi\nabla_y\psi\Big(\frac{x}{\varepsilon}\Big)$ we get

$$\begin{aligned} &\nu\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\Big(\varepsilon\psi\Big(\frac{x}{\varepsilon}\Big)\nabla\Phi + \Phi\nabla_y\psi\Big(\frac{x}{\varepsilon}\Big)\Big)dx + \alpha\varepsilon^2\int_{\partial B_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}\varepsilon\Phi\psi\Big(\frac{x}{\varepsilon}\Big)d\sigma(x) \\ &-\varepsilon\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_{\varepsilon\delta_1\delta_2}\mathrm{div}\,\Phi\psi\Big(\frac{x}{\varepsilon}\Big)dx = \varepsilon\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon\cdot\Phi\psi\Big(\frac{x}{\varepsilon}\Big)dx + \varepsilon\int_{\partial B_{\varepsilon\delta_1}} g_0\cdot\Phi\psi\Big(\frac{x}{\varepsilon}\Big)d\sigma(x) \\ &\qquad + \varepsilon\int_{\partial B_{\varepsilon\delta_1}} g_\varepsilon\Phi\psi\Big(\frac{x}{\varepsilon}\Big)d\sigma(x) \end{aligned}$$

When $\varepsilon \longrightarrow 0$ we obtain

$$\nu\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2}\Phi(x)\nabla_y\psi\Big(\frac{x}{\varepsilon}\Big)dxdy = 0$$

By unfolding with $\mathcal{T}_\varepsilon$ one has

$$\nu\int_{\Omega\times Y} \mathcal{T}_\varepsilon\Big(\nabla u_{\varepsilon\delta_1\delta_2}\Big)\mathcal{T}_\varepsilon(\Phi(x))\mathcal{T}_\varepsilon\Big(\nabla_y\psi\Big(\frac{x}{\varepsilon}\Big)\Big)dxdy = 0$$

By passing to limit we obtain (4.23)

$$\nu \int_{\Omega \times Y} \Big(\nabla u(x) + \nabla_y \widehat{u}(x,y) \Big) \nabla_y \varPsi(x,y) dx dy = 0$$

In order to describe the contribution of the perforations, we use the function $w_{\varepsilon\delta_2}$ introduced in lemma 4.4. For ψ in $D(\Omega)$, chooe $w_{\varepsilon\delta_2}\psi$ as a test function in (4.6). We get

$$\begin{aligned} &\nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} \nabla w_{\varepsilon\delta_2} \psi dx + \nu \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2} \nabla \psi dx + \alpha\varepsilon^2 \int_{\partial B_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2} d\sigma(x) - \\ &\int_{\Omega_{\varepsilon\delta_1\delta_2}} p_{\varepsilon\delta_1\delta_2} \mathrm{div}\,(w_{\varepsilon\delta_2}\psi) dx = \int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon w_{\varepsilon\delta_2} \psi dx + \int_{\partial B_{\varepsilon\delta_1}} g_0 w_{\varepsilon\delta_2} \psi d\sigma_{\varepsilon\delta_1}(x) + \int_{\partial B_{\varepsilon\delta_1}} g_\varepsilon w_{\varepsilon\delta_2} d\sigma_{\varepsilon\delta_1}(x) \end{aligned} \tag{4.41}$$

By unfolding the first term in (4.41) with $\mathcal{T}_{\varepsilon\delta_2}$, we get

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} \nabla w_{\varepsilon\delta_2} \psi dx \overset{\mathcal{T}_{\varepsilon\delta_2}}{\simeq} \delta_2^N \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2}) \mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2}) \mathcal{T}_{\varepsilon\delta_2}(\psi) dx dz \tag{4.42}$$

since by lemma 4.4, one has:

$$\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2}) = -\frac{1}{\varepsilon\delta_2} \nabla_z v$$

then

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} \nabla w_{\varepsilon\delta_2} \psi dx \overset{\mathcal{T}_{\varepsilon\delta_2}}{\simeq} \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \int_{\Omega \times \mathbb{R}^N} \delta_2^{\frac{N}{2}} \mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2}) (-\nabla_z v) \mathcal{T}_{\varepsilon\delta_2}(\psi) dx dz$$

Convergence (4.40) as well as hypothesis (4.3), allows us to pass to the limit in (4.42) to obtain:

$$\lim_{\varepsilon \longrightarrow 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} \nabla w_{\varepsilon\delta_2} \psi dx = -k_2 \int_{\Omega \times (\mathbb{R}^N \setminus T)} \nabla_z U(x,z) \nabla_z v(z) \psi(x) dx dz \tag{4.43}$$

The second term in (4.41) is unfolded with $\mathcal{T}_\varepsilon$ and we have,

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2} \nabla \psi dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta_1\delta_2}) \mathcal{T}_\varepsilon(w_{\varepsilon\delta_2}) \mathcal{T}_\varepsilon(\nabla \psi) dx dy$$

Using proposition 4.2 and convergence (4.36), we can pass to the limit with respect to ε in tne above equality to get :

$$\lim_{\varepsilon \longrightarrow 0} \int_{\Omega_{\varepsilon\delta_1\delta_2}} \nabla u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2} \nabla \psi dx = v(T) \int_{\Omega \times Y} (\nabla u + \nabla_y \widehat{u}) \nabla_x \psi(x) dx dy \tag{4.44}$$

For the third term in (4.41) by using (4.18) we get

$$\lim_{\varepsilon \longrightarrow 0} \alpha \int_{\partial B_{\varepsilon\delta_1}} \varepsilon^2 u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2} \psi d\sigma(x) = \lim_{\varepsilon \longrightarrow 0} \frac{\delta_1^{N-1}}{\varepsilon} \alpha \int_{\mathbb{R}^N \times \partial B} \mathcal{T}^b_{\varepsilon\delta_1\delta_2}(\varepsilon^2 u_{\varepsilon\delta_1\delta_2}) \mathcal{T}^b_{\varepsilon\delta_1\delta_2}(w_{\varepsilon\delta_2}) \mathcal{T}^b_{\varepsilon\delta_1\delta_2}(\psi) dx d\sigma(z)$$

Passing to the limit yields

$$\lim_{\varepsilon\longrightarrow 0}\alpha\int_{\partial B_{\varepsilon\delta_1}}\varepsilon^2 u_{\varepsilon\delta_1\delta_2}w_{\varepsilon\delta_2}\psi d\sigma(x)=\alpha k_1 v(T)\int_{\mathbb{R}^N\times\partial B}u\psi dxd\sigma(z) \tag{4.45}$$

For the fourth term in (4.41) we have

$$\int_{\Omega_{\varepsilon\delta_1\delta_2}}p_{\varepsilon\delta_1\delta_2}\mathrm{div}\,(w_{\varepsilon\delta_2}\psi)dx=\int_{\Omega_{\varepsilon\delta_1\delta_2}}p_{\varepsilon\delta_1\delta_2}w_{\varepsilon\delta_2}\mathrm{div}\,(\psi)dx+\int_{\Omega_{\varepsilon\delta_1\delta_2}}p_{\varepsilon\delta_1\delta_2}\nabla w_{\varepsilon\delta_2}\psi dx \tag{4.46}$$

For the second term of right-hand sid of this equation we apply the operator $\mathcal{T}_{\varepsilon\delta_2}$ and we use the lemma 4.4 we get

$$\begin{aligned}\int_{\Omega_{\varepsilon\delta_1\delta_2}}p_{\varepsilon\delta_1\delta_2}\nabla w_{\varepsilon\delta_2}\psi dx&=\delta_2^N\int_{\Omega\times\mathbb{R}}\mathcal{T}_{\varepsilon\delta_2}(p_\varepsilon)\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon\delta_2}(\psi)dxdz\\&=\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\delta_2^{\frac{N}{2}}\int_{\Omega\times\mathbb{R}}\mathcal{T}_{\varepsilon\delta_2}(p_\varepsilon)(-\nabla_z v_z)\mathcal{T}_{\varepsilon\delta_2}(\psi)dxdz\end{aligned}$$

For $\varepsilon\to 0$ this integral go to zero. Passing to the limit in (4.46) we get

$$\lim_{\varepsilon\longrightarrow 0}\int_{\Omega_{\varepsilon\delta_1\delta_2}}p_{\varepsilon\delta_1\delta_2}\mathrm{div}\,(w_{\varepsilon\delta_2}\psi)dx=v(T)\int_\Omega p\ \mathrm{div}\,(\psi)dx \tag{4.47}$$

Similarly, for the fifth, sixth and seventh terms, in view again (4.18), we get

$$\lim_{\varepsilon\longrightarrow 0}\int_{\Omega_{\varepsilon\delta_1\delta_2}}f_\varepsilon w_{\varepsilon\delta_2}\psi dx=v(T)\int_\Omega f\psi dx \tag{4.48}$$

$$\lim_{\varepsilon\longrightarrow 0}\int_{\partial B_{\varepsilon\delta_1}}g_0 w_{\varepsilon\delta_2}\psi d\sigma(x)=k_1 v(T)\int_{\mathbb{R}^N\times\partial B}g_0(z)\psi dxd\sigma(z) \tag{4.49}$$

$$\lim_{\varepsilon\longrightarrow 0}\int_{\partial B_{\varepsilon\delta_1}}g_\varepsilon w_{\varepsilon\delta_2}\psi d\sigma(x)=k_1 v(T)\int_{\mathbb{R}^N\times\partial B}g(z)\psi dxd\sigma(z) \tag{4.50}$$

Passing to limit in (4.41) and using (4.42), (4.44), (4.45), (4.47), (4.48), (4.49), and (4.50) we obtain

$$\begin{aligned}v(T)\int_{\Omega\times Y}(\nabla u+\nabla_y\widehat{u})\nabla_x\psi(x)dxdy-k_2\int_{\Omega\times(\mathbb{R}^N\backslash T)}\nabla_z U(x,z)\nabla_z v(z)\psi(x)dxdz\\+\alpha k_1 v(T)\int_{\mathbb{R}^N\times\partial B}u\psi dxd\sigma(z)+v(T)\int_\Omega\nabla p\psi dx=v(T)\int_\Omega f\psi dx\\+k_1 v(T)\int_{\mathbb{R}^N\times\partial B}g_0(z)\psi dxd\sigma(z)+k_1 v(T)\int_{\mathbb{R}^N\times\partial B}g(z)\psi dxd\sigma(z)\end{aligned}$$

Which by density, holds true for all $\psi\in H_0^1(\Omega)$ and $v\in K_T$. With $v(T)=0$ above we obtain the limit equation (4.24). Equations (4.25) follow simply by integrating by parting the above equation.

If $\gamma>2$ by similar argument as those used in all termes of exeprition (4.41) except the third term the bihaviour at the limit for $\gamma > 2$, this term goes to zero, in view of (4.42), (4.44), (4.47), (4.48), (4.49), and (4.50) we obtain

$$\begin{aligned} & v(T)\int_{\Omega\times Y}(\nabla u+\nabla_y\widehat{u})\nabla_x\psi(x)dxdy - k_2\int_{\Omega\times(\mathbb{R}^N\setminus T)}\nabla_z U(x,z)\nabla_z v(z)\psi(x)dxdz \\ & + v(T)\int_\Omega \nabla p\psi dx = v(T)\int_\Omega f\psi dx + k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g_0(z)\psi dxd\sigma(z) \\ & + k_1 v(T)\int_{\mathbb{R}^N\times\partial B} g(z)\psi dxd\sigma(z) \end{aligned}$$

Equation (4.26) follow simply by integrating by parting the above equation. □

To finish let us give the classical "strong" formulation of the homogenized problem obtained in theorem (4.4). We skip the proof since the strong formulation from the unfolded problem is standard, we refer the reader for instance to [27] or [46]

Remark 4.3. *For the case $\gamma \geqslant 2$, a tensor $B = b_{ijkh}$ is introduced. Its form is found following standard argument, as shown in [21]*

The next theorem gives the classical (standard) from of homogenized system (4.23)-(4.26). To state it, we follow the procedure from [19] , where more details can be found.

Theoreme 4.5. *(u, p) is the unique solution of the homogenized problem*
If $\gamma = 2$

$$\begin{cases} -b_{ijkh}\dfrac{\partial^2 u_k}{\partial x_j\partial x_h} + \alpha k_1|\partial B|u_i + \nabla p + k_2^2\Theta u = k_1|\partial B|g_{0i} + k_1|\partial B|\mathcal{M}_{g_i} + f_i \ \text{in } \Omega, 1\le i\le N \\ u = 0 \ \text{on } \partial\Omega \end{cases} \tag{4.51}$$

if $\gamma > 2$

$$\begin{cases} -b_{ijkh}\dfrac{\partial^2 u_k}{\partial x_j\partial x_h} + \nabla p + k_2^2\Theta u = k_1|\partial B|g_{0i} + k_1|\partial B|\mathcal{M}_{g_i} + f_i \ \text{in } \Omega, 1\le i\le N \\ u = 0 \ \text{on } \partial\Omega \end{cases} \tag{4.52}$$

In both systems the function Θ giving rise to a "strange term", is given by

$$\Theta(x) = \int_{\mathbb{R}^N\setminus T}\nabla_z\chi(x,z)dz \tag{4.53}$$

Introduce first the classical correctors $\widehat{\chi}_j, j = 1, ..., n$, for the homogenization in fixed domains (see for instance [9]). They are defined by the cell problems

$$\begin{cases} \widehat{\chi}_j \in L^\infty(\Omega; H^1_{per}(Y)), \\ \quad \displaystyle\int_Y \nabla(\widehat{\chi}_j - y_j)\nabla\phi \ = \ 0 \qquad a.e. x\in\Omega, \\ \quad \forall\phi\in H^1_{per}(Y). \end{cases} \tag{4.54}$$

Here χ is the solution of the cell problem corresponding

$$\left\{\begin{array}{l} \chi \in L^{\infty}(\Omega; K_T) \quad (\chi, T) \equiv 1 \\ \displaystyle\int_{\mathbb{R}\setminus T} \nabla_z \chi(x,z) \nabla_z \Psi(z) dz \;\; = \;\; 0 \quad a.e. x \in \Omega, \forall \Psi \in K_T, \Psi(T) = 0. \end{array}\right. \tag{4.55}$$

To do so, observe that (4.23) gives $\widehat{u}$ in terms of ∇u and a tensor $B = (b_{ijkh})$ expressed as integrals of function defined on cell problems. The procedure is standard, for the Stokes problem.The procedure is standard, for the Stokes problem the details can be found in [21]. We will just recall here the definition of B.
For $k, h = 1, ..., N$ let $\Pi^{kh} = (\Pi_i^{kh})_i$ with $\Pi_i^{kh} = \delta_{ki} y_h$ (δ_{ki} being the Kronecker symbols) and introduce the solution (χ^{kh}, q^{kh}) of the Stokes cell system

$$\left\{\begin{array}{l} -\Delta \chi^{kh} + \nabla q^{kh} = 0 \text{ in } Y^*, \\ \operatorname{div}(\chi^{kh} - \mathit{\Pi}^{kh}) = 0 \text{ in } Y^*, \\ -\dfrac{\partial(\chi^{kh} - \mathit{\Pi})}{\partial n} + q^{kh}.n = 0 \text{ on } \partial B, \\ M_{Y^*}(\chi^{KH}) = 0 \; \chi^{kh} \text{ Y-periodic} \end{array}\right.$$

the tensor $B = (b_{ijkh})$ is defined as fllow:

$$b_{ijkh} = \int_{Y^*} \frac{\partial(\chi^{kh} - \mathit{\Pi}^{kh})_l}{\partial y_m} \frac{\partial(\chi^{ij} - \mathit{\Pi}^{ij})_l}{\partial y_m} dy \tag{4.56}$$

Proof. The proof follows the reasoning from [19], we just emphasize the main points. The correctors defined by (4.54) enable us to express $\widehat{u}(x, y)$ in equation (4.23) in terms of u as

$$\widehat{u} = \sum_{ijkh} b_{ijkh} \frac{\partial u_k}{\partial x_l} \widehat{\chi}_j^{hl}(x, y)$$

replacing this expression (4.25) it is easily seen that the limit function u is solution of

$$\int_{\Omega} b_{ijkh} \nabla u \nabla \psi dx dy - \nu k_2 \int_{\Omega \times \partial T} \nabla_z U n_T \psi dx d\sigma + \alpha k_1 |\partial B| \int_{\Omega} u \psi dx + \int_{\Omega} \nabla p \psi dx = \int_{\Omega} f \psi dx + k_1 |\partial B| \int_{\Omega} g_0 \psi dx + k_1 |\partial B| \; M_{\partial B} \int_{\Omega} \psi dx \tag{4.57}$$

with b_{ijkh} given by (4.56). Now, by integrating by parts in (4.24), one easily gets

$$\int_{\partial T} \nabla_z U v_T d\sigma = \int_{\partial T} \nabla_z (U - k_2 u) v_T d\sigma = -k_2 u \Big(\int_{\partial T} \nabla_z \chi v_T d\sigma \Big)$$

Wich, replaced into (4.57) gives (4.51) with Θ defined by (4.53)
By similar method we get the equation (4.52)

□

Remark 4.4. *For $k_1 = 0, k_2 \neq 0$ and $\gamma \geq 2$, arises the following equation*

$$\begin{cases} -b_{ijkh}\dfrac{\partial^2 u_k}{\partial x_j \partial x_h} + \nabla p + k_2^2 \Theta u = f_i \text{ in } \Omega, 1 \leq i \leq N \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Remark 4.5. *For $k_2 = 0$ and $k_1 \neq 0$ two following cases arise:*
If $\gamma = 2$ we get

$$\begin{cases} -b_{ijkh}\dfrac{\partial^2 u_k}{\partial x_j \partial x_h} + \alpha k_1 |\partial B| u_i + \nabla p = k_1 |\partial B| g_{0i} + k_1 |\partial B| \mathcal{M}_{g_i} + f_i \text{ in } \Omega, 1 \leq i \leq N \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

If $\gamma > 2$ we get

$$\begin{cases} -b_{ijkh}\dfrac{\partial^2 u_k}{\partial x_j \partial x_h} + \nabla p = k_1 |\partial B| g_{0i} + k_1 |\partial B| \mathcal{M}_{g_i} + f_i \text{ in } \Omega, 1 \leq i \leq N \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Remark 4.6. *For $k_1 = 0$ and $k_2 = 0$ and $\gamma \geq 2$, we get the following problem*

$$\begin{cases} -b_{ijkh}\dfrac{\partial^2 u_k}{\partial x_j \partial x_h} + \nabla p = f_i \text{ in } \Omega, 1 \leq i \leq N \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Remark 4.7. *If $k_1 = 1$ and $k_2 = 0$ we become in the case of classical homogenization and we find all the results established in [55].*

Chapter 5

Homogenization of a Class of Hyperbolic-Parabolic Problems in Domains with Small Holes

Abstract

We propose to study the asymptotic behaviour of a class of Hyperbolic-Parabolic problem in a perforated domain in $\mathbb{R}^N, N \geq 3$, with small holes ε-periodically distributed. The size of the holes is of the order $(\varepsilon\delta(\varepsilon))$ with $\delta(\varepsilon) \longrightarrow 0$ as ε goes to zero. On the boundary of some holes we prescribe a Dirichlet-type condition on the boundary of the others, a Neumann type condition is assumed . To do so, we use the periodic unfolding method introduced by Cioranescu, Damlamian and Griso in [15] and [16] which allows to consider a general geometric framework.

5.1 Introduction

Let Ω be a bounded open domain of $\mathbb{R}^N,\ N \geq 3$ with $|\partial\Omega| = 0$ and ε a small positive parameter. To define the perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$, we introduce the following sets: Let $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N$ be connected and it's boundary ∂Y is a null set ;S and B are compacts subsets of Y . We assume that the holes B and S have Lipschitz boundary. For three small parameters ε, δ_1 and δ_2, we define the following sets

$$S_{\varepsilon\delta_1} = \bigcup_{\xi\in\Pi_\varepsilon} (\varepsilon\delta_1 S + \varepsilon\xi) \qquad B_{\varepsilon\delta_2} = \bigcup_{\xi\in\Pi_\varepsilon} (\varepsilon\delta_2 B + \varepsilon\xi),$$

$$S^{int}_{\varepsilon\delta_1} = \bigcup_{\xi\in\Xi_\varepsilon} (\varepsilon\delta_1 S + \varepsilon\xi) \qquad B^{int}_{\varepsilon\delta_2} = \bigcup_{\xi\in\Xi_\varepsilon} (\varepsilon\delta_2 B + \varepsilon\xi) \tag{5.1}$$

$$\Pi_\varepsilon = \{\xi \in \mathbb{Z}^N : \varepsilon(\xi + Y) \cap \Omega \neq \emptyset\}$$
$$\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N : \varepsilon(\xi + Y) \subset \Omega\}; \qquad Y_\varepsilon = \bigcup_{\xi \in \Pi_\varepsilon} \{\varepsilon(Y + \xi)\};$$

such that $\forall\ (\xi_1, \xi_2) \in \Pi_\varepsilon^2,\ \xi_1 \neq \xi_2, \Rightarrow (\xi_1 + Y) \cap (\xi_2 + Y) = \emptyset$.
Then, we set $\Omega_{\varepsilon\delta_1\delta_2} = \Omega \setminus (S_{\varepsilon\delta_1} \cup B_{\varepsilon\delta_2})$. The main feature of the perforated domain $\Omega_{\varepsilon\delta_1\delta_2}$ is that the size $r(\varepsilon) = \varepsilon\delta_1, \varepsilon\delta_2,\ 0 < \delta_1, \delta_2 < 1$ of the holes is not proportional to the size of the reference cell εY and $r(\varepsilon)/\varepsilon$ tends to zero as ε is taking values in a decreasing sequence of positive numbers which tends to zero.
We consider the following Hyperbolic-parabolic systeme

$$\begin{cases} \alpha_{\varepsilon\delta_1\delta_2} u''_{\varepsilon\delta_1\delta_2}(x,t) + \beta_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x,t) - div(A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)) = f_\varepsilon(x,t) \\ \qquad in\ \Omega_{\varepsilon\delta_1\delta_2} \times]0,T[\\ A^\varepsilon \nabla u_{\varepsilon\delta_1\delta_2} \eta_\varepsilon + \nu\varepsilon^\gamma u_{\varepsilon\delta_1\delta_2} = 0 \quad on\ \ \partial S_{\varepsilon\delta_1} \times]0,T[\\ u_{\varepsilon\delta_1\delta_2} = 0\ on\ \partial\Omega_{\varepsilon\delta_1\delta_2} \times]0,T[\\ u_{\varepsilon\delta_1\delta_2}(x,0) = u^0_{\varepsilon\delta_1\delta_2}\ on\ \Omega_{\varepsilon\delta_1\delta_2} \\ \alpha_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x,0) = \sqrt{\alpha_{\varepsilon\delta_1\delta_2}} u^1_{\varepsilon\delta_1\delta_2}\ on\ \Omega_{\varepsilon\delta_1\delta_2} \end{cases} \tag{5.2}$$

η_ε is the outword unit normal vector field defined on $\partial S_{\varepsilon\delta_1}$, $\alpha_{\varepsilon\delta_1\delta_2}$ and $\beta_{\varepsilon\delta_1\delta_2}$ two coefficients such that:

$$\begin{cases} \alpha_{\varepsilon\delta_1\delta_2}, \beta_{\varepsilon\delta_1\delta_2} \in L^\infty(\Omega) \\ \alpha_{\varepsilon\delta_1\delta_2} \geq 0 \qquad \text{a.e in } \Omega \\ \beta_{\varepsilon\delta_1\delta_2} \geq c > 0 \qquad \text{a.e in } \Omega \end{cases}$$

For the initial data, we always assume $u^0_{\varepsilon\delta_1\delta_2} \in H^1_0(\Omega), u^1_{\varepsilon\delta_1\delta_2} \in L^2(\Omega_{\varepsilon\delta_1\delta_2})$ and $f_\varepsilon \in L^2(0,T,L^2(\Omega_{\varepsilon\delta_1\delta_2}))$.
The problem (5.2) models many kinds of phenomena arising in electricity and magnetism, in the theory of elasticity, in hydrodynamics and in vibration theory [49] and [51].
In 1977, Bensoussan, J. L. Lionand papanicolau [9] considered the hyperbolic-parabolic system in the fixed domains. The problem (5.2) was studied by Migorski in 1996 [44] in perforated domain subsequently in 2000 [52] Timofte further extended the homogenization to the nonlinear case. Again in 2016 Zhanying Yang and Xianhe Zhao [54] considered the problem (5.2) under some condition weaker than usual. In [37] Gaveau stadied the classical homogenization of the wave equation in periodically perforated domains and more recently in [35, 36] by Donato and Yong. In 2016 [32] Cabarrubias adn Donato studied the asymptotic behavior of the wave and heat equation in perforated domains with small holes and Dirichlet conditions on the boundary of the holes.
The Dirichlet problem for the Poisson equation in perforated domains with small holes of size $\varepsilon^\alpha, \alpha > 0$ was treated in Cioranescu and Murat [25]. They showed that

the size $\varepsilon^{N/N-2}$ is "critical" in the following sense: the limit problem contains an additional zero order term, called in [25] "strange term", depending on the limit of the capacity of the set of holes.

Our aime here is to apply the periodic infolding methode due to cioranescu, Damlamian and Griso [15, 16] and extended to perfated domains in Cioranescu, Damlamian, Donato, Griso and Zaki [20].

For the Laplace equation an homogeneous Dirichlet boundary condition in domains with small holes, first applications of the unfolding method have been done in [19, 45]. With small holes of size $\varepsilon^{N/N-1}$ and non homogeneous Neumann condition in Ould Hammouda [46] and in Cioranescu and Ould Hammouda for mixed boundary conditons [27].

The paper is organized as follows. in the section 2 we give some notations and definitions for fixed domains. In the section 3 we recall briefly the different definitions of the time-dependent unfolding operator in fixed domains and their properties. In Section 3.1, we list some notations and define the time-dependent unfolding operator in perforated domains. In section 4, we recall the different definitions of the periodic unfolding operators $\mathcal{T}_{\varepsilon\delta}$ depending on two parameter ε *and* δ as introduced in [19] . Section 5 we extend the operator $\mathcal{T}_{\varepsilon\delta}$ to time-dependent function and thier properties

Finally, the homogenization result concerning problem (5.2) is stadied and proved in Section 6.

5.2 Notation and definitions

We recall here some notations and definitions as given in [15] for fixed domains.

Let Ω de a bounded open set in $\mathbb{R}^N$, such that $|\partial\Omega|= 0$ and $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N$, be the reference periodicity cell. Let us now introduce the sets

$$\widehat{\Omega_\varepsilon} = \text{ interior } \bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi+\overline{Y}), \Xi_\varepsilon = \xi \in \mathbb{Z}^n : \varepsilon(\xi+Y) \subset \Omega$$

$$\Lambda_\varepsilon = \Omega \setminus \Omega_\varepsilon. \tag{5.3}$$

By construction, $\widehat{\Omega_\varepsilon}$ is the interior of the largest union of $\varepsilon(\xi+\overline{Y})$ cells fully contained in Ω, while Λ_ε is the subset of Ω containing the parts from the $\varepsilon(\xi+Y)$ cells intersecting the boundary $\partial\Omega$.

As in [15], for every $z \in \mathbb{R}^N$, we denote by $[z]_Y$ the unique integer combination of periods such that

$$\{z\}_Y = z - [z]_Y \in Y \tag{5.4}$$

Then, because of the periodicity and recalling 5.4, each $x \in \mathbb{R}^N$ can be uniquely written as

$$x = \varepsilon(\{\frac{x}{\varepsilon}\}_Y + [\frac{x}{\varepsilon}]_Y). \tag{5.5}$$

5.3 Time-dependent unfolding operator in fixed domains

Throughout this paper, T will be a given positive number. This section recalls the time-dependent unfolding operator for fixed domains as introduced in [37].

definition 5.1. *([37]) Let $\varphi \in L^q(0,T;L^p(\Omega))$ where $p \in [1,\infty[$ and $q \in [1,+\infty]$ the unfolding operator $\mathcal{T}_\varepsilon$: $L^q(0,T;L^p(\Omega)) \longmapsto L^q(0,T;L^p(\Omega \times Y))$ is defined as*

$$\mathcal{T}_\varepsilon(\varphi)(x,y,t) = \begin{cases} \varphi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y, t\Big) & a.e. \;\; for\ (x,y,t) \in \widehat{\Omega}_\varepsilon \times Y \times]0,T[, \\ 0 & a.e. \;\; for\ (x,y,t) \in \;\Lambda_\varepsilon \times Y \times]0,T[. \end{cases}$$

Some of the properties of this operator which were stated in [37] are listed below. For perforated domains with holes of the same size as the period and for detailed proofs (in definition 5.1 obviously true for fixed domains), we refer to [36]

Remark 5.1. *Notice that if in definition 5.1 we take φ in $L^p(\Omega)$ independent of time, we recover the definition of the unfolding operator for fixed domains from [15]*

Proposition 5.1. *([36, 37]) Let $p \in [1,+\infty[$ and $q \in [1,+\infty]$. Suppose that u and v are functions in $L^q(0,T;L^p(\Omega))$. Then*
(1) $\mathcal{T}_\varepsilon$ is linear and continuous from $L^q(0,T;L^p(\Omega))$ to $L^q(0,T;L^p(\Omega \times Y))$;
(2)$\mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v) \qquad \forall u,v \in L^q(0,T;L^p(\Omega_\varepsilon))$
(3) If $u \in L^q(0,T;W^{1,p}(\Omega))$ then $\mathcal{T}_\varepsilon(u) \in L^q(0,T;L^p(\Omega;W^{1,p}(Y)))$ and

$$\nabla_y(\mathcal{T}_\varepsilon(u) = \varepsilon\mathcal{T}_\varepsilon(\nabla(u)) \text{ in } \Omega \times Y \times]0,T[$$

(4) For almost every $t \in]0,T[$,

$$\begin{aligned} \frac{1}{|Y|}\int_{\Omega \times Y} \mathcal{T}_\varepsilon(u)(x,y,t)dxdydt = \int_\Omega u(x,t)dx - \int_{\Lambda_\varepsilon} u(x,t)dx \\ = \int_{\widehat{\Omega}_\varepsilon} u(x,t)dx \end{aligned}$$

Proposition 5.2. *([37, 36]). Let $p,q \in [1,+\infty[$. Suppose that $\phi \in L^q(0,T;L^p(\Omega))$ and $\{\phi_\varepsilon\}$ is a sequence in $L^q(0,T;L^p(\Omega))$.*
(1)$\mathcal{T}_\varepsilon(\phi) \longrightarrow \phi$ strongly in $L^q(0,T;L^p(\Omega \times Y))$.
(2)If $\phi_\varepsilon \longrightarrow \phi$ strongly in $L^q(0,T;L^p(\Omega))$, then $\mathcal{T}_\varepsilon(\phi_\varepsilon) \longrightarrow \phi$ strongly in the space $L^q(0,T;L^p(\Omega \times Y))$

Proposition 5.3. *([37, 36]). Let* $p \in]1,+\infty[$ *and* $\{\varphi_\varepsilon\}$*be a sequence in the space* $L^\infty(0,T;W_0^{1,p}(\Omega))$ *such that*

$$\|\nabla\varphi_\varepsilon\|_{L^\infty(0,T;L^p(\Omega))} \leq C.$$

Then there exist $\varphi \in L^\infty(0,T;W_0^{1,p}(\Omega))$ *and* $\widehat{\varphi} \in L^\infty(0,T;L^p(\Omega;W_{per}^{1,p}(Y)))$ *such that up to a subsequence,*
(i) $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi$ *weakly* in* $L^\infty(0,T;L^p(\Omega;W^{1,p}(Y)))$,
(ii) $\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon) \rightharpoonup \nabla_x\varphi + \nabla_y\widehat{\varphi}$ *weakly* in* $L^\infty(0,T;L^p(\Omega\times Y))$.

We end this section by recalling the definition of the mean value operator M_Y and that of the local average operator M_Y^ε and give some of their properties that will be useful in the sequel.

definition 5.2. *Let* $p \in [1,+\infty[$ *and* $q \in [1,+\infty]$. *The mean value operator* M_Y:$L^q(0,T;L^p(\Omega\times Y)) \longmapsto L^q(0,T;L^p(\Omega))$ *is defined by*

$$M_Y(u)(x,t) = \frac{1}{|Y|}\int_Y u(x,y,t)dy$$

for every $u \in L^q(0,T;L^p(\Omega\times Y))$.

definition 5.3. *Let* $p \in [1;+\infty[$ *and* $q \in [1,+\infty]$. *The local average operator* $M_Y^\varepsilon : L^q(0,T;L^p(\Omega)) \longmapsto L^q(0,T;L^p(\Omega))$ *is defined by*

$$M_Y^\varepsilon(\varphi)(x,t) = \frac{1}{|Y|}\int_Y \mathcal{T}_\varepsilon(\varphi)(x,y,t)dy,$$

for any $\varphi \in L^q(0,T;L^p(\Omega))$.

Remark 5.2. *In connection with remark 5.1, some of properties of* $\mathcal{T}_\varepsilon$ *(in the case of dependence on time) can be derived directly for those of the unfolding operator for fixed domains from [15] with the time* t *as a mere parameter.*

As a consequence, we have the following result.

Proposition 5.4. *[11] Let* $p \in [1,+\infty[$ *and* $q \in [1,+\infty]$.
(1) For $\varphi \in L^q(0,T;L^p(\Omega))$, *one has*

$$\mathcal{T}_\varepsilon(M_Y^\varepsilon(\varphi))(x,y,t) = M_Y(\mathcal{T}_\varepsilon(\varphi))(x,t) = M_Y^\varepsilon(\varphi)(x,t) \text{ in } \Omega\times]0,T[$$

(2) Let $\{w_\varepsilon\}$ *be a sequence in* $L^q(0,T;L^p(\Omega))$ *such that*

$$w_\varepsilon \longrightarrow w \text{ strongly in } L^q(0,T;L^p(\Omega)).$$

Then

$$M_Y^\varepsilon(w_\varepsilon) \longrightarrow M_Y(w) = w \text{ strongly in } L^q(0,T;L^p(\Omega)).$$

(3) For any $\varphi \in L^q(0,T;L^p(\Omega))$,

$$\|M_Y^\varepsilon(\varphi)\|_{L^q(0,T;L^p(\Omega))} \leq |Y|^{\frac{1-p}{p}}\|\varphi\|_{L^q(0,T;L^p(\Omega))}.$$

Proof. see [11] □

5.3.1 The time-dependent unfolding operator $\mathcal{T}_\varepsilon^*$ in perforated domains

In this subsection, we present the unfolding operator $\mathcal{T}_\varepsilon^*$ wich maps functions defined on the oscillating domain $\Omega_\varepsilon^* \times (0,T)$ into functions defined on the fixed domain $\Omega \times Y^* \times (0,T)$. This avoids the use of any extension operator. We also give some properties (see [20, 35] for more delails).

definition 5.4. *Let $\varphi \in L^q(0,T;L^p(\Omega_\varepsilon^*))$ where $p \in [1,\infty[$ and $q \in [1,+\infty]$ the unfolding operator $\mathcal{T}_\varepsilon^*$: $L^q(0,T;L^p(\Omega_\varepsilon^*)) \longmapsto L^q(0,T;L^p(\Omega \times Y^*))$ is defined as*

$$\mathcal{T}_\varepsilon^*(\varphi)(x,y,t) = \begin{cases} \varphi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y, t\Big) & \text{a.e. for } (x,y,t) \in \widehat{\Omega}_\varepsilon \times Y^* \times]0,T[, \\ 0 & \text{a.e. for } (x,y,t) \in \Lambda_\varepsilon \times Y^* \times]0,T[. \end{cases}$$

From this definition, the following properties are immediate:

(i) $\mathcal{T}_\varepsilon^*(vw) = \mathcal{T}_\varepsilon^*(v)\mathcal{T}_\varepsilon^*(w)\ \forall w,v \in L^q(0,T;L^p(\Omega_\varepsilon^*))$

(ii)$\mathcal{T}_\varepsilon^*(\psi\varphi) = \varphi\mathcal{T}_\varepsilon^*(\psi),\ \forall \psi \in L^p(\Omega_\varepsilon^*)$ and $\varphi \in L^q(0,T)$

(iii) $\nabla_y(\mathcal{T}_\varepsilon^*(\phi)) = \varepsilon\mathcal{T}_\varepsilon^*(\nabla\phi),\ \forall \phi \in L^q(0,T;W^{1,p}(\Omega_\varepsilon^*))$

Remark 5.3. *Concerning $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$, we have the following:*

$$\mathcal{T}_\varepsilon^*(w)|_{\Omega_\varepsilon^* \times (0,T)} = \mathcal{T}_\varepsilon(w)|_{\Omega \times Y^* \times (0,T)},$$

$$\mathcal{T}_\varepsilon^*(\psi) = \mathcal{T}_\varepsilon(\widetilde{\psi})|_{\Omega \times Y^* \times (0,T)},$$

where w and ψ are defined on $\Omega \times (0,T)$ and $\Omega_\varepsilon^* \times (0,T)$, respectively.

Next we list some properties of the unfolding operator $\mathcal{T}_\varepsilon^*$. the proofs are fond in [35]

Proposition 5.5. *[35] Let $p \in [1,+\infty[$ and $q \in [1,+\infty]$, the operator $\mathcal{T}_\varepsilon^*$ is linear and continuous from $L^q(0,T;L^p(\Omega_\varepsilon^*))$ to $L^q(0,T;L^p(\Omega \times Y^*))$. Let $\phi \in L^q(0,T;L^1(\Omega_\varepsilon^*))$ and $w,v \in L^q(0,T;L^p(\Omega_\varepsilon^*))$.*
For $t \in (0,T)$ we have
(i) $\dfrac{1}{|Y|}\displaystyle\int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y,t)dxdy = \int_{\widehat{\Omega}_\varepsilon} \phi(x,t)dx = \int_{\Omega_\varepsilon^*} \phi(x,t)dx - \int_{\Lambda\varepsilon^*} \phi(x,t)dx$
(ii) $\|\mathcal{T}_\varepsilon^*(w)\|_{L^p(\Omega \times Y^*)} = |Y|^{\frac{1}{p}}\|w\|_{L^p(\widehat{\Omega}_\varepsilon^*)} \leq |Y|^{\frac{1}{p}}\|w\|_{L^p(\Omega_\varepsilon^*)}$

Proposition 5.6. *[35] Let $q \in [1,+\infty]$, let ϕ_ε be in $L^q(0,T;L^1(\Omega_\varepsilon^*))$ and satisfy*

$$\int_0^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| dxdt \longrightarrow 0$$

Then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon dxdt - \frac{1}{|Y|}\int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)dxdydt \longrightarrow 0$$

As usual, this convergence is denoted by

$$\int_0^T \int_{\Omega^*_\varepsilon} \phi_\varepsilon dxdt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(\phi_\varepsilon) dxdydt$$

Moreover, we have the following convergence:

Proposition 5.7. *[35] For $p, q \in (1, +\infty]$, let $\phi_\varepsilon \in L^q(0, T; L^p(\Omega^*_\varepsilon))$ and $\psi \in L^{q'}(0, T; L^{p'}(\Omega^*_\varepsilon))$*
$(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1)$ such that

$$\|\phi_\varepsilon\|_{L^q(0,T;L^p(\Omega^*_\varepsilon))} \leq C \text{ and } \|\psi\|_{L^{q'}(0,T;L^{p'}(\Omega^*_{\varepsilon\varepsilon}))} \leq C \tag{5.6}$$

Then

$$\int_0^T \int_{\Omega^*_\varepsilon} \phi_\varepsilon \psi dxdt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(\phi_\varepsilon) \mathcal{T}^*_\varepsilon(\psi) dxdydt.$$

(ii) For $p, q \in (1, +\infty]$, let $\phi_\varepsilon \in L^q(0, T; L^p(\Omega^*_\varepsilon))$ and $\psi_\varepsilon \in L^{q'}(0, T; L^{p_0}(\Omega^*_\varepsilon))$
$(\frac{1}{p} + \frac{1}{p_0} < 1, \frac{1}{q} + \frac{1}{q'} = 1)$ such that

$$\|\phi_\varepsilon\|_{L^q(0,T;L^p(\Omega^*_\varepsilon))} \leq C \text{ and } \|\psi_\varepsilon\|_{L^{q'}(0,T;L^{p_0}(\Omega^*_{\varepsilon\varepsilon}))} \leq C \tag{5.7}$$

Then

$$\int_0^T \int_{\Omega^*_\varepsilon} \phi_\varepsilon \psi_\varepsilon dxdt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(\phi_\varepsilon) \mathcal{T}^*_\varepsilon(\psi_\varepsilon) dxdydt.$$

Proof. (i) For $x \in \Omega$, notice that $1_{\Lambda^*_\varepsilon} \to 0$ holds true. For a.e. $t \in [0, T]$, in view of (5.6), it follows from the Lebegue dominated convergence theorem that

$$\int_{\Lambda^*_\varepsilon} |\psi|^{p'} dx \to 0$$

and

$$\int_0^T \left(\int_{\Lambda^*_\varepsilon} |\psi|^{p'} dx \right)^{q'/p'} dt \to 0$$

This implies

$$\int_0^T \int_{\Lambda^*_\varepsilon} |\phi_\varepsilon \psi| dxdt \to 0$$

Hence statement (i) holds true due to Proposition 5.6
(ii) Let $1/p' + 1/p = 1$. By (5.7) and the Holder inequality, we have

$$\begin{aligned}
\int_0^T \int_{\Lambda^*_\varepsilon} |\phi_\varepsilon \psi_\varepsilon| dxdt &\leq C \left(\int_0^T \left[\int_{\Lambda^*_\varepsilon} |\psi_\varepsilon|^{p'} dx \right]^{q'/p'} dt \right)^{1/q'} \\
&\leq C \left(\int_0^T \left[\int_{\Lambda^*_\varepsilon} |\psi_\varepsilon|^{p_0} dx \right]^{q'/p_0} dt \right)^{1/q'} . |\Lambda^*_\varepsilon|^{1-(1/p+1/p_0)} \\
&\leq C |\Lambda^*_\varepsilon|^{1-(1/p+1/p_0)}.
\end{aligned}$$

As ε tends to zero, we get

$$\int_0^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon \psi_\varepsilon| dxdt \to 0$$

which implies that statement (ii) holds true, due to Proposition 5.6

□

Proposition 5.8. *[35](Some convergence properties)*
(i) For $p, q \in [1, +\infty)$, let $w \in L^q(0, T; L^p(\Omega))$. Then

$$\mathcal{T}_\varepsilon^*(w) \longrightarrow w \text{ strongly in } L^q(0, T; L^p(\Omega \times Y^*)).$$

(ii) For $p, q \in [1, +\infty)$, let $\{w_\varepsilon\}$ be a sequence in $L^q(0, T; L^p(\Omega))$such that

$$w_\varepsilon \longrightarrow w \text{ strongly in } L^q(0, T; L^p(\Omega))$$

then

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \longrightarrow w \text{ strongly in } L^q(0, T; L^p(\Omega \times Y^*)).$$

(iii) For $p \in (1, +\infty)$ and $q \in (1, +\infty]$, let $\{w_\varepsilon\}$ be a sequence in $L^q(0, T; L^p(\Omega_\varepsilon^*))$ such that

$$\|w_\varepsilon\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq\leq C$$

If

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup \widehat{w} \text{ weakly in } L^q(0, T; L^p(\Omega \times Y^*))$$

then we have

$$\widetilde{w_\varepsilon} \rightharpoonup \theta M_{Y^*}(\widehat{w}) \text{ weakly in } L^q(0, T; L^p(\Omega)).$$

For $q = \infty$, the weak convergences above are replaced by the weak* convergences, respectively.

Proof. (i) Using Proposition 5.5, we have

$$\begin{aligned}\|\mathcal{T}_\varepsilon^*(w) - w\|_{L^q(0,T;L^p(\Omega\times Y^*))} &= \|\mathcal{T}_\varepsilon^*(w - \phi\varphi) + \mathcal{T}_\varepsilon^*(\phi\varphi) - \phi\varphi + \phi\varphi - w\|_{L^q(0,T;L^p(\Omega\times Y^*))} \\ &\leq 2|Y|^{1/p}\|\phi\varphi - w\|_{L^q(0,T;L^p(\Omega))} + \|(\mathcal{T}_\varepsilon^*(\phi) - \phi)\varphi\|_{L^q(0,T;L^p(\Omega\times Y^*))}\end{aligned}$$

for any $\phi \in \mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Since

$$\mathcal{T}_\varepsilon^*(\phi) \to \phi \quad \text{strongly} \quad \text{in} \quad L^q(0, T; L^p(\Omega \times Y^*))$$

then

$$\|(\mathcal{T}_\varepsilon^*(\phi) - \phi)\varphi\|_{L^q(0,T;L^p(\Omega\times Y^*))} \to 0$$

Consequently, we have

$$\limsup_{\varepsilon\to 0} \|(\mathcal{T}_\varepsilon^*(w) - w)\|_{L^q(0,T;L^p(\Omega\times Y^*))} \leq 2|Y|^{1/p}\|\phi\varphi - w\|_{L^q(0,T;L^p(\Omega))}$$

which implies statement (i) due to the density of $\mathcal{D}(0,T)\otimes\mathcal{D}(\Omega)$ in $L^q(0,T;L^p(\Omega))$.
(ii) From Proposition 5.5 we have

$$\|\mathcal{T}_\varepsilon^*(w_\varepsilon - w)\|_{L^q(0,T;L^p(\Omega\times Y^*))} \leq |Y|^{1/p}\|w_\varepsilon - w\|_{L^q(0,T;L^p(\Omega))}$$

Hence statement (ii) follows from statement (i).
(iii) Let $\psi \in L^{q'}(0,T;L^{p'}(\Omega))(1/p+1/p'=1, 1/q+1/q'=1)$. From Proposition 5.7(i)

$$\lim_{\varepsilon\to0}\int_0^T\int_\Omega \widetilde{w}_\varepsilon\psi dxdt = \lim_{\varepsilon\to0}\int_0^T\int_{\Omega_\varepsilon^*} w_\varepsilon\psi dxdt = \lim_{\varepsilon\to0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(w_\varepsilon)\mathcal{T}_\varepsilon^*(\psi)dxdydt.$$

Since statement (1) gives

$$\mathcal{T}_\varepsilon^*(w) \to w \;\; strongly \;\; in \;\; L^{q'}(0,T;L^{p'}(\Omega\times Y^*)),$$

then we get

$$\lim_{\varepsilon\to0}\int_0^T\int_\Omega \widetilde{w}_\varepsilon\psi dxdt = \int_0^T\left\{\frac{1}{|Y|}\int_{Y^*}\widehat{w}dy\right\}\psi dxdt = \theta\int_0^T\int_\Omega \mathcal{M}_{Y^*}(\widehat{w})\psi dxdt$$

which implies the desired result.

□

Proposition 5.9. *[35] For $p\in(1,+\infty)$ and $q\in(1,+\infty]$, let ϕ_ε belong to $L^q(0,T;W^{1,p}(\Omega_\varepsilon^*))$ and satisfy*

$$\|\phi_\varepsilon\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} + \varepsilon\|\nabla\phi_\varepsilon\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq C$$

Then, there exists $\phi\in L^q(0,T;L^p(\Omega;W^{1,p}_{per}(Y^*)))$, such that, up to a susequence,

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightharpoonup \phi \; weakly \; in \; L^q(0,T;L^p(\Omega;W^{1,p}(Y^*))),$$

$$\varepsilon\mathcal{T}_\varepsilon^*(\nabla\phi_\varepsilon) \rightharpoonup \nabla_y\phi \; weakly \; in \; L^q(0,T;L^p(\Omega\times Y^*)).$$

For $q=\infty$, the weak convergences above are replaced by the weak* convergences, respectively.

Theoreme 5.1. *([35])*
For $p\in(0,+\infty)$ let $\{w_\varepsilon\}$ be a sequence in $L^\infty(0,T;W_0^{1,p}(\Omega_\varepsilon^*;\partial\Omega))$ such that

$$\|\nabla w_\varepsilon\|_{L^\infty(0,T;L^p(\Omega_\varepsilon^*))} \leq C \; and \; \|\frac{\partial w_\varepsilon}{\partial t}\|_{L^\infty(0,T;L^p(\Omega_\varepsilon^*))} \leq C$$

then there exist $w\in L^\infty(0,T;W_0^{1,p}(\Omega))$ with $\frac{\partial w}{\partial t}\in L^\infty(0,T;L^p(\Omega))$ and $\widehat{w}\in L^\infty(0,T;L^p(\Omega;W^{1,p}_{per}(Y^*)))$ with $M_{Y^*}(\widehat{w})\equiv 0$, such that, up to a subsequence,

(i) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w$ weakly* in $L^\infty(0,T;L^p(\Omega;W^{1,p}(Y^*)))$

(ii) $\mathcal{T}^*_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla \widehat{w}$ *weakly* in* $L^\infty(0,T;L^p(\Omega \times Y^*))$

(iii) $\mathcal{T}^*_\varepsilon\left(\dfrac{\partial w_\varepsilon}{\partial t}\right) \rightharpoonup \dfrac{\partial w}{\partial t}$ *weakly* in* $L^\infty(0,T;L^p(\Omega \times Y^*))$

(iv) $\mathcal{T}^*_\varepsilon(w_\varepsilon) \longrightarrow w$ *strongly in* $L^q(0,T;L^p(\Omega;W^{1,p}(Y^*)))$,

(v)$\|w_\varepsilon - w\|_{L^q(0,T;L^p(\Omega^*_\varepsilon))} \to 0$

5.4 Unfolding operator in domains depending in two parameters

in this section we recall the definition and some of this properties of the unfolding operator $\mathcal{T}_{\varepsilon\delta}$ depending on two parametres ε and δ, as introduced in [19].

definition 5.5. *([19]) Let* $p \in [1,+\infty[$. *For* $\phi \in L^p(\Omega)$, *the unfolding operator* $\mathcal{T}_{\varepsilon\delta}$ *is the function* $\mathcal{T}_{\varepsilon\delta} : L^p(\Omega) \longrightarrow L^p(\Omega \times \mathbb{R}^N)$ *defined by*

$$\mathcal{T}_{\varepsilon\delta}(\varphi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)\Big(x,\delta z\Big) & \text{if } (x,z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathcal{T}_\varepsilon$ *is the operator for fixed domains as introduced in [15]*
To go further, let us introduce what is called a perforated domain with small holes, denoted here $\Omega^*_{\varepsilon\delta}$. *Let* $B \subset\subset Y$ *and denote* $Y^*_\delta = Y/\delta\overline{B}$. *Then* $\Omega^*_{\varepsilon\delta}$ *is defined as*

$$\Omega^*_{\varepsilon\delta} = \Big\{x \in \Omega \text{ such that } \{\frac{x}{\varepsilon}\}_Y \in Y^*_\delta\Big\}$$

Where $\delta \longrightarrow 0$ *with* ε.

Remark 5.4. *As shown in [19], it turns out that the operator* $\mathcal{T}_{\varepsilon\delta}$ *is well-adapted for domains with small holes when dealing with functions which vanish on the boundary of* $\Omega^*_{\varepsilon\delta}$. *It is precisely the case we we treat in this work. We will deal wiyh functions belonging in particular, to* $H^1_0(\Omega^*_{\varepsilon\delta})$. *The extensions of these functions by zero to the wholr of* Ω, *belong to* $H^1_0(\Omega)$. *Consequently, we will not distinguish the elements of* $H^1_0(\Omega^*_{\varepsilon\delta})$ *and their extensions from* $H^1_0(\Omega)$.

Proposition 5.10. *([19])*
(1) For any $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon\delta}(vw) = \mathcal{T}_{\varepsilon\delta}(u)\mathcal{T}_{\varepsilon\delta}(w)$
(2) Eor any $u \in L^1(\Omega)$,

$$\delta^N \int_{\Omega\times\mathbb{R}^N} |\mathcal{T}_{\varepsilon\delta}(u)|dxdz \leq \int_\Omega |u|dx$$

(3) For any $u \in L^2(\Omega)$,

$$\|\mathcal{T}_{\varepsilon\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leq \frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}$$

(4) For any $u \in L^1(\Omega)$,

$$\left|\int_\Omega u dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(u) dx dz\right| \leq \int_{\Lambda_*} |u| dx.$$

(5) Let $u \in H^1(\Omega)$. *Then*

$$\mathcal{T}_{\varepsilon\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon\delta}(u)) \text{ in } \Omega \times \frac{1}{\delta}Y$$

(6) Suppose $N \geq 3$ *and let* $w \subset \mathbb{R}^N$ *be open and bounded. The following estimates hold:*

$$\|\nabla_z(\mathcal{T}_{\varepsilon\delta}(u))\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

$$\|\mathcal{T}_{\varepsilon\delta}(u - M^\varepsilon_Y(u))\|^2_{L^2(\Omega;L^{2*}(\mathbb{R}^N))} \leq \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

$$\|\mathcal{T}_{\varepsilon\delta}(u)\|^2_{L^2(\Omega\times w)} \leq \frac{2C\varepsilon^2}{\delta^{N-2}}|w|^{2/N}\|\nabla u\|^2_{L^2(\Omega)} + 2|w|\|u\|^2_{L^2(\Omega)},$$

where C *is the Sobolev-Poincaré-Wirtinger constant* $H^1(Y)$.

(7)Suppose $N \geq 3$ *and let* $w_{\varepsilon\delta}$ *be a sequence in* $H^1(\Omega)$ *which is uniformly bounded as both* ε *and* δ *approach* 0. *Then there exists* W *in* $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^2(\Omega\times\mathbb{R}^N)$*such that, up to a subsequence,*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\left(\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta}) - M^\varepsilon_Y(w_{\varepsilon\delta})1_{\frac{1}{\delta}Y}\right) \rightharpoonup W \text{ weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)),$$

and

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\left(\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta})\right)1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \text{ weakly in } L^2(\Omega\times\mathbb{R}^N).$$

Furthermore, if

$$\limsup_{(\varepsilon,\delta)\longrightarrow(0^+,0^+)} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty,$$

then one can choose the subsequence above and some $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta}) \rightharpoonup U \text{weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)).$$

definition 5.6. *A sequence* $\{v_{\varepsilon\delta}\}$ *in* $L^1(\Omega)$ *satisfies the unfolding criterion for integrals (u.c.i) if*

$$\int_\Omega v_{\varepsilon\delta} dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(v_{\varepsilon\delta}) dx dz \longrightarrow 0,$$

for every sequence $(\varepsilon, \delta) \longrightarrow (0^+, 0^+)$. *This property is denoted*

$$\int_\Omega v_{\varepsilon\delta} dx \overset{\mathcal{T}_{\varepsilon\delta}}{\cong} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(v_{\varepsilon\delta}) dx dz,$$

Proposition 5.11. *([19])(u.c.i). If $\{v_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_*} |u_\varepsilon| dx \longrightarrow 0$$

then it satisfies u.c.i.

Corollaire 5.1. *([19]). Let $\{u_\varepsilon\}$ be bounded in $L^2(\Omega)$ with $p > 2$. Then $\{u_\varepsilon v_\varepsilon\}$ satisfies u.c.i.*

Remark 5.5. *As observed in [19], for any $\psi \in \mathcal{D}(\Omega)$, one has*

$$\|\mathcal{T}_{\varepsilon\delta}(\psi) - \psi\|_{L^\infty(\widehat{\Omega}_\varepsilon \times \frac{1}{\delta} Y)} \longrightarrow 0$$

5.5 Time-dependent unfolding operator in domains with two parameters

In this section, we extend the operator $\mathcal{T}_{\varepsilon\delta}$ defined in the previous section to time-dependent functions by adapting what is done in [36]. We start by $\Omega^*_{\varepsilon\delta} \times]0, T[$, depending on ε and δ.

definition 5.7. *Let $p \in [0, \infty[$ and $q \in [1, +\infty]$Let $\varphi \in L^q(0, T; L^p(\Omega))$. The unfolding operator $\mathcal{T}_{\varepsilon\delta} : L^q(0, T; L^p(\Omega)) \longrightarrow L^q(0, T; L^p(\Omega \times \mathbb{R}^N))$*

$$\mathcal{T}_{\varepsilon\delta}(\varphi)(x, z, t) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(x, \delta z, t) & \text{a.e. for } (x, z, t) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta} Y \times]0, T[, \\ 0 & \text{otherwise .} \end{cases}$$

that is

$$\mathcal{T}_{\varepsilon\delta}(\varphi)(x, z, t) = \begin{cases} \varphi\Big(\varepsilon \Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\delta z, t\Big) & \text{if } (x, z, t) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta} Y \times]0, T[, \\ 0 & \text{otherwise .} \end{cases}$$

As mentioned above , for $\delta = 1$ we are in presence of the unfolding operator for fixed domains introduced in [15] Being defined by means of the the operator $\mathcal{T}_\varepsilon$, the unfolding operator $\mathcal{T}_{\varepsilon\delta}$ inherits most of the general properties of it. In particular, the following proposition is straightforward:

Proposition 5.12. *[11] Let $p \in [0, +\infty[$ and $q \in [1, +\infty]$.*
(1) $\mathcal{T}_{\varepsilon\delta}$ is linear and continuous from $L^q(0, T; L^p(\Omega))$ to $L^q(0, T; L^p(\Omega \times \mathbb{R}^N))$.
(2) $\mathcal{T}_{\varepsilon\delta}(uv) = \mathcal{T}_{\varepsilon\delta}(u)\mathcal{T}_{\varepsilon\delta}(v)$ for every $u, v \in L^q(0, T; L^p(\Omega))$.
(3) $\nabla_z(\mathcal{T}_{\varepsilon\delta}(\varphi)) = \varepsilon\delta\mathcal{T}_{\varepsilon\delta}(\nabla\varphi)$ in $\Omega \times \frac{1}{\delta} Y \times]0, T[$ for all $\varphi \in L^q(0, T; H^1(\Omega))$.

The proofs of the following theorem is finde in [11]

Theoreme 5.2. *[11] Let $p \in [0, +\infty[$ and $q \in [1, +\infty]$.*
(1) Let $\varphi \in L^q(0, T; L^p(\Omega))$

$$\frac{\delta^N}{|Y|} \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(\varphi)(x, z, t)dxdz = \int_{\widehat{\Omega}_\varepsilon} \varphi(x, t)dx = \int_{\Omega} (\varphi)(x, t)dx - \int_{\Lambda_\varepsilon} \varphi(x, t)dx$$

for a.e. $t \in]0, T[$
(2) The continuity of the operator $\mathcal{T}_{\varepsilon\delta}$ from proposition 5.12 reads as follow:

$$\|\mathcal{T}_{\varepsilon\delta}(\varphi)\|_{L^q(0,T;L^p(\Omega))} \leq \left(\frac{|Y|}{\delta^N}\right)^{1/p} \|\varphi\|_{L^q(0,T;L^p(\Omega))}. \tag{5.8}$$

(3) Let $\varphi \in L^q(0, T; H^1(\Omega))$ and $N \geq 3$. Then, for a.e. $t \in]0, T[$,

$$\|\nabla_z(\mathcal{T}_{\varepsilon\delta}(\varphi))\|_{L^p(\Omega \times \frac{1}{\delta}Y)} \leq \frac{\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}} \|\nabla\varphi\|_{L^p(\Omega)}$$

Regarding the integral formulas, one still has an unfolding criterion for integrals, which is very useful in homogenization problems.

Proposition 5.13. *[11] Let $q \in [1, +\infty]$ and $\varphi_\varepsilon \in L^q(0, T; L^1(\Omega))$ satisfying*

$$\int_0^T \int_{\Lambda_\varepsilon} \varphi_\varepsilon dxdt \longrightarrow 0$$

then

$$\int_0^T \int_{\Omega} \varphi_\varepsilon dxdt \overset{\mathcal{T}_{\varepsilon\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(\varphi_\varepsilon)dxdzdt$$

The proof of the following proposition is essentially the same as that of [36]

Proposition 5.14. *[11] Let $p, q \in]1, +\infty]$. Let $\{\varphi_\varepsilon\}$ be a sequence in $L^q(0, T; L^p(\Omega))$ and $\{\psi_\varepsilon\}$ be a sequence in $L^{q'}(0, T; L^{p_0}(\Omega))$, such that*

$$\|\varphi_\varepsilon\|_{L^q(0,T;L^p(\Omega))} \leq C \text{ and } \|\psi_\varepsilon\|_{L^{q'}(0,T;L^{p_0}(\Omega))} \leq C,$$

where $\dfrac{1}{p} + \dfrac{1}{p_0} < 1$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Then,

$$\int_0^T \int_{\Omega} \varphi_\varepsilon \psi_\varepsilon dxdt \overset{\mathcal{T}_{\varepsilon\delta}}{\backsimeq} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega \times \frac{1}{\delta}Y} \mathcal{T}_{\varepsilon\delta}(\varphi_\varepsilon \psi_\varepsilon)dxdzdt$$

Proposition 5.15. *[11] Let $u \in L^q(0, T; H^1(\Omega))$. For $q \in [1, +\infty[$, one has the estimates*

$$\|\mathcal{T}_{\varepsilon\delta}(u - M_Y^\varepsilon(u))\|_{L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N)))} \leq \frac{C\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}} \|\nabla u\|_{L^q(0,T;L^p(\Omega))},$$

and for w an open and bounded subset of $\mathbb{R}^N$,

$$\|\mathcal{T}_{\varepsilon\delta}(u)\|_{L^q(0,T;L^p(\Omega \times w))} \leq \frac{2C\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}} \|\nabla u\|_{L^q(0,T;L^p(\Omega))} + 2|w||Y|^{\frac{1-p}{p}} \|u\|_{L^q(0,T;L^p(\Omega))},$$

where C is the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$

Theoreme 5.3. *([11]) Let* $p \in [1,+\infty[, q \in [1,+\infty], N \geq 3, \{w_{\varepsilon\delta}\}$ *be a sequence in* $L^q(0,T;H^1(\Omega))$ *which is uniformly bounded with respect to* ε *and* δ *as* $(\varepsilon,\delta) \longrightarrow (0,0)$. *Then up to a subsequence, there exists* W *in* $L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N)))$ *with* $\nabla_z W$*in* $L^q(0,T;L^p(\Omega\times\mathbb{R}^N))$

$$\frac{\delta^{\frac{N}{P}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta}) - M^{\varepsilon}_{Y}(w_{\varepsilon\delta})1_{\frac{1}{\delta}Y}) \rightharpoonup W \ \textit{weakly in}\ L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N))) \tag{5.9}$$

and

$$\frac{\delta^{\frac{N}{P}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta})1_{\frac{1}{\delta}Y}) \rightharpoonup \nabla_z W \ \textit{weakly in}\ L^q(0,T;L^p(\Omega\times\mathbb{R}^N)) \tag{5.10}$$

Furthermore, if

$$k = \limsup_{(\varepsilon,\delta)\longrightarrow(0^+,0^+)} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon} < +\infty \tag{5.11}$$

then one can choose the subsequence above and some $U \in L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N)))$ *with*

$$\frac{\delta^{\frac{N}{P}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta}) \rightharpoonup U \ \textit{weakly in}\ L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N))) \tag{5.12}$$

5.5.1 The time-dependent boundary unfolding operator in domains with two parameters

definition 5.8. *Let* $p \in [0,\infty[$ *and* $q \in [1,+\infty]$*Let* $\varphi \in L^q(0,T;L^p(\partial T_{\varepsilon\delta}))$. *The unfolding operator* $\mathcal{T}_{\varepsilon\delta}{}^b : L^q(0,T;L^p(\partial T_{\varepsilon\delta})) \longrightarrow L^q(0,T;L^p(\mathbb{R}^N\times\partial T))$

$$\mathcal{T}^b_{\varepsilon\delta}(\varphi)(x,z,t) = \begin{cases} \varphi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon\delta z, t\Big) & \textit{if } (x,z,t)\in\mathbb{R}^N\times\partial T\times]0,T[, \\ 0 & \textit{otherwise .} \end{cases}$$

Proposition 5.16. *The linear and continuous boundary unfolding operator* $\mathcal{T}^b_{\varepsilon\delta}$ *has the following propertie: for every* $\phi \in L^q(0,T;L^1(\partial T_{\varepsilon\delta}))$

$$\int_0^T\int_{\partial T_{\varepsilon\delta}} \phi(x,t)d\sigma(x) = \frac{\delta^{N-1}}{\varepsilon}\int_0^T\int_{\mathbb{R}^N\times\partial T} \mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z,t)dxd\sigma(z)dt$$

5.6 Homogenization of a class of hyperbolic-parabolic problems

In this section, we suppose that $N \geqslant 3$ and that ε and $\delta_1 = \delta_1(\varepsilon), \delta_2 = \delta_2(\varepsilon)$ are such that (5.12) holds, that is, there exists the following limit and is finite:

$$k_1 = \lim_{\varepsilon\longrightarrow 0}\frac{\delta_1^{N-1}}{\varepsilon} < +\infty \quad \textit{and} \quad k_2 = \lim_{\varepsilon\longrightarrow 0}\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} < +\infty \tag{5.13}$$

We also denote by $M(\alpha,\beta,\Omega)$ the set of $N\times N$ matrices $A = (a_{ij})_{1\leqslant i,j\leqslant N}$ in $(L^\infty(\Omega))^{N\times N}$ such that

(i) $(A(x)\lambda,\lambda) \geqslant \alpha|\lambda|^2$,

(ii)$|A(x)\lambda| \leqslant \beta|\lambda|$

5.6.1 Hyperbolic-parabolic equation

We want to study the asymptotic behavior as $\varepsilon \longrightarrow 0$, of the problem (5.2) We suppose that the data satisfy the following assumptions:

$$\begin{aligned}
&\text{(i) } A^{\varepsilon} \in M(\alpha,\beta,\Omega), A^{\varepsilon} \text{ symmetric}\\
&\text{(ii) } f_{\varepsilon} \in L^2(0,T;L^2(\Omega_{\varepsilon\delta_1\delta_2})),\\
&\text{(iii) } u^0_{\varepsilon\delta_1\delta_2} \in H^1_0(\Omega_{\varepsilon\delta_1\delta_2}),\\
&\text{(iv) } u^1_{\varepsilon\delta_1\delta_2} \in L^2(\Omega)
\end{aligned} \tag{5.14}$$

For the coefficients $\alpha_{\varepsilon\delta_1\delta_2}$ and $\beta_{\varepsilon\delta_1\delta_2}$, we assume that $\alpha_{\varepsilon\delta_1\delta_2} > 0$ and

$$\begin{aligned}
&(1) \|\alpha_{\varepsilon\delta_1\delta_2}\|_{L^\infty(\Omega)} \leq c \text{ and } \|\beta_{\varepsilon\delta_1\delta_2}\|_{L^\infty(\Omega)} \leq c\\
&(2)\ \mathcal{T}_{\varepsilon}(\alpha_{\varepsilon\delta_1\delta_2}) \longrightarrow \alpha(x,y) \text{ strongly in } L^2(\Omega\times Y) \text{ and } \alpha^* = \mathcal{M}_Y(\alpha)\\
&(3)\ \mathcal{T}_{\varepsilon}(\beta_{\varepsilon\delta_1\delta_2}) \longrightarrow \beta(x,y) \text{ strongly in } L^2(\Omega\times Y) \text{ and } \beta^* = \mathcal{M}_Y(\beta)\\
&(4)\ \sqrt{\alpha_{\varepsilon\delta_1\delta_2}} \rightharpoonup \rho \text{ weakly } in L^2(\Omega)
\end{aligned} \tag{5.15}$$

Moreover, we assume that

$$\begin{aligned}
&\text{(i) } u^0_{\varepsilon\delta_1\delta_2} \rightharpoonup u^0 \text{ weakly in } H^1_0(\Omega)\\
&\text{(ii) } u^1_{\varepsilon\delta_1\delta_2} \rightharpoonup u^1 \text{ weakly in } L^2(\Omega),\\
&\text{(iii) } f_{\varepsilon} \rightharpoonup f \text{ weakly in } L^2(0,T;L^2(\Omega))
\end{aligned} \tag{5.16}$$

the set

$$\begin{aligned}
\mathcal{W}_{\varepsilon\delta_1\delta_2} = \{v_{\varepsilon\delta_1\delta_2} \in L^2(0,T;H^1_0(\Omega_{\varepsilon\delta_1\delta_2})) :& v'_{\varepsilon\delta_1\delta_2} \in L^2(0,T;L^2(\Omega_{\varepsilon\delta_1\delta_2})),\\
&\alpha_{\varepsilon\delta_1\delta_2} v''_{\varepsilon\delta_1\delta_2} \in L^2(0,T;H^{-1}(\Omega_{\varepsilon\delta_1\delta_2})\}
\end{aligned}$$

The variational formulation of problem (5.2) is:

$$\begin{cases}
\qquad\qquad \text{Find } u_{\varepsilon\delta_1\delta_2} \in \mathcal{W}_{\varepsilon\delta_1\delta_2} \text{ such that for all } v \in H^1_0(\Omega_{\varepsilon\delta_1\delta_2}),\\
\langle \alpha_{\varepsilon\delta_1\delta_2} u''_{\varepsilon\delta_1\delta_2}(x,t), v(x)\rangle_{(H^1_0(\Omega_{\varepsilon\delta_1\delta_2}))', H^1_0(\Omega^*_{\varepsilon\delta_1\delta_2})} + \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} \beta_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x) v(x)dx\\
+ \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^{\varepsilon}(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla v(x)dx + \nu\varepsilon^{\gamma}\int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}(x,t)v(x)d\sigma(x)\\
= \displaystyle\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_{\varepsilon}(x,t)v(x)dx \text{ in } \mathcal{D}'(0,T)\\
\qquad\qquad u_{\varepsilon\delta_1\delta_2}(x,0) = u^0_{\varepsilon\delta_1\delta_2}(x) \qquad \text{in } \Omega_{\varepsilon\delta_1\delta_2}\\
\qquad \alpha_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x,0) = \sqrt{\alpha_{\varepsilon\delta_1\delta_2}} u^1_{\varepsilon\delta_1\delta_2}(x) \qquad \text{in } \Omega_{\varepsilon\delta_1\delta_2}
\end{cases} \tag{5.17}$$

For every fixed ε, following the classical arguments [44], we know that the problem (5.2) has a unique solution $u_{\varepsilon\delta_1\delta_2}$ such that

$u_{\varepsilon\delta_1\delta_2} \in L^\infty(0,T;H^1_0(\Omega_{\varepsilon\delta_1\delta_2})) u'_{\varepsilon\delta_1\delta_2} \in L^2(0,T;L^2(\Omega_{\varepsilon\delta_1\delta_2}))$ and $\alpha_{\varepsilon\delta_1\delta_2} u''_{\varepsilon\delta_1\delta_2} \in L^2(0,T;(H^1_0(\Omega_{\varepsilon\delta_1\delta_2}))')$

Lemma 5.1. *Suppose that the assumptions* (5.14), (5.15) *and* (5.16) *are satisfied. For every* ε, *we have the following uniform estimates:*

$$\|u_{\varepsilon\delta_1\delta_2}\|_{L^\infty(0,T;H_0^1(\Omega_{\varepsilon\delta_1\delta_2}))}+\|u'_{\varepsilon\delta_1\delta_2}\|_{L^2(0,T;L^2(\Omega_{\varepsilon\delta_1\delta_2}))}+\|\sqrt{\alpha_{\varepsilon\delta_1\delta_2}}u'_{\varepsilon\delta_1\delta_2}\|_{L^\infty(0,T;L^2(\Omega_{\varepsilon\delta_1\delta_2}))}\leq c \tag{5.18}$$

where c *is independent of* ε *and* δ

We adapt here for the evolution problem some argument introduced in [19]. Let us introduce the functional space

$$K_B=\left\{\Phi\in L^2(0,T;L^{2^*}(\mathbb{R}^N)):\nabla\Phi\in L^2(0,T;L^2(\mathbb{R}^N)),\Phi \text{ is constant on } B\right\} \tag{5.19}$$

We also need the following lemmas from [19] in order to pass to the limit in equation (5.17)

Lemma 5.2. *([19]).*
Let $N\geq 3$. *Then, for every* $\delta_0>0$, *the set*

$$\bigcup_{0<\delta<\delta_0}\left\{\phi\in H^1_{per}(Y):\phi=0 \ on \ \delta B\right\},$$

is dense in $H^1_{per}(Y)$

Lemma 5.3. *([19]).*
Let $v\in\mathcal{D}(\mathbb{R}^N)\cap K_B$*(i.e.,* $v=v(B)$ *is constant on* B*) and set*

$$w_{\varepsilon\delta}(x)=v(B)-v\left(\frac{1}{\delta}\{\frac{x}{\varepsilon}\}_Y\right) \ for \ x\in\mathbb{R}^N.$$

then

$$w_{\varepsilon\delta}\rightharpoonup v(B) \ weakly \ in \ H^1(\Omega) \tag{5.20}$$

Remark 5.6. *(1) From the definition of* $w_{\varepsilon\delta}$ *above, one has*

$$\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta})(x,z)=v(B)-v(z) \ in \ \widehat{\Omega_\varepsilon}\times\frac{1}{\delta}Y,$$

and consequently (see [19]),

$$\mathcal{T}_{\varepsilon\delta}(\nabla w_{\varepsilon\delta})=-\frac{1}{\varepsilon\delta}\nabla_z v \ in \ \widehat{\Omega_\varepsilon}\times\frac{1}{\delta}Y \tag{5.21}$$

(2)Let $\{w_{\varepsilon\delta}\}$ *be a sequence satisfying* (5.20). *We have,*

$$\mathcal{T}_{\varepsilon\delta}(w_{\varepsilon\delta})\longrightarrow v(B) \ strongly \ in \ L^2(\Omega\times Y). \tag{5.22}$$

Indeed, it was shown in [19] that $\{w_{\varepsilon\delta}\}$ is bounded in $H^1(\Omega)$ so that together with (5.20) and Rellich compactness theorem, one has $w_{\varepsilon\delta}\longrightarrow v(B)$ strongly in $L^2(\Omega)$; that is

$$\|w_{\varepsilon\delta}-v(B)\|_{L^2(\Omega)}\longrightarrow 0$$

(see [19]) This, together with proposition5.1 (2) gives (5.22)
We state now a homogenization theorem for system (5.2)

Theoreme 5.4. *Under assumptions* (5.14), (5.15) *and* (5.16), *suppose that as* $\varepsilon \longrightarrow 0$ *there is a matrix field* A *such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,y) \longrightarrow A(x,y) \text{ a.e. in } \quad \Omega \times Y, \tag{5.23}$$

and as both $\varepsilon, \delta_2 \longrightarrow 0$, *there a matrix field* A^0 *such that*

$$\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon)(x,z) \longrightarrow A^0(x,z) \text{ a.e. in } \quad \Omega \times (\mathbb{R}^N \setminus B). \tag{5.24}$$

Let $u_{\varepsilon\delta_1\delta_2}$ *be the solution of* (5.17). *Then there exists* u *in* $L^\infty(0,T;H_0^1(\Omega))$ *and* $\widehat{u}$ *in* $L^\infty(0,T;L^2(\Omega;H^1_{per}(Y)))$ *such that*

$$\begin{aligned} &\text{(i) } u_{\varepsilon\delta_1\delta_2} \rightharpoonup u \text{ weakly* in } L^\infty(0,T;H_0^1(\Omega)),\\ &\text{(ii) } u'_{\varepsilon\delta_1\delta_2} \rightharpoonup u' \text{ weakly* in } L^\infty(0,T;L^2(\Omega)),\\ &\text{(iii) } \mathcal{T}_\varepsilon(u_{\varepsilon\delta_1\delta_2}) \rightharpoonup u \text{ weakly* in } L^\infty(0,T;L^2(\Omega;H^1(Y))),\\ &\text{(iv) } \mathcal{T}_\varepsilon(u'_{\varepsilon\delta_1\delta_2}) \rightharpoonup u' \text{ weakly* in } L^\infty(0,T;L^2(\Omega\times Y)),\\ &\text{(v) } \mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta_1\delta_2}) \rightharpoonup \nabla_x u + \nabla_y \widehat{u} \text{ weakly* in } L^\infty(0,T;L^2(\Omega\times Y)), \end{aligned} \tag{5.25}$$

and $U \in L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N)))$, *such that*

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}) \rightharpoonup U \text{ weakly in } L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N))), \tag{5.26}$$

with U *vanishing on* $\Omega \times B \times]0,T[$ *and* $U - k_2 u \in L^2(0,T;L^2(\Omega;K_B))$ *(*K_B *being defined by* (5.19)*). While the function* U *obeys*

$$\int_{\mathbb{R}^N\setminus B} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)dz = 0 \tag{5.27}$$

for a.e. $x \in \Omega$, *a.e* $t \in]0,T[$ *and for all* $v \in K_B$, *with* $v(B) = 0$.
The ordered triplet $(u,\widehat{u},U)$ *satisfies the limit equations*
For $\gamma = 0$

$$\begin{aligned} &\langle \alpha u''(x,t),\psi\rangle_{(H_0^1(\Omega))',H_0^1(\Omega)} + \int_\Omega \beta u'(x,t)\psi(x)dx\\ &\quad+ \int_{\Omega\times Y} A(x,y)((\nabla_x u(x,t) + \nabla_y\widehat{u}(x,y,t))\nabla\psi(x)dxdy\\ &\quad- k_2\int_{\mathbb{R}^N\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B\psi(x)dxd\sigma z + \nu k_1\int_{\mathbb{R}^N\times\partial S} u(x,t)\psi(x)dxd\sigma(z)\\ &\quad= \int_\Omega f(x,t)\psi(x)dx; \text{ for a.e. } t \in]0,T[\text{ and for all } \psi \in H_0^1(\Omega) \end{aligned} \tag{5.28}$$

$$u(x,0) = u^0,\ u'(x,0) = u^1 \text{ in } \Omega$$

For $\gamma > 0$

$$\begin{aligned} &\langle \alpha u''(x,t), \psi \rangle_{(H_0^1(\Omega))', H_0^1(\Omega)} + \int_\Omega \beta u'(x,t)\psi(x)dx \\ &+ \int_{\Omega\times Y} A(x,y)((\nabla_x u(x,t) + \nabla_y \widehat{u}(x,y,t))\nabla\psi(x)dxdy \\ &-k_2 \int_{\mathbb{R}^N\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B \psi(x)dxd\sigma(z) \\ &= \int_\Omega f(x,t)\psi(x)dx; \ \textit{for a.e. } t \in]0,T[\textit{ and for all } \psi \in H_0^1(\Omega) \\ &u(x,0) = u^0, \ u'(x,0) = u^1 \textit{ in } \Omega \end{aligned} \tag{5.29}$$

where ν_B *is the inward normal to* ∂B *and* $d\sigma(z)$ *its surface measure.*
For $\gamma < 0$

$$\nu k_1 \int_{\mathbb{R}^N\times\partial S} u(x,t)\psi(x)dxd\sigma(z) = 0 \tag{5.30}$$

In what follows, we will use the notation $m_Y(.)$ for the average over Y defined as

$$m_Y(v) = \frac{1}{|Y|}\int_Y v(y)dy, \ \forall v \in L^1(Y).$$

The result below describes now homogenized problem in the variable (x,t) in $\Omega\times]0,T[$. To this aim, let us consider the correctors $\widehat{\chi}_j, j = 1, ..., N$ solutions of the cell problem, they are the same domains without holeys (see[9, 18]).

$$\begin{gathered} \widehat{\chi}_j \in L^\infty(\Omega; H^1_{per}(Y)), \\ \int_Y A\nabla(\widehat{\chi}_j - y_j)\nabla\varphi dy = 0 \text{ a.e. } x \in \Omega, \forall\varphi \in H^1_{per}(Y) \\ m_Y(\widehat{\chi}_j) = 0 \end{gathered} \tag{5.31}$$

where A is give by (5.23)
We consider also the cell problem corresponding to the holes B defining the corrector θ for small holes, introduced in [19]

$$\begin{gathered} \theta \in L^\infty(\Omega; K_B), \theta(x,B) \equiv 1, \\ \int_{\mathbb{R}^N\setminus B} {}^tA^0(x,z)\nabla_z\theta(x,z)\nabla_z\Psi(z)dz = 0 \\ \text{a.e. for } x \in \Omega, \forall\Psi \in K_B \text{ with } \Psi(B) = 0 \end{gathered} \tag{5.32}$$

Corollaire 5.2. *Under assumption* (5.14), (5.15) *and* (5.16), $u \in H_0^1(\Omega)$ *is the unique solution of the limit problems*
For $\gamma = 0$

$$\left\{ \begin{array}{rcl} \alpha^* u'' + \beta^* u' - div(A^{hom}\nabla u) + \nu k_1 u + k_2^2\Theta u &=& f \textit{ in } \Omega\times]0,T[\\ u &=& 0 \textit{ on } \partial\Omega\times]0,T[\\ u(x,0) &=& u^0 \textit{ on } \Omega \\ u'(x,0) &=& \dfrac{\rho}{\alpha^*}u^1 \textit{ on } \Omega \end{array} \right. \tag{5.33}$$

For $\gamma > 0$

$$\begin{cases} \alpha^* u'' + \beta^* u' - div(A^{hom}\nabla u) + k_2^2 \Theta u &= f \ in \ \Omega\times]0,T[\\ u &= 0 \ on \ \partial\Omega\times]0,T[\\ u(x,0) &= u^0 \ on \ \Omega \\ u'(x,0) &= \dfrac{\rho}{\alpha^*} u^1 \ on \ \Omega \end{cases} \tag{5.34}$$

For $\gamma < 0$ *In this case we have*

$$u = 0$$

where the homogenized matrix field is

$$A^{hom} = m_Y \left(a_{ij} + \sum_{k=1}^{N} a_{ik} \frac{\partial \widehat{\chi}_j}{\partial y_k} \right) \tag{5.35}$$

and

$$\Theta = \int_{\partial B} {}^t A^0 \nabla_z \theta \nu_B d\sigma z \tag{5.36}$$

Remark 5.7. *As shown in [19], Θcan be interpreted as the local capacity of B. (See also [25, 24]) Moreover, from* (5.32) *it is easily seen that Θ is non-negative, i.e*

$$\Theta(x) = \int_{\mathbb{R}^N \setminus B} A^0(x,z) \nabla_z \theta(x,z) dz \geq 0.$$

that is essential for the existence of the solution of the homogenized system (5.33)

5.6.2 Proof of theorem 5.4

In view of (5.18) and theorem 5.1 , we get that ther exist $u \in L^\infty(0,T;H_0^1(\Omega))$ and $\widehat{u} \in L^\infty(0,T;L^2(\Omega, H^1_{per}(Y^*)))$ with $M_{Y^*}(\widehat{u}) = 0$ such that

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta_1\delta_2}) \rightharpoonup \nabla u + \nabla_y \widehat{u} \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega\times Y)) \tag{5.37}$$

Moreover by theorem 5.3 there exist $U \in L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N)))$ and $W \in L^2(0,T;L^2(\Omega;L^{2^*}(\mathbb{R}^N)))$ with $\nabla_z W \in L^2(0,T;L^2(\Omega\times\mathbb{R}^N))$ such that

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2}) \rightharpoonup U \text{ weakly in } L^2(0,T;L^2(\Omega, L^2_{loc}(\mathbb{R}^N))) \tag{5.38}$$

on the other hand and from (i) and (ii) of (5.25) we have

$$u_{\varepsilon\delta_1\delta_2} \longrightarrow u \text{ strongly in } L^2(0,T;L^2(\Omega)) \tag{5.39}$$

then

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \mathcal{M}_Y^\varepsilon(u_{\varepsilon\delta_1\delta_2}) \longrightarrow k_2 u \ \ strongly \ in \ L^2(0,T;L^2(\Omega;L^2_{loc}(\mathbb{R}^N))) \tag{5.40}$$

from (5.39) and (5.40) we have

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\left(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2})-\mathcal{M}_Y^{\varepsilon}(u_{\varepsilon\delta_1\delta_2})\right)\longrightarrow U-k_2u$$

this and from theorem 5.3 eqaution (5.9) we have

$$W\rightharpoonup U-k_2u \text{ imply } \nabla_zW\rightharpoonup\nabla_zU$$

by definition we have

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\left(\nabla_z\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2})\right)=\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\varepsilon\delta_2\mathcal{T}_{\varepsilon\delta_2}\left(\nabla u_{\varepsilon\delta_1\delta_2}\right))=\delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}\left(\nabla u_{\varepsilon\delta_1\delta_2}\right) \tag{5.41}$$

on the other hand by (5.10) we have

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2})1_{\frac{1}{\delta}Y})\rightharpoonup\nabla_zW=\nabla_zU \text{ weakly in } L^2(0,T;L^2(\Omega\times\mathbb{R}^N))$$

by identification with (5.41) we get

$$\delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}\left(\nabla u_{\varepsilon\delta_1\delta_2}\right)\rightharpoonup\nabla_zU \tag{5.42}$$

also we have

$$\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta_1\delta_2})=0 \text{ in } \Omega\times B\times]0,T[$$

This mean that

$$W=U-k_2u \text{ in } L^2(0,T,L^2(\Omega\times K_B))$$

Now we have

For $\gamma=0$

Let $\psi\in\mathcal{D}(\Omega)$ and $w_{\varepsilon\delta_2}$ is the function defined in lemma 5.3 then we have $w_{\varepsilon\delta_2}\rightharpoonup v(B)$ and by remark 5.6 We have

$$\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2})=\frac{1}{\varepsilon\delta_2}\nabla_z\mathcal{T}_{\varepsilon\delta_2}(w_{\varepsilon\delta_2})=-\frac{1}{\varepsilon\delta_2}\nabla_zv(z) \tag{5.43}$$

Now, to take into account the effect of the function of the perforation, let use $w_{\varepsilon\delta_2}\psi$ as a test function in (5.17), we get

$$\langle\alpha_{\varepsilon\delta_1\delta_2}u''_{\varepsilon\delta_1\delta_2}(x,t),w_{\varepsilon\delta_2}(x)\psi(x)\rangle_{(H_0^1(\Omega_{\varepsilon\delta_1\delta_2}))',H_0^1(\Omega_{\varepsilon\delta_1\delta_2})}+\int_{\Omega_{\varepsilon\delta_1\delta_2}}\beta_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x)w_{\varepsilon\delta_2}(x)\psi(x)dx$$
$$+\int_{\Omega_{\varepsilon\delta_1\delta_2}}A^{\varepsilon}(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla w_{\varepsilon\delta_2}(x)\psi(x)dx+\int_{\Omega_{\varepsilon\delta_1\delta_2}}A^{\varepsilon}(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)w_{\varepsilon\delta_2}(x)\nabla\psi(x)dx$$
$$+\nu\int_{\partial S_{\varepsilon\delta_1}}u_{\varepsilon\delta_1\delta_2}(x,t)w_{\varepsilon\delta_2}(x)\psi(x)d\sigma(x)=\int_{\Omega_{\varepsilon\delta_1\delta_2}}f_{\varepsilon}(x,t)w_{\varepsilon\delta_2}(x)\psi(x)dx$$

Let $\varphi \in \mathcal{D}(0,T)$ and multiply the intergrands in this equation and integrate over $]0,T[$ then

$$\begin{aligned}
&\int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2} u_{\varepsilon\delta_1\delta_1}(x,t) w_{\varepsilon\delta_2}(x)\psi(x)\varphi''(t)dxdt\\
&+\int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} \beta_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_1}(x) w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dxdt\\
&+\int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} A^{\varepsilon}(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dxdt\\
&+\int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} A^{\varepsilon}(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t) w_{\varepsilon\delta_2}(x)\nabla\psi(x)\varphi(t)dxdt\\
&+\nu\int_0^T \int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2} w_{\varepsilon\delta_2}(x)\psi(x)d\sigma(x)dt = \int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} f_{\varepsilon}(x,t) w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dtdx
\end{aligned} \tag{5.44}$$

For the first and second term on the left-hand side of this equation, we apply the operator $\mathcal{T}_{\varepsilon}$.
Thus, from propositin 5.1 (4) , remark 5.6 (2) , (5.15) (2,3) and (5.25) we obtain

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2} u_{\varepsilon\delta_1\delta_2}(x,t) w_{\varepsilon\delta_2}(x)\psi(x)\varphi''(t)dxdt\\
&=\lim_{\varepsilon\longrightarrow 0}\int_0^T \int_{\Omega\times Y} \mathcal{T}_{\varepsilon}(\alpha_{\varepsilon\delta_1\delta_2})\mathcal{T}_{\varepsilon}(u_{\varepsilon\delta_1\delta_2})\mathcal{T}_{\varepsilon}(w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon}(\psi)\varphi''(t)dxdydt\\
&=v(B)\int_0^T \int_{\Omega\times Y} \alpha(x,y)u(x,t)\psi(x)\varphi''(t)dxdydt
\end{aligned} \tag{5.45}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T \int_{\Omega_{\varepsilon\delta_1\delta_2}} \beta_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x,t) w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\longrightarrow 0}\int_0^T \int_{\Omega\times Y} \mathcal{T}_{\varepsilon}(\beta_{\varepsilon\delta_1\delta_2})\mathcal{T}_{\varepsilon}(u'_{\varepsilon\delta_1\delta_2})\mathcal{T}_{\varepsilon}(w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon}(\psi)\varphi(t)dxdydt\\
&=v(B)\int_0^T \int_{\Omega\times Y} \beta(x,y)u'(x,t)\psi(x)\varphi(t)dxdydt
\end{aligned} \tag{5.46}$$

For the third term on the left-hand side of equation (5.44) we use the operator $\mathcal{T}_{\varepsilon\delta_2}$

then proposition 5.13 ,(5.13), (5.24),(5.42), (5.43),and remark 5.6,we get

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\longrightarrow 0}\delta_2^N\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon\delta_\in}(A^\varepsilon)\mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2})\mathcal{T}_{\varepsilon\delta_2}(\nabla w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon\delta_2}(\psi)\varphi(t)dxdzdt\\
&=\lim_{\varepsilon\longrightarrow 0}\delta_2^N\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon)\mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2})\frac{1}{\varepsilon\delta_2}\nabla_z\mathcal{T}_{\varepsilon\delta_2}(w_{\varepsilon\delta_2})\mathcal{T}_{\varepsilon\delta_2}(\psi)\varphi(t)dxdzdt\\
&=\lim_{\varepsilon\longrightarrow 0}\frac{\delta_2^{N-1}}{\varepsilon}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon)\mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2})(-\nabla_z v(z))\mathcal{T}_{\varepsilon\delta_2}(\psi)\varphi(t)dxdzdt\\
&=\lim_{\varepsilon\longrightarrow 0}\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon\delta_1}(A^\varepsilon)\delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2})(-\nabla_z v(z))\mathcal{T}_{\varepsilon\delta_2}(\psi)\varphi(t)dxdzdt\\
&=\lim_{\varepsilon\longrightarrow 0}\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon)\delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla u_{\varepsilon\delta_1\delta_2})(-\nabla_z v(z))\mathcal{T}_{\varepsilon\delta_2}(\psi)\varphi(t)dxdzdt\\
&=-k_2\int_0^T\int_{\Omega\times\mathbb{R}^N}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\
&=-k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\backslash B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt
\end{aligned}$$

now we have

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dxdt\\
&=-k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\backslash B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt
\end{aligned}\tag{5.47}$$

For the forth term on the left-hand side of (5.44) ,we use $\mathcal{T}_\varepsilon$ from proposition 5.1 together with remark 5.6, passing to limit gives

$$\begin{aligned}
&\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)w_{\varepsilon\delta_2}(x)\nabla\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta_1\delta_2})\mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})\mathcal{T}_\varepsilon(\nabla\psi)\varphi(t)dxdydt\\
&=v(B)\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt
\end{aligned}\tag{5.48}$$

For the fifth term on the left-hand side (5.44), we use $\mathcal{T}^b_{\varepsilon\delta_1}$ we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\nu\int_0^T\int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}w_{\varepsilon\delta_2}\psi(x)\varphi(t)d\sigma(x)dt\\
&=\lim_{\varepsilon\to 0}\nu\frac{\delta_1^{N-1}}{\varepsilon}\int_0^T\int_{\mathbb{R}^N\times\partial S}\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta_1\delta_2})\mathcal{T}^b_{\varepsilon\delta_1}(w_{\varepsilon\delta_2})\mathcal{T}^b_{\varepsilon\delta_1}(\psi(x))\varphi(t)dxd\sigma(z)dt\\
&=\nu k_1 v(B)\int_0^T\int_{\mathbb{R}^N\times\partial S}u(x,t)\psi(x)\varphi(t)dxd\sigma(z)dt
\end{aligned}\tag{5.49}$$

For the term on the right-hand side of equation (5.44) we apply $\mathcal{T}_\varepsilon$ from proposition 5.1 together with remark 5.6 yields

$$\begin{aligned}\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon(x,t)w_{\varepsilon\delta_2}(x)\psi(x)\varphi(t)dtdx &= \lim_{\varepsilon\longrightarrow 0}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(f_\varepsilon)\mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})\mathcal{T}_\varepsilon(\psi)\varphi(t)dxdydt\\ &= v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)dxdydt\end{aligned} \tag{5.50}$$

Thus combining (5.45),(5.46), (5.47), (5.48), (5.49) and (5.50), the limit equation of (5.44) is

$$\begin{aligned}&v(B)\int_0^T\int_{\Omega\times Y}\alpha(x,y)u(x,t)\psi(x)\varphi''(t)dxdydt\\ &+v(B)\int_0^T\int_{\Omega\times Y}\beta(x,y)u'(x,t)\psi(x)\varphi(t)dxdydt\\ &-k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\ &+v(B)\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt\\ &+\nu k_1 v(B)\int_0^T\int_{\mathbb{R}^N\times\partial S}u(x,t)\psi(x)\varphi(t)dxd\sigma(z)dt = v(B)\int_0^T\int_{\Omega\times Y}f(x,t)\psi(x)\varphi(t)dxdydt\end{aligned} \tag{5.51}$$

which is true for all $\varphi\in\mathcal{D}(0,T)$, $\psi\in H_0^1(\Omega)$ and $v\in K_B$
If $v(B)=0$ we get:

$$k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt=0$$

this imply equation (5.27)
if $v(B)\neq 0$, by applying Stoke's formula we have

$$\begin{aligned}&\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\ &=v(B)\int_0^T\int_{\Omega\times\partial B}A^0(x,z)\nabla_z U(x,z,t)\nu_B\psi(x)\varphi(t)dxd\sigma zdt\end{aligned}$$

remplaced in (5.51) we get

$$\begin{aligned}&v(B)\int_0^T\int_{\Omega\times Y}\alpha(x,y)u(x,t)\psi(x)\varphi''(t)dxdydt\\ &+v(B)\int_0^T\int_{\Omega\times Y}\beta(x,y)u'(x,t)\psi(x)\varphi(t)dxdydt\\ &-k_2 v(B)\int_0^T\int_{\Omega\times\partial B}A^0(x,z)\nabla_z U(x,z,t)\nu_B\psi(x)\varphi(t)dxd\sigma zdt\\ &+v(B)\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt\\ &+\nu k_1 v(B)\int_0^T\int_{\mathbb{R}^N\times\partial S}u(x,t)\psi(x)\varphi(t)dxd\sigma(z)dt = v(B)\int_0^T\int_{\Omega\times Y}f(x,t)\psi(x)\varphi(t)dxdydt\end{aligned}$$

gives the equation of problem (5.29)
Now, in order to chek the initial condition, let $v_\varepsilon = w_{\varepsilon\delta_2}\psi$ where $w_{\varepsilon\delta}$ is given by lemma 5.3 and $\varphi \in C^\infty(]0,T[)$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Take $v_\varepsilon\varphi$ as a test function in (5.17) we have

$$\begin{aligned}
&\int_0^T \langle \alpha_{\varepsilon\delta_1\delta_2} u''_{\varepsilon\delta_1\delta_2}(x,t), v_\varepsilon(x)\rangle_{(H_0^1(\Omega_{\varepsilon\delta_1\delta_2}))', H_0^1(\Omega_{\varepsilon\delta_1\delta_2})}\varphi(t)dt\\
&+\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} \beta_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x)v_\varepsilon(x)\varphi(t)dxdt + \int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla v_\varepsilon(x)\varphi(t)dxdt\\
&+\alpha\int_0^T\int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}v_\varepsilon(x)\varphi(t)d\sigma(x)dt = \int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon(x,t)v_\varepsilon(x)\varphi(t)dxdt
\end{aligned}$$

using the initial condition in (5.17) and by integration by parts, we have

$$\begin{aligned}
&\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon(x,t)v_\varepsilon(x)\varphi(t)dxdt - \int_0^T\int_{\Omega_{\varepsilon\epsilon}} \beta_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x)v_\varepsilon(x)\varphi(t)dxdt\\
&-\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla v_\varepsilon(x)\varphi(t)dxdt - \nu\int_0^T\int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}v_\varepsilon(x)\varphi(t)d\sigma(x)dt\\
&= \int_0^T \langle \alpha_{\varepsilon\delta_1\delta_2}u''_{\varepsilon\delta_1\delta_2}(x,t), v_\varepsilon(x)\rangle_{(H_0^1(\Omega_{\varepsilon\delta_1\delta_2})), H_0^1(\Omega^*_{\varepsilon\delta_1\delta_2})}\varphi(t)dt\\
&= \int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi(t)\mid_0^T v_\varepsilon(x)dx - \int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)v_\varepsilon(x)dxdt\\
&= -\int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,0)v_\varepsilon(x)dx - \int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)v_\varepsilon(x)dxdt\\
&= -\int_{\Omega_{\varepsilon\delta_1\delta_2}} \sqrt{\alpha_{\varepsilon\delta_1\delta_2}}u^1_{\varepsilon\delta_1\delta_2}(x)v_\varepsilon(x)dx - \int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} \alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)v_\varepsilon(x)dxdt
\end{aligned}$$

In view of (5.45),(5.46), (5.47), (5.48), (5.49) (5.50), (5.15) and (5.16), passing to the limit in this equation yields:

$$\begin{aligned}
&v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)dxdydt\\
&+k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\backslash B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\
&-v(B)\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt\\
&-\nu k_1 v(B)\int_0^T\int_{\mathbb{R}^N\times\partial S} u(x,t)\psi(x)\varphi(t)dxd\sigma(z)dt\\
&-v(B)\int_0^T\int_{\Omega\times Y} \beta(x,y)u'(x,t)\psi(x)\varphi(t)dxdydt\\
&= -v(B)\int_\Omega \rho u^1(x)\psi(x)dx - v(B)\int_0^T\int_\Omega \alpha u'(x,t)\psi(x)\varphi'(t)dxdt\\
&= -v(B)\int_\Omega \rho u^1(x)\psi(x)dx + v(B)\int_\Omega \alpha^* u'(x,0)\psi(x)\varphi(t)dxdt\\
&+v(B)\int_0^T\int_{\Omega\times Y} \alpha(x,y)u''(x,t)\psi(x)\varphi(t)dxdydt
\end{aligned} \tag{5.52}$$

Combining this equation with (5.51) yields

$$-v(B)\int_{\Omega}\rho u^1(x)\psi(x)dx+v(B)\int_{\Omega}\alpha^* u'(x,0)dx=0 \tag{5.53}$$

imply

$$u'(x,0)=\frac{\rho}{\alpha^*}u^1(x)$$

Now choosing $\varphi\in C^\infty([0,T])$ with $\varphi(0)=\varphi(T)=\varphi'(T)=0$ and $\varphi'(0)=1$ and taking $v_\varepsilon\varphi$ as test function in the variational formulation (5.17) we have

$$\begin{aligned}
&\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_\varepsilon(x,t)v_\varepsilon(x)\varphi(t)dxdt-\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}}\beta_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x)v_\varepsilon(x)\varphi(t)dxdt\\
&-\nu\int_0^T\int_{\partial S_{\varepsilon\delta_1}}u_{\varepsilon\delta_1\delta_2}v_\varepsilon\varphi(t)d\sigma(x)dt-\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}}A^\varepsilon(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla v_\varepsilon(x)\varphi(t)dxdt\\
&=\int_0^T\langle\alpha_{\varepsilon\delta_1\delta_2}u''_{\varepsilon\delta_1\delta_2}(x,t),v_\varepsilon(x)\rangle_{(H^1_0(\Omega_{\varepsilon\delta_1\delta_2})),H^1_0(\Omega_{\varepsilon\delta_1\delta_2})}\varphi(t)dt\\
&=\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi(t)\big|_0^T\,v_\varepsilon(x)dx-\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)v_\varepsilon(x)dxdt\\
&=-\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)\big|_0^T\,v_\varepsilon(x)dx-\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)v_\varepsilon(x)dxdt\\
&=\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u_{\varepsilon\delta_1\delta_2}(x,0)v_\varepsilon dx-\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u'_{\varepsilon\delta_1\delta_2}(x,t)\varphi'(t)v_\varepsilon(x)dxdt\\
&=\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u_{\varepsilon\delta_1\delta_2}(x,0)v_\varepsilon dx+\int_0^T\int_{\Omega_{\varepsilon\delta_1\delta_2}}\alpha_{\varepsilon\delta_1\delta_2}u_{\varepsilon\delta_1\delta_2}(x,t)\varphi''(t)v_\varepsilon(x)dxdt
\end{aligned}$$

we use $\mathcal{T}_\varepsilon$ passing to the limit gives

$$\begin{aligned}
&v(B)\int_0^T\int_{\Omega\times Y}f(x,t)\psi(x)\varphi(t)dxdydt\\
&+k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\
&-v(B)\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla_x u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt\\
&-v(B)\int_0^T\int_{\Omega\times Y}\beta(x,y)u'(x,t)\psi(x)\varphi(t)dxdydt\\
&-\alpha k_1 v(B)\int_0^T\int_{\mathbb{R}^N\times\partial B}u(x,t)\psi(x)\varphi(t)dxd\sigma(z)dt\\
&=v(B)\int_{\Omega\times Y}\alpha u^0(x)\psi(x)dxdy+v(B)\int_0^T\int_{\Omega}\alpha u(x,t)\psi(x)\varphi''(t)dxdt
\end{aligned} \tag{5.54}$$

For the second term on right-hand side of this equation we apply the stok's formula we get

$$\begin{aligned} v(B)\int_0^T\int_{\Omega\times Y}\alpha u(x,t)\psi(x)\varphi''(t)dxdydt &= v(B)\int_{\Omega\times Y}\alpha u(x,t)\varphi'(t)|_0^T\psi(x)dxdy\\ -\int_0^T\int_{\Omega\times Y}\alpha u'(x,t)\varphi'(t)dxdydt &= -v(B)\int_{\Omega\times Y}\alpha u(x,0)\psi(x)dxdy\\ -\int_0^T\int_{\Omega\times Y}\alpha u'(x,t)\varphi'(t)dxdydt &= -v(B)\int_{\Omega\times Y}\alpha u(x,0)\psi(x)dxdy\\ -\int_{\Omega\times Y}\alpha u'(x,t)\varphi(t)|_0^T\psi(x)dxdy &+ \int_0^T\int_{\Omega\times Y}\alpha u''(x,t)\varphi(t)dxdydt \end{aligned}$$

finaly we have

$$\begin{aligned} v(B)\int_0^T\int_{\Omega\times Y}\alpha u(x,t)\psi(x)\varphi''(t)dxdydt = -v(B)\int_{\Omega\times Y}\alpha u(x,0)\psi(x)dxdy\\ +\int_0^T\int_{\Omega\times Y}\alpha u''(x,t)\varphi(t)dxdydt \end{aligned} \tag{5.55}$$

remplace in (5.54) we get

$$\begin{aligned} & v(B)\int_0^T\int_{\Omega\times Y}f(x,t)\psi(x)\varphi(t)dxdydt\\ & + k_2\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)}A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\ & - v(B)\int_0^T\int_{\Omega\times Y}A(x,y)(\nabla_x u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt\\ & - v(B)\int_0^T\int_{\Omega\times Y}\beta(x,y)u'(x,t)\psi(x)\varphi(t)dxdydt\\ & - \nu k_1 v(B)\int_0^T\int_{\mathbb{R}^N\times\partial B}u(x,t)\psi(x)\varphi(t)dxd\sigma(z)dt\\ & = v(B)\int_{\Omega\times Y}\alpha u^0(x)\psi(x)dxdy - v(B)\int_{\Omega\times Y}\alpha u(x,0)\psi(x)dxdy\\ & + \int_0^T\int_{\Omega\times Y}\alpha u''(x,t)\varphi(t)dxdydt \end{aligned}$$

combining this equation with (5.51) gives

$$v(B)\int_{\Omega\times Y}\alpha u^0(x)\psi(x)dxdy - v(B)\int_{\Omega\times Y}\alpha u(x,0)\psi(x)dxdy = 0$$

imply

$$v(B)\int_{\Omega}\alpha^* u^0(x)\psi(x)dxdy - v(B)\int_{\Omega}\alpha^* u(x,0)\psi(x)dxdy = 0 \ \forall\psi\in\mathcal{D}(\Omega)$$

which implies $u(x,0)=u^0(x)$.

For $\gamma > 0$ System (5.29) is obtained by similar arguments observing that in this case the boundary term has no contribution at the limit.

For $\gamma < 0$

We mutltiply (5.17) by $\varepsilon^{-\gamma}$ we have

$$\varepsilon^{-\gamma}\langle \alpha_{\varepsilon\delta_1\delta_2} u''_{\varepsilon\delta_1\delta_2}(x,t), v(x)\rangle_{(H^1_0(\Omega_{\varepsilon\delta_1\delta_2}))', H^1_0(\Omega_{\varepsilon\delta_1\delta_2})} + \varepsilon^{-\gamma}\int_{\Omega_{\varepsilon\delta_1\delta_2}} \beta_{\varepsilon\delta_1\delta_2} u'_{\varepsilon\delta_1\delta_2}(x)v(x)dx$$
$$+ \varepsilon^{-\gamma}\int_{\Omega_{\varepsilon\delta_1\delta_2}} A^{\varepsilon}(x)\nabla u_{\varepsilon\delta_1\delta_2}(x,t)\nabla v(x)dx + \nu\int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}(x,t)v(x)d\sigma(x)$$
$$= \varepsilon^{-\gamma}\int_{\Omega_{\varepsilon\delta_1\delta_2}} f_{\varepsilon}(x,t)v(x)dx$$

Passing to the limit we observe that all the integrals go to zero, except for the forth integral, wich gives

$$\nu\int_{\partial S_{\varepsilon\delta_1}} u_{\varepsilon\delta_1\delta_2}(x,t)v(x)d\sigma(x) = 0$$

by unfolding $\mathcal{T}^b_{\varepsilon\delta_1}$ we get

$$\nu\frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial S} \mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta_1\delta_2}(x,t))\mathcal{T}^b_{\varepsilon\delta_1}(v(x))dxd\sigma(z) = 0$$

Let us now pass to the limit we get

$$\nu k_1\int_{\mathbb{R}^N\times\partial S} u(x,t)v(x)dxd\sigma(z) = 0$$

we get $u = 0$, which ends the proof.

Proof. of corollary 5.2

Let us show first that $\widehat{u}$ can be expressed as function of u.

This is a standard procedure in homogenization, see for instance [9] or [18]. Recalling the cell problem (5.31) defining the function $\widehat{\chi}_j, j = 1,...,N$, this equation allows to write $\widehat{u}$ in the from

$$\widehat{u}(x,y) = -\sum_{j=0}^{n}\widehat{\chi}_j\frac{\partial u}{\partial x_j}$$

Pllugging this formula in the second integral from (5.29) yields

$$\begin{aligned}&\langle \alpha^* u'', \psi\rangle_{(H^1_0(\Omega))', H^1_0(\Omega)} + \int_{\Omega}\beta^* u'\psi dx + \nu k_1\int_{\partial S} u\psi dx\\ &-k_2\int_{\Omega\times\partial B} A^0\nabla_z U\nu_B\psi dxd\sigma z + \int_{\Omega} A^{hom}\nabla u\nabla\psi dx = \int_{\Omega} f\psi dx\end{aligned} \tag{5.56}$$

for $t \in]0,T[$ and A^{hom} given by (5.35). Now, by integrating by parts in (5.31), one easily gets

$$\int_{\partial B} A^0\nabla_z U\nu_B d\sigma z = \int_{\partial B} A^0\nabla_z(U - k_2 u)\nu_B d\sigma z = -k_2 u\left(\int_{\partial B} {}^tA_0\nabla_z\theta\nu_B d\sigma z\right)$$
$$= -k_2 u\Theta(x)$$

which, replaced into (5.56) gives (5.33) □

Remark 5.8. *1. If $\delta = 1$ and $k_2 = 0$ we find the classical homogenization $y = \dfrac{x}{\varepsilon}$ " and we the results stablished in [54]*

2. If $k_1 = 0$ and $k_2 = 0$ we have

For $\gamma < 0$

$$u = 0$$

For $\gamma \geq 0$

$$\begin{cases} \alpha^* u'' + \beta^* u' - div(A^{hom}\nabla u) &= f \text{ in } \Omega\times]0,T[\\ u &= 0 \text{ on } \partial\Omega\times]0,T[\\ u(x,0) &= u^0 \text{ on } \Omega \\ u'(x,0) &= \dfrac{\rho}{\alpha^*}u^1 \text{ on } \Omega \end{cases}$$

3. If $k_1 = 0$ and $k_2 \neq 0$ we have

For $\gamma < 0$

$$u = 0$$

For $\gamma \geq 0$

$$\begin{cases} \alpha^* u'' + \beta^* u' - div(A^{hom}\nabla u) + k_2^2\Theta u &= f \text{ in } \Omega\times]0,T[\\ u &= 0 \text{ on } \partial\Omega\times]0,T[\\ u(x,0) &= u^0 \text{ on } \Omega \\ u'(x,0) &= \dfrac{\rho}{\alpha^*}u^1 \text{ on } \Omega \end{cases}$$

4. If $k_1 \neq 0$ and $k_2 = 0$ we have

For $\gamma < 0$

$$u = 0$$

For $\gamma = 0$

$$\begin{cases} \alpha^* u'' + \beta^* u' - div(A^{hom}\nabla u) + \nu k_1 u &= f \text{ in } \Omega\times]0,T[\\ u &= 0 \text{ on } \partial\Omega\times]0,T[\\ u(x,0) &= u^0 \text{ on } \Omega \\ u'(x,0) &= \dfrac{\rho}{\alpha^*}u^1 \text{ on } \Omega \end{cases}$$

For $\gamma > 0$

$$\begin{cases} \alpha^* u'' + \beta^* u' - div(A^{hom}\nabla u) &= f \text{ in } \Omega\times]0,T[\\ u &= 0 \text{ on } \partial\Omega\times]0,T[\\ u(x,0) &= u^0 \text{ on } \Omega \\ u'(x,0) &= \dfrac{\rho}{\alpha^*}u^1 \text{ on } \Omega \end{cases}$$

Conclusion and Perspectives

In this thesis, we studied the asymptotic analysis of some boundary value problem in domains with small holes.
In chapters tree and four, we treated the homogenization of the Stokes system with two different kinds of non homogeneous slip boundary conditions on the boundary of the holes. The problem describe the flow of an incompressible viscous fluid through a porous medium under the action of an external electrical field. To do so, we used the periodic unfolding method introduced by Doina Cioranescu, Alain Damlamian and George Griso in (2002)[15].
We obtained at the limit, following the values of a real parameter γ, a Darcy or Brinkmann type laws.
In the last chapter, we considered an hyperbolic parabolic problem for which we got at the limit the homogenized problem.
For all the problems considered, appear some additional terms depending on two constants k_1 and k_2 representing the critical size of the Neumann and Dirichlet holes.

Perspectives: In the near future, I want to focus on the stochastic homogenization which use the probabilistic framework for mechanical modeling in the presence of uncertainties.

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Abstract

In this thesis we consider the Stokes system with two different kinds of boundary condition on the boundary of the holes, which are non-homogeneous slip boundary conditions depending a parametre γ, we consider the flow of an incompressible viscous fluid through a porous medium, under the action of an exterior electrical field. In this work we apply the periodic unfolding methode. The aim is to give the asymptotic behavior of the velocity and of the pressure of the fluid as ε goes to zero. We use the periodic unfolding method in perforated domains. We give the limit problems corresponding to different values of γ (Darcy, Brinkmann or Stokes type problems).
As we have already studied a class of hyperbolic-parabolic problems in periodically perforated domains with a homogeneous Neumann condition on the boundary of holes depending a parametre γ. We always apply the periodic unfolding method for study this problem we get at the limit three defferent cases.

Key words : Stokes problem, periodic unfolding, Robin conditions, perforated domain, homognisation, Hyperbolic-parabolic equation.

Résumé

Nous considérons dans cette thése le Problème de Stokes dans un domaine perforé avec des petits trous périodiquement distribués de période $r(\varepsilon)$. Sur le bord des trous, nous imposons une condition de type Robin dépendant d' un paramétre γ . Notre objectif est d étudier le comportement asymptotique de la vitesse d' écoulement et de la pression du fluide quand ε tend vers zéro par la méthode de l'éclatement périodique. Nous obtenons plusieurs cas différents (loi de Darcy, Brinkmann et Stokes). Par ailleurs, dans cette thése, nous avons également abordé homogénéisation de problème hyperbolique-parabolique. Ce problème modélise de nombreux types de phénomènes apparaissant dans l'électricité et magnétisme, en théorie de l'élasticité, en hydrodynamique et en théorie de la vibration. Nous étudions le comportement asymptotique du problème hyperbolique-parabolique dans un domaines perforé avec des tropes ε-périodiquement distribuée on pose une condition de Neumann homogéne dépendent á paramétre γ sur les tropes $S_{\varepsilon\delta_1}$. Sous les valeurs de γ Nous présentons trois types de contributions dans les systémes de limites.

Mots clés : problème de Stokes, éclatement périodique, conditions de Robin, domaine perforé, homogénéisation, problème Hyperbolique-parabolique

ملخص

في هذا العمل نعتبر سائل لزج غير قابل للضغط يتدفق في وسط مسامي تحت تأثير حقل كهربائي و المتمثل في مشكلة ستوكس. قمنا بدراسة مشكلة ستوكس مع نوعين مختلفين من الشروط الحدية الغير متجانسة متعلقة ب γ الهدف من هذا العمل هو دراسة السلوك التقاربي لسرعة وضغط المائع عندما يؤول ε إلى الصفر و من اجل ذلك نستخدم طريقة الانفجار الدوري في مجالات مسامية . في النهاية نحصل على عدة حالات حسب قيم γ و المتمثلة في قوانين دارسي ، قانون برينكمان و ستكوس.
كما قمنا بدراسة مشكلة القطع الزائد في وسط مساميءحيث وضعنا على الثقوب ذات الحجم $\varepsilon\delta_1$ شروط ديريكلي المتجانسة و على الثقوب ذات الحجم $\varepsilon\delta_2$ نضع شروط روبن المتجانسة ، هدفنا هو دراسة السلوك التقاربي بتطبيق طريقة الانفجار الدوري، نصل في الأخير إلى عدة مشاكل حدية و ذلك حسب قيم γ .

كلمات مفتاحية: مسائل ستوكس، انفجار دورية ، شروط روبن، وسط مسامي، التجانس ، مشكلة القطع الزائدي