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Regularization methods for a class of ill-posed inverse problem

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Regularization methods for a class of ill-posed inverse problems

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Abstract

In the present memory, we investigate two classes of inverse problems. In the first class, we study the inverse problem of identifying the unknown source in the Poisson equation. The second class is devoted to the study of inverse problem for identifying the initial value on the heat equation on the columnar symmetric domain. These problems are ill posed problem in the sense of Hadamard. By using the quasi-boundary value method, we show that the solutions of approximate problems have a stable character as well as their convergences towards to the solutions of the original problems. Moreover, some convergence results are established for the proposed methods.

Key words nverse problems, ill-posed problems, regularization, quasi-boundary value method.

Résumé

Dans le présent mémoire, on étudie deux classes de problèmes inverses. La première classe, est consacrée à l'étude d'un problème inverse qui consiste à l'identification de source dans l'équation de Poisson. La deuxième classe, est consacrée à l'étude du problème inverse d'identification de condition initiale de l'équation de la chaleur sur le domaine symétrique en colonnes. En se basant sur la méthode des conditions aux limites auxiliaires, on démontre la stabilité des solutions ainsi que leurs convergences vers les solutions des problèmes originaux. On établit aussi des estimations d'erreurs entre les solutions originales et les solutions approchées sous certaines hypothèses de régularité sur les données.

Mots clés : Problèmes inverses, Problèmes mal-posés, régularisation, méthode des conditions Auxiliaires.

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I thank God Almight first and foremost for the great grace that he has bestowed upon me,

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dedication

"To my incredible parents, your unwavering love and endless support have shaped me into who I am today. Your guidance and encouragement have been my pillars of strength. Thank you for believing in me and always being there. I am forever grateful for your love and sacrifices."

"To my awesome brothers, your presence in my life brings joy and laughter. The bond we share is a treasure that I hold dear. Thank you for always being there, for your support, and for the unforgettable memories we've created together. I am grateful to have you by my side."

Contents

Introduction

The terms "inverse problems" and "ill-posed problems" have been steadily and surely gaining popularity in modern science since the middle of the 20th century. A little more than fifty years of studying problems of this kind have shown that a great number of problems from various branches of classical mathematics (computational algebra, differential and integral equations, partial differential equations, functional analysis) can be classified as inverse or ill-posed, and they are among the most complicated ones (since they are unstable and usually nonlinear).[28]

Il posed problem

In 1923, the French mathematician J.Hadamard wrote his famous book on partial differential equations and their physical meaning [17]. This work was the starting point for the development of the concept of a well-posed problem in mathematical physics. It conserns a problem whose solution exists, is unique, and continuously depends on the data (stability). Otherwise it is called ill-posed problem.

In the same book. J.Hadamrd suggested (and it was also an opinion shared with I.G. Petrovsky) that only a well-posed problem could accurately model a physical phenomenon. Later, in 1950's and early 1960's, a group of Russian mathematicians led by A. N. Tikhonov appeared a lot of new approaches and some methods tha became fundamental for the theory of ill posed problems and drew attention of mathematicians all over the world to this theory.[24]

Inverse problem

Following Keller we call two problems inverses of one another, if the formulations of each involves all or part of the other. From this definition it is arbithrary which one of the two problems we call the direct and which one the inverse problem. However, for historical or other reasons one of the two problems has been studied extensively for some time and is better understood than the other. This one we would call the direct problem.

Most difficulties in solving ill-posed problems are caused by solution instability. Therefore, the term "ill-posed problem" is often used for unstable problems. In the majority of cases, inverse problems turn out to be ill-posed and, conversely, an ill-posed problem can usualy be reduced to a problem that is inverse to some direct (well-posed) problem.[25]

Inverse and ill-posed problems began to be studied and applied systematically in physics, geophysics, medicine, astronom, Tomography (CT), Positron Emission Tomography (PET), Magnetic Resonance Imaging (MRI), Electron Tomography (ET), microscopic imaging, geophysical imaging), signal- and image-processing, and all other areas of knowledge where mathematical methods are used. The reason is that solutions to inverse problems describe important properties of media under study, such as density and velocity of wave propagation, elasticity parameters, conductivity, dielectric permittivity and magnetic permeability, and properties and location of inhomogeneities in inaccessible areas, etc.[25]

Regularization

In mathematics, regularization is a procedure that involves modifying a nonregular problem into another problem that is close to it (in a certain sense) and possesses good properties, making its theoretical and numerical study easier.

In mathematical literature[40], several regularization methods have been used to solve certain ill-posed problems in the sense of Hadamard. Among them, we can mention:

- **The alternative iterative method** initially proposed by Kozlov et al. in 1991 [41]. This method involves solving a sequence of well-posed problems whose solutions converge, for data belonging to certain admissible classes, to the solution of the original problem.

- **The quasi-reversibility method,** initially introduced by Lattès & Lions in 1967, [20] , consists of transforming the ill-posed second-order Cauchy problem into a well-posed higher-order problem (fourth-order) by introducing a certain parameter (correction term). This method was subsequently adopted by several authors to solve some elliptic inverse problems, notably Kilbanov & Santosa in 1992 and Bourgeois in 2005.

- **The Tikhonov regularization method** : is the oldest regularization method. It involves transforming the original ill-posed problem into a minimization problem. [44]

- **The regularization method by auxiliary boundary conditions (Q.B.V. method):** Was introduced by Abdulkerimov in 1977. [1]. The idea behind this method is to replace the ill-posed problem with a wellposed problem, where we perturb the final condition by replacing it with a non-local condition dependent on a small parameter. It has been utilized by several authors including D.N. Hào, V.D. Nguyen, and H. Sahli, Samariski.

Content

In the first chapter: The first chapter is dedicated to presenting the basic concepts as well as recalling the functional analysis tools necessary for the proposed study, such as the spectral theory of, linear operators and the Riesz-Fredholm theory.

In the second chapter: The concept of inverse problems is introduced, along with the analysis tools for ill-posed problems, illustrated with several examples.

In the third chapter: We are studying an inverse problem in which we try to determine an unknown source appearing in an elliptic equation from an internal measurement. To regularize the problem in question, we propose the method of quasi-boundary conditions (Q.B.V. Method) and

demonstrate their convergence.

In the fourth chapter: We consider an inverse problem to determine an initial value for homogeneous heat equation on a columnar symmetric domain. We construct the quasi boundary value method to solve this inverse problem and obtain regularization solution

Finally, we conclude with a summary that encompasses the main results of this work.

Chapter 1

Element of theory

In this chapter we recall some of the basic concepts and results from Functional Analysis and Operator Theory. Well-known results are stated without proofs. These concepts and results are available in standard textbooks on Functional Analysis, However, we do give detailed proofs of some of the results which are particularly interested to us in the due course

1.1 Spaces

1.1.1 vector(linear)

Definition 1.1.1 [26] *a* ***vector space*** *is a set V along with an addition on V and the scalar multiplication on V such that the followin propeties hold:*

commutativity

$$u + v = v + u \text{ for all } u, v \in V$$

associativity

$$(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv) \text{ for all } u, v, w \in V; \text{ and all } a, b \in F,$$

additive identity

$$\text{there exite an element } 0 \in V \text{ such that } v + 0 = v\text{for all } v \in V$$

additive inverse

$$\text{for evry } v \in V,\text{there existes } w \in V \text{ such that } v + w = 0;$$

multiplicative identity

$$1v = v \text{ for all } v \in V;$$

distributive properties

$$a(u+v) = au + av \text{ and } (a+b)v = av + bv \text{ for all } a, b \in F \text{ and all } u, v \in V$$

1.1.2 Normed linear space

Definition 1.1.2 *[23] A linear space endowed with norm is called a* ***Normed linear space.***

recall that norm on a linear space X is a non-negative real-valued function

$$x \to \|x\|, x \in X$$

which satisfies the following condition:

1. $\forall x \in X, \|x\| = 0$
2. $\|\alpha x\| = |\alpha| \|x\|, \forall x \in X, \forall \alpha \in \Bbbk$
3. $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$

1.1.3 Banach space

Definition 1.1.3 *[23] if a* ***normed space*** *is* ***complete*** *with respect to the induced metric,then it is called a* ***Banach space****.*

1.1.4 Hilbert space

Definition 1.1.4 *a hilbert space is a vector space provided with a scalar product and which is complete for the norm associated with this product.*

1.2 Linear operator

Definition 1.2.1 *[23] Recall that a function $T : X \to X$ between linear spaces X and Y is called a linear operator if*

$$\begin{aligned} T(x+y) &= T(x)+T(y), \qquad \forall x, y \in X, \\ T(\alpha x) &= \alpha T(x), \quad \forall x \in X, \quad \forall \alpha \in \mathbb{k}. \end{aligned}$$

If $T : X \to Y$ is a linear operator, then we write Tx instead of $T(x)$ for $x \in X$. We note that

$$N(T) = \{x \in X : Tx = 0\},$$

is a subset of X, called **the null space of** T,

and

$$R(T) = \{Tx : x \in X\},$$

is a subspace of Y, called **range of** T, it is immediate that a linear operator $T : X \to Y$ is one-to-one or surjective if and only if $N(T) = \{0\}$, and it is onto or surjective if and only if $R(T) = Y$.

1.3 Bounded linear operator

Definition 1.3.1 *[23] Let $T : X \to Y$ be a linear operator between normed linear spaces X and Y. It can be seen that T is continuous if and only if there exists $c > 0$ such*

that

$$\| Tx \| \leq c \| x \| \ \ \forall x \in X,$$

and in that case

$$inf\{c > 0 : \| Tx \| \leq \| x \| \ , \ \forall x \in X\} = sup\{\| Tx \| : x \in X, \ \| x \| \geq 1\}.$$

Moreover, T is continuous if and only if T maps every bounded subsets of X onto bounded subsets of Y. Therefore, a continuous linear operator is also called a **bounded linear operator** or simply a **bounded operator.**

Definition 1.3.2 *Let X and Y be normed linear spaces. The set of all bounded operators from X to Y is denote by $B(X,Y)$, if $Y = X$ then we write $B(X,Y)$ as $B(X)$. Element of $B(X,Y)$ are also called **bounded linear operator.***

Proposition 1.3.1 *Let $T : X \longrightarrow Y$ be linear operator where X and Y are normed linear spaces over $\mathbb{k}(\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C})$. Then the following statements are equivalent:*

- T is continuous at 0.
- T is continuous on X.
- There is a number $c > 0$ such that $\|Tx\| \leq c$ for all $x \in X$ with $\|x\| \leq 1$.
- There is a number $c > 0$ such that $\|Tx\| \leq c\,\|x\|$ for all $x \in X$.

Definition 1.3.3 *For a bounded linear operator $T : X \rightarrow Y$ where X and Y are normed linear spaces, define the operator norm*

$$\|T\| = \sup_{x\in X, \|x\|\leq 1} \|Tx\| < \infty.$$

Theorem 1.3.1 (Uniform boundedness principle) *Let U, X be Banach space. Suppose that S is a non-empty set and , for each $s \in S$,$T_s \in B(U,\ X)$ if for each $u \in U$ the set $\{\|T_s(u)\| : s \in S\}$ is bounded then the set $\{\|T_s\| : s \in S\}$ is bounded.*

Theorem 1.3.2 *(RISZ) [23] Each linear functional ϕ in Hilbert space H can expressed in the form*

$$\phi(h) = \langle h, f\rangle \,,$$

where f is an element of H which is uniquely determined by the functional ϕ, furthermore

$$\| \phi \| = \| f \| \ .$$

1.3.1 Orthoganle Operator

Definition 1.3.4 *[23] A linear operator $P : X \to X$ on a linear space X called a* ***projection operator*** *or simply a* ***projection*** *if*

$$Px = x, \forall x \in R(P),$$

thus, a linear operator $P : X \to X$ is a projection operator if and only if

$$P^2 = P.$$

1.3.2 Invertibility Of Operator

Proposition 1.3.2 *[23] Let X and Y be normed linear space, X_0 be a subspace of X and $T : X_0 \to Y$ be a linear operator. Then there existe $c > 0$ such that*

$$\|Tx\| \geq c\,\|x\| \quad \forall x \in X,$$

if and only if T is injective and $T^{-1} : R(T) \to Y$ is continuous, and in that case

$$\left\|T^{-1}\right\| \leq \frac{1}{c}.$$

Corollary 1.3.1 *Let X and Y be a normed linear space X_0 be a subspace of X and $T : X \to Y$ be a closed linear operator . If X is Banach space and T is bounded bellow, then $R(T)$ is closed subspace of Y.*

Corollary 1.3.2 (Banach's Isomorphism theorem) *If X, Y are Banach spaces, and $T \in B(X, Y)$ is bijective, then T is invertible.*

1.4 Unbounded operator

Definition 1.4.1 *Let X and Y be a real or complex Banach space. an* ***unbounded (complex) linear operator from*** *X to Y is a pair (A,dom(A)), where $dom(A) \subset X$ is a (complex) linear subspace and $A : dom(A) \to Y$ is a (complex) linear map. An unbounded operator $A : dom(A) \to Y$ is called* ***densly defined*** *if its domain is a dense subspace of X . it is caled* ***closed if its graph*** *defined by $graph(A) := \{(x, Ax)\ ,\ x \in dom(A)\}$ is a closed linear subspace of $X \times Y$ with respact to the product topology.*

1.5 Compact operator

Definition 1.5.1 *Let X and Y be normed spaces . A linear operator $T \in L(X,Y)$ is compact if for every bounded sequance $\{x_n\}_{n\geq 1}$ in X, the sequance $\{Tx_n\}_{n\geq 1}$ in Y has a convergent subsequence . The set of compact operator in $L(X,Y)$ will be denoted by $K(X,Y)$.*

We note that every compact operator T is bounded.

Theorem 1.5.1 *Let $(T_n)_{n\geq 1}$ be a sequence of compact linear operators from a normed space X into a Banach spaces Y. If $T_n \longrightarrow T$ (that is, $\|T_n - T\| \longrightarrow 0$), then the limit operator T is compact.*

Theorem 1.5.2 *Let X, Y, Z normed linear spaces.*

1. If $S, T \in K(X,Y)$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha S + \beta T$ is compact , thus $K(X,Y)$ is linear subspace of $B(X,Y)$.
2. If $S \in B(X,Y), T \in B(Y,Z)$ and at least on of the operators S, T is compact, then $TS \in B(X,Z)$ is compact

Theorem 1.5.3 *Let X, Y be normed spaces and $T \in B(X,Y)$*

1. if T has finite rank then T is compact
2. if either $\dim(K)$ or $\dim(Y)$ is finite then T is compact.

Theorem 1.5.4 *If X is an infinit-demensional normed space then the identity operator I on X is not compact*

Corollary 1.5.1 *If X is an infinite-dementional normed space and $T \in K(X,Y)$ then T is not invertible*

Theorem 1.5.5 *Let X, Y be normed space and let $T \in L(X,Y)$*

1. T is compact if and only if , for every bounded subset $A \subset X$ the set $T(A) \subset Y$ is relatively compact.
2. If T is compact the $\mathrm{Im}(T)$ and $\mathrm{Im}\bar{(}T)$ are separable

1.6 Closed operator

1.6.1 Closed graph

Definition 1.6.1 [23]*Let X and Y be normed linear space, X_0 be a subspace of X and let $T : X_0 \to Y$ be a linear operator. Then T is called a* ***closed linear operator*** *or a* ***closed operator*** *if the* ***graph*** *of T, namely*

$$G(T) = (x, Tx); \;\; x \in X_0,$$

is a ***closed*** *subspace of the product space $X \times Y$.*

Definition 1.6.2 [23] *A linear operator $T : X_0 \to Y$ is called a* ***closed linear operato****r if and only if for every sequence (x_n) in X_0*

$$x_n \to x \text{ in } X, \;\; Tx_n \to y \text{ in } Y \Longrightarrow x \in X_0, \;\; Tx = y.$$

Proposition 1.6.1 *Let X and Y be normed linear space , X_0 be a subspace of X and let $T : X_0 \to Y$ be a closed operator. Then we have the following:*

1. $N(T)$ is closed subspace of X.
2. If T is injective the $T^{-1} : R(T) \to X$ is a closed operator .
3. If Y is a Banach space and T is bounded operator , then X_0 is closed in X.

Theorem 1.6.1 *(closed graph) Let X and Y be Banach space and X_0 be a subspace of X . Then a closed linear operator $T : X_0 \to Y$ is continuous if and only if X_0 is closed in X .*

Corollary 1.6.1 *(Bounded Inverse Theorem) Let X and Y be Banach spaces and $T \in B(X, Y)$. If T is bijective, then $T^{-1} \in B(Y, X)$.*

1.7 Spectral notion

Definition 1.7.1 [23] *An important set of scalar associated with a linear operator id its spectrum .*

Let X_0 be a subspace of a normed linear space X and let $A : X_0 \to X$ be a linear operator then the set

$$\rho(A) := \{\lambda \in K : A - \lambda I : X_0 \to X \text{ bijective and } (A - \lambda I)^{-1} \in B(X)\}$$

is called **the resolvent set**, and its compliment in $\Bbbk$,

$$\sigma(A) = \{\lambda \in K : \lambda \notin \rho(A)$$

is called **spectrum of A** ,element of $\sigma(A)$ called **spectral value** of A and the quantity

$$r_\sigma(A) := s\acute{u}p\{|\lambda| : \lambda \in \sigma(A)$$

is called **the spectral raduis of A.**

The set of all scalar λ for which the operator $A - \lambda I$ is **not bijective** is called **the eigen spectrum of A** , and it is denoted by $\sigma_e(A)$, thus

$$\sigma_e(A) \subseteq \sigma(A)$$

element of the eigen spectrum are called **eigenvalue**. Thus, a scalar λ is an eigenvalue of A if and only if there exists a nonzero $x \in X_0$ such that

$$Ax = \lambda x$$

and in that case x is called an **eigen vector** of A corresponding to the eigenvalue λ .

the set of all eigen vectors corresponding to an eigenvalue λ together with the zero vector, that is , the **subspace** $N(A - \lambda I)$, is called **the eigen space** of A corresponding to the eigenvalue λ.

1.7.1 Adjoint

Definition 1.7.2 *[23] Let X and Y be inner product spaces, and let $T : X \to Y$ be a linear operator. A linear operator $T^* : Y \to X$ is called an adjoint of T if*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \forall x \in X,\ y \in Y.$$

Properties of the adjoint

Proposition 1.7.1 *Let X be a Hilbert space, $S : X \longrightarrow X$ and $T : X \longrightarrow X$ be bounded linear operators and $\alpha, \beta \in \Bbbk$ any two scalars. We then have:*

a $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$,

b $(ST)^* = T^* S^*$,

c $(T^*)^* = T$,

d $\|T^*\| = \|T\|$,

e $\|TT^*\| = \|T^*T\| = \|T\|^2$.

1.7.2 Self adjoint operator

Definition 1.7.3 *A bounded linear operators $T \in L(X)$ is called* ***self-adjoint*** *if $T = T^*$ in other words, $T \in L(X)$ is self-adjoint if and only if*

$$\langle Tv, w \rangle = \langle v, Tw \rangle ,$$

for every $v, w \in V$.

1.7.3 Normal operator

Definition 1.7.4 *An operator on an inner product space is caled* ***Normal operator*** *if it commutes with its adjoint, in other words $T \in L(X, Y)$ is normal if*

$$TT^* = T^*T.$$

1.8 Riesz-Fredholm theory

1.8.1 Diagonalization of compact self-adjoint operators

Definition 1.8.1 *We sad an operator $K \in L(H_1, H_2)$ is compact if $K(B_{H_1}(0, 1))$ is relatively compact fo the strong topology . We designate by $K(H_1, H_2)$ the set of compact operator of H_1 in H_2 and we pose $K(H_1, H_2) = K(H_1)$*

- the compactness of the operator $T \in L(H_1, H_2)$ is characterized as follows

$$T \in K(H_1, H_2) \Longleftrightarrow \forall (x_n) \subset H_1, x_n \in 0(weakly) \Longrightarrow Tx_n \to 0(strongly)$$

- Let E, F and G three Banach spaces , if $S_1 \in L(E, F)$ and $S_2 \in K(F, G)$

$$(resp. S_1 \in K(E, F) \text{ and } S_2 \in L(F, G))\text{so } S_2 S_1 \in K(E, G).$$

- [Shauder theory] if K is compact so K^* is compact , and reciprocally.

Theorem 1.8.1 *Let* $K \in \Bbbk(H)$ *with* $\dim(H) = \infty$ *so we have*

(a) $0 \in \sigma(K)$

(b) $\sigma(K)\backslash\{0@ = \sigma_p(K)\backslash\{0\}$

(c) *one of the follwing situation*

- $or\sigma(K) = \{0\}$
- *or* $\sigma(K)\backslash\{0\}$ *is finite*
- *or* $\sigma(K)\backslash\{0\}$ *is a sequence that tends to* 0.

We assume H is separable. Let $T \in L(H)$ a compact self-adjoint operator, so H admits a Hilbertian basis formed by an eigenvector of T

$$\forall x \in H, \qquad x = x_0 + \sum_{k \geq 1} (x, e_k) e_k,\ x_0 \in N(A), \qquad Tx = \sum_{k \geq 1} (x, e_k) \lambda_k e_k\ .$$

1.9 Spectral family and identity resolution

- **version described**

Definition 1.9.1 *let* $A \in D(A) \subset H \to H$ *An unbounded operator , so* A *is sad a* ***compact resolving*** *if*

$$\forall \lambda \in \rho(A),\ R(\lambda; A) \in K(H).$$

we have the next result

Theorem 1.9.1 *The operator $A : D(A) \subset H \to H$ is a* ***resolvent*** *if and only if there exsit $\mu \in \rho(A)$ such that $R(\mu; A) \in K(H)$.*

Theorem 1.9.2 *Let $A : D(A) \in H \to H$ a self-adjoint operator, so*

1. $\sigma_r(A) = \phi$,
2. $\sigma(A) = \sigma_p(A) \cup \sigma_c \subseteq \mathbb{R}$,
3. $A \geq \theta \Longleftrightarrow \sigma(A) \subset [\theta, \infty[$.

Theorem 1.9.3 *let $A : D(A) \in H \to H$ a self-adjoint operator bounded lower and a compact resolving , so is diagonalizable i ,e., there exsit a hilbertian basis in $H, (e_m)_{m\geq 1}$ such that*

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m \to +\infty \qquad Ae_m = \lambda_m e_m,\ m = 1, 2....$$

Remark 1.9.1 *If $A : D(A) \in H \to H$ is a self-adjoint operator with $A \geq \theta > 0 (\Longrightarrow 0 \in \varrho(A))$ and injection $H_1 = (D(A), |.|_G) \hookrightarrow H$ is compact so A is compact resolving and so is diagonalisable.*

Chapter 2

ILL-Posed Problem

In this chapter, we introduce a wide range of technological applications modelled by inverse problems. We also give an introductory insight into the techniques for modelling and classifying these particular problems. Moreover, we present the most difficult challenge in the inverse problem theory, namely the **ill-posedness.**

2.1 Well-posed problem

Definition 2.1.1 *Hadamard's definition of well-posedness occurs if and only if a problem has the following characteristics:*

1. For all admissible data, a solution exists.
2. For all admissible data, the solution is unique.
3. The solution depend continuously on the data.

To understand this definition, we must further examine the well-posedness characteristics. Suppose a linear operator, T is given in the form

$$Tx = y \tag{2.1}$$

where $T : X \to Y$ is a bounded linear operator between the normed vector spaces X and Y. The first condition is met if all $y \in Y$ are also in $R(T)$. The second condition is met if and only if $N(T) = \{0\}$. In other words we need to provide sufficient evidence that the solution is unique. Suppose that $y_1 = y_2$.

By equation (2.1) we get, $Tx_1 - Tx_2 = 0$ and through linearity we obtain, and through linearity we obtain,

$$T(x_1 - x_2) = 0 \Longleftrightarrow x_1 = x_2.$$

Thus the solution is unique if $N(T) = \{0\}$.

Remark If the linear operator, T, satisfies both condition one and condition two of Hadamard's condition, then T^{-1} exists.

if T^{-1} exists and is bounded the problem is well-posed and if T^{-1} does not exist or does not bounded the problem becomes ill-posed.

Definition 2.1.2 *A problem is ill-posed if one, or more, of the conditions for well-posedness are not satisfied, that is, a problem is ill-posed if it is not well-posed.*

2.2 Inverse problem

Definition 2.2.1 *Following Keller we call tow problems inverses of one another, if the formulations of each involves all or part of the solution of the other. From this definition it is arbitrary which one of the two problems we call the direct and which one the inverse problem. However, for historical-or other- reasons one of the two problems has been studied extensively for some times and is better understood than the other. This one we would call the direct problem.*

We formulate the direct problem as the evaluation of an operator a acting on a known "model" x, and the inverse problem as the solution of an equation:

Direct problem: given x (and A), evaluate $A(x)$.

Inverse problem: given y (and A), solve $A(x) = y$ for x.

Examples of familiar inverse problems include medical imaging (CT scans, ultrasound imaging), image enhancement in image processing and detection of rain with weather radars, In this course, we introduce ourselves with mathematical inverse problem and solve in practice some simple inverse problem.[29]

Definition 2.2.2 *the mapping F that tastes the unknown to the corresponding data is called the direct theory (also forward mapping).*

The inverse problem is to seek for x that has produced y. In layman terms

- **Direct problem:** from cause to consequence.
- **Inverse problem:** from consequences to cause.

In mathematical terms, the question is plainly about the determination of inverse mapping F^{-1}, but we will see later that inaccurate data makes things more complicated.

2.2.1 Direct Problems and Inverse Problems in PDE

In the case of direct problems, given a domain $\Omega \subset \mathbb{R}^N$, we are interested in the solutions

$$\begin{cases} u: \ (x,t) \in \Omega \times [0,+\infty[\longmapsto u(x,t) \in E \\ u_t + F\left(t, x, \partial_{x_1}^{\alpha_1} u, ..., \partial_{x_p}^{\alpha_p} u\right) = f \quad \text{in } \Omega \\ \left\{B_{i=1}^{q}\right\} u = j_i \text{ on } \partial\Omega \times [0,+\infty[\\ u(x,0) = u_0(x) \ \text{ in } \Omega \end{cases}$$

In the inverse problem: from partial knowledge of the solution u of the PDE

We must find for example:

1. f, $g_1, ..., g_q \longrightarrow$ source identification problem.
2. $u_0 \longrightarrow$ initial data identification problem.
3. $F \longrightarrow$ coeficient identification problem.
4. $\Omega \longrightarrow$ geometric identiÖcation problem.

The main difficulty of inverse problems is their general ill-posed nature.

2.3 Examples of ill-posed problem

Example 1:

Another classic inverse problem is differentiation. Assume we are given a function $f \in L^\infty([0,1])$ with $f(0) = 0$ for which we want to compute $u = f'$. These conditions are satisfied if and only if u and f satisfy the operator equation

$$f(y) = \int_0^y u(x)dx$$

which can be written as the operator equation $Ku = f$ with $Ku\,(y) = \int_0^y u(x)dx$. As in the previous section, we assume that instead of f we observe a noisy version f^δ for which we further assume that the perturbation is additive, i.e. $f^\delta = f + n^\delta$ with $f \in C^1\left([0,1]\right)$ and $n^\delta \in L^\infty\left([0,1]\right)$. It is obvious that the derivative u only exists if the noise n^δ is differentiable. However, even in case n^δ is differentiable the error in the derivative can become arbitrarily large.

Consider for instance the sequence $\left(\delta_j\right)_{j\in\mathbb{N}}$ with $\delta_j \longrightarrow 0$ and the sequence of corresponding noise functions $n^{\delta_j} \in C^1\left([0,1]\right) \hookrightarrow L^\infty\left([0,1]\right)$ with

$$n^{\delta_j}\left(x\right) = \delta_j \sin\left(\frac{kx}{\delta_j}\right)$$

for a fixed but arbitrary number k. We on the one hand observe $\left\|\left(n^{\delta_j}\right)'\right\|_{L^\infty([0,1])} = \delta_j \longrightarrow 0$, but on the other hand have

$$u^{\delta_j}\left(x\right) = f'\left(x\right) + k\cos\left(\frac{kx}{\delta_j}\right),$$

and therefore obtain the estimate

$$\left\|u - u^j\right\|_{L^\infty([0,1])} = \left\|\left(n^{\delta_j}\right)'\right\|_{L^\infty([0,1])} = k$$

Thus, despite the noise in the data becoming arbitrarily small, the error in the derivative can become arbitrarily big (depending on k).

Note that considering a decreasing error in the norm of the Banach space $C^1\left([0,1]\right)$ will yield a different result. If we have a sequence of noise functions with $\left\|n^{\delta_j}\right\|_{C^1([0,1])} = \delta_j \longrightarrow 0$ instead, we can conclude

$$\left\|u - u^{\delta_j}\right\|_{L^\infty([0,1])} \leq \left\|n^{\delta_j}\right\|_{C^1([0,1])} \longrightarrow 0,$$

due to $C^1\left([0,1]\right)$ being embedded in $L^\infty\left([0,1]\right)$. In contrast to the previous example the sequence of functions $n^{\delta_j} = \delta_j \sin\left(kx\right)$ for instance satisfies

$$\left\|n^{\delta_j}\right\|_{C^1([0,1])} = \sup_{x[0,1]}\left|n^{\delta_j}(x)\right| + \sup_{x[0,1]}\left|\left(n^{\delta_j}\right)'(x)\right| = (1+k)\,\delta_j \longrightarrow 0.$$

However, for a fixed δ the bound on $\left\|u - u^{\delta_j}\right\|_{L^\infty([0,1])}$ can obviously still become fairly large compared to δ , depending on how large k is.

Example 2: Backward Heat Conduction Problem

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ 0 < t < \tau \tag{2.2}$$

with boundary condition

$$u(0,t) = 0 = u(\pi,t),\ 0 \leq t \leq \tau \tag{2.3}$$

Suppose the initial condition is given by

$$u(x,0) = f(x),\ 0 \leq x \leq \pi. \tag{2.4}$$

Here $u(x,t)$ represents the temperature at time t at the point x of a thin metallic wire of length.

The direct problem related to the heat equation is to determine the temperature $u(x,t)$ at time $t \in]0,\tau]$ from the knowledge of the initial temperature $f(x) = u(x,0)$, where as, the inverse problem is to determine $u(x,t)$ for $t \in [0,\tau[$ from the knowledge of the final temperature $u(x,t)$.

It can be shown, by using the method of separation of variables, that $u(.,.)$ defined by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx),$$

satisfies the equations 2.2 – 2.4 if

$$a_n = \frac{2}{\pi}\int_0^\pi f(y)\sin(ny)\,dy,\ n \in \mathbb{N}.$$

Note that

$$f\left(x\right)=u\left(x,0\right)=\sum_{n=1}^{\infty}a_n\sin\left(nx\right),$$

the Fourier series expansion of f. Now, using the inner product

$$\left\langle \varphi,\psi\right\rangle =\int_0^{\pi}\varphi\left(x\right)\overline{\psi\left(x\right)}dx,$$

it follows that the functions un defined by $u_n\left(x\right)=\sqrt{\frac{2}{\pi}}\sin\left(nx\right)$, $n\in\mathbb{N}$, satisfy

$$\left\langle u_n,u_m\right\rangle =\begin{cases}1, & n=m\\ 0, & n\neq m\end{cases}$$

i.e., $\{u_n\ :\ n\in\mathbb{N}\}$ is an orthonormal set with respect to the inner product $\langle .,.\rangle$ Thus, we see that $a_n=\sqrt{\frac{2}{\pi}}\left\langle f,u_n\right\rangle$ so that

$$f\left(x\right)=\sum_{n=1}^{\infty}\left\langle f,u_n\right\rangle u\left(x\right)\ \ ,\ u\left(x,t\right)=\sum_{n=1}^{\infty}e^{-n^2t}\left\langle f,u_n\right\rangle u_n\left(x\right).$$

For each $t\in[0,\tau]$, let us denote $u\left(x,t\right)$ by $\varphi_t\left(x\right)$.Then we have

$$\varphi_t\left(x\right)=\sum_{n=1}^{\infty}e^{-n^2t}\left\langle f,u_n\right\rangle u_n.$$

Note that

$$\left\langle \varphi_t,u_n\right\rangle =e^{-n^2t}\left\langle f,u_n\right\rangle \qquad \forall n\in\mathbb{N}.$$

Thus,

$$\varphi_\tau=\sum_{n=1}^{\infty}e^{-n^2\tau}\left\langle f,u_n\right\rangle u_n=\sum_{n=1}^{\infty}e^{-n^2(\tau-t)}e^{-n^2t}\left\langle f,u_n\right\rangle u_n=\sum_{n=1}^{\infty}e^{-n^2(\tau-t)}\left\langle \varphi_t,u_n\right\rangle u_n$$

For $t\in[0,\tau]$, let A_t be defined by

$$A_t\varphi=\sum_{n=1}^{\infty}e^{-n^2(\tau-t)}\left\langle \varphi,u_n\right\rangle u_n.$$

Since $\sigma_n=e^{-n^2(\tau-t)}\longrightarrow 0$ as $n\longrightarrow\infty$, we see that $A_t:L^2\left[0,\pi\right]\longrightarrow L^2\left[0,\pi\right]$ is a compact operator with singular values σ_n , $n\in\mathbb{N}$. Thus, the problem of finding $\varphi_t=u\left(.,t\right)$ from the knowledge $\varphi_\tau=u\left(.,\tau\right)$ is same as solving the compact operator equation

$$A_t\varphi = \varphi_\tau.$$

We may also observe that the above equation is an integral equation of the first kind. Hence, this backward heat conduction problem is ill-posed.

Theorem 2.3.1 *Let H_1 be a Hilbert space and $A \in B(H_1)$ be a compact self adjoint operator. Let $\sigma(A) = \{\lambda_j \ : j \in \Lambda\}$ where Λ is either $\{1, 2, ..., n\}$ for some $n \in \mathbb{N}$ or $\Lambda = \mathbb{N}$ according as $\sigma(A)$ is finite or infinite. For each nonzero eigenvalue λ_j, let $\left\{u_1^{(j)}, ..., u_{m_j}^{(j)}\right\}$ be an orthonormal basis of $N(A - \lambda_j I)$ and P_j be the orthogonal projection onto $N(A - \lambda_j I)$. Then*

$$Ax = \sum_{j\in\Lambda}\sum_{i=1}^{m_j} \lambda_j \left\langle x, u_i^{(j)}\right\rangle u_i^{(j)} \qquad \forall x \in H_1, \tag{2.5}$$

and

$$A = \sum_{j\in\Lambda} \lambda_j P_j \ .$$

Moreover $\cup_{j\in\Lambda}\left\{u_1^{(j)}, ..., u_{m_j}^{(j)}\right\}$ is an orthonormal basis of $N(A)^\perp$. From Theorem 2.27 we see that if A is a compact self adjoint operator on a Hilbert space then there exists a sequence (μ_n) of scalars and an orthonormal set $\{u_n : \ n \in \mathbb{N}\}$ in H_1 satisfying

$$Ax = \sum_{n=1}^{\infty} \mu_n \langle x, u_n\rangle u_n \qquad \forall x \in H_1.$$

From the above representation of A, it follows that, for every polynomial p,

$$p(A)x = \sum_{n=1}^{\infty} p(\mu_n)\langle x, u_n\rangle u_n, \qquad \forall x \in H_1.$$

and hence, for every continuous real valued function f on $[a, b]$,

$$f(A)x = \sum_{n=1}^{\infty} f(\mu_n)\langle x, u_n\rangle u_n, \qquad \forall x \in H_1.$$

2.3.1 Singular value representation

One may ask whether a representation similar to 2.5 is possible if A is a compact operator which is not a self adjoint. There are compact operators having no nonzero eigenvalues.

Therefore, a representation as in 2.5, in terms of eigenvalues, is not possible for a general compact operator. But we do obtain a representation in terms of singular values of A.
We have observed in Theorem (**2.3.1**) that if $A, B \in B(X)$ and one of them is compact, then their products, AB and BA, are also compact operators.
Therefore, for any compact operator A on a Hilbert space H_1, the operators A^*A and AA^* are also compact. Moreover they are self adjoint operators.
Since A^*A is also a positive operator, there exist non-negative scalars σ_n and orthonormal basis $\{u_n : n \in \mathbb{N}\}$ for $N(A^*A)^\perp$ such that

$$A^*Ax = \sum_{n=1}^{\infty} \sigma_n^2 \langle x, u_n \rangle u_n \quad \forall x \in H_1.$$

Note that

$$A^*Au_n = \sigma_n^2 u_n, \quad n \in \mathbb{N}.$$

The scalars σ_n, $n \in \mathbb{N}$, are called the singular values of the compact operator A.
Let $v_n = Au_n/\sigma_n$ for $n \in \mathbb{N}$. Then we see that

$$Au_n = \sigma_n v_n \ \text{ and } A^*v_n = \sigma_n u_n$$

for every $n = 1, 2, ...$ The set $\{(\sigma_n, u_n, v_n) : \ n \in \mathbb{N}\}$ be a singular system for A, is called a singular system for the compact operator A.

Theorem 2.3.2 *Let $A : H_1 \longrightarrow H_2$ be a compact operator between Hilbert spaces H_1 and H_2 and $\{(\sigma_n, u_n, v_n) : \ n \in \mathbb{N}\}$ be a singular system for A. Then $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ are orthonormal bases of $N(A)^\perp$ and $\overline{R(A)}$ respectively, and for $x \in H_1$ and $y \in H_2$,*

$$Ax = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle v_n \ , \ A^*y = \sum_{n=1}^{\infty} \sigma_n \langle y, v_n \rangle u_n \ .$$

2.4 Regularization of inverse problem

Suppose again the equation (2.1), where H_1 and H_2 are normed linear spaces, A is bounded linear operator and is(2.1) the ill-posed problem. Now we deal with the case which often arises in practical situations, i.e; the inverse of A is unbounded. The stability of the solution is obtained by approximation of the given ill-posed problem by certain well-posed one. These procedures are called regularization methods.

2.4.1 Regularization strategy

Definition 2.4.1 *The regularization strategy is a family of bounded linear operators* $R_\alpha : H_2 \longrightarrow H_1$, $\alpha > 0$, *for which*

$$\lim_{\alpha \longrightarrow 0^+} \|R_\alpha Ax - x\|$$

for any $x \in X$. The regularization strategy may not uniformly bounded. In fact, there is a sequence of positive numbers $\alpha_n \longrightarrow 0$ such that $\|R_{\alpha_n}\| \longrightarrow \infty$.
assume the converse: there is a constant c such that for any $\alpha > 0$ it holds $\|R_{\alpha_n}\| < c$. Then for any $y \in R(A)$ the following inequalities holds:

$$\begin{aligned} \left\|A^{-1}y\right\| &\leq \left\|A^{-1}y - R_\alpha y\right\| + \|R_\alpha y\| \\ &\leq \|x - R_\alpha Ax\| + \|R_\alpha\| \|y\| \\ &\leq \|x - R_\alpha Ax\| + c\|y\| . \end{aligned}$$

The first term on the right side tends to zero (the definition of the regularization strategy) and hence we obtain the contradiction with A^{-1} is unbounded.
Another moment in the regularization is the fact that the right hand side in (1) may involve some noise. Suppose that $y_\delta \in V$ is a perturbation of y, $\|y - y_\delta\| \leq \delta$. We define

$$x_{\alpha,\delta} = R_\alpha y_\delta$$

The regularization strategy may not be uniformly bounded. In fact, there is a sequence of positive numbers $\alpha_n \longrightarrow 0$ such that $\|.\|$

Definition 2.4.2 *[46] Let* $T : H_1 \rightarrow H_2$ *be a bounded linear operator between the normed vector spaces* F *and* G *and let* $\alpha_0 \in]0, \infty[$ *for every* $\alpha \in]0, \infty[$ *let:*

$$R_\alpha ; H_2 \rightarrow H_1 \tag{2.1}$$

be a continuous operator, The family $\{R_\alpha\}$ is called a regularization operator if there exists a parameter choice rule $\alpha = \alpha(\delta)$ such that

$$\lim_{\delta \to 0} \alpha(\delta) \text{ and } \lim_{\delta \to 0} \sup_{g^\delta \in H_2} \{ \left\| R_{\alpha(\delta, g^\delta)} g^\delta \text{-} g \right\| : g^\delta \in H_2, \left\| g^\delta - g \right\| \leq \delta \} = 0, \tag{2.2}$$

holds.

Chapter 3

A Quasi Boundary value regularization method for identifying an unknown source in the poisson equation

3.1 Introduction

Inverse source problem arise in many branches of science and enineering, e,g. heat conduction, crack identification, electromagnetic theory, geophysical prospecting, and pollutant detection. For the heat source identification, there have been a large number of research results for different forms of heat source [[4].][[7].] [[35].] [[10].] [[32].] [[33].] [[34].] [[35].] [[36].]. To the author's knowledge, there were few papers for identifying an unkown source in the poisson equation using the regularization method in [[37].], the authors identified the unknown point source with the logarithmic potential, in [[38].], the author identified the unknown poit source using the projective method. [[39].], the authors identified the unknown point source using the Green's function . in [[42].], the authors identified the unknown source dependent only on one variable using the dual reciprocity method, in [[43].] the autors identified the unknown source dependent only on one variable using the method of fundamental solution ,but y the regulaization method, there are a few papers with strict theorical analysis on identifying the unknown source. Our work is a reading of the article [[13].]

In this chapter we consider the following inverse problem: to find a pair of functions $(u(x, y)$,

$f(x)$) which satisfy

$$\begin{cases} -u_{xx} - u_{yy} = f(x), & -\infty < x < \infty, \quad y > 0 \\ u(x,0) = 0, & -\infty < x < \infty \\ u(x,y)|_{y\to\infty} \text{ bounded}, & -\infty < x < \infty \\ u(x,1) = g(x), & -\infty < x < \infty \end{cases} \tag{3.1}$$

Where $f(x)$ is the unknown source depending only on one spatial variable and $u(x,1) = g(x)$ is the supplementary condition . In application, input data $g(x)$ can only be measured there will be measured data function $g_\delta(x)$ which is merely in $l^2(\mathbb{R})$. and it satisfies

$$\|g - g_\delta\|_{L^2(\mathbb{R})} \leq \delta \tag{3.2}$$

Were the constant $\delta > 0$ represents a noise level of input data.

The problem (3.1) is ill posed, i.e., the solution (if it exists) does not depend continuously on the data. One way to solve an ill-posed problem is by perturbing it into a well-posed one. Many perturbing techniques have been proposed, including a biharmonic regularization developed by Lattés and Lion in [[20].] a pseudo-parabolic regularization proposed by Showalter and Ting in [[30].], a stabilized quasi-reversibility proposed by Miller in [[22].], the method of quasi-reversibility proposed by Mel'nikova in [[21]], a hyperbolic regularization proposed by Ames and Cobb in [[3]] the Gajewski and Zacharias quasi-reversibility proposed by Huang and Zheng in [[16]] a quasi -boundary value method by Denche and Bessila in [[9]] and an optimal regularization proposed by Boussetila and Rebbani in [[6]] it appears that Showalter in [[27]] was the first who used the quasi-boundary value reagularization method to consider the backward heat conduction problem. in [[15].] the authors used the quasi-boundary-value method to consider the Cauchy problem for elliptic equation with nonhomogeneous Neumann data. in this paper, we use the quasi-boundary value regularization method to indentify the unknown source for the poison equation .

3.2 Ill-posdness of the problem and some auxiliary results

The ill-posedness be seen by solving the problem in frequency domain. Let

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx \tag{3.3}$$

be the fourier transform of the function $f(x)$.

The problem(3.1) can be formulated in frequency space as follows:

$$\begin{cases} \xi^2\hat{u}\,\hat{}\,(\xi, y) - \hat{u}(\xi, y) = \quad \hat{f}(\xi), \xi \in \mathbb{R}, y > 0 \\ \hat{u}(\xi, 0) = 0, \qquad\qquad\qquad\qquad \xi \in \mathbb{R} \\ \hat{u}(\xi, y)\big|_{y\to\infty} \text{ bounded }, \qquad\qquad \xi \in \mathbb{R} \\ \qquad\qquad \hat{u}(\xi, y) = g(\xi), \xi \in \mathbb{R} \end{cases} \tag{3.4}$$

$$-u_{xx} - u_{yy} = f(x)$$

we fixed x and by Fourier transform

$$\begin{aligned} \xi^2\hat{u}(\xi, y) - \hat{u}(\xi, y) &= \hat{f}(\xi) \\ u(\xi, y &= u_h(\xi, y) + u_p(\xi, y) \\ \xi^2\hat{u}(\xi, y) - \hat{u}(\xi, y) &= 0 \end{aligned}$$

by characteristic polynomial

$$\begin{aligned} \xi^2 - r^2 &= 0 \Longrightarrow r = |\xi| \\ \hat{u}_h(\xi, y) &= c_1 e^{|\xi| y} + c_2 e^{-|\xi| y} \\ \hat{u}_h(0) &= c_1 e^{|\xi| y} = 0 \Longrightarrow c_1 = 0 \\ \hat{u}_h(\xi, y) &= c_2 e^{-|\xi| y} \\ \hat{u}_p(\xi, y) &= A(\xi) \end{aligned}$$

$$\begin{aligned}
\xi^2 A(\xi) &\Longrightarrow A(\xi) = \frac{\hat{f}(\xi)}{\xi^2} \\
\hat{u}(\xi, y) &= c_2 e^{-|\xi| y} + \frac{\hat{f}(\xi)}{\xi^2} \\
\hat{u}(\xi, 0) &= c_2 + \frac{\hat{f}(\xi)}{\xi^2} = 0 \\
&\Longrightarrow c_2 = \frac{-\hat{f}(\xi)}{\xi^2} \\
\hat{u}(\xi, y) &= \frac{-\hat{f}(\xi)}{\xi^2} e^{-|\xi| y} + \frac{\hat{f}(\xi)}{\xi^2} \\
&= [1 - e^{-|\xi| y}] \frac{\hat{f}(\xi)}{\xi^2} \\
\hat{u}(\xi, 1) &= [1 - e^{-|\xi|}] \frac{\hat{f}(\xi)}{\xi^2} = \hat{g}(\xi) \\
\hat{f}(\xi) &= \frac{\xi^2}{1 - e^{-|\xi| y}} \hat{g}(\xi)
\end{aligned}$$

the solution of the problem (3.1) is given by

$$\hat{f} = \frac{\xi^2}{1 - e^{-|\xi|}} \hat{g}(\xi) \tag{3.5}$$

so

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\xi^2}{1 - e^{-|\xi|}} \hat{g}(\xi) d\xi, \tag{3.6}$$

the unbounded function $\frac{\xi^2}{1-e^{-|\xi|}}$in (3.5) or (3.6) can be seen as an amplification factor of $g\hat{}\,(\xi)$ when $\xi \to \infty$. Therefore when we consider our problem in $L^2(\mathbb{R})$, the exact data function $g\hat{}\,(\xi)$ must decay rapidly as $\xi \to \infty$. But in applications, the input data $g(x)$ can only be measured and never exact .We assume the measured data function $g_\delta(x) \in L^2(\mathbb{R})$. Thus if we try to obtain the unknown source $f(x)$, high frequency components in the error are magnified and can destroy the solution . So it is impossible to solve the problem (3.1) by using the classical method .In the following section, we will use a quasi-boundary value regularization method to deal with the ill-posed problem. Before doing that, we impose an a priori bound on the input data i,e.,

$$\|f(.)\|_{H^p} \leq E \qquad p > 0 \tag{3.7}$$

where $E > 0$ is a constant , $\|.\|_{H^p}$ denotes the norm in Sobolev space $H^P(\mathbb{R})$ defined by

$$\|f(.)\|_{H^p} := (\int_{-\infty}^{\infty} \left|f\hat{}(\xi)\right|^2 (1+\xi^2)^p d\xi)^{\frac{1}{2}} \tag{3.8}$$

Now we give some lemmas which are very useful for our main conclusion.

Lemma 3.2.1 (1) *if $x > 1$, we have the following inequality*

$$\frac{1}{1-e^{-x}} \leq \frac{e}{e-1} \tag{3.9}$$

Lemma 3.2.2 (2) *for $0 < \beta < 1$ the following inequality hold:*

$$\sup_{\xi\in\mathbb{R}} \left| \frac{\beta^2\xi^2}{\beta^2\xi^2 + 1 - e^{-|\xi|}} (1+\xi^2)^{\frac{-p}{2}} \right| \leq 2\max\{\beta^p, \beta^2\}. \tag{3.10}$$

Proof Let

$$A(\xi) = \frac{\beta^2\xi^2}{\beta^2\xi^2 + 1 - e^{-|\xi|}} (1+\xi^2)^{\frac{-P}{2}} \tag{3.11}$$

The proof of (3.11) is separated into three cases.

Case if $|\xi| \geq |\xi_0| = \frac{1}{\beta}$, we have

$$\frac{\beta^2\xi^2}{\beta^2\xi^2 + 1 - e^{-|\xi|}} \leq 1 \tag{3.12}$$

and

$$1 + \xi^2 > \xi^2 = |\xi|^2 \tag{3.13}$$

so

$$(\frac{1}{1+\xi^2})^{\frac{P}{2}} \leq (\frac{1}{|\xi^2|})^{\frac{P}{2}} \tag{3.14}$$

so

$$A(\xi) \leq (1+\xi^2)^{\frac{-p}{2}} \leq |\xi|^{-p} \leq \xi_0^{-p} \leq (\frac{1}{\beta})^{-P} \leq \beta^P \tag{3.15}$$

Case 2 if $1 < |\xi| < \xi_0$, we get

$$A(\xi) \leq \frac{\beta^2\xi^2}{\beta^2\xi^2 + 1 - e^{-|\xi|}} \leq \frac{\beta^2\xi^{2-P}}{1 - e^{-|\xi|}} \leq \frac{e}{e-1}\beta^2\xi^{2-P} \leq 2\beta^2\xi^{2-P}$$

if $0 < P \leq 2$

$$\begin{aligned} A(\xi) &\leq \frac{\beta^2 \xi^{2-p}}{1 - e^{-|\xi|}} \leq 2\beta^2 \xi^{2-p} \leq 2\beta^2 \xi_0^{2-p} \\ &\leq 2\beta^2 (\frac{1}{\beta})^{2-p} \leq 2\beta^P \end{aligned} \tag{3.16}$$

If $p > 2$

$$A(\xi) \leq \frac{\beta^2 \xi^{2-p}}{1 - e^{-|\xi|}} \leq 2\beta^2 \xi^{2-p} \leq 2\beta^2 \xi_0^{2-p} \leq 2\beta^2 (\frac{1}{\beta})^{2-p} \leq 2\beta^P \tag{3.17}$$

Case 3 if $|\xi| \leq 1$, we obtain

$$A(\xi) \leq \frac{\beta^2 \xi^2}{1 - e^{-|\xi|}} \leq \beta^2 \frac{e}{e-1} \leq 2\beta^2 \tag{3.18}$$

Combining (3.15)with(3.16), (3.17) and (3.18), the inequalty(3.10) hold

3.3 The conditional stability result

Since the problem (3.1) is linear, stability estimates can be derived by estimating the size of solution to the correponding homogeneous probelem. We establish the stability estimate for the problem (3.1).

Theorem 3.3.1 *Suppose that $f(x)$ is the solution of the problem (3.1) with the exact data $g(x)$ and suppose that (3.7) holds, then the following estimate holds*

$$\|f(.)\| = \frac{e}{e-1} \|g(.)\| + E^{\frac{p}{p-2}} (\frac{e}{e-1})^{\frac{p}{p+2}} \|g(.)\|^{\frac{p}{p+2}} \tag{3.19}$$

Proof According to (3.1) and using the Parseval formula, we have

$$\|f(.)\|^2 = \left\|\hat{f}(.)\right\|^2 = \int\limits_{|\xi| \leq 1} \left|\frac{\xi^2}{1 - e^{-|\xi|}}\right|^2 \left|\hat{g}(.)\right|^2 d\xi + \int\limits_{|\xi| \geq 1} \left|\hat{f}(\xi)\right|^2 d\xi = A_1 + A_2$$

According to (3.9) when $|\xi| \leq 1$, we obtain

$$\left|\frac{\xi^2}{1 - e^{-|\xi|}}\right| \leq \frac{e}{e-1}.$$

Hence

$$A_1 \leq (\frac{e}{e-1})^2 \left\| g(.) \right\|^2$$

Now for A_2, using the Hölder inequality, we have

$$\begin{aligned}
A_2 &= \int\limits_{|\xi|\geq 1} \left| \hat{f}(\xi) \right|^2 d\xi = \int_{|\xi|\geq 1} [(1+|\xi|^2)^p \left| \hat{f}(\xi) \right|^2]^{\frac{2}{P=2}} [(1+|\xi|^2)^{-2} \left| \hat{f}(\xi) \right|^2]^{\frac{p}{p+2}} d\xi \\
&\leq (\int_{|\xi|\geq 1} (1+|\xi|^2)^p \left| \hat{f}(\xi) \right|^2)^{\frac{2}{P=2}} (\int_{|\xi|\geq 1} (1+|\xi|^2)^{-2} \left| \hat{f}(\xi) \right|^2 d\xi)^{\frac{p}{p+2}} \\
&\leq \left\| f(.) \right\|_{H^P}^{\frac{4}{P+2}} (\int (1+|\xi|^2)^{-2} \left| \frac{\xi^2}{1-e^{-|\xi|}} \hat{g}(\xi) \right|^2 d\xi)^{\frac{p}{p+2}} \\
&\leq \left\| f(.) \right\|_{H^P}^{\frac{4}{P+2}} (\sup \frac{1}{(1+|\xi|^2)^2} \left| \frac{\xi^2}{1-e^{-|\xi|}} \right|^2)^{\frac{p}{p+2}} (\int_{-\infty}^{\infty} \left| \hat{g}(\xi) \right|^2 d\xi)^{\frac{p}{p+2}} \\
&= \left\| f(.) \right\|_{H^P}^{\frac{4}{P+2}} (\sup \frac{|\xi|^2}{(1+|\xi|^2)^2} \left| \frac{1}{1-e^{-|\xi|}} \right|^2)^{\frac{p}{p+2}} \left\| g(.) \right\|^{\frac{p}{p+2}} \\
&\leq \left\| f(.) \right\|_{H^P}^{\frac{4}{P+2}} (\frac{e}{e-1})^{\frac{2p}{p+2}} \left\| g(.) \right\|^{\frac{2P}{P+2}}
\end{aligned}$$

So we obtain

$$\begin{aligned}
\left\| f(.) \right\| &\leq \sqrt[2]{A_1 + A_2} \leq \sqrt[2]{A_1} + \sqrt[2]{A_2} \\
&\leq \frac{e}{e-1} \left\| g(.) \right\| + E^{\frac{P}{P+2}} (\frac{e}{e-1})^{\frac{p}{p+2}} \left\| g(.) \right\|^{\frac{p}{p+2}}
\end{aligned}$$

Remark 3.3.1 *Suppose the function $f_1(x)$ and $f_2(x)$ are the solution of the problem (3.1) with the exact data $g_1(x)$and $g_2(x)$, respectively , then we have the estimate*

$$\left\| f_1(.) - f_2(.) \right\| \leq \frac{e}{e-1} \left\| g_1(.) - g_2(.) \right\| + E^{\frac{P}{P+2}} (\frac{e}{e-1})^{\frac{p}{p+2}} \left\| g_1(.) - g_2(.) \right\|^{\frac{p}{p+2}} \tag{3.20}$$

From (3.20), it is obvious that $\left\| f_1(.) - f_2(.) \right\| \to 0$ when $\left\| g_1(.) - g_2(.) \right\| \to 0$. However, this conditional stability result cannot ensure the stability of numerical computation with noisy data , we must use the regularization method to deal with this ill-posed problem.

3.4 The quasi boundary value regularization method and the error estimate

To obtain a stable approximate solution of the problem (3.1), we make a modification of the boundary value of the (3.1) as follows :

$$u(x,1) + \beta^2 f(x) = g_\delta(x)$$

we can obtain the regularization solution of the problem (3.1) by solving the following problem:

$$\begin{cases} -u_{xx} - u_{yy} = f(x) - \infty < x < \infty, y > 0 \\ u(x,0) = 0 - \infty < x < \infty \\ u(x,y)|_{y\to\infty} \text{ bounded, } -\infty < x < \infty \\ u(x,1) + \beta^2 f(x) = g_\delta(x) - \infty < x < \infty \end{cases} \tag{3.21}$$

where the parameter β is regarded as a regularization parameter. The problem (3.21) can be formulated in frequency space as follows:

$$\begin{cases} \xi^2 \hat{u}(\xi,y) - \hat{u}(\xi,y) = \hat{f}(\xi), \epsilon \in \mathbb{R}, y > 0 \\ \hat{u}(\xi,0) = 0, \xi \in \mathbb{R} \\ \hat{u}(\xi,y)\Big|_{y\to\infty} \text{ bounded } \xi \in \mathbb{R} \\ \hat{u}(\xi,1) + \beta^2 \hat{f}(\xi) = g_\delta(\xi), \xi \in \mathbb{R} \end{cases} \tag{3.22}$$

the solution of the problem (3.22) is given by

$$f_{\beta,\delta}(\xi) = \frac{\xi^2}{\beta^2\xi^2 + 1 - e^{-|\xi|}} \hat{g}_\delta(\xi) \tag{3.23}$$

i,e

$$f_{\beta,\delta}(x) = \frac{1}{\sqrt[2]{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\xi^2}{\beta^2\xi^2 + 1 - e^{-|\xi|}} \hat{g}_\delta(\xi) d\xi \tag{3.24}$$

note that for all small β, $\frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}$ is closed to $\frac{\xi^2}{1-e^{-|\xi|}}$. on the other hand, if $|\xi|$ becomes larger $\frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}$ is bounded , so $f_{\beta,\delta}$ is considered as an approximation of $f(x)$.

now we will give an error estimate between the regularization solution and the exact solution by the following theorem.

Theorem 3.4.1 *Let $f(x)$ given by (3.22) be the exact solution of (3.1) and let $f_{\beta,\delta}(x)$ given by (3.24) be its regularization solution. Let assumptions (3.1) and an priori condition (3.7) hold if we select*

$$\beta = (\frac{\delta}{E})^{\frac{1}{P+2}}.$$

Then we have the following error estimate :

$$\|f(.) - f_{\beta,\delta}(.)\| \leq \delta^{\frac{p}{P+2}} E^{\frac{2}{P+2}} (1 + 2\max\{1, (\frac{\delta}{E})^{\frac{2-P}{P+2}}\}),$$

Proof Using the Parseval formula, the triangle inequality

$$\|f(.) - f_{\beta,\delta}(.)\| = \left\|\hat{f}(.) - \hat{f}_{\beta,\delta}(.)\right\| = \left\|\frac{\xi^2}{1-e^{-|\xi|}}\hat{g}(\xi) - \frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}_\delta(\xi)\right\|$$

then

$$\begin{aligned}\|f(.) - f_{\beta,\delta}(.)\| &= \left\|\tfrac{\xi^2}{1-e^{-|\xi|}}\hat{g}(\xi) - \tfrac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}(\xi) + \tfrac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}(\xi) - \tfrac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}_\delta(\xi)\right\| \\ &\leq \left\|\tfrac{\xi^2}{1-e^{-|\xi|}}\hat{g}(\xi) - \tfrac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}(\xi)\right\| + \left\|\tfrac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\hat{g}(\xi) - \tfrac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}_\delta(\xi)\right\|\end{aligned}$$

We first consider

$$\begin{aligned} & \left\|\frac{\xi^2}{1-e^{-|\xi|}}\hat{g}(\xi) - \frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g}(\xi)\right\| \\ \leq & \left\|\left(\frac{\xi^2}{1-e^{-|\xi|}} - \frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\right)\hat{g}(\xi)\right\| \\ \leq & \left\|\frac{\xi^2(\beta^2\xi^2+1-e^{-|\xi|} - \xi^2(1-e^{-|\xi|}))}{(1-e^{-|\xi|})\left(\beta^2\xi^2+1-e^{-|\xi|}\right)}\hat{g}(\xi)\right\| \\ \leq & \left\|\left(\frac{\xi^2}{1-e^{-|\xi|}} - \frac{\xi^2\left((1-e^{-|\xi|})\right)}{(1-e^{-|\xi|})\left(\beta^2\xi^2+1-e^{-|\xi|}\right)}\right)\hat{g}(\xi)\right\| \\ \leq & \left\|\left(1 - \frac{(1-e^{-|\xi|})}{\beta^2\xi^2+1-e^{-|\xi|}}\right)\left(\frac{\xi^2}{1-e^{-|\xi|}}\right)\hat{g}(\xi)\right\| \\ \leq & \left\|\left(\frac{\xi^2}{1-e^{-|\xi|}}\right)\hat{g}(\xi)(1+\xi^2)^{\frac{P}{2}}(1+\xi^2)^{\frac{-p}{2}}\left(1 - \frac{(1-e^{-|\xi|})}{\beta^2\xi^2+1-e^{-|\xi|}}\right)\right\| \\ \leq & \left\|\left(\frac{\xi^2}{1-e^{-|\xi|}}\right)\hat{g}(\xi)(1+\xi^2)^{\frac{P}{2}}\left(\frac{\beta^2\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\right)(1+\xi^2)^{\frac{-p}{2}}\right\| \end{aligned}$$

$$\begin{aligned}
&\left\|\hat{f}(\xi)(1+\xi^2)^{\frac{P}{2}}\left(\frac{\beta^2\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\right)(1+\xi^2)^{\frac{-p}{2}}\right\| \\
\le\ &\left\|\hat{f}(\xi)(1+\xi^2)^{\frac{P}{2}}\right\| \sup\left|\left(\frac{\beta^2\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\right)(1+\xi^2)^{\frac{-p}{2}}\right| \\
\le\ & E.2\max\{\beta^P,\beta^2\}
\end{aligned}$$

Now we consider

$$\begin{aligned}
\left\|\frac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\hat{g}(\xi)-\frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g_\delta}(\xi)\right\| &= \left\|\frac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\hat{g}(\xi)-\hat{g_\delta}(\xi)\right\| \\
&\le \left\|\frac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\right\|\left\|\hat{g}(\xi)-\hat{g_\delta}(\xi)\right\|
\end{aligned}$$

we have

$$\left\|\frac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\right\| \le \sup\left|\frac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\right| = \frac{1}{\beta^2}$$

and

$$\left\|\hat{g}(\xi)-\hat{g_\delta}(\xi)\right\| \to \delta$$

so

$$\left\|\frac{\xi^2}{\beta^2\xi^2 1+-e^{-|\xi|}}\hat{g}(\xi)-\frac{\xi^2}{\beta^2\xi^2+1-e^{-|\xi|}}\hat{g_\delta}(\xi)\right\| \le \frac{1}{\beta^2}\delta$$

then

$$\|f(.)-f_{\beta,\delta}(.)\| = \left\|\hat{f}(.)-\hat{f}_{\beta,\delta}(.)\right\| \le 2\max\{\beta^P,\beta^2\}E+\frac{1}{\beta^2}\delta$$

Using $\beta=\left(\frac{\delta}{E}\right)^{\frac{1}{P+2}}$, we obtain

$$\begin{aligned}
\|f(.)-f_{\beta,\delta}(.)\| &\le 2\max\{\beta^P,\beta^2\}E+\frac{1}{\beta^2}\delta \\
&\le 2\max\{\left(\frac{\delta}{E}\right)^{\frac{P}{P+2}},\left(\frac{\delta}{E}\right)^{\frac{2}{P+2}}\}E+\frac{1}{\left(\frac{\delta}{E}\right)^{\frac{2}{P+2}}}\delta \\
&\le \delta^{\frac{P}{P+2}}(1+2\max\{1,\left(\frac{\delta}{E}\right)^{\frac{P}{P+2}}\})
\end{aligned}$$

Chapter 4

The Quasi-Boundary value method for Identifying the Initial Value Problem on a Columnar Symmetric Domain

4.1 Introduction

In many industrial applications, it is difficult for one to determine the temperature on the surface of a body .The backward heat conduction problem is a model of this situation. In general, the solution which satisfies the heat conduction equation with final data and the boundary conditions do not exist. Even if a solution exists , it will not be continuously dependent on the final data, i.e., any small perturbation in the input data may cause large change to the solution. In the past, many regularization methods have been proposed for the BHCP, such as the method fundamental solution [[8]], the Tikhonov regularization method [[45].] the quasi-reversibility method [[46].] the wavelet and wavelet-Galerkin method [[47].] the Fourier regularization method [[48].] and so on .

Many articles analyzed a columnar symmetric heat conduction problems which have been obtained some achievements on research. For example, in [[49].] Cheng utilized a wavelet dual least squares method to determine the surface temperature from a fixed location inside a cylinder. Our work is a reading of the article [[14].]

In this chapter we consider homogeneous heat equation on a symmetric domain as follows:

$$\begin{cases} u_t - \frac{1}{r}u_r - u_{rr} = 0,\ 0 < t < T,\ 0 < r < r_0 \\ u(r,0) = \varphi(r), 0 \le r \le r_0 \\ u(r_0,t) = 0, 0 \le t \le T \\ \lim\limits_{r \to 0} u(r,t) \text{ is bounded } 0 < t < T, 0 < r < r_0 \\ u(r,T) = g(r), 0 \le r \le r_0 \end{cases} \tag{4.1}$$

Where r_0 is the radius, $\varphi(r)$ is the initial value. We use the additional condition $u(r,T) = g(r)$ to determine the initial value $\varphi(r)$. The measured data of $g(r)$ is $g^{\delta}(r)$, which satisfies

$$\left\| g^{\delta}(.) - g(.) \right\|_{L^2[0,r_0;r]} \le \delta \tag{4.2}$$

Where the consant $\delta > 0$ represents a noise level of input data. Throughout this paper, $L^2[0, r_0 : r]$ denotes the Hilbert space of Lebesgue measurable function φ with weight r on $[0, r]$, $\langle ., . \rangle$ and $\|.\|$ denote the inner and norm on $L^2[0, r_0; r]$ respectively. The norm of φ is dfined as follows

$$\|\varphi\| = (\int_0^{r_0} r \left|\varphi(r)\right|^2 dr)^{\frac{1}{2}} \tag{4.3}$$

This problem is ill-posed, we use the quasi-boundary value regularization to solve this problem. The quasi boundray value method, also called nonlocal boundary value method is a regularization technique by replacing the final condition or boundary condition by a new condition. This method has been used to solve some inverse problems [[50].].
Using the separation of variables, we obtain the solution of the problem as follows

$$u(r,t) = \sum_{n=1}^{\infty} e^{-(\frac{\mu_n}{r_0})^2 t} \varphi_n \omega_n(r) \tag{4.4}$$

where

$$\omega_n(r) = \frac{\sqrt[2]{2}}{r_0 J_1(\mu_n)} J_0(\frac{\mu_n r}{r_0}) \tag{4.5}$$

the eigenfunction system $\omega_n(r)$ is orthonormal with weight r on $[0, r_0]$. it is also a complete system in $L^2[0, r_0; r]$, $J_0(x)$ and $J_1(x)$ denote the zero order and first order Bessel functions, respectively , and $\{\mu_n\}_{n=1}^{\infty}$ are the sequence of root of the equation $J_0(x) = 0$ which satisfy

$$0 < \mu_1 < \mu_2 < \mu_3 < ..... < \mu_n < ..... < \lim_{n \to \infty} \mu_n = \infty \tag{4.6}$$

now let $\varphi_n = (\varphi(r), \omega_n(r))$ and $g_n = (g(r), \omega_n(r))$. Using $u(r,T) = g(r)$, we have

$$g(r) = \sum_{n=1}^{\infty} e^{-(\frac{\mu_n}{r_0})^2 T} \varphi_n \omega_n(r) \tag{4.7}$$

.

$$g_n = e^{-(\frac{\mu_n}{r_0})^2 T} \varphi_n \tag{4.8}$$

defined operator $K : \varphi(r) \to g(r)$ then

$$g(r) = K\varphi(r) = \sum_{n=1}^{\infty} e^{-(\frac{\mu_n}{r_0})^2 T} \varphi_n \omega_n(r) \tag{4.9}$$

the operator K is linear self adjoint compact operator , k_n are corresponding eigenvalue of K

$$k_n = e^{-(\frac{\mu_n}{r_0})^2 T} \tag{4.10}$$

using (4.1)and (4.7) , equation (4.8) can be rewritten as

$$(g(r), \omega_n(r)) = (\varphi(r), \omega_n(r)) e^{-(\frac{\mu_n}{r_0})^2 T} \tag{4.11}$$

so $\varphi(r) = \sum_{n=1}^{\infty} \frac{g_n}{e^{-(\frac{\mu_n}{r_0})^2 T}} \omega_n(r)$

when $n \to \infty$, $\mu_n \to \infty$ so $(e^{-(\frac{\mu_n}{r_0})^2 T})^{-1} \to \infty$, thus problem (4.1) is ill-posed.

We give a priori bound in the initial value i,e.,

$$\|\varphi(.)\|_p \leq E,\ p > 0, \tag{4.12}$$

where $E > 0$ is a constant and $\|.\|_p$ denotes the norm in Sobolev space which is defined as follows

$$\|\varphi(.)\|_p := (\sum_{n=1}^{\infty} (\frac{\mu_n}{r_0})^p \left|\varphi(.), \omega_n(.)\right|^2)^{\frac{1}{2}} \tag{4.13}$$

Lemma 4.1.1 $\left\{ e^{-(\frac{\mu_n}{r_0})^2 T} \right\}_{n=1}^{\infty}$ *mentioned from [?,?]*μ_n *are the infinite number real roots of the equation* $J_0(r) = 0$*, then*

$$\frac{C_1}{\mu} \leq e^{-(\frac{\mu_n}{r_0})^2 T} \leq \frac{C_2}{\mu_n} \tag{4.14}$$

Where C_1, C_2 are constants.

Lemma 4.1.2 *for any positive constantp* $> 0, 0 < \mu < 1, s \geq \mu_1 \geq 0$ *, we have*

$$F(s) = \frac{\mu s^{1-\frac{p}{2}}}{\mu s + C_1} \leq \begin{cases} C_5 \mu^{\frac{P}{2}} 0 < P < 2 \\ C_6 \mu, P \geq 2 \end{cases} \tag{4.15}$$

here $C_5 = C_5(p, C_1), C_6 = C_6(p, \mu, C_1)$

4.2 Regularization method and convergence estimate

in this section, through modifying $u(r,T) \approx g(r)$ as $u(r,T) + \mu u(r,0) = g^\delta(r)$, we use the quasi-boundary value method to solve the following problem

$$\begin{cases} u_t^{\mu,\delta} - \frac{1}{r} u_r^{\mu,\delta} - u_{rr}^{\mu,\delta} = 0, 0 < t < T, 0 < r, r_0 \\ u^{\mu,\delta}(r,0) = \varphi(r), 0 \leq r \leq r_0 \\ u^{\mu,\delta}(r_0,t) = 0, 0 \leq t \leq T \\ \lim\limits_{r \to \to 0} u^{\mu,\delta}(r,t) \text{ is bounded } 0 < t < T, 0 < r < r_0 \\ u^{\mu,\delta}(r,T) + \mu u^{\mu,\delta}(r,0) = g^\delta(r), 0 \leq r \leq r_0 \end{cases} \tag{4.16}$$

where μ is regularization parameter. By the separation of variables, we obtain the solution of the problem ((4.16)) as follows

$$u^{\mu,\delta}(r,t) = \sum_{n=1}^{\infty} e^{-(\frac{\mu_n}{r_0})^2 t} \varphi_n^{\mu,\delta} \omega_n(r) \tag{4.17}$$

using $u^{\mu,\delta}(r,T) + \mu u^{\mu,\delta}(r,0) = g^\delta(r)$ we obtain

$$g^\delta(r) = \sum_{n=1}^{\infty} (\mu + e^{-(\frac{\mu_n}{r_0})^2 T}) \varphi_n^{\mu,\delta} \omega_n(r) \tag{4.18}$$

Hence

$$\varphi^{\mu,\delta}(r) = \sum_{n=1}^{\infty} \frac{g_n^\delta}{\mu + e^{-(\frac{\mu_n}{r_0})^2 t}} \omega_n(r) \tag{4.19}$$

which is called the quasi-boundary regularized solution of the problem (4.16) and μ is a constant which will be selected appropriately as regularization parameter.

Before giving the main conclusion of this paper , we first introduce two important lemmas

Lemma 4.2.1 $\left\{e^{-(\frac{\mu_n}{r_0})^2T}\right\}_{n=1}^{\infty}$ *mentioned from [?,?]*μ_n *are the infinite number real roots of the equation* $J_0(r)=0$*, then*

$$\frac{C_1}{\mu} \leq e^{-(\frac{\mu_n}{r_0})^2T} \leq \frac{C_2}{\mu_n} \tag{4.20}$$

Where C_1, C_2 are constants.

Lemma 4.2.2 *for any positive constant* $p>0$, $0<\mu<1, s \geq \mu_1 \geq 0$ *, we have*

$$F(s)=\frac{\mu s^{1-\frac{p}{2}}}{\mu s+C_1} \leq \begin{cases} C_5\mu^{\frac{P}{2}}, & 0<P<2 \\ C_6\mu, & P \geq 2 \end{cases} \tag{4.21}$$

here $C_5=C_5(p,C_1), C_6=C_6(p,\mu,C_1)$.

Theorem 4.2.1 *Let* $\varphi^{\mu,\delta}(r)$ *be the regularized solution and* $\varphi(r)$ *be the exact solution . Let* $g_\delta(r)$ *be the measured data at* $t=T$ *satisfying (4.1)) and priori condition 4.13) holds for* $p>0$*. Then we obtain*

1. *As* $0<p<2$*, if we select* $\mu=(\frac{\delta}{E})^{\frac{2}{P+2}}$ *then*

$$\left\|\varphi^{\mu,\delta}(.)-\varphi(.)\right\| \leq (1+C_1)E^{\frac{1}{P+2}}\delta^{\frac{P}{P+2}} \tag{4.22}$$

2. *As* $p \geq 2$*, if we select* $\mu=(\frac{\delta}{E})^{\frac{1}{2}}$ *, then*

$$\left\|\varphi^{\mu,\delta}(.)-\varphi(.)\right\| \leq (1+C_1)E^{\frac{1}{2}}\delta^{\frac{1}{2}} \tag{4.23}$$

Proof

$$\left\|\varphi^{\mu,\delta}(.)-\varphi(.)\right\| = \left\|\varphi^{\mu,\delta}(.)-\varphi^{\mu}(.)+\varphi^{\mu}(.)-\varphi(.)\right\|$$

By utilizing triangle inequality, we obtain

$$\left\|\varphi^{\mu,\delta}(.)-\varphi(.)\right\| \leq \left\|\varphi^{\mu,\delta}(.)-\varphi^{\mu}(.)\right\| + \left\|\varphi^{\mu}(.)-\varphi(.)\right\|$$

We first consider$\left\|\varphi^{\mu,\delta}(.)-\varphi^{\mu}(.)\right\|$

$$\begin{aligned}
\left\|\varphi^{\mu,\delta}(.)-\varphi^{\mu}(.)\right\|^2 &= \left\|\sum_{n=1}^{\infty}(\frac{g_n^{\delta}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}-\frac{g_n}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}})\omega_n^2\right\| \\
&= \sum_{n=1}^{\infty}(\frac{g_n^{\delta}-g_n}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}})^2 \le \sum_{n=1}^{\infty}(\frac{g_n^{\delta}-g_n}{\mu})^2=\frac{\delta^2}{\mu^2}
\end{aligned}$$

So

$$\left\|\varphi^{\mu,\delta}(.)-\varphi^{\mu}(.)\right\| \le \frac{\delta}{\mu}$$

now we consider $\|\varphi^{\mu}(.)-\varphi(.)\|$

$$\begin{aligned}
\|\varphi^{\mu}(.)-\varphi(.)\| &= \left\|\sum_{n=1}^{\infty}(\frac{g_n}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}-\frac{g_n}{e^{-(\frac{\mu_n}{r_0})^2T}})\omega_n^2\right\| \\
& \sum_{n=1}^{\infty}(\frac{g_n(e^{-(\frac{\mu_n}{r_0})^2T})-g_n\mu-g_n(e^{-(\frac{\mu_n}{r_0})^2T})}{(\mu+e^{-(\frac{\mu_n}{r_0})^2T})(e^{-(\frac{\mu_n}{r_0})^2T})} \\
& (\sum_{n=1}^{\infty}(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\frac{g_n}{e^{-(\frac{\mu_n}{r_0})^2T}})^2\omega_n)^{\frac{1}{2}} \\
& (\sum_{n=1}^{\infty}(\frac{\mu(\mu_n)^{\frac{-P}{2}}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}(\mu_n)^{\frac{P}{2}}\frac{g_n}{e^{-(\frac{\mu_n}{r_0})^2T}})^2\omega_n)^{\frac{1}{2}} \\
&\le \sup(\frac{\mu(\mu_n)^{\frac{-P}{2}}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}})=\sup(A_n) \\
\text{and } \left\|(\mu_n)^{\frac{P}{2}}\frac{g_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right\| &\le E
\end{aligned}$$

So

$$\|\varphi^{\mu}(.)-\varphi(.)\| \le \sup(A_n)E$$

Using Lemma we obtain

$$A_n \le \frac{\mu\mu_n^{1-\frac{P}{2}}}{\mu\mu_n+C} \le \left\{\begin{array}{ll} C_1\mu^{\frac{P}{2}}; & 0<P<2 \\ C_2\mu, & P\ge 2\end{array}\right.$$

so

$$\left\|\varphi^{\mu,\delta}(.)-\varphi(.)\right\| \le \frac{\delta}{\mu}+\left\{\begin{array}{ll} C_1\mu^{\frac{P}{2}}E, & 0<P<2 \\ C_2\mu E; & P\ge 2\end{array}\right.$$

Using $\mu=\left(\frac{\delta}{E}\right)^{\frac{2}{P+2}}$ $(0<P<2)$ and $\mu=\left(\frac{\delta}{E}\right)^{\frac{1}{2}}$ $(p\ge 2)$

$$\left\|\varphi^{\mu,\delta}(.)-\varphi(.)\right\| \le \left\{\begin{array}{ll} (1+C_1)E^{\frac{2}{P+2}}\delta^{\frac{P}{P+2}}, & 0<P<2 \\ (1+C_2)E^{\frac{1}{2}}\delta^{\frac{1}{2}}; & P\ge 2\end{array}\right. .$$

Conclusion

In this memory, we studied two classes of inverse problems

- In the first problem, we identify an unknown source term depending only on one variable in two dimensional Poisson equation. This problem is ill-posed, i.e., the solution (if it exists) does not depend on the input data. We obtained the stability estimate using the conditional stability. Moreover, using the quasi-boundary value regularization method, we obtain the regularization solution and the Hölder type error estimate between the exact solution and the regularization solution.

- In the second problem, we consider an inverse problem to determine an initial value for homogeneous heat equation on a columnar symmetric domain. We construct the quasi boundary value method to solve this inverse problem and obtain regularization solution. Moreover, we obtain the Hölder type error estimate under a priori parameter choice rule.

Bibliography

[1] L. S. Abdulkerimov. Regularization of an ill-posed Cauchy problem for evolution equations in a Banach space, Azerbaidzan. Gos. Univ. Ucen. Zap. Fiz. Mat., 1(1977), 32-36 (MR0492645) (in Russian).

[2] A.A.Buryki and A.M. Denisov, Determination of the unknown sources in the heat-conduction equation,Comput.Math.Model.8(1997), pp.

[3] Ames, KA, Cobb, SS: Continuous dependence on modeling for related Cauchy problems of a class of evolution equations. J. Math. Anal. Appl. 215(1), 15-31 (1997).

[4] Ahmadabadi, MN, Arab, M, Malek Ghaini, FM: The method of fundamental solutions for the inverse space-dependent heat source problem. Eng. Anal. Bound. Elem. 33, 1231-1235 (2009).

[5] Bryan P, Rynne and Martin A. Youngson Linear Functioal Analysis , Second Edition .

[6] Boussetila, N, Rebbani, F: Optimal regularization method for ill-posed Cauchy problems. Electron. J. Differ. Equ. 2006, 147 (2006).

[7] Cannon, JR, Duchateau, P: Structural identification of an unknown source term in a heat equation. Inverse Probl. 14, 535-551 (1998).

[8] C.S. Liu, The mehod of fundaental solution for solving the backward heat conduction problem with conditioning by a new post-conditioner, Numer. Heat. Tr. B-fund., 60 (2011), no. 1, 57-72. https://doi.org/10.1080/10407790.2011.588134.

[9] Denche, M, Bessila, K: A modified quasi-boundary value method for ill-posed problems. J. Math. Anal. Appl. 301(2), 419-426 (2005).

[10] Dou, FF, Fu, CL, Yang, FL: Optimal error bound and Fourier regularization for identifying an unknown source in the heat equation. J. Comput. Appl. Math. 230, 728-737 (2009).

[11] Eldén, L, Berntsson, F, Reginska, T: Wavelet and Fourier methods for solving the sideways heat equation. SIAM J. Sci. ‘ Comput. 21, 2187-2205 (2000).

[12] Engl, HW, Hanke, M, Neubauer, A: Regularization of Inverse Problem. Kluwer Academic, Boston, 1996.

[13] Fan Yang, Miao Zhang and Xiao-Xiao Li. A quasi-boundary value regularization method for identifying an unknown source in the Poisson equation, . Journal of Inequalities and Applications 2014, 2014:117.

[14] Feng Tian and Dun-Gang Li 1 School of Science, The Quasi-Boundary Value Method for Identifying the Initial Value Problem on a Columnar Symmetric Domain. International Journal of Mathematical Analysis Vol. 16, 2022, no. 2, 49 - 55.

[15] Feng, XL, Eldén, L, Fu, CL: A quasi-boundary-value method for the Cauchy problem for elliptic equations with nonhomogeneous Neumann data. J. Inverse Ill-Posed Probl. 18(6), 617-645 (2010).

[16] Huang, Y, Zheng, Q: Regularization for a class of ill-posed Cauchy problems. Proc. Am. Math. Soc. 133(10), 3005-3012 (2005).

[17] J. Hadamard. Lecture note on Cauchy’s problem in linear partial differential equations,Yale Uni Press, New Haven, (1923).

[18] Herman Singh. Regularization An ill-posed problem with Tikhonov’s Regularization , depatment of mathematics , Linkoping University LiTH-MAT-EX-2021:09-SE.

[19] Kirsch, A: An Introduction to the Mathematical Theory of Inverse Problems. Springer, New York 1996.

[20] R. Lattès, J.L. Lions. Méthode de Quasi-réversibilité et Applications, Dunod, (1967).

[21] Mel'nikova, IV: Regularization of ill-posed differential problems. Sib. Math. J. 33(2), 289-298 (1992).

[22] Miller, K: Stabilized quasireversibility and other nearly best possible methods for non-well posed problems. In:Symposium on Non-Well-Posed Problems and Logarithmic Convexity. Lecture Notes in Math., vol. 316, pp. 161-176.Springer, Berlin (1973).

[23] M. T. Nair. Linear operator equations: approximation and regularization. World Scientific, (2009).

[24] Richard Horsky, Ill-posed problems in time series analysis, the 9th International Days of Statistics and Economics, Prague, September 10-12, 2015.

[25] Ramiz Tapdigoglu, Thèse, Inverse problems for fractional order differential equations, université de la Rochelle, 2019. NNT: 2019LAROS004. tel-02326244.

[26] SHELDON AXLER Linear Algebra Done Right Third Edition ,springer cham heidelberg New York Dordrecht London.

[27] 4. Showalter, RE: Cauchy problem for hyperparabolic partial differential equations. In: Trends in the Theory and Practice of Non-Linear Analysis, pp. 421-425. Elsevier, Amsterdam (1983).

[28] S.I.Kabanikhin , J.INV.Ill-Posed P roblems definitin and examples of inverse and ill-posed problems 16 (2008)

[29] SARI LESANEN introduction to inverse problem (Spring 2014).

[30] Showalter, RE, Ting, TW: Pseudo-parabolic partial differential equations. SIAM J. Math. Anal. 1(1), 1-26 (1970).

[31] Yan, L, Fu, CL, Yang, FL: The method of fundamental solutions for the inverse heat source problem. Eng. Anal. Bound. Elem. 32, 216-222 (2008).

[32] Dou, FF, Fu, CL, Yang, FL: Optimal error bound and Fourier regularization for identifying an unknown source in the heat equation. J. Comput. Appl. Math. 230, 728-737 (2009).

[33] Li, GS: Data compatibility and conditional stability for an inverse source problem in the heat equation. Appl. Math. Comput. 173, 566-581 (2006).

[34] Ma, YJ, Fu, CL, Zhang, YX: Identification of an unknown source depending on both time and space variables by a variational method. Appl. Math. Model. 36, 5080-5090 (2012).

[35] Yang, F, Fu, CL: Two regularization methods to identify time-dependent heat source through an internal measurement of temperature. Math. Comput. Model. 53, 793-804 (2011).

[36] Ohe, T, Ohnaka, K: A precise estimation method for locations in an inverse logarithmic potential problem for point mass models. Appl. Math. Model. 18, 446-452 (1994).

[37] Nara, T, Ando, S: A projective method for an inverse source problem of the Poisson equation. Inverse Probl. 19, 355-369 (2003).

[38] Farcas, A, Elliott, L, Ingham, DB, Lesnic, D, Mera, NS: A dual reciprocity boundary element method for the regularized numerical solution of the inverse source problem associated to the Poisson equation. Inverse Probl. Sci. Eng. 11(2), 123-139 (2003).

[39] Kagawa, Y, Sun, Y, Matsumoto, O: Inverse solution of Poisson equation using DRM boundary element models -identification of space charge distribution. Inverse Probl. Sci. Eng. 1(2), 247-265 (1995).

[40] Khelili Besma, Sur quelques méthodes de régularisation appliquées à une classe de problèmes de Cauchy inverses. Thèse de Doctorat,université Badji Mokhtar annaba, 2018.

[41] V.A. Kozlov, V.G. Maz'ya, A.V. Fomin. An iterative method for solving the Cauchy problem for elliptic equation, Comput. Math. Phys. 31, 45-52, (1991).

[42] . Sun, Y, Kagawa, Y: Identification of electric charge distribution using dual reciprocity boundary element models. IEEE Trans. Magn. 33(2), 1970-1973 (1997).

[43] . Jin, BT, Marin, L: The method of fundamental solutions for inverse source problems associated with the steady-state heat conduction. Int. J. Numer. Methods Eng. 69, 1570-1589 (2007).

[44] A.N. Tikhonov, V.Y. Arsenin. Solutions of Ill-posed Problems, Winston and Sons, (1977).

[45] Z.Y. Zhao, Z.H. Meng, A modified Tikhonov regularization method for a backward heat equation, Inv. Probl. Sci. En., 19 (2011), no. 8, 1175-1182. https://doi.org/10.1080/17415977.2011.605885.

[46] F. Yang, Y.P. Ren, X.X. Li, The quasi-reversibility method for a final value problem of the time-fractional diffusion equation with inhomoge neous sourc, Math. Method Appl. Sci., 41 (2018), no. 5, 1774-1795. https://doi.org/10.1002/mma.4705.

[47] H. Hashish, S.H. Behiry, B. Abdel-Hamid, A wavelet-Galerkin solution for the backward heat conduction equation, Int. J. Appl. Math., 11 (2002), no. 2, 197-207.

[48] C.L. Fu, X.T. Xiong, Z. Qian, Fourier regularization for a backward heat equation, J. Math. Anal. Appl., 331 (2007), no. 1, 472-480.https://doi.org/10.1016/j.jmaa.2006.08.040.

[49] W. Cheng, C.L. Fu, Solving the axisymmetric inverse heat conduction problem by a wavelet dual least squares method, Comput. Math. Appl.,59 (2010), no. 1, 142-148.

[50] F. Yang, H.H. Wu, X.X. Li, Three regularization methods for identifying the initial value of time fractional advection-dispersion equation, Comput. Appl. Math., 41, (2022). https://doi.org/10.1007/s40314-022-01762-0.