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Thèse

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Option : ***Mathématique***

Etude d'une classe d'équation parabolique et application en traitement d'image

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Study of a class of parabolic equation and application in image processing

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Abstract

The aim of this thesis is the study of the existence of weak solutions for two types of problems. The first one is a new model for image restoration. The proposed model is an interpolation of two classical models, Perona-Malik and heat equation. By using the compactness method and the monotonicity arguments, we prove the existence of weak solutions. We also give some numerical study and show experimental results on examples images which prove the efficiency and effectiveness of our model. Then we generalize this problem to the system and we use the same techniques for obtain the existence results. For the second problem, we establish the existence of solutions of a system of convection-diffusion-reaction by using the Leray-Schauder degree theorys.

Keywords:

Boundary value problems, topological degree, homotopy, image restoration, smoothing images, Perona-Malik model.

العنوان: دراسة فئة من المعادلات المكافئية و تطبيقاتها في معالجة الصور

ملخص

الهدف من هذه الأطروحة هو دراسة وجود الحلول الضعيفة لنوعين من المسائل، حيث قمنا في المسألة الأولى باقتراح نموذج جديد لتحسين الصورة، الذي هو عبارة عن استكمال لنموذجين كلاسيكيين نموذج بيرونا ـ ماليك و معادلة الحرارة، وذلك باستعمال طريقة الإكتناز و حجة الرتابة ، كما نقدم دراسة عددية لهذا النموذج مع بعض التطبيقات على مجموعة من الصور لتحسين نوعيتها والتي تبين كفاءة و فعالية هذا النموذج. ثم عممنا هذه المسألة إلى جملة معادلات و نستعمل نفس التقنيات للحصول على نتائج الوجود. أما في المسألة الثانية فقد قمنا بدراسة وجود الحلول الضعيفة لجملة معادلات التفاعل و الإنتشار باستخدام نظرية النقطة الثابثة لـ لوري ـ شودر.

الكلمات المفتاحية: درجة طوبولوجية، تشوه مستمر، تحسين الصورة، نموذج بيرونا ـ ماليك.

Titre: **Etude d'une classe d'équation parabolique et application en traitement d'image**

Résumé

L'objectif de cette thèse est l'étude de l'existence des solutions faibles pour deux types de problémes . Le premier est un nouveau modèle pour la restauration d'image. Le modèle proposé est une interpolation de deux modèles classiques, Perona-Malik et l'equation de la chaleur. En utilisant la méthode de compacité et l'argument de la monotonicité pour montrer l'existence de la solution faible. On fournit également quelques études numériques et on montre des résultats expérimentaux sur des exemples des images qui illustrant l'efficacité et l'efficience de notre modèle. Ensuite, on généralise ce problème au systéme et on utilise les mêmes techniques pour obtenir les résultats d'existence. Pour le deuxième problème étudié, ont établit l'existence de solution d'un système de convection-diffusion-réaction en utilisant le théorème du degré topologique de Leray-Schauder.

Mots Clés:

Problèmes aux limites, degré topologique, homotopie, restauration des image, lissage des images, modèle de Perona-Malik.

Notations

- $\to$ designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- ∇ stands for the gradient operator.
- div is the divergence operator.
- $\dfrac{\partial}{\partial x}$ partial derivative.
- $\dfrac{\partial}{\partial n}$ outward normal derivative.
- $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}$, $\alpha = (\alpha_1, \alpha_2 \cdots, \alpha_N)$, $|\alpha| = \sum_{i=1}^{N} \alpha_i$.
- Δ is the Laplace operator.
- $\sigma(-\Delta)$ denotes the spectrum of $-\Delta$.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^N$ = real space of dimension N.
- $[a, b]$ order interval in X; $[a, b] = \{x \in X : a \leq x \leq b\}$.
- $\Omega \subset \mathbb{R}^N$ open set in $\mathbb{R}^N$.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.

- $x \cdot y$ denotes the Euclidean scalar product of $x, y \in \mathbb{R}^N$.

- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.

- $C^\infty(\Omega) = \bigcap\limits_{m \in \mathbb{N}} C^m(\Omega)$.

- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.

- $L^p_{loc}(\Omega)$ the space of local p-integrable functions on Ω.

- $W^{m,p}(\Omega)$ Sobolev space with norm $\|\cdot\|_{m,p}$.

- $W^{m,p}_{loc}(\Omega)$ the local Sobolev space.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$, $H^{-1}(\Omega) = W^{-1,2}(\Omega)$.

- $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H_0^1(\Omega)}$ denote the usual norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively.

- $\mathrm{U} = H_0^1(\Omega) \times H_0^1(\Omega)$, which is a Banach space endowed with the norm

$$\|(u,v)\|_{\mathrm{U}}^2 = \|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2,$$

  such as U^* its dual.

- $\widetilde{\mathrm{U}} = L^\infty(\Omega) \times L^\infty(\Omega)$.

- $\widetilde{\mathrm{V}} = H^1(\Omega) \times H^1(\Omega)$.

- $\mathrm{V} = L^2(\Omega) \times L^2(\Omega)$.

- $B(R)$ open ball centered at the point 0 with the radius R.

- $\partial B(R)$ sphere centered at the point 0 with the radius R.

Contents

Introduction

In mathematical modeling, partial differential equations (PDEs) have been studied extensively for many years and applied to a large number of application in fluid dynamics, elastic mechanics, physics, biology, chemistry and image processing etc. The topic of the application of PDEs in image processing has become *hot* in recent years and a very active field of computer applications and research [22]. Several techniques have been developed in image processing during recent decades, these techniques are now used in all areas: medical visualization, human computer interfaces, artistic effects, industrial inspection, etc. This is due to the mathematical formulation that sets the framework for any approach based on PDEs and that can provide a good justification and an explanation of the results obtained through these techniques in image processing. One of the most active topics in this field has been restoration of images, see for instance [3–5, 41, 42, 51, 53, 54]. To be more precise, the important problem is the restoration of degraded images because it allow to recovery lost information from the observed degraded image data. Noise removal is one of the most oldest problems in image processing and it is still a necessary preprocessing step for many applications. In this setting the answer of the question *how to eliminate the noise without losing the contours of an image* is given by Perona and Malik in their model which is one of the first attempts to derive a model that incorporates local information from an image within a PDE framework [51]. It has attracted the attention of many researchers in image processing. But this model has a disadvantage called *staircasing phenomena.* In this work, we present a new model for image restoration. The main goal of this model is to remove staircase during the image denoising and preserve the edges. In this setting, numerous algorithms have been proposed recently to tackle the problems of noise removal and image restoration in real images ([9, 13, 16, 18, 20, 21, 38, 49–52])

Many methods have been proposed to deal with nonlinear problems: fixed point method, semigroups method, sub-supersolution method Brouwer degee and Leray-Schauder degree, etc. The last one method is an important topological tool introduced by Leray and Schauder in the study of nonlinear partial differential equations in the early 1930s. The nontriviality of the degree ensures the existence of a fixed point of the compact mapping in the domain. It enjoys the properties of homotopy invariance and additivity, which make the topological tool more convenient in application, and provides more information on fixed points.

The Leray-Schauder degree is an extension of the Brouwer degree from finite-dimensional spaces to infinite-dimensional Banach spaces. For more details, we refer the reader to [6, 19, 23, 27].

This thesis is mainly devoted to the study of a class of problems of nonlinear partial differential equations with a divergence operator and a Carathéodory function. By using the topological degree method and some functional analysis tools, we prove the existence of weak solutions of this problems. To be more precise, we are interested in the study of two classes of problems:

- The first class of problems is a new model for image restoration. The proposed model is an interpolation of two classical models, Perona-Malik and heat equation as following

$$\begin{cases} -\mathrm{div}\Big(g(|\nabla u|)\nabla u\Big) - \dfrac{1}{\lambda^2}\Delta u = f(x) - uk(x) \ \text{ in } \Omega, \\ \\ \Big(g(|\nabla u|) + \dfrac{1}{\lambda^2}\Big)\nabla u \cdot \vec{\eta} = 0. \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

  where $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary $\partial\Omega$ and the function $k \in L^\infty(\Omega)$ is the probability density of noise. We also study the case where the right hand side of the above problem is $f(u)$ (see [36]). In this setting, it is well known that in the case where the problem is an interpolation of two classical models, Perona-Malik and p-Laplacian with $k(x) = 1$ was studied by A. Atlas et al in [7] which is the origin of our model.

- The second class is the system of convection-diffusion-reaction. Our aim in this part is to investigate the existence of weak solution of the problem

$$\begin{cases} -\mathrm{div}(a_1(x, v(x))\nabla u) - \mathrm{div}(G_1(x)\varphi_1(v)) = f(x, u, v) \quad \text{ in } \Omega, \\ \\ -\mathrm{div}(a_2(x, u(x))\nabla v) - \mathrm{div}(G_2(x)\varphi_2(u)) = g(x, u, v) \quad \text{ in } \Omega, \\ \\ u = v = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{2}$$

  where Ω is a bounded domain in $\mathbb{R}^N \quad (N \geq 1)$ with smooth boundary $\partial\Omega$.
  The corresponding scalar case was solved by Thierry Gallouet [23]. This type of systems are widely studied in the literature. We can cite the leading works of some authors. A. Moussaoui and B. Khodja [44] treated the system (2) in the case where $a_1 = a_2 = 1$, $\varphi_1 = \varphi_2 = 0$ and λ is not eigenvalue i.e. $\lambda \notin \sigma(-\Delta)$. H. Lakhal et al [33] studied the system (2) in the case where $a_1 = a_2 = 1$, $\varphi_1 = \varphi_2 = 0$ and λ is a simple eigenvalue of $(-\Delta)$. They also considered the case of resonance (see [32]). Far from being complete, we refer readers to [37, 43, 46, 47].

This thesis can be roughly divided into four chapters. In the first chapter, we will discuss some preliminary material we will use throughout the thesis. This chapter has the following structure: In the first, we recall some definitions and important results in the $L^p(\Omega)$ spaces, Sobolev spaces and compact operators that has an essential role in the supsequent chapters. Finally, we introduce the method of the topological degree which is very useful tool for solving nonlinear partial differential equations.

In the second chapter we investigate the existence of weak solution of the problem (1). Our model is well-posed and substantially reduces the staircase effect. We use the compactness method and the monotonicity arguments, for obtained the existence result under some classical conditions on the function g. In the second section we consider the parabolic type of our model and we give some numerical implementation details and show experimental results on examples images which prove the efficiency and effectiveness of our model.

In the third chapter we extend the problem (1) to the system and we use the same technique for obtain the existence results.

The last chapter is devoted to the study of the existence of weak solution of boundary values problem for the nonlinear elliptic system of convection-diffusion-reaction. The existence results are obtained by using the Leray-Schauder degree theory under suitable assumptions on the nonlinearities.

Chapter 1

Preliminaries

In this chapter, we start by recalling some basic knowledge in functional analysis, most of which will be used in this thesis. Then, we introduce the method of the topological degree. The reader can easily find the detailed in the related literature see, e.g., [2, 6, 11, 23, 27, 30].

1.1 Functional Spaces

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. For any nonnegative integer m let $C^m(\Omega)$ be the space of all functions f which, together with all thier partial derivatives $D^\alpha f$ of orders $|\alpha| \leq m$, are continuous on Ω. We abbreviate $C^0(\Omega) \equiv C(\Omega) = \{f : \Omega \to \mathbb{R}; f \text{ continuous }\}$. Let $C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega)$,

$$C_c(\Omega) = \{f \in C(\Omega); f(x) = 0 \quad \forall x \in \Omega \backslash K, \text{ where } K \text{ is copmact}\},$$

and $D(\Omega)$ the space of C^∞ functions with compact support on Ω (also called the space of test functions).

1.1.1 The $L^p(\Omega)$ spaces

Definition 1. *Let $1 \leq p < \infty$, and let Ω be an open subset in $\mathbb{R}^N$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \ f \textit{ is measurable and } \int_\Omega |f(x)|^p \,\mathrm{d}x < \infty \right\},$$

with

$$\|f\|_{L^p} = \|f\|_p = \Big(\int_\Omega |f(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \leq C \text{ a.e. on } \Omega \right\},$$

with
$$\|f\|_\infty = \text{Inf}\Big\{C; |f(x)| \leq C \text{ a.e. on } \Omega\Big\}.$$

Remark 1.1. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*
$$(f, g) = \int_\Omega f(x)g(x)\,\mathrm{d}x,$$
is a Hilbert space.

Recall that
$$L^1_{loc}(\Omega) = \{f : f \in L^1(K) \text{ for every compact } K \text{ of } \Omega\}.$$

Proposition 1.1. *[12]*

1. *For $1 \leq p \leq \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space.*
2. *For $1 < p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a reflexive space.*
3. *For $1 \leq p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a separable space.*

Notation: Let $1 \leq p \leq \infty$; we denote by p' the conjugate exponent,
$$\frac{1}{p} + \frac{1}{p'} = 1.$$

1.1.2 Some useful inequalities

In this subsection, we shall recall some inqualities which will be used in the supsequent chapters.

Young inequalities

Theorem 1.2. *[11] Let $1 \leq p \leq \infty$, then*
$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \forall a \geq 0, b \geq 0.$$

The Holder inequalities

Theorem 1.3. *[11] Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$. Then*
$$fg \in L^1(\Omega) \text{ and } \|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

The Minkowski inequality

Theorem 1.4. *[11] Let $f, g \in L^p(\Omega)$ and $p \geq 1$, then*
$$f + g \in L^p(\Omega) \text{ and } \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

The Poincar inequality

Theorem 1.5. *[12] Let Ω be a bounded domain in $\mathbb{R}^N$ and $f \in H_0^1(\Omega)$. Then there is a positive constant C_Ω such that*

$$\|f\|_{L^2(\Omega)} \leq C_\Omega \|\nabla f\|_{L^2(\Omega)}, \quad \forall f \in H_0^1(\Omega).$$

1.1.3 Some Results about Integration and duality

Theorem 1.6 (**dominated convergence theorem, Lebesgue** [11])**.** *Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy*

1. *$f_n(x) \to f(x)$ a.e. on Ω,*
2. *there is a function $g \in L^1(\Omega)$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω.*

Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

Theorem 1.7. *[11] Let (f_n) be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$ be such that $\|f_n - f\|_p \to 0$.*
Then, there exist a subsequence (f_{n_k}) and a function $h \in L^p(\Omega)$ such that

1. *$f_{n_k}(x) \to f(x)$ a.e. on Ω,*
2. *$|f_{n_k}(x)| \leq h(x)$ $\forall k$, a.e. on Ω.*

Theorem 1.8 (**Riesz representation theorem** [11])**.** *Let $1 < p < \infty$ and let $\varphi \in (L^p(\Omega))'$. Then there exists a unique function $u \in L^{p'}(\Omega)$ such that*

$$\langle \varphi, f \rangle = \int_\Omega uf \, \mathrm{d}x, \forall f \in L^p(\Omega).$$

Moreover,

$$\|u\|_{p'} = \|\varphi\|_{(L^p)'}.$$

Definition 2 (**Weak derivative**)**.** *Let $f \in L^1_{loc}(\Omega)$. We say that f is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f \partial_i \phi \, \mathrm{d}x = -\int_\Omega g_i \phi \, \mathrm{d}x, \quad \text{for all } \phi \in C_c^\infty.$$

The function g_i is called the weak ith partial derivative of f, and is denoted by $\partial_i f$.

1.1.4 Sobolev spaces

Sobolev spaces are functional spaces, that is spaces containing functions, and these functions are such that their powers and the powers of their derivatives (in the sense of transposition, or in a weak sense which we shall give later) are Lebesgue-integrable. Similarly to the Lebesgue spaces, these spaces are Banach spaces. The fact that the Sobolev spaces are complete is very important to prove the existence of solutions to some partial differential equations.

Definition 3.
- *We define a functional* $\|\cdot\|_{m,p}$, *where* m *is a nonnegative integer and* $1 \leq p \leq \infty$, *as follows;*

$$\|f\|_{m,p} = \left\{ \sum_{0\leq|\alpha|\leq m} \|D^\alpha f\|_p^p \right\}^{1/p}, \quad \text{if } 1 \leq p < \infty$$

$$\|f\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^\alpha f\|_\infty;$$

for any function f *for which the right side makes sense,* $\|\cdot\|_p$ *being, of course, the* $L^p(\Omega)$*-norm.*

- $W^{m,p}(\Omega) = \{f \in L^p(\Omega)$ *such that* $D^\alpha f \in L^p(\Omega)$ *for* $0 \leq |\alpha| \leq m\}$.
- $W_0^{m,p}(\Omega) =$ *the closure of* $C_0^\infty(\Omega)$ *in the space* $W^{m,p}(\Omega)$.

Remark 1.2. *If* $p = 2$, *we usually write*

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Proposition 1.9. *[12] Let* Ω *be an open subset of* $\mathbb{R}^N$*;*

1. *For* $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ *is a Banach space.*
2. *For* $1 < p < \infty$, $W^{m,p}(\Omega)$ *is a reflexive space.*
3. *For* $1 \leq p < \infty$, $W^{m,p}(\Omega)$ *is a separable space.*

Theorem 1.10. *[30] Let us assume that:* Ω *is an open subset of* $\mathbb{R}^N$, $N \geq 1$ *and* $m \in \mathbb{N}$, $1 \leq p < \infty$ *and* $p^* = \dfrac{Np}{N-p}$. *Then*

1. *If* $\dfrac{1}{p} - \dfrac{m}{N} > 0$, *we have* $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, *with* $q \in [p, p^*]$, $\dfrac{1}{p} - \dfrac{1}{p^*} = \dfrac{m}{N}$.
2. *If* $\dfrac{1}{p} - \dfrac{m}{N} = 0$, *we have* $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, +\infty[$.
3. *If* $\dfrac{1}{p} - \dfrac{m}{N} < 0$, *we have* $W_0^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Notation: The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Proposition 1.11 (**Integration by parts formula**[30]). *Let $u, v \in H^1(\Omega)$ and $\partial\Omega \in C^1$, then for every $1 \leq i \leq N$ we have*

$$\int_\Omega \frac{\partial u(x)}{\partial x_i} v(x)\,\mathrm{d}x = -\int_\Omega u(x)\frac{\partial v(x)}{\partial x_i}\,\mathrm{d}x + \int_{\partial\Omega} u(s)v(s)n_i ds.$$

If $v \in H^1(\Omega)$ and if $u_i \in H^1(\Omega)$ where u_i is the components of the vector $\vec{u}$, then we have

$$\int_\Omega div(\vec{u}(x)) \cdot v(x)\,\mathrm{d}x = -\int_\Omega (\bar{u}(x), \nabla v(x))\,\mathrm{d}x + \int_{\partial\Omega} (\vec{u}, \vec{n}) v ds.$$

Finally, noticing that

$$\Delta u = div(\nabla \vec{u}),$$

we get Green's formula.

Proposition 1.12. *[30] For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have:*

$$\int_\Omega \Delta u(x)v(x)\,\mathrm{d}x = -\int_\Omega \nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds.$$

Theorem 1.13 (**Rellich** [23]). *Let Ω be an open subset of $\mathbb{R}^N$, and let $1 \leq p < +\infty$. Any bounded part of $W_0^{1,p}(\Omega)$ is relatively compact in $L^p(\Omega)$. This amounts to saying that for any bounded sequence of $W_0^{1,p}(\Omega)$, we can extract a subsequence which converges in $L^p(\Omega)$.*

The previous theorem remains true on $W^{1,p}(\Omega)$ with the assumption of the Lipschitzian boundary condition.

Theorem 1.14 (**Stampacchia** [30]). *Let Ω be an open subset of $\mathbb{R}^N$ and $1 \leq p < \infty$. Let $u \in W_0^{1,p}(\Omega)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function, C^1 piecewise such that $\varphi(0) = 0$ and φ' bounded on $\mathbb{R}$. Then $\varphi(u) \in W_0^{1,p}(\Omega)$ and $\nabla\varphi(u) = \varphi'(u)\nabla u$ a.e..*

Theorem 1.15 (**Lax-Milgram** [11]). *Assume that $a(u, v)$ is a continuous coercive bilinear form on H. Then, given any $\varphi \in H^*$, there exists a unique element $u \in H$ such that*

$$a(u, v) = \langle \varphi, v \rangle \quad \forall v \in H.$$

Moreover, if a is symmetric, then u is characterizedby the property

$$u \in H \quad \text{and} \quad \frac{1}{2}a(u, u) - \langle \varphi, u \rangle = \min_{v \in H}\Big\{\frac{1}{2}a(v, v) - \langle \varphi, v \rangle\Big\}.$$

Definition 4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$. If $u : \Omega \to \mathbb{R}$ is a measurable real valued function, we can consider the map $u \mapsto f(u)$, where $f(u)$ is the real valued function defined on Ω by setting*

$$f(u)(x) = f(x, u(x)),$$

such a map is called the Nemitski operator associated to f.

Theorem 1.16. *[6] Let α, $\beta \geq 1$. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies*

1. *$f(x,t)$ is measurable with respect to $x \in \Omega$ for all $t \in \mathbb{R}$ and is continuous with respect $t \in \mathbb{R}$ for a.e. $x \in \Omega$.*

2. *There exists $a_1 \in L^\beta(\Omega)$ and $a_2 > 0$ such that*

$$|f(x,u)| \leq a_1(x) + a_2|u|^{\alpha/\beta}, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \quad (\alpha, \beta \geq 1).$$

Remark 1.3. *Condition* 1 *is also called the Caratheodory condition and a function $f(x,t)$ satisfying* 1 *is usually called a Caratheodory function.*

1.2 Topological degree

The topological degree of a map is a classical tool which is very useful for solving functional equations. It was introduced by L. Brouwer for finite dimension and extended by J. Leray and J. Schauder to infinite dimension.

1.2.1 The Brouwer degree and its properties

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. Let $f \in C(\bar{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N$. We study the existence of $x \in \bar{\Omega}$ such that $f(x) = y$. We start by giving the existence and the uniqueness of an application, called topological degree, in finite dimension then we extend it to the infinite dimension. This application sometimes allows us to obtain the results of existence of desired solution.

Definition 5. *Let Ω be an open subset of $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}^N$, $f \in C^1(\Omega) \cap C(\bar{\Omega})$, $y \in \Omega$ is said to be a regular value for f, if the Jacobian $J_f(x)$ is different from zero for every $x \in f^{-1}(y)$. In the case where $J_f(x) = 0$, y called critical point or singular point.*
We denote by S_f the set of singular points of f,

$$S_f = \{x \in \Omega : \; J_f(x) = 0\}.$$

If y is a regular value then the set $f^{-1}(y)$ is finite and one can define the degree by setting

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} sgn(J_f(x)),$$

such that

$$f^{-1}(y) = \{x \in \Omega : \; f(x) = y\}.$$

Theorem 1.17 (Brouwer, 1933 [23]**).** *Let $N \geq 1$ and let asume that*

1. *Ω is an open bounded set in $\mathbb{R}^N$, with boundary $\partial\Omega$;*

2. *$f \in C(\bar{\Omega}, \mathbb{R}^N)$;*

3. $y \in \mathbb{R}^N$ *such that* $y \notin \{f(x), x \in \partial\Omega\}$.

To each triple (f, Ω, y) *satisfying* $1-3$ *one can associate an integer* $\deg(f, \Omega, y) \in \mathbb{Z}$, *called the degree of* f *(with respect to* Ω *and* y*), with the following basic properties:*

(P_1) **Normalization:** $\deg(Id, \Omega, y) = 1$ *if* $y \in \Omega$.

(P_2) **Decomposition:** *If* $\Omega_1 \cup \Omega_2 \subset \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $y \notin \{f(x), x \in \bar{\Omega} \setminus \Omega_1 \cup \Omega_2\}$ *then*

$$\deg(f, \Omega, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y).$$

(p_3) **Homotopy invariance:** *If* $h \in C([0,1] \times \bar{\Omega}, \mathbb{R}^N)$, $y \in C([0,1], \mathbb{R}^N)$ *and* $y(t) \notin \{h(t,x), x \in \partial\Omega\}$ *(for every* $t \in [0,1]$*), then we have*

$$\deg(h(t,\cdot), \Omega, y(t)) = \deg(h(0,\cdot), \Omega, y(0)), \text{ for every } t \in [0,1].$$

1.2.2 The Brouwer fixed point theorem

Here, we will assume that a degree $\deg(f, \Omega, y)$ satisfying the properties listed in the previous subsection has been defined, and we will give the Brouwer fixed point theorem which is the first consequence of topological degree method.

Theorem 1.18. *[23] Let* $N \geq 1$, $R > 0$ *and* $f \in C(B_R, B_R)$ *with* $B_R = \{x \in \mathbb{R}^N, \|x\| \leq R\}$ *(we denote by* $\|\cdot\|$ *the norm in* $\mathbb{R}^N$*). Then* f *has a fixed point, that is to say there exists* $x \in B_R$ *such that* $f(x) = x$.

1.2.3 The Leray-Schauder degree and its properties

For arbitrary continuous maps in infinite dimensions, it is certainly impossible to define a degree with the same properties as in finite dimensions. To this end the above theorem has been generalized by Leray-Schauder under a compactness assumptions.

Definition 6. *Let* E *be a Banach space,* B *be a part of* E *and* f *an application from* B *to* E. *We said that* f *is compact (Leray-Schauder use the expression "completely continuous") if* f *satisfies the following two properties*

1. f *is continuous,*

2. $\{f(x), x \in C\}$ *is relatively compact in* E *for any bounded* C *of* B.

We give now the main result of this part

Theorem 1.19 (Leray-Schauder, 1934 [23]**).** *Let us assume that:*

1. E *is a Banach space;*

2. Ω *is an open bounded set in* E, *with boundary* $\partial\Omega$;

3. *f is a compact map from $\bar{\Omega}$ to E;*

4. *y is a point in E such that $y \notin \{x - f(x),\ x \in \partial\Omega\}$.*

To each triple $(I - f, \Omega, y)$ satisfying $(1) - (4)$, one can associate an integer $\deg(I - f, \Omega, y) \in \mathbb{Z}$, called "Topological degree", with the following basic properties:

(P_1) **Normalization:** $\deg(Id, \Omega, y) = 1$ *if $y \in \Omega$.*

(P_2) **Decomposition:** *If $\Omega_1 \cup \Omega_2 \subset \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $y \notin \{x - f(x), x \in \bar{\Omega} \setminus \Omega_1 \cup \Omega_2\}$ then*

$$\deg(I - f, \Omega, y) = \deg(I - f, \Omega_1, y) + \deg(I - f, \Omega_2, y).$$

(p_3) **Homotopy invariance:** *If h is a compact map from $[0, 1] \times \bar{\Omega}$ to E, $y \in C([0, 1], E)$ and $y(t) \notin \{x - h(t, x),\ x \in \partial\Omega\}$ (for every $t \in [0, 1]$) , then we have*

$$\deg(I - h(t, \cdot), \Omega, y(t)) = \deg(I - h(0, \cdot), \Omega, y(0)),\ \textit{for every } t \in [0, 1].$$

Definition 7. *Let h be a compact map and I is a identity map, the map f such that*

$$f = I - h$$

is a compact perturbation of the identity I (or the map of Leray-Schauder).

Remark 1.4. *The essential property of the topological degree is:*
If the triple $(I - f, \Omega, y)$ satisfying the assumptions of the Theorem 1.19 and $\deg(I - f, \Omega, y) \neq 0$, then there exists $x \in \Omega$ such that $x - f(x) = y$.

Chapter 2

Existence of the solution of a quasilinear equation and its application to image denoising

The aim of this chapter is to establish the existence of weak solution for a new model for image restoration. The proposed model is an inerpolation of two classical models, Perona-Malik and heat equation. This existence is obtained by using the compactness method and the monotonicity arguments. The corresponding method has been first introduced by Vishik and called the compacteness method by J.L. Lions [39]. Then we give some numerical implementation details and show experimental results on examples images which prove the efficiency and effectiveness of our model.

2.1 Position of the problem

Here we investigate the existence of solution for the following problem

$$\begin{cases} -\mathrm{div}\Big(g(|\nabla u|)\nabla u\Big) - \dfrac{1}{\lambda^2}\Delta u = \chi(u) \ \text{ in } \Omega, \\ \\ \Big(g(|\nabla u|) + \dfrac{1}{\lambda^2}\Big)\nabla u \cdot \vec{\eta} = 0. \ \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary $\partial\Omega$, $0 < \lambda \leq 1$ and the function $g(\cdot)$ is defined by the following expression:

$$g(z) = \frac{1}{1 + (\frac{z}{\lambda})^2} \quad \text{or} \quad g(z) = \exp\big(-\frac{z^2}{2\lambda^2}\big). \tag{2.2}$$

It is clear, that the function $g(z)$ is a decreasing non-negative function satisfies the following conditions

$$\begin{cases} \lim\limits_{z\to 0} g(z) = 1, \\ \\ \lim\limits_{z\to +\infty} g(z) = 0. \end{cases} \tag{2.3}$$

In the case where the Euler-Lagrange equation equal to $u - \operatorname{div}\Big(g(|\nabla u|)\nabla u\Big) - \frac{1}{\lambda^p}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x)$ was treated in [7]. The problem (2.1) is equivalent to solve the Perona-Malik problem in the region where the norm of the gradient is less than λ and equivalent to solve the heat equation if not.

This chapter is organized as follows. In the next section, we prove the existence of the solution of the problem (2.1). And the last section is devoted to numerical results and comments.

2.2 Proposed model

2.2.1 Case where the right hand side is $f(x) - uk(x)$

Given $f \in L^2(\Omega)$ and $k \in L^\infty(\Omega)$, in the application k is the probability density of noise (we speak here on the gaussian noise because the probability density of this variable is the gaussian law). We are interested in finding weak solution of the problem (2.1), we need the following definition

Definition 8. *We say that $u \in H^1(\Omega)$ is a weak solution for the problem* (2.1) *if for any $\varphi \in H^1(\Omega)$ we have*

$$\int_\Omega \Big(g(|\nabla u|) + \frac{1}{\lambda^2}\Big)\nabla u \nabla\varphi \,\mathrm{d}x = \int_\Omega f(x)\varphi\,\mathrm{d}x - \int_\Omega uk(x)\varphi\,\mathrm{d}x. \tag{2.4}$$

Our main result of this chapter is the:

Theorem 2.1. *Under condition* (2.3), *problem* (2.1) *has at least one solution.*

Proof. Let W be a finite-dimensional subspace of $H^1(\Omega)$ endowed with the H^1-norm, and W* its dual. Define the mappings $H : W \times [0,1] \longrightarrow W^*$ by

$$\begin{aligned} &\langle H(u,t), \varphi\rangle_H \\ &= \int_\Omega \Big(g(t|\nabla u|) + \frac{1}{\lambda^2}\Big)\nabla u\nabla\varphi\,\mathrm{d}x - \int_\Omega f(x)\varphi\,\mathrm{d}x \\ &+ \int_\Omega uk(x)\varphi\,\mathrm{d}x, \end{aligned} \tag{2.5}$$

for all $\varphi \in W$, H is well defined. Let us show now that

$$\Big\{u \in W : H(u,t) = 0, \quad \text{for some } t \in [0,1]\Big\} \subset \bar{B}(m^{-1}\|f\|_{L^2}).$$

Indeed, if $H(u,t) = 0$ for some $(u,t) \in W \times [0,1]$, then

$$\begin{aligned} 0 = \langle H(u,t), u\rangle \geq \ & \frac{1}{\lambda^2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega u^2 k(x)\,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \\ & \geq \min(\frac{1}{\lambda^2}, k)\|u\|_{H^1}^2 - \|f\|_{L^2}\|u\|_{H^1}, \end{aligned}$$

then

$$\|f\|_{L^2}\|u\|_{H^1} \geq \ \min(\frac{1}{\lambda^2}, k)\|u\|_{H^1}^2,$$

which implies that

$$\|u\|_{H^1} \leq m^{-1}\|f\|_{L^2}, \tag{2.6}$$

where $m = \min\left(\frac{1}{\lambda^2}, k\right)$. Consequently, for any $R > m^{-1}\|f\|_{L^2}$ we have

$$H(u,t) \neq 0 \quad \text{ if } (u,t) \in \partial B^W(R) \times [0,1].$$

Now, if $(u,t) \in \bar{B}^W \times [0,1]$, we have

$$\begin{aligned} |\langle H(u,t), \varphi\rangle| \leq \ & \max\left(1 + \frac{1}{\lambda^2}, \|k\|_{L^\infty}\right)\|u\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)} + \|f\|_{L^2}\|\varphi\|_{H^1(\Omega)} \\ & \leq \Bigg(\underbrace{\max\left(1 + \frac{1}{\lambda^2}, \|k\|_{L^\infty}\right)R + \|f\|_{L^2}}_{\widetilde{R}}\Bigg)\|\varphi\|_{H^1(\Omega)}, \end{aligned}$$

for all $\varphi \in H^1(\Omega)$, and hence

$$H\Big(\bar{B}^{\mathrm{W}}(R) \times [0,1]\Big) \subset \bar{B}^{\mathrm{W}^*}(\widetilde{R}). \tag{2.7}$$

We now show that H is continuous on $\bar{B}^{\mathrm{W}}(R) \times [0,1]$.
Let $(u_n, t_n) \in \bar{B}^{\mathrm{W}}(R) \times [0,1]$ converge to (u,t) in $W \times [0,1]$, i.e in $H^1 \times [0,1]$. Since $(H(u_n,t_n))$ is bounded because of (2.7), to prove that

$$H(u_n, t_n) \to H(u,t),$$

it is sufficient to show that $H(u,t)$ is the unique cluster point of $(H(u_n,t_n))$. Let $\widetilde{k} \in \mathrm{W}^*$ be such a cluster point, still we denote by $(t_n), (u_n)$ a subsequence of $(t_n), (u_n)$ respectively such that

$$H(u_n, t_n) \to \widetilde{k} \text{ in } \mathrm{W}^*.$$

Since $u_n \to u$ in $H^1(\Omega)$, it follows that $u_n \to u$ in $L^2(\Omega)$, and hence, going if necessary to a subsequence, we may assume that $u_n \to u$ a.e. in Ω. On the other hand, $\partial_i u_n \to \partial_i u$ in $L^2(\Omega)$, therefore $\nabla u_n \to \nabla u$ a.e. in Ω. This implies that

$$g(t_n|\nabla u_n|) \to g(t|\nabla u|) \quad \text{a.e in } \Omega, \tag{2.8}$$

and hence, for any $\varphi \in W$,

$$g(t_n|\nabla u_n|)\nabla\varphi \to g(t|\nabla u|)\nabla\varphi \text{ in } L^2(\Omega).$$

We conclude that

$$\begin{aligned}
&\langle H(t_n, u_n), \varphi\rangle_H \\
&= \int_\Omega u_n k(x)\varphi \,\mathrm{d}x + \int_\Omega \Big(g(t_n|\nabla u_n|) + \frac{1}{\lambda^2}\Big)\nabla u_n \nabla\varphi \,\mathrm{d}x \\
&\to \int_\Omega u k(x)\varphi \,\mathrm{d}x + \int_\Omega \Big(g(t|\nabla u|) + \frac{1}{\lambda^2}\Big)\nabla u \nabla\varphi \,\mathrm{d}x \\
&= \langle H(t, u), \varphi\rangle_H.
\end{aligned}$$

Thus $\widetilde{k} = H(t, u)$.

It is clear that

$$H : W \times [0, 1] \to W^*,$$

is a continuous homotopy and the existence of at least one solution of the problem (2.1) would follow from

$$\deg_B\Big(H(\cdot, 1), B(R), 0\Big) \neq 0.$$

All those properties allow us to apply the homotopy invariance property to

$$\deg_B\Big(H(\cdot, 1), B(R), 0\Big) = \deg_B\Big(H(\cdot, 0), B(R), 0\Big). \tag{2.9}$$

But $H(u, 0) = 0$ is equivalant to the problem

$$(1 + \frac{1}{\lambda^2})\int_\Omega \nabla u \nabla\varphi \,\mathrm{d}x - \int_\Omega f(x)\varphi \,\mathrm{d}x + \int_\Omega u k(x)\varphi \,\mathrm{d}x = 0,$$

for all $\varphi \in$ W, whose solution is unique because of the boundedness of the set of its possible solutions. Consequently,

$$\deg_B\Big(H(\cdot, 0), B(R), 0\Big) = \pm 1,$$

and, from (2.9) and the existence property of degree, there exists $u \in B^{\mathrm{W}}(R)$ which satisfies

$$\begin{aligned} &\int_\Omega \Big(g(|\nabla u|) + \frac{1}{\lambda^2}\Big)\nabla u \nabla\varphi \,\mathrm{d}x \\ &= \int_\Omega f(x)\varphi\,\mathrm{d}x - \int_\Omega uk(x)\varphi\,\mathrm{d}x, \quad \|u\|_{H^1} \le m^{-1}\|f\|_{L^2}, \end{aligned} \tag{2.10}$$

for all $\varphi \in W$.

We now show the passage to the limit.
Consider the function $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a(\xi) = \Big(g(\xi) + \frac{1}{\lambda^2}\Big)\xi \quad \text{for any } \xi \in \mathbb{R}^N.$$

To prove the passage to the limit, we need the following two lemmas:

Lemma 2.1. *Let $0 < \lambda \le 1$, for any $\xi, \eta \in \mathbb{R}^N$ such that $\xi \ne \eta$ we have*

$$\big(a(\xi) - a(\eta)\big)(\xi - \eta) > 0.$$

Proof. The proof of this lemma remains to prove that F_λ is a nondecreasing function defined by

$$F_\lambda(s) = sg(s) + \frac{s}{\lambda^2} \quad \text{for } s > 0.$$

We compute for this the derivative of $F_\lambda(s)$, and then we find

$$F'_\lambda(s) = \widetilde{g}(s) + \frac{1}{\lambda^2},$$

where

$$\widetilde{g}(s) = \frac{\lambda^2 - s^2}{\lambda^2(1 + (\frac{s}{\lambda})^2)^2} \text{ or } \widetilde{g}(s) = \frac{\lambda^2 - s^2}{\lambda^2}\exp\Big(-\frac{s^2}{2\lambda^2}\Big).$$

For $s \le \lambda$, we have $1 - \Big(\dfrac{s}{\lambda}\Big)^2 \ge 0$ and $\dfrac{\lambda^2 - s^2}{\lambda^2} \ge 0$ then $F'_\lambda(s) \ge 0$.
For $s \ge \lambda$, using the fact that $\lambda^2 \le 1$, we have $\dfrac{s^4}{\lambda^6} \ge \dfrac{s^2}{\lambda^2}$, we deduce that $F'_\lambda(s) \ge 0$ and we find the desired result. □

Lemma 2.2.

$$\begin{cases} \textit{If } a \in C(\mathbb{R}^N, \mathbb{R}^N),\ a(\xi) \leq (1 + \frac{1}{\lambda^2})\xi \quad \textit{for all } \xi \in \mathbb{R}^N \\ \textit{and} \\ \textit{if } u_n \to u \textit{ in } H^1(\Omega) \\ \textit{then} \\ a(\nabla u_n) \to a(\nabla u) \textit{ in } L^2(\Omega). \end{cases}$$

Lemma (2.2) is proved by the dominated convergence theorem of Lebesgue.

Now, it is well known that one can write $H^1(\Omega) = \overline{\bigcup_{n\geq 1} W_n}$ where $W_n \subset W_{n+1}(n \geq 1)$ and W_n has dimension n. Consequently, given any $\varphi \in H^1(\Omega)$, there exists a sequence φ_n with $\varphi_n \in W_n$ which converges to φ. On the other hand, by (2.10) applied to $W = W_n$, there exists, for each $n \geq 1$, some $u_n \in W_n$ such that

$$\int_\Omega a(\nabla u_n)\nabla\psi \,\mathrm{d}x = \int_\Omega f(x)\psi \,\mathrm{d}x - \int_\Omega u_n k(x)\psi \,\mathrm{d}x, \quad \|u_n\|_{H^1} \leq m^{-1}\|f\|_{L^2},$$

for all $\psi \in W_n$. In particular, taking $\psi = \varphi_n$ introduced above,

$$\begin{aligned} &\int_\Omega a(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x = \int_\Omega f(x)\varphi_n \,\mathrm{d}x - \int_\Omega u_n k(x)\varphi_n \,\mathrm{d}x, \\ &\|u_n\|_{H^1(\Omega)} \leq m^{-1}\|f\|_{L^2}, \end{aligned} \tag{2.11}$$

for all $n \geq 1$. The estimate in (2.11) implies that, going if necessary to subsequences, we can assume that there exists $u \in H^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and $u_n \to u$ a.e. in Ω. As $\big(a(\nabla u_n)\big)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, then there exists $\zeta \in L^2(\Omega)$ such that

$$a(\nabla u_n) \to \zeta \text{ weakly in } L^2(\Omega),$$

and $\nabla\varphi_n \to \nabla\varphi$ strongly in $L^2(\Omega)$, one can let $n \to \infty$ in (2.11) to obtain

$$\int_\Omega \zeta\nabla\varphi \,\mathrm{d}x = \int_\Omega f(x)\varphi \,\mathrm{d}x - \int_\Omega uk(x)\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in H^1(\Omega). \tag{2.12}$$

It remains to show that

$$\int_\Omega \zeta\nabla\varphi \,\mathrm{d}x = \int_\Omega a(\nabla u)\nabla\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in H^1(\Omega),$$

for it using the trick of Minty [23]; we begin by studying the limit of $\int_\Omega a(\nabla u_n)\nabla u_n \,\mathrm{d}x$. Indeed

$$\begin{aligned}\int_\Omega a(\nabla u_n)\nabla u_n \,\mathrm{d}x &= \int_\Omega f(x)u_n \,\mathrm{d}x - \int_\Omega u_n^2 k(x)\,\mathrm{d}x \\ &\to \int_\Omega f(x)u \,\mathrm{d}x - \int_\Omega u^2 k(x)\,\mathrm{d}x,\end{aligned}$$

because $u_n \to u$ weakly in $H^1(\Omega)$. But we know that u satisfies (2.12), and hence

$$\int_\Omega f(x)u \,\mathrm{d}x - \int_\Omega u^2 k(x)\,\mathrm{d}x = \int_\Omega \zeta \nabla u \,\mathrm{d}x.$$

Therefore

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega a(\nabla u_n)\nabla u_n \,\mathrm{d}x &= \int_\Omega f(x)u \,\mathrm{d}x - \int_\Omega u^2 k(x)\,\mathrm{d}x \\ &= \int_\Omega \zeta \nabla u \,\mathrm{d}x.\end{aligned} \tag{2.13}$$

Let $\varphi \in H^1(\Omega)$; it exists $(\varphi_n)_{n\in\mathbb{N}}$ such that $\varphi_n \in W_n$ for all $n \in \mathbb{N}$ and $\varphi_n \to \varphi$ in $H^1(\Omega)$ when $n \to +\infty$. We will pass to the limit in the term $\int_\Omega a(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x$ through the Lemma 2.1. Indeed,

$$\begin{aligned}&0 \le \int_\Omega (a(\nabla u_n) - a(\nabla\varphi_n))(\nabla u_n - \nabla\varphi_n)\,\mathrm{d}x = \\ &\int_\Omega a(\nabla u_n)\nabla u_n \,\mathrm{d}x - \int_\Omega a(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x \\ &- \int_\Omega a(\nabla\varphi_n)\nabla u_n \,\mathrm{d}x + \int_\Omega a(\nabla\varphi_n)\nabla\varphi_n \,\mathrm{d}x \\ &= \quad L_{1,n} \quad - \quad L_{2,n} \quad - \quad L_{3,n} \quad + \quad L_{4,n},\end{aligned}$$

where

$$L_{1,n} = \int_\Omega a(\nabla u_n)\nabla u_n \,\mathrm{d}x, \quad L_{2,n} = \int_\Omega a(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x,$$

and

$$L_{3,n} = \int_\Omega a(\nabla\varphi_n)\nabla u_n \,\mathrm{d}x, \quad L_{4,n} = \int_\Omega a(\nabla\varphi_n)\nabla\varphi_n \,\mathrm{d}x,$$

we saw in (2.13) that $L_{1,n} \to \displaystyle\int_\Omega \zeta\nabla u \,\mathrm{d}x$ when $n \to \infty$.
We have

$$\lim_{n\to+\infty} L_{2,n} = \int_\Omega \zeta\nabla\varphi \,\mathrm{d}x,$$

by product of strong convergence in $L^2(\Omega)$ and weak convergence in $L^2(\Omega)$.
Similarly,

$$\lim_{n\to+\infty} L_{3,n} = \int_\Omega a(\nabla\varphi)\nabla u \,\mathrm{d}x.$$

Finally, we also have

$$\lim_{n\to+\infty} L_{4,n} = \int_\Omega a(\nabla\varphi)\nabla\varphi \,\mathrm{d}x,$$

when $n \to +\infty$ and this last term is the simplest because we have the product of a strong convergence in $L^2(\Omega)$ and a strong convergence in $L^2(\Omega)$.
The passage to the limit gives:

$$\int_\Omega (\zeta - a(\nabla\varphi))(\nabla u - \nabla\varphi)\,\mathrm{d}x \geq 0 \text{ for all } \varphi \in H^1(\Omega).$$

We now choose judicious test function φ. We take $\varphi = u + \frac{1}{n}v$, with $v \in H^1(\Omega)$ and $n \in \mathbb{N}^*$. We thus obtain:

$$-\frac{1}{n}\int_\Omega \big(\zeta - a(\nabla u + \frac{1}{n}\nabla v)\big)\nabla v \,\mathrm{d}x \geq 0,$$

then

$$\int_\Omega \big(\zeta - a(\nabla u + \frac{1}{n}\nabla v)\big)\nabla v \,\mathrm{d}x \leq 0.$$

But $u + \frac{1}{n}v \to u$ in $H^1(\Omega)$, thanks to Lemma (2.2) we obtain

$$a\big(\nabla u + \frac{1}{n}\nabla v\big) \to a(\nabla u) \text{ in } L^2(\Omega).$$

Passing to the limit when $n \to +\infty$, we then obtain

$$\int_\Omega (\zeta - a(\nabla u))\nabla v \,\mathrm{d}x \leq 0, \quad \forall v \in H^1(\Omega).$$

By linearity (can change v into $-v$), we have:

$$\int_\Omega (\zeta - a(\nabla u))\nabla v \,\mathrm{d}x = 0, \quad \forall v \in H^1(\Omega).$$

We deduce that

$$\int_\Omega \zeta\nabla v \,\mathrm{d}x = \int_\Omega a(\nabla u)\nabla v \,\mathrm{d}x, \quad \forall v \in H^1(\Omega).$$

Hence we have to show that u is a solution of (2.1).

□

2.2.2 Case where the right hand side is $f(u)$

In this case we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous function satisfying the caratheodory conditions, and verifying also the growth restriction defined below:

$$|f(x,s)| \leq d(x) + \frac{1}{2\lambda^2}|s|, \tag{2.14}$$

where $d \in L^2(\Omega)$ and $\lambda > 0$ is real positive constants.
This case aims to generalize the work already done in the previous case, for more general data than $f(x) - uk(x)$.

We give now a definition of weak solution.

Definition 9. *We say that* $u \in H^1(\Omega)$ *is a weak solution for the problem* (2.1) *if for any* $v \in H^1(\Omega)$ *we have*

$$\int_\Omega \Big(g(|\nabla u|) + \frac{1}{\lambda^2}\Big)\nabla u \nabla v \,\mathrm{d}x = \int_\Omega f(u) v \,\mathrm{d}x. \tag{2.15}$$

Our main result is formulated in the following theorem.

Theorem 2.2. *Assume that the assumption of the previous theorem hold. Assume further that* (2.14) *is fulfilled. Then there exists at least one solution of the problem* (2.1)

Proof. We use the same techniques used in the previous case for obtain the existence results. □

2.3 Numerical aspects and results

2.3.1 Case where the right hand side is $f(x) - uk(x)$

The artificial time discretisation associated with the functional (2.4) can be rewritten as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(\Psi(|\nabla u|)\nabla u\big) = f - ku & \text{in } \Omega \times (0,T), \\ \\ \Psi(|\nabla u|)\nabla u \cdot \vec{\eta} = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.16}$$

where Ψ is defined by

$$\Psi(t) = \frac{1}{1 + (\frac{t}{\lambda})^2} + \frac{1}{\lambda^2} \quad \text{or } \Psi(t) = \exp\Big(-\frac{t^2}{2\lambda^2}\Big) + \frac{1}{\lambda^2}. \tag{2.17}$$

We apply finite differences method to this problem. We denote respectively by h and Δt the spatial and time steps sizes. In what follows, we take $h = 1$ and we introduce a discrete gradient operator. We can write

$$(\nabla u)_{i,j} = \Big((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j}\Big),$$

with

$$(\nabla u)^1_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{si } i < N, \\ 0 & \text{si } i = N, \end{cases}$$

$$(\nabla u)^2_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{si } j < N, \\ 0 & \text{si } j = N, \end{cases}$$

and

$$|(\nabla u)_{i,j}| = \sqrt{((\nabla u)^1_{i,j})^2 + ((\nabla u)^2_{i,j})^2},$$

for $i, j = 1, \cdots, N$. Other choices of discretization are of course possible for the gradient, as long as it is a linear operator.
For every field $p = (p_1, p_2) \in \mathbb{R}^2$, we define the discrete divergence as

$$(\text{div}\, \mathbf{p})_{i,j} = \begin{cases} \mathbf{p}^1_{i,j} - \mathbf{p}^1_{i-1,j} & \text{si } 1 < i < N, \\ \mathbf{p}^1_{i,j} & \text{si } i = 1, \\ -\mathbf{p}^1_{i-1,j} & \text{si } i = N, \end{cases} + \begin{cases} \mathbf{p}^2_{i,j} - \mathbf{p}^2_{i,j-1} & \text{si } 1 < j < N, \\ \mathbf{p}^2_{i,j} & \text{si } j = 1, \\ -\mathbf{p}^2_{i,j-1} & \text{si } j = N, \end{cases}$$

where N is an integer greater than 2. One can write the following scheme:

$$\begin{aligned} & u^{n+1}(i,j) \\ &= u^n(i,j) + \Delta t\Big(\text{div}(\Phi(|\nabla u^n(i,j)|)\nabla u^n(i,j))\Big), \quad 1 \le n \le M \end{aligned}$$

where $u^n(i,j) = u(x_i, y_j, t_n),\ x_i = ih,\ y_j = jh,\ t_n = n\Delta t$ and $\Delta t = \dfrac{T}{M}$.
As shown in the figures below for images 2.1 . The choice for our numerical tests are: for noise we use the gaussian noise (the probability density of noise k with zero mean and variance $\sigma = 0,5$), the time step size $\Delta t = 10^{-4}$ and the number of iterations is equal to 3.5×10^3. We start by the improvements tests (Fig. 2.3) in the restorations provided by our approach and we choose the parameter $\lambda = 1$. In the second experiment, we illustrate the difference between our proposed method and the method of the total variation and the Perona-Malik model (see Figs. 2.4, 2.5, 2.6 and 2.7) and we will show that our method is effective in reducing the staircasing effect and preserving fine details. We also compared the result of PSNR for the three models as shown in the Table 2.1, 2.2 and 2.8, where the PSNR is the Peak signal-to-noise ratio defined as

$$\text{PSNR } = 10\log_{10}\left[\frac{255^2}{\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\Big[I(x,y) - I_0(x,y)\Big]^2}\right],$$

where I and I_0 are original image and reconstructed image and M, N represents the size of the image respectively, in horisontal and in vertical direction.

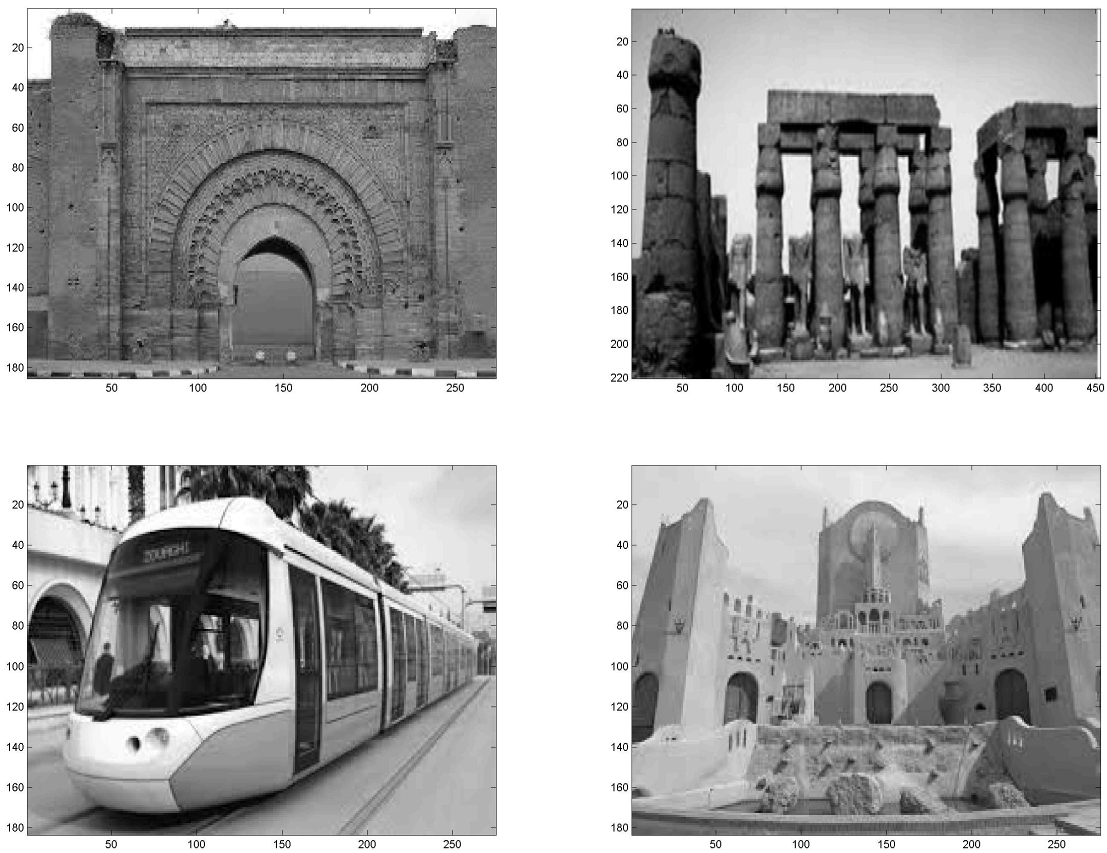


Figure 2.1: Original image.

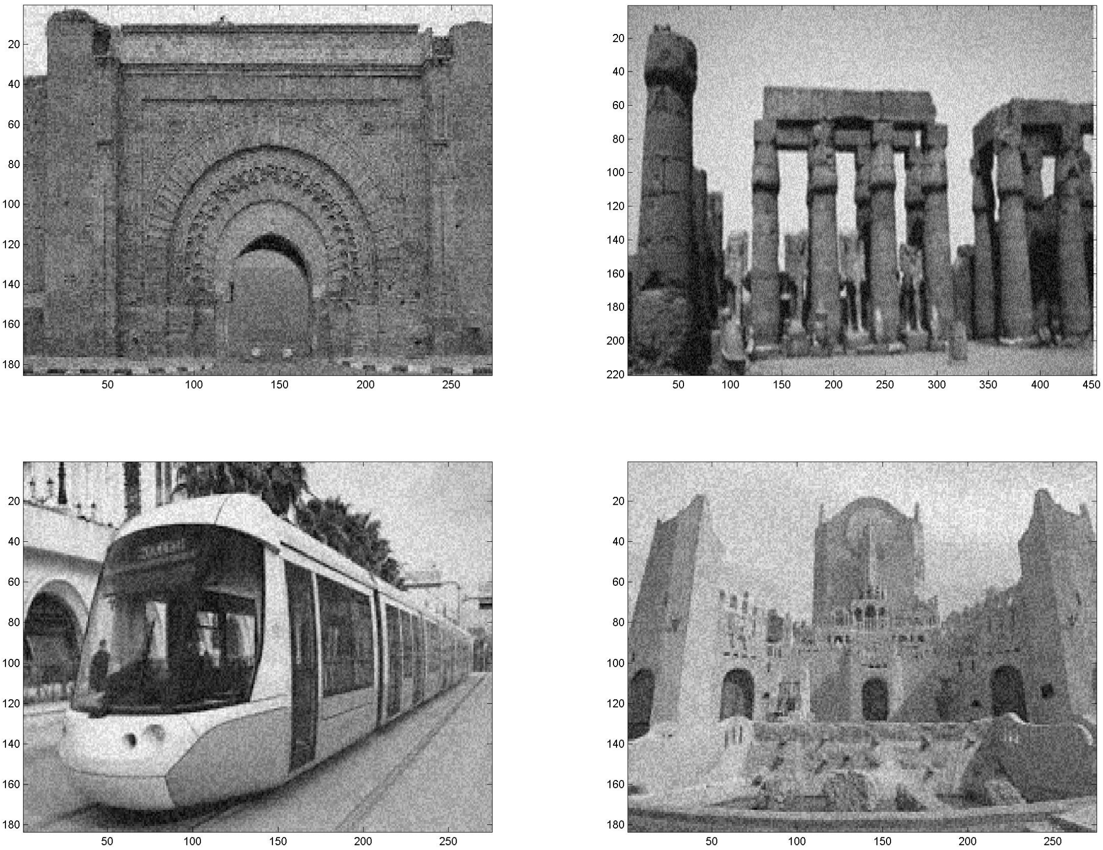
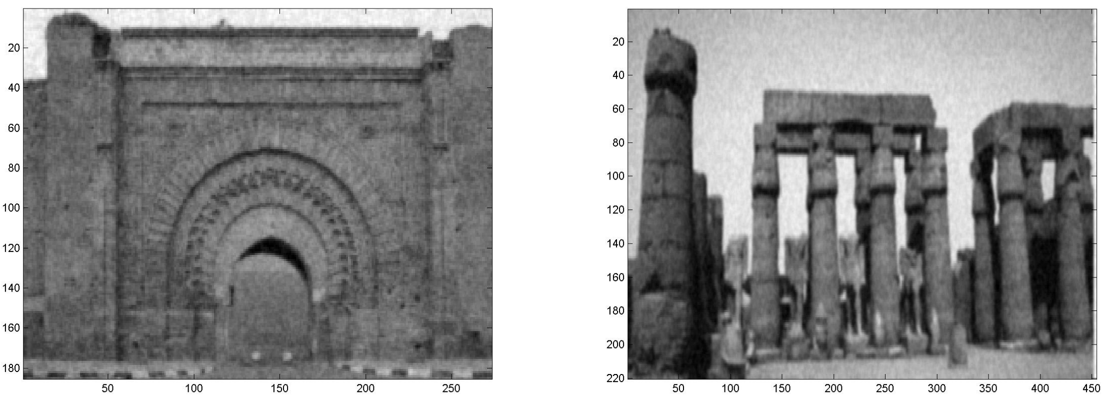


Figure 2.2: Noisy image by Gaussian noise with $\sigma = 0.5$.



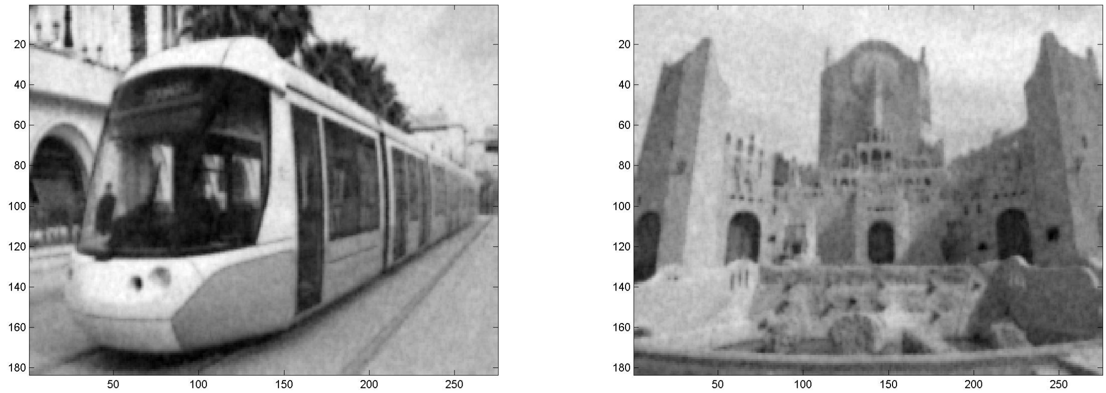
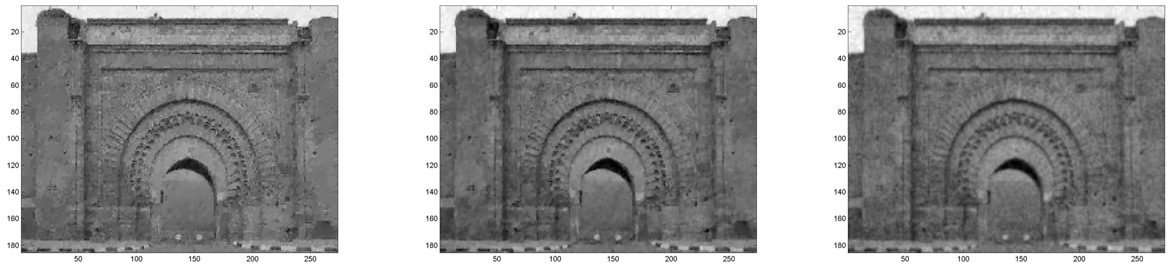
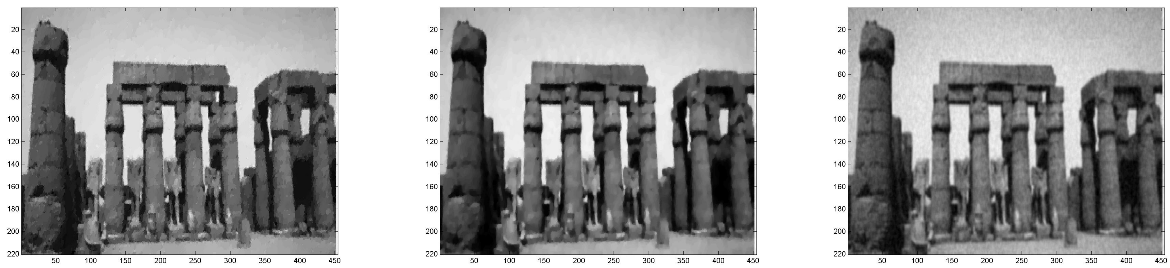


Figure 2.3: Restored image by using our model.



(a) (b) (c)

Figure 2.4: (a) The image restored by using the Perona-Malik model (PSNR=17.9571), (b) the image restored by using the method of the total variation (PSNR=17.9714) and (c) is obtained by our model (PSNR=18.0285).

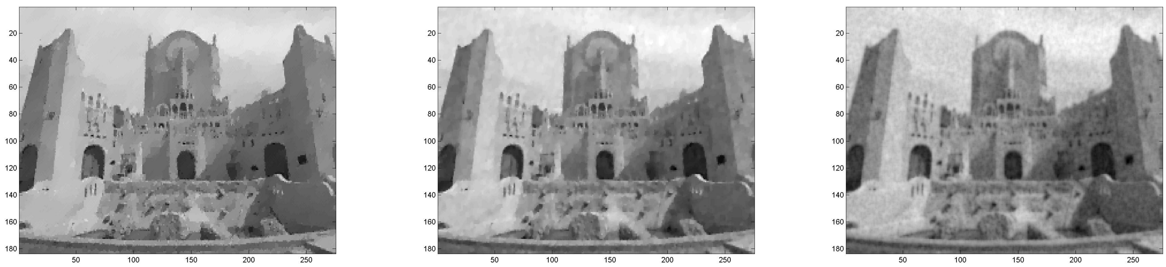


(a) (b) (c)

Figure 2.5: (a) The image restored by using the Perona-Malik model (PSNR=18.2266), (b) the image restored by using the method of the total variation (PSNR=18.2295) and (c) is obtained by our model (PSNR=18,3190).

(a) (b) (c)

Figure 2.6: (a) The image restored by using the Perona-Malik model (PSNR=19,6138), (b) the image restored by using the method of the total variation (PSNR=19.6166) and (c) is obtained by our model (PSNR=19,9261).



(a) (b) (c)

Figure 2.7: (a) The image restored by using the Perona-Malik model (PSNR=19,8125), (b) the image restored by using the method of the total variation (PSNR=19,8469) and (c) is obtained by our model (PSNR=20,6295).

Iteration	$PSNR_{OM}$	$PSNR_{TV}$	$PSNR_{PM}$
1	17.4285	17.3714	17.3357
50	17.7285	17.6500	17.5642
100	17.9071	17.8142	17.74258
150	17.9892	17.9285	17.8928
200	18.0285	17.9714	17.9571

Table 2.1: PSNR comparisons for the three models (for image 01).

Iteration	$PSNR_{OM}$	$PSNR_{TV}$	$PSNR_{PM}$
1	17.4190	17.3523	17.3523
100	17.9952	17.8380	17.6285
200	18.2476	18.1619	17.8666
300	18.2904	18.2095	18.0666
400	18.3142	18.2285	18.1904
500	18.3190	18.2295	18.2266

Table 2.2: PSNR comparisons for the three models (for image 02).

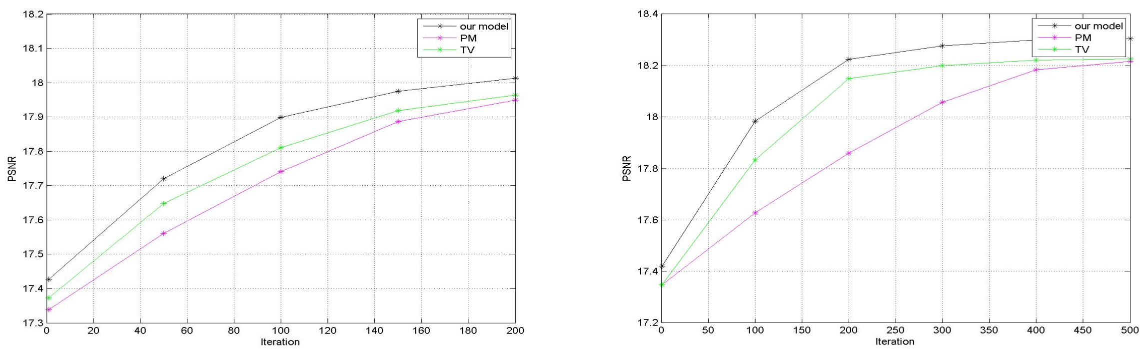


Figure 2.8: The PSNR for different numbers of iterations for image 1,2.

2.3.2 Case where the right hand side is $f(u)$

We consider the following model problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \mathrm{div}(\Psi(|\nabla u|)\nabla u) = f(u) \text{ in } (0,T)\times\Omega, \\ \Phi(|\nabla u|)\nabla u \cdot \vec{\eta} = 0 \text{ in } (0,T)\times\partial\Omega, \end{cases} \tag{2.18}$$

where Ψ is defined in (2.17) and f by $f(s) = \exp(-\frac{s}{\lambda})$. The discretization of the Problem (2.18) is given by the finite difference method (see [15]). Let us $h = 1$ the space step and Δt the time step, one can write the following scheme:

$$\begin{aligned} &u^{k+1}(i,j) \\ &= u^k(i,j) + \Delta t\Big(\mathrm{div}(\Phi(|\nabla u^k(i,j)|)\nabla u^k(i,j))\Big) - f(i,j), \end{aligned}$$

where $u^k(i,j) = u(x_i, y_j, t_k)$, $x_i = ih$, $y_j = jh$, $t_k = k\Delta t$ and $\Delta t = \frac{T}{M}$.
Most algorithm parameters are chosen heuristically for the algorithms to problem their best. We choose the Gaussian noise 50%, $\Delta t = 0.3$, $\varepsilon = 2 \times 10^{-3}$ and the number of iterations is 700. We given firstly restore of noisy images (Figs. 2.11) by our approach and we choose the parameter $\lambda = 0.2$. In the second experiment, we give the difference between our results, the method of the total variation [14, 15, 17, 40] and the Perona-Malik model [7, 28, 45, 51] (see Fig 2.12, 2.14), and we give also zoom in these results (see Fig, 2.13, 2.15). We can notice from the results of PSNR in the figure 2.16 that our model is better then the model of Perona-Malik and the total variation method.

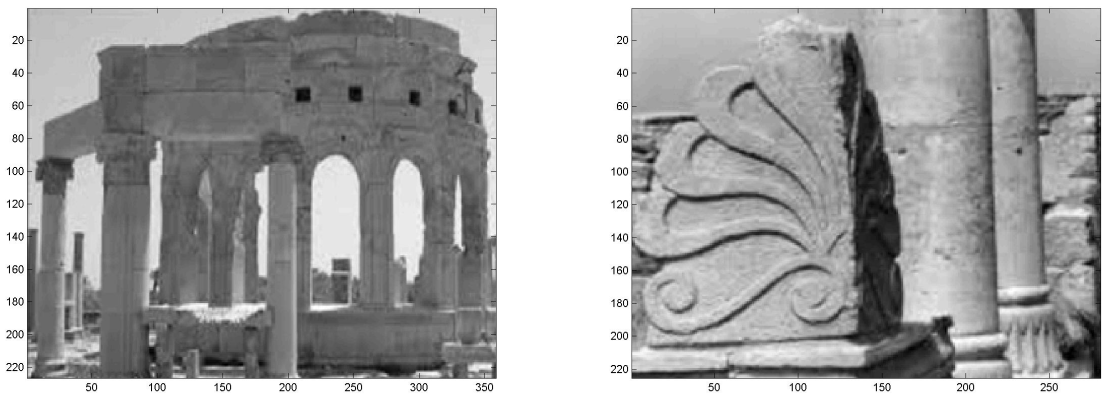


Figure 2.9: Original image.

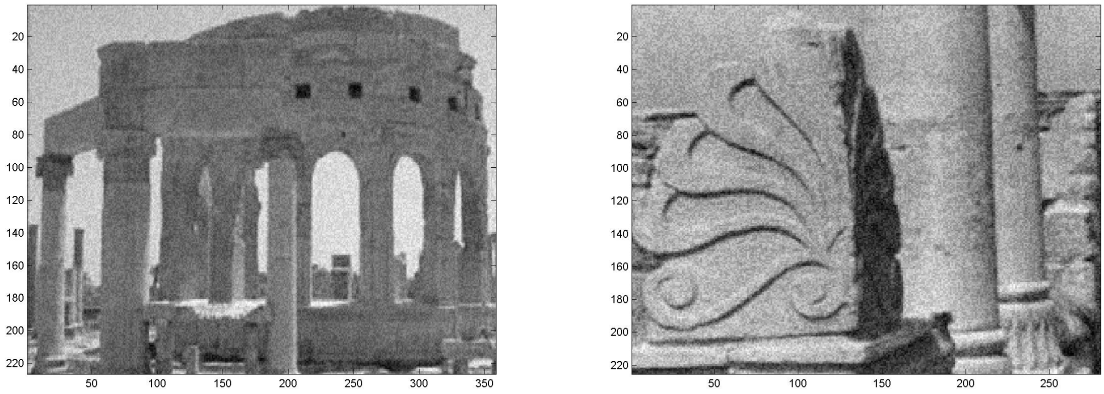


Figure 2.10: Noisy image by Gaussian noise with $\sigma = 0.5$.

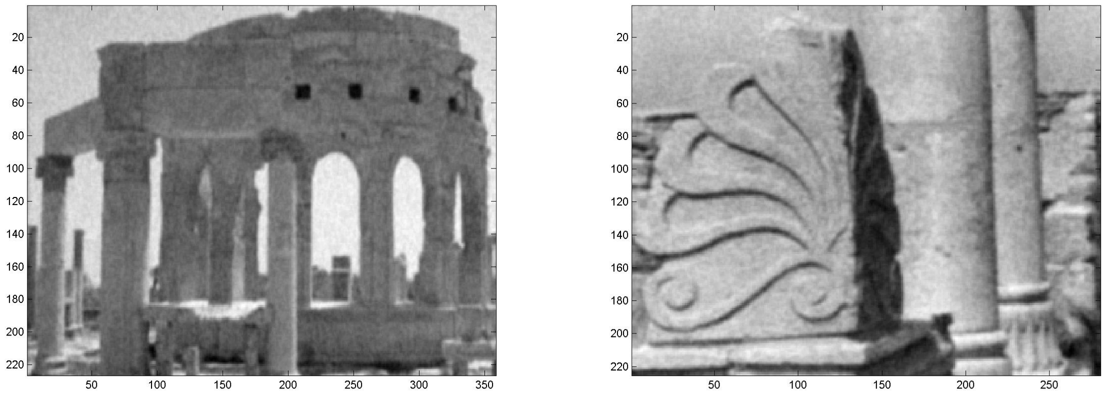
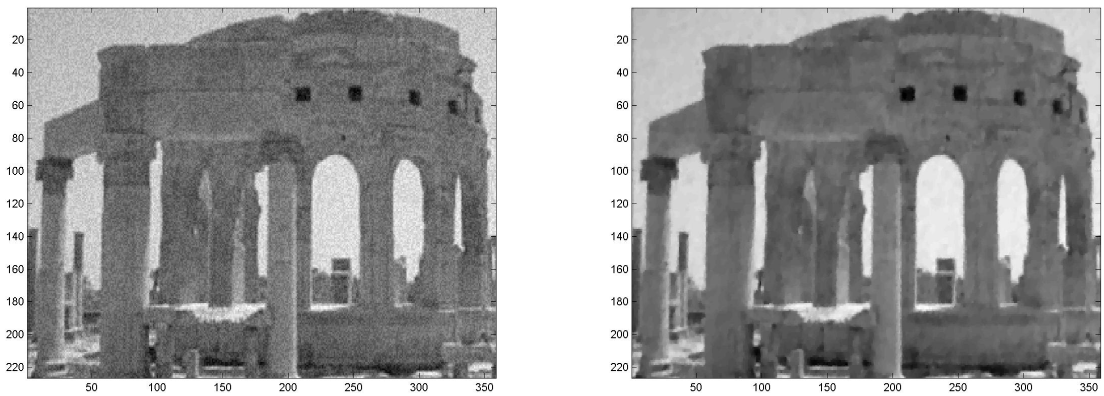
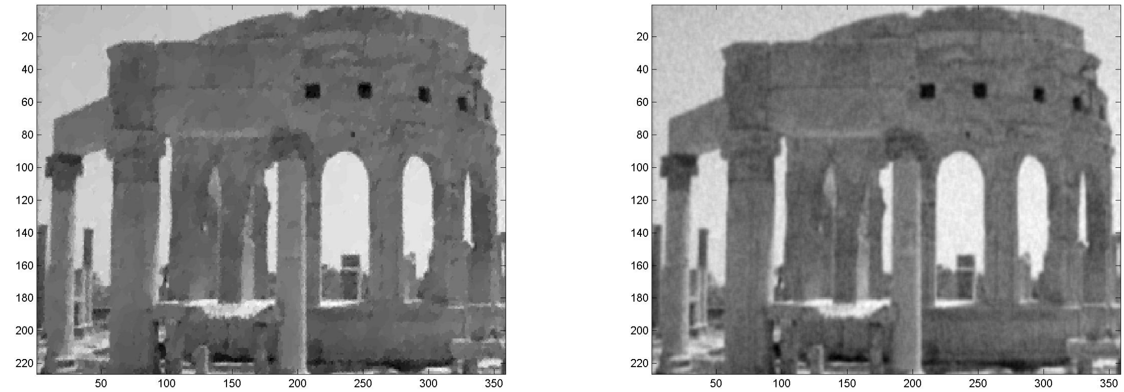


Figure 2.11: Restored image by using our model with $g(k) = \dfrac{1}{1+(\frac{k}{\lambda})^2}$.

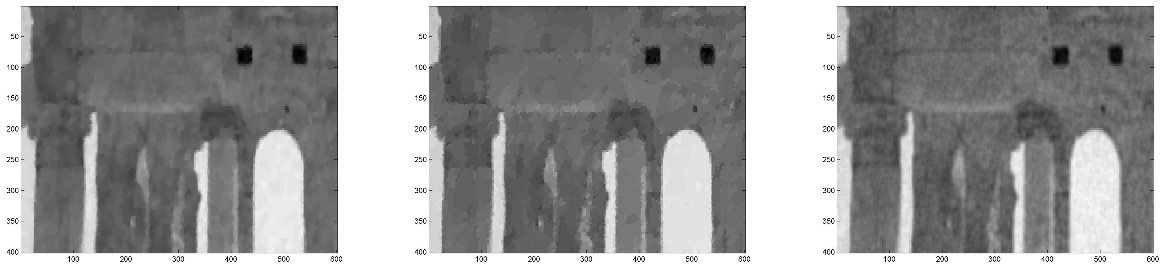


(a) (b)



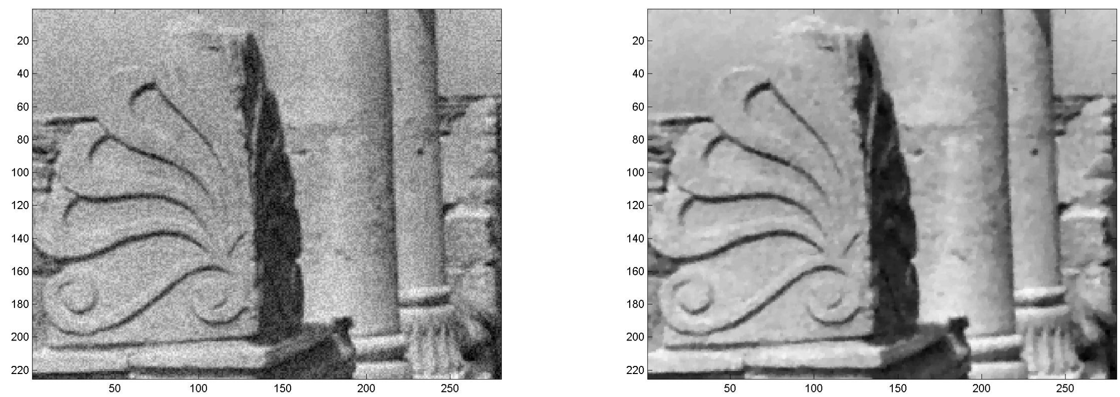
(c) (d)

Figure 2.12: (a) The noisy image, (b) the image restored by using the method of the total variation, (c) the image restored by using the Perona-Malik model and (c) is obtained by our model.

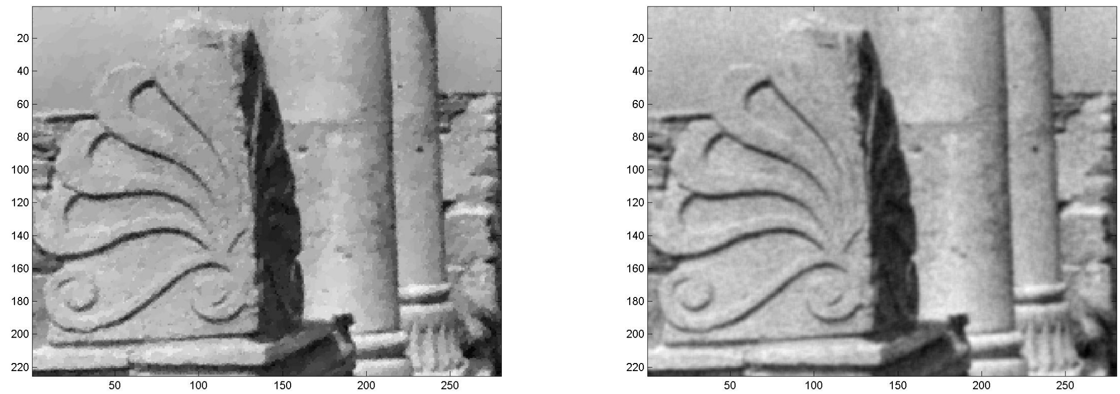


(a) (b) (c)

Figure 2.13: (a) Zoom in of the image restored by TV, (b) zoom in of the image restored by PM and (c) zoom in of the our model.

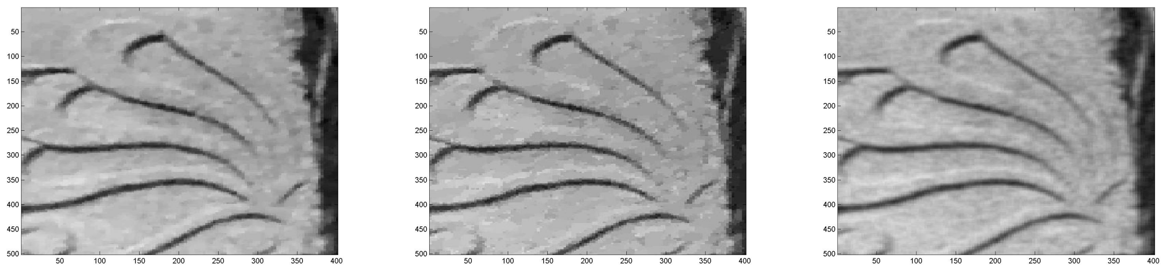


(a) (b)



(c) (d)

Figure 2.14: (a) The noisy image, (b) the image restored by using the method of the total variation, (c) the image restored by using the Perona-Malik model and (d) is obtained by our model.



(a) (b) (c)

Figure 2.15: (a) Zoom in of the image restored by TV, (b) zoom in of the image restored by PM and (c) zoom in of the our model.

Iteration	$PSNR_{OM}$	$PSNR_{TV}$	$PSNR_{PM}$
1	20.8668	20.8529	20.8520
100	21.2799	21.1378	21.1190
200	21.5155	21.3650	21.3607
300	21.6672	21.5327	21.5300
400	21.7134	21.6251	21.6340
500	21.7720	21.7184	21.7105
600	21.7989	21.7502	21.6962

Table 2.3: PSNR comparisons for the three models (for image 01).

Iteration	$PSNR_{OM}$	$PSNR_{TV}$	$PSNR_{PM}$
1	20.8694	20.8591	20.8381
100	21.2846	21.1174	21.0856
200	21.5200	21.3370	21.2816
300	21.6341	21.4951	21.4376
400	21.6781	21.6149	21.5518
500	21.7456	21.6787	21.6234
600	21.7628	21.7153	/

Table 2.4: PSNR comparisons for the three models (for image 02).

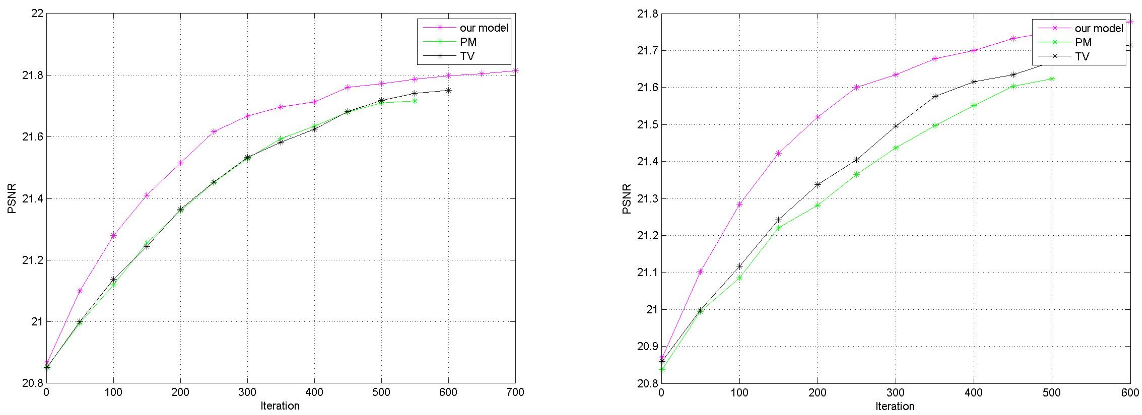


Figure 2.16: The PSNR for different numbers of iterations for image 1,2.

Chapter 3

Existence of solutions of a quasilinear problem with Neumann boundary conditions

This chapter is devoted to the study of the existence of weak solutions of a quasilinear system of partial differential equations which are a combination of the Perona-Malik equation and the heat equation. This result can be seen as a generalization of the results obtained in the previous chapter i.e. we extend the problem (2.1) to the system (3.1).

3.1 Position of the problem

In the present chapter, we study the existence of the solutions for the following problem

$$\begin{cases} -\mathrm{div}\Big(g_1(|\nabla v|)\nabla u\Big) - \dfrac{1}{\lambda_1^2}\Delta u = f_1(x) - uh_1(x) \quad \text{in } \Omega, \\ -\mathrm{div}\Big(g_2(|\nabla u|)\nabla v\Big) - \dfrac{1}{\lambda_2^2}\Delta v = f_2(x) - vh_2(x) \quad \text{in } \Omega, \\ \Big(g_1(|\nabla v|) + \dfrac{1}{\lambda_1^2}\Big)\nabla u \cdot \vec{\eta} = \Big(g_2(|\nabla u|) + \dfrac{1}{\lambda_2^2}\Big)\nabla v \cdot \vec{\eta} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f = (f_1, f_2)$ is function in $(L^2(\Omega))^2$ and $0 < \lambda \leq 1$ suth that $\lambda = (\lambda_1, \lambda_2)$, $h = (h_1, h_2)$ is function in $(L^\infty(\Omega))^2$ satisfy $h_i > 0$, $i = 1, 2$.
The function $g = (g_1, g_2)$ is defined by (2.2) in the previous chapter and satisfying the conditions (2.3). We remark that, if $g_i = 1$ for $i = 1, 2$ we recover the linear diffusion.

In 2014, A. Atlas et al [7] proved the existence and the uniqueness of solutions of the problem

$$\begin{cases} -\operatorname{div}(g(|\nabla u|)\nabla u) - \dfrac{1}{\lambda^p}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f - u \ \text{ in } \Omega, \\ \Big(g(|\nabla u|) + \dfrac{1}{\lambda^2}\Big)\nabla u \cdot \vec{\eta} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3.2}$$

they also studied the asymptotic behavior of the solution as $p \to \infty$.

In this chapter, we extend the results obtained in the chapter 2 to the system (3.1). This type of systems has been extensively studied by several authors. In 2009, A. Moussaoui and B. Khodja [44] studied the existence of nontrivial solutions of semilinear elliptic systems. In 2013, H. Lakehal et al [31] proved the existence of solution for a nonlinear elliptic system through the Schauder's fixed point theorem and an appropriate choice of homotopy. Far from being complete, we refer readers to [24, 32, 37, 43].

The aim of this part is to investigate the existence of solutions to the quasilinear system (3.1) with zero Neumann boundary conditions. This existence is obtained by using the same method used in the previous chapter. Our problem is a combination of the Perona-Malik equation [7, 26, 28, 45] and the heat equation [10].

The chapter is organized as follows. After stating the definition of the weak solution and the main result, we prove the existence of weak solutions of the system (3.1).

3.2 Existence results

In this section, we discuss the notions of weak solutions and the existence result. First, we give the:

Definition 10. *We say that* $(u, v) \in \widetilde{V}$ *is a weak solution for the system* (3.1) *if for any* $(\varphi, \psi) \in \widetilde{V}$ *we have*

$$\begin{aligned} &\int_\Omega (g_1(|\nabla v|) + \frac{1}{\lambda_1^2})\nabla u \nabla \varphi \, \mathrm{d}x + \int_\Omega (g_2(|\nabla u|) + \frac{1}{\lambda_2^2})\nabla v \nabla \psi \, \mathrm{d}x \\ = &\int_\Omega f_1 \varphi \, \mathrm{d}x + \int_\Omega f_2 \psi \, \mathrm{d}x - \int_\Omega u h_1(x) \varphi \, \mathrm{d}x - \int_\Omega v h_2(x) \psi \, \mathrm{d}x. \end{aligned} \tag{3.3}$$

Our main aim now is to prove the Theorem 2.1 for obtain the existence results of the system (3.1).

3.2.1 Proof of the main theorem

Let Z be a finite-dimensional subspace of $\widetilde{V}$ endowed with the $\widetilde{V}$-norm, and Z^* its dual. Define the mappings $H : Z \times [0,1] \longrightarrow Z^*$ by

$$\begin{aligned}\langle H(u,v,t),(\varphi,\psi)\rangle_{\widetilde{V}} &= \int_\Omega \Big(g_1(t|\nabla v|)+\frac{1}{\lambda_1^2}\Big)\nabla u\nabla\varphi\,\mathrm{d}x + \int_\Omega \Big(g_2(t|\nabla u|)+\frac{1}{\lambda_2^2}\Big)\nabla v\nabla\psi\,\mathrm{d}x\\ &-\int_\Omega f_1(x)\varphi\,\mathrm{d}x - \int_\Omega f_2(x)\psi\,\mathrm{d}x + \int_\Omega uh_1(x)\varphi\,\mathrm{d}x + \int_\Omega vh_2(x)\psi\,\mathrm{d}x,\end{aligned}$$

for all $(\varphi,\psi)\in$ Z, H is well defined.

We shall prove the theorem in several steps.

Step 1. A priori bounds.

Let us show now that

$$\Big\{(u,v)\in Z : H(u,v,t)=0, \quad \text{for some } t\in[0,1]\Big\} \subset \bar{B}\Big(\frac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_{\widetilde{V}}\Big).$$

Indeed, if $H(u,v,t)=0$ for same $(u,v,t)\in$ Z $\times[0,1]$, then

$$\begin{aligned}0=\langle H(u,v,t),(u,v)\rangle_{\widetilde{V}} &\geq \frac{1}{\lambda_1^2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{\lambda_2^2}\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_\Omega u^2h_1(x)\,\mathrm{d}x + \int_\Omega v^2h_2(x)\,\mathrm{d}x\\ &-\int_\Omega f_1u\,\mathrm{d}x - \int_\Omega f_2v\,\mathrm{d}x\\ &\geq \min(\frac{1}{\lambda_1^2},h_1)\|u\|_{H^1}^2 + \min(\frac{1}{\lambda_2^2},h_2)\|v\|_{H^1}^2 - \|f_1\|_{L^2}\|u\|_{H^1} - \|f_2\|_{L^2}\|v\|_{H^1}\\ &\geq c_1\|(u,v)\|_{\widetilde{V}}^2 + c_2\|(u,v)\|_{\widetilde{V}}^2 - \|(f_1,f_2)\|_V\|(u,v)\|_{\widetilde{V}} - \|(f_1,f_2)\|_V\|(u,v)\|_{\widetilde{V}}\\ &\geq \min(c_1,c_2)\|(u,v)\|_{\widetilde{V}}^2 - 2\|(f_1,f_2)\|_V\|(u,v)\|_{\widetilde{V}},\end{aligned}$$

which implies that

$$\|(u,v)\|_{\widetilde{V}} \leq \frac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_V.$$

Consequently, for any $R > \dfrac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_V$, we have

$$H(u,v,t)\neq 0 \quad \text{if } (u,v,t)\in\partial B^Z(R)\times[0,1]. \tag{3.4}$$

Step 2. H is bounded.

Now, if $(u, v, t) \in \bar{B}^Z(R) \times [0, 1]$, we have

$$\begin{aligned} |\langle H(u,v,t),(\varphi,\psi)\rangle| &\leq \left(\max\left(1+\frac{1}{\lambda_1^2}, 1+\frac{1}{\lambda_2^2}, 2\|(h_1,h_2)\|_{\widetilde{U}}\right)\|(u,v)\|_{\widetilde{V}} + 2\|(f_1,f_2)\|_V \right)\|(\varphi,\psi)\|_{\widetilde{V}} \\ &\leq \Bigg(\underbrace{\max\left(1+\frac{1}{\lambda_1^2}, 1+\frac{1}{\lambda_2^2}, 2\|(h_1,h_2)\|_{\widetilde{U}}\right)R + 2\|(f_1,f_2)\|_V}_{\widetilde{R}} \Bigg)\|(\varphi,\psi)\|_{\widetilde{V}} \\ &\leq \widetilde{R}\|(\varphi,\psi)\|_{\widetilde{V}}, \end{aligned}$$

for all $(\varphi, \psi) \in \widetilde{\mathrm{V}}$, and hence

$$H\left(\bar{B}^{\mathrm{Z}}(R) \times [0,1]\right) \subset \bar{B}^{\mathrm{Z}^*}(\widetilde{R}). \tag{3.5}$$

Step 3. H is continuous.

Let $(u_n, v_n, t_n) \in \bar{B}^{\mathrm{Z}}(R) \times [0, 1]$ converge to (u, v, t) in $Z \times [0, 1]$, i.e in $\widetilde{V} \times [0, 1]$. Since $(H(u_n, v_n, t_n))$ is bounded because of (3.5), to prove that

$$H(u_n, v_n, t_n) \to H(u, v, t),$$

it is sufficient to show that $H(u, v, t)$ is the unique cluster point of $(H(u_n, v_n, t_n))$. Let $M \in \mathrm{Z}^*$ be such a cluster point, still we denote by $(t_n), (u_n)$ and (v_n) a subsequence of $(t_n), (u_n)$ and (v_n) respectively such that

$$H(u_n, v_n, t_n) \to M \text{ in } \mathrm{Z}^*.$$

Since $(u_n, v_n) \to (u, v)$ in $\widetilde{V}$, it follows that $(u_n, v_n) \to (u, v)$ in V, and hence, going if necessary to a subsequence, we may assume that $(u_n, v_n) \to (u, v)$ a.e. in Ω. On the other hand, $(\partial_i u_n, \partial_i v_n) \to (\partial_i u, \partial_i v)$ in V, therefore $(\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v)$ a.e. in Ω. This implies that

$$g_1(t_n|\nabla v_n|) \to g_1(t|\nabla v|) \quad \text{a.e. in } \Omega,$$

$$g_2(t_n|\nabla u_n|) \to g_2(t|\nabla u|) \quad \text{a.e. in } \Omega,$$

and hence, for any $(\varphi, \psi) \in Z$,

$$g_1(t_n|\nabla v_n|)\nabla\varphi \to g_1(t|\nabla v|)\nabla\varphi \text{ in } L^2(\Omega),$$

$$g_2(t_n|\nabla u_n|)\nabla\psi \to g_2(t|\nabla u|)\nabla\psi \text{ in } L^2(\Omega).$$

We conclude that

$$\begin{aligned}&\langle H(u_n,v_n,t_n),(\varphi,\psi)\rangle_{\widetilde{V}}\\&=\int_\Omega u_n h_1(x)\varphi\,\mathrm{d}x+\int_\Omega v_n h_2(x)\psi\,\mathrm{d}x+\int_\Omega\Big(g_1(t_n|\nabla v_n|)+\frac{1}{\lambda_1^2}\Big)\nabla u_n\nabla\varphi\,\mathrm{d}x\\&+\int_\Omega\Big(g_2(t_n|\nabla u_n|)+\frac{1}{\lambda_2^2}\Big)\nabla v_n\nabla\psi\,\mathrm{d}x\\&\to\int_\Omega u h_1(x)\varphi\,\mathrm{d}x+\int_\Omega v h_2(x)\psi\,\mathrm{d}x+\int_\Omega\Big(g_1(t|\nabla v|)+\frac{1}{\lambda_1^2}\Big)\nabla u\nabla\varphi\,\mathrm{d}x\\&+\int_\Omega\Big(g_2(t|\nabla u|)+\frac{1}{\lambda_2^2}\Big)\nabla v\nabla\psi\,\mathrm{d}x=\langle H(u,v,t),(\varphi,\psi)\rangle_{\widetilde{V}}.\end{aligned}$$

Thus $M=H(u,v,t)$. All those properties allow us to apply the homotopy invariance property to

$$\deg_B\Big(H(\cdot,\cdot,1),B(R),0\Big)=\deg_B\Big(H(\cdot,\cdot,0),B(R),0\Big).\tag{3.6}$$

But $H(u,v,0)=0$ is equivalant to the problem

$$\begin{aligned}&(1+\frac{1}{\lambda_1^2})\int_\Omega\nabla u\nabla\varphi\,\mathrm{d}x+(1+\frac{1}{\lambda_2^2})\int_\Omega\nabla v\nabla\psi\,\mathrm{d}x\\&=\int_\Omega f_1(x)\varphi\,\mathrm{d}x+\int_\Omega f_2(x)\psi\,\mathrm{d}x-\int_\Omega u h_1(x)\varphi\,\mathrm{d}x-\int_\Omega v h_2(x)\psi\,\mathrm{d}x,\end{aligned}$$

for all $(\varphi,\psi)\in \mathrm{Z}$, whose solution is unique because of the boundedness of the set of its possible solutions. Consequently,

$$\deg_B\Big(H(\cdot,\cdot,0),B(R),0\Big)=\pm1,$$

and, from (3.6) and the existence property of the degree, there exists $(u,v)\in B^{\mathrm{Z}}(R)$ which satisfies

$$\begin{cases}\displaystyle\int_\Omega\Big(g_1(|\nabla v|)+\frac{1}{\lambda_1^2}\Big)\nabla u\nabla\varphi\,\mathrm{d}x+\int_\Omega\Big(g_2(|\nabla u|)+\frac{1}{\lambda_2^2}\Big)\nabla v\nabla\psi\,\mathrm{d}x\\ \displaystyle=\int_\Omega f_1(x)\varphi\,\mathrm{d}x+\int_\Omega f_2(x)\psi\,\mathrm{d}x-\int_\Omega u h_1(x)\varphi\,\mathrm{d}x-\int_\Omega v h_2(x)\psi\,\mathrm{d}x,\\ \\ \displaystyle\|(u,v)\|_{\widetilde{V}}\le\frac{2}{\min(c_1,c_2)}\|(f_1,f_2)\|_V,\end{cases}\tag{3.7}$$

for all $(\varphi,\psi)\in Z$.

Step 4. Passing to the limit.

Consider the function $a_i : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a_i(\xi_i) = \Big(g_i(\xi_i) + \frac{1}{\lambda_i^2}\Big)\xi_i \quad \text{for any } \xi_i \in \mathbb{R}^N \text{ and } i = 1, 2.$$

To prove this step we use the two Lemmas 2.1 and 2.2 mentioned in the previous chapter. Now, it is well known that one can write $\widetilde{V} = \overline{\bigcup_{n\geq 1} Z_n}$ where $Z_n \subset Z_{n+1}(n \geq 1)$ and Z_n has dimension n. Consequently, given any $(\varphi, \psi) \in \widetilde{V}$, there exists a sequence (φ_n, ψ_n) with $(\varphi_n, \psi_n) \in Z_n$ which converges to (φ, ψ). On the other hand, by (3.7) applied to $Z = Z_n$, there exists, for each $n \geq 1$, some $(u_n, v_n) \in Z_n$ such that

$$\begin{cases} \displaystyle\int_\Omega a_1(\nabla u_n)\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega a_2(\nabla v_n)\nabla\widetilde{\psi}\,\mathrm{d}x \\ \displaystyle= \int_\Omega f_1(x)\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega f_2(x)\widetilde{\psi}\,\mathrm{d}x - \int_\Omega u_n h_1(x)\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega v_n h_2(x)\widetilde{\psi}\,\mathrm{d}x, \\ \\ \displaystyle\|(u_n, v_n)\|_{\widetilde{V}} \leq \frac{2}{\min(c_1, c_2)}\|(f_1, f_2)\|_V, \end{cases}$$

for all $(\widetilde{\varphi}, \widetilde{\psi}) \in Z_n$. In particular, taking $(\widetilde{\varphi}, \widetilde{\psi}) = (\varphi_n, \psi_n)$ introduced above,

$$\begin{cases} \displaystyle\int_\Omega a_1(\nabla u_n)\nabla\varphi_n\,\mathrm{d}x + \int_\Omega a_2(\nabla v_n)\nabla\psi_n\,\mathrm{d}x \\ \displaystyle= \int_\Omega f_1(x)\varphi_n\,\mathrm{d}x + \int_\Omega f_2(x)\psi_n\,\mathrm{d}x - \int_\Omega u_n h_1(x)\varphi_n\,\mathrm{d}x - \int_\Omega v_n h_2(x)\psi_n\,\mathrm{d}x, \\ \\ \displaystyle\|(u_n, v_n)\|_{\widetilde{V}} \leq \frac{2}{\min(c_1, c_2)}\|(f_1, f_2)\|_V, \end{cases} \tag{3.8}$$

for all $n \geq 1$. The estimate in (3.8) implies that, going if necessary to subsequences, we can assume that there exists $(u, v) \in \widetilde{V}$ such that $u_n \to u$ weakly in $\widetilde{V}$, $u_n \to u$ strongly in V and $u_n \to u$ a.e. in Ω. As $\big(a_1(\nabla u_n)\big)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, then there exists $\zeta_1 \in L^2(\Omega)$ such that

$$a_1(\nabla u_n) \to \zeta_1 \text{ weakly in } L^2(\Omega).$$

Similarly, we obtain

$$a_2(\nabla v_n) \to \zeta_2 \text{ weakly in } L^2(\Omega),$$

and $(\nabla\varphi_n, \nabla\psi_n) \to (\nabla\varphi, \nabla\psi)$ strongly in V, one can let $n \to \infty$ in (3.8) to obtain

$$\begin{aligned} &\int_\Omega \zeta_1\nabla\varphi\,\mathrm{d}x + \int_\Omega \zeta_2\nabla\psi\,\mathrm{d}x \\ &= \int_\Omega f_1(x)\varphi\,\mathrm{d}x + \int_\Omega f_2(x)\psi\,\mathrm{d}x - \int_\Omega u h_1(x)\varphi\,\mathrm{d}x - \int_\Omega v h_2(x)\psi\,\mathrm{d}x. \end{aligned} \tag{3.9}$$

It remains to show that
$$\int_\Omega \zeta_1 \nabla\varphi \,\mathrm{d}x = \int_\Omega a_1(\nabla u)\nabla\varphi \,\mathrm{d}x, \tag{3.10}$$
and
$$\int_\Omega \zeta_2 \nabla\psi \,\mathrm{d}x = \int_\Omega a_2(\nabla v)\nabla\psi \,\mathrm{d}x. \tag{3.11}$$
To prove the two equalities, we use the trick of Minty[23]; we begin by studying the limit of
$$\int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x,$$
and
$$\int_\Omega a_2(\nabla v_n)\nabla v_n \,\mathrm{d}x.$$
Indeed
$$\int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x = \int_\Omega f_1(x)u_n \,\mathrm{d}x - \int_\Omega u_n^2 h_1(x)\,\mathrm{d}x \to \int_\Omega f_1(x)u \,\mathrm{d}x - \int_\Omega u^2 h_1(x)\,\mathrm{d}x,$$
and
$$\int_\Omega a_2(\nabla v_n)\nabla v_n \,\mathrm{d}x = \int_\Omega f_2(x)v_n \,\mathrm{d}x - \int_\Omega v_n^2 h_2(x)\,\mathrm{d}x \to \int_\Omega f_2(x)v \,\mathrm{d}x - \int_\Omega v^2 h_2(x)\,\mathrm{d}x,$$
because $(u_n, v_n) \to (u, v)$ weakly in $\widetilde{V}$. But we know that (u, v) satisfies (3.9), and hence
$$\int_\Omega f_1(x)u \,\mathrm{d}x - \int_\Omega u^2 h_1(x)\,\mathrm{d}x = \int_\Omega \zeta_1 \nabla u \,\mathrm{d}x,$$
and
$$\int_\Omega f_2(x)v \,\mathrm{d}x - \int_\Omega v^2 h_2(x)\,\mathrm{d}x = \int_\Omega \zeta_2 \nabla v \,\mathrm{d}x.$$
Therefore
$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega a_1(\nabla u_n)\nabla u_n \,\mathrm{d}x &= \int_\Omega f_1(x)u \,\mathrm{d}x - \int_\Omega u^2 h_1(x)\,\mathrm{d}x \\ &= \int_\Omega \zeta_1 \nabla u \,\mathrm{d}x,\end{aligned} \tag{3.12}$$
and
$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega a_2(\nabla v_n)\nabla v_n \,\mathrm{d}x &= \int_\Omega f_2(x)v \,\mathrm{d}x - \int_\Omega v^2 h_2(x)\,\mathrm{d}x \\ &= \int_\Omega \zeta_2 \nabla v \,\mathrm{d}x.\end{aligned} \tag{3.13}$$
Let $(\varphi, \psi) \in \widetilde{V}$, it exists $(\varphi_n, \psi_n)_{n\in\mathbb{N}}$ such that $(\varphi_n, \psi_n) \in Z_n$ for all $n \in \mathbb{N}$ and $(\varphi_n, \psi_n) \to (\varphi, \psi)$ in $\widetilde{V}$ when $n \to +\infty$. Thanks to Lemma 2.1, we will pass to the limit in the two terms
$$\int_\Omega a_1(\nabla u_n)\nabla\varphi_n \,\mathrm{d}x,$$

and

$$\int_\Omega a_2(\nabla v_n)\nabla\psi_n\,\mathrm{d}x.$$

Indeed, for the first equation

$$0 \le \int_\Omega (a_1(\nabla u_n) - a_1(\nabla\varphi_n))(\nabla u_n - \nabla\varphi_n)\,\mathrm{d}x =$$

$$\int_\Omega a_1(\nabla u_n)\nabla u_n\,\mathrm{d}x - \int_\Omega a_1(\nabla u_n)\nabla\varphi_n\,\mathrm{d}x - \int_\Omega a_1(\nabla\varphi_n)\nabla u_n\,\mathrm{d}x + \int_\Omega a_1(\nabla\varphi_n)\nabla\varphi_n\,\mathrm{d}x$$

$$= \quad F_{1,n} - F_{2,n} - F_{3,n} + F_{4,n},$$

where

$$F_{1,n} \int_\Omega a_1(\nabla u_n)\nabla u_n\,\mathrm{d}x, \quad F_{2,n} = \int_\Omega a_1(\nabla u_n)\nabla\varphi_n\,\mathrm{d}x,$$

and

$$F_{3,n} = \int_\Omega a_1(\nabla\varphi_n)\nabla u_n\,\mathrm{d}x, \quad F_{4,n} = \int_\Omega a_1(\nabla\varphi_n)\nabla\varphi_n\,\mathrm{d}x,$$

we saw in (3.12) that $F_{1,n} \to \int_\Omega \zeta_1\nabla u\,\mathrm{d}x$ when $n \to \infty$.
We have

$$\lim_{n\to+\infty} F_{2,n} = \int_\Omega \zeta_1\nabla\varphi\,\mathrm{d}x,$$

by product of strong convergence in $L^2(\Omega)$ and weak convergence in $L^2(\Omega)$.
Similarly

$$\lim_{n\to+\infty} F_{3,n} = \int_\Omega a_1(\nabla\varphi)\nabla u\,\mathrm{d}x.$$

Finally, we also have

$$\lim_{n\to+\infty} F_{4,n} = \int_\Omega a_1(\nabla\varphi)\nabla\varphi\,\mathrm{d}x,$$

when $n \to +\infty$. and this last term is the simplest because we have the product of a strong convergence in $L^2(\Omega)$ and a strong convergence in $L^2(\Omega)$. The passage to the limit therefore gives:

$$\int_\Omega (\zeta_1 - a_1(\nabla\varphi))(\nabla u - \nabla\varphi)\,\mathrm{d}x \ge 0 \text{ for all } \varphi \in H^1(\Omega).$$

Similarly, we obtain

$$\int_\Omega (\zeta_2 - a_2(\nabla\psi))(\nabla u - \nabla\psi)\,\mathrm{d}x \ge 0 \text{ for all } \psi \in H^1(\Omega).$$

We now choose judicious test functions φ and ψ. We take

$$\varphi = u + \frac{1}{n}\varphi^*, \text{ with } \varphi^* \in H^1(\Omega) \text{ and } n \in \mathbb{N}^*,$$

and

$$\psi = v + \frac{1}{n}\psi^*, \text{ with } \psi^* \in H^1(\Omega) \text{ and } n \in \mathbb{N}^*.$$

We thus obtain:

$$-\frac{1}{n}\int_\Omega \big(\zeta_1 - a_1(\nabla u + \frac{1}{n}\nabla\varphi^*)\big)\nabla\varphi^* \,\mathrm{d}x \geq 0,$$

and

$$-\frac{1}{n}\int_\Omega \big(\zeta_2 - a_2(\nabla v + \frac{1}{n}\nabla\psi^*)\big)\nabla\psi^* \,\mathrm{d}x \geq 0,$$

then

$$\int_\Omega \big(\zeta_1 - a_1(\nabla u + \frac{1}{n}\nabla\varphi^*)\big)\nabla\varphi^* \,\mathrm{d}x \leq 0,$$

and

$$\int_\Omega \big(\zeta_2 - a_2(\nabla v + \frac{1}{n}\nabla\psi^*)\big)\nabla\psi^* \,\mathrm{d}x \leq 0.$$

But

$$u + \frac{1}{n}\varphi^* \to u \text{ in } H^1(\Omega),$$
$$v + \frac{1}{n}\psi^* \to v \text{ in } H^1(\Omega),$$

thanks to Lemma 2.2, we obtain

$$a_1\big(\nabla u + \frac{1}{n}\nabla\varphi^*\big) \to a_1(\nabla u) \text{ in } L^2(\Omega),$$

and

$$a_2\big(\nabla v + \frac{1}{n}\nabla\psi^*\big) \to a_2(\nabla v) \text{ in } L^2(\Omega).$$

Passing to the limit when $n \to +\infty$, we then obtain

$$\int_\Omega (\zeta_1 - a_1(\nabla u))\nabla\varphi^* \,\mathrm{d}x \leq 0, \quad \forall \varphi^* \in H^1(\Omega),$$

and

$$\int_\Omega (\zeta_2 - a_2(\nabla v))\nabla\psi^* \,\mathrm{d}x \leq 0, \quad \forall \psi^* \in H^1(\Omega).$$

By linearity (can change φ^* into $-\varphi^*$ and ψ^* into $-\psi^*$), we have

$$\int_\Omega (\zeta_1 - a_1(\nabla u))\nabla\varphi^* \,\mathrm{d}x = 0, \quad \forall \varphi^* \in H^1(\Omega),$$

and

$$\int_\Omega (\zeta_2 - a_2(\nabla v))\nabla\psi^* \,\mathrm{d}x = 0, \quad \forall \psi^* \in H^1(\Omega).$$

We deduce that

$$\int_\Omega \zeta_1 \nabla\varphi^* \,\mathrm{d}x = \int_\Omega a_1(\nabla u)\nabla\varphi^* \,\mathrm{d}x, \quad \forall\varphi^* \in H^1(\Omega),$$
$$\int_\Omega \zeta_2 \nabla\psi^* \,\mathrm{d}x = \int_\Omega a_2(\nabla v)\nabla\psi^* \,\mathrm{d}x, \quad \forall\psi^* \in H^1(\Omega).$$

Hence we have showed that (u, v) is a solution of (3.1).

Chapter 4

Study of a System of Convection-Diffusion-Reaction

In this chapter, we are interested in the study of the existence of weak solutions of boundary value problem for the nonlinear elliptic system of convection-diffusion-reaction with the right-hand side terms having a growth conditions. We use the Leray-Schauder degree theory under suitable assumptions on the nonlinearities.

4.1 Position of the problem

Here, we aim to presenting the results of existence of solution for a nonlinear elliptic systems of partial differential equations, in a bounded domain of $\mathbb{R}^N$, with zero Dirichlet boundary conditions. These results are obtained by using Leray-Schauder's topological degree and some tools of functional analysis. It is well known that the corresponding scalar case was solved in [25]. In this chapter, we establish the existence of weak solutions for the problem

$$\begin{cases} -\operatorname{div}(a_1(x, v(x))\nabla u) - \operatorname{div}(G_1(x)\varphi_1(v)) = f(x, u, v) \quad \text{in } \Omega, \\ -\operatorname{div}(a_2(x, u(x))\nabla v) - \operatorname{div}(G_2(x)\varphi_2(u)) = g(x, u, v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$ and a_1, a_2 are continuous functions satisfying the condition below:

$$\begin{cases} \alpha_i, \beta_i > 0, \\ \alpha_i \leq a_i(x, s) \leq \beta_i \quad \forall s \in \mathbb{R} \quad \text{a.e } x \in \Omega \text{ for } i = 1, 2. \end{cases} \tag{4.2}$$

In the case where $a_1 = a_2 = 1$, $\varphi_1 = \varphi_2 = 0$, we have the existence of the solution if λ is a simple eigenvalue of $(-\Delta)$, see [31]. And in [44] studied the case where λ is not an eigenvalue

i.e $\lambda \notin \sigma(-\Delta)$. The case of resonance was treated by Lakhal and Khodja (see [32]).
For the rest of this work, we suppose that

$$\begin{cases} G_i \in C^1(\overline{\Omega}, \mathbb{R}^N), \\ \\ \text{div} G_i = 0 \text{ for } i = 1, 2, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \varphi_i \in C(\mathbb{R}, \mathbb{R}), \\ \\ |\varphi_i(s)| \leq C_i |s| \text{ for } i = 1, 2, \end{cases} \tag{4.4}$$

where C_i are real positive constants.
We assume that $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the carathéodory conditions, and verifying also the growth restriction defined below

$$\begin{cases} |\ f(x, s, t)| \leq C_1'(1 + |s| + |t|), \\ \\ |\ g(x, s, t)| \leq C_2'(1 + |s| + |t|), \end{cases} \tag{4.5}$$

where C_1', C_2' are real positive constants.

$$\begin{cases} \lim\limits_{s,|t| \to \infty} f(x, s, t) = \zeta_1^+, \qquad \lim\limits_{-s,|t| \to \infty} f(x, s, t) = \zeta_1^-, \\ \\ \zeta_1^-, \zeta_1^+ \in L^2(\Omega), \qquad \zeta_1^- \leq f(x, t, s) \leq \zeta_1^+, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \lim\limits_{s,|t| \to \infty} g(x, s, t) = \zeta_2^+, \qquad \lim\limits_{-s,|t| \to \infty} g(x, s, t) = \zeta_2^-, \\ \\ \zeta_2^-, \zeta_2^+ \in L^2(\Omega), \qquad \zeta_2^- \leq g(x, t, s) \leq \zeta_2^+. \end{cases} \tag{4.7}$$

Theorem 4.1. *Under the assumptions* (4.2), (4.3), (4.4), (4.5), (4.6) *and* (4.7), *there exists a solution to the problem* (4.1).

4.2 Preliminaries

In this section, we give some preliminary knowledge and some important proposition which will be used to prove our theorem.
First, we give a definition of weak solution.

Definition 11. *We say that* $(u, v) \in U$ *is a weak solution for the system* (4.1) *if for any* $(\varphi, \psi) \in \mathrm{U}$ *we have*

$$\begin{cases} \int_\Omega a_1(v)\nabla u\nabla\varphi\,\mathrm{d}x + \int_\Omega G_1\varphi_1(v)\nabla\varphi\,\mathrm{d}x = \int_\Omega f(u,v)\varphi\,\mathrm{d}x, \\ \int_\Omega a_2(u)\nabla v\nabla\psi\,\mathrm{d}x + \int_\Omega G_2\varphi_2(u)\nabla\psi\,\mathrm{d}x = \int_\Omega g(u,v)\psi\,\mathrm{d}x. \end{cases} \tag{4.8}$$

We write the problem in the form

$$\begin{cases} (u,v) \in U, \\ \int_\Omega a_1(v)\nabla u\nabla\varphi\,\mathrm{d}x + \int_\Omega a_2(u)\nabla v\nabla\psi\,\mathrm{d}x = \langle L(u,v),(\varphi,\psi)\rangle_{U^*,U}, \forall(\varphi,\psi) \in U, \end{cases}$$

where $L(u,v)$ is, for $(u,v) \in V$, the element of U^* defined by

$$\begin{aligned} \langle L(u,v),(\varphi,\psi)\rangle_{U^*,U} &= -\int_\Omega G_1\varphi_1(v)\nabla\varphi\,\mathrm{d}x - \int_\Omega G_2\varphi_2(u)\nabla\psi\,\mathrm{d}x, \\ &+ \int_\Omega f(u,v)\varphi\,\mathrm{d}x + \int_\Omega g(u,v)\psi\,\mathrm{d}x. \end{aligned}$$

From (4.3), (4.4) and (4.5), it is clear that the application $L(u,v)$ is continuous from V into U^*. For $S \in U^*$, the linear problem

$$\begin{cases} (w_1,w_2) \in U, \\ \int_\Omega a_1(v)\nabla w_1\nabla\varphi\,\mathrm{d}x + \int_\Omega a_2(u)\nabla w_2\nabla\psi\,\mathrm{d}x = \langle S,(\varphi,\psi)\rangle_{U^*,U}, \end{cases} \tag{4.9}$$

has a unique solution $(w_1,w_2) \in (H_0^1(\Omega))^2$. We note the operator $B_{u,v}$ who at S in U^* associates (w_1,w_2) solution of (4.9). The operator $B_{u,v}$ is linear continuous from U^* into U and $U \hookrightarrow V$ is compact. Then, we deduced that the operator $B_{u,v}$ is compact from U^* into V.
The problem (4.1) is equivalent to solving the fixed point problem $(u,v) = B_{u,v}(L(u,v))$. Therefore, we will show, by topological degree, that the following problem has a solution

$$\begin{cases} (u,v) \in V, \\ (u,v) = B_{u,v}(L(u,v)). \end{cases}$$

For $t \in [0,1]$, we put $H(t,u,v) = B_{u,v}(tL(u,v)) \in V$. The mapping H is defined from $[0,1]\times V$ into V. For $R > 0$, let us put

$$B(R) = \{(u,v) \in V \text{such that } \|(u,v)\|_V \le R\}.$$

Let us show now that

$$\Big\{H(t,u,v), t \in [0.1], (u,v) \in \bar{B}(R)\Big\} \quad \text{is relatively compact in } V.$$

Let $R > 0$, we assume that $\|(u,v)\|_V \le R$. We have $L(u,v) \in U^*$, and

$$\begin{aligned}\langle L(u,v),(\varphi,\psi)\rangle_{U^*,U} &= -\int_\Omega G_1\varphi_1(v)\nabla\varphi\,\mathrm{d}x - \int_\Omega G_2\varphi_2(u)\nabla\psi\,\mathrm{d}x,\\ &+\int_\Omega f(u,v)\varphi\,\mathrm{d}x + \int_\Omega g(u,v)\psi\,\mathrm{d}x.\end{aligned}$$

We want to estimate $\langle L(u,v),(\varphi,\psi)\rangle_{U^*,U}$.

$$\begin{aligned}\langle L(u,v),(\varphi,\psi)\rangle_{U^*,U} \le\ & \||G_1|\|_\infty\|\varphi_1(v)\|_{L^2(\Omega)}\|\varphi\|_{H_0^1(\Omega)} + \||G_2|\|_\infty\|\varphi_2(u)\|_{L^2(\Omega)}\|\psi\|_{H_0^1(\Omega)}\\ +\ & \|f(u,v)\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \|g(u,v)\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}\\ \le\ & \||G_1|\|_\infty C_1\|v\|_{L^2(\Omega)}\|\varphi\|_{H_0^1(\Omega)} + \||G_2|\|_\infty C_2\|u\|_{L^2(\Omega)}\|\psi\|_{H_0^1(\Omega)}\\ +\ & C_\Omega C_1'(1+\|u\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)})\|\varphi\|_{H_0^1(\Omega)} + C_\Omega C_2'(1+\|u\|_{L^2(\Omega)}\\ +\ & \|v\|_{L^2(\Omega)})\|\psi\|_{H_0^1(\Omega)}\\ \le\ & \||G_1|\|_\infty C_1\|(u,v)\|_V\|(\varphi,\psi)\|_U + \||G_2|\|_\infty C_2\|(u,v)\|_V\|(\varphi,\psi)\|_U\\ +\ & C_\Omega C_1'(1+\|(u,v)\|_V+\|(u,v)\|_V)\|(\varphi,\psi)\|_U + C_\Omega C_2'(1+\|(u,v)\|_V\\ +\ & \|(u,v)\|_V)\|(\varphi,\psi)\|_U\\ \le\ & \begin{pmatrix} \||G_1|\|_\infty C_1R + \||G_2|\|_\infty C_2R\\ +C_1'C_\Omega(1+2R)+C_2'C_\Omega(1+2R)\end{pmatrix}\|(\varphi,\psi)\|_U,\end{aligned}$$

where C_Ω depends only on Ω (and is given by the Poincar inequality). So $\forall t \in [0,1]$

$$t\|L(u,v)\|_{U^*} \le \||G_1|\|_\infty C_1R + \||G_2|\|_\infty C_2R + C_1'C_\Omega(1+2R) + C_2'C_\Omega(1+2R).$$

Let $H(t,u,v) = B_{u,v}(tL(u,v)) = (w_1,w_2)$ and show that there $\bar{R}$ depending only on R, G_i, C_Ω, C_i, C_i', α_i, for $i = 1,2$ such that

$$\|H(t,u,v)\|_U \le \bar{R}.$$

By definition, (w_1,w_2) is a solution of

$$\begin{cases} (w_1,w_2) \in U,\\ \displaystyle\int_\Omega a_1(v)\nabla w_1\nabla\varphi\,\mathrm{d}x + \int_\Omega a_2(u)\nabla w_2\nabla\psi\,\mathrm{d}x\\ = \langle tL(u,v),(\varphi,\psi)\rangle_{U^*,U} \quad \forall(\varphi,\psi)\in U.\end{cases} \tag{4.10}$$

Taking $\varphi = w_1$, $\psi = w_2$, in (4.10) we obtain

$$\min(\alpha_1, \alpha_2)\|(w_1, w_2)\|_U^2 \le \|tL(u,v)\|_{U^*}\|(w_1, w_2)\|_U \le \widetilde{R}\|(w_1, w_2)\|_U,$$

with $\widetilde{R} = \||G_1|\|_\infty C_1 R + \||G_2|\|_\infty C_2 R + C_1' C_\Omega(1+2R) + C_2' C_\Omega(1+2R)$.
We have

$$\|H(t,u,v)\|_U = \|(w_1, w_2)\|_U \le \frac{\widetilde{R}}{\min(\alpha_1, \alpha_2)} = \bar{R}.$$

By Rellich theorem, we deduce that the set

$$\{H(t,u,v), t \in [0,1], (u,v) \in \bar{B}(R)\} \text{ is relatively compact in } V. \tag{4.11}$$

We now show that H is continous.

Proposition 4.2. *The mapping H is continuous from $[0.1] \times \bar{B}(R)$ into $\bar{B}(R)$.*

Proof. Let (t_n, u_n, v_n) converge to (t,u,v) in $[0,1] \times V$.
We want to show that

$$H(t_n, u_n, v_n) \to H(t,u,v) \text{ in } V.$$

Let

$$(w_{n,1}, w_{n,2}) = H(t_n, u_n, v_n),$$

and

$$(w_1, w_2) = H(t,u,v).$$

To show that

$$(w_{n,1}, w_{n,2}) \to (w_1, w_2) \text{ in } V,$$

seeking to pass to the limit on the following equation:

$$\begin{cases} (w_{n,1}, w_{n,2}) \in U, \\ \displaystyle\int_\Omega a_1(v_n)\nabla w_{n,1}\nabla\varphi\,\mathrm{d}x + \int_\Omega a_2(u_n)\nabla w_{n,2}\nabla\psi\,\mathrm{d}x \\ \displaystyle= -t_n\int_\Omega G_1\varphi_1(v_n)\nabla\varphi\,\mathrm{d}x - t_n\int_\Omega G_2\varphi_2(u_n)\nabla\psi\,\mathrm{d}x \\ \displaystyle+t_n\int_\Omega f(u_n,v_n)\varphi\,\mathrm{d}x + t_n\int_\Omega g(u_n,v_n)\psi\,\mathrm{d}x. \end{cases} \tag{4.12}$$

We know that $(w_{n,1}, w_{n,2})$ is bounded in U, because (u_n, v_n) is bounded in V (this is shown in the previous step: $\|(u_n, v_n)\|_V \le R$ then $\|(w_{n,1}, w_{n,2})\|_U \le \bar{R}$).
The sequence $(w_{n,1}, w_{n,2})_{n\in\mathbb{N}}$ is bounded in U, therefore

$$\begin{aligned} &(w_{n,1}, w_{n,2}) \to (\widetilde{w}_1, \widetilde{w}_2) \text{ in } U \text{ weak and } (w_{n,1}, w_{n,2}) \to (\widetilde{w}_1, \widetilde{w}_2) \text{ in } V, \\ &(u_n, v_n) \to (u,v) \text{ a.e. and } \exists\, H, G \in L^2(\Omega); |u_n| \le H \text{ a.e. and } |v_n| \le G \text{ a.e..} \end{aligned} \tag{4.13}$$

Let $(\varphi, \psi) \in U$; as $a_1(v_n) \to a_1(v)$a.e. therefore

$$a_1(v_n)\nabla\varphi \to a_1(v)\nabla\varphi \text{ a.e.,}$$

and

$$|a_1(v_n)\nabla\varphi| \leq \beta_1\nabla\varphi,$$

we have

$$a_1(v_n)\nabla\varphi \to a_1(v)\nabla\varphi \text{ in } L^2(\Omega).$$

But $\nabla w_{n,1} \to \nabla\widetilde{w}_1$ in $(L^2(\Omega))^N$ weak. We then have

$$\int_\Omega a_1(v_n)\nabla w_{n,1}\nabla\varphi \,\mathrm{d}x \to \int_\Omega a_1(v)\nabla\widetilde{w}_1\nabla\varphi \,\mathrm{d}x \text{ when } n \to +\infty.$$

Similarly we have

$$\int_\Omega a_2(u_n)\nabla w_{n,2}\nabla\psi \,\mathrm{d}x \to \int_\Omega a_2(u)\nabla\widetilde{w}_2\nabla\psi \,\mathrm{d}x \text{ when } n \to +\infty.$$

Then we notice that

$$\varphi_1(v_n) \to \varphi_1(v) \text{ a.e. },$$

and

$$|\varphi_1(v_n)| \leq C_1|v_n| \leq C_1 G,$$

thanks to Lebesgue's dominated convergence theorem, we deduce that

$$\varphi_1(v_n) \to \varphi_1(v) \text{ in } L^2(\Omega),$$

and

$$\int_\Omega G_1\varphi_1(v_n)\nabla\varphi \,\mathrm{d}x \to \int_\Omega G_1\varphi_1(v)\nabla\varphi \,\mathrm{d}x \text{ when } n \to +\infty.$$

Similarly we have

$$\int_\Omega G_2\varphi_1(u_n)\nabla\psi \,\mathrm{d}x \to \int_\Omega G_2\varphi_2(u)\nabla\psi \,\mathrm{d}x \text{ when } n \to +\infty.$$

Finally for the last term,

$$f(u_n, v_n) \to f(u, v) \text{ a.e..}$$

By dominated convergence (from (4.5) and (4.13)) we have

$$f(u_n, v_n) \to f(u, v) \text{ in } L^2(\Omega),$$

and consequently

$$\int_\Omega f(u_n, v_n)\varphi \,\mathrm{d}x \to \int_\Omega f(u, v)\varphi \,\mathrm{d}x \text{ when } n \to +\infty.$$

Similarly we have

$$\int_\Omega g(u_n, v_n)\psi \,\mathrm{d}x \to \int_\Omega g(u, v)\psi \,\mathrm{d}x \text{ when } n \to +\infty.$$

Passing to the limit in (4.12), we obtain

$$\begin{aligned}\int_\Omega a_1(v)\nabla\widetilde{w}_1\nabla\varphi \,\mathrm{d}x + \int_\Omega a_2(u)\nabla\widetilde{w}_2\nabla\psi \,\mathrm{d}x &= -t\int_\Omega G_1\varphi_1(v)\nabla\varphi \,\mathrm{d}x - t\int_\Omega G_2\varphi_2(u)\nabla\psi \,\mathrm{d}x,\\ &+t\int_\Omega f(u, v)\varphi \,\mathrm{d}x + t\int_\Omega g(u, v)\psi \,\mathrm{d}x,\end{aligned}$$

and therefore $(\widetilde{w}_1, \widetilde{w}_2) = H(t, u, v) = (w_1, w_2)$.
As $(w_{n,1}, w_{n,2}) \to (w_1, w_2)$ in V where

$$(w_{n,1}, w_{n,2}) = H(t_n, u_n, v_n) \text{ and } (w_1, w_2) = H(t, u, v), \text{ then } H \text{ is continuous.} \tag{4.14}$$

□

From (4.11) and (4.14), we deduce that $H(t, u, v) : [0, 1] \times V \to V$ is compact.

It remains to show the step of a priori bounds.

4.3 A priori bounds for solutions of the problem 4.1

This is the main section in this chapter, where we show that a priori bounds of the solution of the system (4.1). Let $t \in [0, 1]$, and $(u, v) = H(t, u, v) = tB_{u,v}(L(u, v))$, that is to say

$$\begin{cases}(u, v) \in U,\\ \int_\Omega a_1(v)\nabla u\nabla\varphi \,\mathrm{d}x + \int_\Omega a_2(u)\nabla v\nabla\psi \,\mathrm{d}x = -t\int_\Omega G_1\varphi_1(v)\nabla\varphi \,\mathrm{d}x\\ -t\int_\Omega G_2\varphi_2(u)\nabla\psi \,\mathrm{d}x + t\int_\Omega f(u, v)\varphi \,\mathrm{d}x + t\int_\Omega g(u, v)\psi \,\mathrm{d}x, \quad \forall(\varphi, \psi) \in U.\end{cases} \tag{4.15}$$

For $s \in \mathbb{R}$, we put $\Phi_i(s) = \int_0^s \varphi_i(\xi)\,\mathrm{d}\xi$ (Φ_i is a primitive of φ_i, for $i = 1, 2$). As $(u, v) \in (H_0^1(\Omega))^2$, It is not difficult to show that $\Phi_1(u) \in W_0^{1,1}(\Omega)$ (resp $\Phi_2(v) \in W_0^{1,1}(\Omega)$) and

$$\begin{cases}\int_\Omega G_1\varphi_1(u)\nabla u \,\mathrm{d}x = \int_\Omega G_1\nabla\Phi_1(u)\,\mathrm{d}x,\\ \int_\Omega G_2\varphi_2(v)\nabla v \,\mathrm{d}x = \int_\Omega G_2\nabla\Phi_2(v)\,\mathrm{d}x.\end{cases}$$

For (4.3), we have

$$\begin{cases} \displaystyle\int_\Omega G_1\varphi_1(u)\nabla u\,\mathrm{d}x = \int_\Omega G_1\nabla\Phi_1(u)\,\mathrm{d}x = -\int_\Omega \mathrm{div}G_1\Phi_1(u)\,\mathrm{d}x = 0, \\ \displaystyle\int_\Omega G_2\varphi_2(v)\nabla v\,\mathrm{d}x = \int_\Omega G_2\nabla\Phi_2(v)\,\mathrm{d}x = -\int_\Omega \mathrm{div}G_2\Phi_2(v)\,\mathrm{d}x = 0. \end{cases}$$

Taking $\varphi = u, \ \psi = v$ in (4.15) . By assumptions (4.2), (4.3), (4.4), (4.5) and (4.6), we have

$$\underbrace{\min(\alpha_1,\alpha_2)}_{\alpha}\|(u,v)\|_U^2 \le \int_\Omega |f(u,v)u|\,\mathrm{d}x + \int_\Omega |g(u,v)v|\,\mathrm{d}x.$$

Lemma 4.1. *There exists $R > 0$ such that for all $t \in [0,1]$ and all $(u,v) \in V$,*

$$(u,v) - H(t,u,v) = 0 \Longrightarrow \|(u,v)\|_V < R.$$

Proof. To prove this lemma we assume by contradiction, that for all $R > 0$ there exists $(t,u,v) \in [0,1]\times V$ such that

$$(u,v) - H(t,u,v) = 0 \quad \text{and} \quad \|(u,v)\|_V > R.$$

In other words, we can find a sequence $(t_n,u_n,v_n) \in [0,1]\times V$ such that

$$(u_n,v_n) - H(t_n,u_n,v_n) = 0 \quad \text{and} \quad \|(u_n,v_n)\|_V > n. \tag{4.16}$$

Taking

$$(\widetilde{u}_n,\widetilde{v}_n) = \Big(\frac{u_n}{\|(u_n,v_n)\|_V}, \frac{v_n}{\|(u_n,v_n)\|_V}\Big),$$

we have

$$\|(\widetilde{u}_n,\widetilde{v}_n)\|_V = 1,$$

and

$$\alpha\|(\widetilde{u}_n,\widetilde{v}_n)\|_U^2 \le \int_\Omega \Big|\frac{f(u_n,v_n)}{\|(u_n,v_n)\|_V}\widetilde{u}_n\Big|\,\mathrm{d}x + \int_\Omega \Big|\frac{g(u_n,v_n)}{\|(u_n,v_n)\|_V}\widetilde{v}_n\Big|\,\mathrm{d}x.$$

For (4.5), we have

$$\begin{aligned}
\alpha\|(\widetilde{u}_n,\widetilde{v}_n)\|_U^2 &\leq \int_\Omega \frac{C_1'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V}|\widetilde{u}_n|\,\mathrm{d}x + \int_\Omega \frac{C_2'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V}|\widetilde{v}_n|\,\mathrm{d}x \\
&\leq \int_\Omega \frac{C_1'|\widetilde{u}_n|}{\|(u_n,v_n)\|_V}\,\mathrm{d}x + C_1'\int_\Omega |\widetilde{u}_n||\widetilde{u}_n|\,\mathrm{d}x + C_1'\int_\Omega |\widetilde{v}_n||\widetilde{u}_n|\,\mathrm{d}x + \int_\Omega \frac{C_2'|\widetilde{v}_n|}{\|(u_n,v_n)\|_V}\,\mathrm{d}x \\
&+C_2'\int_\Omega |\widetilde{u}_n||\widetilde{v}_n|\,\mathrm{d}x + C_2'\int_\Omega |\widetilde{v}_n||\widetilde{v}_n|\,\mathrm{d}x \\
&\leq \frac{1}{\|(u_n,v_n)\|_V}\int_\Omega (C_1'|\widetilde{u}_n| + C_2'|\widetilde{v}_n|)\,\mathrm{d}x + 2\max(C_1',C_2')\int_\Omega |(\widetilde{u}_n,\widetilde{v}_n)|^2\,\mathrm{d}x \\
&\leq \frac{1}{\|(u_n,v_n)\|_V}(C_1'\|\widetilde{u}_n\|_{L^2(\Omega)} + C_2'\|\widetilde{v}_n\|_{L^2(\Omega)}) + 2\max(C_1',C_2')\|(\widetilde{u}_n,\widetilde{v}_n)\|_V^2 \\
&\leq \frac{1}{\|(u_n,v_n)\|_V}(C_1'\|(\widetilde{u}_n,\widetilde{v}_n)\|_V + C_2'\|(\widetilde{v}_n,\widetilde{v}_n)\|_V) + 2\max(C_1',C_2') \\
&\leq \frac{\max(C_1',C_2')}{\|(u_n,v_n)\|_V} + 2\max(C_1',C_2'),
\end{aligned}$$

Moreover, by (4.16) we have

$$\frac{\max(C_1',C_2')}{\|(u_n,v_n)\|_V} \leq \frac{\max(C_1',C_2')}{n} \leq \max(C_1',C_2'),$$

Then

$$\|(\widetilde{u}_n,\widetilde{v}_n)\|_U^2 \leq k,\ \text{such that}\ k = \frac{3\max(C_1',C_2')}{\alpha},$$

that is, $\|(\widetilde{u}_n,\widetilde{v}_n)\|_U^2$ is bounded in U.
Since $(\widetilde{u}_n,\widetilde{v}_n) \in (H_0^1(\Omega))^2$ and the embedding $(H_0^1(\Omega) \hookrightarrow L^2(\Omega))$ is compact, we can extract a subsequence $(\widetilde{u}_n,\widetilde{v}_n)$, still denoted by $(\widetilde{u}_n,\widetilde{v}_n)$, which converges in V. Let $(\widetilde{u},\widetilde{v})$ be the limit of $(\widetilde{u}_n,\widetilde{v}_n)$ in V. We have threrfore $\|(\widetilde{u},\widetilde{v})\|_V = 1$ (which give $\widetilde{u} \neq 0$, and $\widetilde{v} \neq 0$). We also have

$$\begin{gathered}
(\widetilde{u}_n,\widetilde{v}_n) \to (\widetilde{u},\widetilde{v}) \quad \text{a.e. }, \\
|\widetilde{u}_n| \leq H \text{ and } |\widetilde{v}_n| \leq G, \quad \text{with } H,G \in L^2(\Omega).
\end{gathered} \tag{4.17}$$

Finally, using the Poincare inequality, there is C_Ω , depending only on Ω such that

$$\frac{\alpha}{C_\Omega^2} = \frac{\alpha}{C_\Omega^2}\|(\widetilde{u}_n,\widetilde{v}_n)\|_V^2 \leq \alpha\|(\widetilde{u}_n,\widetilde{v}_n)\|_U^2 \leq \int_\Omega \frac{|f(u_n,v_n)|}{\|(u_n,v_n)\|_V}|\widetilde{u}_n|\,\mathrm{d}x + \int_\Omega \frac{|g(u_n,v_n)|}{\|(u_n,v_n)\|_V}|\widetilde{v}_n|\,\mathrm{d}x.$$

Let us put

$$Z_n = \int_\Omega \frac{|f(u_n,v_n)||\widetilde{u}_n|}{\|(u_n,v_n)\|_V}\,\mathrm{d}x + \int_\Omega \frac{|g(u_n,v_n)||\widetilde{v}_n|}{\|(u_n,v_n)\|_V}\,\mathrm{d}x,$$

and now we show that $Z_n \to 0$ when $n \to \infty$, which is impossible since Z_n is reduced by constant α/C_Ω^2 which is strictly positive.
Show that

$$\begin{cases} \dfrac{f(u_n,v_n)|\widetilde{u}_n|}{\|(u_n,v_n)\|_V} \to 0 \quad \text{a.e. in } \Omega, \\ \\ \dfrac{g(u_n,v_n)|\widetilde{v}_n|}{\|(u_n,v_n)\|_V} \to 0 \quad \text{a.e. in } \Omega, \end{cases}$$

with domination (in $L^1(\Omega)$), we have by the dominated convergence theorem that $Z_n \to 0$ when $n \to \infty$.
We first show dominance. From (4.5) and (4.17), we have

$$\begin{cases} \dfrac{|f(u_n,v_n)|}{\|(u_n,v_n)\|_V} \le \dfrac{C_1'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V} \le C_1' + C_1'|\widetilde{u}_n| + C_1'|\widetilde{v}_n| \le C_1' + C_1'H + C_1'G, \\ \\ \dfrac{|g(u_n,v_n)|}{\|(u_n,v_n)\|_V} \le \dfrac{C_2'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V} \le C_2' + C_2'|\widetilde{u}_n| + C_2'|\widetilde{v}_n| \le C_2' + C_2'H + C_2'G, \end{cases}$$

Then

$$\begin{cases} \left| \dfrac{f(u_n,v_n)}{\|(u_n,v_n)\|_V} \widetilde{u}_n \right| \le (C_1' + C_1'H + C_1'G)H \in L^1(\Omega), \\ \\ \left| \dfrac{g(u_n)}{\|(u_n,v_n)\|_V} \widetilde{v}_n \right| \le (C_2' + C_2'H + C_2'G)G \in L^1(\Omega). \end{cases}$$

We now show the convergence a.e.. We have

$$(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}, \widetilde{v}) \text{ a.e..}$$

Let $x \in A$. From the hypothesis (4.6) and (4.7) it follows that

Case I

if $\widetilde{u}(x) > 0$ and $\widetilde{v}(x) > 0$, therefore $\widetilde{u}(x)\cdot\widetilde{v}(x) > 0$ (resp if $\widetilde{u}(x) < 0$ and $\widetilde{v}(x) < 0$), $\widetilde{u}_n(x) \to \widetilde{u}(x)$ and $\widetilde{v}_n(x) \to \widetilde{v}(x)$ but

$$\lim_{n\to\pm\infty} \|(u_n,v_n)\|_V = \pm\infty,$$

therefore $u_n(x) = \widetilde{u}_n(x)\|(u_n,v_n)\|_V \to \pm\infty$ and $v_n(x) = \widetilde{v}_n(x)\|(u_n,v_n)\|_V \to \pm\infty$.

$$\begin{cases} \dfrac{f(u_n(x),v_n(x))}{\|(u_n,v_n)\|_V}\widetilde{u}_n(x) = \dfrac{f(u_n(x),v_n(x))v_n(x)}{v_n(x)\|(u_n,v_n)\|_V}\widetilde{u}_n(x) = \dfrac{f(u_n(x),v_n(x))}{v_n(x)}\widetilde{u}_n(x)\cdot\widetilde{v}_n(x) \underset{n\to\infty}{\to} 0, \\ \\ \dfrac{g(u_n(x),v_n(x))}{\|(u_n,v_n)\|_V}\widetilde{v}_n(x) = \dfrac{g(u_n(x),v_n(x))u_n(x)}{u_n(x)\|(u_n,v_n)\|_V}\widetilde{v}_n(x) = \dfrac{g(u_n(x),v_n(x))}{u_n(x)}\widetilde{u}_n(x)\cdot\widetilde{v}_n(x) \underset{n\to\infty}{\to} 0\,. \end{cases}$$

Case II

if $\widetilde{u}(x) > 0$ and $\widetilde{v}(x) < 0$, therefore $\widetilde{u}(x)\cdot\widetilde{v}(x) < 0$ (resp if $\widetilde{u}(x) < 0$ and $\widetilde{v}(x) > 0$), $\widetilde{u}_n(x) \to \widetilde{u}(x)$ and $\widetilde{v}_n(x) \to \widetilde{v}(x)$ but

$$\lim_{n\to\pm\infty} \|(u_n, v_n)\|_V = \pm\infty,$$

therefore $u_n(x) = \widetilde{u}_n(x)\|(u_n, v_n)\|_V \to \pm\infty$ and $v_n(x) = \widetilde{v}_n(x)\|(u_n, v_n)\|_V \to \pm\infty$.

$$\begin{cases} \dfrac{f(u_n(x), v_n(x))}{\|(u_n, v_n)\|_V}\widetilde{u}_n(x) = \dfrac{f(u_n(x), v_n(x))\widetilde{v}_n(x)}{\widetilde{v}_n(x)\|(u_n, v_n)\|_V}\widetilde{u}_n(x) = \dfrac{f(u_n(x), v_n(x))}{v_n(x)}\widetilde{u}_n(x)\cdot\widetilde{v}_n(x) \underset{n\to\infty}{\to} 0, \\ \dfrac{g(u_n(x), v_n(x))}{\|(u_n, v_n)\|_V}\widetilde{v}_n(x) = \dfrac{g(u_n(x), v_n(x))\widetilde{u}_n(x)}{\widetilde{u}_n(x)\|(u_n, v_n)\|_V}\widetilde{v}_n(x) = \dfrac{g(u_n(x), v_n(x))}{u_n(x)}\widetilde{u}_n(x)\cdot\widetilde{v}_n(x) \underset{n\to\infty}{\to} 0. \end{cases}$$

Case III

if $\widetilde{u}(x) = \widetilde{v}(x) = 0$

$$\begin{aligned} \Big|\frac{f(u_n(x), v_n(x))}{\|(u_n, v_n)\|_V}\widetilde{u}_n(x)\Big| &\le \frac{C_1'(1 + |u_n(x)| + |v_n(x)|)}{\|(u_n, v_n)\|_V}|\widetilde{u}_n(x)| \\ &\le (C_1' + C_1'|\widetilde{u}_n(x)| + C_1'|\widetilde{v}_n(x)|)|\widetilde{u}_n(x)| \\ &\to 0 \quad \text{because } \widetilde{u}(x) = 0, \\ \Big|\frac{g(u_n(x), v_n(x))}{\|(u_n, v_n)\|_V}\widetilde{v}_n(x)\Big| &\le \frac{C_2'(1 + |u_n(x)| + |v_n(x)|)}{\|(u_n, v_n)\|_V}|\widetilde{v}_n(x)| \\ &\le (C_2' + C_2'|\widetilde{u}_n(x)| + C_2'|\widetilde{v}_n(x)|)|\widetilde{v}_n(x)| \\ &\to 0 \quad \text{because } \widetilde{v}(x) = 0. \end{aligned}$$

In summary we have

$$\begin{cases} \dfrac{f(u_n, v_n)}{\|(u_n, v_n)\|_V}\widetilde{u}_n \to 0 \quad \text{a.e.,} \\ \\ \dfrac{g(u_n, v_n)}{\|(u_n, v_n)\|_V}\widetilde{v}_n \to 0 \quad \text{a.e..} \end{cases}$$

It was also shown that $\lim_{n\to+\infty} Z_n = 0$, this is contradiction with $Z_n \ge \alpha/C_\Omega^2$ for all $n \in \mathbb{N}^*$. We have shown that there exists $R > 0$ suth that

$$(u, v) = H(t, u, v) \Rightarrow \|(u, v)\|_V < R.$$

□

Now, we give the proof of our main result.

Proof. of Theorem (4.1). We have no solution to the equation $(u, v) - H(t, u, v) = 0$ on the edge of the ball $B(R)$ such that $B(R) = \{(u, v) \in V, \|(u, v)\|_V < R\}$.
Then, we can define the degree $\deg(H(t, \cdot, \cdot), B(R), 0)$. This degree does not depend on t, then we have

$$\begin{aligned} \deg(Id - H(t, \cdot, \cdot), B(R), 0) = & \deg(Id - H(0, \cdot, \cdot), B(R), 0) \\ & = \deg(Id, B(R), 0) = 1. \end{aligned}$$

We deduce the existence of $(u, v) \in B(R)$ such that $(u, v) - H(1, u, v) = 0$, that is to say

$$(u, v) = B_{u,v}(L(u, v)).$$

Therefore (u, v) is a solution of (4.1) (and Theorem(4.1) is shown). □

Conclusion

In this thesis, by using the compactness method and some functional analysis tools, we have studied the existence of solutions for some classes of differential equations with Neumann boundary conditions and systems of differential equations with Neumann boundary conditions or Dirichlet boundary conditions.

The first class of equations is a new model for image denoising while the second is the system of convection-diffusion-reaction. The results of existence of this system can be looked upon as an extension of the work in [23].

The study was crowned with the publication of four articles:

1. S. Lecheheb, H. Lakhal, M. Maouni, K. Slimani, *Study of a system of Diffusion-Convection-Reaction*, International Journal of Partial Differential Equations and Applications. **4**(2), 32-37, 2016.

2. S. Lecheheb, M. Maouni, H. Lakhal, *Existence of the solution of a quasilinear equation and its application to image denoising*, International Journal of Computer Science, Computer Science, Communication and Information Technology (CSCIT) 7 (2), 1-6, 2019.

3. S. Lecheheb, M. Maouni, H. Lakhal, *Image restoration using nonlinear elliptic Equaion*, International Journal of Computer Science, Computer Science, Communication and Information Technology (CSCIT) 6 (2), 32-37, 2019.

4. S. Lecheheb, M. Maouni, H. Lakhal, *Existence of solutions of a quasilinear problem with Neumann boundary conditions*, Boletim da Sociedade Paranaense de Matemática, 2019. In press.

Therefore, these studies can be extend to more general boundary value problems involving fractional derivatives models and find numerical methods suitable to our results. Do the same study in the spaces of Orlicz and making an application of these models in image processing.

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