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جامعة 20 أوت 1955 -سكيكدة
كلية العـلوم
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Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques

Option : ***Mathématiques***

Approches Numériques par des Volumes Finis de l'équation de Bürgers Stochastique.

Présentée par :

DIB Nidal

Devant la commission du jury composé de:

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20 Août 1955 University -Skikda
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جامعة 20 أوت 1955 -سكيكدة
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Thesis

Doctorate (LMD) in Mathematics

Option: ***Mathematics***

Numerical Approximation by Finit Volume Methods To the Stochastic Burgers Equation

Presented by:

DIB Nidal

Before the commission of the jury:

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University year: 2019/2020

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DIB Nidal

Abstract

We are interested in a one dimensional nonlinear stochastic partial differential equation: the generalized Burgers equation with homogeneous Dirichlet boundary conditions, perturbed by additive space-time white noise. We present a result of existence and uniqueness of the solution to the viscous equation using fixed point argument. We prove also a result of existence and uniqueness of the entropy solution to the inviscid equation using the concept of measure valued solutions and Kruzhkov's entropy formulation.

Then, we study the finite volume methods for the discretization of the inviscid equation with additive stochastic force defined on a bounded domain D of $\mathbb{R}$ with Dirichlet boundary conditions and a given initial data in $L^{\infty}(D)$. We prove that the numerical solution converges to the unique stochastic entropy weak solution of the continuous problem under a stability condition on the time and space steps. This also yields a new proof of the existence of an entropy weak solution. Finally, We present the simulations of stochastic and deterministic solutions for a particular initial condition by finite volume methods with the software Matlab.

Keywords: Stochastic Burgers equation; Viscosity coefficient; Space-time white noise; Finite volume method; Monotone scheme; Dirichlet boundary conditions.

Résumé

On s' intéress à l'équation aux dérivées partielles stochastiques non linéaires unidimensionnelles: l'équation de Burgers généralisée avec des conditions aux limites de Dirichlet homogènes, perturbées par un bruit blanc additif du type espace-temps. On présente un résultat d'existence et d'unicité d'une solution à l'équation avec viscosité en utilisant l'argument du point fixe. On donne aussi un résultat d'existence et d'unicité de la solution entropique à l'équation non visqueuse en utilisant le concept de la solutions à valeurs mesures et la formulation d'entropie de Kruzhkov.

Ensuite, nous étudions la méthode de volumes finis pour la discrétisation de l'équation de Burgers stochastique généralisée non visqueuse définie sur un domaine borné D de $\mathbb{R}$ avec des conditions aux limites de Dirichlet et une donnée initiale dans $L^{\infty}(D)$. On montre que la solution numérique converge vers la solution entropique stochastique unique du problème continu sous une condition de stabilitée. Ceci donne une nouvelle preuve d'existence d'une solution faible entropique. Finallement, on termine par quelques simulations numériques de la solution approchée stochastique et deterministe, pour une conditions initiales particulièreréalisées sous le logiciel Matlab.

Mots-clefs: Equation de Burgers stochastique; Coefficient de Viscosité; Bruit blanc espace-temps; Méthode des volumes finis; Schéma monotone; Conditions de Dirichlet sur le bord.

ملخص

نهتم في هذه الأطروحة بالمعادلة التفاضلية الجزئية العشوائية أحادية البعد غير الخطية
: معادلة برغر المعممة مع شروط حدية ديريشلي المتجانسة ، مضطربة بضجة بيضاء الزمان والمكان.

بداية نقدم نتيجة لوجود ووحدانية الحل للمعادلة المحتوات على معامل اللزوجة باستخدام نظرية النقطة الثابتة.
نبرهن ايضا وجود ووحدانية الحل الضعيف الانتروبي للمعادلة بدون معامل اللزوجة باستخدام مفهوم الحلول المحسوبة للقياس وصياغة إنتروبيا لكروزكوف.
ثم ندرس طريقة التقريب بالأحجام المحدودة لتقريب حل معادلة برغر المعممة بدون معامل اللزوجة مع اضافة حد ثاني عشوائي. المعرفة على مجال محدود D من $\mathbb{R}$ بشروط حدية لديريشلي ومعطيات أولية في $L^{\infty}(D)$. يمكننا اثبات أن الحل العددي يتقارب نحو الحل الضعيف الانتروبي العشوائي الوحيد للمعادلة المستمرة تحت شرط الاستقرار . هذا يعطي برهانا جديدا على وجود حل ضعيف انتروبي.
و في الاخير نعطي بعض التوضيحات العددية لمعادلة برغر العشوائية أحادية البعد. بإعطاء الطريقة المستعملة و تقديم توضيحات للحل المقرب العشوائي والعادي، باستعمال شرط اولي خاص، بطريقة التقريب بالأحجام المحدودة ، باستعمال برنامج ماتلاب.

الكلمات المفتاحية :

معادلة برغر العشوائية; معامل اللزوجة; ضجة بيضاء الزمان والمكان; طريقة التقريب بالأحجام المحدودة; شروط حدود ديريشلي المتجانسة.

Contents

Introduction

We study the generalized Burgers equation with a stochastic source term. In every natural system, all kinds of perturbative effects may be added to basic equations; this is what we model by noise. When a stochastic term W_ω is introduced in an equation, the solution $u(t,x)$ becomes a random variable $u(t,x,\omega)$, which leads to revisit the mathematical studies already carried out in the deterministic case.

Spatiotemporal systems in physics, chemistry, and biology are often subject to random phenomena that can occur in the form of fluctuating forces, uncertain parameters, and random source terms or boundary conditions. It is therefore of essential importance to be able to introduce uncertainty in the reality that the model must represent, and this is often done by means of stochastic partial differential equations. As a concrete example [19], where the author model the ocean and the atmosphere, introduce a multiplicative noise to represent uncertainties due to the complexity of the earth's climate system. Another example is given by the stochastic Burgers equation studied in this thesis. One of the main problems in the field of complex systems is to achieve a sufficiently deep understanding of turbulence. However, one still lacks a complete theory of turbulence that would permit to predict important phenomena such as turbulent mixing, turbulent convection and turbulent combustion starting from the basic equations of fluid dynamics.

The Burgers equation is due to the Dutch physicist J.M. Burgers [9] who in 1939 simplified the Navier-Stokes equation by neglecting the pressure term, obtaining the following Burgers equation

$$\frac{\partial u}{\partial t} = \rho \frac{\partial^2 u\left(t,x\right)}{\partial x^2} - \partial_x \frac{\left(u\left(t,x\right)\right)^2}{2} \tag{1}$$

which can be studied in one space dimension, with or without a term source.

In the statistical theory of turbulence, a random force method was proposed in the hope of finding a physically meaningful stationary distribution of turbulence.

The study of Burgers' equation with a forcing term is interesting in view of the phenomenological character of this equation. Since it represents an incomplete description of a system, a forcing term can provide a good model of the neglected effects. Thus equation (1), become

$$\frac{\partial u\left(t,x,\omega\right)}{\partial t}=\rho\frac{\partial^{2}u\left(t,x,\omega\right)}{\partial x^{2}}-\partial_{x}\frac{\left(u\left(t,x\omega\right)\right)^{2}}{2}+\xi\left(t,x,\omega\right) \tag{2}$$

where $\xi\left(t,x,\omega\right)$, is a space time white noise.

Motivations for studying the stochastic Burgers equation (2) are manifold. Just to name a few, it is used to model vortex lines in high-temperature superconductors, dislocations in disordered solids and kinetic roughening of interfaces in epitaxial growth, the formation of large-scale structures in the universe, constructive quantum field theory [7], etc. Since in the deterministic viscous case, this equation is furthermore explicitly solvable via the Hopf–Cole transform ($u=\partial_x v/v$, where v solves the heat equation), it comes as no surprise that a wealth of numerical and analytical results are available. From a purely mathematical point of view, let us mention for example the well-posedness results from [8], [7], [17]. One remarkable achievement was the construction of a stationary solution in the inviscid limit with non-vanishing noise (dissipation then occurs purely through shocks). From a more quantitative perspective, the scaling exponents of the solutions in the small viscosity limit have attracted considerable interest, both in the physics and the applied mathematics literature.

For the numerical solution, we apply the finite volume method, this method has first been developed by engineers in order to study complex coupled physical phenomena where the conservation of extensive quantities (such as masses, energy, momentum...) must be carefully respected by the approximate solution.
Another advantage of such schemes is that a large variety of meshes can be used. From the industrial point of view, the finite volume method is known as a robust and cheap method for the discretization of conservation laws (by robust, we mean a scheme which behaves well even for particularly difficult equations, such as nonlinear systems of hyperbolic equations and which can easily be extended to more realistic and physical contexts than the classical academic problems). The finite volume method is cheap thanks to short and reliable computational coding for complex problems. It may be more adequate than the finite difference method (which in particular requires a simple geometry). However, in some cases, it is difficult to design schemes which give enough precision. Indeed, the finite element method can be much more precise than the finite volume method when using higher order polynomials, but it requires an adequate functional framework which is not always available in industrial problems.
The basic idea of the finite volume method is the following: one integrates the partial differential equations in each volume element and then approximates the fluxes

across the volume boundaries. It is the method which we apply to solve the stochastic Burgers equation in one space dimension.
The main aim of this thesis is to study the finite volume methods for the discretization of the stochastic Burgers equation, perturbed by additive space-time white noise. This thesis is composed of four chapters. Chapter 1 contains various theories and results that are required in the sequel. We present in Chapter 2 a result of existence and uniqueness of the local solution to the viscous equation using fixed point argument, then if we impose a condition to the viscosity coefficient we can prove that this solution is global. In Chapter 3, we propose a result of existence and uniqueness of the entropy solution to the inviscid stochastic Burgers equation using the concept of measure valued solutions and Kruzhkov's entropy formulation. In Chapter 4, we study the finite volume methods for the discretization of the inviscid stochastic Burgers equation with additive stochastic force defined on a bounded domain D of $\mathbb{R}$ with Dirichlet boundary conditions and a given initial data in $L^{\infty}(D)$. Then we prove that the numerical solution converges to the unique stochastic entropy weak solution of the continuous problem under a stability condition on the time and space steps. Finally, we propose a numerical application to the stochastic Burgers equation in the one-dimensional case. We introduce the scheme used and present the simulations of stochastic and deterministic solutions for a particular initial condition by finite volume method with the software Matlab.

Chapter 1

Preliminaries

We introduce in this chapter some basic elements of probability theory, Stochastic process and functional analysis for which we refer to [34] for further details.

1.1 Fondamental notion of probability theory

Think of an experiment, with an outcome that changes randomly with each repetition. As the experiment is repeated, the frequencies of the outcomes vary and statistical and probabilistic tools are needed to analyze the results. In particular, we assign probabilities to each outcome as a limit of the frequency of occurrence relative to the total number of trials. These ideas are simple and intuitive in an experiment with a finite number of possibilities. To express these concepts in an uncountable setting, such as the stochastic processes and random fields arising in the study of stochastic PDEs, we use an abstract measure space $(\Omega, \mathcal{F}, \mathbb{P})$, called a probability space.

1.1.1 Probability spaces and random variables

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of the sample space Ω, the σ-algebra (the set of events) $\mathcal{F}$, and the probability measure $\mathbb{P}$.

Definition 1.1 (the sample space Ω). *In an experiment, the sample space Ω is the set of all possible outcomes. In general, the sample space is simply an abstract set, one large enough to parameterize all possible outcomes of the experiment.*

Definition 1.2 (σ-algebra). *A set $\mathcal{F}$ of subsets of a set A is a σ-algebra if*

(i) the empty set $\{\} \in \mathcal{F}$,

(ii) the complement $F^c := \{x \in A : x \notin F\} \in \mathcal{F}$ *for all* $F \in \mathcal{F}$, *and*

(iii) the union $\cup_{j \in \mathbb{N}} F_j \in \mathcal{F}$ *for* $F_j \in \mathcal{F}$.

Thus, a σ-algebra is a collection of subsets of A that contains the empty set and is closed under forming complements and countable unions. Any $F \in \mathcal{F}$ is known as a measurable set and the pair $(A, \mathcal{F})$ is known as a measurable space.

Definition 1.3 (measure)**.** *A measure* μ *on a measurable space* $(A, \mathcal{F})$ *is a mapping from* $\mathcal{F}$ *to* $\mathbb{R}^{+} \cup \{\infty\}$ *such that*

(i) the empty set has measure zero, $\mu(\{\}) = 0$*, and*

(ii) $\mu\left(\cup_{j\in\mathbb{N}} F_j\right) = \sum_{j\in\mathbb{N}} \mu(F_j)$ *if* $F_j \in \mathcal{F}$ *are disjoint (i.e.,* $F_k \cap F_j = \{\}$ *for* $k \neq j$*).*

Together $(A, \mathcal{F}, \mu)$ form a measure space.

Definition 1.4 (probability measure)**.** *A measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *is a probability measure if it has unit total mass,* $\mathbb{P}(\Omega) = 1$*.*

Definition 1.5 (almost surely)**.** *An event* F *happens almost surely (a.s. or* $\mathbb{P}$*-a.s.) with respect to a probability measure* $\mathbb{P}$ *if* $\mathbb{P}(F) = 1$*.*

Definition 1.6 (Random variables, realisation)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $(\Psi, \mathcal{G})$ *be a measurable space. Then,* X *is a* Ψ*-valued random variable if* X *is a measurable function from* $(\Omega, \mathcal{F})$ *to* $(\Psi, \mathcal{G})$*. To emphasise the* σ*-algebra on* Ω*, we may write that* X *is an* $\mathcal{F}$*-measurable random variable. The observed value of* $X(\omega)$ *for a given* $\omega \in \Omega$ *belongs to* Ψ *and is called a realisation of* X*.*

1.1.2 Expectation

Let X be a Banach space-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If X is integrable, the expectation of X is

$$E[X] := \int_{\Omega} X(\omega)\, dp(\omega),$$

1.1.3 Gaussian distribution

A random variable X is said to follow the Gaussian or normal distribution on $D = \mathbb{R}$ if, for some $\mu \in \mathbb{R}$ and $\sigma > 0$, its probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right).$$

We write $X \sim N\left(\mu, \sigma^2\right)$ (the random variable X follow the Gaussian distribution with mean μ and variance σ^2).

1.2 Notion of stochastic process

1.2.1 Stochastic process

Given a set $\mathcal{I} \subset \mathbb{R}$, and a Hilbert space H,a H-valued stochastic process is a set of H-valued random variables $\{X(t) : t \in \mathcal{I}\}$. We some times drop the set $\mathcal{I}$ and simply write $X(t)$ to denote the process. This should not be interpreted as a simple function of t and, to emphasise the dependence on ω and that $X : \mathcal{I} \times \Omega \to H$, we may write $X(t, \omega)$ or $X_t(\omega)$. For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a family of random processes $X_t(\omega)$ on this space with $t \in [0, T]$, where t usually is understood as time variable, and T is a fixed parameter, called also terminal time. For a fixed $\omega \in \Omega$, the time function $X_t(\omega)$, $t \in [0, T]$ is called a trajectory or realization corresponding to an elementary event ω.

1.2.2 Brownian motion

The name Brownian refers to Robert Brown, who identified Brownian motion in the movement of pollen particles. Brownian motion is frequently called the Wiener process, after Norbert Wiener, who made a significant contribution to the mathematical theory. In this section, we focus on real-valued processes and use the name Brownian motion.

Definition 1.7 (Brownian motion). *Let $T \in \mathbb{R}^+$ given. A Brownian Motion is a stochastic process $(t, \omega) \in [0, T] \times \Omega \mapsto \beta_t(\omega)$, such that:*

1. $\beta_0 = 0$ *a.s.;*
2. *the increments are stationary: for any* $0 \leq s \leq t \leq T$, $\beta_t - \beta_s \sim \beta_{t-s}$*;*
3. *the increments are independent: for any* $n \in \mathbb{N}^*$ *, for any* $0 = t_0 \leq t_1 \leq ... \leq t_n \leq T$, $\left(\beta_{t_{i+1}} - \beta_{t_i}\right)_{0 \leq i \leq n-1}$ *are independent;*
4. *the process is Gaussian: for any* $n \in \mathbb{N}^*$ *, for any* $0 \leq t_1 \leq ... \leq t_n \leq T$ *and any* $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$, $\lambda_1 \beta_{t_1} + ... + \lambda_n \beta_{t_n}$ *is a real Gaussian random variable;*
5. *for any* $t \geq 0$, $\beta_t \sim \mathcal{N}(0, t)$*;*
6. *the trajectories are almost surely continuous: for* $\mathbb{P}$*-almost every* $\omega \in \Omega$, $t \in [0, T] \mapsto \beta_t(\omega)$ *is continuous.*

Usually, the variable ω is not written.

1.2.3 Filtration

To handle the idea of past and future more generally, we use sub σ-algebras and introduce a sub σ-algebra $\mathcal{F}_t$ for each time t. Intuitively, the events in $\mathcal{F}_t$ are those observable by time t and, because we have more observations as time passes,

the σ-algebras $\mathcal{F}_t$ contain more events as t increases. We make this precise with the notion of filtration.

Definition 1.8 (filtration). *(i) A filtration* $\{\mathcal{F}_t \ : t \geq 0\}$ *is a family of sub* σ*-algebras of* $\mathcal{F}$ *that are increasing; that is,* $\mathcal{F}_s$ *is a sub* σ*-algebra of* $\mathcal{F}_t$ *for* $s \leq t$.

(ii) A filtered probability space is a quadruple $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$*, where* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* $\{\mathcal{F}_t \ : t \geq 0\}$ *is a filtration of* $\mathcal{F}$.

Stochastic processes that conform to the notion of time described by the filtration $\mathcal{F}_t$ are known as *adapted* processes.

Definition 1.9 (adapted). *Let* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *be a filtered probability space. A stochastic process* $\{X(t) \ : t \in [0,T]\}$ *is* $\mathcal{F}_t$ *-adapted if the random variable* $X(t)$ *is* $\mathcal{F}_t$*-measurable for all* $t \in [0,T]$.

1.2.4 The cylindrical Wiener process and Stochastic integration in Hilbert spaces.

In this section, we recall the definition of the cylindrical Wiener process and of stochastic integral on a separable Hilbert space H (its norm is denoted by $|.|_H$ or just $|.|$).

We first fix a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. A *cylindrical Wiener process* or *cylindrical Brownian motion* on H is defined with two elements:

- a complete orthonormal system of H, denoted by $(e_i)_{i\in I}$, where I is a subset of $\mathbb{N}$;
- a family $(\beta_i)_{i\in I}$of independent real Wiener processes with respect to the filtration $\left((\mathcal{F}_t)_{t\geq 0}\right)$;

then W is defined by

$$W(t) = \sum_{i\in I} \beta_i(t)\, e_i. \tag{1.1}$$

When I is a finite set, we recover the usual definition of Wiener processes in the finite dimensional space $\mathbb{R}^{|I|}$.

However the subject here is the study of some Stochastic Partial Differential Equations, so that in the sequel the underlying Hilbert space H is infinite dimensional; for instance when $H = L^2\left(0,1\right)$, an example of complete orthonormal system is $\left(e_k\right) = \left(\sqrt{2}\sin\left(k\pi.\right)\right)_{k\geq 1}$.
The formal sum $dW\left(t\right) = \sum_{i\in I} d\beta_i\left(t\right)e_i\left(x\right)$ is called *space-time white noise.*

A fundamental remark is that the series in (1.1) does not converge in H; but is convergent in any larger Hilbert spaces K such that $H \subset K$ and the embedding is Hilbert-Schmidt. We recall now the definition of such operators.

Definition 1.10 (Hilbert-Schmidt operator)**.** *Let H, K two separable Hilbert spaces, and $\phi \in \mathcal{L}\left(H,K\right)$ a bounded linear operator. We say that ϕ is a Hilbert-Schmidt operator if there exists a complete orthonormal system $\left(e_k\right)_{k\in I}$ of H, $I \subset \mathbb{N}^*$, such that $\sum_{k\in I}\left|\phi e_k\right|_K^2 < \infty$.*

In this situation, the value $\sum_{k\in I}\left|\phi e_k\right|_K^2 < \infty$ is finite for any complete orthonormal system, and does not depend on the choice of such a basis.

A typical example of Hilbert-Schmidt operator is the inclusion map $i : L^2\left(D\right) \to H^{-s}\left(\Omega\right)$ for $s > 1/2$.

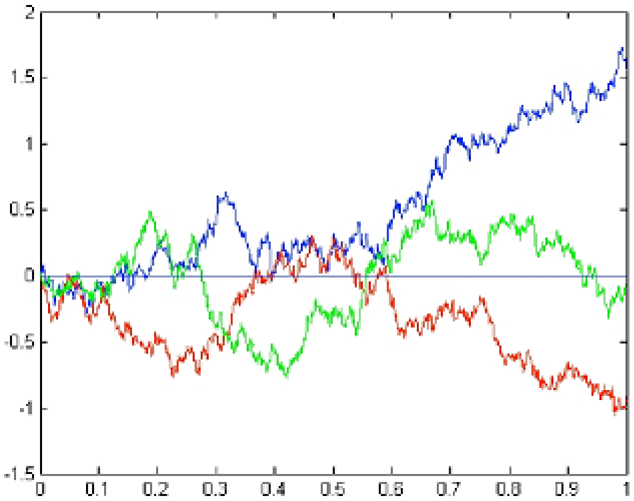

Figure 1.1: Some realizations of Brownian motion

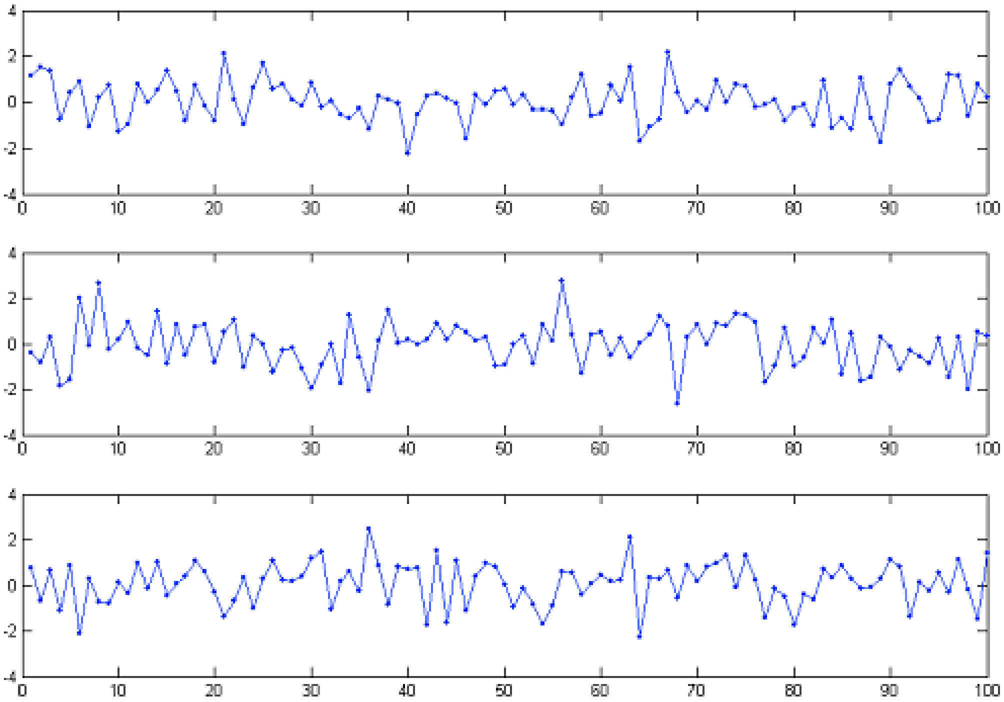

Figure 1.2: Some realizations of white noise.

1.3 Stochastic partial differential equations and their applications

Let (V, H, V') be a triple of Hilbert spaces and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a stochastic basis.

Suppose that $\mathcal{A}(t) : V \to V'$, $\mathcal{M}_k(t) : V \to H$ are linear bounded operators for all $k \in \mathbb{N}$, $t \in [0, T]$, and for some finite terminal time T. A general Linear Stochastic Partial Differential Equation is written as

$$du(t) = (\mathcal{A}u(t) + f(t))\,dt + (\mathcal{M}_k u(t) + g_k(t))\,d\beta_k(t), \qquad u(0) = u_0, \tag{1.2}$$

where $t \in [0, T]$, $u_0 \in L^2(\Omega; H)$ and u_0 is $\mathcal{F}_0$-measurable. We assume that β_k $(k \in \mathbb{N})$ are independent standard Wiener processes, f, g_k $(k \in \mathbb{N})$ are $\mathcal{F}_t$-adapted random processes such that $f \in L^2(\Omega \times (0, T); V')$ and $g_k \in L^2(\Omega \times (0, T); H)$, $k \in \mathbb{N}$. Depending on the noise term the equation (1.2) is classified as follows:

- Equations with *additive noise*, if $\mathcal{M}_k = 0$;
- Equations with *multiplicative noise*, if $\mathcal{M}_k \neq 0$.

Similar to classical deterministic PDEs, the solution of equation (1.2) can be specified in different ways. Since the solution is a stochastic process, besides of PDE part, where we have classical solution, strong/weak generalized solution and mild solution, also we can specify strong and weak probabilistic solution. In this work we consider only solution which are strong in probabilistic sense and weak/strong/mild in PDE sense.

We say that $\mathcal{F}_t$-adapted function $u \in L^2(\Omega \times (0, T); V)$ is a weak solution of equation (1.2), if for every $\varphi \in V$ and all $t \in [0, T]$, the equality

$$\begin{aligned}
\langle u(t), \varphi\rangle_H &= \langle u_0, \varphi\rangle_H + \int_0^t \langle \mathcal{A}u(s) + f(s), \varphi\rangle_H\,ds \\
&\quad + \int_0^t \langle \mathcal{M}_k u(s) + g_k(s), \varphi\rangle_H\,d\beta_k(s)
\end{aligned} \tag{1.3}$$

holds with probability one.

We say that $\mathcal{F}_t$-adapted function $u \in L^2(\Omega \times (0, T); V)$ is a mild solution of equation (1.2) if

$$\begin{aligned}
u(t) &= S(t)u_0 + \int_0^t S(t-s) f(s)\,ds \\
&\quad + \int_0^t S(t-s)(\mathcal{M}_k u(s) + g_k(s))\,d\beta_k(s), \quad t \in [0, T],
\end{aligned}$$

holds true with probability one, where S is a strongly continuous semigroup with infinitesimal generator $\mathcal{A}$. As we observed, stochastic evolution equations in infinite dimensions are natural generalizations of classical PDEs and Systems of Stochastic Ordinary Differential Equations. The theory related to all these equations has motivations coming from physics, chemistry, biology, medicine, finance etc. Although, the theory of SPDEs is already established and widely developed field in mathematics, and problems arising in this theory represent an interest for mathematics itself.

1.3.1 Solutions of linear SPDEs perturbed by space-time white noise: stochastic convolution

In an abstract form, we want to solve SPDEs written in the Hilbert space H

$$du(t) = Au(t)dt + dW(t), \tag{1.4}$$

with an initial condition $u(0) = u_0 \in H$, $(W(t))_{t\in[0,T]}$ a cylindrical Wiener process in H (it could be in another space U), and $A : H \to H$. We assume that A generates a strongly continuous semi-group $(S(t))_{t\in[0,T]}$.

Definition 1.11. *u is a weak solution of the SPDE* (1.4) *if for any $\xi \in D(A^*)$ and any $t > 0$ we have*

$$\langle u(t), \xi\rangle = \langle u_0, \xi\rangle + \int_0^t \langle u(s), A^*\xi\rangle \, ds + \langle W(t), \xi\rangle .$$

Theorem 1.1. *Assume*

$$\int_0^t |S(s)|^2_{\mathcal{L}_2(H,H)} \, ds < \infty. \tag{1.5}$$

Then (1.4) *admits a unique weak solution, which satisfies:*

$$u(t) = S(t)\, u_0 + \int_0^t S(t-s)\, dW(s) .$$

A function u satisfying such a formula is a mild (or integral) solution. Moreover $\mathbb{E}\,|u(t)|^2_H = \int_0^t \|S(s)\|^2_{\mathcal{L}_2(H,H)}\, ds$.

When $u_0 = 0$, the solution is denoted by W_A and is called the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) .$$

The condition (1.5) is precisely the one required to be able to define the stochastic integral in H. If it is removed, there exists no H-valued solution.

1.4 Some notion of functional analysis

Definition 1.12 (Sobolev spaces)**.** *Let* D *be a domain and* Y *be a Banach space. For* $p \geq 1$*, the Sobolev space* $W^{r,p}(D,Y)$ *is the set of functions whose weak derivatives up to order* $r \in \mathbb{N}$ *are in* $L^p(D,Y)$*. That is,*

$$W^{r,p}(D,Y) := \{u : \mathcal{D}^\alpha u \in L^p(D,Y) \ \ if \ |\alpha| \leq r\}.$$

If $p = 2$ *and* H *is a Hilbert space,* $H^r(D,H)$ *is used to denote* $W^{r,2}(D,H)$*.* $W^{r,p}(D,Y)$ *is a Banach space with norm*

$$\|u\|_{W^{r,p}(D,Y)} := \left(\sum_{0 \leq |\alpha| \leq r} \|\mathcal{D}^\alpha u\|^P_{L^p(D,Y)} \right)^{1/p}$$

Definition 1.13 ($L^p(\Omega,H)$ spaces)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* H *be a Hilbert space with norm* $\|.\|$*. Then,* $L^p(\Omega,H)$ *with* $1 \leq p < \infty$ *is the space of* H*-valued* $\mathcal{F}$*-measurable random variables* $X : \Omega \to H$ *with* $E[\|X\|^p] < \infty$ *and is a Banach space with the norm*

$$\|X\|_{L^p(\Omega,H)} := \left(\int_\Omega \|X(\omega)\|^p \, d\mathbb{P} \right)^{1/p} = E\left[\|X\|^p\right]^{1/p}.$$

We recall now some Gronwall's lemma.

Lemma 1.1 (Gronwall's inequality (differential form))**.** *(i) Let* $f(.)$ *be a nonnegative, absolutely continuous function on* $[0,T]$*, which satisfies for a.e.* t *the differential inequality*

$$f'(t) \leq g(t) f(t) + h(t),$$

where $g(t)$ *and* $h(t)$ *are nonnegative, summable functions on* $[0,T]$*. Then*

$$f(t) \leq e^{\int_0^t g(s)ds} \left[f(0) + \int_0^t h(s) \, ds \right]$$

for all $0 \leq t \leq T$*.*

(ii) In particular, if

$$f' \leq gf \quad on \ [0,T] \quad and \ \ f(0) = 0,$$

then

$$f = 0 \quad on \ \ [0,T].$$

We also require the discrete Gronwall inequality

Lemma 1.2 (discrete Gronwall inequality)**.** *Consider* $z_n \geq 0$ *such that*

$$z_n \leq a + b \sum_{k=0}^{n-1} z_k, \quad for \ n = 0, 1, ...$$

and constants $a, b \geq 0$*. If* $b = 1$*, then* $z_n \leq z_0 + na$*. If* $b \neq 1$*, then*

$$z_n \leq b^n z_0 + \frac{a}{1-b}(1 - b^n).$$

Chapter 2

Existence and uniqueness of the solution to the viscous stochastic generalized Burgers equation

2.1 Introduction

It is well known that the Burgers equation is not a good model for turbulence. It does not display any chaos; even when a force is added to the right hand side all solutions converge to a unique stationary solution as time goes to infinity.

However, the situation is totally different when the force is a random one. Several authors have, indeed, suggested to use the stochastic Burgers equation as a simple model for turbulence ([14], [15], [16], [31]). The equation has also been proposed in ([32]) to study the dynamics of interfaces.

Here we consider the generalized Burgers equation with a random force which is a space-time white noise

$$\frac{\partial u\left(t,x\right)}{\partial t}=\rho\frac{\partial^{2}u\left(t,x\right)}{\partial x^{2}}-\partial_{x}f\left(u\left(t,x\right)\right)+\frac{\partial^{2}\tilde{W}}{\partial t\partial x}. \tag{2.1}$$

where ρ is the viscosity coefficient and, $\tilde{W}\left(t,x\right)$, $t\geq 0$, $x\in\mathbb{R}$ is a zero mean Gaussian process whose covariance function is given by

$$E\left[\tilde{W}\left(t,x\right)\tilde{W}\left(s,y\right)\right]=\left(t\wedge s\right)\left(x\wedge y\right),t,s\geq 0,x,y\in\mathbb{R}.$$

Alternatively we can consider a cylindrical Wiener process W by setting

$$W\left(t\right)=\frac{\partial\tilde{W}}{\partial x}=\sum_{h=1}^{\infty}\beta_{h}e_{h}, \tag{2.2}$$

where $\{e_h\}$ is an orthonormal basis of $L^2\left(0,2\pi\right)$ and $\{\beta_h\}$ is a sequence of mutually independent real Brownian motions in a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The series (2.2) defining W does not converge in $L^2(0,2\pi)$ but it is convergent in any Hilbert space U such that the embedding

$$L^2(0,2\pi) \subset U$$

is Hilbert-Schmidt ([18]).

In the following we shall write (2.1) as follows:

$$du(t,x) = \left(\rho \frac{\partial^2 u(t,x)}{\partial x^2} - \partial_x f(u(t,x))\right) dt + dW, x \in [0,2\pi], t > 0, \tag{2.3}$$

where W is defined by (2.2). We assume that $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function.

Equation (2.3) is supplemented with Dirichlet boundary conditions

$$u(t,0) = u(t,2\pi) = 0, \tag{2.4}$$

and the initial condition

$$u(0,x) = u_0(x), \ x \in [0,2\pi]. \tag{2.5}$$

Our aim in this paper is to prove that problem (2.3), with boundary and initial conditions (2.4), (2.5) has a unique global solution.

In the next paragraph, we set the notations, introduce the stochastic convolution and prove local existence in time.

2.2 Local existence in time

Define the unbounded self-adjoint operator A on $L^2(0,2\pi)$ by

$$Au = \rho \frac{\partial^2 u}{\partial x^2},$$

for u on the domain

$$D(A) = \{u \in H^2(0,2\pi) : u(0) = u(2\pi) = 0\}.$$

Denote e^{tA}, $t \geq 0$ the semigroup on $L^2(0,2\pi)$ generated by A. It is well known that e^{tA}, $t \geq 0$, has a natural extension, that we still denote by e^{tA}, $t \geq 0$, as a contraction semigroup on $L^2(0,2\pi)$ for any $p \geq 1$. Finally we denote by $\{e_k\}$ the complete orthonormal system on $L^2(0,2\pi)$ which diagonalizes A and $\{\lambda_k\}$ the corresponding eigenvalues. We have

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin k\pi x, k = 1, 2, ...$$

and

$$\lambda_k = -\pi^2 k^2, k = 1, 2, ...$$

Now we rewrite (2.3), (2.4), (2.5) as the abstract differential stochastic equation

$$\begin{cases} du = \left(Au - \partial_x f\left(u\right)\right) dt + dW, \\ u\left(0\right) = u_0. \end{cases} \tag{2.6}$$

Recall that the solution to the linear problem

$$\begin{cases} du = Audt + dW, \\ u\left(0\right) = u_0. \end{cases} \tag{2.7}$$

is unique and given by the so-called stochastic convolution

$$W_A\left(t\right) = \int_0^t e^{(t-s)A} dW\left(s\right). \tag{2.8}$$

It can be shown that W_A is a Gaussian process and it is mean square continuous with values in $L^2\left(0, 2\pi\right)$. Moreover W_A has a version which is, a.s. for $\omega \in \Omega$, α-Hölder continuous with respect to (t, x) for any $\alpha \in [0, 1/4[$ ([18]).

We set

$$v\left(t\right) = u\left(t\right) - W_A\left(t\right), t \geq 0,$$

then u satisfies (2.6) if and only if v is a solution of

$$\begin{cases} \dfrac{dv}{dt} = Av - \partial_x f\left(v + W_A\right), \\ v\left(0\right) = u_0. \end{cases} \tag{2.9}$$

From now we will study equation (2.9) a. s. $\omega \in \Omega$ and consider for the moment that W_A is an α- Hölder continuous function with respect to (t, x) for any $\alpha \in [0, 1/4[$. We will return to the stochastic point of view (and to equation (2.6)) at the end of §3.

Let us write (2.9) as

$$v\left(t\right) = e^{tA} u_0 - \int_0^t e^{(t-s)A} \partial_x f\left(v + W_A\right) ds; \tag{2.10}$$

then if v satisfies (2.10) we say that it is a mild solution of (2.9).

We are going to solve equation (2.10) by a fixed point argument in the space $C\left([0, T^*]; L^p\left(0, 2\pi\right)\right)$ for $p > 1$ and for some $T^* > 0$. We set

$$\Sigma_p\left(m, T^*\right) = \left\{ v \in C\left([0, T^*]; L^p\left(0, 2\pi\right)\right) : \left|v\left(t\right)\right|_{L^p(0,2\pi)} \leq m, \forall t \in [0, T^*] \right\},$$

and consider an initial datum u_0 $\mathcal{F}_0$-measurable and belonging to $L^p(0,2\pi)$, $\omega \in \Omega$ a. s. We will see, in the proof of the Lemma 2.1 below that if $z(t)$ is, a bounded function from $[0,T]$ into $L^p(0,2\pi)$, then, for $t>0$, the function $e^{tA}\dfrac{\partial}{\partial x}f(z)$ is also in $L^p(0,2\pi)$. Hence the integral in (2.10) is convergent in $L^p(0,2\pi)$ a.s. Thus (2.10) has a meaning as an equality in $L^p(0,2\pi)$.

Lemma 2.1 ([20]). *For any $p \geq 2$ and $m > |u_0|_{L^p(0,2\pi)}$, there exists a stopping time $T^* > 0$ such that* (2.10) *has a unique solution in $\Sigma_p(m,T^*)$.*

Proof- Take any v in $\Sigma_p(m,T^*)$ and define $z = Gv$ by

$$z(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A}\partial_x f(v+W_A)\,ds,$$

where $G : C([0,T^*];L^p(0,2\pi)) \to C([0,T^*];L^p(0,2\pi))$ is a non-linear operator. Then

$$|z(t)|_{L^p(0,2\pi)} \leq \left|e^{tA}u_0\right|_{L^p(0,2\pi)} + \int_0^t \left|e^{(t-s)A}\partial_x f(v+W_A)\right|_{L^p(0,2\pi)} ds.$$

As we noticed before, e^{tA}, $t \geq 0$ is a contraction semigroup on $L^p(0,2\pi)$ which has a regularizing effect and, for any $s_1 \leq s_2$ in $\mathbb{R}$, and $r \geq 1$, e^{tA} maps $W^{s_1,r}(0,2\pi)$ into $W^{s_2,r}(0,2\pi)$, for all $t>0$. Moreover the following estimate holds

$$\left|e^{tA}z\right|_{W^{s_2,r}(0,2\pi)} \leq C_1\left(t^{\frac{s_1-s_2}{2}}+1\right)|z|_{W^{s_1,r}(0,2\pi)} \tag{2.11}$$

for all $z \in W^{s_1,r}(0,2\pi)$. The constant C_1 depends only on s_1, s_2 and r, see for instance ([36]).

Using the Sobolev embedding theorem we have

$$\left|e^{(t-s)A}\partial_x f(v+W_A)\right|_{L^p(0,2\pi)} \leq C_2\left|e^{(t-s)A}\partial_x f(v+W_A)\right|_{W^{\frac{1}{p},\frac{p}{2}}(0,2\pi)},$$

and, thanks to (2.11) with $s_1=-1$, $s_2=1/p$, $r=p/2$

$$\begin{aligned}\left|e^{(t-s)A}\partial_x f(v+W_A)\right|_{L^p(0,2\pi)} &\leq C_1C_2\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)|\partial_x f(v+W_A)|_{W^{-1,\frac{p}{2}}(0,2\pi)}\\ &\leq C_1C_2\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)|f(v+W_A)|_{L^{\frac{p}{2}}(0,2\pi)}\end{aligned}$$

Therefore

$$\begin{aligned} & |z(t)|_{L^p(0,2\pi)} \\ \le\ & |u_0|_{L^p(0,2\pi)} + C_1C_2\int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right) |f(v+W_A)|_{L^{\frac{p}{2}}(0,2\pi)}\,ds \\ \le\ & |u_0|_{L^p(0,2\pi)} + C_1C_2Lip_1\int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right) \left(1+|v+W_A|_{L^{\frac{p}{2}}(0,2\pi)}\right) ds \\ \le\ & |u_0|_{L^p(0,2\pi)} + C_1C_2Lip_1\int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right) \left(1+|v|_{L^{\frac{p}{2}}(0,2\pi)}+|W_A|_{L^{\frac{p}{2}}(0,2\pi)}\right) ds \\ \le\ & |u_0|_{L^p(0,2\pi)} + C_1C_2Lip_1\left(1+(2\pi)^{\frac{1}{p}}\,m+\mu_p\right)\int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right) ds \\ \le\ & |u_0|_{L^p(0,2\pi)} + C_1C_2Lip_1\left(1+(2\pi)^{\frac{1}{p}}\,m+\mu_p\right)\left(\frac{2p}{p-1}t^{\frac{1}{2}-\frac{1}{2p}}+t\right), \end{aligned}$$

where Lip_1 is the Lipschitz constant of f which depend on $m+\mu_p$, and

$$\mu_p = \sup_{t\in[0,T]} |W_A(t)|_{L^{\frac{p}{2}}(0,2\pi)}\,.$$

Hence $|z(t)|_{L^p(0,2\pi)} \le m$ for all $t\in[0,T^*]$ provided

$$|u_0|_{L^p(0,2\pi)} + C_1C_2Lip_1\left(1+(2\pi)^{\frac{1}{p}}\,m+\mu_p\right)\left(\frac{2p}{p-1}\left(T^*\right)^{\frac{1}{2}-\frac{1}{2p}}+T^*\right)\le m \tag{2.12}$$

It is clear that for any $m > |u_0|_{L^p(0,2\pi)}$ there exists a T^* satisfying (2.12). Now consider $v_1, v_2 \in \Sigma_p(m,T^*)$ and set $z_i = Gv_i$, $i=1,2$ and $z=z_1-z_2$. Then

$$z(t) = \int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left[f(v_1+W_A)-f(v_2+W_A)\right]ds,$$

and we derive as above

$$|z(t)|_{L^p(0,2\pi)} \le C_1C_2\int_0^t\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)|f(v_1+W_A)-f(v_2+W_A)|_{L^{\frac{p}{2}}(0,2\pi)}\,ds.$$

According to the hypothesis on f, we have

$$\begin{aligned} |f(v_1+W_A)-f(v_2+W_A)|_{L^{\frac{p}{2}}(0,2\pi)} &\le Lip_2\,|v_1-v_2|_{L^{\frac{p}{2}}(0,2\pi)} \\ &\le Lip_2\,(2\pi)^{\frac{1}{p}}\,|v_1-v_2|_{L^p(0,2\pi)} \\ &= C_3\,|v_1-v_2|_{L^p(0,2\pi)} \end{aligned}$$

where Lip_2 is the Lipschitz constant of f which depend on $m+\mu_p$, and
$C_3 = Lip_2\,(2\pi)^{\frac{1}{p}}$, hence

$$\begin{aligned} |z(t)|_{L^p(0,\ 2\pi)} &\le C_1C_2C_3\int_0^t\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)|v_1-v_2|_{L^p(0,\ 2\pi)}\,ds \\ &\le C\max_{0\le s\le t}|v_1(s)-v_2(s)|_{L^p(0,\ 2\pi)}\int_0^t\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)ds \\ &\le C\left(\frac{2p}{p-1}\left(T^*\right)^{\frac{1}{2}-\frac{1}{2p}}+T^*\right)|v_1-v_2|_{C([0,\ T^*];\ L^p(0,\ 2\pi))} \end{aligned}$$

for all $t \in [0, T^*]$ provided

$$|Gv_1 - Gv_2|_{C([0,\ T^*];\ L^p(0,\ 2\pi))} \leq C \left(\frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^* \right) |v_1 - v_2|_{C([0,\ T^*];\ L^p(0,\ 2\pi))}$$

We take T^* such that

$$C \left(\frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^* \right) < 1$$

and (2.12) holds so that G is a strict contraction on $\Sigma_p (m, T^*)$. ■

Remark 2.1. *([17]) As mentioned before all the previous results are valid a.s. for $\omega \in \Omega$; in particular μ_p and T^* depend on ω. In the next section we will show that $T^* = T$ a.s. for $\omega \in \Omega$ and hence remove the dependence on ω for the time interval on which the solution exists.*

2.3 Global existence

We are still considering equation (2.10) as a deterministic one, working a. s. for $\omega \in \Omega$.

Theorem 2.1. *(Global existence)*
Let u_0 be given which is $\mathcal{F}_0$-measurable and such that for some $p \geq 2$, $u_0 \in L^p (0, 2\pi)$ a.s. If $\rho \geq \frac{Lip_1 c}{2}$ then there exists a unique mild solution of equation (2.6)*, which belongs a. s. to $C ([0, T] ; L^p (0, 2\pi))$.*

In the following Lemma, we derive an a priori estimate which yields global existence.

Lemma 2.2 ([20])**.** *If $v \in C ([0, T] ; L^p (0, 2\pi))$ satisfies* (2.10) *and $\rho \geq \frac{Lip_1 c}{2}$ then*

$$|v (t)|_{L^p(0,2\pi)} \leq e^{tLip_1 c \frac{(p-1)}{2} \mu_\infty^2} |u_0|_{L^p(0,2\pi)} .$$

where $c = (2\pi)^{\frac{2}{p(p-2)}}$ and $\mu_\infty = \sup_{t \in [0,T]} |W_A (t)|_{L^\infty(0,2\pi)}$.

Proof Let $\{u_0^n\}$ be a sequence in $C^\infty(0,2\pi)$ such that

$$u_0^n \longrightarrow u_0, \text{ in } L^p(0,2\pi),$$

and let $\{W^n\}$ be a sequence of regular processes such that

$$W_A^n(t) = \int_0^t e^{(t-s)A} dW^n(s) \longrightarrow W_A(t)$$

in $C([0,T]\times[0,2\pi])$ a. s. for $\omega \in \Omega$.

Let v^n be the solution of

$$v^n(t) = e^{tA} u_0^n - \int_0^t e^{(t-s)A} \partial_x f(v^n + W_A^n)\, ds$$

given by Lemma 2.1 It is easy to see that v^n does exist on an interval of time $[0,T_n]$ such that $T_n \longrightarrow T^*$ a.s. and that v^n converges to v in $C([0,T^*]; L^p(0,2\pi))$ a. s. Moreover v^n is regular a.s. and satisfies

$$\frac{\partial v^n}{\partial t} - \rho \frac{\partial^2 v^n}{\partial x^2} + \partial_x f(v^n + W_A^n) = 0. \tag{2.13}$$

Multiplying (2.13) by $|v^n|^{p-2} v^n$ and integrating over $[0,2\pi]$ we find

$$\begin{aligned} \frac{1}{p}\frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} + \rho(p-1) \int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial v^n}{\partial x}\right)^2 dx \\ + \int_0^{2\pi} \frac{\partial}{\partial x} f(v^n + W_A^n) |v^n|^{p-2} v^n dx = 0. \end{aligned} \tag{2.14}$$

We integrate by parts the last integral

$$\begin{aligned} & \int_0^{2\pi} \frac{\partial}{\partial x} f(v^n + W_A^n) |v^n|^{p-2} v^n dx \\ = & -(p-1) \int_0^{2\pi} f(v^n + W_A^n) |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx \end{aligned}$$

then

$$\begin{aligned} \left| \int_0^{2\pi} \frac{\partial}{\partial x} f(v^n + W_A^n) |v^n|^{p-2} v^n dx \right| &= (p-1) \left| \int_0^{2\pi} f(v^n + W_A^n) |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx \right| \\ &\le (p-1) \int_0^{2\pi} \left| f(v^n + W_A^n) |v^n|^{p-2} \frac{\partial}{\partial x} v^n \right| dx \\ &\le (p-1) \int_0^{2\pi} Lip_1 (1 + |v^n + W_A^n|) |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx \\ &\le (p-1) \int_0^{2\pi} Lip_1 |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx \\ &\quad + (p-1) \int_0^{2\pi} Lip_1 |v^n|^{p-1} \frac{\partial}{\partial x} v^n dx \\ &\quad + (p-1) \int_0^{2\pi} Lip_1 |W_A^n| |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx \end{aligned}$$

The first term is zero, indeed

$$\int_0^{2\pi} |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx = -\int_0^{2\pi} (p-2) |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx$$

hence

$$(p-1) \int_0^{2\pi} |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx = 0$$

In the same way, we can prove that The second term is also zero.

According to the Hölder's and Cauchy's inequalities we bound the third term as follows

$$\begin{aligned}
& Lip_1 (p-1) \int_0^{2\pi} |W_A^n| |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx \\
\leq\ & Lip_1 (p-1) |W_A^n|_{L^\infty(0,2\pi)} |v^n|^{\frac{p-2}{2}}_{L^{p-2}(0,2\pi)} \left(\int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial}{\partial x} v^n \right)^2 dx \right)^{\frac{1}{2}} \\
\leq\ & Lip_1 c (p-1) \mu_{n,\infty} |v^n|^{\frac{p-2}{2}}_{L^p(0,2\pi)} \left(\int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial}{\partial x} v^n \right)^2 dx \right)^{\frac{1}{2}} \\
\leq\ & Lip_1 c \frac{(p-1)}{2} \mu_{n,\infty}^2 |v^n|^{p-2}_{L^p(0,2\pi)} + Lip_1 c \frac{(p-1)}{2} \int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial}{\partial x} v^n \right)^2 dx
\end{aligned}$$

where $c = (2\pi)^{\frac{2}{p(p-2)}}$ and $\mu_{n,\infty} = \sup\limits_{t\in[0,T]} |W_A^n(t)|_{L^\infty(0,2\pi)}$ for a. s. $\omega \in \Omega$.

Going back to (2.14) we obtain

$$\frac{1}{p} \frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} + \rho (p-1) \int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial v^n}{\partial x} \right)^2 dx$$

$$\leq Lip_1 c \frac{(p-1)}{2} \mu_{n,\infty}^2 |v^n|^{p-2}_{L^p(0,2\pi)} + Lip_1 c \frac{(p-1)}{2} \int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial v^n}{\partial x} \right)^2 dx$$

It follows

$$\frac{1}{p} \frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} + (p-1) \left(\rho - \frac{Lip_1 c}{2} \right) \int_0^{2\pi} |v^n|^{p-2} \left(\frac{\partial v^n}{\partial x} \right)^2 dx$$

$$\leq Lip_1 c \frac{(p-1)}{2} \mu_{n,\infty}^2 |v^n|^{p-2}_{L^p(0,2\pi)}$$

if we take ρ and Lip_1 such that

$$\rho \geq \frac{Lip_1 c}{2}$$

we obtain

$$\frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} \leq Lip_1 c \frac{p(p-1)}{2} \mu_{n,\infty}^2 |v^n|^p_{L^p(0,2\pi)}$$

and, according to Gronwall's lemma

$$|v^n|^p_{L^p(0,2\pi)} \leq e^{t Lip_1 c \frac{p(p-1)}{2} \mu_{n,\infty}^2} |u_0^n|^p_{L^p(0,2\pi)} .$$

Taking the limit as $n \longrightarrow \infty$, we see that a.s.

$$|v|^p_{L^p(0,2\pi)} \leq e^{tLip_1 c\frac{p(p-1)}{2}\mu^2_\infty} |u_0|^p_{L^p(0,2\pi)} .$$

It follows

$$|v|_{L^p(0,2\pi)} \leq e^{tLip_1 c\frac{(p-1)}{2}\mu^2_\infty} |u_0|_{L^p(0,2\pi)} .$$

and the assertion of the lemma follows. ■

Proof (Theorem 2.1)

It is easily deduced from Lemma 2.1 and Lemma 2.2 ■

Chapter 3

Existence and uniqueness of the entropy solution to the inviscid stochastic generalized Burgers equation

3.1 Introduction

In this chapter, we are interested in the stochastic generalised Burgers equation of the type :

$$du + \partial_x \left(f\left(u \right) \right) dt = dW\left(t \right) \qquad \text{in} \quad \Omega \times]0, T[\times D, \tag{3.1}$$

with an initial condition u_0 and homogeneous "Dirichlet" boundary condition.

In the sequel, one assumes that $D = \,]0, 2\pi[$, that T is a positive number, $Q =]0, T[\times D$ and that W is a cylindrical Wiener process over $L^2\left(D, \mathbb{R}\right)$. Our aim is to adapt known methods of first-order nonlinear PDE to noise perturbed ones.

On the one hand, Note that, even in the deterministic case, the weak solution to such a problem is not unique in general. One needs to introduce the notion of entropy solution in order to discriminate the physical solution. On the other hand, the stochastic perturbation will not simplify the situation.

Many papers on the viscous Burgers type stochastic problem (i.e. usually in 1D with $f\left(x\right) = x^2$ and a Laplacian) can be found in the literature, where, usually, the stochastic convolution is used. Let us mention, without exhaustiveness,[17], [18], [20], [26] and [29].

Few papers exist concerning the stochastic perturbation of nonlinear first-order hyperbolic problems. Most of them are interested in the Cauchy problem in the

1D case. Let us cite the paper of H. Holden et al [30]. where an operator splitting method is proposed to prove the existence of a weak solution to the Cauchy problem

$$du + (f(u))_x \, dt = g(u) \, d\beta \qquad \text{in } \mathbb{R}.$$

Where β denotes a standard adapted one-dimensional continuous Brownian motion, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the paper of J. U. Kim [33], a method of compensated-compactness is presented to prove the existence of a stochastic weak entropy solution to the Cauchy problem

$$du + (\varphi(u))_x \, dt = g(t,x) \, d\beta \qquad \text{in } \mathbb{R}.$$

Then, a Kruzhkov-type method is used to prove the uniqueness.

In our main result, we propose a result of existence and uniqueness of the stochastic entropy solution to Problem (3.1). A one-dimensional bounded domain with homogeneous Dirichlet conditions is considered. A method of artificial viscosity is proposed to prove the existence of a solution. An adaptation of the classical method of Kruzhkov is proposed to prove the uniqueness of the entropy measure-valued solution. The existence of such a solution follows as usual from the theorem of Prohorov for Young measures.

After giving the assumptions on the data and the definition of an entropy solution, we devote a section to the existence of an entropy measure-valued solution in the sense of Young measures. The uniqueness of the entropy measure-valued solution is proved by using the doubling-variable method of Kruzhkov in a following section. Then, the result of existence of the entropy solution comes from the properties of Young measures connected to weak convergence. And for a basic reminder on Young measures see the appendix.

As mentioned by J. U. Kim [33] for example, the equation has to be understood in the following way:

$$\partial_t \left[u - \int_0^t dW(s) \right] + \partial_x (f(u)) = 0,$$

where $\displaystyle\int_0^t dW(s) = \int_0^t \sum_{k\geq 1} e_k d\beta_k(s) = \sum_{k\geq 1} e_k \beta_k(t) = W(t,.)$.

Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function, [1]as well as f', and $f(0) = 0$

Our aim is to prove a result of existence and uniqueness of the stochastic entropy solution to the above-mentioned problem. Let us fix in what sense such a solution is understood.

---

[1] Some information are given in Sec. 3.4 about locally-Lipschitz f

Notation. In the sequel, for any bounded interval $I \subset \mathbb{R}$, one denotes by $H^1(I)$ the usual Sobolev space and by $H_0^1(I)$ the space of Sobolev functions null on the boundary of I. Note that $H_0^1(I)$ is also the closure in $H^1(I)$ of the test function space $\mathcal{D}(I)$: the space of $C^\infty(\mathbb{R})$ with compact support in I. Then, one denotes by $H^{-1}(I)$ the dual space of $H_0^1(I)$ (see for example R. A. Adams [1] or L. C. Evans and R. Gariepy [21]). In general, if I is not assumed to be an open set ($I = \bar{D}$ or $[0,T[\times D$), $\mathcal{D}(I)$ denotes the restriction to I of $\mathcal{D}(\mathbb{R})$ functions u such that support(u)$\cap$ I is compact. Then, $\mathcal{D}^+(I)$ will denote any non-negative element of $\mathcal{D}(I)$.

For convenience, for any function u of $L^2(\Omega \times Q)$, any real κ and any function φ in $H^1(Q)$, denote by :

(1) $\int_0^t dW(s) = W(t,.)$ and $u_W = u - W$.

(2) $sgn_0^+(x) = 1$ if $x > 0$, 0 else; $x^+ = x\, sgn_0^+(x)$; $F^+(a,b) = [f(a) - f(b)]\, sgn_0^+(a-b)$. Note, in particular, that F^+ is a Lipschitz-continuous function.

(3) d$\mathbb{P}$-a.s. in Ω, denote by $\mu_{u,\kappa}^+$ the distribution in $\mathbb{R}^2$,

$$\begin{aligned}
\varphi \to \mu_{u,\kappa}^+(\varphi) &= \int_{\{u>W+\kappa\}} \{(u - W - \kappa)\,\partial_t\varphi + [f(u) - f(W+\kappa)]\,\partial_x\varphi\}\, dtdx \\
&\quad - \int_{\{u>W+\kappa\}} \varphi\partial_x f(W+\kappa)\, dtdx + \int_D (u_0-\kappa)^+\varphi(0,.)\, dx \\
&= \int_Q \left\{(u_W-\kappa)^+\partial_t\varphi + F^+(W+u_W, W+\kappa)\,\partial_x\varphi\right\} dtdx \\
&\quad - \int_Q \varphi sgn_0^+(u_W-\kappa)\,\partial_x f(W+\kappa)\, dtdx + \int_D (u_0-\kappa)^+\varphi(0,.)\, dx.
\end{aligned}$$

(4) d$\mathbb{P}$-a.s. in Ω, denote by $\mu_{u,\kappa}^-$ the distribution in $\mathbb{R}^2$,

$$\begin{aligned}
\varphi \to \mu_{u,\kappa}^-(\varphi) &= \int_{\{u<W+\kappa\}} \{(W+\kappa-u)\,\partial_t\varphi + [f(W+\kappa) - f(u)]\,\partial_x\varphi\}\, dtdx \\
&\quad + \int_{\{u<W+\kappa\}} \varphi\partial_x f(W+\kappa)\, dtdx + \int_D (\kappa-u_0)^+\varphi(0,.)\, dx \\
&= \int_Q \left\{(\kappa-u_W)^+\partial_t\varphi + F^+(W+\kappa, W+u_W)\,\partial_x\varphi\right\} dtdx \\
&\quad + \int_Q \varphi sgn_0^+(\kappa-u_W)\,\partial_x f(W+\kappa)\, dtdx + \int_D (\kappa-u_0)^+\varphi(0,.)\, dx.
\end{aligned}$$

Then, one would say that

Definition 3.1. *Any function u of $L^2(\Omega \times Q)$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is an entropy solution if*

1. *For any $(\kappa, \varphi) \in \mathbb{R}\times H^1(Q)$ such that $\kappa \geq 0$ and $\varphi \geq 0$, and for any $(\kappa, \varphi) \in \mathbb{R}\times \left[H^1(Q) \cap L^2(0,T; H_0^1(D))\right]$ such that $\varphi \geq 0$,*

$$0 \leq \mu^{+}_{u,\kappa}(\varphi), \qquad d\mathbb{P}\text{-}a.s.$$

2. *For any$(\kappa, \varphi) \in \mathbb{R}\times H^1(Q)$ such that $\kappa \leq 0$ and $\varphi \geq 0$, and for any $(\kappa, \varphi) \in \mathbb{R}\times \left[H^1(Q) \cap L^2(0,T; H_0^1(D))\right]$ such that $\varphi \geq 0$,*

$$0 \leq \mu^{-}_{u,\kappa}(\varphi), \qquad d\mathbb{P}\text{-}a.s.$$

For technical reasons, one also needs to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we are able to deduce the existence of an entropy solution in the sense of Definition 3.1.

Definition 3.2. *Any function $\mathbf{u}$ of $L^2(\Omega \times Q \times]0,1[)$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is a Young measured-valued entropy solution if*

1. *For any $(\kappa, \varphi) \in \mathbb{R}\times H^1(Q)$ such that $\kappa \geq 0$ and $\varphi \geq 0$, and for any $(\kappa, \varphi) \in \mathbb{R}\times \left[H^1(Q) \cap L^2(0,T; H_0^1(D))\right]$ such that $\varphi \geq 0$,*

$$0 \leq \int_0^1 \mu^{+}_{\mathbf{u},\kappa}(\varphi)\, d\alpha, \qquad d\mathbb{P}\text{-}a.s.$$

2. *For any $(\kappa, \varphi) \in \mathbb{R}\times H^1(Q)$ such that $\kappa \leq 0$ and $\varphi \geq 0$, and for any $(\kappa, \varphi) \in \mathbb{R}\times \left[H^1(Q) \cap L^2(0,T; H_0^1(D))\right]$ such that $\varphi \geq 0$,*

$$0 \leq \int_0^1 \mu^{-}_{\mathbf{u},\kappa}(\varphi)\, d\alpha, \qquad d\mathbb{P}\text{-}a.s.$$

Note that in this definition the measures $\mu^{+}_{\mathbf{u},\kappa}$, $\mu^{-}_{\mathbf{u},\kappa}$ also depend on α because $\mathbf{u}$ does.

Therefore, immediate consequences are:

Remark 3.1. *Any entropy solution is also a.s. a weak solution. Following ([11]) $d\mathbb{P}$-a.s., for any $\varphi \in \mathcal{D}^+\left([0,T[\times D\right)$, note that*

$$
\begin{aligned}
\mu^+_{u,\kappa}(\varphi) = {} & \int_Q \left\{(u-W)\,\partial_t\varphi + f(u)\partial_x\varphi\right\} dtdx + \int_D u_0\varphi\,(0,.)\,dx && (:= I_1)\\
& - \int_Q \left\{\kappa\partial_t\varphi + f\,(W+\kappa)\,\partial_x\varphi + \varphi\partial_x f\,(W+\kappa)\right\} dtdx - \int_D \kappa\varphi\,(0,.)\,dx && \\
& && (:= I_2)\\
& - \int_{\{u\le W+\kappa\}} \left\{(u-W-\kappa)\,\partial_t\varphi + \left[f\,(u) - f\,(W+\kappa)\right]\partial_x\varphi\right\} dtdx && (:= I_3)\\
& + \int_{\{u\le W+\kappa\}} \varphi\partial_x f\,(W+\kappa)\,dtdx + \int_D (u_0-\kappa)^-\,\varphi\,(0,.)\,dx && (:= I_4)
\end{aligned}
$$

If $\kappa < 0$, then

$$
|I_3| \le \int_{\{u\le W+\kappa\}} \left\{(|u| + |W|)\,|\partial_t\varphi| + \int_u^W |f'\,(s)|\,ds\,|\partial_x\varphi|\right\} dxdt \underset{\kappa\to-\infty}{\to 0}\ ,
$$

$$
|I_4| \le \int_{\{u\le W+\kappa\}} \left[\|f"\|_\infty (|u|+|W|) + |f'\,(u)|\right] |\partial_x W|\,|\varphi|\,dtdx + \int_{\{u_0\le\kappa\}} |u_0|\,|\varphi\,(0,.)|\,dx\ .
$$

Then, I_4 tends to 0 with κ to $-\infty$ and, since $I_2 = 0$, one concludes that for any $\varphi \in \mathcal{D}^+\left([0,T[\times D\right)$,

$$
0 \le \int_Q \left\{(u-W)\,\partial_t\varphi + f(u)\partial_x\varphi\right\} dxdt + \int_D u_0\varphi\,(0,.)\,dx\ .
$$

Since the opposite inequality can be proved by using $\mu^-_{u,\kappa}$ for large values of κ, u is a solution in the sense of distributions.

Remark 3.2. *The unique solution obtained in this paper satisfies the initial condition in the following sense:*

$$
ess \lim_{t\to0^+} E \int_D |u_W - u_0|\,dx = 0.
$$

Indeed, by the existence proof, the solution u will be in $L^\infty\left(]0,T[, L^2\,(\Omega\times D)\right)$.

3.2 Existence of a Solution

The aim of this section is to give a result on the existence of a measure-valued entropy solution to the problem. The technique is based on the notion of narrow convergence of Young measures (or entropy processes) (cf. the Appendix). Then, thanks to the uniqueness result of the next section, one is able to prove that the measure-valued solution is an entropy weak solution and that the sequence of approximation proposed to prove the existence of the solution converges in L^p_{loc} for any $p < 2$.

Let us set, in the sequel of this section, for any positive integer n, u_n the unique weak solution to the stochastic viscous parabolic equation:

$$\partial_t \left[u_n - W\right] - \frac{1}{n}\partial_x^2 u_n + \partial_x \left(f\left(u_n\right)\right) = 0;$$

i.e. u_n exists in $L^2\left(]0,T[\times\Omega; H_0^1(D)\right)$, adapted to the filtration $\mathcal{F}_t$ as an $L^2(D)$-valued function, with moreover $\partial_t\left[u - W\right] \in L^2\left(]0,T[\times\Omega; H^{-1}(D)\right)$ and, a.s. in Ω, a.e. in $]0,T[$, for any v in $H_0^1(D)$,

$$\left\langle \partial_t\left[u_n - W\right], v\right\rangle_{H^{-1}(D),\, H_0^1(D)} + \int_D \frac{1}{n}\partial_x u_n v'(x) - f\left(u_n\right) v'(x) dx = 0. \tag{3.2}$$

We admit such a result and refer e.g. to ([18], [26] and [38]) for further information on the viscous stochastic parabolic equation. Then, thanks to the stochastic energy equality [26], the following estimate holds:

$$\begin{aligned}\|u_n(t)\|_{L^2(D)}^2 &+ 2\int_0^t\int_D \left[\frac{1}{n}\left(\partial_x u_n\right)^2 - f\left(u_n\right)\partial_x u_n\right] dxds \\ &= \|u(0)\|_{L^2(D)}^2 + 2\int_0^t\int_D u_n dxdW\left(t,.\right) + \pi t.\end{aligned} \tag{3.3}$$

Since $\displaystyle\int_0^t\int_D f\left(u_n\right)\partial_x u_n dxds = 0$, one gets that

Proposition 3.1. *There exists a positive constant C such that,*

$$\forall n \in \mathbb{N}^*, \qquad \|u_n\|_{L^\infty(]0,T[;L^2(\Omega\times D))}^2 + \frac{1}{n}\|u_n\|_{L^2(]0,T[\times\Omega;H_0^1(D))}^2 \leq C.$$

In particular, u_n is a bounded sequence in $L^2(]0,T[\times\Omega\times D)$ and the associated Young measure sequence $\mathbf{u}_n$ converges (up to a subsequence still indexed in the same way) narrowly to an entropy process denoted by $\mathbf{u}$ (see the Appendix).

Consider η, a non-decreasing Lipschitz-continuous function satisfying the assumptions that supp η' is compact and $\eta(0) = 0$, κ an integer and φ a positive element of $\mathcal{D}\left(\bar{Q}\right)$ such that a.s. in Ω and a.e. in $]0,T[$, $v = \eta\left(u_n - W - \kappa\right)\varphi$ belongs to $H_0^1(D)$. Therefore, v is an admissible test-function in (3.2).

(i) Thanks to the chain rule lemma of Alt–Bamberger–Luckhaus–Mignot (see ([4] , [2])) based on convex inequalities, if Ψ denotes the primitive of η such that

$\Psi(0)=0$, one has that

$$\begin{aligned}
I_{1,\eta} &= \int_0^T \langle \partial_t\left[u_n - W\right], \eta\left(u_n - W - \kappa\right)\varphi\rangle_{H^{-1}(D),\ H_0^1(D)}\, dt \\
&= \int_D \Psi\left[u_n\left(T,.\right) - W\left(T,.\right) - \kappa\right]\varphi\left(T,.\right) - \Psi\left[u\left(0,.\right) - \kappa\right]\varphi\left(0,.\right) dx \\
&\quad -\int_Q \Psi[u_n - W - \kappa]\partial_t\varphi dxdt \\
&\geq -\int_D \Psi\left[u\left(0,.\right) - \kappa\right]\varphi\left(0\right) dx - \int_Q \Psi[u_n - W - \kappa]\partial_t\varphi dxdt\ ;
\end{aligned}$$

(ii) Concerning the viscous term, one gets that

$$\begin{aligned}
I_{2,\eta} &= \frac{1}{n}\int_Q \partial_x u_n \partial_x\left[\eta\left(u_n - W - \kappa\right)\varphi\right] dxdt \\
&= \frac{1}{n}\int_Q \left[\partial_x\left(u_n - W\right)\right]^2 \varphi\ \eta'\left(u_n - W - \kappa\right) dxdt + \frac{1}{n}\int_Q \partial_x u_n \eta\left(u_n - W - \kappa\right)\partial_x\varphi dxdt \\
&\quad +\frac{1}{n}\int_Q \eta'\left(u_n - W - \kappa\right)\varphi\partial_x W\partial_x\left(u_n - W\right) dxdt \\
&\geq \frac{1}{n}\int_Q \eta'\left(u_n - W - \kappa\right)\varphi\partial_x W\partial_x\left(u_n - W\right) dxdt \\
&\quad +\frac{1}{n}\int_Q \partial_x u_n \eta\left(u_n - W - \kappa\right)\partial_x\varphi dxdt
\end{aligned}$$

(iii) Then, for the flux term, the chain rule lead to

$$\begin{aligned}
I_{3,\eta} &= \int_Q f\left(u_n\right)\partial_x\left[\eta\left(u_n - W - \kappa\right)\varphi\right] dxdt \\
&= \int_Q \left[f\left(u_n\right) - f\left(W + \kappa\right)\right]\partial_x\left[\eta\left(u_n - W - \kappa\right)\varphi\right] dxdt \\
&\quad -\int_Q \partial_x f\left(W + \kappa\right)\eta\left(u_n - W - \kappa\right)\varphi dxdt \\
&= \int_Q \left[f\left(u_n\right) - f\left(W + \kappa\right)\right]\varphi\eta'\left(u_n - W - \kappa\right)\partial_x\left[u_n - W\right] dxdt \\
&\quad +\int_Q \left[f\left(u_n\right) - f\left(W + \kappa\right)\right]\eta\left(u_n - W - \kappa\right)\partial_x\varphi dxdt \\
&\quad -\int_Q \eta\left(u_n - W - \kappa\right)\varphi f'\left(W + \kappa\right)\partial_x W dxdt
\end{aligned}$$

Let us note that

$$\begin{aligned}
&\partial_x \left\{\int_\kappa^{u_n - W} \left[f\left(r + W\right) - f\left(W{+}\kappa\right)\right]\eta'\left(r - \kappa\right) dr\right\} \\
&= \eta'\left(u_n - W - \kappa\right)\left[f\left(u_n\right) - f\left(W + \kappa\right)\right]\partial_x\left(u_n - W\right) \\
&\quad + \int_\kappa^{u_n - W} \partial_x\left[f(r + W) - f\left(W + \kappa\right)\right]\eta'\left(r - \kappa\right) dr \\
&= \eta'\left(u_n - W - \kappa\right)\left[f\left(u_n\right) - f\left(W + \kappa\right)\right]\partial_x\left(u_n - w\right) \\
&\quad + \int_\kappa^{u_n - W} \eta'\left(r - \kappa\right)\left[f'(r + W) - f'\left(W + \kappa\right)\right]\partial_x W dr
\end{aligned}$$

Thus, it yields

$$\begin{aligned}
I_{3,\eta} &= \int_Q \left[f\left(u_n\right) - f\left(W+\kappa\right)\right] \eta\left(u_n - W - \kappa\right) \partial_x \varphi dxdt \\
&\quad - \int_Q \eta\left(u_n - W - \kappa\right) \varphi f'\left(W+\kappa\right) \partial_x W dxdt \\
&\quad - \int_Q \varphi \left[\int_\kappa^{u_n - W} \eta'\left(r-\kappa\right)\left[f'(r+W) - f'\left(W+\kappa\right)\right] \partial_x W dr\right] dxdt \\
&\quad + \int_Q \partial_x \left\{\int_\kappa^{u_n-W} \left[f\left(r+W\right) - f\left(W+\kappa\right)\right] \eta'\left(r-\kappa\right) dr\right\} \varphi dxdt \\
&= \int_Q \left[f\left(u_n\right) - f\left(W+\kappa\right)\right] \eta\left(u_n - W - \kappa\right) \partial_x \varphi dxdt \\
&\quad - \int_Q \eta\left(u_n - W - \kappa\right) \varphi f'\left(W+\kappa\right) \partial_x W dxdt \\
&\quad - \int_Q \varphi \left[\int_\kappa^{u_n - W} \eta'\left(r-\kappa\right)\left[f'(r+W) - f'\left(W+\kappa\right)\right] \partial_x W dr\right] dxdt \\
&\quad + \int_0^T \left[\int_\kappa^0 \left[f\left(r\right) - f\left(\kappa\right)\right] \eta'\left(r-\kappa\right) dr\right] \left(\varphi\left(.,2\pi\right) - \varphi\left(.,0\right)\right) dt \\
&\quad - \int_Q \left[\int_\kappa^{u_n-W} \left[f\left(r+W\right) - f\left(W+\kappa\right)\right] \eta'\left(r-\kappa\right) dr\right] \partial_x \varphi dxdt
\end{aligned} \tag{3.4}$$

Since η is a non-decreasing Lipschitz-continuous function with supp η' compact, Ψ is a Lipschitz-continuous function and, for any $A \in \mathcal{F}$, $\Psi\left(u_n - W - \kappa\right) \partial_t \varphi 1_A$ is uniformly integrable. Then, one concludes that

$$\begin{aligned}
\liminf_{n\to\infty} E\left[I_{1,\eta} 1_A\right] &\geq -E\left[1_A \int_{Q\times]0,1[} \Psi[\mathbf{u}\left(.,\alpha\right) - W - \kappa] \partial_t \varphi dxdtd\alpha\right] \\
&\quad -E\left[1_A \int_D \Psi\left[u\left(0,.\right) - \kappa\right] \varphi\left(0,.\right) dx\right].
\end{aligned}$$

As $\frac{1}{\sqrt{n}} \left\|u_n\right\|^2_{L^2(]0,T[\times\Omega;H_0^1(D))}$ is bounded and since η is bounded function, the following result holds :

$$\liminf_{n\to\infty} E\left[1_A I_{2,\eta}\right] \geq 0.$$

As η is a bounded Lipschitz-continuous function with supp η' compact, f is a Lipschitz-continuous function (for the first term of $I_{3,\eta}$), the integrands involved in the first term of $E\left[I_{3,\eta} 1_A\right]$ are uniformly integrable and the convergence in the sense of Young measures holds. Noting that the fourth term is independent of n, one needs to take care of the last one. As f is not a bounded function, the uniform integrability of the integrand is ensured by the hypothesis of compact support for η'.

Conclusion: testing (3.2) with $v = \eta\left(u_n - W - \kappa\right)\varphi$, estimating all terms as above, yields for any $A \in \mathcal{F}$,

$$\begin{aligned}
0 \;\geq\; & -E\left[1_A \int_{Q\times]0,1[} \Psi[\mathbf{u}(.,\alpha) - W - \kappa]\partial_t\varphi dxdtd\alpha\right] \;-E\left[1_A\int_D \Psi\left[u\left(0,.\right) - \kappa\right]\varphi\left(0,.\right)dx\right] \\
& +E\left[1_A \int_{Q\times]0,1[} \left[f\left(\mathbf{u}\left(.,\alpha\right)\right) - f\left(W+\kappa\right)\right]\eta\left(\mathbf{u}\left(.,\alpha\right) - W - \kappa\right)\partial_x\varphi dxdtd\alpha\right] \\
& -E\left[1_A \int_{Q\times]0,1[} \eta\left(\mathbf{u}\left(.,\alpha\right) - W - \kappa\right)\varphi f'\left(W+\kappa\right)\partial_x W dxdtd\alpha\right] \\
& -E\left[1_A \int_{Q\times]0,1[} \varphi\left[\int_\kappa^{\mathbf{u}(.,\alpha)-W} \eta'\left(r-\kappa\right)\left[f'(r+W) - f'\left(W+\kappa\right)\right]\partial_x W dr\right] dxdtd\alpha\right] \\
& +E\left[1_A \int_0^T \left[\int_\kappa^0 \left[f\left(r\right) - f\left(\kappa\right)\right]\eta'\left(r-\kappa\right)dr\right]\left(\varphi\left(.,2\pi\right) - \varphi\left(.,0\right)\right)dt\right] \\
& -E\left[1_A \int_{Q\times]0,1[} \left[\int_\kappa^{\mathbf{u}(.,\alpha)-W} \left[f\left(r+W\right) - f\left(W+\kappa\right)\right]\eta'\left(r-\kappa\right)dr\right]\partial_x\varphi dxdtd\alpha\right] \\
= \;& J_1 + J_2 + J_3 + J_4 + J_5 + J_6
\end{aligned}$$

Assume now that $\eta\left(x\right) = \eta_\epsilon\left(x\right) = \min\left(1, \dfrac{x^+}{\epsilon}\right)$. Then, in order to be compatible with the boundary assumption for $\eta_\epsilon\left(u_n - W - \kappa\right)\varphi$, $\varphi \in \mathcal{D}\left(\bar{Q}\right)$ if $k \geq 0$, $\varphi \in \mathcal{D}\left([0,T]\times D\right)$ otherwise.

Obviously,

$$\lim_{\epsilon\to 0} J_1 = -E\left[1_A \int_{Q\times]0,1[} [\mathbf{u}\left(.,\alpha\right) - W - \kappa]^+\partial_t\varphi dxdtd\alpha\right] - E\left[1_A \int_D \left[u\left(0,.\right) - \kappa\right]^+\varphi\left(0,.\right)dx\right],$$

$$\lim_{\epsilon\to 0} J_2 = -E\left[1_A \int_{Q\times]0,1[} \left[f\left(\mathbf{u}\left(.,\alpha\right)\right) - f\left(W+\kappa\right)\right] sgn_0^+\left(\mathbf{u}\left(.,\alpha\right) - W - \kappa\right)\partial_x\varphi dxdtd\alpha\right],$$

$$\lim_{\epsilon\to 0} J_3 = -E\left[1_A \int_{Q\times]0,1[} sgn_0^+\left(\mathbf{u}\left(.,\alpha\right) - W - \kappa\right)\varphi f'\left(W+\kappa\right)\partial_x W dxdtd\alpha\right]$$

$$\lim_{\epsilon\to 0} J_4 = 0$$

since f' is a Lipschitz-continuous function,

$$\lim_{\epsilon\to 0} J_5 = 0$$

since f is a Lipschitz-continuous function.

Then,

$$J_6 = E\left[1_A \int_{Q\times]0,1[} \left[\int_0^{\mathbf{u}(.,\alpha)-W-\kappa} \int_0^r f'\left(W+\kappa+\sigma\right)d\sigma\eta'\left(r\right)dr\right]\partial_x\varphi dxdtd\alpha\right]$$

vanishes as ε goes to 0^+ thanks to the hypothesis on f' and one gets that:

(i) For any $(\kappa,\varphi)\in\mathbb{R}\times H^1(]0,T[\times D)$ such that $\kappa\geq 0$ and $\varphi\geq 0$, and for any $(\kappa,\varphi)\in\mathbb{R}\times[H^1(]0,T[\times D)]\cap L^2(]0,T[;H_0^1(D))$ such that $\varphi\geq 0$,

$$\begin{aligned}0 \;\leq\; & \int_{\{\mathbf{u}>W+\kappa\}}\left\{(\mathbf{u}-W-\kappa)\,\partial_t\varphi+\left[f(\mathbf{u})-f(W+\kappa)\right]\partial_x\varphi\right\}dxdtd\alpha \\ & -\int_{\{\mathbf{u}>W+\kappa\}}\varphi\partial_x f(W+\kappa)\,dxdtd\alpha+\int_D (u_0-\kappa)^+\varphi(0,.)\,dx, \qquad d\mathbb{P}\text{-a.s.},\end{aligned}$$

i.e.

$$\begin{aligned}0 \;\leq\; & \int_{Q\times]0,1[}\left\{(\mathbf{u}-W-\kappa)^+\partial_t\varphi+F^+(\mathbf{u},W+\kappa)\,\partial_x\varphi\right\}dxdtd\alpha \\ & -\int_{Q\times]0,1[} sgn_0^+(\mathbf{u}-W-\kappa)\,\varphi\partial_x f(W+\kappa)\,dxdtd\alpha+\int_D (u_0-\kappa)^+\varphi(0,.)\,dx, \qquad d\mathbb{P}\text{-a.s.}\end{aligned}$$

where $F^+(x,y)=f(x)-f(y)$ if $x>y$ and 0 else.

In the same way, one can prove
(ii) For any $(\kappa,\varphi)\in\mathbb{R}\times H^1(]0,T[\times D)$ such that $\kappa\leq 0$ and $\varphi\geq 0$, and for any $(\kappa,\varphi)\in\mathbb{R}\times[H^1(]0,T[\times D)]\cap L^2(]0,T[;H_0^1(D))$ such that $\varphi\geq 0$,

$$\begin{aligned}0 \;\leq\; & \int_{\{\mathbf{u}<W+\kappa\}}\left\{(W+\kappa-\mathbf{u})\,\partial_t\varphi+\left[f(W+\kappa)-f(\mathbf{u})\right]\partial_x\varphi\right\}dxdtd\alpha \\ & +\int_{\{\mathbf{u}<W+\kappa\}}\varphi\partial_x f(W+\kappa)\,dxdtd\alpha+\int_D (\kappa-u_0)^+\varphi(0,.)\,dx, \qquad d\mathbb{P}\text{-a.s.}\end{aligned}$$

i.e.

$$\begin{aligned}0 \;\leq\; & \int_{Q\times]0,1[}\left\{(W+\kappa-\mathbf{u})^+\partial_t\varphi+F^+(W+\kappa,\mathbf{u})\,\partial_x\varphi\right\}dxdtd\alpha \\ & +\int_{Q\times]0,1[} sgn_0^+(W+\kappa-\mathbf{u})\,\varphi\partial_x f(W+\kappa)\,dxdtd\alpha+\int_D (\kappa-u_0)^+\varphi(0,.)\,dx, \qquad d\mathbb{P}\text{-a.s.}\end{aligned}$$

This proves that an entropy measure-valued solution exists. One needs to use the uniqueness result to conclude that this Young measure is associated to a function that should be the unique entropy solution. Moreover, u belongs to $L^\infty\left(]0,T[,L^2(\Omega\times D)\right)$ and the strong convergence in L^p_{loc} would be obtained too, for any $p\in[1,2[$.

Remark 3.3. *Note that, for any* $(\kappa,\varphi)\in\mathbb{R}\times[H^1(]0,T[\times D)]\cap L^2(]0,T[;H_0^1(D))$

such that $\varphi \geq 0$, *we also have*

$$\begin{aligned} 0 \;\leq\; & \int_{Q\times]0,1[} \left[|\mathbf{u}-W-\kappa|\,\partial_t\varphi + F\left(\mathbf{u}, W+\kappa\right)\partial_x\varphi\right] dxdtd\alpha \\ & -\int_{Q\times]0,1[} sgn_0\left(\mathbf{u}-W-\kappa\right)\varphi\partial_x f\left(W+\kappa\right) dxdtd\alpha + \int_D |u_0-\kappa|\,\varphi\left(0,.\right)dx, \qquad d\mathbb{P}\text{-}a.s. \end{aligned}$$

where $F\left(x,y\right) = sgn_0\left(x-y\right)\left[f\left(x\right)-f\left(y\right)\right]$ *and* $sgn_0\left(x\right)=0$ *if* $x=0$ *and* $\dfrac{x}{|x|}$ *else.*

3.3 Uniqueness

Let us denote by $\mathbf{u}_1$ and $\mathbf{u}_2$ two admissible Young measure-valued solutions associated to two initial conditions $u_{1,0}$ and $u_{2,0}$.

3.3.1 Interior inequality

Consider φ in $\mathcal{D}^+\left([0,T]\times D\right)$ and $G\left(t,x,s,y\right)=\varphi\left(s,y\right)\rho_n\left(x-y\right)\rho_l\left(s-t\right)$ where ρ_n and ρ_l denote the usual mollifier sequences in $\mathbb{R}$, with supp $\rho_l \subset \left[-\dfrac{2}{l},0\right]$. We assume moreover that n and l are large enough for G to belong to $\mathcal{D}\left([0,T]\times D\times]0,T]\times D\right)$.

Proposition 3.2. *For any non-negative* φ *in* $H^1\left(Q\right)\cap L^2\left(]0,T[,H_0^1\left(D\right)\right)$,

$$\begin{aligned} 0 \;\leq\; & E\int_{Q\times]0,1[^2} \left(\mathbf{u}_1\left(t,x,\alpha\right)-\mathbf{u}_2\left(t,x,\beta\right)\right)^+ \partial_t\varphi dxdtd\alpha d\beta \\ & +E\int_{Q\times]0,1[^2} F^+\left(\mathbf{u}_1\left(t,x,\alpha\right),\mathbf{u}_2\left(t,x,\beta\right)\right)\partial_x\varphi dxdtd\alpha d\beta \\ & +\int_D \left(u_{1,0}-u_{2,0}\right)^+\varphi\left(0,x\right)dx. \end{aligned}$$

For convenience set $p=(t,x,\alpha)$, $q=(s,y,\beta)$, $u_W=\mathbf{u}_1-W$, $\hat{u_W}=\mathbf{u}_2-W$. Since $\mathbf{u}_1$ is a solution, for $\kappa=\hat{u_W}\left(q\right)$, the following inequality holds d$\mathbb{P}$-a.s. :

$$\begin{aligned} 0 \;\leq\; & \int_{Q^2\times]0,1[^2} \left(u_W\left(p\right)-\hat{u_W}\left(q\right)\right)^+\partial_t G dpdq \\ & +\int_{Q^2\times]0,1[^2} F^+\left(W\left(t,x\right)+u_W\left(p\right), W\left(t,x\right)+\hat{u_W}\left(q\right)\right)\partial_x G dpdq \\ & -\int_{Q^2\times]0,1[^2} G sgn_0^+\left(u_W\left(p\right)-\hat{u_W}\left(q\right)\right) f'\left[W\left(t\right)+\hat{u_W}\left(q\right)\right]\partial_x W\left(t,x\right)dpdq \end{aligned}$$

Similarly, since $\mathbf{u}_2$ is a solution, for $\kappa=u_W\left(p\right)$, one has d$\mathbb{P}$-a.s. :

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \left(u_W(p)-\hat{u_W}(q)\right)^+ \partial_s G dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(s,y)+u_W(p), W(s,y)+\hat{u_W}(q)\right)\partial_y G dpdq \\
& + \int_{Q^2\times]0,1[^2} G sgn_0^+\left(u_W(p)-\hat{u_W}(q)\right) f'\left[W(s,y)+u_W(p)\right]\partial_y W(s,y)\, dpdq \\
& + \int_{Q\times]0,1[\times D} \left(u_W(p)-u_{2,0}(y)\right)^+ \varphi(0,y)\,\rho_n(x-y)\,\rho_l(-t)\, dpdy.
\end{aligned}$$

Summing up the preceding two inequalities, we obtain

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \left(u_W(p)-\hat{u_W}(q)\right)^+ \left(\partial_t+\partial_s\right) G dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(t,x)+u_W(p), W(t,x)+\hat{u_W}(q)\right)\partial_x G dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(s,y)+u_W(p), W(s,y)+\hat{u_W}(q)\right)\partial_y G dpdq \\
& - \int_{\{u_W(p)>\hat{u_W}(q)\}} G f'\left[W(t,x)+\hat{u_W}(q)\right]\partial_x W(t,x)\, dpdq \\
& + \int_{\{u_W(p)>\hat{u_W}(q)\}} G f'\left[W(s,y)+u_W(p)\right]\partial_y W(s,y)\, dpdq \\
& + \int_{Q\times]0,1[\times D} \left(u_W(p)-u_{2,0}(y)\right)^+ \varphi(0,y)\,\rho_n(x-y)\,\rho_l(-t)\, dpdy, \qquad d\mathbb{P}\text{-a.s.}
\end{aligned}$$

For convenience, denote by $A=\{u_W(p)>\hat{u_W}(q)\}$. Then we can rewrite the preceding inequality as

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \left(u_W(p)-\hat{u_W}(q)\right)^+ \left(\partial_t+\partial_s\right) G dpdq \\
& + \int_A \left(f[W(t,x)+u_W(p)]-f[W(t,x)+\hat{u_W}(q)]\right)\partial_x G dpdq \\
& + \int_A \left(f[W(s,y)+u_W(p)]-f[W(s,y)+\hat{u_W}(q)]\right)\partial_y G dpdq \\
& - \int_A G f'\left[W(t,x)+\hat{u_W}(q)\right]\partial_x W(t,x)\, dpdq \\
& + \int_A G f'\left[W(s,y)+u_W(p)\right]\partial_y W(s,y)\, dpdq \\
& + \int_{Q\times]0,1[\times D} \left(u_W(p)-u_{2,0}(y)\right)^+ \varphi(0,y)\,\rho_n(x-y)\,\rho_l(-t)\, dpdy, \qquad d\mathbb{P}\text{-a.s.,}
\end{aligned}$$

which yields

$$\begin{aligned}
0 \leq\; & \int_{Q^2\times]0,1[^2} \left(u_W(p) - \widehat{u_W}(q)\right)^+ \partial_s\varphi(s,y)\,\rho_n(x-y)\,\rho_l(s-t)\,dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(s,y) + u_W(p), W(s,y) + \widehat{u_W}(q)\right) \\
& \times\partial_y\varphi(s,y)\,\rho_n(x-y)\,\rho_l(s-t)\,dpdq \\
& + \int_A \left(f[W(t,x) + u_W(p)] - f[W(t,x) + \widehat{u_W}(q)]\right)\partial_x\rho_n(x-y)\,\varphi(s,y)\,\rho_l(s-t)\,dpdq \\
& - \int_A \left(f[W(s,y) + u_W(p)] - f[W(s,y) + \widehat{u_W}(q)]\right)\partial_x\rho_n(x-y)\,\varphi(s,y)\,\rho_l(s-t)\,dpdq \\
& - \int_A Gf'\left[W(t,x) + \widehat{u_W}(q)\right]\partial_x W(t,x)\,dpdq \\
& + \int_A Gf'\left[W(s,y) + u_W(p)\right]\partial_y W(s,y)\,dpdq \\
& + \int_{Q\times]0,1[\times D} \left(u_W(p) - u_{2,0}(y)\right)^+ \varphi(0,y)\,\rho_n(x-y)\,\rho_l(-t)\,dpdy, \qquad \text{d}\mathbb{P}\text{-a.s.,}
\end{aligned}$$

i.e.

$$\begin{aligned}
0 \leq\; & \int_{Q^2\times]0,1[^2} \left(u_W(p) - \widehat{u_W}(q)\right)^+ \partial_s\varphi(s,y)\,\rho_n(x-y)\,\rho_l(s-t)\,dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(s,y) + u_W(p), W(s,y) + \widehat{u_W}(q)\right)\partial_y\varphi(s,y)\,\rho_n(x-y)\,\rho_l(s-t)\,dpdq \\
& - \int_A \left(f[W(s,y) + u_W(p)] - f[W(t,x) + u_W(p)]\right)\partial_x\rho_n(x-y)\,\varphi(s,y)\,\rho_l(s-t)\,dpdq \\
& - \int_A \left(f[W(t,x) + \widehat{u_W}(q)] - f[W(s,y) + \widehat{u_W}(q)]\right)\partial_x\rho_n(x-y)\,\varphi(s,y)\,\rho_l(s-t)\,dpdq \\
& - \int_A Gf'\left[W(t,x) + \widehat{u_W}(q)\right]\partial_x W(t,x)\,dpdq \\
& + \int_A Gf'\left[W(s,y) + u_W(p)\right]\partial_y W(s,y)\,dpdq \\
& + \int_{Q\times]0,1[\times D} \left(u_W(p) - u_{2,0}(y)\right)^+ \varphi(0,y)\,\rho_n(x-y)\,\rho_l(-t)\,dpdy, \qquad \text{d}\mathbb{P}\text{-a.s.} \\
=\; & I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7
\end{aligned}$$

Thanks to the properties of Lebesgue sets, the following convergence holds:

$$\begin{aligned}
\lim_{n\to\infty}\lim_{l\to\infty} E\left(I_1 + I_2\right) =\; & E\int_{Q\times]0,1[^2} \left(\mathbf{u}_1(t,x,\alpha) - \mathbf{u}_2(t,x,\beta)\right)^+ \partial_t\varphi\,dxdtd\alpha d\beta \\
& + E\int_{Q\times]0,1[^2} F^+\left(\mathbf{u}_1(t,x,\alpha), \mathbf{u}_2(t,x,\beta)\right)\partial_x\varphi\,dxdtd\alpha d\beta.
\end{aligned}$$

Note that

$$\begin{aligned}
I_3 &= -\int_A \left(\int_0^1 f'[\mathbf{u}_1(p) + \sigma(W(s,y) - W(t,x))]d\sigma \right) \partial_x \rho_n(x-y)(W(s,y) - W(t,x)) \\
&\quad \varphi(s,y)\rho_l(s-t)\,dpdq \\
&= -\int_A \left(\int_0^1 \int_{u_W(p)+W(s,y)}^{\mathbf{u}_1(p)+\sigma(W(s,y)-W(t,x))} f''[\eta]d\eta d\sigma \right) \partial_x \rho_n(x-y)(W(s,y) - W(t,x)) \\
&\quad \varphi(s,y)\rho_l(s-t)\,dpdq \\
&\quad -\int_A f'[u_W(p) + W(s,y)]\partial_x \rho_n(x-y)(W(s,y) - W(t,x))\varphi(s,y)\rho_l(s-t)\,dpdq \\
&= I_{3,1} + I_{3,2}
\end{aligned}$$

Since f' is assumed to be a Lipschitz-continuous function, one has the following estimate

$$|I_{3,1}| \le \|f''\|_\infty \int_A |\partial_x \rho_n(x-y)| (W(s,y) - W(t,x))^2 \varphi(s,y)\rho_l(s-t)\,dpdq.$$

Lemma 3.1. $\limsup_{n\to\infty} \limsup_{l\to\infty} |EI_{3,1}| \le 0.$

Proof. Starting from the above inequality, using the classical properties of the Itô integral, we deduce that

$$\begin{aligned}
|EI_{3,1}| &\le \|f''\|_\infty E \int_{Q^2} |\partial_x \rho_n(x-y)| (W(s,y) - W(t,x))^2 \varphi(s,y)\rho_l(s-t)\,dxdtdyds \\
&\le CE \int_{Q^2} |\partial_x \rho_n(x-y)| (W(s,y) - W(t,y))^2 \rho_l(s-t)\,dxdtdyds \\
&\quad + CE \int_{Q^2} |\partial_x \rho_n(x-y)| (W(t,y) - W(t,x))^2 \rho_l(s-t)\,dxdtdyds \\
&\le 2C \int_{Q^2} |\partial_x \rho_n(x-y)| \left[E(W(s,y))^2 + E(W(t,x))^2 \right] \rho_l(s-t)\,dxdtdyds \\
&\quad + 2C \int_{Q^2} |\partial_x \rho_n(x-y)| \left[E(W(t,y))^2 + E(W(t,x))^2 \right] \rho_l(s-t)\,dxdtdyds \\
&= 2C \int_{Q^2} |\partial_x \rho_n(x-y)| \left[E\left(\sum_{k=1}^\infty \beta_k(s) e_k(y) \right)^2 + E\left(\sum_{k=1}^\infty \beta_k(t) e_k(x) \right)^2 \right] \\
&\quad \times \rho_l(s-t)\,dxdtdyds \\
&\quad + 2C \int_{Q^2} |\partial_x \rho_n(x-y)| \left[E\left(\sum_{k=1}^\infty \beta_k(t) e_k(y) \right)^2 + E\left(\sum_{k=1}^\infty \beta_k(t) e_k(x) \right)^2 \right] \\
&\quad \times \rho_l(s-t)\,dxdtdyds
\end{aligned}$$

since The Brownian motions β_k and β_j are independent and the cross terms for $k \ne j$ have mean zero and the series defining W does not converge in $L^2(0,2\pi)$ but it is convergent in any Hilbert space U such that the embedding $L^2(0,2\pi) \subset U$ is Hilbert-Schmidt, one hase the following estimate

$$\begin{aligned}
|EI_{3,1}| \;&\leq\; 2C\int_{Q^2}|\partial_x\rho_n(x-y)|\,(s+t)\underbrace{\left(\sum_{k=1}^{\infty}\|e_k\|_U^2\right)}_{<\infty}\rho_l(s-t)\,dxdtdyds \\
&\quad +2C\int_{Q^2}|\partial_x\rho_n(x-y)|\,t\underbrace{\left(\sum_{k=1}^{\infty}\|e_k\|_U^2\right)}_{<\infty}\rho_l(s-t)\,dxdtdyds
\end{aligned}$$

Thus, one has that

$$\begin{aligned}
|EI_{3,1}| \;&\leq\; C(n)\left(\sum_{k=1}^{\infty}\|e_k\|_U^2\right)\int_{\mathbb{R}}(s+t)\rho_l(s-t)\,dsdt \\
&\quad +C(n)\left(\sum_{k=1}^{\infty}\|e_k\|_U^2\right)\int_{\mathbb{R}}t\rho_l(s-t)\,dsdt \\
&\leq\; \frac{C(n)}{l}\left(\sum_{k=1}^{\infty}\|e_k\|_U^2\right)
\end{aligned}$$

and the assertion of the lemma follows. ■

In a similar way,

$$\begin{aligned}
I_4 \;&=\; -\int_A\left(\int_0^1 f'[\mathbf{u}_2(q)+\sigma(W(t,x)-W(s,y))]d\sigma\right)\partial_x\rho_n(x-y)(W(t,x)-W(s,y)) \\
&\quad \varphi(s,y)\,\rho_l(s-t)\,dpdq \\
&=\; -\int_A\left(\int_0^1\int_{W(t,x)+\widehat{u_W}(q)}^{\mathbf{u}_2(q)+\sigma(W(t,x)-W(s,y))} f''[\eta]d\eta d\sigma\right)\partial_x\rho_n(x-y)(W(t,x)-W(s,y)) \\
&\quad \varphi(s,y)\,\rho_l(s-t)\,dpdq \\
&\quad -\int_A f'\left[W(t,x)+\widehat{u_W}(q)\right]\partial_x\rho_n(x-y)(W(t,x)-W(s,y))\,\varphi(s,y)\,\rho_l(s-t)\,dpdq \\
&=\; I_{4,1}+I_{4,2}.
\end{aligned}$$

Since f' is assumed to be a Lipschitz-continuous function, one has the following estimate

$$|I_{4,1}|\leq\|f''\|_\infty\int_A|\partial_x\rho_n(x-y)|\,(W(s,y)-W(t,x))^2\,\varphi(s,y)\,\rho_l(s-t)\,dpdq.$$

and, in the same way as for $EI_{3,1}$, we can prove

Lemma 3.2. $\limsup\limits_{n\to\infty}\limsup\limits_{l\to\infty}|EI_{4,1}|\leq 0.$

Next, note that,

$$\begin{aligned}
I_{3,2}+I_6 &= -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[u_W(p)+W(s,y)\right] \\
&\quad \times\left[\partial_x\rho_n(x-y)\left(W(s,y)-W(t,x)\right)-\partial_y W(s,y)\,\rho_n(x-y)\right]dpdq \\
&= \int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[u_W(p)+W(s,y)\right] \\
&\quad \times\partial_y\left[\rho_n(x-y)\left(W(s,y)-W(t,x)\right)\right]dpdq
\end{aligned}$$

and

$$\begin{aligned}
I_{4,2}+I_5 &= -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(t,x)+\widehat{u_W}(q)\right] \\
&\quad \times\left[\partial_x\rho_n(x-y)\left(W(t,x)-W(s,y)\right)+\partial_x W(t,x)\,\rho_n(x-y)\right]dpdq \\
&= -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(t,x)+\widehat{u_W}(q)\right] \\
&\quad \times\partial_x\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{3,2}+I_6+I_{4,2}+I_5 &= -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(t,x)+\widehat{u_W}(q)\right] \\
&\quad \times\partial_x\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq \\
&\quad +\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(s,y)+u_W(p)\right] \\
&\quad \times\partial_y\left[\rho_n(x-y)\left(W(s,y)-W(t,x)\right)\right]dpdq \\
&= -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(t,x)+\widehat{u_W}(q)\right] \\
&\quad \times\partial_x\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq \\
&\quad -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(s,y)+u_W(p)\right] \\
&\quad \times\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq \\
&= -\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(t,x)+\widehat{u_W}(q)\right] \\
&\quad \times\left[\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]+\partial_x\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]\right] \\
&\quad -\int_A \varphi(s,y)\,\rho_l(s-t)\left[f'\left[W(s,y)+u_W(p)\right]-f'\left[W(t,x)+\widehat{u_W}(q)\right]\right] \\
&\quad \times\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq \\
&= J_1+J_2
\end{aligned}$$

Note that

$$\begin{aligned}
|J_1| &= \left|\int_A \varphi(s,y)\,\rho_l(s-t)\,f'\left[W(t,x)+\widehat{u_W}(q)\right].\left[\partial_x W(t,x)-\partial_y W(s,y)\right]\rho_n(x-y)\,dpdq\right| \\
&\le C\int_A \varphi(s,y)\,\rho_l(s-t)\left[\left|W(t,x)+\widehat{u_W}(q)\right|+1\right]\left|\partial_x W(t,x)-\partial_y W(s,y)\right|\rho_n(x-y)\,dpdq
\end{aligned}$$

since f' is a Lipschitz-continuous function.

Lemma 3.3. $\limsup_{n\to\infty}\limsup_{l\to\infty}|EJ_1|\le 0.$

Proof. Note that one has:

$$\begin{aligned}
&|EJ_1| \leq CE\int_A \rho_l(s-t)\left|\partial_x W(t,x) - \partial_x W(s,x)\right|[|W(t,x)+\hat{u_W}(q)|+1]\rho_n(x-y)\,dpdq\\
&+CE\int_A \left|\partial_x W(s,x) - \partial_y W(s,y)\right|[|W(t,x)+\hat{u_W}(q)|+1]\rho_n(x-y)\rho_l(s-t)\,dpdq\\
&\leq C\left[\int_{D\times]0,T[^2}\rho_l(s-t)\,|t-s|\,dxdtds\right]^{\frac{1}{2}}\\
&\times\left[E\int_A \rho_l(s-t)\rho_n(x-y)[|W(t,x)+\hat{u_W}(q)|+1]^2 dpdq\right]^{\frac{1}{2}}\\
&+C\left[\int_{D^2\times]0,T[} s\rho_n(x-y)\,dxdyds\right]^{\frac{1}{2}}\\
&\times\left[E\int_A \rho_l(s-t)\rho_n(x-y)[|W(t,x)+\hat{u_W}(q)|+1]^2 dpdq\right]^{\frac{1}{2}}
\end{aligned}$$

Therefore, $\limsup_{n\to\infty}\limsup_{l\to\infty}|EJ_1| \leq 0$.

Moreover,

$$\begin{aligned}
J_2 &= -\int_A \left[f'[u_W(p)+W(s,y)] - f'[W(t,x)+\hat{u_W}(q)]\right]\varphi(s,y)\rho_l(s-t)\\
&\quad\times\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq\\
&= -\int_A \varphi(s,y)\rho_l(s-t)\left[f'[u_1(p)+W(s,y)-W(t,x)] - f'[u_1(p)+W(t,y)-W(t,x)]\right]\\
&\quad\times\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq\\
&\quad-\int_A \varphi(s,y)\rho_l(s-t)\left[f'[u_1(p)+W(t,y)-W(t,x)] - f'[u_2(q)+W(t,x)-W(s,x)]\right]\\
&\quad\times\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq\\
&\quad-\int_A \varphi(s,y)\rho_l(s-t)\left[f'[u_2(q)+W(t,x)-W(s,x)] - f'[u_2(q)+W(t,x)-W(s,y)]\right]\\
&\quad\times\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]dpdq\\
&= J_{2,1}+J_{2,2}+J_{2,3}
\end{aligned}$$

Then, one has that

$$\begin{aligned}
|J_{2,1}| &\leq \|f''\|_\infty\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|\\
&\quad\times\left|\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]\right|dxdtdyds\\
&\leq \|f''\|_\infty\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|\left|W(t,x)-W(s,y)\right|\\
&\quad\times\left|\partial_y\rho_n(x-y)\right|dxdtdyds\\
&\quad+\|f''\|_\infty\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|\rho_n(x-y)\left|\partial_y W(s,y)\right|dxdtdyds
\end{aligned}$$

Thus,

Lemma 3.4. $\limsup_{n\to\infty}\limsup_{l\to\infty}|EJ_{2,1}| \leq 0.$

Proof. Indeed,

$$\begin{aligned}
0 &\leq E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|\left|W(t,x)-W(s,y)\right|\left|\partial_y\rho_n(x-y)\right|dxdtdyds\\
&\leq \frac{1}{2}E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|^2\left|\partial_y\rho_n(x-y)\right|dxdtdyds\\
&\quad +\frac{1}{2}E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(t,x)-W(s,y)\right|^2\left|\partial_y\rho_n(x-y)\right|dxdtdyds
\end{aligned}$$

and the terms on the right tend to 0 as first l then n tend to ∞, as has been shown already in the study of integral $I_{3,1}$. Moreover,

$$\begin{aligned}
&E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|\rho_n(x-y)\left|\partial_yW(s,y)\right|dxdtdyds\\
&\leq\left[E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|^2\rho_n(x-y)\,dxdtdyds\right]^{\frac{1}{2}}\\
&\times\left[E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\rho_n(x-y)\left|\partial_yW(s,y)\right|^2dxdtdyds\right]^{\frac{1}{2}}\\
&\leq\left[E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(t,y)\right|^2\rho_n(x-y)\,dxdtdyds\right]^{\frac{1}{2}}\\
&\times C\left\|W(s,y)\right\|_{L^2(\Omega\times Q)}
\end{aligned}$$

a terme of the same nature as the one studied already in connection with EJ_1. Next, observe that

$$\begin{aligned}
\left|J_{2,3}\right| &\leq \left\|f''\right\|_\infty\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(s,x)\right|\\
&\quad\times\left|\partial_y\left[\rho_n(x-y)\left(W(t,x)-W(s,y)\right)\right]\right|dxdtdyds\\
&= \left\|f''\right\|\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(s,x)\right|\\
&\quad\times\left|\partial_y\rho_n(x-y)\right|\left|W(t,x)-W(s,y)\right|dxdtdyds\\
&\quad+\left\|f''\right\|\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(s,x)\right|\\
&\quad\times\rho_n(x-y)\left|\partial_yW(s,y)\right|dxdtdyds
\end{aligned}$$

and thus, we can prove

Lemma 3.5. $\limsup\limits_{n\to\infty}\limsup\limits_{l\to\infty}\left|EJ_{2,3}\right|\leq 0.$

Proof. Indeed,

$$\begin{aligned}
0 &\leq \int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(s,x)\right|\\
&\times\left|\partial_y\rho_n(x-y)\right|\left|W(t,x)-W(s,y)\right|dxdtdyds\\
&\leq\frac{1}{2}E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(s,y)-W(s,x)\right|^2\left|\partial_y\rho_n(x-y)\right|dxdtdyds\\
&+\frac{1}{2}E\int_{Q^2}\varphi(s,y)\rho_l(s-t)\left|W(t,x)-W(s,y)\right|^2\left|\partial_y\rho_n(x-y)\right|dxdtdyds
\end{aligned}$$

whose limite have been studied in the treatement of integral $I_{3,1}$.
Moreover,

$$\begin{aligned}
& E\int_{Q^2} \varphi\left(s,y\right)\rho_l\left(s-t\right)\left|W\left(s,y\right)-W\left(s,x\right)\right|\rho_n\left(x-y\right)\left|\partial_y W\left(s,y\right)\right| dxdtdyds \\
& \leq \left[E\int_{Q^2} \varphi\left(s,y\right)\rho_l\left(s-t\right)\left|W\left(s,y\right)-W\left(s,x\right)\right|^2\rho_n\left(x-y\right)dxdtdyds\right]^{\frac{1}{2}} \\
& \times\left[E\int_{Q^2} \varphi\left(s,y\right)\rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|\partial_y W\left(s,y\right)\right|^2 dxdtdyds\right]^{\frac{1}{2}} \\
& \leq \left[E\int_{Q^2} \varphi\left(s,y\right)\rho_l\left(s-t\right)\left|W\left(s,y\right)-W\left(s,x\right)\right|^2\rho_n\left(x-y\right)dxdtdyds\right]^{\frac{1}{2}} \\
& \times C\left\|W(s,y)\right\|_{L^2(\Omega\times Q)}
\end{aligned}$$

whose limit is similar to the one studied in the treatement of integral EJ_1.

Since $G(t,x,.,.)\in\mathcal{D}\left([0,T[\times D\right)$, by chain rule, one has that

$$\begin{aligned}
J_{2,2} &= -\int_{(Q\times]0,1[)^2} \varphi\left(s,y\right)\rho_l\left(s-t\right)F'^{+}\left[u_1\left(p\right),u_2\left(q\right)+W\left(t,x\right)-W\left(s,y\right)\right] \\
&\quad \times\partial_y\rho_n\left(x-y\right)\left(W\left(t,x\right)-W\left(s,y\right)\right)dpdq \\
&= -\int_{(Q\times]0,1[)^2} \varphi\left(s,y\right)\rho_l\left(s-t\right)\partial_y\rho_n\left(x-y\right)\left(W\left(t,x\right)-W\left(s,y\right)\right) \\
&\quad \times\left\{F'^{+}\left[u_1\left(p\right),u_2\left(q\right)+W\left(t,x\right)-W\left(s,y\right)\right]-F'^{+}\left[u_1\left(p\right),u_2\left(t,x,\beta\right)+W\left(t,x\right)-W\left(s,y\right)\right]\right\} \\
&\quad +\int_{(Q\times]0,1[)^2} \rho_l\left(s-t\right)\rho_n\left(x-y\right)\left(W\left(t,x\right)-W\left(s,y\right)\right)\partial_y\varphi\left(s,y\right) \\
&\quad \times F'^{+}\left[u_1\left(p\right),u_2\left(t,x,\beta\right)+W\left(t,x\right)-W\left(s,y\right)\right]dpdq.
\end{aligned}$$

Since f' and F'^{+} are Lipschitz-continuous functions, with a Lipschitz-constant C depending on $\|f''\|_\infty$,

$$\begin{aligned}
\left|J_{2,2}\right| \;\leq\; & C\int_{Q^2\times]0,1[} \varphi\left(s,y\right)\rho_l\left(s-t\right)\left|\partial_y\left[\rho_n\left(x-y\right)\left(W\left(t,x\right)-W\left(s,y\right)\right)\right]\right| \\
& \times\left\{2\left|W(t,y)-W(t,x)\right|+\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|\right\}dxdtdydsd\beta \\
& +C\int_{(Q\times]0,1[)^2} \rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|W\left(t,x\right)-W\left(s,y\right)\right| \\
& \times\left|\mathbf{u}_1\left(p\right)-\mathbf{u}_2\left(t,x,\beta\right)-W\left(t,x\right)+W\left(s,x\right)\right|dpdq.
\end{aligned}$$

Now, we can prove

Lemma 3.6. $\lim\sup_{n\to\infty}\lim\sup_{l\to\infty}\left|EJ_{2,2}\right|\leq 0.$

Proof. On the one hand, in the same way as for Lemma 3.1, we can prove that the last integral vanishes since it is lower than

$$C\left[\int_{Q^2}\rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|W\left(t,x\right)-W\left(s,y\right)\right|^2 dxdtdyds\right]^{\frac{1}{2}}$$
$$\times\left[\int_{(Q\times]0,1[)^2}\rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|\mathbf{u}_1\left(p\right)-\mathbf{u}_2\left(t,x,\beta\right)-W\left(t,x\right)+W\left(s,y\right)\right|^2 dpdq\right]^{\frac{1}{2}}.$$

On the other hand, the first part of the first integral is similar to the one already studied with $J_{2,3}$ so we consentrate on

$$E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\left|\partial_y\left[\rho_n\left(x-y\right)\left(W\left(t,x\right)-W\left(s,y\right)\right)\right]\right|\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|dxdtdq$$

$$\begin{aligned}\leq\ & E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\left|\partial_y\rho_n\left(x-y\right)\right|\left|W\left(t,x\right)-W\left(s,y\right)\right|\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|dxdtdq\\ &+E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|\partial_y W\left(s,y\right)\right|\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|dxdtdq\end{aligned}$$

because $\left|\rho_n'\left(x\right)\right|=\dfrac{2n^2\left|x\right|}{\left(\left|nx\right|^2-1\right)^2}\rho_n\left(x\right)$,one has

$$\begin{aligned}&E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\left|\partial_y\rho_n\left(x-y\right)\right|\left|\left(W\left(t,x\right)-W\left(s,y\right)\right)\right|\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|dxdtdydsd\beta\\ &=E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\left|\partial_x\rho_n\left(x-y\right)\right|\left|W\left(t,x\right)-W\left(s,y\right)\right|\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|dxdtdq\\ &\leq 4\left[E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|^2 dxdtdq\right]^{\frac{1}{2}}\\ &\times\left[E\int_{Q^2}\varphi\left(s,y\right)\rho_l\left(s-t\right)\frac{n^4\left|x-y\right|^2}{\left(\left|n\left(x-y\right)\right|^2-1\right)^4}\rho_n\left(x-y\right)\left|W\left(t,x\right)-W\left(s,y\right)\right|^2 dxdtdyds\right]^{\frac{1}{2}}\\ &+\left[E\int_{Q^2}\varphi\left(s,y\right)\rho_l\left(s-t\right)\frac{n^4\left|x-y\right|^2}{\left(\left|n\left(x-y\right)\right|^2-1\right)^4}\rho_n\left(x-y\right)\left|W\left(s,x\right)-W\left(s,y\right)\right|^2 dxdtdyds\right]^{\frac{1}{2}}\\ &:=4A.B\end{aligned}$$

Note that, if one still denotes by $\mathbf{u}_2$ the same function extended by 0 outside Q, we have

$$A\leq\left[E\int_{\mathbb{R}^2}\rho_l\left(r\right)\rho_n\left(z\right)\int_{\mathbb{R}^2\times]0,1[}\left|\mathbf{u}_2\left(t+r,x+z,\beta\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|^2 dxdtdzdrd\beta\right]^{\frac{1}{2}},$$

which tends to 0 thanks to the continuity of translations in the Lebesgue spaces. Let us prove that B is bounded. In order to do so,note that

$$
\begin{aligned}
B &\leq C\left[\int_{Q^2}\rho_l\left(s-t\right)\frac{n^4\left|x-y\right|^2}{\left(\left|n\left(x-y\right)\right|^2-1\right)^4}\,\rho_n\left(x-y\right)\left(t-s\right)dxdtdyds\right]^{\frac{1}{2}}\\
&+C\left[\int_{Q^2}\rho_l\left(s-t\right)\frac{n^4\left|x-y\right|^2}{\left(\left|n\left(x-y\right)\right|^2-1\right)^4}\,\rho_n\left(x-y\right)sdxdtdyds\right]^{\frac{1}{2}}\\
&\leq C\left[\int_{D\times\mathbb{R}^2}(-r)\rho_l\left(r\right)\int_D\frac{n^4\left|x-y\right|^2}{\left(\left|n\left(x-y\right)\right|^2-1\right)^4}\,\rho_n\left(x-y\right)dydxdtdr\right]^{\frac{1}{2}}\\
&+C\left[\int_{Q^2}\frac{n^4\left|z\right|^2}{\left(\left|n\left(z\right)\right|^2-1\right)^4}\,\rho_n\left(z\right)sdxdtdyds\right]^{\frac{1}{2}}\\
&\leq n^2C\int_{D\times\mathbb{R}^2}(-r)\rho_l\left(r\right)dxdtdr\\
&+C\left[\int_{Q^2}\frac{n^4\left|z\right|^2}{\left(\left|n\left(z\right)\right|^2-1\right)^4}\,\rho_n\left(z\right)sdxdtdyds\right]^{\frac{1}{2}}\\
&\leq\frac{n^2C}{l}\\
&+C\left[\int_{Q^2}\frac{n^4\left|z\right|^2}{\left(\left|n\left(z\right)\right|^2-1\right)^4}\,\rho_n\left(z\right)sdxdtdyds\right]^{\frac{1}{2}}
\end{aligned}
$$

Therefore, $\limsup\limits_{l\to\infty} B$ becomes uniformly bounded with respect to n and the result holds. One has that

$$
\begin{aligned}
&E\int_{Q^2\times]0,1[}\varphi\left(s,y\right)\rho_l\left(s-t\right)\rho_n\left(x-y\right)\left|\partial_yW\left(s,y\right)\right|\left|\mathbf{u}_2\left(q\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|dxdtdq\\
\leq\;& C\left[E\int_{Q^2\times]0,1[}\rho_l\left(r\right)\rho_n\left(z\right)\left|\mathbf{u}_2\left(t+r,x+z,\beta\right)-\mathbf{u}_2\left(t,x,\beta\right)\right|^2dxdtd\beta dzdr\right]^{\frac{1}{2}}
\end{aligned}
$$

Again, the result follows from the continuity of translations in the Lebesgue spaces. Finally, let us show

Lemma 3.7. $\limsup\limits_{n\to\infty}\limsup\limits_{l\to\infty}\left|EI_7\right|\leq\int_D\left(u_{1,0}-u_{2,0}\right)^+\varphi\left(0,x\right)dx.$

Proof. Denote by $\phi\left(t,x,y\right)=\int_t^T\rho_l\left(-r\right)dr\rho_n\left(x-y\right)\varphi\left(0,y\right)$ $=\int_{\inf\left(t,\frac{2}{l}\right)}^{\frac{2}{l}}\rho_l\left(-r\right)dr\rho_n\left(x-y\right)\varphi\left(0,y\right)$. Since it is a non-negative function of $\mathcal{D}\left([0,T[\times D\right)$

for any y in D as soon as n is large enough, and as $\mathbf{u}_1$ is a solution, one gets that

$$\begin{aligned}
&\int_{D\times Q\times]0,1[}\left\{(u_W(p)-u_{2,0}(y))^+\varphi(0,y)\rho_n(x-y)\rho_l(-t)\right\}dpdy\\
\le\ &\int_{D\times Q\times]0,1[}\int_{\inf\left(t,\frac{2}{l}\right)}^{\frac{2}{l}}\rho_l(-r)\,dr\left\{F^+(W+u_W,W+u_{2,0}(y))\,\partial_x\rho_n(x-y)\,\varphi(0,y)\right\}dpdy\\
&-\int_{D\times Q\times]0,1[}\int_{\inf\left(t,\frac{2}{l}\right)}^{\frac{2}{l}}\rho_l(-r)\,drsgn_0^+(u_W-u_{2,0}(y))\partial_x f[W+u_{2,0}(y)]\rho_n(x-y)\,\varphi(0,y)\,dpdy\\
&+\int_{D^2}(u_{1,0}(x)-u_{2,0}(y))^+\int_0^{\frac{2}{l}}\rho_l(-r)\,dr\rho_n(x-y)\,\varphi(0,y)\,dxdy.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\limsup_{n\to\infty}\ \limsup_{l\to\infty}E\int_{D\times Q\times]0,1[}\left\{(u_W-u_{2,0}(y))^+\varphi(0,y)\rho_n(x-y)\rho_l(-t)\right\}dpdy\\
\le\ &\int_D(u_{1,0}-u_{2,0})^+\varphi(0,.)\,dx.
\end{aligned}$$

■

3.3.2 Global inequality

Proposition 3.3. *For any non-negative φ in $H^1(Q)$,*

$$\begin{aligned}
0\ \le\ &E\int_{Q\times]0,1[^2}(\mathbf{u}_1(t,x,\alpha)-\mathbf{u}_2(t,x,\beta))^+\partial_t\varphi dxdtd\alpha d\beta\\
&+E\int_{Q\times]0,1[^2}F^+(\mathbf{u}_1(t,x,\alpha),\mathbf{u}_2(t,x,\beta))\,\partial_x\varphi dxdtd\alpha d\beta\\
&+\int_D(u_{1,0}-u_{2,0})^+\varphi(0,.)\,dx.
\end{aligned}$$

Following ([10], [12]), choose a partition of unity subordinate to a covering of $\bar{D}$ by intervals I_i, $i=0,....,k$ satisfying $I_0\cap\partial D=\emptyset$, and, for $i>0$, $I_i\subset I_i'$ with $I_i'\cap\partial D\neq\emptyset$. Consider φ in $\mathcal{D}^+([0,T[\times\mathbb{R})$ with supp $\varphi\subset I:=I_i$ for some $i>0$. Moreover, we choose a sequence of mollifiers ρ_l in $\mathbb{R}$ with supp $\rho_l\subset\ ]-2/l,0[$ and a sequence of mollifiers ρ_n in $\mathbb{R}$ such that $y\to\rho_n(y-x)\in\mathcal{D}(D)$ for all $x\in I$, $\sigma_n(y)=\displaystyle\int_D\rho_n(y-x)\,dx$ is an increasing sequence for $y\in I$, and $\sigma_n(y)=1$ for any $y\in I$ such that $d(y,\mathbb{R}\backslash D)>c/n$ (with $c=C(i)$ depending on I). Denote $G(t,x,s,y)=\varphi(s,y)\,\rho_n(y-x)\,\rho_l(s-t)$.

Note that, for l, n suffisamment large, $(t,x)\to G(.,.,s,y)\in\mathcal{D}\left(]0,T[\times\bar{D}\right)$ for any $(s,y)\in Q$, and $(s,y)\to G(t,x,.,.)\in\mathcal{D}([0,T[\times D)$ for any $(t,x)\in Q$. Moreover, the function

$$\begin{aligned}
G_n(s,y)\ &=\ \int_Q G(t,x,s,y)\,dxdt=\varphi(s,y)\int_D\rho_n(y-x)\,dx\int_0^T\rho_l(s-t)\,dt\\
&=\ \varphi(s,y)\,\sigma_n(y)\,,
\end{aligned}$$

satisfies: $G_n \in \mathcal{D}\left([0,T[\times D\right)$, $0 \leq G_n \leq G_{n+1} \leq \varphi$. Therefore, a non-negative Borel function ψ exists such that the monotonically increasing sequence G_n converges to ψ everywhere in I and $0 \leq \psi \leq \varphi$. For convenience set $p = (t,x,\alpha)$, $q = (s,y,\beta)$, $u_W = \mathbf{u}_1 - W$, $\hat{u_W} = \mathbf{u}_2 - W$. Since $k = \hat{u_W}^+(q) \geq 0$, using that $\mathbf{u}_1$ is a solution and $G\left(t=0\right) = 0$, d$\mathbb{P}$-a.s. leads to

$$\begin{aligned}
0 \leq & \int_{Q^2\times]0,1[^2} \left(u_W(p) - \hat{u_W}^+(q)\right)^+ \partial_t G dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(t,x) + u_W(p), W(t,x) + \hat{u_W}^+(q)\right) \partial_x G dpdq \\
& - \int_{Q^2\times]0,1[^2} G sgn_0^+\left(u_W(p) - \hat{u_W}^+(q)\right) f'\left[W(t,x) + \hat{u_W}^+(q)\right] \partial_x W(t,x)\, dpdq \\
= & \int_{Q^2\times]0,1[^2} \left(u_W^+(p) - \hat{u_W}^+(q)\right)^+ \partial_t G dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(t,x) + u_W^+(p), W(t,x) + \hat{u_W}^+(q)\right) \partial_x G dpdq \\
& - \int_{Q^2\times]0,1[^2} G sgn_0^+\left(u_W^+(p) - \hat{u_W}^+(q)\right) f'\left[W(t,x) + \hat{u_W}^+(q)\right] \partial_x W(t,x)\, dpdq
\end{aligned}$$

Using that $\mathbf{u}_2$ is also a solution, with $k = u_W(p)^+$and $y \to G(t,s,x,y) \in \mathcal{D}(D)$, one gets, d$\mathbb{P}$-a.s.,

$$\begin{aligned}
0 \quad \leq \quad & \int_{Q^2\times]0,1[^2} \left(u_W^+(p) - \hat{u_W}(q)\right)^+ \partial_s G dpdq \\
& + \int_{Q^2\times]0,1[^2} F^+\left(W(s,y) + u_W^+(p), W(s,y) + \hat{u_W}(q)\right) \partial_y G dpdq \\
& + \int_{Q\times]0,1[} \int_D \left(u_W^+(p) - u_{2,0}(y)\right)^+ G(t,x,0,y)\, dydp \\
& + \int_{Q^2\times]0,1[^2} G sgn_0^+\left(u_W^+(p) - \hat{u_W}(q)\right) f'\left[W(s,y) + u_W^+(p)\right] \partial_y W(s,y)\, dpdq.
\end{aligned}$$

Next note

Lemma 3.8. *For any real a, b and c,*

$$\begin{aligned}
\left(a^+ - b\right)^+ = \left(a - b^+\right)^+ + (-b)^+ = \left(a^+ - b^+\right)^+ + (-b)^+, \\
F^+\left(c + a^+, c + b\right) = F^+\left(c + a^+, c + b^+\right) + F^+\left(c, c + b\right). \\
f'\left(c + a^+\right) 1_{\{b<a^+\}} = f'(c)\, 1_{\{b<0\}} + f'\left(c + a^+\right) 1_{\{b^+<a^+\}} - f'(c)\, 1_{\{b<0<a\}}.
\end{aligned}$$

We have,

$$
\begin{aligned}
0 \leq\ & \int_{Q^2\times]0,1[^2} \left(u_W^+ (p) - \hat{u_W}^+ (q)\right)^+ \partial_s G \\
& + F^+ \left(W(s,y) + u_W^+ (p), W(s,y) + \hat{u_W}^+ (q)\right) \partial_y G dpdq \\
& + \int_{Q^2\times]0,1[^2} Gsgn_0^+ \left(u_W^+ (p) - \hat{u_W} (q)\right) f'[W(s,y) + u_W^+ (p)]\partial_y W(s,y) dpdq \\
& + \int_{Q\times]0,1[} \int_D \left(u_W^+ (p) - u_{2,0} (y)\right)^+ G(t,x,0,y)\, dydp \\
& + \int_{Q^2\times]0,1[^2} \left\{(0 - \hat{u_W} (q))^+ \partial_s G + F^+ (W(s,y) + 0, W(s,y) + \hat{u_W} (q))\, \partial_y G\right\} dpdq \\
=\ & \int_{Q^2\times]0,1[^2} \left(u_W^+ (p) - \hat{u_W}^+ (q)\right)^+ \partial_s G \\
& + F^+ \left(W(s,y) + u_W^+ (p), W(s,y) + \hat{u_W}^+ (q)\right) \partial_y G dpdq \\
& + \int_{Q^2\times]0,1[^2} Gsgn_0^+ \left(u_W^+ (p) - \hat{u_W}^+ (q)\right) f'[W(s,y) + u_W^+ (p)]\partial_y W(s,y) dpdq \\
& + \int_{Q^2\times]0,1[^2} \left\{(0 - \hat{u_W} (q))^+ \partial_s G + F^+ (W(s,y) + 0, W(s,y) + \hat{u_W} (q))\, \partial_y G\right\} dpdq \\
& + \int_{Q\times]0,1[} \int_D \left(u_W^+ (p) - u_{2,0} (y)\right)^+ G(t,x,0,y)\, dydp \\
& + \int_{Q^2\times]0,1[^2} Gsgn_0^+ (0 - \hat{u_W} (q))\, f'[W(s,y)]\partial_y W(s,y) dpdq \\
& - \int_{Q^2\times]0,1[^2} Gsgn_0^+ (u_W (p))\, sgn_0^+ (0 - \hat{u_W} (q))\, f'[W(s,y)]\partial_y W(s,y) dpdq \\
=\ & \int_{Q^2\times]0,1[^2} \left(u_W^+ (p) - \hat{u_W}^+ (q)\right)^+ \partial_s G \\
& + F^+ \left(W(s,y) + u_W^+ (p), W(s,y) + \hat{u_W}^+ (q)\right) \partial_y G dpdq \\
& + \int_{Q^2\times]0,1[^2} Gsgn_0^+ \left(u_W^+ (p) - \hat{u_W} (q)\right) f'[W(s,y) + u_W^+ (p)]\partial_y W(s,y) dpdq \\
& + \int_{Q\times]0,1[} \left\{(0 - \hat{u_W} (q))^+ \partial_s G_n + F^+ (W(s,y) + 0, W(s,y) + \hat{u_W} (q))\, \partial_y G_n\right\} dq \\
& + \int_{Q\times]0,1[} G_n sgn_0^+ (0 - \hat{u_W} (q))\, f'[W(s,y)]\partial_y W(s,y) dq \\
& - \int_{\hat{u_W} \leq 0 \leq u_W} Gf'[W(s,y)]\partial_y W(s,y) dpdq \\
& + \int_{Q\times]0,1[} \int_D \left(u_W^+ (p) - u_{2,0} (y)\right)^+ G(t,x,0,y)\, dydp \\
=\ & \int_{Q^2\times]0,1[^2} \left(u_W^+ (p) - \hat{u_W}^+ (q)\right)^+ \partial_s G \\
& + F^+ \left(W(s,y) + u_W^+ (p), W(s,y) + \hat{u_W}^+ (q)\right) \partial_y G dpdq \\
& + \int_{Q^2\times]0,1[^2} Gsgn_0^+ \left(u_W^+ (p) - \hat{u_W} (q)\right) f'[W(s,y) + u_W^+ (p)]\partial_y W(s,y) dpdq \\
& + \int_{Q\times]0,1[} \int_D \left[\left(u_W^+ (p) - u_{2,0} (y)\right)^+ - (0 - u_{2,0} (y))^+\right] G(t,x,0,y)\, dydp \\
& + \int_0^1 \left\langle \hat{\mu}_0^-, G_n \right\rangle d\beta - \int_{\hat{u_W} \leq 0 \leq u_W} Gf'[W(s,y)]\partial_y W(s,y) dpdq
\end{aligned}
$$

Therefore, with the first inequality, one gets that

$$\begin{aligned}
0 \;\leq\; & \int_{Q^2\times]0,1[^2} \Big\{ \left(u_W^{+}(p) - \hat{u_W}^{+}(q)\right)^{+} (\partial_t + \partial_s)\, G \Big\}\, dpdq \\
& + \int_{Q^2\times]0,1[^2} \{ F^{+}\left(W(s,y) + u_W^{+}(p), W(s,y) + \hat{u_W}^{+}(q)\right) \partial_y G \\
& + F^{+}\left(W(t,x) + u_W^{+}(p), W(t,x) + \hat{u_W}^{+}(q)\right) \partial_x G \} dpdq \\
& - \int_{Q^2\times]0,1[^2} G sgn_0^{+}\left(u_W^{+}(p) - \hat{u_W}^{+}(q)\right) \\
& \times \left[f'[W(t,x) + \hat{u_W}^{+}(q)]\partial_x W(t,x) - f'[W(s,y) + u_W^{+}(p)]\partial_y W(s,y) \right] dpdq \\
& + \int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0}, G_n \right\rangle d\beta \\
& + \int_{Q\times]0,1[} \int_D \left(u_W^{+}(p) - u_{2,0}^{+}(y)\right)^{+} G(t,x,0,y)\, dydp \\
& - \int_{\hat{u_W} \leq 0 \leq u_W} G f'[W(s,y)]\partial_y W(s,y) dpdq,
\end{aligned}$$

i.e.

$$\begin{aligned}
0 \;\leq\; & \int_{Q^2\times]0,1[^2} \Big\{ \left(u_W^{+}(p) - \hat{u_W}^{+}(q)\right)^{+} \partial_s\varphi(s,y)\, \rho_n(y-x)\, \rho_l(s-t) \Big\}\, dpdq \\
& + \int_{Q^2\times]0,1[^2} F^{+}\left(W(s,y) + u_W^{+}(p), W(s,y) + \hat{u_W}^{+}(q)\right) \partial_y\varphi(s,y)\, \rho_n(y-x)\, \rho_l(s-t)\, dpdq \\
& + \int_{Q^2\times]0,1[^2} F^{+}\left(W(s,y) + u_W^{+}(p), W(s,y) + \hat{u_W}^{+}(q)\right) \varphi(s,y)\, \partial_y\rho_n(y-x)\, \rho_l(s-t)\, dpdq \\
& - \int_{Q^2\times]0,1[^2} F^{+}\left(W(t,x) + u_W^{+}(p), W(t,x) + \hat{u_W}^{+}(q)\right) \varphi(s,y)\, \partial_y\rho_n(y-x)\, \rho_l(s-t)\, dpdq \\
& - \int_{Q^2\times]0,1[^2} G sgn_0^{+}\left(u_W^{+}(p) - \hat{u_W}^{+}(q)\right) \\
& \times \left[f'[W(t,x) + \hat{u_W}^{+}(q)]\partial_x W(t,x) - f'[W(s,y) + u_W^{+}(p)]\partial_y W(s,y) \right] dpdq \\
& + \int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0}, G_n \right\rangle d\beta \\
& + \int_{Q\times]0,1[} \int_D \left(u_W^{+}(p) - u_{2,0}^{+}(y)\right)^{+} G(t,x,0,y)\, dydp - \int_{\hat{u_W} \leq 0 \leq u_W} G f'[W(s,y)]\partial_y W(s,y) dpdq.
\end{aligned}$$

1. First, the Lebesgue set properties ensure that

$$\begin{aligned}
& \lim_{n\to\infty} \lim_{l\to\infty} [E \int_{Q^2\times]0,1[^2} \Big\{ \left(u_W^{+}(p) - \hat{u_W}^{+}(q)\right)^{+} \partial_s\varphi(s,y)\, \rho_n(y-x)\, \rho_l(s-t) \Big\}\, dpdq \\
& + E \int_{Q^2\times]0,1[^2} F^{+}\left(W(s,y) + u_W^{+}(p), W(s,y) + \hat{u_W}^{+}(q)\right) \partial_y\varphi(s,y)\, \rho_n(y-x)\, \rho_l(s-t)\, dpdq \\
& = E \int_{Q\times]0,1[^2} \Big\{ \left(u_W^{+} - \hat{u_W}^{+}\right)^{+} \partial_t\varphi + F^{+}\left(W + u_W^{+}, W + \hat{u_W}^{+}\right) \partial_x\varphi \Big\}\, dpd\beta.
\end{aligned}$$

2. The expectation of the third and fourth terms vanish following the same arguments as the one proposed in the previous section.

3. Then, since $\hat{\mu}^{-}_{\mathbf{u},0}$ is a Radon measure in $\mathbb{R}^2$, the theorem of monotone convergence ensures that

$$\lim_{n\to\infty}\lim_{l\to\infty}\int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0}, G_n\right\rangle d\beta = \int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0}, \psi\right\rangle d\beta.$$

4. Denote by $\phi(t,x,y) = \int_t^T \rho_l(-r)\,dr\rho_n(y-x)\varphi(0,y) = \int_{\inf\left(t,\frac{2}{l}\right)}^{\frac{2}{l}} \rho_l(-r)\,dr\rho_n(y-x)\varphi(0,y)$. Since ϕ is a non-negative function of $\mathcal{D}([0,T[\times D)$ for any y in D as soon as n is large enough, and as $\mathbf{u}_1$ is a solution, one gets that

$$\begin{aligned}
&\int_{D\times Q\times]0,1[} \left\{\left(u^{+}_{W} - u^{+}_{2,0}(y)\right)^{+} \rho_l(-t)\rho_n(y-x)\varphi(0,y)\right\} dpdy \\
=\ &\int_{D\times Q\times]0,1[} \left\{\left(u_{W} - u^{+}_{2,0}(y)\right)^{+} \rho_l(-t)\rho_n(y-x)\varphi(0,y)\right\} dpdy \\
\leq\ &\int_{D\times Q\times]0,1[}\int_{\inf\left(t,\frac{2}{l}\right)}^{\frac{2}{l}} \rho_l(-r)\,dr\left\{-F^{+}\left(W+u_W, W+u^{+}_{2,0}(y)\right)\partial_y\rho_n(y-x)\varphi(0,y)\right\} dpdy \\
&+\int_{D\times Q\times]0,1[}\int_{\inf\left(t,\frac{2}{l}\right)}^{\frac{2}{l}} \rho_l(-r)\,dr\rho_n(y-x)\varphi(0,y)\,sgn^{+}_{0}\left(u_W - u^{+}_{2,0}(y)\right) f'[W+u_{2,0}(y)]dpdy \\
&+\int_{D^2}\left(u_{1,0}(x) - u^{+}_{2,0}(y)\right)^{+}\int_0^{\frac{2}{l}} \rho_l(-r)\,dr\rho_n(y-x)\varphi(0,y)\,dxdy.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\limsup_{n\to\infty}\ \limsup_{l\to\infty} E\int_{D\times Q\times]0,1[} \left\{\left(u^{+}_{W} - u^{+}_{2,0}(y)\right)^{+}\rho_l(-t)\rho_n(y-x)\varphi(0,y)\right\} dpdy \\
\leq\ &\int_D \left(u_{1,0} - u^{+}_{2,0}\right)^{+}\varphi(0,.)\,dx = \int_D \left(u^{+}_{1,0} - u^{+}_{2,0}\right)^{+}\varphi(0,.)\,dx
\end{aligned}$$

5.

$$\begin{aligned}
&\int_{u\hat{_W}<0<u_W} Gf'[W(s,y)]\partial_y W(s,y)dpdq \\
&= \int_{Q\times]0,1[} 1_{u\hat{_W}(q)<0} f'[W(s,y)]\partial_y W(s,y)\varphi(s,y) \\
&\times\int_{Q\times]0,1[} 1_{u_W(p)>0}\rho_l(s-t)\rho_n(y-x)\,dpdq \\
&\underset{l\to\infty,n\to\infty}{\rightarrow}\int_{Q\times]0,1[^2} 1_{u\hat{_W}<0<u_W} f'[W]\partial_x W\varphi dpd\beta.
\end{aligned}$$

Therefore we may conclude

$$\begin{aligned}
0\ \leq\ & E\int_{Q\times]0,1[^2}\left\{\left(u^{+}_{W} - u\hat{_W}^{+}\right)^{+}\partial_t\varphi + F^{+}\left(W+u^{+}_{W}, W+u\hat{_W}^{+}\right)\partial_x\varphi\right\} dpd\beta \\
&+\int_0^1\left\langle \hat{\mu}^{-}_{\mathbf{u},0},\psi\right\rangle d\beta + E\int_{u\hat{_W}<0<u_W} f'[W]\partial_x W\varphi dpd\beta + \int_D\left(u^{+}_{1,0} - u^{+}_{2,0}\right)^{+}\varphi(0,.)\,dx.
\end{aligned}$$

Now, repeating the same arguments with $\mathbf{u}_1$ replaced by $-\mathbf{u}_2$, $\mathbf{u}_2$ by $-\mathbf{u}_1$, f by $-f\left(-.\right)$, W by $-W$ and for the initial conditions $-u_{2,0}$ by $-u_{1,0}$, lead to the inequality:

$$\begin{aligned} 0 &\leq E\int_{Q\times]0,1[^2} \left(u_{\hat{W}}^{-} - u_{W}^{-}\right)^{+}\partial_t\varphi + F^{+}\left(W - u_{W}^{-}, W - u_{\hat{W}}^{-}\right)\partial_x\varphi dpd\beta \\ &+ \int_0^1 \left\langle \mu^{+}_{\mathbf{u},0},\psi\right\rangle d\alpha + E\int_{u_{\hat{W}}<0<u_W} Gf'[W]\partial_x W dpd\beta + \int_D \left(u_{2,0}^{-} - u_{1,0}^{-}\right)^{+}\varphi\left(0,.\right)dx. \end{aligned}$$

Summing up these two inequalities, one gets that

$$\begin{aligned} 0 &\leq E\int_{Q\times]0,1[^2} \left\{\left(u_W - u_{\hat{W}}\right)^{+}\partial_t\varphi + F^{+}\left(W + u_W, W + u_{\hat{W}}\right)\partial_x\varphi\right\} dpd\beta \\ &+ \int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0},\psi\right\rangle d\beta + \int_0^1 \left\langle \mu^{+}_{\mathbf{u},0},\psi\right\rangle d\alpha + \int_D \left(u_{1,0} - u_{2,0}\right)^{+}\varphi\left(0,.\right)dx. \end{aligned}$$

Note that $\left(\varphi\sigma_m\right)_m \subset \mathcal{D}\left([0,T[\times D\right)$ with $\varphi\sigma_n = \varphi\sigma_m\sigma_n$ for m large enough. Thus, on the one hand, thanks to Proposition 3.2, one has that

$$\begin{aligned} 0 &\leq E\int_{Q\times]0,1[^2} \left(\mathbf{u}_1\left(t,x,\alpha\right) - \mathbf{u}_2\left(t,x,\beta\right)\right)^{+}\partial_t\varphi\sigma_m dxdtd\alpha d\beta \\ &+ E\int_{Q\times]0,1[^2} F^{+}\left(\mathbf{u}_1\left(t,x,\alpha\right),\mathbf{u}_2\left(t,x,\beta\right)\right)\partial_x\left[\varphi\sigma_m\right] dxdtd\alpha d\beta. \end{aligned}$$

On the other hand, one has

$$\begin{aligned} 0 &\leq E\int_{Q\times]0,1[^2} \{\left(\mathbf{u}_1\left(t,x,\alpha\right) - \mathbf{u}_2\left(t,x,\beta\right)\right)^{+}\partial_t\varphi\left(1-\sigma_m\right) \\ &+ F^{+}\left(\mathbf{u}_1\left(t,x,\alpha\right),\mathbf{u}_2\left(t,x,\beta\right)\right)\partial_x\left[\varphi\left(1-\sigma_m\right)\right]\} dpd\beta \\ &+ \int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0},\psi\left(1-\sigma_m\right)\right\rangle d\beta + \int_0^1 \left\langle \mu^{+}_{\mathbf{u},0},\psi\left(1-\sigma_m\right)\right\rangle d\alpha \\ &+ \int_D \left(u_{1,0} - u_{2,0}\right)^{+}\varphi\left(0,x\right)\left(1-\sigma_m\right)dx. \end{aligned}$$

Thus, for any n,

$$\begin{aligned} 0 &\leq E\int_{Q\times]0,1[^2} \{\left(\mathbf{u}_1\left(t,x,\alpha\right) - \mathbf{u}_2\left(t,x,\beta\right)\right)^{+}\partial_t\varphi \\ &+ F^{+}\left(\mathbf{u}_1\left(t,x,\alpha\right),\mathbf{u}_2\left(t,x,\beta\right)\right)\partial_x\varphi\} dpd\beta \\ &+ \int_0^1 \left\langle \hat{\mu}^{-}_{\mathbf{u},0},\psi\left(1-\sigma_m\right)\right\rangle d\beta + \int_0^1 \left\langle \mu^{+}_{\mathbf{u},0},\psi\left(1-\sigma_m\right)\right\rangle d\alpha \\ &+ \int_D \left(u_{1,0} - u_{2,0}\right)^{+}\varphi\left(0,x\right)\left(1-\sigma_m\right)dx. \end{aligned}$$

Since $G_n(1-\sigma_m) = \varphi\sigma_n - \varphi\sigma_m\sigma_n = 0$ for m large,

$$\begin{aligned}
&\int_0^1 \left\langle \hat{\mu}^-_{\mathbf{u},0}, \psi(1-\sigma_m)\right\rangle d\beta \\
=\ & \lim_{n\to\infty} \int_{Q\times]0,1[} \{(0-\hat{u_W}(q))^+ \partial_s G_n(1-\sigma_m) \\
&+ F^+(W(s,y)+0, W(s,y)+\hat{u_W}(q))\,\partial_y [G_n(1-\sigma_m)]\}dq \\
&+ \int_{Q\times]0,1[} G_n(1-\sigma_m)\, sgn_0^+(0-\hat{u_W}(q))\, f'[W(s,y)]\partial_y W(s,y) dq \\
&+ \int_D (-u_0)^+ G_n(0,x)(1-\sigma_m)\,dx \\
=\ & 0.
\end{aligned}$$

Then, using the partition of unity, the result holds.

3.4 Uniqueness of the Measure-Valued Solution: Existence of Entropy Solution

Proposition 3.4. *The measure-valued solution is unique. Moreover, it is the unique entropy solution.*

Proof. Since for any non-negative φ in $H^1(Q)$,

$$\begin{aligned}
0 \ \leq\ & E\int_{Q\times]0,1[^2} (\mathbf{u}_1(t,x,\alpha) - \mathbf{u}_2(t,x,\beta))^+ \partial_t\varphi dxdtd\alpha d\beta \\
&+ E\int_{Q\times]0,1[^2} F^+(\mathbf{u}_1(t,x,\alpha), \mathbf{u}_2(t,x,\beta))\,\partial_x\varphi dxdtd\alpha d\beta \\
&+ \int_D (u_{1,0} - u_{2,0})^+ \varphi(0,.)\,dx.
\end{aligned}$$

if $u_{1,0} = u_{2,0}$ and $\varphi(t,x) = T-t$, one gets that

$$0 \geq E\int_{Q\times]0,1[^2} (\mathbf{u}_1(t,x,\alpha) - \mathbf{u}_2(t,x,\beta))^+ dxdtd\alpha d\beta,$$

and, by permutation of the solutions,

$$0 \geq E\int_{Q\times]0,1[^2} (\mathbf{u}_2(t,x,\alpha) - \mathbf{u}_1(t,x,\beta))^+ dxdtd\alpha d\beta.$$

Therefore, on the one hand, the uniqueness of the measure-valued solution is proved

and, on the other hand, $\mathbf{u}_1(t,x,\alpha) = \mathbf{u}_2(t,x,\beta)$ for a.e. α and β ensures that the solution does not depend on α or β. ■

Proposition 3.5. *Moreover, entropy solutions satisfy a comparison and a contraction principle:*

1. *If* $u_{1,0} \leq u_{2,0}$ *then* $u_1 \leq u_2$ *a.e. on* Q*, a.s. on* Ω.
2. $E \int_Q |u_1 - u_2| \, dxdt \leq \int_D |u_{1,0} - u_{2,0}| \, dx$.

Proof. The first part of the Proposition is proved in the same way as Proposition 3.4

If $u_{1,0} \leq u_{2,0}$ and $u_1 > u_2$,$\varphi(t,x) = T - t$, one gets that

$$0 \geq E \int_Q (u_1 - u_2) \, dxdt,$$

Therefore,

$$0 \geq u_1 - u_2$$

contradiction with $u_1 > u_2$. and the first assertion follows.

(2) For any non-negative φ in $H^1(]0,T[)$ with $\varphi(T) = 0$, one has that

$$\begin{aligned} 0 &\leq \int_0^T E \int_D (u_1 - u_2)^+ \, dx \varphi'(t) \, dt + \int_D (u_{1,0} - u_{2,0})^+ \varphi(0) \, dx \\ &= \int_0^T E \int_D \left[(u_1 - u_2)^+ - (u_{1,0} - u_{2,0})^+ \right] dx \varphi'(t) \, dt, \end{aligned}$$

and the second assertion follows. ■

3.4.1 A remark about locally Lipschitz f

Assume in this section that f is merely a locally Lipschitz-continuous function with a Lipschitz-continuous derivative f' . Then, one has in particular:

1. $\exists c(f)$, $\forall x \in \mathbb{R}$, $|f(x)| \leq c(f)\left(x^2 + 1\right)$
2. By Sobolev embedding, $\forall w \in H^1(D)$, $f(w) \in L^p(D)$ for some $p > 1$.
3. By truncation arguments, $\forall w \in H^1(D)$, $f'(w) w'(x) \in L^p(D)$ for some $p > 1$ and the chain rule holds: : $\partial_x f(w) = f'(w) w'(x)$.

In this case, the definition of a solution has to be slightly modified in order to give sense to the integrals: the test-functions φ need to belong to $\mathcal{D}\left(\bar{Q}\right)$ instead of $H^1(Q)$ or to $\mathcal{D}([0,T] \times D)$ instead of $L^2\left(0,T; H_0^1(D)\right)$. The result of uniqueness holds in the same way with such f, as well as the main part of the demonstrations of the existence section. If one assumes again the existence of the solution to the viscous problem (3.2), it remains to prove the property of uniform integrability of the sequence $(f(u_n))$ needed when one passes to the limit in the first term of $I_{3,\eta}$ in (3.4). The aim of the following lemma is to propose a possible assumption for that.

Lemma 3.9. *If* $\exists\delta \in \,]0,2[\,, \exists c > 0,$ *such that* $\forall x \in \mathbb{R},\ |f(x)| \leq c(|x|^{\delta} + 1)$ *then* $(f\,(u_n))$ *is uniformly integrable.*

Proof. If $\exists\delta \in \,]0,2[\,, \exists c > 0$, such that $\forall x \in \mathbb{R}$, $|f(x)| \leq c(|x|^{\delta} + 1)$, the sequence is uniformly integrable since it is bounded in $L^{\frac{2}{\delta}}\,(]0,T[\, \times \Omega \times D)$ with $\dfrac{2}{\delta} > 1$. ■

Chapter 4

Numerical Approximation of Stochastic Generalized Burgers Equation on a Bounded Domains

4.1 Introduction

We wish to find an approximate solution to the following nonlinear scalar conservation law with a stochastic additive force, posed over a bounded domain D with initial condition and Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u(\omega,t,x)}{\partial t} + \dfrac{\partial\left(f\left(u(\omega,t,x)\right)\right)}{\partial x} & = \dfrac{\partial W}{\partial t} \quad \text{in } \Omega \times \left]0,T\right[\times D, \\ u(\omega,0,x) & = u_0(x) \quad \omega \in \Omega,\ x \in D, \\ u(\omega,t,x) & = 0 \quad \omega \in \Omega,\ t \in \left]0,T\right[,\ x \in \partial D \end{cases} \tag{4.1}$$

where $D = \left]0,2\pi\right[$, $T > 0$ and W is a cylindrical Wiener process.
Recall that the cylindrical Wiener process can be written as

$$W(t,x) = \sum_{k=1}^{\infty} \beta_k(t)\, e_k(x) \tag{4.2}$$

where $\{e_k\}$ is any orthonormal basis of $L^2(0,2\pi)$ and $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions in a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The series (4.2) does not converge in $L^2(0,2\pi)$ but in any larger Hilbert space U such that the embedding $L^2(0,2\pi) \subset U$ is Hilbert-Schmidt. In order to make the lecture more fluent, we omit in the sequel the variables ω, x, t and write u instead of $u(\omega,t,x)$.

4.2 New formulation of the problem

We take advantage of the fact that the noise is additive (independent on u) to rewrite the equation in problem 4.1 as

$$\frac{\partial\left(u-W\right)}{\partial t}+\frac{\partial\left(f\left(u\right)\right)}{\partial x}=0 \qquad \text{in } \mathcal{D}'\left(Q\right), \tag{4.3}$$

equation (4.3) can be formulated, via the change of variable $u_W := u - W$, as the random problem

$$\frac{\partial u_W}{\partial t}+\frac{\partial\left(B\left(\omega,t,x,u_W\right)\right)}{\partial x}=0$$

with a flux function $B\left(\omega,t,x,u_W\right) := f\left(u_W+W\right)$. In this equation, the stochastic variable ω, at least formally, only plays the role of a parameter and thus essentially deterministic techniques can be applied (though it is not possible to use exclusively pathwise arguments). Thus problem (4.1), become

$$\begin{cases} \dfrac{\partial u_W}{\partial t}+\dfrac{\partial\left(B\left(\omega,t,x,u_W\right)\right)}{\partial x} & =0 \qquad \text{in } \Omega\times D\times\left]0,T\right[, \\ u_W\left(\omega,0,x\right) & =u_0\left(x\right) \qquad \omega\in\Omega,\ x\in D, \\ u_W\left(\omega,t,x\right) & =-W\left(\omega,t,x\right)=u_b \qquad \omega\in\Omega,\ t\in\left]0,T\right[,\ x\in\partial D \end{cases} \tag{4.4}$$

Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. The mathematical challenge consists in introducing a selection criterion in order to identify a unique solution. The notion of entropy solution was first introduced in the 70s by S.N. Kruzkhov in the case where the domain was the whole space. In the present work we consider a stochastic version of the entropy condition proposed by F. Otto in his PhD (see [35]) to take into account our non-homogeneous Dirichlet boundary conditions. We assume the following hypotheses:

H_1: $u_0\in L^\infty\left(D\right)$.

H_2: $u_b\in L^\infty\left(\left]0,T\right[\times\partial D\right)$.

H_3: $B:\Omega\times\left]0,T\right[\times D\times\mathbb{R}\to\mathbb{R}$ is locally Lipschitz-continuous.

H_4: $\partial_x B\left(\omega,t,x,u_W\right)=0$ for a.e. $\left(t,x,u_W\right)\in\left]0,T\right[\times D\times\mathbb{R}$.

4.2.1 Goal of the study and outline of the chapter

The aim of this chapter is to propose a semi-discrete finite volume approximation for Problem (4.4). Then, discretize the resulting system of stochastic ordinary differential equations in time using the explicit Euler schemes and studying the convergence of this scheme.

The chapter is organized as follows. In section 2 we define a weak stochastic entropy solution and a measure-valued entropy solution of Problem (4.4); the class of entropy-flux pairs considered in the definition of stochastic entropy solution can be reduced to the one of the so-called “semi Kruzkov” entropies. It is one of the keys of the result of convergence of the scheme.
In Section 3 we define the finite volume scheme with monotone fluxes used to approximate the stochastic entropy solution of (4.4). Then, in section 4 we give the main result of this chapter, which states the convergence of the approximate solution towards the unique stochastic entropy solution of the equation. The remainder of the chapter is devoted to the proof of this convergence result. In Section 5, several preliminary results satisfied by the finite volume approximate solution denoted $u_{\mathcal{T},k}$ are stated. In Section 6 we show the convergence of $u_{\mathcal{T},k}$ towards the unique stochastic entropy solution of Problem (4.1), and therefore we get a new proof for the existence of such a solution. And in section 7 we give some numerical simulation.

4.2.2 General notations

- $Q =]0,T[\times [0,2\pi]$.
- $E[.]$ denotes the expectation, i.e. the integral over Ω with respect to the probability measure $\mathbb{P}$.
- We denote by $LipB$ the Lipschitz constants of B.
- sgn_0^+ denote the application $\mathbb{R} \to \mathbb{R}$ defined by

$$sgn_0^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases}$$

  and $sgn_0^-(s)$ the application $s \mapsto -sgn_0^+(-s)$. As usual, we set $s^+ = sgn_0^+(s)\, s$ and $s^- = (-s)^+$.

Let $\kappa \in \mathbb{R}$. The entropy-flux pair $(\eta_\kappa^+, \phi_\kappa^+)$ (respectively $(\eta_\kappa^-, \phi_\kappa^-)$) is defined by

$$\begin{cases} \eta_\kappa^+(s) = (s-\kappa)^+, \\ \phi_\kappa^+(\omega,t,x,s) = sgn_0^+(s-\kappa)\left(B(\omega,t,x,s) - B(\omega,t,x,\kappa)\right), \end{cases}$$

$$\left(\text{respectively} \begin{cases} \eta_\kappa^-(s) = (s-\kappa)^-, \\ \phi_\kappa^-(\omega,t,x,s) = sgn_0^-(s-\kappa)\left(B(\omega,t,x,s) - B(\omega,t,x,\kappa)\right) \end{cases}\right).$$

4.3 The continuous problem

Next we define a weak stochastic entropy solution of Problem (4.4) as it has been done in [40] :

Definition 4.1 (Stochastic entropy solution). *A function u_W of $L^2(\Omega \times Q)$, adapted to the filltration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is an entropy solution of the stochastic scalar conservation law (4.4) with the initial condition $u_0 \in L^\infty(D)$, if $\mathbb{P}$-a.s in Ω, For any $\kappa \in \mathbb{R}$ and for any $\varphi \in C_c^\infty\left([0,T[\times [0,2\pi], \mathbb{R}^+\right)$,*

$$\begin{aligned}
0 \leq & \int\int_Q \left\{(u_W-\kappa)^+ \partial_t\varphi + \left[B(\omega,t,x,u_W) - B(\omega,t,x,\kappa)\right] sgn_0^+(u_W-\kappa)\,\partial_x\varphi\right\} dtdx \\
& -\int_Q \varphi sgn_0^+(u_W-\kappa)\,\partial_x B(\omega,t,x,\kappa)\,dtdx + \int_D (u_0-\kappa)^+\varphi(0,x)\,dx \\
& + LipB \int_0^T \left[(-W(t,x)-\kappa)^+\varphi(t,x)\right]_0^{2\pi} dt
\end{aligned}$$

and

$$\begin{aligned}
0 \leq & \int\int_Q \left\{(u_W-\kappa)^- \partial_t\varphi + \left[B(\omega,t,x,u_W) - B(\omega,t,x,\kappa)\right] sgn_0^-(u_W-\kappa)\,\partial_x\varphi\right\} dtdx \\
& -\int_Q \varphi sgn_0^-(u_W-\kappa)\,\partial_x B(\omega,t,x,\kappa)\,dtdx + \int_D (u_0-\kappa)^-\varphi(0,x)\,dx \\
& + LipB \int_0^T \left[(-W(t,x)-\kappa)^-\varphi(t,x)\right]_0^{2\pi} dt
\end{aligned}$$

For technical reasons, as in [5] for the case $D = \mathbb{R}^n$ and as in [40] for the deterministic case, we also need to consider a more general notion of solution.

In fact, in a first step, we will only prove the convergence of the finite volume approximate solution $u_{\tau,k}$ to a stochastic measure-valued entropy solution.

Then, thanks to the result of uniqueness stated in Proposition 3.4, we will be able to deduce the convergence of $u_{\tau,k}$ to the unique stochastic entropy solution of Problem (4.1).

Definition 4.2 (Stochastic measure-valued entropy solution). *A function $\mathbf{u}_W$ of $L^2(\Omega \times Q \times]0,1[)$, adapted to the filltration $\mathcal{F}_t$ as an $L^2(D)$-valued function, is a measure-valued entropy solution of the stochastic scalar conservation law (4.4) with the initial condition $u_0 \in L^\infty(D)$, if $\mathbb{P}$-a.s in Ω, for any $\kappa \in \mathbb{R}$ and for any $\varphi \in C_c^\infty\left([0,T[\times [0,2\pi], \mathbb{R}^+\right)$,*

$$
\begin{aligned}
0 \ \leq\ & \int\int_Q\int_0^1 \left\{(\mathbf{u}_W-\kappa)^+\partial_t\varphi+\left[B(\omega,t,x,\mathbf{u}-\mathbf{W})-B(\omega,t,x,\kappa)\right] sgn_0^+(\mathbf{u}_W-\kappa)\,\partial_x\varphi\right\} d\alpha dt dx \\
& -\int_Q \varphi sgn_0^+(\mathbf{u}_W-\kappa)\,\partial_x B(\omega,t,x,\kappa)\,dtdx+\int_D (u_0-\kappa)^+\varphi(0,x)\,dx \\
& +LipB\int_0^T \left[(-\mathbf{W}(t,x)-\kappa)^+\varphi(t,x)\right]_0^{2\pi} dt
\end{aligned}
$$

The same entropy inequality holds when the negative entropy-flux is selected as an entropy-flux pair.

4.4 Main Result

In the sequel, assume that assumptions H_1 to H_4 hold. Let us first give a definition of the admissible meshes for the finite volume scheme.

4.4.1 Meshes and scheme

Definition 4.3 (Admissible mesh). *An admissible mesh of* $D = \left]0,2\pi\right[$, *denoted by* τ,*for the discretization of Problem* (4.4), *is given by a family* $(K_i)_{i=1,...N}$, $N\in\mathbb{N}^*$, *such that* $K_i = \left]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right[$, *and a family* $(x_i)_{i=0,...N+1}$ *such that*

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < .... < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < .... < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 2\pi.$$

The mesh τ *is the set* $\tau = \{K_i, i\in\{1,...N\}\}$ *of subsets of* $\mathbb{R}$. *The length of* K_i *is denoted by* h_i, *so that* $h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ *for all* $i \in \{1,...N\}$. *It is assumed that* $h = size(\tau) = \sup\{h_i, i\in\{1,...N\}\} < +\infty$ *and that, for some* $\alpha\in\mathbb{R}_+^*$, *we have*

$$\alpha h \leq m(K_i) \qquad, \forall K_i\in\tau, \tag{4.5}$$

where we denote by

- $m(K_i)$ the onedimensional Lebesgue measure of K_i.
- $\mathcal{N}(K_i)$ the set of control volumes neighbors of the control volume K_i.
- $\sigma_{K_iL_j}$ the common interface between K_i and L_j for any $L_j\in\mathcal{N}(K_i)$.

Remark 4.1. *Since* $m(D) = \sum_{i=1}^{N} m(K_i)$, *Assumption* (4.5) *yields the following estimate on the number of control volumes:*

$$card(\tau) \leq \frac{m(D)}{\alpha h} \tag{4.6}$$

We now define the general monotone scheme. Consider an admissible mesh τ in the sense of Definition 4.3.
We take $M \in \mathbb{N}^*$and define the time step $k = \dfrac{T}{M} \in \mathbb{R}_+^*$ and $t^m = mk$ for all $m \in \{0, ..., M-1\}$. The equations satisfied by the discrete unknowns denoted by $u_{W,i}^m$, $m \in \{0, ..., M-1\}$, $i = 1...N$, are obtained by discretizing Problem (4.4). For the discretization of such a problem, we need to define the numerical flux.

Definition 4.4 (Monotone numerical flux)**.** *We say that a function* $F_{K_iL_j}^m : \mathbb{R}^2 \to \mathbb{R}$ *(for* $(K_i, L_j) \in \tau^2$, $L_j \in \mathcal{N}(K_i)$, *for all* $m \in \{0, ..., M-1\}$*) is a monotone numerical flux if it satisfies the following properties of monotony, conservativity, regularity and consistency :*

(i) $F_{K_iL_j}^m(a,b)$ *is nondecreasing with respect to a and nonincreasing with respect to b, for* $(a,b) \in \mathbb{R}^2$*,*

(ii) $F_{K_iL_j}^m(a,b) = -F_{L_jK_i}^m(b,a)$ *for all* $(a,b) \in \mathbb{R}^2$

(iii) There exists $F_1, F_2 > 0$ *such that for any* $a, b \in \mathbb{R}$ *we have*

$$\left|F_{K_iL_j}^m(b,a) - F_{K_iL_j}^m(a,a)\right| \leq F_1 |a-b| \quad \textit{and} \quad \left|F_{K_iL_j}^m(a,b) - F_{K_iL_j}^m(a,a)\right| \leq F_2 \ |a-b|.$$

(iv) $F_{K_iL_j}^m(a,a) = \dfrac{1}{k}\displaystyle\int_{t^m}^{t^{m+1}} B\left(\omega, \sigma_{K_iL_j}, t, a\right) dt$ *for all* $a \in \mathbb{R}$*.*

Notice that the property of consistency (iv) above gives, owing to $\partial_x B(\omega, t, x, u_W) = 0$,

$$\sum_{L_j \in \mathcal{N}(K_i)} F_{K_iL_j}^m(a,a) = 0 \quad \forall K_i \in \tau, \quad \forall m \in \{0, ..., M-1\}, \ \forall a \in \mathbb{R}.$$

The set $\left\{u_i^0, i = 1, ...N\right\}$ is given by the initial condition

$$u_i^0 = \frac{1}{m(K_i)} \int_{K_i} u_0(x)\, dx, \forall K_i \in \tau. \tag{4.7}$$

The discrete unknowns are denoted by $u_{W,i}$, $i = 1...N$, and are expected to be an approximation of the mean value of $u - W$ over K_i, that is

$$u_{W,i}(t) = \frac{1}{m(K_i)} \int_{K_i} u_W(\omega, t, x)\, dx, \forall K_i \in \tau.$$

and

$$\frac{\partial u_{W,i}(t)}{\partial t} = \frac{1}{m(K_i)} \int_{K_i} \frac{\partial u_W(\omega, t, x)}{\partial t} dx, \forall K_i \in \tau.$$

Where, $u_{W,i}(t) = u_i(t) - W_i^J(t)$, u_i, $i = 1...N$ are the discrete unknowns of problem (4.1), and are expected to be an approximation of the mean value of u over K_i, $W_i^J(t) = \displaystyle\sum_{j=1}^{J} \beta_j(t)\, e_i^j$ for $J \in \mathbb{N}^*$, and $e_i^j = \dfrac{1}{m(K_i)} \displaystyle\int_{K_i} e_j(x)\, dx$.

The integration of equation (4.4) over the control volume $K_i = \left]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right[$ results

$$m\left(K_i\right)\frac{\partial u_{W,i}\left(t\right)}{\partial t} + \left(B\left(\omega, x_{i+\frac{1}{2}}, t, u_W\right) - B\left(\omega, x_{i-\frac{1}{2}}, t, u_W\right)\right) = 0\ . \tag{4.8}$$

The equation (4.8) can be written as

$$m\left(K_i\right)\frac{\partial u_{W,i}\left(t\right)}{\partial t} + \left(F^m_{K_iL_j}\left(u_{W,i}, u_{W,i+1}\right) - F^m_{K_iL_j}\left(u_{W,i-1}, u_{W,i}\right)\right) = 0 \tag{4.9}$$

Notice that the semi-discrete equation (4.9) can be formulated as a system of stochastic differential equations (SDEs) rewritten in a compact form as

$$\begin{aligned} \frac{du_W}{dt} &= H\left(t, u_W\right) \\ u\left(0\right) &= u_0 \end{aligned}$$

where

$$H_i\left(t, u_W\right) = \frac{1}{m\left(K_i\right)}\left(F^m_{K_iL_j}\left(u_{W,i-1}, u_{W,i}\right) - F^m_{K_iL_j}\left(u_{W,i}, u_{W,i+1}\right)\right)\ .$$

We then discretize the semi-discrete equation (4.9) in time using the explicit Euler schemes, we obtain

$$\frac{m\left(K_i\right)}{k}\left(u^{m+1}_{W,i} - u^m_{W,i}\right) + \left(F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right) - F^m_{K_iL_j}\left(u^m_{W,i-1}, u^m_{W,i}\right)\right) = 0 \tag{4.10}$$

for any $i = 1, ...N$, and any $m \in \{0, ..., M-1\}$,

where,

$$u^m_{W,0} = -\sum_{j=1}^{J}\beta_j\left(t^m\right)e_j\left(0\right), \quad \text{and} \quad u^m_{W,2\pi} = -\sum_{j=1}^{J}\beta_j\left(t^m\right)e_j\left(2\pi\right), \tag{4.11}$$

The approximate finite volume solution $u_{W,\tau,k}$ may be defined on $\Omega \times D \times [0, T[$ from the discrete unknowns $u^m_{W,i}$, $i = 1, ...N$, $m \in \{0, ..., M-1\}$ which are computed in (4.10) by:

$$u_{W,\tau,k}\left(\omega, t, x\right) = u^m_{W,i} = u^m_i - W^{J,m}_i = u_{\tau,k} - W_{J,\tau,k} \tag{4.12}$$

for $\omega \in \Omega$, $x \in K_i$ and $t \in [mk, (m+1)k[$. Where u^m_i is the discrete unknowns of problem (4.1), and $W_{J,\tau,k} = W^{J,m}_i\left(t, x\right) = \sum_{j=1}^{J}\beta_j\left(t^m\right)e^j_i$ is the discrete noise terms.

4.4.2 Main result

We now state the main result of this chapter.

Theorem 4.1. *[40] [Convergence to the stochastic entropy solution] Assume that hypotheses H_1 to H_4 hold.*

Let τ be an admissible mesh in the sense of Definition 4.3, $M \in \mathbb{N}^*$ and $k = \dfrac{T}{M} \in \mathbb{R}_+^*$ be the time step. Let $u_{W,\tau,k}$ be the finite volume approximation defined by the monotone finite volume scheme (4.10) *and* (4.12)*. Then $u_{W,\tau,k}$ converges to the unique stochastic entropy solution of* (4.4) *in the sense of Definition 4.1, in $L^p_{loc}(\Omega \times Q)$ for any $1 \leq p < 2$ as $h \to 0$.*

4.5 Preliminary results on the finite volume approximation

Let us state in this section several results satisfied by the finite volume approximate solution $u_{W,\tau,k}$ defined by (4.10) and (4.12).

4.5.1 Stability estimates

Proposition 4.1. $\left(L_t^\infty L_{\omega,x}^2\right.$ *estimate*$\left.\right)$ *Let $T > 0$, $u_0 \in L^2(D)$, τ be an admissible mesh in the sense of Definition* (4.3)*, $M \in \mathbb{N}^*$ and $k = \dfrac{T}{M} \in \mathbb{R}_+^*$ satisfying the Courant-Friedrichs-Levy (CFL) Condition*

$$k \leq \frac{\alpha h}{2(F_1 + F_2)}. \tag{4.13}$$

Let $u_{W,\tau,k}$ be the finite volume approximate solution defined by (4.10) *and* (4.12)*. Then we have the following bound*

$$\|u_{W,\tau,k}(\omega,t,x)\|_{L^\infty(0,T;L^2(\Omega\times D))} \leq C,$$

where

$$C = \left(\|u_0\|^2_{L^2(D)} + \pi T\right)^{\frac{1}{2}} + (2\pi T J)^{\frac{1}{2}}.$$

As a consequence we get

$$\|u_{W,\tau,k}(\omega,t,x)\|^2_{L^2(\Omega\times Q)} \leq TC^2,$$

Proof. First one has

$$\begin{aligned}\sum_{i=1}^{N} m(K_i) E\left[\left(u_i^0\right)^2\right] &= \sum_{i=1}^{N} m(K_i) E\left[\left(\frac{1}{m(K_i)}\int_{K_i} u_0(x)\,dx\right)^2\right] \\ &\leq \|u_0\|_{L^2(D)}^2 .\end{aligned}$$

Set $m \in \{0, ..., M-1\}$. Let us multiply the numerical scheme (4.10) by u_i^m, we thus get

$$\begin{aligned}\frac{m(K_i)}{k}\left[u_i^{m+1} - u_i^m\right] u_i^m &= \left\{F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right) - F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right)\right\} u_i^m \\ &\quad + \frac{m(K_i)}{k}\left[W_i^{m+1} - W_i^m\right] u_i^m\end{aligned}$$

And by using the formula $ab = \frac{1}{2}\left[(a+b)^2 - a^2 - b^2\right]$ with $a = u_i^{m+1} - u_i^m$ and $b = u_i^m$ we obtain

$$\begin{aligned}&\frac{m(K_i)}{2}\left[\left(u_i^{m+1}\right)^2 - \left(u_i^m\right)^2 - \left(u_i^{m+1} - u_i^m\right)^2\right] \\ = \;& k\left\{F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right) - F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right)\right\} u_i^m \\ &+ m(K_i)\left[W_i^{m+1} - W_i^m\right] u_i^m,\end{aligned}$$

and then

$$\begin{aligned}\frac{m(K_i)}{2}\left[\left(u_i^{m+1}\right)^2 - \left(u_i^m\right)^2\right] &= \frac{m(K_i)}{2}\left(u_i^{m+1} - u_i^m\right)^2 \\ &\quad + k\left\{F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right) - F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right)\right\} u_i^m \\ &\quad + m(K_i)\left[W_i^{m+1} - W_i^m\right] u_i^m.\end{aligned}$$

Using the finite volume scheme (4.10) we can replace $\left(u_i^{m+1} - u_i^m\right)^2$ and we take then the expectation. Thanks to the independance between the random variables $\left(W_i^{m+1} - W_i^m\right)$ and u_i^m, together with the equality $E\left[\left(W_i^{m+1} - W_i^m\right)^2\right] = k$, we get

$$\begin{aligned}\frac{m(K_i)}{2} E\left[\left(u_i^{m+1}\right)^2 - \left(u_i^m\right)^2\right] &= E\left[\frac{k^2}{2m(K_i)}\left(F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right) - F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right)\right)^2\right] \\ &\quad + \frac{m(K_i)}{2} E\left[\left(W_i^{m+1} - W_i^m\right)^2\right] \\ &\quad - kE\left[\left(F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right) - F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right)\right)\left(W_i^{m+1} - W_i^m\right)\right] \\ &\quad + kE\left[\left\{F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right) - F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right)\right\} u_i^m\right] \\ &\quad + m(K_i) E\left[\left(W_i^{m+1} - W_i^m\right) u_i^m\right] \\ &= \frac{k^2}{2m(K_i)} E\left[\left(F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right) - F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right)\right)^2\right] + \frac{km(K_i)}{2} \\ &\quad + kE\left[\left\{F_{K_iL_j}^m\left(u_{i-1}^m, u_i^m\right) - F_{K_iL_j}^m\left(u_i^m, u_{i+1}^m\right)\right\} u_i^m\right].\end{aligned}$$

This equality can be rewritten as, after summing for $i = 1, ..., N$,

$$\sum_{i=1}^{N} \frac{m(K_i)}{2} E\left[\left(u_i^{m+1}\right)^2 - \left(u_i^m\right)^2\right] = B_1 - B_2 + \sum_{i=1}^{N} \frac{km(K_i)}{2}, \tag{4.14}$$

where

$$\begin{aligned} B_1 &= \sum_{i=1}^{N} \frac{k^2}{2m(K_i)} E\left[\left(F^m_{K_iL_j}\left(u^m_{i-1}, u^m_i\right) - F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right)\right)^2\right] \\ \text{and} \quad B_2 &= \sum_{i=1}^{N} kE\left[\left\{F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_{i-1}, u^m_i\right)\right\} u^m_i\right]. \end{aligned}$$

- Study of B_1: Using Cauchy-Schwarz inequality, we get the following estimate

$$\begin{aligned} B_1 &= \sum_{i=1}^{N} \frac{k^2}{2m(K_i)} E\left[\left(\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{i-1}, u^m_i\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right) \\ -\left(F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right) \end{array}\right)^2\right] \\ &\le \sum_{i=1}^{N} \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{i-1}, u^m_i\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right)^2 \end{array}\right] \\ &\le \sum_{i=1}^{N-1} \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_{i+1}, u^m_{i+1}\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right)^2 \end{array}\right] \\ &\quad + \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_0, u^m_1\right) - F^m_{K_iL_j}\left(u^m_1, u^m_1\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_N, u^m_{N+1}\right) - F^m_{K_iL_j}\left(u^m_N, u^m_N\right)\right)^2 \end{array}\right] \end{aligned} \tag{4.15}$$

This finally gives:

$$B_1 \le B_{1,1} + B_{1,2},$$

where

$$B_{1,1} = \sum_{i=1}^{N-1} \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_{i+1}, u^m_{i+1}\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right)^2 \end{array}\right]$$

and

$$B_{1,2} = \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_0, u^m_1\right) - F^m_{K_iL_j}\left(u^m_1, u^m_1\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_N, u^m_{N+1}\right) - F^m_{K_iL_j}\left(u^m_N, u^m_N\right)\right)^2 \end{array}\right].$$

- Study of B_2: We introduce the term $B_{2,1}$ defined by

$$B_{2,1} = kE\left[\begin{array}{c} \left\{F^m_{K_iL_j}\left(u^m_N, u^m_{N+1}\right) - F^m_{K_iL_j}\left(u^m_N, u^m_N\right)\right\} u^m_N \\ -\left\{F^m_{K_iL_j}\left(u^m_0, u^m_1\right) - F^m_{K_iL_j}\left(u^m_1, u^m_1\right)\right\} u^m_1 \end{array}\right].$$

We have then

$$B_2 - B_{2,1} = k\sum_{i=1}^{N-1} E\left[\begin{array}{c} \left\{F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_i, u^m_i\right)\right\} u^m_i \\ -\left\{F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right) - F^m_{K_iL_j}\left(u^m_{i+1}, u^m_{i+1}\right)\right\} u^m_{i+1} \end{array}\right].$$

Denoting by $\Psi^m_{K_iL_j}$ the function defined for any $a \in \mathbb{R}$ by $\Psi^m_{K_iL_j}(a) = \int_0^a s\frac{d}{ds}\left(F^m_{K_iL_j}(s,s)\right)ds$, an integration by parts yields, for all $(a,b) \in \mathbb{R}^2$

$$\begin{aligned}\Psi^m_{K_iL_j}(b) - \Psi^m_{K_iL_j}(a) &= b\left(F^m_{K_iL_j}(b,b) - F^m_{K_iL_j}(a,b)\right) - a\left(F^m_{K_iL_j}(a,a) - F^m_{K_iL_j}(a,b)\right) \\ &\quad - \int_a^b \left(F^m_{K_iL_j}(s,s) - F^m_{K_iL_j}(a,b)\right)ds.\end{aligned}$$

Using this formula, we define $B_{2,2}$ and $B_{2,3}$ by

$$B_{2,2} = E\left[\sum_{i=1}^{N-1} k\int_{u^m_i}^{u^m_{i+1}} \left(F^m_{K_iL_j}(s,s) - F^m_{K_iL_j}\left(u^m_i, u^m_{i+1}\right)\right)ds\right]$$

and

$$B_{2,3} = -E\left[\sum_{i=1}^{N-1} k\left(\Psi^m_{K_iL_j}\left(u^m_i\right) - \Psi^m_{K_iL_j}\left(u^m_{i+1}\right)\right)\right] = E\left[k\left(\Psi^m_{K_iL_j}\left(u^m_N\right) - \Psi^m_{K_iL_j}\left(u^m_1\right)\right)\right].$$

We have then split B_2 into three terms:

$$B_2 = B_{2,1} + B_{2,2} + B_{2,3}.$$

we first introduce the following technical lemma from [22], which will be used several times in the sequel :

Lemma 4.1. *Let $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ be a monotone Lipschitz-continuous function with a Lipschitz constant $C_{\mathcal{G}} > 0$. then:*

$$\left|\int_c^d \mathcal{G}(t) - \mathcal{G}(c)\,dt\right| \geq \frac{1}{2C_{\mathcal{G}}}\left(\mathcal{G}(d) - \mathcal{G}(c)\right)^2, \ \ \forall c, d \in \mathbb{R}.$$

Thanks to this lemma, we estimate $B_{2,1}$ by treating separately the terms $u_N^m\left(F_{K_iL_j}^m\left(u_N^m,u_{N+1}^m\right)-F_{K_iL_j}^m\left(u_N^m,u_N^m\right)\right)$
and $-u_1^m\left(F_{K_iL_j}^m\left(u_0^m,u_1^m\right)-F_{K_iL_j}^m\left(u_1^m,u_1^m\right)\right)$:

· Study of $u_N^m\left(F_{K_iL_j}^m\left(u_N^m,u_{N+1}^m\right)-F_{K_iL_j}^m\left(u_N^m,u_N^m\right)\right)$: by using the nonincreasing and F_2-Lispchitz continuous function φ_d defined by

$$\varphi_d(s)=F_{K_iL_j}^m\left(u_N^m,s\right),\forall s\in\mathbb{R},$$

we have

$$u_N^m\left(F_{K_iL_j}^m\left(u_N^m,u_{N+1}^m\right)-F_{K_iL_j}^m\left(u_N^m,u_N^m\right)\right)=u_N^m\left(\varphi_d\left(u_{N+1}^m\right)-\varphi_d\left(u_N^m\right)\right).$$

We now introduce the function ϕ_d defined by $\phi_d(a)=a\varphi_d(a)-\displaystyle\int_0^a\varphi_d(s)\,ds$ for any $a\in\mathbb{R}$, one has then for any $a,b\in\mathbb{R}$:

$$\phi_d(b)-\phi_d(a)=b\left(\varphi_d(b)-\varphi_d(a)\right)-\int_a^b\varphi_d(s)-\varphi_d(a)\,ds.$$

With $a=u_{N+1}^m$ and $b=u_N^m$, we deduce from this last equality that

$$\begin{aligned}
u_N^m\left(F_{K_iL_j}^m\left(u_N^m,u_{N+1}^m\right)-F_{K_iL_j}^m\left(u_N^m,u_N^m\right)\right) &= -u_N^m\left(\varphi_d\left(u_N^m\right)-\varphi_d\left(u_{N+1}^m\right)\right)\\
&= \phi_d\left(u_{N+1}^m\right)-\phi_d\left(u_N^m\right)\\
&\quad -\int_{u_{N+1}^m}^{u_N^m}\varphi_d(s)-\varphi_d\left(u_{N+1}^m\right)ds\\
&\geq \phi_d\left(u_{N+1}^m\right)-\phi_d\left(u_N^m\right)\\
&\quad +\frac{1}{2F_2}\left(\varphi_d\left(u_N^m\right)-\varphi_d\left(u_{N+1}^m\right)\right)^2\\
&\geq \phi_d\left(u_{N+1}^m\right)-\phi_d\left(u_N^m\right)\\
&\quad +\left(F_{K_iL_j}^m\left(u_N^m,u_N^m\right)-F_{K_iL_j}^m\left(u_N^m,u_{N+1}^m\right)\right)^2\\
&\quad \times\frac{1}{2\left(F_1+F_2\right)}.
\end{aligned}\tag{4.16}$$

· Study of $-u_1^m\left(F_{K_iL_j}^m\left(u_0^m,u_1^m\right)-F_{K_iL_j}^m\left(u_1^m,u_1^m\right)\right)$: by using the nondecreasing and F_1-Lispchitz continuous function φ_g defined by

$$\varphi_g(s)=F_{K_iL_j}^m\left(s,u_1^m\right),\forall s\in\mathbb{R},$$

we have

$$\begin{aligned}
-u_1^m\left(F_{K_iL_j}^m\left(u_0^m,u_1^m\right)-F_{K_iL_j}^m\left(u_1^m,u_1^m\right)\right) &= u_1^m\left(F_{K_iL_j}^m\left(u_1^m,u_1^m\right)-F_{K_iL_j}^m\left(u_0^m,u_1^m\right)\right)\\
&= u_1^m\left(\varphi_g\left(u_1^m\right)-\varphi_g\left(u_0^m\right)\right).
\end{aligned}$$

We now introduce the function ϕ_g defined by $\phi_g(a) = a\varphi_g(a) - \int_0^a \varphi_g(s)\, ds$ for any $a \in \mathbb{R}$, one has then for any $a, b \in \mathbb{R}$:

$$\phi_g(b) - \phi_g(a) = b\left(\varphi_g(b) - \varphi_g(a)\right) - \int_a^b \varphi_g(s) - \varphi_g(a)\, ds.$$

With $a = u_0^m$ and $b = u_1^m$, we deduce from this last equality that

$$\begin{aligned}
-u_1^m\left(F^m_{K_iL_j}(u_0^m, u_1^m) - F^m_{K_iL_j}(u_1^m, u_1^m)\right) &= u_1^m\left(\varphi_g(u_1^m) - \varphi_g(u_0^m)\right) \\
&= \phi_g(u_1^m) - \phi_g(u_0^m) + \int_{u_0^m}^{u_1^m} \varphi_g(s) - \varphi_g(u_0^m)\, ds \\
&\geq \phi_g(u_1^m) - \phi_g(u_0^m) + \frac{1}{2F_1}\left(\varphi_g(u_1^m) - \varphi_g(u_0^m)\right)^2 \\
&\geq \phi_g(u_1^m) - \phi_g(u_0^m) \\
&\quad + \left(F^m_{K_iL_j}(u_1^m, u_1^m) - F^m_{K_iL_j}(u_0^m, u_1^m)\right)^2 \\
&\quad \times \frac{1}{2(F_1 + F_2)}. \qquad (4.17)
\end{aligned}$$

Thanks to (4.16) and (4.17) we get

$$\begin{aligned}
B_{2,1} &= kE\begin{bmatrix} \left\{F^m_{K_iL_j}\left(u_N^m, u_{N+1}^m\right) - F^m_{K_iL_j}\left(u_N^m, u_N^m\right)\right\} u_N^m \\ -\left\{F^m_{K_iL_j}\left(u_0^m, u_1^m\right) - F^m_{K_iL_j}\left(u_1^m, u_1^m\right)\right\} u_1^m \end{bmatrix} \\
&\geq kE\left[\phi_d\left(u_{N+1}^m\right) - \phi_d\left(u_N^m\right)\right] \\
&\quad + kE\left[\frac{1}{2(F_1+F_2)}\left(F^m_{K_iL_j}\left(u_N^m, u_N^m\right) - F^m_{K_iL_j}\left(u_N^m, u_{N+1}^m\right)\right)^2\right] \\
&\quad + kE\left[\phi_g(u_1^m) - \phi_g(u_0^m)\right] \\
&\quad + kE\left[\frac{1}{2(F_1+F_2)}\left(F^m_{K_iL_j}\left(u_1^m, u_1^m\right) - F^m_{K_iL_j}\left(u_0^m, u_1^m\right)\right)^2\right] \\
&= kE\left[\phi_d\left(u_{N+1}^m\right) - \phi_d\left(u_N^m\right) + \phi_g(u_1^m) - \phi_g(u_0^m)\right] \\
&\quad + \frac{\alpha h}{2k(F_1+F_2)} B_{1,2} \qquad (4.18)
\end{aligned}$$

Let us now turn to an estimate of $B_{2,2}$. To do this, we use again Lemma 4.1 which gives us for all $a, b \in \mathbb{R}$ the following inequalities:

$$\begin{aligned}
\int_a^b F^m_{K_iL_j}(t,t) - F^m_{K_iL_j}(a,b)\, dt &\geq \int_a^b F^m_{K_iL_j}(a,t) - F^m_{K_iL_j}(a,a)\, dt \\
&\geq \frac{1}{2F_2}\left(F^m_{K_iL_j}(a,b) - F^m_{K_iL_j}(a,a)\right)^2 \qquad (4.19)
\end{aligned}$$

and

$$\begin{aligned}\int_a^b F^m_{K_iL_j}(t,t) - F^m_{K_iL_j}(a,b)\,dt &\geq \int_a^b F^m_{K_iL_j}(t,b) - F^m_{K_iL_j}(a,b)\,dt \\ &\geq \frac{1}{2F_1}\left(F^m_{K_iL_j}(b,b) - F^m_{K_iL_j}(a,b)\right)^2. \qquad (4.20)\end{aligned}$$

Multiplying (4.19) (respectively (4.20)) by $\dfrac{F_2}{F_1+F_2}$ (respectively by $\dfrac{F_1}{F_1+F_2}$) and adding the two inequalities yields:

$$\begin{aligned}&\int_a^b F^m_{K_iL_j}(t,t) - F^m_{K_iL_j}(a,b)\,dt \\ \geq\;& \frac{1}{2(F_1+F_2)}\left[\left(F^m_{K_iL_j}(a,a) - F^m_{K_iL_j}(a,b)\right)^2 + \left(F^m_{K_iL_j}(b,b) - F^m_{K_iL_j}(a,b)\right)^2\right]\end{aligned}$$

We can deduce from this last inequality that

$$\begin{aligned}B_{2,2} &= E\left[\sum_{i=1}^{N-1} k\int_{u_i^m}^{u_{i+1}^m}\left(F^m_{K_iL_j}(s,s) - F^m_{K_iL_j}\left(u_i^m,u_{i+1}^m\right)\right)ds\right] \\ &\geq \frac{k}{2(F_1+F_2)}\sum_{i=1}^{N-1} E\left[\begin{array}{c}\left(F^m_{K_iL_j}\left(u_i^m,u_i^m\right) - F^m_{K_iL_j}\left(u_i^m,u_{i+1}^m\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u_{i+1}^m,u_{i+1}^m\right) - F^m_{K_iL_j}\left(u_i^m,u_{i+1}^m\right)\right)^2\end{array}\right] \\ &\geq \frac{\alpha h}{2k(F_1+F_2)}B_{1,1} \qquad (4.21)\end{aligned}$$

In this way, combining (4.18), and (4.21), one gets

$$\begin{aligned}B_2 &= B_{2,1} + B_{2,2} + B_{2,3} \\ &\geq C\left(B_{1,1} + B_{1,2}\right) + kE\left[\phi_d\left(u_{N+1}^m\right) - \phi_d\left(u_N^m\right) + \phi_g\left(u_1^m\right) - \phi_g\left(u_0^m\right)\right] + B_{2,3} \\ &\geq CB_1 + kE\left[\phi_d\left(u_{N+1}^m\right) - \phi_d\left(u_N^m\right) + \phi_g\left(u_1^m\right) - \phi_g\left(u_0^m\right)\right] + B_{2,3}\end{aligned}$$

where $C = \dfrac{\alpha h}{2k(F_1+F_2)}$
which implies thanks to the CFL Condition (4.13) that

$$B_2 \geq B_1 + kE\left[\phi_d\left(u_{N+1}^m\right) - \phi_d\left(u_N^m\right) + \phi_g\left(u_1^m\right) - \phi_g\left(u_0^m\right)\right] + B_{2,3}$$

In summary, we showed that

$$\begin{aligned}
\sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[\left(u_i^{m+1}\right)^2\right] &= \sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[(u_i^m)^2\right]+B_1-B_2+\sum_{i=1}^{N}\frac{km(K_i)}{2}\\
&\leq \sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[(u_i^m)^2\right]\\
&\quad -kE\left[\phi_d\left(u_{N+1}^m\right)-\phi_d\left(u_N^m\right)+\phi_g\left(u_1^m\right)-\phi_g\left(u_0^m\right)\right]\\
&\quad -B_{2,3}+\sum_{i=1}^{N}\frac{km(K_i)}{2}\\
&= \sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[(u_i^m)^2\right]\\
&\quad +kE\left[-\phi_d\left(u_{N+1}^m\right)+\phi_d\left(u_N^m\right)-\phi_g\left(u_1^m\right)+\phi_g\left(u_0^m\right)\right]\\
&\quad +E\left[k\left(-\Psi_{K_iL_j}^m\left(u_N^m\right)+\Psi_{K_iL_j}^m\left(u_1^m\right)\right)\right]+\sum_{i=1}^{N}\frac{km(K_i)}{2}
\end{aligned}$$

Since for any $x\in\mathbb{R}$ we have

$$\begin{aligned}
\phi_g(x) &= xF_{K_iL_j}^m\left(x,u_1^{m+1}\right)-\int_0^x F_{K_iL_j}^m\left(s,u_1^{m+1}\right)ds\\
&= \int_0^x F_{K_iL_j}^m\left(x,u_1^{m+1}\right)-F_{K_iL_j}^m\left(s,u_1^{m+1}\right)ds\\
&\leq F_1\int_0^x (x-s)\,ds\\
&= F_1\frac{x^2}{2},
\end{aligned}$$

and similarly for any $x\in\mathbb{R}$, $\phi_d(x)\geq -F_2\frac{x^2}{2}$, one finally gets that

$$\begin{aligned}
\sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[\left(u_i^{m+1}\right)^2\right] &\leq \sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[(u_i^m)^2\right]\\
&\quad +kE\left[-\phi_d\left(u_{N+1}^m\right)+\phi_d\left(u_N^m\right)-\phi_g\left(u_1^m\right)+\phi_g\left(u_0^m\right)\right]\\
&\quad +kE\left[-\Psi_{K_iL_j}^m\left(u_N^m\right)+\Psi_{K_iL_j}^m\left(u_1^m\right)\right]+\sum_{i=1}^{N}\frac{km(K_i)}{2}\\
&\leq \sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[(u_i^m)^2\right]+kE\left[F_2\frac{\left(u_{N+1}^m\right)^2}{2}+F_1\frac{\left(u_0^m\right)^2}{2}\right]\\
&\quad +\sum_{i=1}^{N}\frac{km(K_i)}{2}\\
&\leq \sum_{i=1}^{N}\frac{m(K_i)}{2}E\left[(u_i^m)^2\right]+\sum_{i=1}^{N}\frac{km(K_i)}{2}
\end{aligned}$$

In this way, we deduce from the discrete Gronwall lemma that for any $m\in\{0,...M\}$

$$\begin{aligned}\sum_{i=1}^{N} m(K_i)\,E\left[(u_i^m)^2\right] &\leq \sum_{i=1}^{N} m(K_i)\,E\left[\left(u_i^0\right)^2\right] + m\sum_{i=1}^{N}\frac{km(K_i)}{2}\\ &\leq \|u_0\|_{L^2(D)}^2 + \pi T \end{aligned}\tag{4.22}$$

we get

$$\sum_{i=1}^{N} m(K_i)\,E\left[(u_i^m)^2\right] \leq \|u_0\|_{L^2(D)}^2 + \pi T.$$

We conclude that

$$\|u_{\tau,k}(\omega,t,x)\|_{L^\infty(0,T;L^2(\Omega\times D))} \leq \sqrt{\|u_0\|_{L^2(D)}^2 + \pi T}.$$

This gives the $L_t^\infty L_{\omega,x}^2$ stability of the approximate solution. As a consequence, we have

$$\begin{aligned}\|u_{\tau,k}(\omega,t,x)\|_{L^2(\Omega\times Q)}^2 &= \sum_{m=0}^{M-1}\sum_{i=1}^{N} km(K_i)\,E\left[(u_i^m)^2\right]\\ &\leq T\left(\|u_0\|_{L^2(D)}^2 + \pi T\right).\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{i=1}^{N} m(K_i)\,E\left[\left(W_i^{J,m}\right)^2\right] &= \sum_{i=1}^{N} m(K_i)\,E\left[\left(\sum_{j=1}^{J}\beta_j(t^m)\frac{1}{m(K_i)}\int_{K_i} e_j(x)\,dx\right)^2\right]\\ &= \sum_{i=1}^{N} m(K_i)\frac{1}{(m(K_i))^2}E\left[\left(\sum_{j=1}^{J}\beta_j(t^m)\int_{K_i} e_j(x)\,dx\right)^2\right]\end{aligned}$$

The Brownian motions β_j and β_l are independent and the cross terms for $j \neq l$ have mean zero and vanish, we thus get

$$\begin{aligned}\sum_{i=1}^{N} m(K_i)\,E\left[\left(W_i^{J,m}\right)^2\right] &= \sum_{i=1}^{N}\frac{1}{m(K_i)}E\left[\sum_{j=1}^{J}\beta_j^2(t^m)\left(\int_{K_i} e_j(x)\,dx\right)^2\right]\\ &= \sum_{i=1}^{N}\frac{1}{m(K_i)}\sum_{j=1}^{J}E\left[\beta_j^2(t^m)\right]\left(\int_{K_i} e_j(x)\,dx\right)^2\\ &= \sum_{i=1}^{N}\frac{1}{m(K_i)}\sum_{j=1}^{J}t^m\left(\int_{K_i} e_j(x)\,dx\right)^2\\ &\leq t^m\sum_{i=1}^{N}\sum_{j=1}^{J}\int_{K_i}\|e_j\|_{L^2(D)}^2\,dx\\ &\leq 2\pi TJ\end{aligned}$$

we thus get

$$\sum_{i=1}^{N} m\left(K_i\right) E\left[\left(W_i^{J,m}\right)^2\right] \leq 2\pi T J$$

We conclude that

$$\left\|W_{J,\tau,k}\left(\omega,t,x\right)\right\|_{L^{\infty}(0,T;L^2(\Omega\times D))} \leq \sqrt{2\pi T J}$$

As a consequence, we have

$$\left\|W_{J,\tau,k}\left(\omega,t,x\right)\right\|^2_{L^2(\Omega\times Q)} \leq 2\pi T^2 J \quad \blacksquare$$

4.5.2 Weak BV estimate

Proposition 4.2 (Weak BV estimate)**.** *Let τ be an admissible mesh in the sense of Definition 4.3, $T > 0$,*

$M \in \mathbb{N}^*$ and let $k = \dfrac{T}{M} \in \mathbb{R}^*_+$ satisfying the CFL Condition

$$k \leq \frac{\alpha h}{2\left(F_1 + F_2\right)},$$

Let $\left\{u^m_{W,i}, i \in \{1, ..., N\}, m \in \{0, ...M-1\}\right\}$ be given by the finite volume scheme (4.10)*.*

Then the following hold:

1.

$$\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N} kE\left[\begin{array}{c}\left(F^m_{K_iL_j}\left(u^m_{W,i},u^m_{W,i+1}\right)-F^m_{K_iL_j}\left(u^m_{W,i},u^m_{W,i}\right)\right)^2\\ +\left(F^m_{K_iL_j}\left(u^m_{W,i-1},u^m_{W,i}\right)-F^m_{K_iL_j}\left(u^m_{W,i},u^m_{W,i}\right)\right)^2\end{array}\right]$$
$$\leq \frac{1}{2}\left\|u_0\right\|^2_{L^2(D)} + \left(F_1+F_2\right)T^2 J.$$

2.

$$\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[\begin{array}{c}\max\limits_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right)\\ +\max\limits_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)\end{array}\right]$$
$$\leq \sqrt{\left(\frac{1}{2\left(F_1+F_2\right)}\left\|u_0\right\|^2_{L^2(D)} + T^2 J\right)}\sqrt{\frac{2\pi T}{\alpha h}}.$$

Proof. By multiplying the numerical scheme (4.10) by $u^m_{W,i}$, taking the expectation and summing for $i = 1, ..., N$, and after summing over $m \in \{0, ...M-1\}$, we have:

$$\begin{aligned} \sum_{m=0}^{M-1} \sum_{i=1}^{N} \frac{m(K_i)}{2} E\left[\left(u^{m+1}_{W,i}\right)^2 - \left(u^m_{W,i}\right)^2\right] &= \sum_{i=1}^{N} \frac{m(K_i)}{2} E\left[\left(u^{M}_{W,i}\right)^2 - \left(u^0_{W,i}\right)^2\right] \\ &= \sum_{m=0}^{M-1} \left(B_3 - B_4\right), \end{aligned}$$

$$\text{where} \quad B_3 = \sum_{i=1}^{N} \frac{k^2}{2m(K_i)} E\left[\left(F^m_{K_iL_j}\left(u^m_{W,i-1}, u^m_{W,i}\right) - F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right)\right)^2\right]$$

$$\text{and} \qquad B_4 = \sum_{i=1}^{N} kE\left[\left\{F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right) - F^m_{K_iL_j}\left(u^m_{W,i-1}, u^m_{W,i}\right)\right\} u^m_{W,i}\right].$$

- Study of B_3: Using Cauchy-Schwarz inequality, we get similarly to (4.15) the following estimate

$$\begin{aligned} B_3 &\leq \sum_{i=1}^{N} \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{W,i-1}, u^m_{W,i}\right) - F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i}\right)\right)^2 \\ + \left(F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right) - F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i}\right)\right)^2 \end{array}\right] \\ &\leq \sum_{i=1}^{N-1} \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right) - F^m_{K_iL_j}\left(u^m_{W,i+1}, u^m_{W,i+1}\right)\right)^2 \\ + \left(F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right) - F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i}\right)\right)^2 \end{array}\right] \\ &\quad + \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{W,0}, u^m_{W,1}\right) - F^m_{K_iL_j}\left(u^m_{W,1}, u^m_{W,1}\right)\right)^2 \\ + \left(F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N+1}\right) - F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N}\right)\right)^2 \end{array}\right] \end{aligned}$$

one gets:

$$B_3 \leq B_{3,1} + B_{3,2}, \tag{4.23}$$

where

$$B_{3,1} = \sum_{i=1}^{N-1} \frac{k^2}{\alpha h} E\left[\begin{array}{c} \max\limits_{u^m_{W,i+1} \leq d \leq c \leq u^m_{W,i}} \left(F^m_{K_iL_j}(c,d) - F^m_{K_iL_j}(d,d)\right)^2 \\ + \max\limits_{u^m_{W,i+1} \leq d \leq c \leq u^m_{W,i}} \left(F^m_{K_iL_j}(c,d) - F^m_{K_iL_j}(c,c)\right)^2 \end{array}\right]$$

$$\text{and } B_{3,2} = \frac{k^2}{\alpha h} E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{W,0}, u^m_{W,1}\right) - F^m_{K_iL_j}\left(u^m_{W,1}, u^m_{W,1}\right)\right)^2 \\ + \left(F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N+1}\right) - F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N}\right)\right)^2 \end{array}\right].$$

- Study of B_4: Using again the notation $\Psi^m_{K_iL_j}(a) = \int_0^a s \frac{d}{ds}\left(F^m_{K_iL_j}(s,s)\right) ds$, B_4 can be decomposed, as in the proof of Proposition 4.1, in the following way

$$B_4 = B_{4,1} + B_{4,2} + B_{4,3}.$$

where

$$B_{4,1} = kE\left[\begin{array}{c} \left\{F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N+1}\right) - F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N}\right)\right\}u^m_{W,N} \\ -\left\{F^m_{K_iL_j}\left(u^m_{W,0}, u^m_{W,1}\right) - F^m_{K_iL_j}\left(u^m_{W,1}, u^m_{W,1}\right)\right\}u^m_{W,1}\end{array}\right],$$

$$B_{4,2} = E\left[\sum_{i=1}^{N-1} k\int_{u^m_{W,i}}^{u^m_{W,i+1}}\left(F^m_{K_iL_j}(s,s) - F^m_{K_iL_j}\left(u^m_{W,i}, u^m_{W,i+1}\right)\right)ds\right],$$

and

$$\begin{aligned} B_{4,3} &= -E\left[\sum_{i=1}^{N-1} k\left(\Psi^m_{K_iL_j}\left(u^m_{W,i}\right) - \Psi^m_{K_iL_j}\left(u^m_{W,i+1}\right)\right)\right] \\ &= E\left[k\left(\Psi^m_{K_iL_j}\left(u^m_{W,N}\right) - \Psi^m_{K_iL_j}\left(u^m_{W,1}\right)\right)\right].\end{aligned}$$

Following the proof of Proposition 4.1 one shows that

$$\begin{aligned} B_{4,1} &\geq \frac{k}{2\left(F_1+F_2\right)}E\left[\begin{array}{c}\left(F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N}\right) - F^m_{K_iL_j}\left(u^m_{W,N}, u^m_{W,N+1}\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_{W,1}, u^m_{W,1}\right) - F^m_{K_iL_j}\left(u^m_{W,0}, u^m_{W,1}\right)\right)^2\end{array}\right] \\ &\quad + kE\left[\phi_g\left(u^m_{W,1}\right) - \phi_g\left(u^m_{W,0}\right) + \phi_d\left(u^m_{W,N+1}\right) - \phi_d\left(u^m_{W,N}\right)\right] \\ &= kE\left[\phi_g\left(u^m_{W,1}\right) - \phi_g\left(u^m_{W,0}\right) + \phi_d\left(u^m_{W,N+1}\right) - \phi_d\left(u^m_{W,N}\right)\right] \\ &\quad + \frac{\alpha h}{2k\left(F_1+F_2\right)}B_{3,2}.\end{aligned}$$

We still follow the proof of Proposition 4.1. In particular we use the fact that F is nondecreasing with respect to its first variable and nonincreasing with respect to its second variable, we deduce that

$$\begin{aligned} B_{4,1} + B_{4,3} &\geq kE\left[\phi_g\left(u^m_{W,1}\right) - \phi_g\left(u^m_{W,0}\right) + \phi_d\left(u^m_{W,N+1}\right) - \phi_d\left(u^m_{W,N}\right)\right] \\ &\quad + \frac{\alpha h}{2k\left(F_1+F_2\right)}B_{3,2} \\ &\quad + kE\left[\left(\Psi^m_{K_iL_j}\left(u^m_{W,N}\right) - \Psi^m_{K_iL_j}\left(u^m_{W,1}\right)\right)\right] \\ &\geq kE\left[-\phi_g\left(u^m_{W,0}\right) + \phi_d\left(u^m_{W,N+1}\right)\right] \\ &\quad + \frac{\alpha h}{2k\left(F_1+F_2\right)}B_{3,2}. \qquad (4.24)\end{aligned}$$

Let us now turn to an estimate of $B_{4,2}$. For this purpose, let $a, b \in \mathbb{R}$ and define

$$\mathcal{C}\left(a,b\right)=\left\{\left(c,d\right)\in\left[\min\left(a,b\right),\max\left(a,b\right)\right]^{2}:\left(d-c\right)\left(b-a\right)\geq 0\right\}.$$

Thanks to the monotonicity of F, the following inequality holds for any $(c,d)\in\mathcal{C}(a,b)$:

$$\begin{aligned}\int_a^b F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(a,b\right)ds &\geq \int_c^d F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(a,b\right)ds\\ &\geq \int_c^d F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(c,d\right)ds.\end{aligned}$$

We now use again Lemma 4.1 and deduce that for all $(c,d)\in\mathcal{C}(a,b)$:

$$\begin{aligned}\int_a^b F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(a,b\right)ds \geq & \int_c^d F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(c,d\right)ds\\ \geq & \int_c^d F_{K_iL_j}^m\left(c,s\right)-F_{K_iL_j}^m\left(c,d\right)ds\\ \geq & \frac{1}{2F_2}\left(F_{K_iL_j}^m\left(c,c\right)-F_{K_iL_j}^m\left(c,d\right)\right)^2 \qquad (4.25)\end{aligned}$$

and

$$\begin{aligned}\int_a^b F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(a,b\right)ds \geq & \int_c^d F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(c,d\right)ds\\ \geq & \int_c^d F_{K_iL_j}^m\left(s,d\right)-F_{K_iL_j}^m\left(c,d\right)ds\\ \geq & \frac{1}{2F_1}\left(F_{K_iL_j}^m\left(c,d\right)-F_{K_iL_j}^m\left(d,d\right)\right)^2. \qquad (4.26)\end{aligned}$$

Multiplying (4.25) (respectively (4.26)) by $\frac{F_2}{F_1+F_2}$ (respectively by $\frac{F_1}{F_1+F_2}$), taking the maximum for $(c,d)\in\mathcal{C}(a,b)$ and adding the two inequalities yields:

$$\int_a^b F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(a,b\right)ds \geq \frac{1}{2\left(F_1+F_2\right)}\left[\begin{array}{c}\max\limits_{(c,d)\in\mathcal{C}(a,b)}\left(F_{K_iL_j}^m\left(c,c\right)-F_{K_iL_j}^m\left(c,d\right)\right)^2\\ +\max\limits_{(c,d)\in\mathcal{C}(a,b)}\left(F_{K_iL_j}^m\left(c,d\right)-F_{K_iL_j}^m\left(d,d\right)\right)^2\end{array}\right].$$

We can deduce from this last inequality that

$$\begin{aligned}B_{4,2} &= E\left[\sum_{i=1}^{N-1}k\int_{u_{W,i}^m}^{u_{W,i+1}^m}\left(F_{K_iL_j}^m\left(s,s\right)-F_{K_iL_j}^m\left(u_{W,i}^m,u_{W,i+1}^m\right)\right)ds\right]\\ &\geq E\left[\sum_{i=1}^{N-1}k\frac{1}{2\left(F_1+F_2\right)}\left[\begin{array}{c}\max\limits_{u_{W,i+1}^m\leq d\leq c\leq u_{W,i}^m}\left(F_{K_iL_j}^m\left(c,c\right)-F_{K_iL_j}^m\left(c,d\right)\right)^2\\ +\max\limits_{u_{W,i+1}^m\leq d\leq c\leq u_{W,i}^m}\left(F_{K_iL_j}^m\left(c,d\right)-F_{K_iL_j}^m\left(d,d\right)\right)^2\end{array}\right]\right].\\ &= \frac{\alpha h}{2k\left(F_1+F_2\right)}B_{3,1}. \qquad (4.27)\end{aligned}$$

In this way, using (4.24) and (4.27), one gets

$$\begin{aligned} B_4 &= B_{4,1}+B_{4,2}+B_{4,3} \\ &\geq kE\left[-\phi_g\left(u^m_{W,0}\right)+\phi_d\left(u^m_{W,N+1}\right)\right] \\ &\quad +\frac{\alpha h}{2k\left(F_1+F_2\right)}\left(B_{3,1}+B_{3,2}\right) \end{aligned} \tag{4.28}$$

Finally, since

$$\begin{aligned} \sum_{m=0}^{M-1}\sum_{i=1}^{N}\frac{m\left(K_i\right)}{2}E\left[\left(u^{m+1}_{W,i}\right)^2-\left(u^m_{W,i}\right)^2\right] &= \sum_{i=1}^{N}\frac{m\left(K_i\right)}{2}E\left[\left(u^M_{W,i}\right)^2-\left(u^0_{W,i}\right)^2\right] \\ &= \sum_{m=0}^{M-1}\left(B_3-B_4\right) \end{aligned}$$

one gets with (4.23), and (4.28)

$$\begin{aligned} \sum_{i=1}^{N}\frac{m\left(K_i\right)}{2}E\left[\left(u^M_{W,i}\right)^2\right] &= \sum_{i=1}^{N}\frac{m\left(K_i\right)}{2}E\left[\left(u^0_{W,i}\right)^2\right]+\sum_{m=0}^{M-1}\left(B_3-B_4\right) \\ &\leq \frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(1-\frac{\alpha h}{2k\left(F_1+F_2\right)}\right)\sum_{m=0}^{M-1}\left(B_{3,1}+B_{3,2}\right) \\ &\quad -kE\left[-\phi_g\left(u^m_{W,0}\right)+\phi_d\left(u^m_{W,N+1}\right)\right], \end{aligned}$$

thus,

$$\begin{aligned} 0 \;\leq\; & \sum_{i=1}^{N}\frac{m\left(K_i\right)}{2}E\left[\left(u^M_{W,i}\right)^2\right]\leq\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(1-\frac{\alpha h}{2k\left(F_1+F_2\right)}\right) \\ & \left(\sum_{m=0}^{M-1}\sum_{i=1}^{N-1}\frac{k^2}{\alpha h}E\left[\begin{array}{c} \displaystyle\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right)^2 \\ \displaystyle+\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)^2 \end{array}\right]\right. \\ & \left.+\sum_{m=0}^{M-1}\frac{k^2}{\alpha h}E\left[\begin{array}{c} \left(F^m_{K_iL_j}\left(u^m_{W,0},u^m_{W,1}\right)-F^m_{K_iL_j}\left(u^m_{W,1},u^m_{W,1}\right)\right)^2 \\ +\left(F^m_{K_iL_j}\left(u^m_{W,N},u^m_{W,N+1}\right)-F^m_{K_iL_j}\left(u^m_{W,N},u^m_{W,N}\right)\right)^2 \end{array}\right]\right) \\ & -kE\left[-\phi_g\left(u^m_{W,0}\right)+\phi_d\left(u^m_{W,N+1}\right)\right] \end{aligned}$$

Then, following again the proof of Proposition 4.1, and using in particular the fact that for any $x\in\mathbb{R}$,

$$\phi_g\left(x\right)\leq F_1\frac{x^2}{2}\ \text{ and }\ \phi_d\left(x\right)\geq -F_2\frac{x^2}{2},$$

we get thanks to the CFL Condition (4.13) that

$$\begin{aligned}
&\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1}kE\left[\begin{array}{c}\max\limits_{u^m_{W,i+1}\le d\le c\le u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right)^2\\ +\max\limits_{u^m_{W,i+1}\le d\le c\le u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)^2\end{array}\right]\\
&+\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}kE\left[\begin{array}{c}\left(F^m_{K_iL_j}\left(u^m_{W,0},u^m_{W,1}\right)-F^m_{K_iL_j}\left(u^m_{W,1},u^m_{W,1}\right)\right)^2\\ +\left(F^m_{K_iL_j}\left(u^m_{W,N},u^m_{W,N+1}\right)-F^m_{K_iL_j}\left(u^m_{W,N},u^m_{W,N}\right)\right)^2\end{array}\right]\\
\le\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\sum_{m=0}^{M-1}kE\left[F_1\frac{\left(u^m_{W,0}\right)^2}{2}+F_2\frac{\left(u^m_{W,N+1}\right)^2}{2}\right]\\
\le\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\frac{F_1+F_2}{2}\sum_{m=0}^{M-1}kE\left[\left(u^m_{W,0}\right)^2+\left(u^m_{W,N+1}\right)^2\right]\\
=\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\frac{F_1+F_2}{2}\sum_{m=0}^{M-1}kE\left[\left(W_0^{J,m}\left(t\right)\right)^2+\left(W_{2\pi}^{J,m}\left(t\right)\right)^2\right]\\
\le\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\frac{F_1+F_2}{2}\sum_{m=0}^{M-1}kE\left[\left(\sum_{j=1}^{J}\beta_j\left(t^m\right)e_0^j\right)^2+\left(\sum_{j=1}^{J}\beta_j\left(t^m\right)e_{2\pi}^j\right)^2\right]\\
=\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\frac{F_1+F_2}{2}\sum_{m=0}^{M-1}kE\left[\sum_{j=1}^{J}\left(\beta_j\left(t^m\right)\right)^2\left\|e_0^j\right\|^2_{L^2}+\sum_{j=1}^{J}\left(\beta_j\left(t^m\right)\right)^2\left\|e_{2\pi}^j\right\|^2_{L^2}\right]\\
=\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(F_1+F_2\right)\sum_{m=0}^{M-1}kJt^m\\
\le\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(F_1+F_2\right)T^2J \qquad (4.29)
\end{aligned}$$

taking $c=u^m_{W,i}$ and $d=u^m_{W,i+1}$ in the maximum, we have in particular

$$\begin{aligned}
&\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N}kE\left[\begin{array}{c}\left(F^m_{K_iL_j}\left(u^m_{W,i-1},u^m_{W,i}\right)-F^m_{K_iL_j}\left(u^m_{W,i},u^m_{W,i}\right)\right)^2\\ +\left(F^m_{K_iL_j}\left(u^m_{W,i},u^m_{W,i+1}\right)-F^m_{K_iL_j}\left(u^m_{W,i},u^m_{W,i}\right)\right)^2\end{array}\right]\\
\le\ &\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(F_1+F_2\right)T^2J,
\end{aligned}$$

which proves the first point of the proposition.

Let us now turn to the second point of the proposition. To do this, we aim to estimate

$$\left\{\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1}kE\left[\begin{array}{c}\max\limits_{u^m_{W,i+1}\le d\le c\le u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right)\\ +\max\limits_{u^m_{W,i+1}\le d\le c\le u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)\end{array}\right]\right\}^2,$$

Let us denote by

$$T_1=\max_{u^m_{W,i+1}\le d\le c\le u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right)+\max_{u^m_{W,i+1}\le d\le c\le u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right),$$

Using Cauchy-Schwarz inequality, one gets

$$
\begin{aligned}
&\left(\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[T_1\right]\right)^2 \\
= \;& \frac{1}{4\left(F_1+F_2\right)^2}\left(\sum_{m=0}^{M-1}\sum_{i=1}^{N-1}\sqrt{k}\sqrt{k}E\left[T_1\right]\right)^2 \\
\leq \;& \frac{1}{4\left(F_1+F_2\right)^2}\left(\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} k\right)\left(\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[T_1^2\right]\right) \\
\leq \;& \frac{TN}{4\left(F_1+F_2\right)^2}\left(\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[\left\{\begin{array}{l}\displaystyle\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right) \\ \displaystyle +\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)\end{array}\right\}^2\right]\right) \\
\leq \;& \frac{TN}{\left(F_1+F_2\right)}\left(\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[\begin{array}{l}\displaystyle\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right)^2 \\ \displaystyle +\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)^2\end{array}\right]\right) \\
\leq \;& \frac{1}{\left(F_1+F_2\right)}\left(T\frac{2\pi}{\alpha h}\right)\left(\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(F_1+F_2\right)T^2J\right)
\end{aligned}
$$

Thus,

$$
\left(\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[T_1\right]\right)^2 \leq \frac{1}{\left(F_1+F_2\right)}\left(\frac{2\pi T}{\alpha h}\right)\left(\frac{1}{2}\left\|u_0\right\|^2_{L^2(D)}+\left(F_1+F_2\right)T^2J\right) \tag{37}
$$

one finally gets

$$
\begin{aligned}
&\frac{1}{2\left(F_1+F_2\right)}\sum_{m=0}^{M-1}\sum_{i=1}^{N-1} kE\left[\begin{array}{l}\displaystyle\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(d,d\right)\right) \\ \displaystyle +\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F^m_{K_iL_j}\left(c,d\right)-F^m_{K_iL_j}\left(c,c\right)\right)\end{array}\right] \\
\leq \;& \sqrt{\left(\frac{1}{2\left(F_1+F_2\right)}\left\|u_0\right\|^2_{L^2(D)}+T^2J\right)}\sqrt{\frac{2\pi T}{\alpha h}},
\end{aligned}
$$

which concludes the proof of the second point of the proposition. ■

4.6 Convergence of the finite volume approximate solution

First of all, note that the estimates stated in Proposition 4.1 only provide (up to a subsequence) weak convergences for $u_{W,\tau,k}$. Moreover, due to the nonlinearity of f, one needs compactness arguments to pass to the limit in the nonlinear terms and these arguments have to be compatible with the random variable.

The concept of Young measures is appropriate here and the technique is based on the notion of narrow convergence of Young measures (or entropy processes), we refer to Balder [3] but also to Eymard-Gallouët-Herbin [23].

In this way, taking a sequence of approximate finite volume solution $u_{W,\tau,k}$, it converges (up to a subsequence still denoted $u_{W,\tau,k}$) in the sense of Young measures to an "entropy process" denoted by $\mathbf{u}-\mathbf{W}$ which belongs to $L^2\left(\Omega\times Q\times(0,1)\right)$. Precisely, given a Carathéodory function $\Psi:\Omega\times Q\times\mathbb{R}\to\mathbb{R}$ such that $\Psi\left(.,u_{W,\tau,k}\right)$ is uniformly integrable, one has:

$$E\left[\int_Q \Psi\left(.,u_{W,\tau,k}\right)dxdt\right]\to E\left[\int_Q\int_0^1 \Psi\left(.,\mathbf{u}\left(.,\alpha\right)-\mathbf{W}\right)dxdt\right].$$

A proof of this result can be found in [6], Section A.3.2. We recall that a function $\Psi:\Omega\times Q\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function if for almost any $(\omega,t,x)\in\Omega\times Q$ the function $\nu\to\Psi\left(\omega,t,x,\nu\right)$ is continuous and for all $\nu\in\mathbb{R}$, the function $(\omega,t,x)\to\Psi\left(\omega,t,x,\nu\right)$ is measurable. We also recall that a sequence $\left(\psi_n\right)_{n\geq 0}$ of functions $\psi_n:\Omega\times Q\to\mathbb{R}$ is said to be uniformly integrable on the domain $\Omega\times Q$ if it satisfies the following properties:

- $\left(\psi_n\right)_{n\geq 0}$ is bounded in $L^1\left(\Omega\times Q\right)$.
- $\left(\psi_n\right)_{n\geq 0}$ is equi-integrable, that is to say that for any $\varepsilon>0$, there exists $\delta>0$ such that for any measurable set A of $\Omega\times Q$ satisfying $\left(\mathcal{L}^2\otimes\mathbb{P}\right)(A)\leq\delta$, we have for any $n\in\mathbb{N}$,

  $$\int_A \left|\psi_n\left(\omega,t,x\right)\right| dxdtd\mathbb{P}\leq\varepsilon$$

  (where $\mathcal{L}^2$ is the 2-dimensional Lebesgue measure).

Remark 4.2 ($L^\infty\left(0,T;L^2\left(\Omega\times D\times(0,1)\right)\right)$ regularity of $\mathbf{u}$)**.** *Since the sequence of approximate solutions $u_{\tau,k}$ is bounded in $L^\infty\left(0,T;L^2\left(\Omega\times D\right)\right)$ according to Proposition 4.1, following [6] we show that $\mathbf{u}\in L^\infty\left(0,T;L^2\left(\Omega\times D\times(0,1)\right)\right)$.*

Note that if one is able to show that $\mathbf{u}-\mathbf{W}$ is a measure-valued entropy solution of Problem (4.4) in the sense of Definition 4.2, then, $\mathbf{u}$ is a measure-valued entropy solution of Problem (4.1) and using Proposition 3.4, we will be able to conclude that all the sequence $u_{\tau,k}$ converges in $L^1\left(\Omega\times Q\right)$ to the unique stochastic entropy solution of (4.1) in the sense of Definition 4.1. Since $\mathbf{u}$ satisfied the regularities required by Definition 4.2, it remains to show that $\mathbf{u}$ satisfies the following entropy inequalities:

$$\begin{aligned}0 \leq\ & E\left[\mathbf{1}_A \int\int_Q \int_0^1 \eta_\kappa^+ \left(\mathbf{u}\left(t,x,\alpha\right) - \mathbf{W}\left(t,x\right)\right)\varphi_t\left(t,x\right)d\alpha dx dt\right] \\ & + E\left[\mathbf{1}_A \int\int_Q \int_0^1 (\phi_\kappa^+\left(\omega,t,x,\mathbf{u}\left(t,x,\alpha\right) - \mathbf{W}\left(t,x\right)\right)\varphi_x) d\alpha dx dt\right] \\ & + E\left[\mathbf{1}_A \int_D \eta_\kappa^+\left(u_0\right)\varphi\left(0\right)dx\right] \\ & + Lip\left(B\right)E\left[\mathbf{1}_A \int_0^\infty \left(\eta_\kappa^+\left(u_b\left(t,2\pi\right)\right)\varphi\left(t,2\pi\right) - \eta_\kappa^+\left(u_b\left(t,0\right)\right)\varphi\left(t,0\right)\right)dt\right].\end{aligned}$$

This is the aim of the next section. We propose in this section entropy inequalities satisfied by the finite volume approximate solution and aim to pass to the limit in these formulations in order to show the convergence of the scheme.

4.6.1 Entropy inequalities for the approximate solution

In this section, we show how the approximate solution $u_{W,\tau,k}$ is close to the entropy solution. First, we derive a discrete entropy inequality which is a consequence of the monotony of the scheme. Then, we prove that $u_{W,\tau,k}$ verifies a continuous entropy inequality, with some error terms.

To this purpose, we will work with the semi Kruzkov entropies; that is one of the keys of the following results (the other key being the weak BV estimate).

We recall some notations about it.

Notations: η_κ^+ denotes the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\eta_\kappa^+\left(s\right) = \left(s-\kappa\right)^+,$$

and ϕ_κ^+ the associated flux-function from $\Omega \times Q \times \mathbb{R}$ to $\mathbb{R}$ defined by

$$\phi_\kappa^+\left(\omega,t,x,s\right) = sgn^+\left(s-\kappa\right)\left(B\left(\omega,t,x,s\right) - B\left(\omega,t,x,\kappa\right)\right).$$

Notice that, if $a\top b = \max\left(a,b\right)$ and $a \perp b = \min\left(a,b\right)$, then we have

$$\eta_\kappa^+\left(s\right) = s\top\kappa - \kappa,$$

and

$$\phi_\kappa^+\left(\omega,t,x,s\right) = B\left(\omega,t,x,s\top\kappa\right) - B\left(\omega,t,x,\kappa\right).$$

Therefore, the associated entropy numerical flux function is defined by the formula

$$\Psi_{K_i,L_j,\kappa}^{+,m}\left(a,b\right) = F_{K_i,L_j}^m\left(a\top\kappa, b\top\kappa\right) - F_{K_i,L_j}^m\left(\kappa,\kappa\right).$$

Discrete entropy inequalities

Lemma 4.2. *Assume that* (4.5),(4.13) *and the properties of Definition 4.4 hold. Let* $u_{W,\tau,k}$ *be the approximate solution of the problem* (4.4) *defined by* (4.7), (4.10) *and* (4.11). *Then, for all* $\kappa \in \mathbb{R}$*, for all* $i \in \{1, ...N\}$, $m \in \{0, ...M-1\}$*, the following local discrete entropy inequality holds:*

$$\frac{\eta_\kappa^+\left(u_{W,i}^{m+1}\right)-\eta_\kappa^+\left(u_{W,i}^{m}\right)}{k}+\frac{1}{m\left(K_i\right)}\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i-1}^{m},u_{W,i}^{m}\right)\right)\leq 0. \tag{4.30}$$

Proof. From the monotony of the scheme we have

$$u_{W,i}^{m+1}\leq u_{W,i}^{m}\top\kappa+\frac{k}{m\left(K_i\right)}\left(F_{K_iL_j}^{m}\left(u_{W,i-1}^{m}\top\kappa,u_{W,i}^{m}\right)-F_{K_iL_j}^{m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\top\kappa\right)\right)$$

we set

$$\kappa=\kappa+\frac{k}{m\left(K_i\right)}\left(F_{K_iL_j}^{m}\left(\kappa,\kappa\right)-F_{K_iL_j}^{m}\left(\kappa,\kappa\right)\right)$$

then

$$\kappa\leq u_{W,i}^{m}\top\kappa+\frac{k}{m\left(K_i\right)}\left(F_{K_iL_j}^{m}\left(u_{W,i-1}^{m}\top\kappa,\kappa\right)-F_{K_iL_j}^{m}\left(\kappa,u_{W,i+1}^{m}\top\kappa\right)\right)$$

such that $u_{W,i}^{m+1}\top\kappa=u_{W,i}^{m+1}$ or κ, thus

$$u_{W,i}^{m+1}\top\kappa\leq u_{W,i}^{m}\top\kappa+\frac{k}{m\left(K_i\right)}\left(F_{K_iL_j}^{m}\left(u_{W,i-1}^{m}\top\kappa,u_{W,i}^{m}\top\kappa\right)-F_{K_iL_j}^{m}\left(u_{W,i}^{m}\top\kappa,u_{W,i+1}^{m}\top\kappa\right)\right)$$

Therefore

$$\frac{\eta_\kappa^+\left(u_{W,i}^{m+1}\right)-\eta_\kappa^+\left(u_{W,i}^{m}\right)}{k}\leq\frac{1}{m\left(K_i\right)}\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i-1}^{m},u_{W,i}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\right)\right)$$

yields the result. ■

Continuous entropy inequality on the discrete solution

Proposition 4.3. *Assume that* (4.5),(4.13) *and the properties of Definition 4.4 hold. Let* $u_{W,\tau,k}$ *be the approximate solution of the problem* (4.4) *defined by* (4.7), (4.10) *and* (4.11), τ *be an admissible mesh in the sense of Definition* (4.3), $M \in \mathbb{N}^*$ *and let* $k=\dfrac{T}{M}\in\mathbb{R}_+^*$ *be the time step. Then,* $\mathbb{P}$*-a.s. in* Ω *:*

$$\text{for all } \kappa\in\mathbb{R} \text{ and } \varphi\in C_c^\infty\left([0,2\pi]\times[0,T[\,,\mathbb{R}^+\right)$$

$$\begin{aligned}
&\int\int_Q \left[\eta_\kappa^+\left(u_{W,i}^m\right)\varphi_t + \phi_\kappa^+\left(\omega,t,x,u_{W,i}^m\right)\varphi_x\right] dxdt \\
&+ \int_D \eta_\kappa^+\left(u_0\right)\varphi\left(0,x\right)dx \\
&+ LipB\int_0^\infty \left[\eta_\kappa^+\left(u_b\left(t,2\pi\right)\right)\varphi\left(t,2\pi\right) - \eta_\kappa^+\left(u_b\left(t,0\right)\right)\varphi\left(t,0\right)\right] dt \\
&\geq -\varepsilon_{\tau,k}\left(\varphi\right),
\end{aligned} \tag{4.31}$$

where for any $\mathbb{P}$*-measurable set* A, $E\left[1_A\varepsilon_{\tau,k}\left(\varphi\right)\right] \to 0$ *as* $h \to 0$.
The same result holds when the negative semi-Kruzkov entropies are considered.

Proof Let φ be in $C_c^\infty\left([0,2\pi]\times[0,T[\,,\mathbb{R}^+\right)$ and $\kappa\in\mathbb{R}$. We fix $T\geq 0$ such that $\varphi\equiv 0$ on $[T,\infty[\,\times D$. We also denote by u_τ^0 the application defined by $u_\tau^0\left(x\right)=u_i^0$ for a.e. $x\in K$, and by $u_{\tau,k}^b$ the application defined by $u_{\tau,k}^b\left(x\right)=-W(t,x_b)$ for a.e. $(t,x_b)\in[mk,(m+1)k[\,\times\{0,2\pi\}$. Multiplying The discrete entropy inequalities (4.30) by $\left|K_i\right|\varphi_i^m=\dfrac{1}{k}\displaystyle\int_{mk}^{(m+1)k}\int_{K_i}\varphi dxdt$, and summing over $i=1,..N$, and $m=0...,M-1$, yields the inequality:

$$T_1+T_2\leq 0, \tag{4.32}$$

where

$$T_1=\sum_{m=0}^{M-1}\sum_{i=1}^{N}\left|K_i\right|\left(\eta_\kappa^+\left(u_{W,i}^{m+1}\right)-\eta_\kappa^+\left(u_{W,i}^m\right)\right)\varphi_i^m,$$

and

$$\begin{aligned}
T_2=&\sum_{m=0}^{M-1}\sum_{i=1}^{N}k\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i}^m\right)\right)\varphi_i^m \\
&-\sum_{m=0}^{M-1}\sum_{i=1}^{N}k\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i-1}^m,u_{W,i}^m\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i}^m\right)\right)\varphi_i^m,
\end{aligned}$$

we can decompose T_2 as
$T_2=T_2^{int}+T_2^b$,
where

$$T_2^{int}=\sum_{m=0}^{M-1}\sum_{i=1}^{N-1}k\left[\begin{array}{c}\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i}^m\right)\right)\varphi_i^m \\ -\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^m,u_{W,i+1}^m\right)\right)\varphi_{i+1}^m\end{array}\right],$$

$$T_2^b=\sum_{m=0}^{M-1}k\left[\begin{array}{c}\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^m,u_{W,N+1}^m\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^m,u_{W,N}^m\right)\right)\varphi_N^m \\ -\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,0}^m,u_{W,1}^m\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,1}^m,u_{W,1}^m\right)\right)\varphi_1^m\end{array}\right].$$

Proving the approximate continuous entropy inequalities comes back to prove

$$T_{10}+T_{20}\leq\varepsilon_{\tau,k}\left(\varphi\right)$$

where T_{10} and T_{20} are defined by

$$T_{10} = -\int\int_Q \eta_\kappa^+\left(u_{W,i}^m\right)\varphi_t dxdt - \int_D \eta_\kappa^+\left(u_0\right)\varphi\left(x,0\right)dx,$$

$$T_{20} = -\int\int_Q \phi_\kappa^+\left(\omega,t,x,u_{W,i}^m\right)\varphi_x dxdt - LipB\int_0^\infty \left[\eta_\kappa^+\left(u_b\left(t,x\right)\right)\varphi\left(t,x\right)\right]_0^{2\pi}dt.$$

From the fact that $\partial_x B = 0$ we can deduce that:

$$\begin{aligned} T_{20} &= -\sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\sum_{i=1}^{N}\left[\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^m\right)\varphi\left(t,x_{i+\frac{1}{2}}\right) - \phi_\kappa^+\left(\omega,t,x_{i-\frac{1}{2}},u_{W,i}^m\right)\varphi\left(t,x_{i-\frac{1}{2}}\right)\right]dt \\ &\quad -LipB\int_0^\infty\left[\eta_\kappa^+\left(u_b\left(t,x\right)\right)\varphi\left(t,x\right)\right]_0^{2\pi}dt \end{aligned}$$

then we can decompose T_{20} as follows $T_{20} = T_{20}^{int} + T_{20}^b$,
where

$$T_{20}^{int} = -\sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\sum_{i=1}^{N-1}\left[\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^m\right) - \phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i+1}^m\right)\right]\varphi\left(t,x_{i+\frac{1}{2}}\right)dt$$

and

$$\begin{aligned} T_{20}^b &= -\sum_{m=0}^{M-1}k\int_{mk}^{(m+1)k}\frac{1}{k}\left[\phi_\kappa^+\left(\omega,t,x_{N+\frac{1}{2}},u_{W,N}^m\right)\varphi\left(t,x_{N+\frac{1}{2}}\right) - \phi_\kappa^+\left(\omega,t,x_{\frac{1}{2}},u_{W,1}^m\right)\varphi\left(t,x_{\frac{1}{2}}\right)\right]dt \\ &\quad -LipB\int_0^\infty\left[\eta_\kappa^+\left(u_b\left(t,x\right)\right)\varphi\left(t,x\right)\right]_0^{2\pi}dt. \end{aligned}$$

To this purpose, we compare T_{10} to T_1 and T_{20} to T_2.

1 Estimate on $T_{10} - T_1$

Using the definitions of u_τ^0 and $u_{\tau,k}$, the quantity T_{10} reads:

$$\begin{aligned} T_{10} &= \sum_{m=0}^{M-1}\sum_{i=1}^{N}\frac{\eta_\kappa^+\left(u_{W,i}^{m+1}\right) - \eta_\kappa^+\left(u_{W,i}^m\right)}{k}\int_{mk}^{(m+1)k}\int_{K_i}\varphi\left(x,\left(m+1\right)k\right)dxdt \\ &\quad +\int_D\left(\eta_\kappa^+\left(u_i^0\right) - \eta_\kappa^+\left(u_0\right)\right)\varphi\left(x,0\right)dx. \end{aligned}$$

From the fact that η_κ^+ is 1-Lipschitz continuous is deduced:

$$\left|T_{10} - T_1\right| \leq \epsilon_\tau^0\left(\varphi\right) + \epsilon_{\tau,k}^1\left(\varphi\right), \tag{4.33}$$

where

$$\epsilon_\tau^0\left(\varphi\right) = \int_D\left|u_i^0 - u_0\right|\varphi\left(x,0\right)dx,$$

and

$$\epsilon_{\tau,k}^1\left(\varphi\right) = \sum_{m=0}^{M-1}\sum_{i=1}^{N}\frac{\left|u_{W,i}^{m+1} - u_{W,i}^m\right|}{k}\int_{mk}^{(m+1)k}\int_{K_i}\left|\varphi\left(x,\left(m+1\right)k\right) - \varphi\left(t,x\right)\right|dxdt.$$

$$\epsilon_\tau^0\left(\varphi\right) = \int_D\left|u_i^0 - u_0\right|\varphi\left(x,0\right)dx \leq \left\|\varphi\right\|_\infty\int_D\left|\sum_{i=1}^{N}u_i^0 - u_0\right|dx$$

Before giving precise estimates on these quantities, we study the difference $T_{20} - T_2$.

2 Comparison of T_{20} and T_2

2.1 Estimate on $\left|T_{20}^{int} - T_2^{int}\right|$

In order to compare T_{20}^{int} to T_2^{int} , let us introduce the average value of φ on an edge, defined by

$$\widetilde{\varphi}_\sigma^m = \frac{1}{k}\int_{mk}^{(m+1)k} \varphi\left(t,\sigma\right)dt.$$

where $\sigma = K\backslash L$.

We have,

$$T_{20}^{int} = -\sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\sum_{i=1}^{N-1}\left[\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^m\right) - \phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i+1}^m\right)\right]\varphi\left(t,x_{i+\frac{1}{2}}\right)dt,$$

then

$$\begin{aligned}T_{20}^{int} = -\sum_{m=0}^{M-1}\sum_{i=1}^{N-1}\Bigg[&\int_{mk}^{(m+1)k}\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^m\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)\widetilde{\varphi}_{i+\frac{1}{2}}^m\\ &-\left(\frac{1}{k}\int_{mk}^{(m+1)k}\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i+1}^m\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)\widetilde{\varphi}_{i+\frac{1}{2}}^m\right)\Bigg],\end{aligned}$$

hence we can rewrite T_{20}^{int} in the following way :

$$\begin{aligned}T_{20}^{int} \;=\; & -\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\Bigg[\frac{1}{k}\int_{mk}^{(m+1)k}\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^m\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\\ & -\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i}^m\right)\widetilde{\varphi}_{i+\frac{1}{2}}^m + \left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i}^m\right) - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)\right)\widetilde{\varphi}_{i+\frac{1}{2}}^m\\ & -\left(\frac{1}{k}\int_{mk}^{(m+1)k}\phi_\kappa^+\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i+1}^m\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^m,u_{W,i+1}^m\right)\widetilde{\varphi}_{i+\frac{1}{2}}^m\right)\\ & -\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^m,u_{W,i+1}^m\right) - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right)\right)\widetilde{\varphi}_{i+\frac{1}{2}}^m\Bigg].\end{aligned}$$

Recall that the quantity T_2^{int} is defined by:

$$\begin{aligned}T_2^{int} = \sum_{m=0}^{M-1}\sum_{i=1}^{N-1}k\Bigg[&\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right) - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i}^m\right)\right)\varphi_i^m\\ &-\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^m,u_{W,i+1}^m\right) - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^m,u_{W,i+1}^m\right)\right)\varphi_{i+1}^m\Bigg].\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|T_{20}^{int}-T_{2}^{int}\right|\leq\\
&\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left[\left|\frac{1}{k}\int_{mk}^{(m+1)k}\left(\phi_{\kappa}^{+}\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i}^{m}\right)\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\right|\right]\\
&+\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left[\left|\frac{1}{k}\int_{mk}^{(m+1)k}\left(\phi_{\kappa}^{+}\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i+1}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^{m},u_{W,i+1}^{m}\right)\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\right|\right]\\
&+\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\right)\right)\left(\varphi_{i}^{m}-\widetilde{\varphi}_{i+\frac{1}{2}}^{m}\right)\right|\\
&+\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^{m},u_{W,i+1}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\right)\right)\left(\varphi_{i+1}^{m}-\widetilde{\varphi}_{i+\frac{1}{2}}^{m}\right)\right|
\end{aligned}$$

Then the following estimate holds:

$$\left|T_{20}^{int}-T_{2}^{int}\right|\leq\varepsilon_{\tau,k}^{c,int}(\varphi)+\varepsilon_{\tau,k}^{int,+}(\varphi)+\varepsilon_{\tau,k}^{int,-}(\varphi). \tag{4.34}$$

Where $\varepsilon_{\tau,k}^{c,int}$, $\varepsilon_{\tau,k}^{int,+}$ and $\varepsilon_{\tau,k}^{int,-}$ are defined by

$$\begin{aligned}
&\varepsilon_{\tau,k}^{c,int}(\varphi)=\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left[\left|\frac{1}{k}\int_{mk}^{(m+1)k}\left(\phi_{\kappa}^{+}\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i}^{m}\right)\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\right|\right]\\
&+\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left[\left|\frac{1}{k}\int_{mk}^{(m+1)k}\left(\phi_{\kappa}^{+}\left(\omega,t,x_{i+\frac{1}{2}},u_{W,i+1}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^{m},u_{W,i+1}^{m}\right)\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\right|\right]
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{\tau,k}^{int,+}(\varphi)&=\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\right)\right)\left(\varphi_{i}^{m}-\widetilde{\varphi}_{i+\frac{1}{2}}^{m}\right)\right|\\
\varepsilon_{\tau,k}^{int,-}(\varphi)&=\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i+1}^{m},u_{W,i+1}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,i}^{m},u_{W,i+1}^{m}\right)\right)\left(\varphi_{i+1}^{m}-\widetilde{\varphi}_{i+\frac{1}{2}}^{m}\right)\right|.
\end{aligned}$$

2.2 Comparison of T_2^b and T_{20}^b

Recall that the quantity T_2^b and T_{20}^b are defined by:

$$\begin{aligned}
T_{2}^{b}=\sum_{m=0}^{M-1}k\Bigg[&\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^{m},u_{W,2\pi}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^{m},u_{W,N}^{m}\right)\right)\varphi_{N}^{m}\\
&-\left(\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,0}^{m},u_{W,1}^{m}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,1}^{m},u_{W,1}^{m}\right)\right)\varphi_{1}^{m}\Bigg],
\end{aligned}$$

$$\begin{aligned}
T_{20}^{b}&=-\sum_{m=0}^{M-1}k\int_{mk}^{(m+1)k}\frac{1}{k}\left[\begin{array}{c}\phi_{\kappa}^{+}\left(\omega,t,x_{N+\frac{1}{2}},u_{W,N}^{m}\right)\varphi\left(t,x_{N+\frac{1}{2}}\right)\\-\phi_{\kappa}^{+}\left(\omega,t,x_{\frac{1}{2}},u_{W,1}^{m}\right)\varphi\left(t,x_{\frac{1}{2}}\right)\end{array}\right]dt\\
&\quad-LipB\int_{0}^{\infty}\left(\eta_{\kappa}^{+}\left(u_{b}(t,2\pi)\right)\varphi(t,2\pi)-\eta_{\kappa}^{+}\left(u_{b}(t,0)\right)\varphi(t,0)\right)dt.
\end{aligned}$$

Now, let us denote by $\widetilde{T}_{20}^{b}$ the following quantity

$$\begin{aligned}\widetilde{T}_{20}^{b} &= -\sum_{m=0}^{M-1} k\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^{m},u_{W,N}^{m}\right)\varphi_N^m + \sum_{m=0}^{M-1} k\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,1}^{m},u_{W,1}^{m}\right)\varphi_1^m \\ &\quad -\sum_{m=0}^{M-1} kLipB\left[\eta_\kappa^+\left(u_{W,2\pi}^{m}\right)\varphi_N^m + \eta_\kappa^+\left(u_{W,0}^{m}\right)\varphi_1^m\right]\end{aligned}$$

Then T_2^b can be compared to $\widetilde{T}_{20}^{b}$:

$$\begin{aligned}T_2^b - \widetilde{T}_{20}^{b} &= \sum_{m=0}^{M-1} k\left[\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^{m},u_{W,2\pi}^{m}\right)\varphi_N^m - \Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,0}^{m},u_{W,1}^{m}\right)\varphi_1^m\right] \\ &\quad +\sum_{m=0}^{M-1} kLipB\left[\eta_\kappa^+\left(u_{W,2\pi}^{m}\right)\varphi_N^m + \eta_\kappa^+\left(u_{W,0}^{m}\right)\varphi_1^m\right],\end{aligned}$$

and this quantity is nonnegative:

$$T_2^b - \widetilde{T}_{20}^{b} \geq 0. \tag{4.35}$$

Indeed, the following lemma holds:

Lemma 4.3. *Assume that the properties of Definition 4.4 holds. Then:* $\forall\kappa \in \mathbb{R}$, $\forall a, b \in \mathbb{R}$,

$$F_{K_i,L_j}^{m}\left(a\top\kappa, b\top\kappa\right) - F_{K_i,L_j}^{m}\left(\kappa,\kappa\right) + \left(F_1+F_2\right)\left(b-\kappa\right)^+ \geq 0.$$

proof. The numerical fluxes being nondecreasing functions with respect to their first variable, and nonincreasing with respect to their second variable, then the following inequality holds:

$$F_{K_i,L_j}^{m}\left(a\top\kappa, b\top\kappa\right) \geq F_{K_i,L_j}^{m}\left(\kappa, b\top\kappa\right),$$

$$F_{K_i,L_j}^{m}\left(a\top\kappa, b\top\kappa\right) - F_{K_i,L_j}^{m}\left(\kappa, b\top\kappa\right) \geq F_{K_i,L_j}^{m}\left(\kappa, b\top\kappa\right) - F_{K_i,L_j}^{m}\left(\kappa,\kappa\right).$$

Moreover, from the fact that F_{K_i,L_j}^{m} is a Lipschitz continuous function, we deduce the inequality

$$\left|F_{K_i,L_j}^{m}\left(\kappa,\kappa\right) - F_{K_i,L_j}^{m}\left(\kappa, b\top\kappa\right)\right| \geq \left(F_1+F_1\right)\left|\kappa - b\top\kappa\right|$$

thus,

$$F_{K_i,L_j}^{m}\left(\kappa, b\top\kappa\right) - F_{K_i,L_j}^{m}\left(\kappa,\kappa\right) + \left(F_1+F_2\right)\left(b-\kappa\right)^+ \geq 0,$$

if we take $LipB = \left(F_1+F_2\right)$, which yields the result. ■

Now, let us estimate the quantity $T_{20}^b - \widetilde{T}_{20}^{b}$

$$
\begin{aligned}
\left|T_{20}^{b}-\widetilde{T}_{20}^{b}\right| \leq\; & \sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\left|\phi_{\kappa}^{+}\left(\omega,t,x_{N+1},u_{W,N}^{m}\right)\varphi\left(t,x_{N+1}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^{m},u_{W,N}^{m}\right)\varphi_{N}^{m}\right|dt\\
&+\sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\left|\phi_{\kappa}^{+}\left(\omega,t,x_{0},u_{W,1}^{m}\right)\varphi\left(t,x_{0}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,1}^{m},u_{W,1}^{m}\right)\varphi_{1}^{m}\right|dt\\
&+\sum_{m=0}^{M-1}k\left(F_1+F_2\right)\eta_{\kappa}^{+}\left(u_{W,2\pi}^{m}\right)\left|\varphi_{N}^{m}-\widetilde{\varphi}_{N+1}^{m}\right|\\
&+\sum_{m=0}^{M-1}k\left(F_1+F_2\right)\eta_{\kappa}^{+}\left(u_{W,0}^{m}\right)\left|\varphi_{1}^{m}-\widetilde{\varphi}_{0}^{m}\right|\\
&+\left(F_1+F_2\right)\int_{0}^{\infty}\left|u_{W,0}^{m}-u_b\left(t,0\right)\right|\varphi\left(t,0\right)dt\\
&+\left(F_1+F_2\right)\int_{0}^{\infty}\left|u_{W,2\pi}^{m}-u_b\left(t,2\pi\right)\right|\varphi\left(t,2\pi\right)dt
\end{aligned}
$$

and get

$$
\left|T_{20}^{b}-\widetilde{T}_{20}^{b}\right| \leq \varepsilon_{\tau,k}^{c,b}\left(\varphi\right)+\widetilde{\varepsilon}_{\tau,k}^{c,b}\left(\varphi\right)+\varepsilon_{\tau,k}^{b}\left(\varphi\right), \tag{4.36}
$$

$$
\text{where}\left\{
\begin{aligned}
\varepsilon_{\tau,k}^{c,b}\left(\varphi\right) &= \sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\left|\phi_{\kappa}^{+}\left(\omega,t,x_{N+1},u_{W,N}^{m}\right)\varphi\left(t,x_{N+1}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^{m},u_{W,N}^{m}\right)\varphi_{N}^{m}\right|dt\\
&+\sum_{m=0}^{M-1}\int_{mk}^{(m+1)k}\left|\phi_{\kappa}^{+}\left(\omega,t,x_{0},u_{W,1}^{m}\right)\varphi\left(t,x_{0}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,1}^{m},u_{W,1}^{m}\right)\varphi_{1}^{m}\right|dt,\\
\widetilde{\varepsilon}_{\tau,k}^{c,b}\left(\varphi\right) &= \sum_{m=0}^{M-1}k\left(F_1+F_2\right)\eta_{\kappa}^{+}\left(u_{W,2\pi}^{m}\right)\left|\varphi_{N}^{m}-\widetilde{\varphi}_{N+1}^{m}\right|\\
&+\sum_{m=0}^{M-1}k\left(F_1+F_2\right)\eta_{\kappa}^{+}\left(u_{W,0}^{m}\right)\left|\varphi_{1}^{m}-\widetilde{\varphi}_{0}^{m}\right|,\\
\varepsilon_{\tau,k}^{b}\left(\varphi\right) &= \left(F_1+F_2\right)\int_{0}^{\infty}\left|u_{W,0}^{m}+u_b\left(t,0\right)\right|\varphi\left(t,0\right)dt\\
&+\left(F_1+F_2\right)\int_{0}^{\infty}\left|u_{W,2\pi}^{m}-u_b\left(t,2\pi\right)\right|\varphi\left(t,2\pi\right)dt.
\end{aligned}
\right.
$$

Eventually, from (4.32), (4.33), (4.34), (4.35) and (4.36) is deduced the approximate continuous entropy inequality (4.31) with

$$
\varepsilon_{\tau,k}\left(\varphi\right)=\epsilon_{\tau}^{0}\left(\varphi\right)+\epsilon_{\tau,k}^{1}\left(\varphi\right)+\varepsilon_{\tau,k}^{c,int}\left(\varphi\right)+\varepsilon_{\tau,k}^{int,+}\left(\varphi\right)+\varepsilon_{\tau,k}^{int,-}\left(\varphi\right)+\varepsilon_{\tau,k}^{c,b}\left(\varphi\right)+\widetilde{\varepsilon}_{\tau,k}^{c,b}\left(\varphi\right)+\varepsilon_{\tau,k}^{b}\left(\varphi\right)
$$

Let us now turn to the study of $\varepsilon_{\tau,k}$.

3 Estimate on $\varepsilon_{\tau,k}$

Convergence of $E\left[1_A\varepsilon^{c,int}_{\tau,k}(\varphi)\right]$:

$$\begin{aligned}
&E\left[1_A\varepsilon^{c,int}_{\tau,k}(\varphi)\right]\\
&=E\left[1_A\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\frac{1}{k}\int_{mk}^{(m+1)k}\left(\phi^+_\kappa\left(\omega,t,x_{i+\frac{1}{2}},u^m_{W,i}\right)-\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i}\right)\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\right|\right]\\
&+E\left[1_A\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\frac{1}{k}\int_{mk}^{(m+1)k}\left(\phi^+_\kappa\left(\omega,t,x_{i+\frac{1}{2}},u^m_{W,i+1}\right)-\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i+1},u^m_{W,i+1}\right)\right)\varphi\left(t,x_{i+\frac{1}{2}}\right)dt\right|\right]
\end{aligned}$$

As the numerical fluxes, the numerical entropy fluxes are consistents: for all $s\in\mathbb{R}$

$$\Psi^{+,m}_{K_i,L_j,\kappa}(s,s)=\frac{1}{k}\int_{mk}^{(m+1)k}\phi^+_\kappa\left(\omega,t,s,x_{i+\frac{1}{2}}\right)dt$$

Therefore,

$$\begin{aligned}
E\left[1_A\varepsilon^{c,int}_{\tau,k}(\varphi)\right]\leq{}&\|\varphi\|_{L^\infty}E\left[1_A\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i}\right)-\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i}\right)\right)\right|\right]\\
&+\|\varphi\|_{L^\infty}E\left[1_A\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i+1},u^m_{W,i+1}\right)-\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i+1},u^m_{W,i+1}\right)\right)\right|\right]\\
&=0.
\end{aligned}$$

we deduce that,

$$E\left[1_A\varepsilon^{c,int}_{\tau,k}(\varphi)\right]\to 0$$

We now study the errors $\varepsilon^{int,+}_{\tau,k}(\varphi)$ and $\varepsilon^{int,-}_{\tau,k}(\varphi)$. Here, the weak BV estimate in Proposition 4.2 is required. Indeed, we have

$$\left|\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i}\right)-\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i+1}\right)\right|\leq\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F(c,d)-F(c,c)\right)$$

Therefore,

$$\begin{aligned}
&E\left[1_A\varepsilon^{int,+}_{\tau,k}(\varphi)\right]\\
={}&E\left[1_A\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}\left|\left(\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i}\right)-\Psi^{+,m}_{K_i,L_j,\kappa}\left(u^m_{W,i},u^m_{W,i+1}\right)\right)\left(\varphi^m_i-\widetilde{\varphi}^m_{i+\frac{1}{2}}\right)\right|\right]\\
\leq{}&h\|\varphi_x\|_{L^\infty}\sum_{m=0}^{M-1}k\sum_{i=1}^{N-1}E\left[\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F(c,d)-F(c,c)\right)+\max_{u^m_{W,i+1}\leq d\leq c\leq u^m_{W,i}}\left(F(c,d)-F(d,d)\right)\right]\\
\leq{}&h\|\varphi_x\|_{L^\infty}\frac{C}{\sqrt{h}}=C\sqrt{h}\|\varphi_x\|_{L^\infty}\underset{h\to 0}{\to}0
\end{aligned}$$

where the constant C is given by Proposition 4.2. We would follow the same lines to get a similar estimate on $\varepsilon_{\tau,k}^{int,-}(\varphi)$.

Convergence of $E\left[1_A\varepsilon_{\tau,k}^{c,b}(\varphi)\right]$:

$$\begin{aligned}
E &\left[1_A\varepsilon_{\tau,k}^{c,b}(\varphi)\right]\\
= &\; E\left[1_A\left(\sum_{m=0}^{M-1}k\int_{mk}^{(m+1)k}\frac{1}{k}\left|\phi_\kappa^+\left(\omega,t,x_{N+1},u_{W,N}^m\right)\varphi\left(t,x_{N+1}\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,N}^m,u_{W,N}^m\right)\varphi_N^m\right|dt\right)\right]\\
&+E\left[1_A\left(\sum_{m=0}^{M-1}k\int_{mk}^{(m+1)k}\frac{1}{k}\left|\phi_\kappa^+\left(\omega,t,x_0,u_{W,1}^m\right)\varphi\left(t,x_0\right)-\Psi_{K_i,L_j,\kappa}^{+,m}\left(u_{W,1}^m,u_{W,1}^m\right)\varphi_1^m\right|dt\right)\right]\\
\leq &\; h\left\|\varphi_x\right\|_{L^\infty}\times 0=0
\end{aligned}$$

Convergence of $E\left[1_A\widetilde{\varepsilon}_{\tau,k}^{c,b}(\varphi)\right]$:

$$\begin{aligned}
&E\left[1_A\widetilde{\varepsilon}_{\tau,k}^{c,b}(\varphi)\right]\\
= &\; E\left[1_A\sum_{m=0}^{M-1}k\left(F_1+F_2\right)\left(\eta_\kappa^+\left(u_{W,2\pi}^m\right)\left|\varphi_N^m-\widetilde{\varphi}_{N+1}^m\right|+\eta_\kappa^+\left(u_{W,0}^m\right)\left|\varphi_1^m-\widetilde{\varphi}_0^m\right|\right)\right]\\
\leq &\; h\left\|\varphi_x\right\|_{L^\infty}\left(F_1+F_2\right)\sum_{m=0}^{M-1}kE\left[\eta_\kappa^+\left(u_{W,2\pi}^m\right)+\eta_\kappa^+\left(u_{W,0}^m\right)\right]\underset{h\to 0}{\to}0
\end{aligned}$$

Convergence of $E\left[1_A\varepsilon_{\tau,k}^{b}(\varphi)\right]$:

$$\begin{aligned}
&E\left[1_A\varepsilon_{\tau,k}^{b}(\varphi)\right]\\
= &\; E\left[1_A\left(F_1+F_2\right)\int_0^\infty\left|u_{W,0}^m+u_b\left(t,0\right)\right|\varphi\left(t,0\right)dt\right]\\
&+E\left[1_A\left(F_1+F_2\right)\int_0^\infty\left|u_{W,2\pi}^m-u_b\left(t,2\pi\right)\right|\varphi\left(t,2\pi\right)dt\right]\\
= &\; E\left[1_A\left(F_1+F_2\right)\int_0^\infty\left|-W^J\left(t^m,0\right)-W\left(t,0\right)\right|\varphi\left(t,0\right)dt\right]\\
&+E\left[1_A\left(F_1+F_2\right)\int_0^\infty\left|-W^J\left(t^m,2\pi\right)+W\left(t,2\pi\right)\right|\varphi\left(t,2\pi\right)dt\right]\\
\leq &\; \left(F_1+F_2\right)\left\|\varphi\right\|_{L^\infty}E\left[1_A\int_0^\infty\left|-W^J\left(t^m,0\right)-W\left(t,0\right)\right|\right]\\
&+\left(F_1+F_2\right)\left\|\varphi\right\|_{L^\infty}E\left[1_A\int_0^\infty\left|-W^J\left(t^m,2\pi\right)+W\left(t,2\pi\right)\right|dt\right]\underset{h\to 0}{\to}0.
\end{aligned}$$

Convergence of $E\left[1_A\epsilon_\tau^0(\varphi)\right]$:

$$E\left[1_A\epsilon_\tau^0(\varphi)\right]=E\left[1_A\int_D\left|u_i^0-u_0\right|\varphi\left(x,0\right)dx\right]\leq\left\|\varphi\right\|_\infty E\left[1_A\int_D\left|\sum_{i=1}^N u_i^0-u_0\right|dx\right]$$

which goes classically to 0 when h tends to 0.

Convergence of $E\left[1_A \epsilon^1_{\tau,k}(\varphi)\right]$:

$$\begin{aligned}
E\left[1_A \epsilon^1_{\tau,k}(\varphi)\right] &= E\left[1_A \sum_{m=0}^{M-1}\sum_{i=1}^{N} \frac{\left|u^{m+1}_{W,i}-u^m_{W,i}\right|}{k}\int_{mk}^{(m+1)k}\int_{K_i}\left|\varphi\left(x,(m+1)k\right)-\varphi\left(t,x\right)\right|dxdt\right]\\
&\leq \|\varphi_t\|_{L^\infty} m\left(K_i\right)k\frac{C}{\sqrt{h}}\\
&\leq \|\varphi_t\|_{L^\infty} k\sqrt{h} \underset{h\to 0}{\to} 0.
\end{aligned}$$

where the constant C is given by Proposition 4.2.

To summarize, we proved in this second step that $E\left[1_A \varepsilon_{\tau,k}(\varphi)\right] \to 0$ as $h \to 0$, which concludes the proof of the proposition. ■

4.6.2 Proof of the convergence

We prove now the convergence theorem (Theorem 4.1) of the finite volume approximation $u_{W,\tau,k}$ to the stochastic entropy solution of Problem (4.4).

We recall now this Theorem

Theorem 4.1[40] (Convergence to the stochastic entropy solution)

Assume that hypotheses H_1 to H_4 hold. Let τ be an admissible mesh in the sense of Definition 4.3, $M \in \mathbb{N}^*$ and $k = \frac{T}{M} \in \mathbb{R}^*_+$ be the time step. Let $u_{W,\tau,k}$ be the finite volume approximation defined by the monotone finite volume scheme (4.10) and (4.12). Then $u_{W,\tau,k}$ converges to the unique stochastic entropy solution of (4.4) in the sense of Definition 4.1, in $L^p_{loc}(\Omega \times Q)$ for any $1 \leq p < 2$ as $h \to 0$.

Proof. Let τ be an admissible mesh in the sense of Definition 4.3, $M \in \mathbb{N}^*$ and let $k = \frac{T}{M} \in \mathbb{R}^*_+$ be the time step such that $h \to 0$. In this way we suppose that (at least for h small enough) the CFL Condition

$$k \leq \frac{\alpha h}{2\left(F_1+F_2\right)}$$

In this manner, the estimates given by Proposition 4.1 and Proposition 4.2 hold. Consider A a $\mathbb{P}$-measurable set, $\varphi \in \mathcal{D}^+\left([0,T[\times\mathbb{R}\right)$. Let us multiply Inequality (4.31) by $\mathbf{1}_A$ and take the expectation. This yields:

$$\begin{aligned}
&E\left[\mathbf{1}_A \int\int_Q \eta^+_\kappa\left(u^m_{W,i}\right)\varphi_t + \phi^+_\kappa\left(\omega,t,x,u^m_{W,i}\right)\varphi_x dxdt\right]\\
&+E\left[\mathbf{1}_A \int_D \eta^+_\kappa\left(u_0\right)\varphi\left(0\right)dx\right]\\
&+LipBE\left[\mathbf{1}_A \int_0^\infty \left(\eta^+_\kappa\left(u_b\left(t,2\pi\right)\right)\varphi\left(t,2\pi\right)-\eta^+_\kappa\left(u_b\left(t,0\right)\right)\varphi\left(t,0\right)\right)dt\right]\\
\geq\ & -E\left[\mathbf{1}_A \varepsilon_{\tau,k}(\varphi)\right]. \qquad (4.37)
\end{aligned}$$

To show the convergence of $u_{W,\tau,k}$ towards the unique stochastic entropy solution of our problem 4.4, we aim to pass to the limit in the above inequality. Thanks to Proposition 4.3 we know that for any $\mathbb{P}$-measurable set A, $E\left[\mathbf{1}_A \varepsilon_{\tau,k}(\varphi)\right] \to 0$ as $h \to 0$. Thus it remains to study the convergence of the left-hand side of (4.37). Recall that thanks to the estimate stated in Proposition 4.1, $u_{W,\tau,k}$ converges (up to a subsequence denoted in the same way) in the sense of Young measures to an "entropy process" denoted by $\mathbf{u}-\mathbf{W}$ in $L^2(\Omega \times Q \times]0,1[)$.

1. Study of $E\left[\mathbf{1}_A \int\int_Q \eta_\kappa^+(u_{W,\tau,k})\varphi_t(t,x)\,dxdt\right]$ Note that $\Lambda : (\omega,t,x,\nu) \in \Omega \times Q \times \mathbb{R} \to \mathbf{1}_A(\omega)\eta_\kappa^+(\nu)\varphi_t(t,x) \in \mathbb{R}$ is a Carathéodory function such that $\Lambda(.,u_{W,\tau,k})$ is bounded in $L^2(\Omega \times Q)$, it is therefore uniformly integrable, thus

$$\begin{aligned} &E\left[\mathbf{1}_A \int\int_Q \eta_\kappa^+(u_{W,\tau,k})\varphi_t(t,x)\,dxdt\right] \\ \to\ &E\left[\mathbf{1}_A \int\int_Q \int_0^1 \eta_\kappa^+(\mathbf{u}(t,x,\alpha) - \mathbf{W}(t,x,\alpha))\,d\alpha\varphi_t(t,x)\,dxdt\right] \end{aligned}$$

We know that the values of $\mathbf{W}(t,x,\alpha) = W(t,x) = \sum_{j=1}^{\infty}\beta_j(t)e_j(x)$ do not depend on α, hence

$$\begin{aligned} &E\left[\mathbf{1}_A \int\int_Q \eta_\kappa^+(u_{W,\tau,k})\varphi_t(t,x)\,dxdt\right] \\ \to\ &E\left[\mathbf{1}_A \int\int_Q \int_0^1 \eta_\kappa^+(\mathbf{u}(t,x,\alpha) - W(t,x))\,d\alpha\varphi_t(t,x)\,dxdt\right] \end{aligned}$$

2. Study of $E\left[\mathbf{1}_A \int\int_Q \phi_\kappa^+\left(\omega,t,x,u_{W,i}^m\right)\varphi_x dxdt\right]$

   Since ϕ_κ^+ is bounded in $L^2(\Omega \times Q)$, using the same arguments as previously, we obtain

$$\begin{aligned} &E\left[\mathbf{1}_A \int\int_Q \phi_\kappa^+\left(\omega,t,x,u_{W,i}^m\right)\varphi_x dxdt\right] \\ \to\ &E\left[\mathbf{1}_A \int\int_Q \int_0^1 (\phi_\kappa^+(\omega,t,x,\mathbf{u}(t,x,\alpha) - W(t,x))\varphi_x)d\alpha dxdt\right] \end{aligned}$$

Finally, by passing to the limit in Inequality (4.37), we obtain:

For any $\mathbb{P}$-measurable set A, for any $\varphi \in \mathcal{D}^+([0,T[\times \mathbb{R})$

$$\begin{aligned}
0 \;\leq\; & E\left[\mathbf{1}_A \int\int_Q \int_0^1 \eta_\kappa^+ \left(\mathbf{u}\left(t,x,\alpha\right) - W\left(t,x\right)\right) \varphi_t\left(t,x\right) d\alpha dx dt\right] \\
& + E\left[\mathbf{1}_A \int\int_Q \int_0^1 (\phi_\kappa^+ \left(\omega,t,x,\mathbf{u}\left(t,x,\alpha\right) - W\left(t,x\right)\right) \varphi_x) d\alpha dx dt\right] \\
& + E\left[\mathbf{1}_A \int_D \eta_\kappa^+ \left(u_0\right) \varphi\left(0\right) dx\right] \\
& + LipBE\left[\mathbf{1}_A \int_0^\infty \left(\eta_\kappa^+ \left(u_b\left(t,2\pi\right)\right) \varphi\left(t,2\pi\right) - \eta_\kappa^+ \left(u_b\left(t,0\right)\right) \varphi\left(t,0\right)\right) dt\right].
\end{aligned}$$

Hence $\mathbf{u} - W$ is a measure-valued entropy solution in the sense of Definition 4.2. Thanks to Proposition 3.4 , $\mathbf{u}$ is independent of α and is hence the unique stochastic entropy solution in the sense of Definition 4.1 and we denote it by u. In this way, all the sequence of approximate solution $u_{\tau,k}$ converges to u in $L^1_{loc}\left(\Omega \times Q\right)$. In addition, since $u_{W,\tau,k}$ is bounded in $L^2\left(\Omega \times Q\right)$ from the dominated in norme convergence theorem, all the sequence converges in $L^p_{loc}\left(\Omega \times Q\right)$ for any $1 \leq p < 2$. ■

4.7 Numerical simulations

Our aim in this section is to numerically approximate solutions of the stochastically forced inviscid Burgers equation

$$\frac{\partial u\left(\omega,t,x\right)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u\left(\omega,t,x\right)^2}{2}\right) = \xi\left(\omega,t,x\right) \qquad \text{in } \Omega \times D \times]0,T[, \tag{4.38}$$

in a bounded domain $D =]0,2\pi[$ with Dirichlet boundary conditions.
We suppose that the stochastic source term in (4.38) is a space time white noise, i.e.

$$E\left(\xi\left(t,x\right)\xi\left(y,s\right)\right) = \delta\left(t-s\right)\delta\left(x-y\right).$$

We realize the white noise ξ as the generalized derivative of the cylindrical Wiener process, i.e.

$$\xi = \frac{\partial W}{\partial t}.$$

The integration of equation (4.38) over the control volume $K_i = \left]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right[$ results

$$\begin{aligned}
& m\left(K_i\right)\frac{\partial u_i\left(t\right)}{\partial t} + \frac{1}{2}u\left(\omega,t,x_{i+\frac{1}{2}}\right)^2 - \frac{1}{2}u\left(\omega,t,x_{i-\frac{1}{2}}\right)^2 \\
& = m\left(K_i\right)\frac{\partial W_i^J\left(t\right)}{\partial t}\;.
\end{aligned} \tag{4.39}$$

Where

$$u_i\left(t\right)=\frac{1}{m\left(K_i\right)}\int_{K_i}u\left(\omega,t,x\right)dx,\forall K_i\in\tau.$$

and, $W_i^J\left(t\right)=\sum_{j=1}^{J}\beta_j\left(t\right)e_i^j$, where $e_i^j=\frac{1}{m\left(K_i\right)}\int_{K_i}e_j\left(x\right)dx.$

The equation (4.39) can be written as

$$\begin{aligned}&m\left(K_i\right)\frac{\partial u_i\left(t\right)}{\partial t}+\Big(F^m_{K_iL_j}\left(u_i,u_{i+1}\right)-F^m_{K_iL_j}\left(u_{i-1},u_i\right)\Big)\\&=m\left(K_i\right)\frac{\partial W_i\left(t\right)}{\partial t}\end{aligned}\tag{4.40}$$

We then discretize the semi-discrete equation (4.40) in time using the explicit Euler schemes, we obtain

$$\begin{aligned}&\frac{m\left(K_i\right)}{k}\left(u_i^{m+1}-u_i^m\right)+\Big(F^m_{K_iL_j}\left(u_i^m,u_{i+1}^m\right)-F^m_{K_iL_j}\left(u_{i-1}^m,u_i^m\right)\Big)\\&=\frac{m\left(K_i\right)}{k}\left(W_i^{m+1}-W_i^m\right)\end{aligned}\tag{4.41}$$

for any $i=1,...N$, and any $m\in\{0,...,M-1\}$,

where,

$$u_0^m=u\left(\omega,t^m,0\right)=0,\quad u_{2\pi}^m=u\left(\omega,t^m,2\pi\right)=0,$$

and

$$u_i^0=\frac{1}{m\left(K_i\right)}\int_{K_i}u_0\left(x\right).$$

For the numerical flux, we use the Godunov scheme , which was introduced in Godunov [25], it seems to be a suitable choice for the Burgers' $1D$-equation, may be summarized by the following expression.

$$F\left(a,b\right)=\begin{cases}\min\left\{f\left(x\right),x\in\left[a,b\right]\right\} & \text{if } a\leq b,\\ \max\left\{f\left(x\right),x\in\left[b,a\right]\right\} & \text{if } b\leq a.\end{cases}\tag{4.42}$$

Because f is a nondecreasing function, the Godunov monotone flux scheme (4.42) reduces to $F(a,b)=f(a)$. Then, the scheme (4.41) reduces to

$$u_i^{m+1}=u_i^m+\frac{k}{2m\left(K_i\right)}\left(u_{i-1}^m\right)^2-\frac{k}{2m\left(K_i\right)}\left(u_i^m\right)^2+\left(W_i^{m+1}-W_i^m\right)\tag{4.43}$$

To illustrate our proposal, we need to revisit the deterministic methods in order to understand the effect of this random source term. In particular, we systematically draw graphs of two quantities: a prescribed single realization of the space time white noise, and an average of a large number of realizations, such that 200 and 80000.

We get the following graphics with the spatial and temporal step

$h = 2\pi/(n+1)$, $dt = 0.01$ respectively, where n is the number of control volume, $T = 0.1$ and the numerical tests are all performed with the deterministic initial condition $u_0(x) = \sin 2\pi x,\ x \in]0, 2\pi[$. These simulations have been implemented with the software Matlab.

Numerical entropy solution by finite volume method for t=0,01 to t=0,1 with dt=0,01 and number of control volume n=10

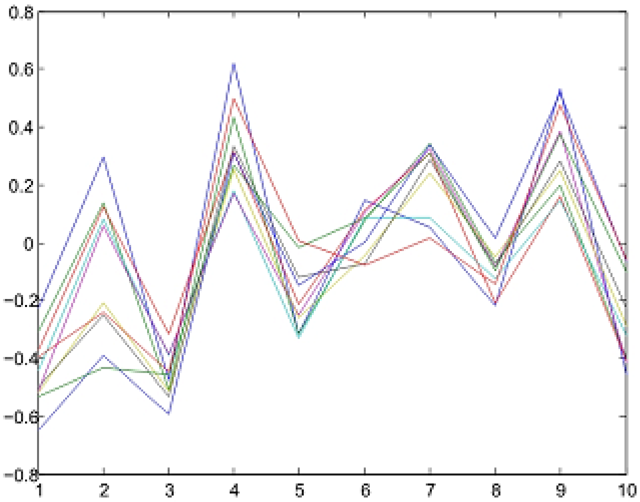

Figure 4.1: Numerical solution by FV method in the stochastic case with one realization of the noise

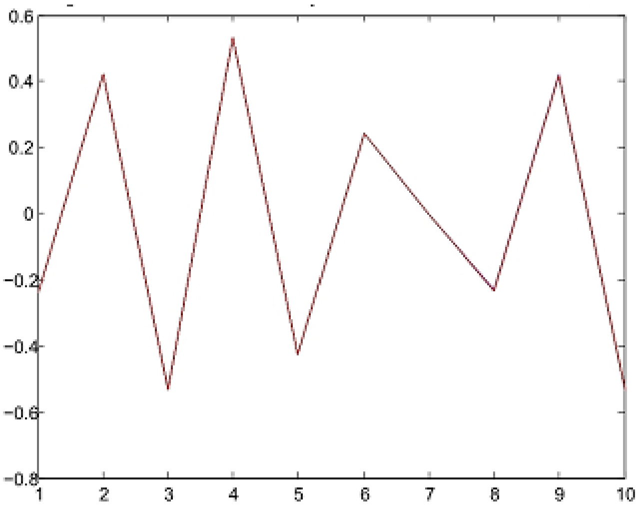

Figure 4.2: Numerical solution by FV method in the deterministic case

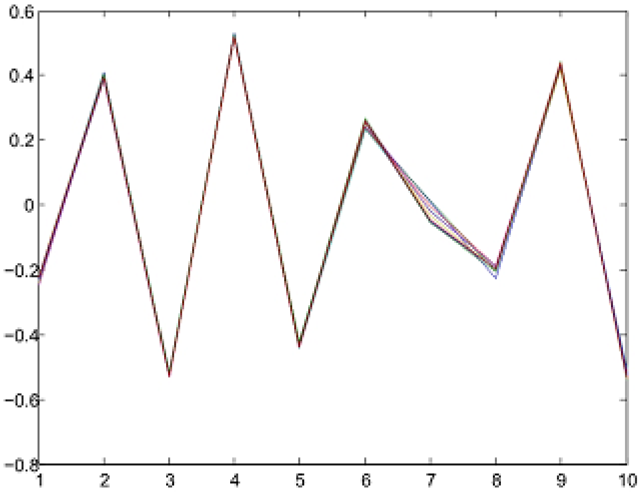

Figure 4.3: Numerical solution by FV method in the stochastic case average of 200 realizations of the noise

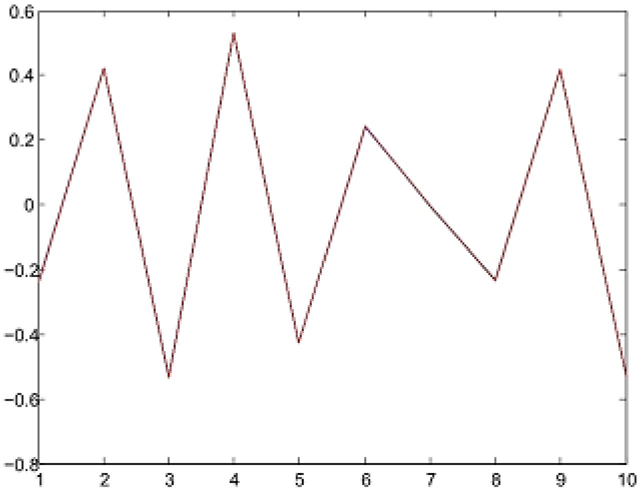

Figure 4.4: Numerical solution by FV method in the stochastic case average of 80000 realizations of the noise

Numerical entropy solution by finite volume method for t=0,01 to t=0,1 with dt=0,01 and number of control volume n=19

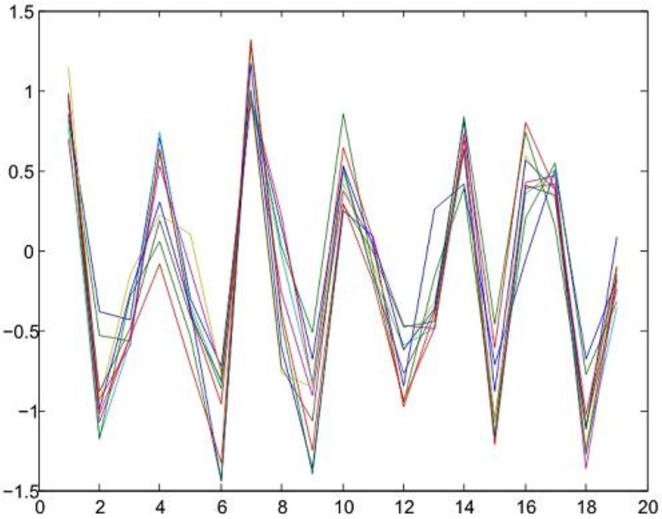

Figure 4.5: Numerical solution by FV method in the stochastic case with one realization of the noise

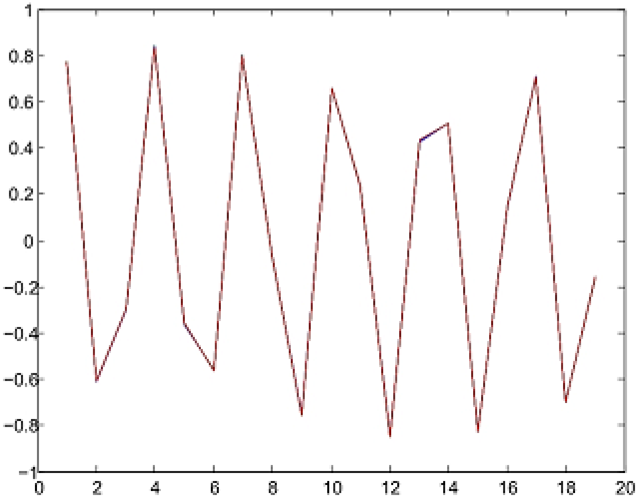

Figure 4.6: Numerical solution by FV method in the deterministic

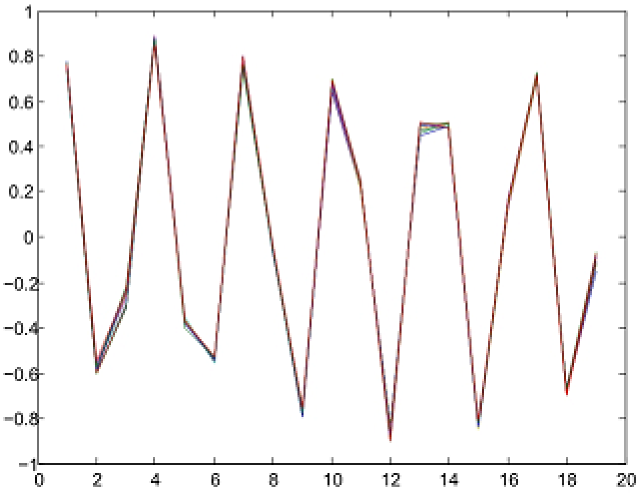

Figure 4.7: Numerical solution by FV method in the stochastic case average of 200 realizations of the noise

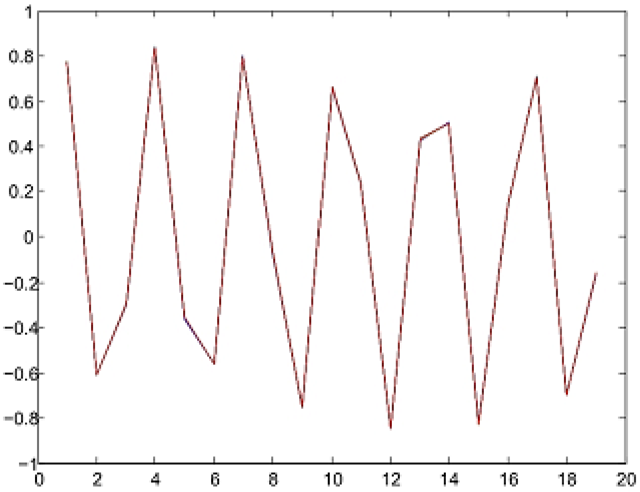

Figure 4.8: Numerical solution by FV method in the stochastic case average of 80000 realizations of the noise

Numerical entropy solution by finite volume method for t=0,01 to t=0,1 with dt=0,01 and number of control volume n=100

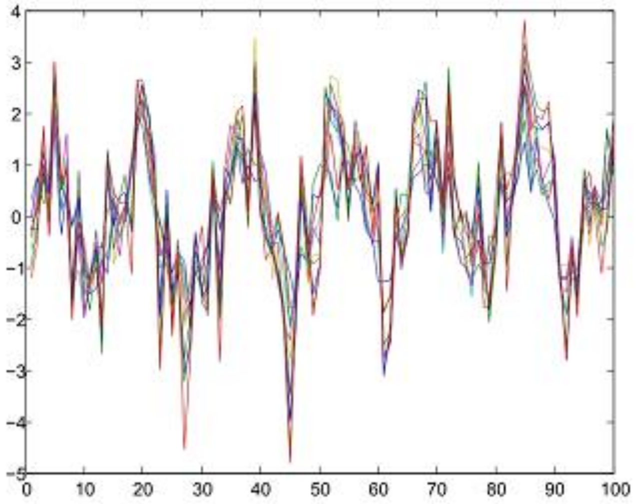

Figure 4.9: Numerical solution by FV method in the stochastic case with one realisation of the noise

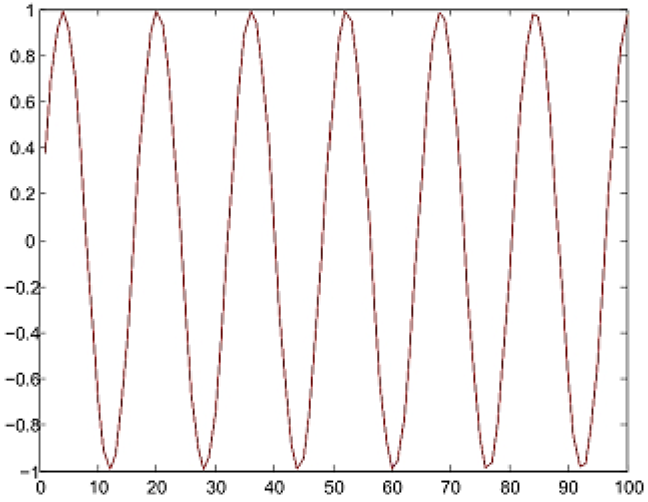

Figure 4.10: Numerical solution by FV method in the deterministic case

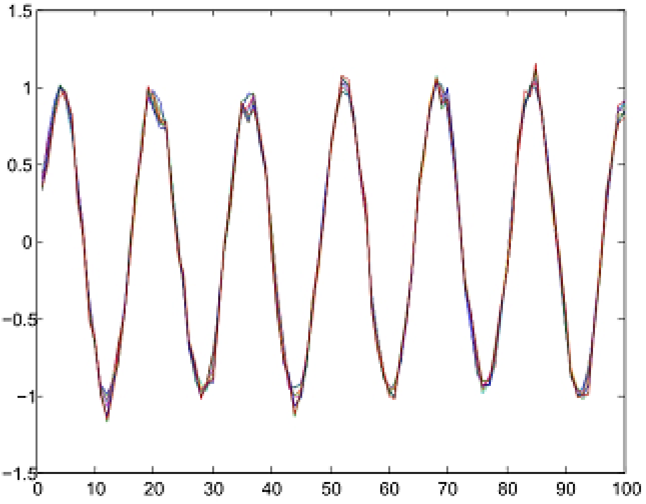

Figure 4.11: Numerical solution by FV method in the stochastic case average of 200 realization of the noise

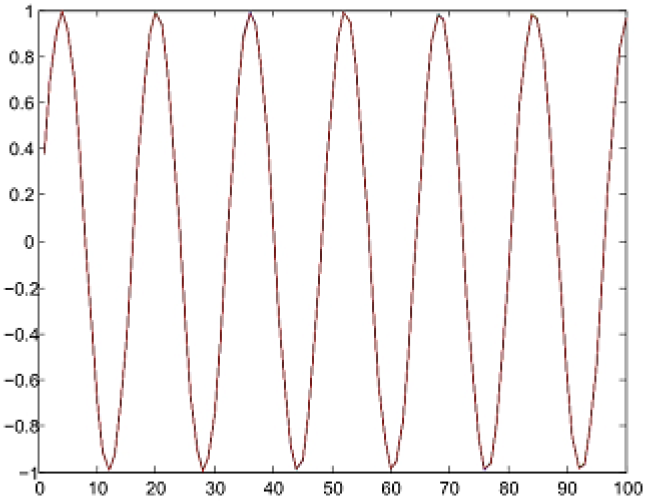

Figure 4.12: Numerical solution by FV method in the stochastic case average of 80000 realization of the noise

Numerical entropy solution by finite volume method for t=0,01 to t=0,1 with dt=0,01 and number of control volume n=200

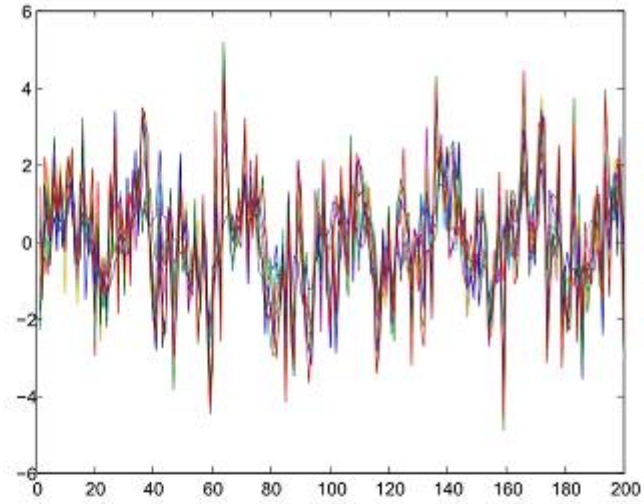

Figure 4.13: Numerical solution by FV method in the stochastic case with one realization of the noise

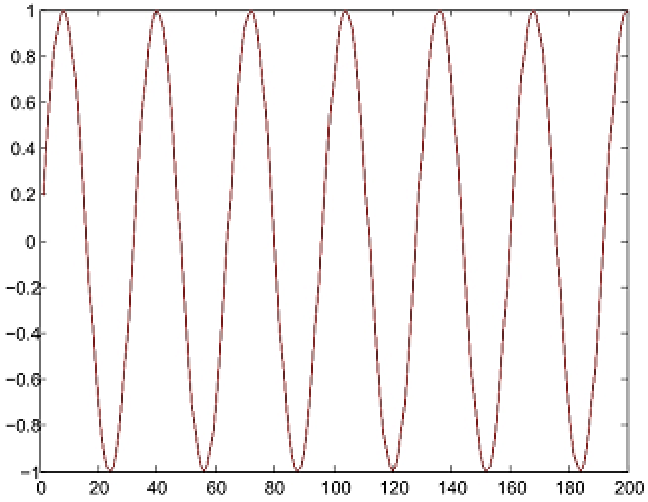

Figure 4.14: Numerical solution by FV method in the deterministic case

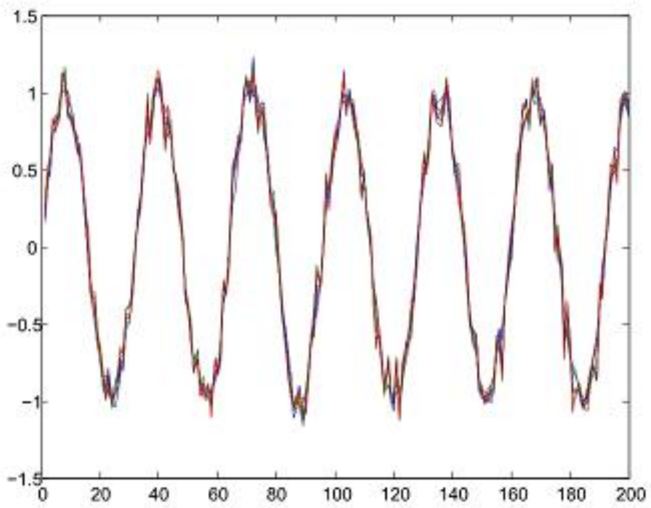

Figure 4.15: Numerical solution by FV method in the stochastic case average of 200 realization of the noise

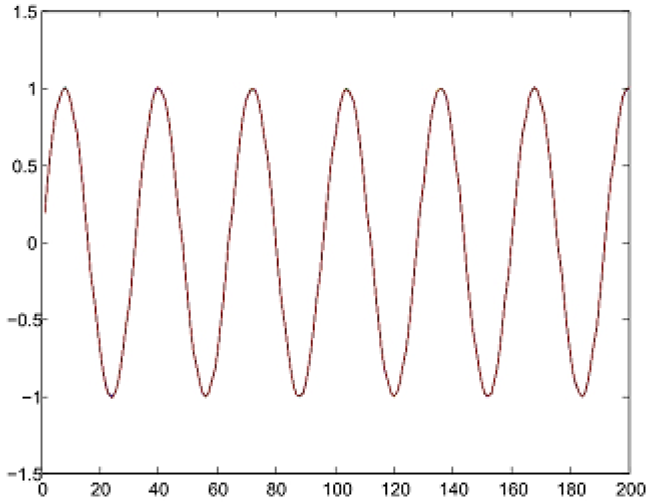

Figure 4.16: Numerical solution by FV method in the stochastic case average of 80000 realization of the noise

Numerical entropy solution by finite volume method for t=0,01 to t=0,1 with dt=0,01 and number of control volume n=800

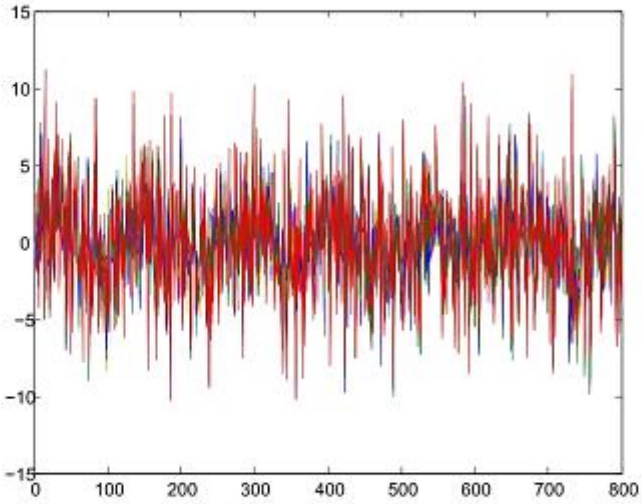

Figure 4.17: Numerical solution by FV method in the stochastic case with one realization of the noise

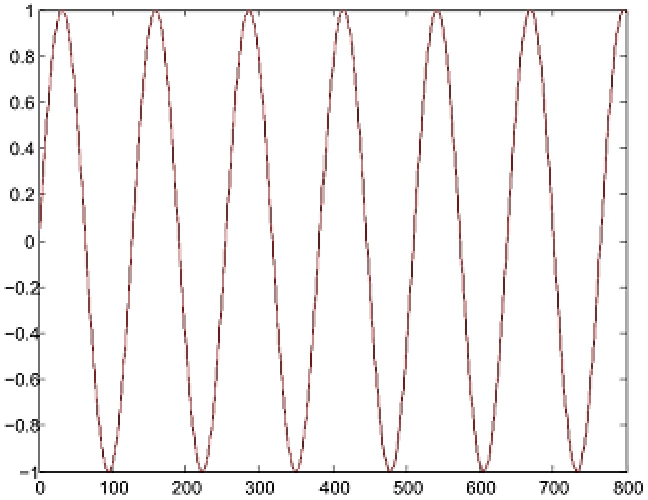

Figure 4.18: Numerical solution by FV method in the deterministic case

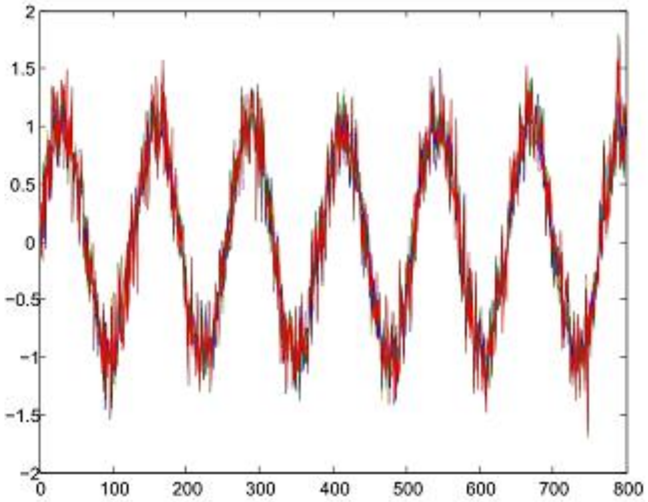

Figure 4.19: Numerical solution by FV method in the stochastic case average of 200 realization of the noise

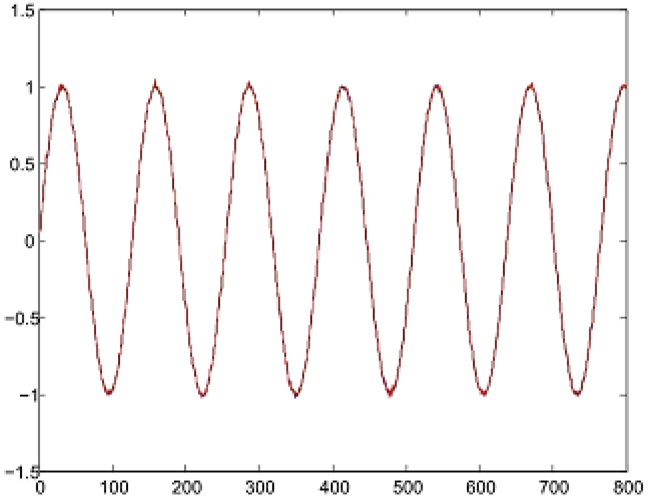

Figure 4.20: Numerical solution by FV method in the stochastic case average of 80000 realization of the noise

Conclusion

The main goal of this thesis was to study the finite volume method for the stochastic generalized Burgers equation with homogeneous Dirichlet boundary conditions in a bounded domain of $\mathbb{R}$. We have considered cases of white noise in space and time. Here, the most important findings and achievements of this work will be summarized.

As explained in the general introduction of this manuscript, the Burgers equation is not a good model for turbulence. It does not display any chaos, Then, adding a stochastic perturbation in the model is a suitable choice to take into account this imperfect knowledge of the studied phenomenon.

First, we establish a result of existence and uniqueness to the viscous equation using fixed point argument. Then, we prove a result of existence and uniqueness of the entropy solution to the inviscid equation using the concept of measure-valued solutions and Kruzhkov's entropy formulation.

Finally, we propose a semi-discrete finite volume approximation of the problem, and discretize the resulting system of SODE in time using the explicit Euler schemes, we prove the convergence of the finite volume approximate solution towards the unique stochastic entropy solution and we give some numerical simulation by the software Matlab that supports the theoretical analysis.

We find that stochastic entropy solution have more dissipation than deterministic solutions. This can be expected because the stochastic entropy inequality contains an additional dissipative term. Our numerical results for one realization are very dispersed. However, the average of a large number of realizations is very close to the deterministic solution (it is not equal to it), namely the solution of the partial differential equation where the source term ξ is such that $\xi = 0$.
So we conclude that our numerical results for the stochastic solution is a good approximation of reality. and the finite volume method is more adequate than other methods for the discritisation of stochastic conservation law.

Directions for Future Research

In view of the importance of the stochastic problem in physics, chemistry, biology and its various applications. We hope to develop these results in future. In particular, the following areas could lead to fruitful research:

(a) Study the problem with other type of noise such as colored noise and fractional Brownian motion.

(b) Study the fractional problem.

(c) Investigate our numerical simulations with different initial conditions.

(d) Compare our numerical simulations with various amplitude of the noise.

Appendix A

Basic Reminder of Young Measures

In this section we recall some basic facts on Young measures and refer to ([3], [13], [23] and [37]) for more information.
Consider the space $L^1(\Omega, \mu, \mathbb{R})$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space with a non-negative bounded measure μ.
For u in $L^1(\Omega, \mu, \mathbb{R})$ the Young measure associated with u is τ_u, the measure on $\Omega \times \mathbb{R}$ image of μ by $x \to (x, u(x))$.
A general Young measure τ is a non-negative measure on $\Omega \times \mathbb{R}$ such that, for any A in $\mathcal{F}$, $\tau(A \times \mathbb{R}) = \mu(A)$.
A Young measure τ is described by its disintegration which is the unique family of probabilities on $\mathbb{R}$, $(d\tau_x)_{x\in\Omega}$, such that for any τ-measurable function ψ,

$$x \to \int_{\mathbb{R}} \psi(x, \lambda)\, d\tau_x(\lambda) \qquad \text{is } \mu\text{-measurable on } \Omega \text{ and}$$

$$\text{if } \psi \geq 0, \quad \int_{\Omega\times\mathbb{R}} \psi d\tau = \int_{\Omega}\int_{\mathbb{R}} \psi(x, \lambda)\, d\tau_x(\lambda)\, \mu(dx).$$

Therefore, if $\tau = \tau_u$ is the Young measure associated with the above function u, then $\tau_x = \delta_{u(x)}$, the Dirac mass at $u(x)$.

Another way to define Young measures on $\Omega \times \mathbb{R}$ is to consider the notion of entropy process proposed by ([23]). For a Young measure τ on $\Omega \times \mathbb{R}$ and F_x the repartition function of τ_x, one considers the function $\mathbf{u}$, defined in $\Omega \times]0,1[$ by:

$$\mathbf{u}(x, \alpha) = \inf\{t \in \mathbb{R},\ F_x(t) > \alpha\}.$$

It is a $\mu \times \mathcal{L}$-measurable function on $\Omega \times]0,1[$ (where $\mathcal{L}$ is a Lebesgue measure on $\mathbb{R}$) and for any non-negative Carathéodory function ψ,

$$\int_{\Omega\times\mathbb{R}} \psi d\tau = \int_{\Omega}\int_{\mathbb{R}} \psi(x, \lambda)\, d\tau_x(\lambda)\, \mu(dx) = \int_{\Omega}\int_0^1 \psi(x, \mathbf{u}(x, \alpha))\, d\alpha\mu(dx).$$

A sequence of Young measure $(\tau^n)_n$ is said to converge narrowly towards τ if $\int_{\Omega\times\mathbb{R}} \psi d\tau^n$ converges towards $\int_{\Omega\times\mathbb{R}} \psi d\tau$ for all bounded Carathéodory function ψ.

Consider now $(u_n)_n \subset L^1(\Omega,\mu,\mathbb{R})$ and denote by τ^n the associated Young measures. If the sequence$(u_n)_n$ is assumed to be bounded in $L^1(\Omega)$, the theorem of Prohorov for Young measures ([3] and [37]) ensures that a subsequence $(\tau^{n_k})_k$ of $(\tau^n)_n$ and a Young measure τ exist such that τ^{n_k} converges narrowly towards τ.

Moreover:

(i) for μ-a.e. x in Ω, supp $(d\tau_x) \subset \cap_{p=1}^{\infty} \overline{\cup_{n\geq p}\{u_n(x)\}}$,

(ii) for any Carathéodory function ψ such that the sequence of functions $\{\psi(.,u_n(.))\}_n$ is uniformly integrable,

$$\int_{\Omega} \psi(x,u_n(x))\,\mu(dx) \to \int_{\Omega\times\mathbb{R}} \psi(x,\lambda)\,d\tau$$

(if the sequence $(u_n)_n$ is uniformly integrable, the above convergence still holds if one assumes that $|\psi(x,\lambda)| \leq \alpha(x) + k|\lambda|$ where $k \geq 0$ and $\alpha \in L^1(\Omega)$),

(iii) for any measurable function ψ, l.s.c. with respect to its second variable and such that $\left\{\psi(.,u_n(.))^-\right\}_n$ is uniformly integrable,

$$\liminf_{n\to\infty} \int_{\Omega} \psi(x,u_n(x))\,\mu(dx) \geq \int_{\Omega\times\mathbb{R}} \psi(x,\lambda)\,d\tau.$$

As a consequence, if u_n converges weakly to some u in L^1, it converges strongly to u in L^1, if and only if τ^n converges narrowly to τ_u.

Bibliography

[1] R. A. ADAMS, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 65 (Academic Press, 1975).

[2] H.W. ALT and S. LUCKHAUS, Quasilinear elliptic-parabolic differential equations, *Math. Z.* 183 (1983) 311–341.

[3] E.J. BALDER, Lectures on Young measure theory and its applications in economics. Workshop on Measure Theory and Real Analysis (Italian), Grado (1997). *Rend. Iftit. Mat. Univ. Trieste*31 (2000) 1–69.

[4] A. BAMBERGER, Etude d'une équation doublement non linéaire, *J. Funct. Anal.* 24 (1977) 148–155.

[5] C. BAUZET, J. CHARRIER, and T. GALLOUËT. Convergence of monotone finite volume schemes for hyperbolic scalar conservation laws with a multiplicative noise. *Accepted for publication in Stochastic Partial Differential Equations : Analysis and Computations,* 2014.

[6] C. BAUZET, G. VALLET and P.WITTBOLD, The Cauchy problem for a conservation law with a multiplicative stochastic perturbation. *J. Hyperbolic Differ. Eq.* 9 (2012) 661–709.

[7] BERTINI, L., CANCRINI, N., JONA-LASINIO, G.: The stochastic Burgers equation. *Commun. Math. Phys.* 165(2), 211–232 (1994)

[8] BRZEŹNIAK, Z., CAPIŃSKI, M., FLANDOLI, F.: Stochastic partial differential equations and turbulence. *Math. Models Methods Appl. Sci.* 1(1), 41–59 (1991).

[9] J.M. BURGERS. Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion. *Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen,* Afdeeling Natuurkunde, 1re série, (ISSN 0373-4668), 1939, vol. 17, cahier no 2, 1-53.

[10] J. CARRILLO, On the uniqueness of the solution of a class of elliptic equations with nonlinear convection, in *Contributions to Nonlinear Partial Differential Equations,* Vol. II (Paris, 1985), Pitman Res. Notes Math. Ser., Vol. 155 (Longman Sci. Tech., 1987), pp. 55–68.

[11] J. CARRILLO and P. WITTBOLD, Renormalized entropy solutions of scalar conservation laws with boundary condition, *J. Differ. Eqns.* 185 (2002) 137–160.

[12] J. CARRILLO and P. WITTBOLD, Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems, *J. Diff. Eqns.* 156 (1999) 93–121.

[13] Ch. CASTAING, P. RAYNAUD DE Fitte and M. VALADIER, *Young Measures on Topological Spaces with Applications in Control Theory and Probability Theory,* Mathematics and its Applications, Vol. 571 (Kluwer, 2004).

[14] D. H. CHAMBERS, R. J. ADRIAN, P. MOIN, D. S. STEWART, H. J. SUNG and L. KARHUNEN, Expansion of Burgers' model of turbulence. *Phys. Fluids*, 31, (9), 2573–2582, (1988).

[15] H. CHOI, R. TEMAM, P. MOIN and J. KIM, Feedback control for unsteady flow and its application to Burgers equation. Center for Turbulence Research, Stanford University, CTR Manuscript 131. To appear on *J. Fluid Mechanics* (1992).

[16] D.T. JENG, Forced Model Equation for Turbulence. *The Physics of Fluids* , 12, 10, 2006-2010 (1969).

[17] G. DA PRATO, A. DEBUSSCHE and R. TEMAM, Stochastic Burgers equation. *NoDEA Nonlinear Differential Equations Appl., vol. 1, no. 4, pp.* 389-402, 1994.

[18] G. DA PRATO and J. ZABCZYK, *Stochastic equations in infinite dimensions,* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.

[19] A. DEBUSSCHE, N. GLATT-HOLTZ, R. TEMAM and M. ZIANE. Global existence and regularity for the 3D stochastic primitive equations of the ocean and atmosphere with multiplicative white noise. *Nonlinearity,* 2012, 25(7):2093–2118.

[20] N. DIB, A. GUESMIA, and N. DAILI, On the Solution of Stochastic Generalized Burgers Equation. *Communications in Mathematics and Applications, Vol. 9, No. 4, pp.* 521-528, 2018.

[21] L. C. EVANS and R. F. GARIEPY, *Measure Theory and Fine Properties of Functions,* (CRC Press, 1992).

[22] R. EYMARD, T. GALLOUËT, and R. HERBIN. Finite volume methods. In *Handbook of numerical analysis, Vol. VII,* Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.

[23] R. EYMARD, T. GALLOUËT and R. HERBIN, Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation. *Chinese Ann. Math. Ser. B* 16 (1995) 1–14. A Chinese summary appears in *Chinese Ann. Math. Ser. A* 16 (1995) 119.

[24] F. FLANDOLI, *Dissipativity and invariant measures for stochastic Navier Stokes equations*, 24 Scuola Normale Superiore di Pisa, 1993.

[25] S. GODUNOV (1976), *Résolution numérique des problèmes multidimensionnels de la dynamique des gaz* (Editions de Moscou).

[26] W. GRECKSCH and C. TUDOR, *Stochastic Evolution Equations: A Hilbert Space Approach*, Mathematical Research, Vol. 85 (Akademie-Verlag, 1995).

[27] A. GUESMIA and N. DAILI, About the existence and uniqueness of solution to fractional Burgers equation, *Acta Universitatis Apulensis* 21(2010), 161-170.

[28] A. GUESMIA and N. DAILI, Existence and Uniqueness of an Entropy Solution for Burgers Equations, Applied Mathematical Sciences (AMS), Vol. 2, 33 (2008), 1635-1664.

[29] I. GYÖNGY and D. NUALART, On the stochastic Burgers' equation in the real line, *Ann. Probab.* 27 (1999) 782–802.

[30] H. HOLDEN and N. H. RISEBRO, A stochastic approach to conservation laws, in *Hyperbolic Problems Theory Numerical Methods and Applications*, Vol. II, Proc. Conf., Uppsala/ Sweden 1990, 575–587 (Studentlitteratur, 1991).

[31] I. HOSOKAWA and K. YAMAMOTO, Turbolence in the randomly forced one dimensional Burgers flow, *J. Stat. Phys.* 13, 245 (1975).

[32] M. KARDAR, M. PARISI and J. C. ZHANG, Dynamical scaling of growing interfaces, *Phys. Rev. Lett.*, 56, 889 (1986).

[33] J. U. KIM, On a stochastic scalar conservation law, *Indiana Univ. Math. J.* 52 (2003) 227–256.

[34] G. J. LORD, C. E. POWELL, T. SHARDLOW, *An introduction to computational stochastic PDEs*, Cambridge texts in applied mathematics, 2014.

[35] F. OTTO. Initial-boundary value problem for a scalar conservation law. *C. R. Acad. Sci. Paris Sér. I Math,322 (1996), no.* 8, 729–734.

[36] F. ROTHE, *Global Solutions of Reaction-Diffusion Systems*, Lecture Notes in Mathematics, 1072, Springer Verlag, 1984.

[37] M. VALADIER, A course on Young measures, *Rend. Istit. Mat. Univ. Trieste* 26, Suppl. (1–2) (1994) 349–394.

[38] G. VALLET, Stochastic perturbation of nonlinear degenerate parabolic problems, *Diff. Int. Eqns.*, to appear.

[39] G. VALLET and P. WITTBOLD, On a stochastic first-order hyperbolic equation in a bounded domain. *Infin. Dimens. Anal. Quantum. Probab. Relat. Top,* vol. 12, No. 4: 613-651, 2009.

[40] J. VOVELLE. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numer. Math. 90 (2002), no. 3, 563–596.*