République Algérienne Démocratique et Populaire
وزارة التعليم العالي والبحث العلمي
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université 20 Août 1955 -Skikda
Faculté des Sciences
Département des Mathématiques
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جامعة 20 أوت 1955 -سكيكدة
كلية العــلوم
قسم الرياضيات
المرجع:...........

Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option: ***Mathématiques***

Etude théorique et numérique d'une classe des équations fortement non linéaires

Présentée par:

Sara Dob

Soutenue publiquement: 28 Février 2021 à 11:00 h

Devant les Jury:

1.	**Guesmia Amar**	**Professeur**	**Université 20 Août 1955, Skikda**	**Président**
2.	**Ellaggoune Fateh**	**Professeur**	**Université 08 Mai 1945, Guelma**	**Examinateur**
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Année Universitaire: 2020/2021

Algerian Democratic and Popular Republic
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

University 20 august 1955-Skikda
Faculty of Sciences
Department of Mathematics
Ref: ………………

جامعة 20 أوت 1955 -سكيكدة
كلية العــلوم
قسم الرياضيات
المرجع:...........

Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics
Option: ***Mathematics***

Theoretical and numerical study of a class of strongly nonlinear equations

Presented by:

Sara Dob

Publicly discussed: February 28, 2021 at 11:00 h

In front of the Jury:

1.	**Guesmia Amar**	**Professor**	**University 20 august 1955, Skikda**	**President**
2.	**Ellaggoune Fateh**	**Professor**	**University 08 May 1945, Guelma**	**Examiner**
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University year: 2020/2021

Thesis *for the fulfilment of the requirements of Doctorate of 3rd cycle degree in Mathematics*
Option: *Mathematics*

Presented by:
SARA DOB

Topic:

Theoretical and numerical study of a class of strongly nonlinear equations

Supervisor: Professor. M. MAOUNI

Co-supervisor: MCA. H. LAKHAL

UNIVERSITY OF 20 AUGUST 1955-SKIKDA, 2021.

بسم الله الرحمن الرحيم

Acknowledgements

In the name of Allah, the most Beneficent, the most Merciful.
I thank Allah Almighty, for without his grace, I could not have finished this work.

This work was carried out at the Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), Mathematics Department, University of August 20th, 1955 Skikda under the direction of Professor ***M. Maouni***, to whom I would like to express my deep gratitude for the honour he gave me by proposing, accepting and directing this work. I thank him for his support and his scientific supervision necessary for the accomplishment of this work, for his availability as well as for his enthusiasm.

I would wholeheartedly like to thank my co-supervisor, Doctor ***H. Lakhal***, to whom we owe the theme of this thesis, for his patience, his support, his precious advice. Quite simply, without him, this work would not have seen the light this day.

Thanks to the jury members who will judge this work,
and I particularly mention Mr. ***A. Guesmia***, Professor at the university of August 20th, 1955 Skikda, for his availability, his orientations and the honour he gave me for agreeing to chair the jury for this work.

I humbly extend my thanks to those who were of help be it near or distant for their collaborations in the development of this work.

To all respected doctors and professors who contributed to our formation.

Dedication

I dedicate this modest work to the flame that lights my life and guides me to the right path, to my very dear mother, Dalila.

To my dear father, Abdelkader, for his unlimited support, his patience and his sacrifices.

To my brothers: Hichem and his wife Khaoula, Fayçal and his wife Amina, and Mohamed Lamine.

To my sisters: Wiem, Amel and Karima along with their husbands Nedjmeddine and Kamel.

To my nephews and my nieces Malak, Rym Djoury, Mohamed Hassan, Mohamed elzin, Lina Yassmin, Anas Zakaria, Djomana, Mohamed Raid, Maria and Meriem.

To all my family and relatives.

To all my friends and colleagues.

To all those who are present in the heart but absent in the lines.

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Abstract

The aim of this thesis is to study the existence of weak solutions for three types of systems. The first is a partial differential system which is considered as a generalization of a work obtained by A. Anane and N. Tsouli (2001). The others are fractional differential systems, we use the fixed point theory to prove the existence and uniqueness of solution of the second type which is a nonlinear fractional elliptic system, the third type are nonlinear elliptic systems in resonance and nonresonance, we use the Leray-Schauder topological degree to solve this kind of systems. Then, in the last part of our thesis, we present the finite differences method for the numerical approximation of the solution of the second system.

Keywords:

Elliptic systems, weak solution, topological degree, homotopy, fractional Laplacian, eigenvalues, finite differences method.

العنوان: **دراسة نظرية وعددية لفئة من المعادلات اللاخطية بقوة**

ملخص

الهدف من هذه الرسالة هو دراسة وجود حلول ضعيفة لثلاثة أنواع من الأنظمة. الأول هو نظام تفاضل جزئي يعتبر بمثابة تعميم للعمل المتحصل عليه من طرف أ. عنان و ن. تسولي (2001). أما الأنظمة الأخرى فهي أنظمة تفاضلية كسرية، نستخدم نظرية النقطة الثابتة لإثبات وجود وتفرد الحل للنوع الثاني والذي هو نظام إهليلجي غير خطي، النوع الثالث عبارة عن أنظمة إهليلجية غير خطية في الرنين وعدم الرنين نستخدم درجة ليراي-شودر الطوبولوجية لحل هذا النوع من الأنظمة. بعد ذلك ، في الجزء الأخير من أطروحتنا، نقدم طريقة الفروق المحدودة لـلحل التقريبي العددي للنظام الثاني.

Titre: **Etude théorique et numérique d'une classe des équations fortement non linéaires**

Résumé

L'objectif de cette thèse est l'étude de l'existence des solutions faibles pour trois types des systemes. Le premier est un système différentiel partiel qui est considèré comme une généralisation d'un travail obtenir par A. Anane et N. Tsouli (2001). Les autres sont des systèmes différentiels fractionnaires, en utilisant la théorie du point fixe pour prouver l'existence et l'unicité de solution du deuxième type qui est un système elliptique fractionnaire non linéaire, le troisième type sont des systèmes elliptiques non linéaires en résonance et non résonance, en utilisant le degré topologique de Leray-Schauder pour résoudre ce genre des systèmes. Ensuite, dans la dernière partie de notre thèse en présente la méthode des différences finies pour l'approximation numérique de la solution du deuxième système.

Mots clés:

Systèmes elliptiques, solution faible, degré topologique, homotopie, Laplacien fractionnaire, valeurs propres, méthode des différences finies.

Notations

- $\to$ designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- $\hookrightarrow$ indicates the continuous embedding.
- ∇ stands for the gradient operator.
- *div* is the divergence operator.
- $\dfrac{\partial}{\partial x}$ partial derivative.
- $\dfrac{\partial}{\partial n}$ outward normal derivative.
- Δ_p is the p-Laplace operator.
- Δ_p^{-1} is the p-Laplace inverse operator.
- Δ^s is the fractional Laplace operator of order s.
- *sp* denotes the spectrum of an operator.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^n$ is the real space of dimension n.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.

- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.

- Ω^c the complement of Ω.

- $< .,. >$ denotes the scalar product.

- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.

- $C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega)$.

- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.

- $L^p_{loc}(\Omega)$ the space of local p-integrable functions on Ω.

- $W^{m,p}(\Omega)$ Sobolev space with norm $\|\cdot\|_{m,p}$.

- $W^{m,p}_{loc}(\Omega)$ the local Sobolev space.

- $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

- $W^{-1,p'}(\Omega)$ is the dual of $W^{1,p}(\Omega)$.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $W^{s,p}(\Omega)$ fractional Sobolev space with norm $\|\cdot\|_{s,p}$.

- $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$.

- $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, $W_0^{s,2}(\mathbb{R}^n) = H_0^s(\mathbb{R}^n)$.

- $U = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ endowed with the norm

$$\|(u,v)\|_U = \|u\|^p_{W_0^{1,p}(\Omega)} + \|v\|^p_{W_0^{1,p}(\Omega)}.$$

- $V = L^p(\Omega) \times L^p(\Omega)$.
- $Y = L^{p'}(\Omega) \times L^{p'}(\Omega)$.
- $Z = W^{-1,p'}(\Omega) \times W^{-1,p'}(\Omega)$.
- $D^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^n),\ \text{such that } u = 0 \text{ in } \mathbb{R}^n \backslash \Omega\}$.
- $\widetilde{U} = D^{s,2}(\Omega) \times D^{s,2}(\Omega)$, with the norm, that we will denote by $\|.\|_U$

$$\|(u,v)\|^2_{\widetilde{U}} = \|u\|^2_{D^{s,2}(\Omega)} + \|v\|^2_{D^{s,2}(\Omega)}$$

.

- $\widetilde{V} = L^2(\Omega) \times L^2(\Omega)$.
- $B(0,R)$ open ball centered at the point 0 with the radius R.
- $\partial B(0,R)$ sphere centered at the point 0 with the radius R.
- $\mathcal{S}$ Schwartz Space.
- Γ is the usual Gamma function.
- I_α is the Riesz potentials.
- $P.V.$ is an abbreviation for “in the principal value sense”.

Contents

Introduction

In recent years, partial differential equations (PDEs) or even fractional differential equations (FDEs) have been the subjects of interest not only among mathematicians, but also among physicists and engineers. For details, see [35, 49]. Systems of strongly nonlinear elliptic equations present some new and interesting phenomena, which are not present in the study of a single equation. Many publications have appeared concerning nonlinear elliptic systems we refer the readers to [29, 64].

It is crucial to mention that fractional calculus is a generalization of ordinary differentiation and integration. The idea of fractional calculus is considered since 1695 when the derivative of arbitrary order was described by Leibniz [46]. After that, many researchers have studied the fractional derivatives and the fractional differential equations like Liouville, Grunwald and Riemann [51] and over the course of time, the renowned mathematicians gave plenty of attention to the fractional calculus see [37, 59, 61]. In the last few years, there has been considerable interest in the nonlinear fractional systems. It's caused by both the intensive development of the theory of fractional calculus itself and by its application in various fields of science such as electrochemistry, biology, viscoelasticity, chaotic systems, biophysics, chemistry...ect (see [34, 38, 62]) and the references therein.

Another aspect in the study of fractional coupled systems is when involving fractional Laplacian and as far as we know, the fractional Laplacian is widely-spread in the modern study of fractional differential systems. It has a variety of definitions, though they can be distilled down to the following two:

$$(-\Delta)^s\varphi = (2\pi|\xi|^{2s}\widehat{\varphi})\check{},$$

and

$$(-\Delta)^s\varphi(z) = C(n,s)P.V.\int_{\mathbb{R}^n} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}dy.$$

The fractional Poisson system is one of the building blocks in the study of fractional systems, the extended homogeneous boundary conditions are imposed on the complement Ω^c, distinguishing from the classical Poisson problem where boundary conditions are given on $\partial\Omega$.

This difference can be explained from probabilistic interpretation of the standard and fractional Laplacian. The standard Laplace operator represents the infinitesimal generator of a Brownian motion with continuous sample paths; thus for a particle in domain Ω, it must leave the domain via the boundary points on $\partial\Omega$. By contrast, the fractional Laplacian is the infinitesimal generator of a symmetric s-stable Lévy process with discontinuous sample paths; particles may "jump" out of the domain without touching any boundary points on $\partial\Omega$. Hence, the solution

on Ω can be determined by the values at $\partial\Omega$ in the context of classical Poisson systems but not in the context of fractional Poisson systems.
Many methods have been proposed to deal with nonlinear systems: fixed point method, semigroups method, sub-supersolution method, Brouwer degree and Leray-Schauder degree, etc. The last method is an important topological tool introduced by Leray and Schauder in the study of nonlinear partial differential equations in the early 1930's. The nontriviality of the degree ensures the existence of a fixed point of the compact mapping in the domain. It combines the properties of homotopy invariance and additivity, which make the topological tool more convenient in application, and provides more information on fixed points. The Leray-Schauder degree is an extension of the Brouwer degree from finite-dimensional spaces to infinite-dimensional Banach spaces. For more details, we refer the readers to [4, 14, 17, 31].
All researchers agree on the difficulty of studying the numerical solution of nonlinear fractional systems, but it has been studied extensively in the last decade see [2, 57, 63]. The finite difference methods for the nonlinear fractional problem were extended in some sense [56] and many authors contributed to develop the finite difference approximations.

This thesis is mainly devoted to the study of a class of strongly nonlinear elliptic systems with the Laplacian operator. We use the topological degree method, the fixed point theorems and some functional analysis tools to prove the existence of weak solutions for these systems and we present a finite differences method for the numerical approximation of the solution to a nonlinear system. To be more precise, we are interested in the study of two classes of systems:

- The first class of systems is an eigenvalue systems for p-Laplacian operators. The main purpose of this work is to prove the existence of solutions for the quasilinear elliptic system
$$\begin{cases} -\Delta_p u(x) = f_1(x, v(x)) + h_1(x) \quad \text{in } \Omega, \\ -\Delta_p v(x) = f_2(x, u(x)) + h_2(x) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$
when the second terms on the two equations $f_i(x, s)$, $(i = 1, 2)$ locates between the first and the second eigenvalue of the p-Laplacian. This result can be seen as a generalization of the result obtained by A. Anane and N. Tsouli in [6].

- The second class of systems are the nonlinear fractional elliptic systems involving the fractional Laplacian. We study the existence of weak solutions to the system
$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) \quad \text{in} \quad \Omega, \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) \quad \text{in} \quad \Omega, \\ u = v = 0 \qquad\qquad\qquad \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2}$$
where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in]0, 1[$, f, g are two continuous functions satisfying the Carathéodory conditions, and also verifying the growth restriction. As far as we know, this result is new and represent fractional version of the classical theorem see [23].

- In the last class of systems, we are interested in the nonlinear fractional elliptic systems at resonance and nonresonance of the type

  $$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) + f_1(x) & \text{in} \quad \Omega, \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) + f_2(x) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3}$$

  with $s \in (0,1)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, f, g are continuous functions. H. Lakhal and B. Khodja [39] studied the system (3) in the classical case.

The thesis is divided into five chapters. In the first chapter, we will discuss some preliminary materials that we will use throughout the thesis, this chapter has the following structure: firstly, we recall definitions and important results in the $L^p(\Omega)$ spaces, Sobolev spaces and Fractional Sobolev spaces that have an essential role in the subsequent chapters, then we introduce the method of the topological degree which is very useful tool for solving nonlinear systems. Finally, we define the phenomena of resonance and nonresonance.

In the second chapter, we investigate the existence of weak solutions of the problem (1). We use the Leray-Schauder degree to obtain the existence result.

The third chapter is devoted for the use of the fixed point theory to prove the existence of a weak solutions to the system (2).

In the fourth chapter, we provide an application of the Leray-Schauder degree theorem to prove the existence of weak solutions to the system (3).

The focus of the last chapter is to prove a general convergence result of the finite difference approximation of the nonlinear fractional elliptic system (2) in two dimension.

Chapter 1

Preliminaries

In this chapter, we cover three parts, we start by recalling a various general results in functional analysis which have served in this thesis. In the second part we introduce the method of topological degree and in the final part we define the resonance and non-resonant phenomenon. The reader can easily find the detailed in the related works see, e.g., [1, 4, 10, 27, 31, 33].

1.1 Functional spaces

Here we recall the essential notions on functional spaces, particularly, L^p spaces, Sobolev spaces and fractional Sobolev spaces. We give, by the same occasion, some definitions and useful results for the following chapters.

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$, we note

$$C(\Omega) = \{f : \Omega \to \mathbb{R}; f \text{ continuous } \},$$

and

$$C^m(\Omega): \text{ The space of functions } m \text{ times continuously differentiable on } \Omega,$$

where

$$C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega),$$

$$C_c(\Omega) = \{f \in C(\Omega); f(x) = 0 \quad \forall x \in \Omega\backslash K, \text{ where } K \text{ is copmact}\},$$

and $D(\Omega)$ the space of functions C^∞ on Ω with compact support in Ω (also called the space of test functions).

1.1.1 The $L^p(\Omega)$ spaces

Let $1 \leqslant p < \infty$ and Ω be an open subset in $\mathbb{R}^n$. We define the standard Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ \text{ f } : \Omega \to \mathbb{R} : \text{ f is measurable and } \int_\Omega |f(x)|^p \, \mathrm{d}x < \infty \right\},$$

with the norm

$$\|f\|_{L^p} = \|f\|_p = \Big(\int_\Omega |f(x)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

If $p = \infty$, we define

$$L^\infty(\Omega) = \Big\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \leqslant C \text{ a.e. on } \Omega \Big\},$$

with the norm

$$\|f\|_\infty = \mathrm{Inf}\Big\{C; |f(x)| \leqslant C \text{ a.e. on } \Omega\Big\}.$$

Remark 1.1.1. *In particularly, when $p = 2$, $L^2(\Omega)$ is a Hilbert space for the inner product*

$$(f, g) = \int_\Omega f(x)g(x)\,\mathrm{d}x.$$

Recall that $L^p_{loc}(\Omega)$ denotes the set of locally integrable functions on Ω, i.e.

$$L^p_{Loc} = \{f : \Omega \to \mathbb{R} \text{ measurable such that} : \forall K \text{ compact} \subset \Omega \int_K |f(x)|^p dx < \infty\}.$$

In particular

$$L^p(\Omega) \subset L^p_{Loc}(\Omega).$$

Proposition 1.1.1. *[12]*

1. *For $1 \leqslant p \leqslant \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space.*
2. *For $1 \leqslant p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a separable space.*
3. *For $1 < p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a reflexive space.*

Notation: Let $1 \leqslant p \leqslant \infty$; we denote by p' the conjugate exponent,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Some useful inequalities

In this part, we give some inequalities which will be used in the subsequent chapters.

Theorem 1.1.1 (Young's inequality). *[10] Let $1 \leqslant p \leqslant \infty$, then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \forall a \geqslant 0, b \geqslant 0.$$

Theorem 1.1.2 (Hölder's inequality). *[10] Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leqslant p \leqslant \infty$. Then*

$$fg \in L^1(\Omega) \text{ and } \|fg\|_1 \leqslant \|f\|_p \|g\|_{p'}.$$

Theorem 1.1.3 (Minkowski inequality). *[10] Let $f, g \in L^p(\Omega)$ and $p \geqslant 1$, then*

$$f + g \in L^p(\Omega) \text{ and } \|f + g\|_p \leqslant \|f\|_p + \|g\|_p.$$

Some results about integration and duality

Theorem 1.1.4 (Dominated convergence of lebesgue). *[10]*
Let (f_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfy*

1. $f_n(x) \to f(x)$ *a.e. on* Ω,
2. *there is a function* $g \in L^1(\Omega)$ *such that for all* $n \geqslant 1$, $|f_n(x)| \leqslant g(x)$ *a.e. on* Ω.

Then $f \in L^1(\Omega)$ *and* $\|f_n - f\|_1 \to 0$.

Lemma 1.1.1. *[10] Let* (f_n) *be a sequence in* $L^p(\Omega)$ *and* $f \in L^p(\Omega)$ *such that* $\|f_n - f\|_p \to 0$. *Then, there exist a subsequence* (f_{n_k}) *and a function* $h \in L^p(\Omega)$ *such that*

1. $f_{n_k}(x) \to f(x)$ *a.e. on* Ω,
2. $|f_{n_k}(x)| \leqslant h(x)$ $\forall k$, *a.e. on* Ω.

Theorem 1.1.5 (Riesz representation theorem). *[10]*
Let $1 < p < \infty$ *and let* $\varphi \in (L^p(\Omega))'$. *Then there exists a unique function* $u \in L^{p'}(\Omega)$ *such that*

$$\langle \varphi, f\rangle = \int_\Omega uf \,\mathrm{d}x, \forall f \in L^p(\Omega).$$

Moreover,

$$\|u\|_{p'} = \|\varphi\|_{(L^p)'}.$$

Definition 1.1.1 (Weak derivative). *[36] Assume that* $f \in L^1_{loc}(\Omega)$ *and let* $\alpha \in \mathbb{N}^n$ *be a multi-index. Then* $g \in L^1_{loc}(\Omega)$ *is the* α*th weak partial derivative of* f, *written* $D^\alpha f = g$, *if*

$$\int_\Omega f D^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g\varphi \,\mathrm{d}x,$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

Warning: If such a g exists, it is unique, however it doesn't always exist.

1.1.2 Sobolev spaces

A Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p-norms of the function together with its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, i.e. a Banach space. Intuitively, a Sobolev space is a space of functions possessing sufficiently many derivatives for some application domain, such as partial differential equations.

Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense.

Let Ω be an open set of $\mathbb{R}^n$ we define a functional $\| \cdot \|_{m,p}$, where m is a nonnegative integer and $1 \leqslant p \leqslant \infty$, as follows;

$$\|f\|_{m,p} = \left\{ \sum_{0 \leqslant |\alpha| \leqslant m} \|D^\alpha f\|_p^p \right\}^{1/p}, \quad \text{if } 1 \leqslant p < \infty$$

$$\|f\|_{m,\infty} = \max_{0 \leqslant |\alpha| \leqslant m} \|D^\alpha f\|_\infty;$$

for any function f that gives meaning to this writing.

We define the space $W^{m,p}(\Omega)$ as being the space of measurable functions $f \in L^p(\Omega)$ such that the derivative in the weak sense $D^\alpha f$, $(0 \leqslant |\alpha| \leqslant m)$ belongs to $L^p(\Omega)$ and the space $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

We associate the space $W^{m,p}(\Omega)$ with the norm $\| \cdot \|_{m,p}$, then have the following proposition

Proposition 1.1.2. *[12] Let Ω be an open subset of $\mathbb{R}^n$;*

1. *For $1 \leqslant p \leqslant \infty$, $W^{m,p}(\Omega)$ is a Banach space.*
2. *For $1 \leqslant p < \infty$, $W^{m,p}(\Omega)$ is a separable space.*
3. *For $1 < p < \infty$, $W^{m,p}(\Omega)$ is a reflexive space.*

Remark 1.1.2. *If $p = 2$, we usually write*

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Theorem 1.1.6. *[33] Let us assume that Ω is an open subset of $\mathbb{R}^n$ $(n \geqslant 1)$, $m \in \mathbb{N}$, $1 \leqslant p < \infty$ and $p^* = \dfrac{np}{n-p}$. Then*

1. *If $\dfrac{1}{p} - \dfrac{m}{n} > 0$, we have $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, with $q \in [p, p^*]$, $\dfrac{1}{p} - \dfrac{1}{p^*} = \dfrac{m}{n}$.*
2. *If $\dfrac{1}{p} - \dfrac{m}{n} = 0$, we have $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, +\infty[$.*
3. *If $\dfrac{1}{p} - \dfrac{m}{n} < 0$, we have $W_0^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.*

Remark 1.1.3. *The space $W^{1,p}(\Omega)$ is equipped with the norm*

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Proposition 1.1.3 (**Integration by parts formula**). *[33] Let $u, v \in H^1(\Omega)$ and $\partial\Omega \in C^1$, then for every $1 \leqslant i \leqslant N$, we have*

$$\int_\Omega \frac{\partial u(x)}{\partial x_i} v(x)\,\mathrm{d}x = -\int_\Omega u(x)\frac{\partial v(x)}{\partial x_i}\,\mathrm{d}x + \int_{\partial\Omega} u(s)v(s)n_i ds,$$

where $n = cos(n, x)$ is the cosine of the angle of the outside normal at $\partial\Omega$ and the axis x_i.

If $v \in H^1(\Omega)$ and if $u_i \in H^1(\Omega)$ where u_i is the components of the vector $\vec{u}$, then we have

$$\int_\Omega div(\vec{u}(x)) \cdot v(x)\,\mathrm{d}x = -\int_\Omega (\bar{u}(x), \nabla v(x))\,\mathrm{d}x + \int_{\partial\Omega} (\vec{u}, \vec{n})vds.$$

Finally, noticing that

$$\Delta u = div(\nabla\vec{u}),$$

we get the 2^{nd} Green's formula

Proposition 1.1.4. *[33] For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \Delta u(x)v(x)\,\mathrm{d}x = -\int_\Omega \nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} \frac{\partial u}{\partial n} vds.$$

Lemma 1.1.2 (**Poincaré inequality**). *[12] Let Ω be a bounded domain in $\mathbb{R}^n$. Then there is a positive constant C_Ω such that*

$$\|f\|_{L^2(\Omega)} \leqslant C_\Omega \|\nabla f\|_{L^2(\Omega)}, \quad \forall f \in H_0^1(\Omega).$$

Theorem 1.1.7 (**Rellich**). *[27] Let Ω be an open subset of $\mathbb{R}^n$ ($n \geqslant 1$) and $1 \leqslant p < +\infty$. Any bounded part of $W_0^{1,p}(\Omega)$ is relatively compact in $L^p(\Omega)$. This amounts to saying that for any bounded sequence of $W_0^{1,p}(\Omega)$, we can extract a subsequence which converges in $L^p(\Omega)$.*

The theorem stay true with $W^{1,p}(\Omega)$ provided the border of Ω is lipschitzian.

Theorem 1.1.8 (**Lax-Milgram**). *[10] Let L be a continuous linear form on Hilbert space H and a is a continuous and coercive bilinear form, then there is one and only one function $u \in H$ such that:*

$$a(u, v) = L(v), \quad \forall v \in H.$$

Moreover, if the bilinear form a is symmetric, then u is the only element of H which minimizes the functional $J : H \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}a(v, v) - L(v), \quad \forall v \in H,$$

i.e.

$$J(u) = \min_{v \in H} J(v) \text{ and } J(u) < J(v) \text{ if } u \neq v.$$

Definition 1.1.2. *[23] Let $u : \Omega \to \mathbb{R}$ be a measurable function with real value. We can consider the map*

$$\begin{array}{rcl} A : \Omega \times \mathbb{R} & \to & \mathbb{R} \\ u(x) & \mapsto & f(u)(x); \end{array}$$

where $A(u)$ is a function with real value defined on Ω by:

$$A(u)(x) = f(x, u(x)),$$

such a map is called the Nemitski operator associated to f.

Theorem 1.1.9. *[4] Let α, $\beta \geqslant 1$. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies*

1. *$f(x,t)$ is measurable with respect to $x \in \Omega$ for all $t \in \mathbb{R}$ and continues with respect $t \in \mathbb{R}$ for a.e. $x \in \Omega$.*
2. *There exists $g \in L^{\beta}(\Omega)$ and $a > 0$ such that*

$$|f(x,u)| \leqslant g(x) + a|u|^{\alpha/\beta}, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \quad (\alpha, \beta \geqslant 1).$$

then the Nemytskii operator A is continuous and compact from $L^{\alpha}(\Omega)$ to $L^{\beta}(\Omega)$.

Remark 1.1.4. *Condition* (1) *is called the Caratheodory condition and a function $f(x,t)$ satisfying* (1) *is called a Caratheodory function.*

1.1.3 Fractional Sobolev spaces

Fractional Sobolev spaces have been a classical topic in functional and harmonic analysis all along, and some important books, such as [40] treat the topic in detail. On the other hand, fractional spaces, and the corresponding nonlocal equations, are now experiencing impressive applications in different subjects, such as, among others, the thin obstacle problem [43], optimization [24], finance [15], elliptic problems with measure data [32] and gradient potential theory [44]...etc.

Let Ω be an open set in $\mathbb{R}^n$. For any real $s > 0$ and for any $p \in [1, \infty)$ we define the fractional Sobolev spaces $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called Aronszajn, Gagliardo or Slobodeckij spaces, by the name of the ones who introduced them, almost simultaneously (see [8, 26, 55])

We start by fixing the fractional exponent s in $(0, 1)$. For any $p \in [1, \infty)$, we define $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}; \tag{1.1}$$

i.e., an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p})^{\frac{1}{p}},$$

where the term

$$[u]_{s,p} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}},$$

is the so-called Gagliardo (semi) norm of u.

Remark 1.1.5. *The definition in (1.1) cannot be plainly extended to the case* $s \geqslant 1$. *Suppose that* Ω *is a connected open set in* $\mathbb{R}^n$, *then any measurable function* $u : \Omega \to \mathbb{R}$ *such that*

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy < +\infty,$$

is actually constant (see [11], Proposition 2). This fact is a matter of scaling and it is strictly related to the following result that holds for any u *in* $W^{s,p}(\Omega)$:

$$\lim_{s \to 1^-} (s - 1)^{\frac{1}{p}} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy = C \int_\Omega |\nabla u|^p dx,$$

for a suitable positive constant C *depending only on* n *and* p.

When $s > 1$ and it is not an integer we write $s = m + \sigma$, where m is an integer and $\sigma \in (0, 1)$. In this case the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, namely

$$W^{s,p}(\Omega) = \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ ffor any } \alpha \text{ s.t. } |\alpha| = \sigma\},$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}}.$$

Clearly, if $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

Theorem 1.1.10. *[18] For any* $s > 0$, *the space* $C_0^\infty(\mathbb{R}^n)$ *of smooth functions with compact support is dense in* $W^{s,p}(\mathbb{R}^n)$.

Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$. Note that, in view of Theorem 1.1.10, we have

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n),$$

but in general, for $\Omega \subset \mathbb{R}^n$, $W^{s,p}(\Omega) \neq W_0^{s,p}(\Omega)$, i.e. $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$.

Remark 1.1.6. *For* $s < 0$ *and* $p \in (1, \infty)$, *we can define* $W^{s,p}(\Omega)$ *as the dual space of* $W^{-s,q}(\Omega)$ *where* $1/p + 1/q = 1$. *Notice that, in this case, the space* $W^{s,p}(\Omega)$ *is actually a space of distributions on* Ω, *since it is the dual of a space having* $C_0^\infty(\Omega)$ *as density subset.*

Proposition 1.1.5. *[16] Let* Ω *be an open set of* $\mathbb{R}^n$ *and* $s \in (0, 1)$, *then*

1. *For* $1 \leqslant p < \infty$, $W^{s,p}(\Omega)$ *is a Banach space.*
2. *For* $1 \leqslant p < \infty$, $W^{s,p}(\Omega)$ *is a separable space.*
3. *For* $1 < p < \infty$, $W^{s,p}(\Omega)$ *is a reflexive space.*

Corollary 1.1.1. *[16] Let* $s \in (0,1)$ *and let* $p \in]1, \infty[$. *Let* Ω *be a Lipschitz open set of* $\mathbb{R}^n$. *Then we have:*

1. *If* $sp < n$, *then* $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for every* $q \leqslant np/(n-sp)$.
2. *If* $n = sp$, *then* $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for every* $q < \infty$.
3. *If* $sp > n$, *then* $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ *and, more precisely,*
$$W^{s,p}(\Omega) \hookrightarrow C^{0,s-n/p}(\Omega).$$

Theorem 1.1.11 (Compact embeddings). *[16] Let* Ω *be a bounded Lipschitz open subset of* $\mathbb{R}^n$. *Let* $s \in [0,1[$, *let* $p > 1$, *and let* $n \geqslant 1$. *Then we have:*

1. *If* $sp < n$, *then the embedding of* $W^{s,p}(\Omega)$ *into* L^k *is compact for every* $k < np/(n-sp)$.
2. *If* $sp = n$, *then the embedding of* $W^{s,p}(\Omega)$ *into* L^q *is compact for every* $q < \infty$.
3. *If* $sp > n$, *then the embedding of* $W^{s,p}(\Omega)$ *into* $C_b^{0,\lambda}(\Omega)$ *is compact for* $\lambda < s - n/p$.

The space H^s and the fractional Laplacian operator

In this part, we focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ and $W_0^{s,2}(\mathbb{R}^n)$ out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^n)$ and $H_0^s(\mathbb{R}^n)$, respectively. Moreover, they are strictly related to the fractional Laplacian operator $(-\Delta)^s$, where, for any $u \in \mathcal{S}$ (Schwartz Space) and $s \in (0,1)$, $(-\Delta)^s$ it is defined as

$$\begin{aligned}(-\Delta)^s u(x) &= C(n,s)\ P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy,\ x \in \mathbb{R}^n \\ &= C(n,s) \lim_{\epsilon \to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy. \qquad (1.2)\end{aligned}$$

Here *P.V.* is a commonly used abbreviation for “in the principal value sense” (as defined by the latter equation) and $C(n,s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n,s) = \frac{4^s \Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}} \frac{s}{\Gamma(1-s)},$$

where Γ is the usual Gamma function (see [9]).

In our thesis, we symbolize the space $H_0^s(\Omega)$ by $D^{s,2}(\Omega)$ where $D^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\|_{H^s}}$ ($D^{s,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ compared to the $H^s(\Omega)$ norm), if Ω is a bounded Lipschitz open set, then

$$D^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^n), \text{ such that } u = 0 \text{ in } \mathbb{R}^n \backslash \Omega\},$$

such that

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\};$$

thus $D^{s,2}(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx.$$

The norm in $D^{s,2}(\Omega)$ is

$$\|u\|_{D^{s,2}(\Omega)} = \left[\iint\limits_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx \right]^{\frac{1}{2}}.$$

Proposition 1.1.6. *[52] Let* $s \in (0,1)$ *and* Ω *be a bounded Lipschitzian subset of* $\mathbb{R}^n$ *such that* $n > 2s$*. Let* $u : \Omega \to \mathbb{R}$ *be a measurable function compactly supported. Then, there exists a positive constant* $c_{emb} > 0$ *(embedding constant) depending on* n *and* s *such that*

$$\|u\|_{L^2(\Omega)} \leqslant c_{emb} \|u\|_{D^{s,2}(\Omega)}.$$

Proposition 1.1.7. *[47] Let* $s \in (0,1)$*,* $n \geqslant 1$*,* $\Omega \in \mathbb{R}^n$ *be a Lipschitz bounded open set and* $\Im$ *be a bounded subset of* $L^2(\Omega)$*. Suppose that*

$$\sup_{u \in \Im} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy < +\infty,$$

then $\Im$ *is precompact in* $L^2(\Omega)$*.*

1.2 Topological degree

In this part, we will present the topological degree which is considered as an important tool for solving nonlinear elliptic problems. It was introduced by L. Brouwer for finite dimension and extended to infinite dimension by J. Leray and J. Schauder, see [27] and [41].

We start by giving the existence and the uniqueness of an application, called topological degree, in finite dimension.

1.2.1 The Brouwer degree and its properties

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$ (or a bounded open of a Banach space E). Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$. The application call topological degree show the existence of solutions of the equation $f(x) = y$ such that $x \in \bar{\Omega}$.

Definition 1.2.1. *[41] Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$, $f \in C^1(\Omega) \cap C(\bar{\Omega})$, $x_0 \in \Omega$ is called regular point if $J_f(x_0) \neq 0$ (or $J_f(x_0) = detDf(x_0)$ with $Df(x_0) = (\frac{\partial f_i}{\partial x_j})_{i,j}(x_0)$), Otherwise, x_0 is called critical point or singular point.*

Let us designate by

$$S_f(\Omega) = \{x_0 \in \Omega : J_f(x_0) = 0\},$$

the set of singular points of f on Ω.

Definition 1.2.2 (Regular case). *[41] Let $\Omega \subset \mathbb{R}^n$ be a bounded open and $f \in C^1(\Omega) \cap C(\bar{\Omega})$ a defined function of Ω with values in $\mathbb{R}^n$, for $y \notin f(\partial\Omega)$ a regular value, we define the degree of f at the point y by*

$$deg(f, \Omega, y) = \sum_{f(x_i)=y; i=\overline{1,n}} sgn(detD_x f(x_i)).$$

Definition 1.2.3. *[27] Let $N \geqslant 1$. We note by A the set of triplets (f, Ω, y) Where Ω is an open bounded of $\mathbb{R}^N$, $f \in C(\bar{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N$ such that $y \notin \{f(x); x \in \partial\Omega\}$.*

Theorem 1.2.1 (Brouwer, 1933). *[27] Let $N \geqslant 1$ and A given by the definition (1.2.3). Then there exists an application d of A in $\mathbb{Z}$ called "topological degree", verifying the following three properties:*

(P1) **Normalization:** *$d(I, \Omega, y) = 1$ if $y \in \Omega$.*

(P2) **Degree of an union:** *$d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ if $\Omega_1 \cup \Omega_2 \subset \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $y \notin \{f(x), x \in \bar{\Omega} \backslash \Omega_1 \cup \Omega_2\}$.*

(P3) **Homotopy invariance:** *If $h \in C([0,1] \times \bar{\Omega}, \mathbb{R}^N)$, $y \in C([0,1], \mathbb{R}^N)$ and $y(t) \notin \{h(t,x), x \in \partial\Omega\}$ (for all $t \in [0,1]$), we have then:*

$$d(h(t,.), \Omega, y(t)) = d(h(0,.), \Omega, y(0)) \text{ for all } t \in [0,1].$$

Brouwer's fixed point theorem, 1912

A first consequence of this "topological degree" method is Brouwer's fixed point theorem that we are giving now.

Theorem 1.2.2. *[27] Let $N \geqslant 1$, $R > 0$ and $f \in C(B_R, B_R)$ with $B_R = \{x \in \mathbb{R}^N, \|x\| \leqslant R\}$ (we provided $\mathbb{R}^N$ with a norm noted $\|.\|$). Then f admits a fixed point, that is to say it exists $x \in B_R$ such that $f(x) = x$.*

Theorem 1.2.1 was generalized (from 1934) in infinite dimension by Leray and Schauder under a hypothesis of compactness that we give now.

1.2.2 The Leray-Schauder degree and its properties

It is certainly impossible to define a degree in infinite dimensions with the same assumptions as in finite dimensions. For this, Leray and Schauder have been generalized under compactness assumptions.

Definition 1.2.4. *[27] Let E be a Banach space (real), B be a part of E and f an application from B to E. We said that f is compact (the terminology of Leray-Schauder is different, they use the expression "completely continuous") if f satisfies the following two properties*

1. *f is continuous.*
2. *$\{f(x), x \in C\}$ is relatively compact in E for any bounded C of B.*

Remark 1.2.1. *We can notice, in the previous definition, that if f is linear (and $B = E$) the second condition leads to the first one. But this is not true for non-linear applications.*

Here is the main result of this part, which states the existence of the Leray-Schauder degree along with its main properties.

Definition 1.2.5. *[27] Let E be a Banach space (real). We note A the set of triplets $(I-f, y, \Omega)$ where Ω is an open bound of E, f is a compact application from Ω into E (which is equivalent to say that f is continuous and $\{f(x), x \in \Omega\}$ is a relatively compact part of E) and $y \in E$ such that::*

$$y \notin \{x - f(x); x \in \partial\Omega\}.$$

Theorem 1.2.3 (Leray-Schauder, 1934). *[27] Let E be a Banach space (real) and A given by the definition (1.2.5). Then there exists an application d of A in $\mathbb{Z}$ called "topological degree", verifying the following three properties:*

(P1) **Normalization:** *$d(Id, \Omega, y) = 1$ if $y \in \Omega$.*

(P2) **Degree of an union:** *$d(I - f, y, \Omega) = d(I - f, y, \Omega_1) + d(I - f, y, \Omega_2)$ if $\Omega_1 \bigcup \Omega_2 \subset \Omega, \Omega_1 \cap \Omega_2 = \varnothing$ and $y \notin \{x - f(x); x \in \overline{\Omega} \backslash \Omega_1 \bigcup \Omega_2\}$.*

(P3) **Homotopy invariance:** *If h is a compact application of $[0,1] \times \overline{\Omega}$ in E, $y \in C([0,1], E)$ and $y(t) \notin \{x - h(t,x), x \in \Omega\}$ (for all $t \in [0,1]$) we have then:*

$$d(I - h(t,.), y, \Omega) = d(I - h(0,.), y(0), \Omega) \text{ for all } t \in [0,1].$$

Definition 1.2.6. *[27] An application of the form*

$$f = I - h,$$

where I is the identity application and h is a compact application is called compact perturbation of the identity (or Leray-Schauder application).

Remark 1.2.2. *The essential property of the topological degree is:*
If $(I - f, \Omega, y) \in A$ and $d(I - f, \Omega, y) \neq 0$, then there exists $x \in \Omega$ such that $x - f(x) = y$.

Schauder's fixed point theorem

As in finite dimension, a first consequence of the existence of the topological degree is the obtaining of a fixed point theorem that we give now.

Theorem 1.2.4. *[27] Let E be a Banach space, $R > 0$, $B(0,R) = \{x \in E, \|x\| \leqslant R\}$ and f a compact application from $B(0,R)$ to $B(0,R)$ (that is to say f continues and $\{f(x), x \in B(0,R)\}$ relatively compact in E). Then f admits a fixed point that is, there exists $x \in B(0,R)$ such that $f(x) = x$.*

1.3 Resonance and non-resonance

Resonance is a phenomenon where some physical systems (Electrical, Mechanical...) are sensitive to certain frequencies.

1.3.1 Electronics: resonance circuits

They are a particular case of RLC electrical circuits (R: resistance, L: coil (inductive reactance X_L), C: capacitor (capacitive reactance X_C)).

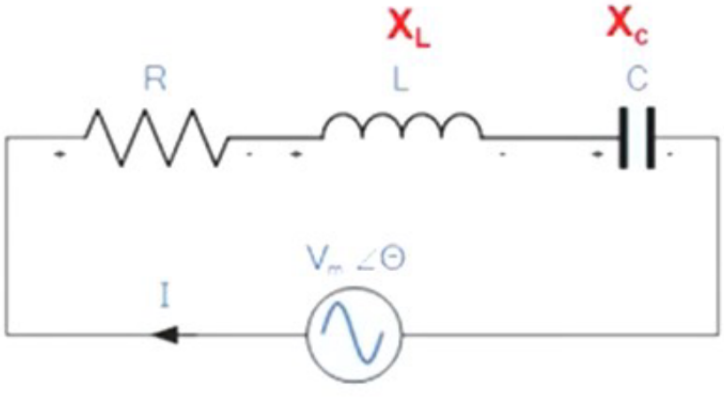

Figure 1.1: Successive RLC circuit

When we apply an alternating voltage, an electric current will initiate through the circuit, but we know that the law of reluctance or total circuit resistance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}.$$

Every time we increase the frequency of the alternating voltage, the X_L increase and X_C reduction, in a given frequency becomes $X_L = X_C$ i.e.

$$Z = R,$$

meaning: if, only the R exists in the circuit, X_L and X_C disappear what has become like a short circuit, this is called a resonant frequency f_r.

In the case of resonance, all the voltage that is coming from the source goes to R only, and the current, which passes, reaches the maximum value.

We take the bandwidth when less than the value of the current or the voltage is about 70% of the maximum value.

Whenever we underestimates the resistance resonance unit getting higher, it means a decrease in bandwidth, and therefore the quality factor increase.

Example on radio receivers

If you pick up the radio and want to hear the channel on the frequency $100MHz$, we turn the knob to the right and to the left i.e. we change the frequency (you change either the L or C) if you get to $100MHz$ which makes the radio frequency $100MHz$ you can listen to the channel so $100MHz$ is the resonant frequency.

Therefore, the circuits used in radios are resonance circuits. These frequency selective circuits pass a specific frequency and behave towards it as a thread and do not pass other frequencies, which makes the radio frequency parallel to the channel frequency.

1.3.2 Mathematical

Let the Dirichlet proleme

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = f(x,u) + h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary, $1 < p < \infty$ and h is given in $W^{-1,p'}(\Omega)$. suppose that

$$F_{\pm} = \lim_{s\to\pm\infty} \sup p\frac{F(x,s)}{|s|^p}, \tag{1.4}$$

where $F(x,s) = \int_0^s f(x,t)dt$.

Definition 1.3.1. *[58] We say that the problem (1.3) is non-resonance under an eigenvalue if $F_+(x) < \lambda_n$ and $F_-(x) < \lambda_n$ a.e. $x \in \Omega$, where F_+ and F_- are defined by (1.4) and λ_n is an eigenvalue of the problem:*

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u \quad in\ \Omega, \\ u = 0 \quad on\ \partial\Omega. \end{cases} \tag{1.5}$$

Definition 1.3.2. *[58] We will say that the problem (1.3) presents a resonance under an eigenvalue* λ_n *if* $F_+(x) \leqslant \lambda_n$ *and* $F_-(x) \leqslant \lambda_n$ *a.e.* $x \in \Omega$ *and if at least one of the two inequalities is an equality on a subset of* Ω *of non-zero measure.*

Remark 1.3.1. *[58] We note:*

$$f_\pm = \lim_{s \to \pm\infty} \sup \frac{f(x,s)}{|s|^{p-2}s}.$$

We establish several existence results under different conditions on the report $p\frac{F(x,s)}{|s|^p}$ *when* $s \to \pm\infty$*. Because we always have* $F_\pm \leqslant f_\pm$*, the conditions we consider are more general than those relating to the report* $\frac{f(x,s)}{|s|^{p-2}s}$*.*

Chapter 2

Existence of solutions for a p-Laplacian system with a nonresonanse condition between the first and the second eigenvalues

In recent years, the eigenvalue problems for p-Laplacian operators have been extensively studied (see [6, 39, 42, 53]). The main purpose of this chapter is to prove the existence of solutions for a quasilinear elliptic system when the second terms on the two equations $f_i(x,s)$, $(i = 1,2)$ locates between the first and the second eigenvalue of the p-Laplacian. This result can be seen as a generalization of the result obtained by A. Anane and N. Tsouli in [6].

2.1 Position of the problem

In this chapter, we study the existence of positive solution for the nonlinear elliptic system

$$\begin{cases} -\Delta_p u(x) = f_1(x, v(x)) + h_1(x) \quad \text{in } \Omega, \\ -\Delta_p v(x) = f_2(x, u(x)) + h_2(x) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with the exponent p, $1 < p < \infty$ and Ω is a smooth bounded region in $\mathbb{R}^n$ for $n \geqslant 1$.

Through this chapter, $h_i \in W^{-1,p'}(\Omega)$ with $i = 1,2$ and p' the Hölder conjugate of p. As to the nonlinearities f_i $(i = 1,2)$, we assume that they are Carathéodory functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that

$$\max_{|s| \leqslant R_i} |f_i(x,s)| \in L^{p'}(\Omega), \quad \forall R_i > 0, \tag{2.2}$$

$$\lambda_1 \underset{\neq}{\leqslant} l_i(x) \leqslant k_i(x) < \lambda_2 \quad \text{a.e. in } \Omega, \tag{2.3}$$

where

$$l_i(x) = \lim_{s\to\pm\infty} \inf \frac{f_i(x,s)}{|s|^{p-2}s}, \quad k_i(x) = \lim_{s\to\pm\infty} \sup \frac{f_i(x,s)}{|s|^{p-2}s},$$

and λ_1 (resp., λ_2) is the first (resp., the second) eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

First inequality in (2.3) means: "less or equal almost everywhere with strict inequality on a set of positive measure". we also assume that the inequalities in (2.3) holds for $i = 1, 2$:

$$\begin{aligned} &\forall \varepsilon_i > 0, \quad \exists \eta(\varepsilon_i) > 0 : \lambda_1 - \varepsilon_i \leqslant \tfrac{f_i(x,s)}{|s|^{p-2}s}, \quad \forall |s| \geqslant \eta(\varepsilon_i), \quad \text{a.e. in } \Omega, \\ &\forall \varepsilon_i > 0, \quad \exists \eta(\varepsilon_i) > 0 : \tfrac{f_i(x,s)}{|s|^{p-2}s} \leqslant \lambda_2 + \varepsilon_i, \quad \forall |s| \geqslant \eta(\varepsilon_i), \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.4}$$

Recently, A. Anane and N. Tsouli [6] study the existence of solutions for the Dirichlet problem $-\Delta_p u = f(x,u) + h(x)$ in Ω, $u = 0$ in $\partial\Omega$, when $f(x,u)$ locates between the first and the second eigenvalues of the p-Laplacian (Δ_p), using Leray-Schauder topological degree.

Their work is based on the absurd reasoning, they arrived at a contradiction by using different lemmas and the variation characterization of λ_2, more precisely the monotonicity of λ_2. Our work is based on the same method of proof.

The main result in this chapter is the following theorem.

Theorem 2.1.1. *For $i = 1, 2$, assume that f_i satisfies (2.2), (2.3) and (2.4). Then for any $h_i \in W^{-1,p'}(\Omega)$, (2.1) admits a weak solution (u,v) in U.*

As usual, a weak solution of system (2.1) is any $(u,v) \in U$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi_1 dx + \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi_2 dx = \int_\Omega f_1(x,v)\varphi_1 dx + \int_\Omega f_2(x,u)\varphi_2 dx + \langle h_1, \varphi_1\rangle + \langle h_2, \varphi_2\rangle,$$

for every $\varphi_i \in W^{-1,p'}(\Omega)$, $(i = 1, 2)$.

Where $\langle .,. \rangle$ denotes the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Next, let us define by $(T_t)_{t\in[0,1]}$ the family of operators from U to U defined by

$$T_t(u,v) = \begin{pmatrix} T_{1t}(u,v) \\ T_{2t}(u,v) \end{pmatrix} = \begin{pmatrix} -\Delta_p^{-1} & 0 \\ 0 & -\Delta_p^{-1} \end{pmatrix} \times \begin{pmatrix} (1-t)\alpha_1 |u|^{p-2}u + tf_1(x,v) + th_1 \\ (1-t)\alpha_2 |v|^{p-2}v + tf_2(x,u) + th_2 \end{pmatrix}, \tag{2.5}$$

where α_i, $i = 1, 2$ are some fixed numbers with $\lambda_1 < \alpha_i < \lambda_2$.

Remark 2.1.1. *Hypotheses (2.2) and (2.4) give us the growth conditions*

$$|f_i(x,s)| \leqslant a_i|s|^{p-1} + b_i(x) \quad \forall |s| \in \mathbb{R},\ a.e.\ in\ \Omega, \tag{2.6}$$

where $a_i > 0$ *and* $b_i(.) \in L^{p'}(\Omega)$.

Remark 2.1.2. *Equations (2.2) and (2.4) imply*

$$\begin{gathered} \forall \varepsilon_i > 0, \quad \exists b_{\varepsilon_i} \in L^{p'}(\Omega)\ such\ that \\ |s|^p(\lambda_1 - \varepsilon_i) - b_{\varepsilon_i}(x) \leqslant s f_i(x,s) \leqslant |s|^p(\lambda_2 + \varepsilon_i) - b_{\varepsilon_i}(x), \\ \forall s \in \mathbb{R}, \quad a.e.\ in\ \Omega. \end{gathered} \tag{2.7}$$

Lemma 2.1.1. T_t *is continuous and compact.*

Proof. We have, $T_t : U \to U$; to prove the Lemma, we have

$$U \hookrightarrow V \underset{A}{\to} Y \hookrightarrow Z \underset{S}{\to} U, \tag{2.8}$$

such that the Nemytskii operator

$$\begin{aligned} A: \quad & V & \to & Y \\ & (u,v) & \mapsto & (f_1(x,v), f_2(x,u)), \end{aligned}$$

and

$$\begin{aligned} S: \quad & Z & \to & U \\ & \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} & \mapsto & \begin{pmatrix} -\Delta_p^{-1} & 0 \\ 0 & -\Delta_p^{-1} \end{pmatrix} \begin{pmatrix} f_1(x,v) \\ f_2(x,u) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

are continuous and compact.

▲ If we take $(u_n, v_n) \to (u,v)$ in U, according to (2.8) $T_t(u_n,v_n) \to T_t(u,v)$ in U, so the operator T_t is continuous $\forall t \in [0,1]$.

▲ If we take a bounded set $M \in U$, according to (2.8) the image of this set by the operator T_t is relatively compact, so the operator T_t is compact. ☐

2.2 A priori estimate

To prove Theorem (2.1.1), we first establish the following estimate:

$$\exists R > 0 \text{ such that } \forall t \in [0,1], \forall (u,v) \in \partial B(0,R) \text{ such that } [I - T_t](u,v) \neq 0,$$

where $B(0,R)$ denotes the ball of center 0 and radius R in U.

For, we assume by contradiction that

$$\begin{gathered}\forall n > 0, \quad \exists t_n \in [0,1], \quad \exists (u_n, v_n) \in U \text{ with} \\ \|(u_n, v_n)\|_{1,p} = n \text{ such that } T_{t_n}(u_n, v_n) = (u_n, v_n).\end{gathered} \tag{2.9}$$

Let $w_n = (w_{1n}, w_{2n}) = (\frac{u_n}{n}, \frac{v_n}{n})$. We still denoted by (w_n) the subsequence of (w_n) which converges weakly in U, strongly in V and a.e. in Ω to w.

We can also suppose that t_n converges to $t \in [0,1]$. That to reach a contradiction, we need the following lemmas.

Lemma 2.2.1. *If the sequence $g_n = (g_{1n}, g_{2n})$ are defined by*

$$g_{in} = \frac{f_i(x, nw_{i+(-1)^{i+1}n})}{n^{p-1}}, \quad i = 1, 2, \tag{2.10}$$

then g_{in} are bounded in $L^{p'}(\Omega)$, and they admit subsequences g_{in} converging weakly to some g_i in $L^{p'}(\Omega)$.

Proof. From (2.6), we have

$$|f_i(x,s)| \leqslant a_i |s|^{p-1} + b_i(x),$$

then

$$\frac{|f_i(x, nw_{i+(-1)^{i+1}n})|}{n^{p-1}} \leqslant a_i \frac{|nw_{i+(-1)^{i+1}n}|^{p-1}}{n^{p-1}} + \frac{b_i(x)}{n^{p-1}},$$

we get

$$|g_{in}(x)| \leqslant a_i |w_{i+(-1)^{i+1}n}|^{p-1} + \frac{b_i(x)}{n^{p-1}};$$

as $b_i(x)$ in $L^{p'}(\Omega)$ and $|w_{i+(-1)^{i+1}n}|^{p-1} \in L^{p'}(\Omega)$, so g_{in} become bounded in $L^{p'}(\Omega)$. Consequently, there exists a subsequence, still denoted by g_{in} converging weakly to g_i in $L^{p'}(\Omega)$. □

Lemma 2.2.2. *$w_i \neq 0$, $i = 1, 2$.*

Proof. We have that

$$\begin{cases} -\Delta_p(u) = (1-t)\alpha_1 |u|^{p-2} u + t f(x, v)) + t h_1(x), \\ -\Delta_p(v) = (1-t)\alpha_2 |v|^{p-2} u + t f(x, u)) + t h_2(x), \end{cases}$$

the variational formulation give

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 dx + \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 dx = (1-t)\big[\alpha_1 \int_\Omega |u|^{p-2} u \varphi_1 dx \\ + \alpha_2 \int_\Omega |v|^{p-2} v \varphi_2 dx\big] + t\big[\int_\Omega f_1(x, v)\varphi_1 dx + \int_\Omega f_2(x, u)\varphi_2 dx \\ + \int_\Omega h_1(x)\varphi_1 dx + \int_\Omega h_2(x)\varphi_2 dx\big],\end{aligned}$$

according to the continuity, we found

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi_{1n}dx+\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla\varphi_{2n}dx &= (1-t)\big[\alpha_1\int_\Omega |u_n|^{p-2}u_n\varphi_{1n}dx\\ &+\alpha_2\int_\Omega |v_n|^{p-2}v_n\varphi_{2n}dx\big]+t\big[\int_\Omega f_1(x,v_n)\varphi_{1n}dx\\ &+\int_\Omega f_2(x,u_n)\varphi_{2n}dx+\int_\Omega h_1(x)\varphi_{1n}dx+\int_\Omega h_2(x)\varphi_{2n}dx\big],\end{aligned}$$

if we take $(\varphi_{1n},\varphi_{2n})=(u_n,v_n)$, we receive to

$$\begin{aligned}\int_\Omega |\nabla u_n|^p dx+\int_\Omega |\nabla v_n|^p dx =&(1-t)\big[\alpha_1\int_\Omega |u_n|^p dx+\alpha_2\int_\Omega |v_n|^p dx\big]\\ &+t\big[\int_\Omega f_1(x,v_n)u_n dx+\int_\Omega f_2(x,u_n)v_n dx\\ &+\int_\Omega h_1(x)u_n dx+\int_\Omega h_2(x)v_n dx\big],\end{aligned}$$

the definition of w_n verify that

$$\begin{aligned}\int_\Omega |\nabla w_{1n}|^p dx+\int_\Omega |\nabla w_{2n}|^p dx =&(1-t_n)\big[\alpha_1\int_\Omega |w_{1n}|^p dx+\alpha_2\int_\Omega |w_{2n}|^p dx\big]\\ &+t_n\big[\int_\Omega g_{1n}(x)w_{1n}dx+\int_\Omega g_{2n}(x)w_{2n}dx\\ &+\frac{1}{n^{p-1}}<h_1,w_{1n}>+\frac{1}{n^{p-1}}<h_2,w_{2n}>\big].\end{aligned} \tag{2.11}$$

We get from Lemma (2.2.1)

$$1=(1-t)\big[\alpha_1\int_\Omega |w_1|^p dx+\alpha_2\int_\Omega |w_2|^p dx\big]+t\big[\int_\Omega g_1(x)w_1dx+\int_\Omega g_2(x)w_2dx\big]; \tag{2.12}$$

from the diffrent properties of the weak and strong convergences we get that $w_i\neq 0$, $i=1,2$. □

Lemma 2.2.3. *Let* $A=\{x\in\Omega : w_i(x)\neq 0, \quad (i=1,2)\}$*, then*

$$g_i=0 \ \textit{a.e. in}\ \Omega\backslash A \ \textit{where}\ i=1,2.$$

Proof. The inequality (2.6) gives us for every i $(i=1,2)$

$$|g_{in}(x)|\leqslant a_i|w_{i+(-1)^{i+1}n}|^{p-1}+\frac{b_i(x)}{n^{p-1}} \quad \text{a.e. in } \Omega\backslash A, \tag{2.13}$$

so

$$\begin{aligned}\|g_{in}\|_{L^{p'}(\Omega\backslash A)} &\leqslant a_i\|w_{i+(-1)^{i+1}n}\|^{p-1}_{L^p(\Omega\backslash A)}+\frac{1}{n^{p-1}}\|b_i\|_{L^{p'}(\Omega\backslash A)}\\ &\leqslant a_i\|w_{i+(-1)^{i+1}n}\|^{\frac{p}{p'}}_{L^p(\Omega\backslash A)}+\frac{1}{n^{p-1}}\|b_i\|_{L^{p'}(\Omega\backslash A)},\end{aligned} \tag{2.14}$$

From Lemma (2.2.2), we have

$$\lim_{n\to+\infty}\|g_{in}\|_{L^{p'}(\Omega\backslash A)} = 0. \quad (i=1,2) \tag{2.15}$$

Let $D = \{x \in \Omega\backslash A : g_i \neq 0, \quad (i=1,2)\}$. By Lemma (2.2.1) we get, for $\phi_i(x) = sign[g_i(x)]\chi_D(x)$ belong to $L^p(D)$ such that

$$\chi_D(x) = \begin{cases} 0 & ; x \notin D, \\ 1 & ; x \in D, \end{cases}$$

that

$$\lim_{n\to+\infty}\int_D g_{in}(x)\phi_i(x)dx = \int_D g_i(x)\phi_i(x)dx = \int_D |g_i(x)|dx, \tag{2.16}$$

but, we have by (2.15)

$$\int_D |g_i(x)|dx = 0, \quad (i=1,2) \tag{2.17}$$

consequently, $meas(D) = 0$ which implies

$$g_i = 0 \text{ a.e. in } \Omega\backslash A \text{ where } i = 1,2.$$

□

Lemma 2.2.4. *Let $i = 1,2$ and*

$$\tilde{g}_i(x) = \begin{cases} \frac{g_i(x)}{|w(x)_{i+(-1)^{i+1}}|^{p-2}w(x)_{i+(-1)^{i+1}}} & \text{on } A, \\ \beta_i & \text{on } \Omega\backslash A, \end{cases} \tag{2.18}$$

where β_i are fixed numbers such that $\lambda_1 < \beta_i < \lambda_2$, then

$$\lambda_1 \underset{\neq}{\leqslant} \tilde{g}_i(x) < \lambda_2 \quad a.e. \ in \ \Omega. \tag{2.19}$$

Proof. For $i = 1,2$, firstly we define new subsets us follow

$$\begin{aligned} B_{l_i} &= \{x \in A : w_{i+(-1)^{i+1}}(x)g_i(x) < l_i(x)|w_{i+(-1)^{i+1}}(x)|^p\}, \\ B_{k_i} &= \{x \in A : w_{i+(-1)^{i+1}}(x)g_i(x) > k_i(x)|w_{i+(-1)^{i+1}}(x)|^p\}, \end{aligned}$$

then we prove that $meas(B_{l_i}) = meas(B_{k_i}) = 0$.

By Remark (2.1.2), we have that $\forall\varepsilon_i > 0, \quad \exists b_{\varepsilon_i} \in L^{p'}(\Omega)$ such that

$$|nw_{i+(-1)^{i+1}n}|^p(l_i - \varepsilon_i) - b_{\varepsilon_i} \leqslant nw_{i+(-1)^{i+1}n}f_i(x, nw_{i+(-1)^{i+1}n}) \leqslant |nw_{i+(-1)^{i+1}n}|^p(k_i + \varepsilon_i) + b_{\varepsilon_i}, \tag{2.20}$$

according to the definition of g_{in}, we find

$$|nw_{i+(-1)^{i+1}n}|^p(l_i - \varepsilon_i) - b_{\varepsilon_i} \leqslant n^p w_{i+(-1)^{i+1}n}g_{in} \leqslant |nw_{i+(-1)^{i+1}n}|^p(k_i + \varepsilon_i) + b_{\varepsilon_i}, \tag{2.21}$$

dividing on n^p

$$|w_{i+(-1)^{i+1}n}|^p(l_i-\varepsilon_i)-\frac{b_{\varepsilon_i}}{n^p}\leqslant w_{i+(-1)^{i+1}n}g_{in}\leqslant |w_{i+(-1)^{i+1}n}|^p(k_i+\varepsilon_i)+\frac{b_{\varepsilon_i}}{n^p}. \tag{2.22}$$

By integrating in the first inequality and letting $n\to\infty$, then $\varepsilon\to 0$, we deduce

$$\int_{B_{l_i}}[w_{i+(-1)^{i+1}}(x)g_i(x)-|w_{i+(-1)^{i+1}}(x)|^p l_i(x)]dx\geqslant 0, \tag{2.23}$$

and from the definition of the subset B_{l_i}, we get

$$\int_{B_{l_i}}[w_{i+(-1)^{i+1}}(x)g_i(x)-|w_{i+(-1)^{i+1}}(x)|^p l_i(x)]dx< 0. \tag{2.24}$$

Whereupon

$$\int_{B_{l_i}}[w_{i+(-1)^{i+1}}(x)g_i(x)-|w_{i+(-1)^{i+1}}(x)|^p l_i(x)]dx= 0, \tag{2.25}$$

which implies $meas(B_{l_i})=0$. The second inequality give us $meas(B_{k_i})=0$.

In the second step, from the definition of $\tilde{g}_i$, we obtain

$$l_i(x)\leqslant \tilde{g}_i(x)\leqslant k_i(x) \text{ a.e. in } A, \tag{2.26}$$

and hypothesis (2.3) allow us to write

$$\lambda_1\leqslant \tilde{g}_i(x)<\lambda_2 \text{ a.e. in } A. \tag{2.27}$$

Since $\tilde{g}_i=\beta_i$ in $\Omega\backslash A$, then

$$\lambda_1<\tilde{g}_i<\lambda_2 \text{ in } \Omega\backslash A. \tag{2.28}$$

The inequalities (2.27) and (2.28) leads to

$$\lambda_1\leqslant \tilde{g}_i(x)<\lambda_2 \text{ a.e. in } \Omega. \tag{2.29}$$

From (2.28), (2.29) and the fact that $mes(\Omega\backslash A)\neq 0$, we obtain

$$\lambda_1 \underset{\neq}{\leqslant} \tilde{g}_i(x)<\lambda_2 \quad \text{a.e. in } \Omega.$$

□

Lemma 2.2.5. *If $i=1,2$, then w_i is a solution of*

$$\begin{cases}-\Delta_p w_i=m_i|w_i|^{p-2}w_i & \textit{in } \Omega,\\ w_i=0 \quad \textit{on } \partial\Omega,\end{cases} \tag{2.30}$$

where $m_i(x)=(1-t)\alpha_i+t\tilde{g}_{i+(-1)^{i+1}}(x)$.

Proof. We first prove that w_i $(i = 1, 2)$ is a solution of

$$\begin{cases} -\Delta_p w_i = (1-t)\alpha_i |w_i|^{p-2} w_i + t g_{i+(-1)^{i+1}} & \text{in } \Omega, \\ w_i = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.31}$$

From [6], we have that w_{in} $(i = 1, 2)$ satisfies

$$\begin{cases} -\Delta_p w_{in} = (1-t_n)|w_{in}|^{p-2} w_{in} + t_n \big[g_{i+(-1)^{i+1}n} + \frac{1}{n^{p-1}} h_i\big] & \text{in } \Omega, \\ w_{in} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.32}$$

We know that for $i = 1, 2$, $(-\Delta_p)(w_{in})$ are bounded in $W^{-1,p'}(\Omega)$, so we can extract from it a subsequence (w_{in}) (for simplicity of the notation), and a distribution $L_i \in W^{-1,p'}$ such that

$$(-\Delta_p)(w_{in}) \xrightarrow[weak]{} L_i,$$

in particular

$$\lim_{n\to+\infty} < -\Delta_p w_{in}, w_i > = < L_i, w_i > .$$

Since

$$\begin{aligned} < -\Delta_p w_{in}, w_{in} - w_i > &= (1-t_n)\alpha_i \int_\Omega |w_{in}|^{p-2} w_{in}(w_{in} - w_i)dx \\ &\quad + t_n \Big[\int_\Omega g_{i+(-1)^{i+1}n}(w_{in} - w_i)dx + \frac{1}{n^{p-1}} < h_i, w_{in} - w_i > \Big], \end{aligned}$$

it holds

$$\lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} - w_i > = 0.$$

But, we have

$$\begin{aligned} \lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} - w_i > &= \lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} > - \lim_{n\to+\infty} < -\Delta_p w_{in}, w_i > \\ &= \lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} > - < L_i, w_i > \\ &= 0, \end{aligned}$$

consequently

$$\lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} > = < L_i, w_i > .$$

We also know that $(-\Delta_p)$ is an operator of type $(M)^1$, so we get

$$L_i = -\Delta_p w_i.$$

Passing to the limit in (2.32) gives (2.31), but by Lemma (2.2.3), we have

$$(1-t)\alpha_i |w_i|^{p-2} + t g_{i+(-1)^{i+1}} = m_i |w_i|^{p-2} w_i \quad \text{a.e. in } \Omega,$$

which implies that w_i is a solution of (2.30) for every i sush that $i = 1, 2$. □

---

[1] We say that $A : V \to V'$ is an operator of type (M) if $[u_n \to u$ weakly in V, $A(u_n) \to z$ weakly in V' and $\limsup < A(u_n), u_n > \leqslant < z, u > \Rightarrow z = A(u)]$.

Now, we can prove our estimate.

To reach the contradiction, we set $\lambda_1(\Omega, m_i(x))$ (resp., $\lambda_2(\Omega, m_i(x))$ to be the first (resp., the second) eigenvalue of the problem with weight

$$\begin{cases} -\Delta_p u = \lambda m_i(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

For $i = 1, 2$, we use Lemma (2.2.4) and the fact that $\lambda_1 < \alpha_i < \lambda_2$, to get

$$\lambda_1 \underset{\neq}{\leqslant} m_i(x) < \lambda_2 \quad \text{a.e. in } \Omega;$$

now, by the strict monotonicity property of the first eigenvalue [60] and the second eigenvalue [5], we have

$$\lambda_1(\Omega, m_i) < \lambda_1(\Omega, \lambda_1) = 1,$$

and

$$1 = \lambda_2(\Omega, \lambda_2) < \lambda_2(\Omega, m_i),$$

so clearly

$$\lambda_1(\Omega, m_i) < 1 < \lambda_2(\Omega, m_i).$$

But by Lemmas (2.2.2) and (2.2.5), for every i (sush that $i = 1, 2$), 1 is an eigenvalue of $(-\Delta_p)$ for the weights m_i, which contradicts the definition of the second eigenvalues[2] $\lambda_2(\Omega, m_i)$. From above we deduce that the estimation holds true.

2.3 Proof of the main result

Using the homotopy invariance of the degree map, which through the homotopy T_t yields

$$deg(I - T_0, B(0, R), 0) = deg(I - T_1, B(0, R), 0).$$

As T_0 is odd, so following the theory of Borsuk, we get that $deg(I - T_0, B(0, R), 0)$ is an odd integer and so nonzero. This implies that there exists $(u, v) \in B(0, R)$ such that $T_1(u, v) = (u, v)$. Hence, system (2.1) has a positive solution.

This completes the proof.

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[2] λ_2 is the second eigenvalue therefore $\lambda_1 < \lambda_2$ and $sp(-\Delta_p) \cap]\lambda_1, \lambda_2[= \phi$.

Chapter 3

Existence and uniqueness of solution for a nonlinear fractional elliptic system

The study of the existence of weak solutions to a system of nonlocal equations involving the fractional Laplacian its our aim in this chapter. To prove the existence and uniqueness of solutions under suitable assumptions on the non linearities, we use the fixed point theory.

3.1 Position of the problem and the main result

Fractional differential equations involve derivatives of fractional order are important mathematical models of some functional ways to some of the problems in several disciplines like in image denoising, natural sciences and different other branches (see [28, 30]). As a result, the subject of fractional differential equations is gaining much importance and attention, for examples see [48, 50, 51, 65] and the references therein.

The Dirichlet problem for the fractional Laplacian has been studied from the point of view of probability, potential theory, and PDEs. It has attracted lots of interest, see [7, 25]. In this chapter, we adopt the fixed point theory in order to confirm the existence of a weak solutions to the coupled system

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) & \text{in } \Omega, \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in]0, 1[$ and $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying the Carathéodory conditions (i.e : $f(., w)$, $g(., z)$ are measurables for each $w, z \in \mathbb{R}^2$ and $f(x, .)$, $g(y, .)$ are continuous for almost every $x, y \in \Omega$), and also verifying the growth restriction defined below

$$\begin{cases} |f(x,\xi_1,\xi_2)| \leqslant r_1(x) + a|\xi_1|^{\delta_1} + b|\xi_2|^{\delta_1}, \\ |g(x,\eta_1,\eta_2)| \leqslant r_2(x) + c|\eta_1|^{\delta_2} + d|\eta_2|^{\delta_2}. \end{cases} \tag{3.2}$$

(We are employed the notation that $|.|$ stands for absolute value in $\mathbb{R}$).
Where δ_1, $\delta_2 \in]0,1[$ and $r = (r_1, r_2) \in \widetilde{V}$ nonnul function; a, b, c and d are nonnegative constants.

We recall that the fractional Laplacian $(-\Delta)^s$ in its nonlocal representation defined as

$$(-\Delta)^s \varphi(x) = C(n,s) P.V. \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}} dy,$$

along $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $s \in]0,1[$, $P.V.$ denotes the integral in the sense of the principal value and

$$C(n,s) = \frac{4^s \Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}} \frac{s}{\Gamma(1-s)}.$$

As far as we know, this result is new and represent fractional version of the classical theorem see [23]. The linear case is already studied in a lot of works.

The rest of this chapter is organized as follows. Section 2 introduces basic definitions and the main result of this paper. In Section 3, a fixed point formulation of the problem (3.1) is presented, and in Section 4, we prove the main result. Finally, in Section 5, we give a particular case.

Throughout the chapter, without futther mention, we always assume that $n > 2s$, and the main result is

Theorem 3.1.1. *Under hypothesis (3.2), problem (3.1) has at least one solution* $(u,v) \in \widetilde{U}$.

3.2 Fixed point formulation of the problem

From the definition of the fractional Laplacian $(-\Delta)^s$, the problem (3.1) is weakly formulated as follows:

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dydx = \int_\Omega f(x,u(x),v(x))\varphi(x)dx, \\ \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}} dydx = \int_\Omega g(x,u(x),v(x))\phi(x)dx, \end{cases}$$

for $(\varphi, \phi) \in \widetilde{U}$.

Let us define

$$\hat{L}_{u,v} : (\varphi, \phi) \mapsto (\hat{L}_u(\varphi), \hat{L}_v(\phi)),$$
$$\hat{S}_{u,v} : (\varphi, \phi) \mapsto (\hat{S}_1(\varphi), \hat{S}_2(\phi)),$$

where

$$\hat{L}_u(\varphi) = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dydx,$$
$$\hat{L}_v(\phi) = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}} dydx,$$

and

$$\hat{S}_1(\varphi) = \int_\Omega f(x,u(x),v(x))\varphi(x)dx,$$
$$\hat{S}_2(\phi) = \int_\Omega g(x,u(x),v(x))\phi(x)dx.$$

Lemma 3.2.1. *The operators $\hat{L}$ and $\hat{S}$ are continuous linear functionals on the space $\widetilde{U}$.*

Since $\widetilde{U}$ is a Hilbert space, by the Riesz representation theorem (see Chapter 1 Theorem 1.1.5) there exists uniquely determined elements $L(u,v)$, $S(u,v) \in \widetilde{U}$ such that

$$L(u,v) = (L(u), L(v)) \text{ and } S(u,v) = (S_1(u,v), S_2(u,v)).$$

We have also

$$\begin{cases} \hat{L}_u(\varphi) = \langle \hat{L}_u, \varphi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L(u), \varphi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{L}_v(\phi) = \langle \hat{L}_v, \phi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L(v), \phi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

and

$$\begin{cases} \hat{S}_1(\varphi) = \langle \hat{S}_1, \varphi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_1(u,v), \varphi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{S}_2(\phi) = \langle \hat{S}_2, \phi \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_2(u,v), \phi \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

for all $(\varphi, \phi) \in \widetilde{U}$.

The standard norms of $L(u,v)$ and $S(u,v)$ are defined by:

$$\begin{cases} \|L(u,v)\|^2_{\widetilde{U}} = \|L(u)\|^2_{D^{s,2}} + \|L(v)\|^2_{D^{s,2}}, \\ \\ \|S(u,v)\|^2_{\widetilde{U}} = \|S_1(u,v)\|^2_{D^{s,2}} + \|S_2(u,v)\|^2_{D^{s,2}}, \end{cases}$$

where

$$\begin{cases} \|L(u)\|_{D^{s,2}} = \|\hat{L}_u\|_{(D^{s,2})'} = \sup_{\|\varphi\| \leqslant 1} |\langle Lu, \varphi \rangle|, \\ \\ \|L(v)\|_{D^{s,2}} = \|\hat{L}_v\|_{(D^{s,2})'} = \sup_{\|\phi\| \leqslant 1} |\langle Lv, \phi \rangle|, \end{cases}$$

and

$$\begin{cases} \|S_1(u,v)\|_{D^{s,2}} = \|\hat{S}_1\|_{(D^{s,2})'} = \sup_{\|\varphi\|\leqslant 1}|\langle S_1(u,v),\varphi\rangle|, \\ \\ \|S_2(u,v)\|_{D^{s,2}} = \|\hat{S}_2\|_{(D^{s,2})'} = \sup_{\|\phi\|\leqslant 1}|\langle S_2(u,v),\phi\rangle|. \end{cases}$$

To prove that the Dirichlet problem (3.1) has at least one weak solution, it is necessary and sufficient to prove that the operator equation

$$L(u,v) = S(u,v), \tag{3.3}$$

has at least one solution in the space $\widetilde{U}$.

There are several equivalent inner products defined on $D^{s,2}(\Omega)$. If we choose

$$\langle u,v\rangle = C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dydx,$$

then L defined by (3.3) is just an identity on $\widetilde{U}$.

Hence (3.3) is equivalent in $\widetilde{U}$ to the operator equation

$$(u,v) = S(u,v). \tag{3.4}$$

3.3 Proof of the main result

In this section, we are stade different lemmas to get the existence and the uniqueness of a weak solution of problem (3.1). Our method of proof is based on the application of the Schauder fixed point theorem.

Lemma 3.3.1. *The operator S is continuous in $\widetilde{U}$.*

Proof. Let $(u_n,v_n) \longrightarrow (u,v)$ in $\widetilde{U}$, then we have

$$\begin{cases} \|S_1(u_n,v_n)-S_1(u,v)\|_{D^{s,2}} = \sup_{\|\varphi\|\leqslant 1}|\langle S_1(u_n,v_n)-S_1(u,v),\varphi\rangle|, \\ \\ \|S_2(u_n,v_n)-S_2(u,v)\|_{D^{s,2}} = \sup_{\|\phi\|\leqslant 1}|\langle S_2(u_n,v_n)-S_2(u,v),\phi\rangle|, \end{cases}$$

since

$$\begin{cases} \sup_{\|\varphi\|\leqslant 1}|\langle S_1(u_n,v_n)-S_1(u,v),\varphi\rangle| \leqslant c_{emb}\|f(x,u_n,v_n)-f(x,u,v)\|_{L^2}, \\ \\ \sup_{\|\phi\|\leqslant 1}|\langle S_2(u_n,v_n)-S_2(u,v),\phi\rangle| \leqslant c_{emb}\|g(x,u_n,v_n)-g(x,u,v)\|_{L^2}, \end{cases}$$

thus

$$\|S(u_n,v_n)-S(u,v)\|^2_{\widetilde{U}} \leqslant c^2_{emb}\|f(x,u_n,v_n)-f(x,u,v)\|^2_{L^2}+c^2_{emb}\|g(x,u_n,v_n)-g(x,u,v)\|^2_{L^2}.$$

The right-hand side approaches zero as $n\to\infty$ it follows from the continuity of the Nemytski operators from $L^2(\Omega)$ in $L^2(\Omega)$. This proves the continuity of S. $\square$

Lemma 3.3.2. *The operator S is compact.*

Proof. Let $M\subset\widetilde{U}$ be a bounded set and $\{w_n\}_{n=1}^{\infty}=\{w_{1,n},w_{2,n}\}_{n=1}^{\infty}\subset S(M)$ be an arbitrary sequence.
Let $\{u_n,v_n\}_{n=1}^{\infty}\subset M$ be such that

$$S(u_n,v_n)=(w_{1n},w_{2n}).$$

The reflexivity of $\widetilde{U}$ implies that $(u_n,v_n)\rightharpoonup(u,v)$ in $\widetilde{U}$ at least for a subsequence. As a result of the compact injection of $D^{s,2}(\Omega)$ in $L^2(\Omega)$ (Proposition 4.15) that $(u_n,v_n)\to(u,v)$ in $\widetilde{V}$. Estimates similar to those from the proof of Lemma 5.1 yield

$$(w_{1n},w_{2n})\longrightarrow S(u,v),\ \text{in } \widetilde{U}$$

(at least for a subsequence). This proves the compactness of $\overline{S(M)}$, i.e., S is a compact operator. $\square$

Lemma 3.3.3. *The operator S maps the closure of the ball $B(0;R)\subset\widetilde{U}$ into itself.*

Proof. For all $(u,v)\in\widetilde{U}$, we have from Section 3.2 that

$$\begin{cases}\|S_1(u,v)\|_{D^{s,2}}=\sup_{\|\varphi\|\leqslant 1}|\langle S_1(u,v),\varphi\rangle|,\\[2ex] \|S_2(u,v)\|_{D^{s,2}}=\sup_{\|\varphi\|\leqslant 1}|\langle S_2(u,v),\varphi\rangle|.\end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{cases}\|S_1(u,v)\|_{D^{s,2}}\leqslant c_{emb}(\int_\Omega|f(x,u(x),v(x))|^2dx)^{\frac{1}{2}},\\[2ex] \|S_2(u,v)\|_{D^{s,2}}\leqslant c_{emb}(\int_\Omega|g(x,u(x),v(x))|^2dx)^{\frac{1}{2}}.\end{cases}$$

From the hypothesis (3.2), we get

$$\begin{cases}\|S_1(u,v)\|_{D^{s,2}}\leqslant c_{emb}(\int_\Omega|r_1(x)+a|u(x)|^{\delta_1}+b|v(x)|^{\delta_1}|^2dx)^{\frac{1}{2}},\\[2ex] \|S_2(u,v)\|_{D^{s,2}}\leqslant c_{emb}(\int_\Omega|r_2(x)+c|u(x)|^{\delta_2}+d|v(x)|^{\delta_2}|^2dx)^{\frac{1}{2}},\end{cases}$$

and

$$\begin{cases}\|S_1(u,v)\|_{D^{s,2}}\leqslant c_{emb}\big(\|r_1\|_{L^2}+a(\int_\Omega|u(x)|^{2\delta_1}dx)^{\frac{1}{2}}+b(\int_\Omega|v(x)|^{2\delta_1}dx)^{\frac{1}{2}}\big),\\[2ex] \|S_2(u,v)\|_{D^{s,2}}\leqslant c_{emb}\big(\|r_2\|_{L^2}+c(\int_\Omega|u(x)|^{2\delta_2}dx)^{\frac{1}{2}}+d(\int_\Omega|v(x)|^{2\delta_2}dx)^{\frac{1}{2}}\big),\end{cases}\tag{3.5}$$

where the last estimate is due to the Minkowski inequality for $p = 2$.

Applying the Hölder inequality, we have

$$\begin{cases} (\int_\Omega |u(x)|^{2\delta_1} dx)^{\frac{1}{2}} \leqslant (\int_\Omega |u(x)|^2 dx)^{\frac{\delta_1}{2}} (mes(\Omega))^{\frac{1-\delta_1}{2}} \leqslant c_{emb}^{\delta_1} (mes(\Omega))^{\frac{1-\delta_1}{2}} \|u\|^{\delta_1}_{D^{s,2}(\Omega)}, \\ \\ (\int_\Omega |v(x)|^{2\delta_2} dx)^{\frac{1}{2}} \leqslant (\int_\Omega |v(x)|^2 dx)^{\frac{\delta_2}{2}} (mes(\Omega))^{\frac{1-\delta_2}{2}} \leqslant c_{emb}^{\delta_2} (mes(\Omega))^{\frac{1-\delta_2}{2}} \|v\|^{\delta_2}_{D^{s,2}(\Omega)}. \end{cases} \tag{3.6}$$

Now, (3.5) and (3.6) yield

$$\begin{cases} \|S_1(u,v)\|^2 \leqslant \left[c_{emb}\|r_1\| + ac_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}\|u\|^{\delta_1} + bc_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}\|v\|^{\delta_1}\right]^2, \\ \\ \|S_2(u,v)\|^2 \leqslant \left[c_{emb}\|r_2\| + cc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}\|u\|^{\delta_2} + dc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}\|v\|^{\delta_2}\right]^2, \end{cases}$$

if we put

$$\begin{cases} k = ac_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ \\ l = bc_{emb}^{\delta_1+1}(mes(\Omega))^{\frac{1-\delta_1}{2}}, \\ \\ j = cc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \\ \\ h = dc_{emb}^{\delta_2+1}(mes(\Omega))^{\frac{1-\delta_2}{2}}, \end{cases}$$

then

$$\begin{cases} \|S_1(u,v)\|^2 \leqslant \left[c_{emb}\|r_1\| + max(k,l)(\|u\|^{\delta_1} + \|v\|^{\delta_1})\right]^2, \\ \\ \|S_2(u,v)\|^2 \leqslant \left[c_{emb}\|r_2\| + max(j,h)(\|u\|^{\delta_2} + \|v\|^{\delta_2})\right]^2, \end{cases}$$

thus

$$\begin{cases} \|S_1(u,v)\|^2 \leqslant 2c_{emb}^2\|r_1\|^2 + 4max^2(k,l)(\|u\|^{2\delta_1} + \|v\|^{2\delta_1}), \\ \\ \|S_2(u,v)\|^2 \leqslant 2c_{emb}^2\|r_2\|^2 + 4max^2(j,h)(\|u\|^{2\delta_2} + \|v\|^{2\delta_2}). \end{cases}$$

Where-upon, we have

$$\|S(u,v)\|^2 \leqslant \underbrace{2c_{emb}^2\|r\|^2}_{=C} + \underbrace{4(max^2(k,l) + max^2(j,h))}_{=D} max(\|(u,v)\|^{2\delta_1}, \|(u,v)\|^{2\delta_2}). \tag{3.7}$$

It follows from (3.7) that for any $(u,v) \in B(0;R) \subset \widetilde{U}$

$$\|S(u,v)\| \leqslant R, \quad \text{with} \quad \sqrt{C + D\ max(R^{2\delta_1}, R^{2\delta_2})} < R,$$

hence S maps $B(0;R)$ into itself if R is large enough. □

Now, we can prove our main result.

Proof. (Theorem 3.1.1) To prove Theorem 3.1.1, we can apply the Schauder fixed point theorem. It follows from Lemmas 5.1, 5.3 and 3.3.3 that there is at least one fixed point $(u, v) \in \widetilde{U}$ of S (which mean the system (3.1) have a weak solution in $\widetilde{U}$). ☐

This completes the proof.

3.4 Particular case

Let us assume that f and g are Lipschitz continuous functions with respect to the second variable, i.e., there exists constants c_1, $c_2 \in \mathbb{R}^+$ for almost every $x \in \Omega$ and for any $w = (w_1, w_2), z = (z_1, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{cases} \|f(x, w_1) - f(x, w_2)\|_{L^2(\Omega)} \leqslant c_1 \|w_1 - w_2\|_{L^2(\Omega)\times L^2(\Omega)}, \\ \|g(x, z_1) - g(x, z_2)\|_{L^2(\Omega)} \leqslant c_2 \|z_1 - z_2\|_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{3.8}$$

We can apply the contraction principle to get the following result.

Theorem 3.4.1. *Let the Carathéodory functions f, g be Lipschitzian continuous with respect to the second variable with constants $c_i > 0$ $(i = 1, 2)$ such that $|c| < c_{emb}^{-2}$ $(c = (c_1, c_2))$. Then, there is a unique fixed point $(u, v) \in \widetilde{U}$, i.e., (u, v) is a unique weak solution of (3.1).*

3.4.1 Proof of the theorem

To prove Theorem 3.4.1 we need the following contraction principle.

Theorem 1 (Contraction principle). *[3] Let ψ be a contraction mapping from X to X. Then ψ admits a unique fixed-point in X.*

So to use the contraction principle we must prove that the operator S is contraction.

Lemma 3.4.1. *The S operator is a contraction.*

Proof. For any $(u, v) \in \widetilde{U}$ we also have $(u, v) \in \widetilde{V}$, then $(f(x, u, v), g(x, u, v)) \in \widetilde{V}$.
Then, for all $(u_1, v_1), (u_2, v_2) \in \widetilde{U}$, we have

$$\begin{cases} \|S_1(u_1, v_1) - S_1(u_2, v_2)\|_{D^{s,2}} = \sup_{\|\varphi\|\leqslant 1} |\langle S_1(u_1, v_1) - S_1(u_2, v_2), \varphi\rangle|, \\ \\ \|S_2(u_1, v_1) - S_2(u_2, v_2)\|_{D^{s,2}} = \sup_{\|\phi\|\leqslant 1} |\langle S_2(u_1, v_1) - S_2(u_2, v_2)\rangle, \phi)|, \end{cases}$$

this means that

$$\begin{cases} \|S_1(u_1, v_1) - S_1(u_2, v_2)\|_{D^{s,2}} = \sup_{\|v\|\leqslant 1} |\int_\Omega [f(x, u_1, v_1) - f(x, u_2, v_2)]\varphi(x)dx|, \\ \\ \|S_2(u_1, v_1) - S_2(u_2, v_2)\|_{D^{s,2}} = \sup_{\|v\|\leqslant 1} |\int_\Omega [g(x, u_1, v_1) - g(x, u_2, v_2)]\phi(x)dx|. \end{cases}$$

Using the hypothesis (5.2), we get

$$\begin{cases} \|S_1(u_1,v_1) - S_1(u_2,v_2)\|^2_{D^{s,2}} \leqslant c_1^2 c_{emb}^4 \|(u_1,v_1) - (u_2,v_2)\|^2_{\widetilde{U}}, \\ \\ \|S_2(u_1,v_1) - S_2(u_2,v_2)\|^2_{D^{s,2}} \leqslant c_2^2 c_{emb}^4 \|(u_1,v_1) - (u_2,v_2)\|^2_{\widetilde{U}}. \end{cases}$$

Consequently S is a contraction if $c_{emb}^2|c| < 1$. □

Now, we can prove Theorem 3.4.1.

Proof. We have according to Lemma 3.4.1 that S is a contraction if $c_{emb}^2|c| < 1$, then we can apply the contraction principle to get that there is a unique fixed point $(u,v) \in \widetilde{U}$ of the operator S. That is, (u,v) is the unique weak solution of (3.1).

□

Chapter 4

Existence of solution for a nonlinear fractional elliptic system at resonance and nonresonance

In this chapter, we study the existence of a weak solutions for the nonlinear fractional elliptic systems with Dirichlet boundary conditions. We use the Leray-Schauder degree to solve a resonance and non-resonance systems.

4.1 Position of the problem

This work is devoted to the study of the existence of solutions to nonlocal equations involving the fractional Laplacian, we give an application of the Leray-Schauder degree theorem to prove the existence of a weak solution to the system

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x), v(x)) + f_1(x) & \text{in} \quad \Omega, \\ (-\Delta)^s v(x) = g(x, u(x), v(x)) + f_2(x) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{4.1}$$

with $s \in (0,1)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, f and g are continuous functions defined in $\Omega \times \mathbb{R} \times \mathbb{R}$ and $h = (f_1, f_2) \in \widetilde{V}$.

To a better understanding, this result is new and represent a fractional version of a classical theorem obtained working with Laplacian equations.

Let us assume now that the nonlinear functions f and g are of the form:

$$\begin{cases} f(x, u, v) = au + f_0(x, v), \\ g(x, u, v) = bv + g_0(x, u), \end{cases}$$

where a, b are real positive constants. In this work we will see 3 class of systems considering the constants a and b.

We recall that the fractional Laplacian $(-\Delta)^s$ is a defined as

$$(-\Delta)^s\varphi(x) = C(n,s)P.V.\int_{\mathbb{R}^n} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}dy,$$

along $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $s \in (0,1)$, *P.V.* denotes the integral in the sense of the principal value, $C(n,s)$ is a positive constant of normalization and its spectrum in $L^2(\Omega)$ is formed by the sequence $(\lambda_k)_k \in \mathbb{R}^*$ such that $|\lambda_k| \to +\infty$.

We assume that $f_0, g_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory functions and we denote by λ_1 the first eigenvalue of $(-\Delta^s)$ and φ_1 is the normalized eigenfunction associated a λ_1.

Let $\lambda_1 \in \mathbb{R}^*$ be defined as

$$\lambda_1 = \inf_{\substack{u\in D^{s,2}(\Omega)\\ u\neq 0}} \frac{C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx}{\int_\Omega |u|^2dx},$$

or equivalently as

$$\lambda_1 = \inf\{C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx : \int_\Omega |u|^2dx = 1, u \in D^{s,2}(\Omega), u \neq 0\},$$

if

$$C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(\varphi_1(x)-\varphi_1(y))(v(x)-v(y))}{|x-y|^{n+2s}}dydx = \lambda_1 \int_\Omega \varphi_1(x)v(x)dx, \text{ for all } v \in D^{s,2}(\Omega).$$

The purpose of the present chapter is to extend the results of [45, 51] to the system (4.1) under the following conditions on the functions f_0 and g_0:

$$\begin{cases} \beta_1^-, \beta_1^+ \in L^2(\Omega) \\ \text{and} \\ \beta_1^-(x) \leqslant f_0(x,t) \leqslant \beta_1^+(x) \text{ a.e. } \Omega \\ \text{where} \\ \lim\limits_{t\to-\infty} f_0(.,t) = \beta_1^-(.) \text{ a.e. } \Omega \\ \lim\limits_{t\to+\infty} f_0(.,t) = \beta_1^+(.) \text{ a.e. } \Omega, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \beta_2^-, \beta_2^+ \in L^2(\Omega) \\ \text{and} \\ \beta_2^-(x) \leqslant g_0(x,t) \leqslant \beta_2^+(x) \text{ a.e. } \Omega \\ \text{where} \\ \lim\limits_{t\to-\infty} g_0(.,t) = \beta_2^-(.) \text{ a.e. } \Omega \\ \lim\limits_{t\to+\infty} g_0(.,t) = \beta_2^+(.) \text{ a.e. } \Omega. \end{cases} \tag{4.3}$$

4.2 Preliminaries and the main results

Firsty, we give a definition of weak solution.

Definition 4.2.1. *We say $(u,v) \in \widetilde{U}$ is a weak solution of the system (4.1) if for any $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \widetilde{U}$, we have*

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}} dydx = \int_\Omega au\tilde{w}_1 dx + \int_\Omega f_0(x,v)\tilde{w}_1 dx + \int_\Omega f_1(x)\tilde{w}_1 dx, \\ \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}} dydx = \int_\Omega bv\tilde{w}_2 dx + \int_\Omega g_0(x,u)\tilde{w}_2 dx + \int_\Omega f_2(x)\tilde{w}_2 dx. \end{cases}$$

For a fixed $(u,v) \in \widetilde{U}$ it is easy to see that

$$\hat{L}_{u,v} : (\tilde{w}_1, \tilde{w}_2) \mapsto (\hat{L}_u(\tilde{w}_1), \hat{L}_v(\tilde{w}_2)),$$

$$\hat{S}_{u,v} : (\tilde{w}_1, \tilde{w}_2) \mapsto (\hat{S}_v(\tilde{w}_1), \hat{S}_u(\tilde{w}_2)),$$

$$\hat{A}_{u,v} : (\tilde{w}_1, \tilde{w}_2) \mapsto (\hat{A}_u(\tilde{w}_1), \hat{A}_v(\tilde{w}_2)),$$

such that

$$\hat{L}_u(\tilde{w}_1) = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}} dydx,$$

$$\hat{L}_v(\tilde{w}_2) = C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}} dydx,$$

$$\hat{S}_v(\tilde{w}_1) = \int_\Omega f_0(x, v(x))\tilde{w}_1(x) dx,$$

$$\hat{S}_u(\tilde{w}_2) = \int_\Omega g_0(x, u(x))\tilde{w}_2(x) dx,$$

and

$$\hat{A}_u(\tilde{w}_1) = \int_\Omega u\tilde{w}_1(x) dx,$$

$$\hat{A}_v(\tilde{w}_2) = \int_\Omega v\tilde{w}_2(x) dx.$$

Lemma 4.2.1. *$\hat{L}$, $\hat{S}$ and $\hat{A}$ are continuous linear functionals on the space $\widetilde{U}$.*

As the previous chapter, since $\widetilde{U}$ is a Hilbert space, by the Riesz Representation Theorem (see chapter 1 Theorem 1.1.5) there exist uniquely determined elements $L(u,v)$, $S(u,v)$ and $A(u,v) \in \widetilde{U}$ such that

$$L(u,v) = (L_1(u), L_2(v)), \quad S(u,v) = (S_1(v), S_2(u)) \text{ and } A(u,v) = (A_1(u), A_2(v)),$$

we have also that

$$\begin{cases} \hat{L}_u(\tilde{w}_1) = \langle \hat{L}_u, \tilde{w}_1 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L_1(u), \tilde{w}_1 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{L}_v(\tilde{w}_2) = \langle \hat{L}_v, \tilde{w}_2 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle L_2(v), \tilde{w}_2 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

$$\begin{cases} \hat{S}_v(\tilde{w}_1) = \langle \hat{S}_v, \tilde{w}_1 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_1(v), \tilde{w}_1 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{S}_u(\tilde{w}_2) = \langle \hat{S}_u, \tilde{w}_2 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle S_2(u), \tilde{w}_2 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

and

$$\begin{cases} \hat{A}_u(\tilde{w}_1) = \langle \hat{A}_u, \tilde{w}_1 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle A_1(u), \tilde{w}_1 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \\ \\ \hat{A}_v(\tilde{w}_2) = \langle \hat{A}_v, \tilde{w}_2 \rangle_{\langle (D^{s,2})', D^{s,2} \rangle} = \langle A_2(v), \tilde{w}_2 \rangle_{\langle D^{s,2}, D^{s,2} \rangle}, \end{cases}$$

for all $(\tilde{w}_1, \tilde{w}_2) \in \widetilde{U}$.

We can define L as an identity on $\widetilde{U}$ and it is clearly that S and A are a compact and continuous operators.

For $t \in [0,1]$ and $(u,v) \in \widetilde{U}$ we define the following homotopy

$$T(t,u,v) = \begin{pmatrix} T_1(t,u,v) \\ T_2(t,u,v) \end{pmatrix} = \begin{pmatrix} u - aA_1(u) - tS_1(v) - (1-t)\varepsilon A_1(u) \\ v - bA_2(v) - tS_2(u) - (1-t)\varepsilon A_2(v) \end{pmatrix}.$$

Equivalently,

$$T(t,u,v) = (u,v) - BA(u,v) - tS(u,v) - (1-t)C(\varepsilon)A(u,v), \forall \varepsilon > 0$$

where

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } C(\varepsilon) = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that

$$BA - tS - (1-t)C(\varepsilon)A : [0,1] \times \widetilde{U} \to \widetilde{U},$$

is a compact and continuous operator, then the existence of at least one solution of the system (4.1) would follow from

$$deg(I - BA - S, B(0,R), 0) \neq 0.$$

Let's now give the main result of this chapter.

Theorem 4.2.1. *Assume that (4.2) and (4.3) are fulfilled. Then (4.1) has at least one solution* $(u, v) \in \widetilde{U}$.

4.3 A first class of systems

The class of systems considered in this section is determined by taking the functions

$$\begin{cases} f(x, u, v) = au + f_0(x, v), \\ g(x, u, v) = bv + g_0(x, u), \end{cases}$$

and

$$\begin{cases} f_1(x) = h_1(x), \\ f_2(x) = h_2(x), \end{cases}$$

where $a, b \notin sp((-\Delta)^s)$.

In light of the notation above (4.1) reads as

$$\begin{cases} (-\Delta)^s u(x) = au + f_0(x, v) + h_1(x) & \text{in} \quad \Omega, \\ (-\Delta)^s v(x) = bv + g_0(x, u) + h_2(x) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega. \end{cases} \tag{4.4}$$

To prove the existence of solutions for (4.4), we have to define an admissible homotopy T, the simplest way to do it is to follow Section 4.2. Define

$$T(t, u, v) = (u, v) - BA(u, v) - tS(u, v) - (1 - t)C(\varepsilon)A(u, v),$$

$\forall \varepsilon > 0$, $t \in [0, 1]$ and $(u, v) \in \widetilde{U}$. T is a compact homotopy connecting $I(.) - BA(.) - S(.)$ and $I(.) - BA(.)$.

4.3.1 A priori bounds for solutions

So, to complete the proof, we have to prove that there exists $R_1 > 0$ such that, for all $(u, v) \in \widetilde{U}$, $\|(u, v)\|_{\widetilde{U}} = R_1$ and $t \in [0, 1]$, we obtain

$$T(t, u, v) \neq 0. \tag{4.5}$$

The usual way to establish (4.5) is an indirect proof.

Lemma 4.3.1. *There exists* $R_1 > 0$ *such that*

$$\begin{cases} \|(u, v)\|_{\widetilde{U}} = R_1, \quad \forall t \in [0, 1], \forall (u, v) \in \widetilde{U} \\ T(t, u, v) \neq 0. \end{cases}$$

Proof. Let $\varepsilon > 0$ such that $[a, a+\varepsilon] \cap sp((-\Delta)^s) = \varnothing$ and $[b, b+\varepsilon] \cap sp((-\Delta)^s) = \varnothing$. Assume that no such $R_1 > 0$ exists, i.e., we can find sequences $\{(u_n, v_n)\}_{n=1}^{\infty} \subset \widetilde{U}$ and $\{t_n\}_{n=1}^{\infty} \subset [0,1]$ such that $\|(u_n, v_n)\|_{\widetilde{U}} \to \infty$ (i.e. $\|(u_n, v_n)\|_{\widetilde{U}} = n$) and

$$(u_n, v_n) - BA(u_n, v_n) - t_n S(u_n, v_n) - (1-t_n)C(\varepsilon)A(u_n, v_n) = 0. \quad \forall \varepsilon > 0 \tag{4.6}$$

Set

$$w_n = (w_{1,n}, w_{2,n}) = \Big(\frac{u_n}{\|(u_n, v_n)\|_{\widetilde{U}}}, \frac{v_n}{\|(u_n, v_n)\|_{\widetilde{U}}}\Big),$$

then it follows that $\|w_n\|_{\widetilde{U}} = 1$.

Divide (4.6) by $\|(u_n, v_n)\|_{\widetilde{U}}$ to get

$$w_n - BA(w_n) - (1-t_n)C(\varepsilon)A(w_n) - t_n \frac{S(u_n, v_n)}{\|(u_n, v_n)\|_{\widetilde{U}}} = 0, \forall \varepsilon > 0 \tag{4.7}$$

this is equivalent to

$$C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(w_{1,n}(x) - w_{1,n}(y))(\tilde{w}_1(x) - \tilde{w}_1(y))}{|x-y|^{n+2s}} dydx = [a + (1-t_n)\varepsilon] \int_\Omega w_{1,n}\tilde{w}_1 dx + \int_\Omega \frac{f_0(x, v_n)\tilde{w}_1}{\|(u_n, v_n)\|} dx + \int_\Omega h_1(x)\tilde{w}_1 dx,$$

$$C(n,s) \iint\limits_{\mathbb{R}^{2n}} \frac{(w_{2,n}(x) - w_{2,n}(y))(\tilde{w}_2(x) - \tilde{w}_2(y))}{|x-y|^{n+2s}} dydx = [b + (1-t_n)\varepsilon] \int_\Omega w_{2,n}\tilde{w}_2 dx + \int_\Omega \frac{g_0(x, u_n)\tilde{w}_2}{\|(u_n, v_n)\|} dx + \int_\Omega h_2(x)\tilde{w}_2 dx,$$

for any $(\tilde{w}_1, \tilde{w}_2) \in \widetilde{U}$. Now, passing to suitable subsequences still denoted by $(t_n, w_{1,n}, w_{2,n})$, we can assume that $t_n \to t \in [0,1]$ and $(w_{1,n}, w_{2,n}) \rightharpoonup (w_1, w_2)$ in $\widetilde{U}$. At the same time

$$\int_\Omega \frac{|f_0(x, v_n)|}{\|(u_n, v_n)\|} |\tilde{w}_1| dx \leqslant \int_\Omega \frac{|\beta_1^+(x)|}{\|(u_n, v_n)\|} |\tilde{w}_1| dx \to 0 \quad n \to \infty,$$

and

$$\int_\Omega \frac{|g_0(x, u_n)|}{\|(u_n, v_n)\|} |\tilde{w}_2| dx \leqslant \int_\Omega \frac{|\beta_2^+(x)|}{\|(u_n, v_n)\|} |\tilde{w}_2| dx \to 0 \text{ as } n \to \infty.$$

To summarize, we have

$$t_n \frac{S(u_n, v_n)}{\|(u_n, v_n)\|_{\widetilde{U}}} \to 0, \tag{4.8}$$

$$A(w_n) \to A(w). \tag{4.9}$$

(by the compactness of A, see [23] Proposition 2.2.4 (iii)). So, putting together (4.7)-(4.9) we also obtain that

$$w_n \to w^* \text{ in } \widetilde{U}.$$

But $w^* = w$ by virtue of $w_n \rightharpoonup w$. Now, passing to the limit in (4.7), we arrive at

$$w - [B + (1-t)C(\varepsilon)]A(w) = 0, \tag{4.10}$$

and $w \in \widetilde{U}$ satisfies $\|w\| = 1$ (it is the strong limit of elements w_n which satisfy $\|w_n\| = 1$).

Then

$$\begin{aligned}(-\Delta)^s w_1 &= [a + (1-t)\varepsilon]w_1,\\ (-\Delta)^s w_2 &= [b + (1-t)\varepsilon]w_2.\end{aligned}$$

However, this contradicts our assumption $[a, a+\varepsilon] \cap sp((-\Delta)^s) = \varnothing$ and $[b, b+\varepsilon] \cap sp((-\Delta)^s) = \varnothing$.
It proves that (4.5) holds, i.e., the homotopy T is admissible. This completes the proof. □

4.4 A second class of systems

In this section, we let

$$\begin{cases} f(x,u,v) = au + f_0(x,v),\\ g(x,u,v) = bv + g_0(x,u),\end{cases}$$

and

$$\begin{cases} f_1(x) = h_1(x),\\ f_2(x) = -h_2(x),\end{cases}$$

where $a = \lambda \in sp((-\Delta)^s)$ and $b = \lambda_1$ which is the first eigenvalue of the fractional Laplace operator.

We obtain the following system

$$\begin{cases} (-\Delta)^s u(x) = \lambda u + f_0(x,v) + h_1(x) & \text{in} \quad \Omega,\\ (-\Delta)^s v(x) = \lambda_1 v + g_0(x,u) - h_2(x) & \text{in} \quad \Omega,\\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega.\end{cases} \tag{4.11}$$

Proposition 4.4.1. *Assume that (4.2) and (4.3) are fulfilled. Let* $(\varphi, \varphi_1) \in N_\lambda \times N_{\lambda_1}$, $\lambda_1 > 0$ *be the first eigenvalue of the fractional Laplace operator subject to the homogeneous Dirichlet boundary condition and* φ_1 *the associate eigenfunction normalized by*

$$\int_\Omega \varphi_1 = 1.$$

Then the problem (4.11) has at least one weak solution if and only if

1. $-\int_\Omega \beta_1^+ \varphi(x)dx \leqslant \int_\Omega h_1(x)\varphi(x)dx \leqslant -\int_\Omega \beta_1^- \varphi(x)dx.$

2. $\beta_2^- \leqslant \int_\Omega h_2(x)\varphi_1(x)dx \leqslant \beta_2^+.$

with $(h_1, h_2) \in \widetilde{V}$.

Proof. We will follow a scheme similar to the proof in Section (4.3). For $\varepsilon > 0$ so small such that $]\lambda, \lambda + \varepsilon] \cap sp((-\Delta)^s) = \varnothing$, $\lambda_1 + \varepsilon < \lambda_2$ we define the homotopy

$$T(t, u, v) = (u, v) - BA(u, v) - tS(u, v) - (1 - t)C(\varepsilon)A(u, v), \quad t \in [0, 1] \quad (u, v) \in \widetilde{U}$$

here

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_1 \end{pmatrix}.$$

Performing all steps as in the proof in Section (4.3) we arrive at an analogue of (4.10), namely,

$$w - [B + (1 - t)C(\varepsilon)]A(w) = 0, \quad \|w\| = 1 \text{ for all } t \in [0, 1] \tag{4.12}$$

This is a contradiction if $t \neq 1$ since $\lambda + (1-t)\varepsilon$ and $\lambda_1 + (1-t)\varepsilon$ are not an eigenvalues and $w \neq 0$.

Let us assume $t = 1$, i.e., $t_n \to 1$. Now, we have no contradiction since λ and λ_1 are eigenvalues and

$$w - BA(w) = 0,$$

has a solution with $\|w\| = 1$. Another step is necessary to reach a contradiction and to prove that the homotopy T is admissible. We have to revise the last step when passing to the limit in

$$w_n - BA(w_n) - (1 - t_n)C(\varepsilon)A(w_n) - t_n \frac{S(u_n, v_n)}{\|(u_n, v_n)\|_{\widetilde{U}}} = 0, \quad t \in [0, 1] \quad (u, v) \in \widetilde{U}$$

and employ special properties of S. We mean,

$$(u_n, v_n) - BA(u_n, v_n) - t_n S(u_n, v_n) - (1 - t_n)C(\varepsilon)A(u_n, v_n) = 0, \quad \forall \varepsilon > 0$$

is equivalent to the integral identity

$$C(n, s) \iint\limits_{\mathbb{R}^{2n}} \frac{(u_n(x) - u_n(y))(\tilde{w}_1(x) - \tilde{w}_1(y))}{|x - y|^{n+2s}} dydx = [\lambda + (1 - t_n)\varepsilon] \int_\Omega u_n \tilde{w}_1 dx + \int_\Omega f_0(x, v_n)\tilde{w}_1 dx + \int_\Omega h_1(x)\tilde{w}_1 dx, \tag{4.13}$$

$$C(n, s) \iint\limits_{\mathbb{R}^{2n}} \frac{(v_n(x) - v_n(y))(\tilde{w}_2(x) - \tilde{w}_2(y))}{|x - y|^{n+2s}} dydx = [\lambda_1 + (1 - t_n)\varepsilon] \int_\Omega v_n \tilde{w}_2 dx + \int_\Omega g_0(x, u_n)\tilde{w}_2 dx - \int_\Omega h_2(x)\tilde{w}_2 dx, \tag{4.14}$$

for all $(\tilde{w_1}, \tilde{w_2}) \in \widetilde{U}$. Taking $(\tilde{w_1}, \tilde{w_2}) = (\varphi, \varphi_1)$ in (4.13)-(4.14) and using the fact that

$$C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dydx = \lambda\int_\Omega u_n\varphi dx,$$

and

$$C(n,s)\iint\limits_{\mathbb{R}^{2n}} \frac{(v_n(x)-v_n(y))(\varphi_1(x)-\varphi_1(y))}{|x-y|^{n+2s}}dydx = \lambda_1\int_\Omega v_n\varphi_1 dx,$$

we obtain

$$(1-t_n)\varepsilon\int_\Omega u_n\varphi dx + \int_\Omega f_0(x,v_n)\varphi dx + \int_\Omega h_1(x)\varphi dx = 0, \tag{4.15}$$

and

$$(1-t_n)\varepsilon\int_\Omega v_n\varphi_1 dx + \int_\Omega g_0(x,u_n)\varphi_1 dx = \int_\Omega h_2(x)\varphi_1 dx. \tag{4.16}$$

We have, $w_n = (w_{1,n}, w_{2,n}) = \left(\frac{u_n}{\|(u_n,v_n)\|_{\widetilde{U}}}, \frac{v_n}{\|(u_n,v_n)\|_{\widetilde{U}}}\right) \to w = (w_1, w_2)$ in $\widetilde{U}$ and $w = km$ with a

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, m = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi_1 \end{pmatrix},$$

such that $k_i \neq 0$. Assume that $k_i > 0$, using Theorem 1.2.26 and Remark 1.2.18 in [23], we get (at least for a subsequence) $(u_n, v_n) \to \infty$. Passing to the limit in (4.15) and (4.16) with $t_n \to -1$ and the Lebesgue dominated convergence theorem, we obtain

1. $\displaystyle\int_\Omega h_1(x)\varphi(x)dx \leqslant -\int_\Omega \beta_1^+\varphi(x)dx,$
2. $\displaystyle\beta_2^+ \leqslant \int_\Omega h_2(x)\varphi_1(x)dx,$

this contradicts ones inequality in (4.4.1). Similarly we proceed if $k < 0$ to get a contradiction with the other inequality in (4.4.1).

To prove that conditions (1) and (2) are also necessary, we take (u_0, v_0) as a week solution of (4.11),i.e. for any $(\tilde{w_1}, \tilde{w_2}) \in \widetilde{U}$, we have

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(u_0(x)-u_0(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}}dydx = \lambda\int_\Omega u_0\tilde{w}_1dx + \int_\Omega f_0(x,v_0)\tilde{w}_1dx + \int_\Omega h_1(x)\tilde{w}_1dx, \\ \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(v_0(x)-v_0(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}}dydx = \lambda_1\int_\Omega v_0\tilde{w}_2dx + \int_\Omega g_0(x,u_0)\tilde{w}_2dx - \int_\Omega h_2(x)\tilde{w}_2dx. \end{cases}$$

Set $(\tilde{w_1}, \tilde{w_2}) = (\varphi, \varphi_1)$. Then

$$\begin{cases} \int_\Omega f_0(x,v_0)\varphi dx + \int_\Omega h_1(x)\varphi dx = 0, \\ \\ \int_\Omega g_0(x,u_0)\varphi_1 dx = \int_\Omega h_2(x)\varphi_1 dx. \end{cases}$$

From conditions (4.2) and (4.3) respectively, we obtain

$$-\int_\Omega \beta_1^+ \varphi(x)dx \leqslant \int_\Omega h_1(x)\varphi(x)dx \leqslant -\int_\Omega \beta_1^- \varphi(x)dx.$$

$$\beta_2^- \leqslant \int_\Omega h_2(x)\varphi_1(x)dx \leqslant \beta_2^+.$$

□

4.5 A third class of systems

In the last system, we put $a = \lambda^1$ and $b = \lambda^2$ where λ^1, $\lambda^2 \in sp((-\Delta)^s)\backslash\{\lambda_1\}$ and

$$\begin{cases} f_1(x) = h_1(x), \\ f_2(x) = h_2(x), \end{cases}$$

we obtain the following system

$$\begin{cases} (-\Delta)^s u(x) = \lambda^1 u + f_0(x,v) + h_1(x) & \text{in} \quad \Omega, \\ (-\Delta)^s v(x) = \lambda^2 v + g_0(x,u) + h_2(x) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \mathbb{R}^n\backslash\Omega. \end{cases} \tag{4.17}$$

Proposition 4.5.1. *Assume that (4.2) and (4.3) are fulfilled. Let* $\theta_i \in N_\lambda^i$ $(i = 1,2)$. *Then the problem (4.17) has at least one weak solution if and only if*

$$-\int_\Omega \beta_i^+ \theta_i(x)dx \leqslant \int_\Omega h_i(x)\theta_i(x)dx \leqslant -\int_\Omega \beta_i^- \theta_i(x)dx,$$

with $h_i \in L^2(\Omega)$ *for* $i = 1,2$.

Proof. Let (u_0, v_0) be a weak solution of (4.17), i.e. for any $\tilde{w} = (\tilde{w_1}, \tilde{w_2}) \in \widetilde{U}$, we have

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(u_0(x)-u_0(y))(\tilde{w}_1(x)-\tilde{w}_1(y))}{|x-y|^{n+2s}}dydx = \lambda^1\int_\Omega u_0\tilde{w}_1dx + \int_\Omega f_0(x,v_0)\tilde{w}_1dx + \int_\Omega h_1(x)\tilde{w}_1dx, \\ \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(v_0(x)-v_0(y))(\tilde{w}_2(x)-\tilde{w}_2(y))}{|x-y|^{n+2s}}dydx = \lambda^2\int_\Omega v_0\tilde{w}_2dx + \int_\Omega g_0(x,u_0)\tilde{w}_2dx + \int_\Omega h_2(x)\tilde{w}_2dx. \end{cases}$$

The choice of $(\tilde{w}_1, \tilde{w}_2) = (\theta_1, \theta_2)$, gives

$$\begin{cases} \int_\Omega f_0(x,v_0)\theta_1dx + \int_\Omega h_1(x)\theta_1dx = 0, \\ \\ \int_\Omega g_0(x,u_0)\theta_2dx + \int_\Omega h_2(x)\theta_2dx = 0. \end{cases}$$

From conditions (4.2) and (4.3) respectively, we obtain

$$-\int_\Omega \beta_i^+ \theta_i(x)dx \leqslant \int_\Omega h_i(x)\theta_i(x)dx \leqslant -\int_\Omega \beta_i^- \theta_i(x)dx.$$

□

4.5.1 A priori bounds for solutions

Lemma 4.5.1. *There exists* $R_2 > 0$ *such that*

$$\begin{cases} \|(u,v)\|_{\widetilde{U}} = R_2, & \forall t \in [0,1], \forall (u,v) \in \widetilde{U} \\ T(t,u,v) \neq 0. \end{cases}$$

Proof. Let $\varepsilon > 0$ so small such that $]\lambda^i, \lambda^i + \varepsilon] \cap sp((-\Delta)^s) = \varnothing$ $(i = 1,2)$.
Recall the proof of the previous lemma with

$$B = \begin{pmatrix} \lambda^1 & 0 \\ 0 & \lambda^2 \end{pmatrix}.$$

Taking $(\tilde{w}_1, \tilde{w}_2) = (\theta_1, \theta_2)$.

By the same method, we get

$$-\int_\Omega \beta_i^- \theta_i(x)dx \leqslant \int_\Omega h_i(x)\theta_i(x)dx \leqslant -\int_\Omega \beta_i^+ \theta_i(x)dx, i = 1,2.$$

This contradicts Proposition 4.5.1. □

Now we present the proof of Theorem 4.2.1, which give the existence of solutions for system (4.1).

Proof. (of Theorem 4.2.1). Let

$$B(0,R) = \{(u,v) \in \widetilde{U}, \|(u,v)\|_{\widetilde{U}} < R\}.$$

We can define the topological degree and by invariance of the homtopy, we have

$$d(T(t,.,.), B(0,R), 0) = d(T(0,.,.), B(0,R), 0),$$

which is equivalent to

$$d(I - BA - tS, B(0,R), 0) = d(I - BA, B(0,R), 0).$$

In particular if $t = 1$:

$$d(I - BA - S, B(0,R), 0) = d(I - BA, B(0,R), 0).$$

On the other hand, for $t = 0$ the system has a unique solution. Then

$$d(I - BA(.) - S(.), B(0,R), 0) = d(I - BA(.), B(0,R), 0) \neq 0,$$

whereupon there exists $(u,v) \in B(0,R)$ such that:

$$T(1,u,v) = 0,$$

therefore there is a weak solution of the system (4.1).

□

Chapter 5

Numerical approximation of a nonlinear fractional elliptic system

In this chapter, we present a finite difference method (FDM) for the numerical approximation of the solution to nonlinear fractional elliptic system, the analysis is performed on two general dimensional domains with homogeneous boundary conditions and it is shown for a general fractional order ($s \in]0,1[$). Convergence of the approximation and the corresponding error estimates are proven. A numerical example with known exact solution is also presented, which confirm the theoretical predictions.

5.1 Position of the problem and mathematical preliminaries

The main motivation of our study is the efficient numerical solution of the boundary value system

$$\begin{cases} (-\Delta)^s u(z) = f(z, u(z), v(z)) & \text{in} \quad \Omega, \\ (-\Delta)^s v(z) = g(z, u(z), v(z)) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \mathbb{R}^n \backslash \Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in]0,1[$ and $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying the Carathéodory conditions (i.e : $f(.,x)$, $g(.,y)$ are measurables for each $x, y \in \mathbb{R}^2$ and $f(z,.)$, $g(w,.)$ are continuous for almost every $z, w \in \Omega$), and Lipschitz continuous functions with respect to the second variable, i.e., there are constants c_1, $c_2 \in \mathbb{R}^+$ for almost every $x \in \Omega$ and for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{cases} \|f(z, x_1) - f(z, x_2)\|_{L^2(\Omega)} \leqslant c_1 \|x_1 - x_2\|_{L^2(\Omega) \times L^2(\Omega)}, \\ \|g(z, y_1) - g(z, y_2)\|_{L^2(\Omega)} \leqslant c_2 \|y_1 - y_2\|_{L^2(\Omega) \times L^2(\Omega)}. \end{cases} \tag{5.2}$$

The fractional Laplacian is widely-spread in the modern study of fractional partial differential systems, it has a variety of definitions. We recall that the Riesz potentials of order α for

$0 < \alpha < n$ and $n \in \mathbb{N}^*$ is defined by

$$I_\alpha \varphi = I_\alpha * \varphi,$$

where

$$I_\alpha(z) = \frac{\gamma(n,\alpha)}{|z|^{n-\alpha}},$$

and the constant

$$\gamma(n,\alpha) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}.$$

Through analytic continuation, the Riesz potential can be extended to negative exponents. Thus the auteurs in [54] arrives at the next formula for the fractional laplacian

$$(-\Delta)^s \varphi = I_{-2s}\varphi,$$

and others propositions which are used in our work.

After the analytic study that the reader can extend into by checking our article [21], we perform numerical study and we close with numerical experiment illustrating the convergence results.

All along the chapter and without further mention, we always assume that $n = 2$ and $\Omega =]0,1[\times]0,1[$.

5.2 Analytical study

The contraction principle is applied to have the following result.

Theorem 5.2.1 (Existence and uniqueness). *Let the Carathéodory functions f, g be Lipschitzian continuous with respect to the second variable with constants $c_i > 0$ $(i = 1, 2)$ such that $|c| < c_{emb}^{-2}$ ($c = (c_1, c_2)$ and c_{emb} is the embedding constant). Then, there is a unique fixed point $(u, v) \in \widetilde{U}$, i.e., (u, v) is a unique weak solution of (5.1).*

For more details and the proof, you can see [21].

5.3 Numerical study

We devote this Section to the description of the numerical scheme that we are going to employ. In order to solve numerically (5.1), we will develop a finite difference scheme on a uniform mesh. To this purpose, let us first introduce a partition of $\Omega =]0,1[\times]0,1[$ as follows:

$$\Omega = \{(x_i, y_j) : 0 = x_0 < ... < x_i < ... < x_{N+1} = 1 \text{ and } 0 = y_0 < ... < y_i < ... < y_{M+1} = 1\},$$

with $x_{i+1} = x_i + h$, $x_i = x_0 + ih$ where $i = 0, ...N$ and $y_{j+1} = y_j + k$, $y_j = j_0 + jk$ where $j = 0, ...M$ (N and M are non-null positive constants), in the rest of this chapter, we take

$N = M$ and $h = k$.

For $n = 2$, we call (i, j) an interior grid point if all of its neighbors $(i-1, j)$, $(i+1, j)$, $(i, j-1)$ and $(i, j+1)$ are in Ω. The matrix $A \in M^{N\times N}$ denotes the approximation of the operator $(-\Delta)^s$ with the standard five-star difference scheme such that for each (u, v) indexed according to the grid points we have

$$
\begin{aligned}
(x_i, y_j) &\to u(x_i, y_j) = u_{ij},\\
(x_i, y_j) &\to v(x_i, y_j) = v_{ij},
\end{aligned}
$$

and from [54], we get that the system (5.1) is equivalent to

$$
\begin{aligned}
-\frac{\partial^s}{\partial x^s}\frac{\partial^s u}{\partial x^s} - \frac{\partial^s}{\partial y^s}\frac{\partial^s u}{\partial y^s} &= f(x, y, u, v),\\
-\frac{\partial^s}{\partial x^s}\frac{\partial^s v}{\partial x^s} - \frac{\partial^s}{\partial y^s}\frac{\partial^s v}{\partial y^s} &= g(x, y, u, v),
\end{aligned}
$$

which lead to

$$
\begin{aligned}
-(I_{2-2s} * \frac{\partial^2 u}{\partial x^2}) - (I_{2-2s} * \frac{\partial^2 u}{\partial y^2}) &= f(x, y, u, v),\\
-(I_{2-2s} * \frac{\partial^2 v}{\partial x^2}) - (I_{2-2s} * \frac{\partial^2 v}{\partial y^2}) &= g(x, y, u, v),
\end{aligned}
$$

but we have

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &\simeq \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2},\\
\frac{\partial^2 u}{\partial y^2} &\simeq \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2},
\end{aligned}
$$

then, we can write our system as follows

$$
\begin{aligned}
I_{2-2s} * \frac{-u_{i-1,j} - u_{i+1,j} + 4u_{ij} - u_{i,j-1} - u_{i,j+1}}{h^2} &= f(x_i, y_j, u_{ij}, v_{ij}),\\
I_{2-2s} * \frac{-v_{i-1,j} - v_{i+1,j} + 4v_{ij} - v_{i,j-1} - v_{i,j+1}}{h^2} &= g(x_i, y_j, u_{ij}, v_{ij}),
\end{aligned}
$$

according to the definition of convolution product for sequences (by replacing the Lebesgue measure by the counting measure), we can write

$$
\begin{aligned}
\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{|(x_i, y_j) - (x_n, y_m)|^{n-2+2s}} \frac{-u_{i-1,j} - u_{i+1,j} + 4u_{ij} - u_{i,j-1} - u_{i,j+1}}{h^2} &= f(x_i, y_j, u_{ij}, v_{ij}),\\
\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{|(x_i, y_j) - (x_n, y_m)|^{n-2+2s}} \frac{-v_{i-1,j} - v_{i+1,j} + 4v_{ij} - v_{i,j-1} - v_{i,j+1}}{h^2} &= g(x_i, y_j, u_{ij}, v_{ij}),
\end{aligned}
$$

and after a simple calculus and the fact that $n = 2$ we arrive to the numerical scheme of the system (5.1)

$$\begin{cases} \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{1}{((i-n)^2+(j-m)^2)^s} \frac{-u_{i-1,j}-u_{i+1,j}+4u_{ij}-u_{i,j-1}-u_{i,j+1}}{h^2} = h^{2-2s} f(x_i, y_j, u_{ij}, v_{ij}), \\ \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{1}{((i-n)^2+(j-m)^2)^s} \frac{-v_{i-1,j}-v_{i+1,j}+4v_{ij}-v_{i,j-1}-v_{i,j+1}}{h^2} = h^{2-2s} g(x_i, y_j, u_{ij}, v_{ij}), \\ u_{i,0} = u_{i,N+1} = u_{0,j} = u_{N+1,j} = v_{i,0} = v_{i,N+1} = v_{0,j} = v_{N+1,j}. \end{cases} \tag{5.3}$$

Hence (5.3) is equivalent to the matrix system

$$BU = b,$$

where $U = (u, v)$, $b = (f, g)$ and

$$B = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

as an example if we take $N = 2$, then

$$A = \begin{pmatrix} -2 & (4 - \frac{1}{2^s}) & (4 - \frac{1}{2^s}) & (\frac{4}{2^s} - 2) \\ (4 - \frac{1}{2^s}) & -2 & (\frac{4}{2^s} - 2) & (4 - \frac{1}{2^s}) \\ (4 - \frac{1}{2^s}) & (\frac{4}{2^s} - 2) & -2 & (4 - \frac{1}{2^s}) \\ (\frac{4}{2^s} - 2) & (4 - \frac{1}{2^s}) & (4 - \frac{1}{2^s}) & -2 \end{pmatrix}$$

and

$$U = \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix},$$

$$b = \begin{pmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ g_{11} \\ g_{12} \\ g_{21} \\ g_{22} \end{pmatrix}.$$

Consequently to get the approaches solution of the system (5.3) it's equivalent to solve the system

$$U = B^{-1}b.$$

5.3.1 Error estimate

In this part, we study the consistency, the stability and the convergence of the scheme (5.3) but first, it is necessary to prove the existence and the unicity of the solution and for that we must prove the next theorem.

Theorem 5.3.1. *Under the assumption (5.2) the system (5.3) admits a unique solution.*

We use the discrete maximum principle to prove that the application BU is injective in finite dimension, than the matrix B is invertible which lead us to the existence and uniqueness of solution.

Consistency

We say that a method is consistent with the differential equation and boundary conditions if

$$\|R\|_\infty \leqslant Ch^t,$$

where R is the rest and C is positive constant.

Remark 5.3.1. *If $\|R\|_\infty \leqslant Ch^t$, we say that the method is consistent of order t where t is a positive real constant.*

Proposition 5.3.1. *The scheme (5.3) is consistent of order $(2-2s)$, moreover*

$$\|R\|_\infty \leqslant Ch^{2-2s}.$$

Proof. We have

$$R_{ij} = \begin{cases} \frac{h^{2-2S}}{4!}(N\times N-1)(S+T), \\ \\ \frac{h^{2-2S}}{4!}(N\times N-1)(L+M), \end{cases}$$

where

$$S = \frac{\partial^4 u}{\partial x^4}(\alpha_1,y) + \frac{\partial^4 u}{\partial x^4}(\alpha_2,y),$$

$$T = \frac{\partial^4 u}{\partial y^4}(x,\beta_1) + \frac{\partial^4 u}{\partial y^4}(x,\beta_2),$$

$$L = \frac{\partial^4 v}{\partial x^4}(\alpha_1,y) + \frac{\partial^4 v}{\partial x^4}(\alpha_2,y),$$

$$M = \frac{\partial^4 v}{\partial y^4}(x,\beta_1) + \frac{\partial^4 v}{\partial y^4}(x,\beta_2),$$

then for every N

$$\|R\|_\infty \leqslant \frac{h^{2-2s}}{6}C_N,$$

such that $C_N = (N \times N - 1)W$, where $W = max(max|\partial^4 u|, max|\partial^4 v|)$.
Consequently

$$\|R\|_\infty \leqslant \frac{h^{2-2s}}{6} min(C_N). \tag{5.4}$$

☐

Stability

A numerical scheme is said to be stable if

$$\|U\|_\infty \leqslant K\|b\|_\infty,$$

where K is positive constant.

The stability result is given in the following statement, which is proved in several steps.

Theorem 5.3.2. *The (5.3) scheme is stable for the* $\|.\|_\infty$ *norm. In particular:*

$$\|B^{-1}\|_\infty \leqslant \frac{1}{8}. \tag{5.5}$$

Proof. Step 1: In the first step, we prove that $B^{-1} \geqslant 0$.
Step 2: We give an exact solution.
Step 3: We calculate the critical points.
Step 4: Finally, we conclude that $\|B^{-1}\|_\infty \leqslant \frac{1}{8}$. ☐

Convergence

A scheme is said to be convergent if $\|e\| \to 0$ as $h \to 0$ where e is the error between the exact solution and the approximate solution. Combining the ideas introduced above, we arrive at the conclusion that if we have the consistency and the stability we get the convergence of the scheme.

Theorem 5.3.3. *The (5.3) scheme is convergent, moreover*

$$\|e\|_\infty \leqslant \frac{h^{2-2s}}{48} min(C_N).$$

Proof. This is easily proved by using (5.4) and (5.5). ☐

These facts will be confirmed by the numerical simulations that we are going to present in Section 5.4 below, by observing the behavior of the approximate solution, the exact solution, and the norm of the error e in the infinity. In this way, as predicted by Theorem 5.3.3, we obtain a numerical evidence of the properties of null and the convergence of system (5.1), in accordance with the theoretical results in Section 5.3.

5.4 Numerical results

In this Section, we present the numerical simulations corresponding to the scheme previously described, and we provide a complete discussion of the results obtained.
First of all, in order to numerically test the accuracy of our method, we use the following system

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u(x,y) = f(x,y,u(x,y),v(x,y)) & \text{in} \quad]0,1[\times]0,1[, \\ (-\Delta)^{\frac{1}{2}}v(x,y) = g(x,y,u(x,y),v(x,y)) & \text{in} \quad]0,1[\times]0,1[, \\ u = v = 0 & \text{on} \quad \mathbb{R}^2\backslash]0,1[\times]0,1[, \end{cases} \tag{5.6}$$

where

$$\begin{aligned} f(x,y,u(x,y),v(x,y)) &= -4x^3(y-1)^4 - 4x^4(y-1)^3, \\ g(x,y,u(x,y),v(x,y)) &= -4y^3(x-1)^4 - 4y^4(x-1)^3, \end{aligned}$$

we can write f and g in terms of u and v as follows

$$\begin{aligned} f(x,y,u(x,y),v(x,y)) &= -4x(y-1)^2u^{\frac{1}{2}} - 4x^2(y-1)v^{\frac{1}{2}} - 4x^2y^2(y-1) \\ &- 8x^2(y-1)y^2(x-1) - 4x^4(y-1) + 8x^4y(y-1), \\ g(x,y,u(x,y),v(x,y)) &= -4y^2(x-1)v^{\frac{1}{2}} - 4y(x-1)^2u^{\frac{1}{2}} - 4yx^2(x-1)^2 \\ &- 8yx^2(x-1)^2(y-1) - 4y^3(x-1)^2 + 8y^3x(x-1)^2. \end{aligned}$$

In this particular case, the solution can be computed exactly and it reads as follows,

$$\begin{aligned} u(x,y) &= x^4(y-1)^4, \\ v(x,y) &= y^4(x-1)^4. \end{aligned}$$

According to the matrix transformation method we proceeded as follows.

- The domain was discretized using a uniform square-grid with the grid size h.
- The standard five-point approximation of the operator $(-\Delta)^s$ was applied to obtain the matrix B.
- Gauss Seidel method was applied.
- To get the approximate solution, we use the solutions of the quadratic equation.

The next results were shown after a lot of mathematical calculations.

In Fig. 5.1, we show a comparison between the exact solution and the computed numerical approximation. Here we consider $N = 5$ then $h = 0.1667$ and $s = \frac{1}{2}$.

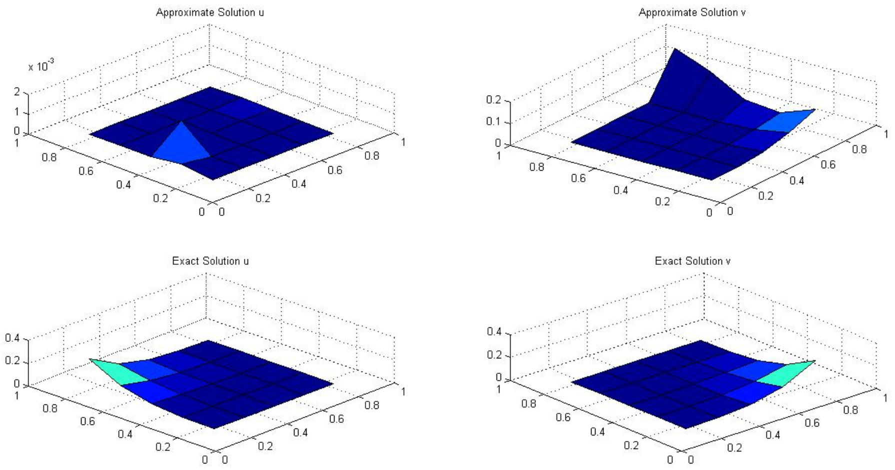

Figure 5.1: The surface graph of the exact solution and the fifth-order approximate solution.

One can notice that the computed solution is to a certain extent not different from the exact solution, Fig. 5.2 where $N = 60$ and $h = 0.0164$ shown that very well (there is a different but we can't detect it by the aye we need to zoom the figure to see it). However, one should be careful with such result and a more precise analysis of the error should be carried.

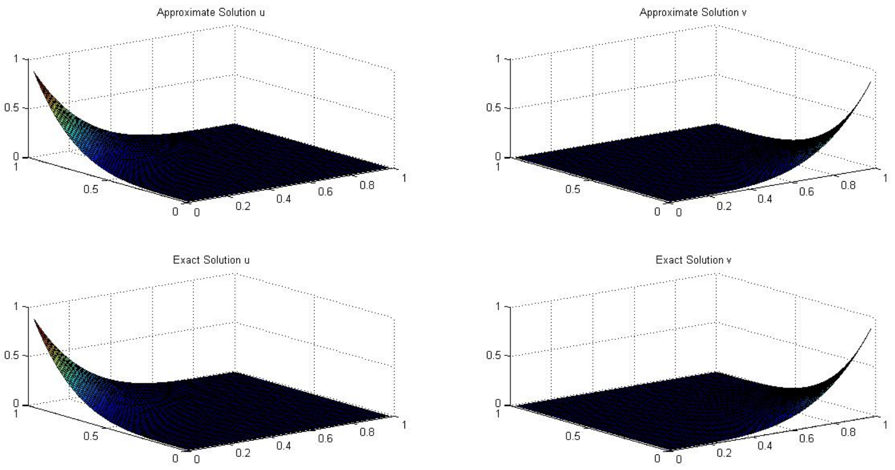

Figure 5.2: The surface graph of the exact solution and the sixty-order approximate solution.

In the same spirit as in [56], the computation of the error can be readily done by using the Theorem 5.3.3, namely

$$\|e\|_\infty \leqslant \frac{h^{2-2s}}{48} min(C_N).$$

The computational results are shown for our model in Table 5.1. While in the two-dimensional case, the predicted convergence rate is reached shortly, in the threedimensional computations a remarkable oscillation can be detected see [56]. Since the computations are lengthy, we have tested our system only with a single parameter $s = \frac{1}{2}$.

		u				v	
N	h	max-error	r	N	h	max-error	r
35	0.0278	1.3650×10^{-11}	0.0417	35	0.0278	1.1657×10^{-14}	0.0417
40	0.0244	3.0754×10^{-12}	0.0366	40	0.0244	3.5565×10^{-15}	0.0366
45	0.0217	1.0939×10^{-12}	0.0326	45	0.0217	1.2084×10^{-15}	0.0326
60	0.0164	5.3331×10^{-13}	0.0246	60	0.0164	6.9390×10^{-17}	0.0246
65	0.0152	3.6230×10^{-13}	0.0227	65	0.0152	2.7303×10^{-17}	0.0227

Table 5.1: Computational error and estimated convergence rate r with respect to the infty-norm for the matrix transformation method applied to the finite difference approximation of (5.6). N: number of steps.

In Fig. 5.3, we present the computational errors evaluated for different values of N and h.

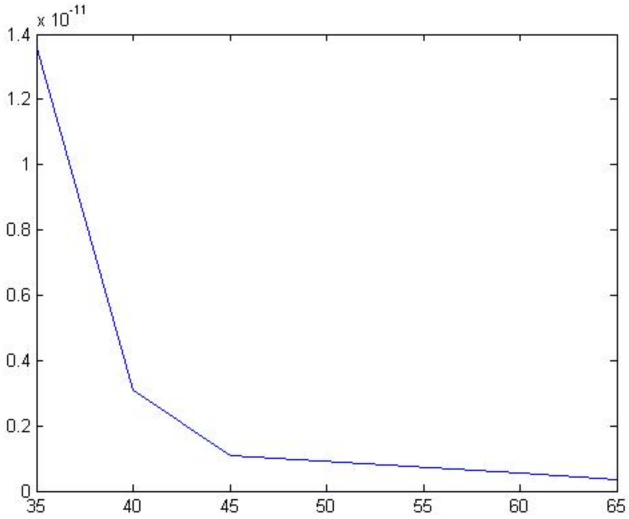

Figure 5.3: Plot of the absolute error.

The rates of convergence shown are of order (in h) of $(2 - 2s)$.

5.5 Summary

We have verified the convergence of the matrix transformation method applied to the fractional elliptic system. The corresponding computation algorithm is difficult: we can't avoid the computation of a full matrix containing involved finite differences. Combined with the Gauss Seidel method, the corresponding method exhibits optimal convergence rate for $s \in (0,1)$ in the infty-norm.

The finite difference method is applied successfully for solving the nonlinear fractional elliptic systems. The fundamental objective of this article is to introduce an algorithmic form and implement a new analytical repeated algorithm derived from the finite difference method to find numerical solutions for the fractional elliptic system. Graphical and numerical consequences are introduced to illustrate the solutions. Thus, it is concluded that we can translate numericaly and find a numerical solutions for a wide class of linear and nonlinear fractional differential systems applied in physics, biologics...ect. From the results, it is clear that the numerical resolution of fractional system yields very accurate and convergent approximate solutions.

Conclusion

In conclusion, our thesis allowed us to answer the question of having a solution to strongly nonlinear elliptic systems in the case of partial derivatives and fractional derivatives in addition to the numerical study of a fractional system. The systems have been treated by the technique of topological degree and the fixed point theory, and we used the finite differences method for the numerical study, we need to mention that we got a big problem with the calculus of the matrices.

The study was crowned with the publication of one article, along with three other articles accepting for reviewing in different journals.

Lastly, these studies can extend to more general boundary value systems involving fractional derivatives such as systems of convection-diffusion-reaction and find the appropriate numerical methods. We can also try to find an application of these models in image processing.

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