وزارة التعليم العالي والبحث العلمي

Faculté des Sciences

كلــــــــية العلــــــوم

N^0 : U.S/F.S/D.M/……/2024.

Faculté des Sciences
Département de Mathématiques

Mémoire

Présenté en vue de l'obtention du diplôme de
Master en Mathématiques

ASYMPTOTIC BEHAVIOR OF SOLUTION FOR A PARTIAL DIFFERENTIAL EQUATION

Option :AFA

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Année : 2023/2024

DEDICACES

I dedicate this modest work :

has

everyone in my family (my dear parent ,my dear brothers newfel et charef eddine and my sisters :Rima et Ritadj and sister'son loukman).

has

my friends (Roukia,Linda,wiam et chaima). And to all those who encouraged me

has

those who were by my side in difficult times

AIACHI MARWA

Remerciments

First of all, I would like to thank Allah for giving me strength, courage

and above all, knowledge.

I would like to express my deep gratitude to Dr. Amar Ouaoua, my supervisor, for his patience, motivation and enthusiastic encouragement. His guidance, advice and constant support throughout my research. I am very grateful to him.

My deepest thanks go to the proposed jury members for agreeing to be part of my jury. I thank them for accepting to examine and evaluate my work.

My sincere gratitude to my family for their moral support and care, and a special thanks to the professors, students, and fellows at the mathematics department of the University of Skikda on August 20, 1955, for their assistance with the preparation of my memory, as well as everyone who has supported me during my academic career.

ملخص

في هذه المذكرة نقوم بدراسة مسألتين الأولى تتعلق بشبه جملة مكافئة خطية مع وجود حد لزج مرن والمسألة الثانية تتعلق بمعادلة الأمواج
أولا برهنا في المسألة الأولى أن وجود الحل العالمي في حيز محدود ومع شروط ديريكلي المتجانسة كما أثبتنا أن هذا الحل يضمحل أسيا أي لما t يؤول إلى الانهاية الحل يؤول إلى الصفر

ثانيا برهنا أن الحل لمعادلة الأمواج كذلك في وجود شروط ديريكلي المتجانسة ينفجر في زمن محدود حيث أن الدراسة اعتمدت على في الأساس على فضاء نهاري.
الكلمات المفتاحية:الوجود العالمي،إضمحلال الحل ،شبه جملة مكافئة ،حد ضعيف مرن ـ لزج .

Abstract

In this memory, we study two problems: the first concerns a quasi-linear parabolic system with a weak visco-elastic term, and the second concerns the wave equation.

In the first problem, we proved the existence of a global solution in a bounded domain with homogeneous Dirichlet conditions. We also proved that this solution decays exponentially, meaning that as time approaches to infinity, the solution approaches to zero.

Second, we proved that the solution to the wave equation, also under homogeneous Dirichlet conditions, blows up in finite time. The study is based on Nehari space.

Key words : Global existence, Decay of solution, Quasi-linear system, Weak visco-elastic term

Résumé

Dans cette mémoire, nous étudions deux problèmes : le premier concerne un système parabolique quasi-linéaire avec un terme viscoélastique faible, et le second concerne une équation des ondes.

Dans le premier problème, nous avons prouvé l'existence d'une solution globale dans un domaine borné avec des conditions de Dirichlet homogènes. Nous avons également prouvé que cette solution décroît de manière exponentielle, c'est-à-dire que lorsque le temps tend vers l'infini, la solution tend vers zéro.

Ensuite, nous avons prouvé que la solution de l'équation des ondes, également sous des conditions de Dirichlet homogènes, explose en un temps fini. L'étude est principalement basée sur l'espace de Nehari.

Mots clés : Existence globale, Décroissance de la solution, Système quasi-linéaire, Terme viscoélastique faible

CONTENTS

0.1 Introduction

On the mathematical analysis of equations which model the motions of materials with memory, we refer to[[3], [16], [19]]and references therein. Yin in[[19]] considered a general equation of the form

$$u_t = \operatorname{div} A\left(x, t, u, u_x\right) + a\left(x, t, u, u_t\right) + d \int_0^t \operatorname{div} B\left(x, t, \tau, u, u_x\right) \tau.$$

Under some conditions on A, B, and a, similarly to the case of parabolic, the existence of a unique weak solution is established Regarding the heat equations without the memory term, study of global existence and finite time blow-up of solutions for the following initial boundary value problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + f\left(u\right) = 0, (x,t) \in \Omega \times (0, \infty) \\ u\left(x,t\right) = 0, (x,t) \in \partial\Omega \times (0, \infty) \\ u\left(x,0\right) = u_0\left(x\right), x \in \Omega \end{cases}$$

has attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on p, the degree of nonlinearity in f, the dimension n and the size of the initial data. See for example, the works of Levine et al. [[10], [11]], Kalantarov and Ladyzhenskaya [[8]] and Messaoudi [[14]]. Also,concerning the asymptotic behavior of the solution, see [[1], [17]]. Pucci and Serrin [[17]] studied the following equation with homogenous Dirichlet boundary condition

$$A\left(t\right) |u_t|^{m-2} u_t - \Delta u + f\left(x, u\right) = 0 \tag{1}$$

They proved that the strong solution tends to zero when t it leads to infinity under the condition$(f\left(x,u\right), u) > 0$ but did not give the decay rate. Berrimi and Messaoudi[[1]] proved that if bounded square matrix $A(t)$ in equation (1) satisfying (3.2), then the solution with small energy decays exponentially for $m = 2$ and polynomially for $m > 2$.

In the presence of the memory term in the heat equations, Messaoudi and Tellab [[15]] considered the following quasi-linear parabolic system

$$A\left(t\right) |u_t|^{m-2} u_t - \Delta u + \int_0^t g\left(t-s\right) \Delta u\left(s\right) = 0, \tag{2}$$

with Dirichlet boundary condition and proved a general decay result which depends on the behavior of the function g. Ferhat and Hakem [[4]] considered the quasi-linear parabolic system

$$A(t)|u_t|^{m-2}u_t - Lu + \int_0^t g(t-s)\,Lu(s) = 0,$$

where

$$Lu = -\operatorname{div}(M\nabla u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right)$$

They improved the result obtained by Messaoudi and Tellab [[15]] and proved a general decay result. Later,Youkana et al.[[20]] studied the equation (2) where the relaxation function satisfies $g\prime(t) \leq -\xi(t)\,g^p(t)$,for all $t \geq 0, 1 \leq p < \frac{3}{2}$ and established a general and optimal decay result. Recently, Youkana and Messaoudi [[21]] considered the equation (2) under a general assumption on the relaxation functions satisfying $g\prime(t) \leq -\xi(t)\,H(g(t))$, where H is an increasing convex function and is a nonincreasing function. In the

case of viscoelastic heat equations with source term, Liu and Chen [[12]] considered the following quasilinear parabolic system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,ds = |u|^{p-2}u,$$

and they proved a general decay of the energy function for the global solution and a blow-up result for the solution with both positive and negative initial energy under suitable conditions on g and p. In another study, Di et al.[[7]] investigated a nonlinear pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\,\Delta u(\tau)\,d\tau = |u|^{p-2}u.$$

They obtained finite-time blow-up results for the solutions with initial data at non-positive energy level as well as arbitrary positive energy level and give some upper bounds for the blow-up time T^*depending on the sign and size of initial energy $E(0)$. For more information in this regards we refer to [[2], [6], [9], [18], [21]].The following equation, with initial and Dirichlet-boundary conditions,

$$u_{tt} - \Delta u + au_t|u_t|^{m-2} = bu|u|^{p-2}, x \in \Omega, t > 0,$$

where $a, b > 0$,$p, m > 2$, and Ω is a a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$ For $b = 0$, has been studied by many researchers.it is well–known that the damping term $au_t|u_t|^{m-2}$assures global existence for arbitrary initial data (see [24], [26]). If $a = 0$ then the source term bu $|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [22],[25],[27],[28]).The interaction between the damping and the source terms was first considered by Levine [[27],[28]]in the linear damping case($m = 2$). He showed that solutions with negative initial energy blow up in finite time. Recently Georgiev and Todorova [[23]]extended Levine's result to the nonlinear case($m > 2$). In their work, the authors

introduced a different method and determined suitable relations between m and p,for which there is global existence or alternatively finite time blow up. Precisely: they showed that solutions with negative energy continue to exist globally "in time" if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative.This result has been lately generalized to an abstract setting and to unbounded domains by Levine and Serrin[[29]] and Levine, Park, and Serrin [[30]]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several noncontinuation theorems. This generalization allowed them also to apply their result to quasilinear situations, of which problem (4.1) is a particular case.Vitillaro [[31]] combined the arguments in [[23]] and [[29]] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy.

CHAPTER 1

Classification of partial differential equation and canonical forms

1.1 second-order partial differential equation

The most general case of second-order linear partial differential equation (PDE) intwo independent variables is given by

$$A\frac{\partial^{2u}}{\partial^2 x}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G \tag{1.1}$$

Where the coefficients A, B and C are functions of x and y and do not vanish simultaneously, because in that case the second-order PDE degenerates to one of first order.

Further the coefficients D, E and F are also assumed to be functions of x and y. We shall assume that the function u(x, y) and the coefficients are twice continuously differentialble in some domain Ω. The classification of second-order PDE depends on the form of the leading part of the equation consisting of the second order terms. So, for simplicity of notation, we combine the lower ordre terms and rewrite the above aquation in the folowing form

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=\phi\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) \tag{1.2}$$

or using the short-hand notations for partial derivatives,

$$A(x,y)u_{x\,x}+B(x,y)u_{xy}+C(x,y)u_{yy}=\phi(x,y,u,u_x,u_y) \tag{1.3}$$

As we shall see, there are fundanmentally theree types of $PDE_{s-}hyperbolic$,and elliptic $PDE_{S.}$From the physical point of view, these PDE_S respectively represents the wave propagation, the time-dipendent diffusion processes and the steady state or equilibrium processes. thus, hyperbolic equations model the transport of some physical quantity, such as fluids or waves. Parabolic problems descrde evolutionary phenomena that lead to a steady state described by an elleptic equation.and elleptic equations are associated to a special state of a system, in principle corresponding to the minimum of the invergy. Mathematically, these classification of second-order PDE_S is based upon the possibility of redesing equation(1.3) by coordinate transformation to canonical or standard from at a point. It may be noted that, for the purposes of classification, it is not necessary to restrict considuration to linear equation. It is applicable to quasilinear second-order PDE as well. Aquasilinear second-ordre PDE is linear in the second derivatives only . The type of second-ordre PDE (1.2) at a point $(x_{0,}y_0)$ depents on the sign of the discriminant defined as

$$\Delta(x_{0,}y_0) = \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B\left(x_{0,}y_0\right)^2 \_4A\left(x_0, y_0\right)C\left(x_{0,}y_0\right) \tag{1.4}$$

The classification of second-order linear PDE_s is given by the following .If $\Delta\left(x_0y_0\right) \succ 0$,the equation is hyperbolic,$\Delta\left(x_0, y_0\right) = 0$ the equation is parabolic, and $\Delta\left(x_0, y_0\right) \prec 0$

the equation is elliptic . In order to illustrate the significance of the discriminant Δ and thus the classification of the $PDE.$(1.3) We try to reduce the given eqution (1.3) to a

canonical form .To do this ,we transform the independent variables x and y to the new independent variables ζ and η through the change of variables

$$\zeta = \zeta\left(x, y\right), \qquad \eta = \eta\left(x, y\right) \tag{1.5}$$

Where both ζ and η are twice continuously differntiable and that the Jacobian

$$J = \frac{\partial\left(\zeta, \eta\right)}{\partial\left(x, y\right)} = \begin{vmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \tag{1.6}$$

in the region under consideration . The nonvanishing of the Jacobian of the transformation ensure that a one to one transformation exists betwrrn the new and old variables . This simply means that the new independent variables can serve as new coordinate variables without any apply the chain rule to compute the terms of the equation (1.3) in terms of ζ and η as follows:

$$\begin{aligned} u_x &= \omega_\zeta \zeta_x + \omega_\eta \eta_x \\ u_y &= \omega_\zeta \zeta_y + \omega_\eta \eta_y \\ u_{xx} &= \omega_{\zeta\zeta}\zeta_x^2 + 2\omega_{\zeta\eta}\zeta_x\eta_x + \omega_{\eta\eta}\eta_x^2 + \omega_\zeta\zeta_{xx} + \omega_\eta\eta_{xx} \\ u_{yy} &= \omega_{\zeta\zeta}\zeta_y^2 + 2\omega_{\zeta\eta}\zeta_y\eta_y + \omega_{yy}\eta_y^2 + \omega_\zeta\zeta_{yy} + \omega\eta_{yy} \\ u_{xy} &= \omega_{\zeta\zeta}\zeta_x\zeta_y + \omega_{\zeta\eta}\left(\zeta_x\eta_y + \zeta_y\eta_x\right) + \omega_{\eta\eta}\eta_x\eta_y + \omega_\zeta\zeta_{xy} + \omega_\eta\eta_{xy} \end{aligned} \tag{1.7}$$

substituting these expressions into equation(1.3) we obtain the transformed PDE as

$$a\omega_{\zeta\zeta} + b\omega_{\zeta\eta} + c\omega_{\eta\eta} = \phi\left(\zeta, \eta, \omega, \omega_\zeta, \omega_\eta\right) \tag{1.8}$$

Where Φ becomes ϕ and the new coefficients of the higher order terms a , b and c are expressed via the original coefficients and the change of variables formulas as follows :

$$\begin{aligned} a &= A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2, \\ b &= 2A\zeta_x\eta_y + B\left(\zeta_x\eta_y + \zeta_y\eta_x\right) + 2C\zeta_y\eta_y, \\ c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned} \tag{1.9}$$

As this stage the form of the EDP (1.8) is no simpler than that of the original EDP(1.2) ,but this is to be expected because so far the choice of the new vaeiable ζ and has been

that equation (1.9) can be written in matrix form as

$$\begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} + \begin{vmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{vmatrix} J^2$$

where J is Jacobian of the change of variables given by (1.6).Expanding the determinant and multiplying by the factor, -4 to obtain

$$b^2 - 4ac = J^2\left(B^2 - 4AC\right) \Rightarrow \delta = J^2\Delta. \tag{1.10}$$

Where $\delta = b^2 - 4ac$ is the discriminant of the equation(1.8) .This shows that the discriminant of(1.2) has the same sign as the discriminant of the transformed equation (1.8) and

therfore it is clear that any real nonsingular $(J \neq 0)$ transformation does not change the type of PDE . Note that the discriminant involoves only the coefficients of second-order derivatives of the corresponding PDE.

1.1. second-order partial differential equation

1.1.1 Canonical forms

Let use now try to contruct transformation, which will make one, or possibly two of the coefficients of the leading second-order terms of equation(1.8) vanish, thus reducing the equation to asimpler form called canonical form .For convenience,we reproduce below the original PDE

$$A(x;y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = \Phi(x,y,u,u_x,u_y) \tag{1.11}$$

and the corresponding transformrd PDE

$$a(\zeta,\eta)\,\omega_{\zeta\zeta} + b(\zeta,\eta)\,\omega_{\eta\eta} + c(\zeta,\eta)\,\omega_{\eta\eta} = \phi(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta)\,. \tag{1.12}$$

We again mention here that for the (1.2) or (1.8) to remain a second-order PDE , the coefficients A,B and C (or a , b and c) do not vanish simultaneously . By definition, a PDE is hyperbolic if the discriminant $\Delta = b^2 - 4ac \succ 0$. Since the sign of dicriminant is invariant under the change of coordinater (see equatio (1.10)) ,it follows that for a hyperboulic PDE,we should have $b^2 - 4ac \succ 0$. The simplest case of satisfying this condition is $a = c = 0$. So, if we try to chose the new variables ζ and η such that the cofficients a and c vanish , we get the following canonical form of hyperbolic euqation :

$$\omega_{\zeta\eta} = \psi(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta) \tag{1.13}$$

Where $\psi = \frac{\phi}{b}$, this form is called the first canonical form of the hyperbolic eqution ,we also have another simple case for which $b^2 - 4ac \succ 0$ condition is satified . This the case when $b = 0$ and $c = -a$. Is this case (1.10) reduces to

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi(\alpha,\beta,\omega,\omega_\alpha,\omega_\beta)\,, \tag{1.14}$$

which is the second canonical form of the hyperbolicequatoin . By definition , a PDE is parabolic if the discriminant $\Delta = b^2 - 4ac = 0$. It follows that for a parabolic PDE, we should have $b^2 - 4ac = 0$ .Thesimplest case of satisfying this condition is a (or c) = 0. In this case another necessary requirement $b = 0$ will follow automatically (sinec $b^2 - 4ac$) . So if we try to chose the new variables ζ and η such that the coefficients a and b vanish, we get the following canonical form of parabolic eqution :

$$\omega_{\eta\eta} = \psi(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta)\,, \tag{1.15}$$

where $\psi = \frac{\phi}{c}$.By definition , a PDE is elliptic if the discriminant $\Delta = b^2 - 4ac \prec 0.It$ follows that for a elliptic PDE, we should have $b^2 - 4ac \prec 0$.The simplesr case of satisfying this condition is $b = 0$ and $c = a$. Soif we tey to chose the new variables ζ and η such that b vanishes and $c = a$, we get the following canonical form of elliptic eqution :

$$\omega_{\zeta\zeta} + \omega_{\eta\eta} = \psi\left(\zeta, \eta, \omega, \omega_{\zeta}, \omega_{\eta}\right), \tag{1.16}$$

where $\psi = \frac{\phi}{a}$ In summary eqution(1.8) can be reduced to a canonical form if the coordinate fransformation $\zeta = \zeta(x, y)$ and $\eta = \eta(x, y)$ can be selected such that:

.$a = c = 0$ corresponds to the first canonical form of hyperbolic PDE given by (1.13) .$b = 0, c = -a$ corresponds to the second canonical foem of hyperbolic PDE given by (1.14).$a = b = 0$ corresponds to the canonical form of parabolic PDE given by (1.15) .$b = 0, c = a$ corresponds to the canonical form of elliptic PDE given by (1.16)

1.1.2 Hyperbolic equtions

For a hyperbolic PDE the discriminant $(\Delta = b^2 - 4ac) \succ 0$.In this case ,we have seen that to reduce this PDE to canonical form we need to choose the new variablrs ζ and η such that the coefficients a and c vanish in(1.8).Thus form (1.9) we have

$$a = A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0, \tag{1.17}$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0. \tag{1.18}$$

Divising eqution(1.7) and (1.8) throughout by ζ_y^2 and η_y^2 respectively to obtain

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0, \tag{1.19}$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0. \tag{1.20}$$

Equation (1.10) is a quadratic euqation for $\left(\frac{\zeta_x}{\zeta_y}\right)$ whose roots are given by

$$\mu_1(x, y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

$$\mu_2(x, y) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

The roots of the equation (1.20) can also be found in an identical manner, so as only two distinct roots are possible between the two equation (1.19) and (1.20). Here we may consider μ_1 as the root of (1.19) and μ_2 as that of (1.20) That is :

$$\mu_1(x, y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \tag{1.21}$$

$$\mu_2(x, y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{1.22}$$

The above equations lead to the following two first-order diferential aquation

$$\zeta_x - \mu_1(x, y)\,\zeta_Y = 0, \tag{1.23}$$

$$\eta_x - \mu_2(x, y)\,\eta_y = 0. \tag{1.24}$$

These are the equations that define the new coordinate variables ζ and η that are neces-sary to make $a = c = 0$ in (1.8) As the total derivative of ζ along the coordinate $\zeta(x, y) =$ constant , $d\zeta = 0$. It follows that

$$d\zeta = \zeta_x dx = \zeta_y dy = 0,$$

and hence the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}$$

We also have a similar result along coordinate line $\eta(x, y) =$ constant ,i.e,

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$$

Using these results equation(1.19) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{1.25}$$

This is called the characteristic polynomial of the PDE (1.2) and its roots are given by

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \lambda_1(x, y), \tag{1.26}$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \lambda_2(x, y). \tag{1.27}$$

The required variables ζ and η are determined by the respective solutions of the two ordinary differential equations (1.26) and (1.27) , known as the characteristic equation of the PDE.(1.2)They are ordinary differential equation for families of curves in the xy-plane along wich $\zeta =$ constant and $\eta =$constant.Clearly these families of curves depend on the coefficients A,B and C in the original PDE (1.2) Integation of equation(1.26) leads to the family of curvilinear coordinates $\zeta(x, y) = c_1$ while the integration of (1.27) gives another family of curvilinear coordinates $\eta(x, y) = c_2$, where c_1 and c_2 are arbitrary

constants of integration. These two families of curvilinear coordinates $\zeta(x, y) = c_1$ and

$\eta(x,y)=c_2$ are called characteristic curves of the hyperbolic equation (1.3) or simply the characteristics of the equation. Hence second-order hyperbolic eqution have twe families of characteristic curves. The fact that $\Delta \succ 0$

means that the characteristic are real curves in xy-plane. If the coefficients A,B and C are constants it is easy to integrate euations (1.26) and (1.27) to obtain the expressions for change of variables formulas for reducing a hyperbolic PDE to the canonical from .

Thus integration of (1.26) produces

$$y=\frac{B+\sqrt{B^2-4AC}}{2A}x+c_1 \qquad \textit{and} \qquad y=\frac{B-\sqrt{B^2-4AC}}{2A}x+c_2 \tag{1.28}$$

Or

$$y=\frac{B+\sqrt{B^2-4AC}}{2A}x=c_1 \qquad \textit{and} \qquad y=\frac{B-\sqrt{B^2-4AC}}{2A}x=c_2 \tag{1.29}$$

Thus when the coefficients A,B and C are two constants the two families of characteristic curves associated with PDE reduces to two distinct families of parallel straight lines .

Since the families of curves ζ =constant and η =constant are the characteristic curves ,the change of variables are given by the following equations :

$$\zeta = y-\frac{B-\sqrt{B^2-4AC}}{2A}x=y-\lambda_1 x, \tag{1.30}$$

$$\eta = y-\frac{B+\sqrt{B^2-4AC}}{2A}x=y-\lambda_2 x. \tag{1.31}$$

The first canonical form of the hyperbolic is :

$$\omega_{\zeta\eta}=\psi\left(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta\right), \tag{1.32}$$

where $\psi=\frac{\phi}{b}$ and b is calculated from (1.9)

$$\begin{aligned} b &= 2A\zeta_x\eta_x+B\left(\zeta_x\eta_x+\zeta_y\eta_y\right)+2C\zeta_y\eta_y \\ &= 2A\left(\frac{B^2-(B^2-4AC)}{4A^2}\right)+B\left(-\frac{B}{2A}-\frac{B}{2A}\right)+2C \\ &= 4C-\frac{B^2}{A}=\frac{\Delta}{A} \end{aligned} \tag{1.33}$$

Each of the familiies $\zeta(x,y)=$ constant and $\eta(x,y)=$ constant forms an envelop of the domain of the xy-plane in which th PDE is hyperbolic .The transformation $\zeta=\zeta(x,y)$

and $\eta = \eta(x,y)$ can be regarded as a mapping from the xy-plane to the $\zeta\eta - plane$. and the curves along which ζ and η are constant in the xy-plane brcome coordinates lines in the $\zeta\eta - plane$. Sinec these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE . In other worde, charactreristic curves of a hyperbolic PDE are those curves for the $PDE.$ must be referred as coordinate curves in order that it take on canonical form .We now determine the Jacobian of transformation definded by(1.30) and (1.31) . we have

$$J = \begin{vmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 1 \end{vmatrix} = \lambda_2 - \lambda_1.$$

We know that $\lambda_1 = \lambda_2$ only if $B^2 - 4AC = 0$. However, for an hyperbolic PDE , $B^2 - 4AC \neq 0$. Hence Jacobian is nonsingular for the given transformation. A consequence of

$\lambda_1 \neq \lambda_2$ is that at no point can the particular curves from each family share a common trangent line . It is easy to show that the hyperbolic PDE has a second canonical form. The following linear change of variables

$$\alpha = \zeta + \eta \qquad\qquad \beta = \zeta - \eta$$

converts (1.32) in to

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi(\alpha, \beta, \omega, \omega_\alpha, \omega_\beta) \tag{1.34}$$

which is the seconde canonical form of the hyperbolic equtions

1.1.3 Parabolic equtions

For a parabolic PDE the discriminant $\Delta = B^2 - 4AC = 0$. In this case , we have seen that to reduce this PDE to canonical form we need to choose the new variables ζ and η such that the coefficients a and b vanish in (1.8) . Thus, from(1.9) we have

$$a = A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0$$

Dividing the above eqution throughhout by ζ_y^2 to obtain

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \tag{1.35}$$

As the total derivative of ζ along the coordinate line $\zeta(x,y)$ =constant , $d\zeta = 0$.It follows that

$$d\zeta = \zeta_x dx + \zeta_y dy = 0,$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}.$$

Using this result , equation (1.35) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.36}$$

This is called the characteristic polynomial of the PDE (1.2). Since $B^2 - 4AC = 0$ in this case the characteristi (1.35) has only root , given by

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda\,(x, y) \tag{1.37}$$

Hence we see that for a parabolic PDE there is only one family of real characteristic curves. The required varibles ζ is determined by the ordinary differential equation (1.37) known as the characteristic equations of thr PDF.(1.2) this is an ordinary differential equation for families of curves in the xy-plane along which $\zeta = const.$ to determine the second transformation variablr η,we set $b = 0$ in (1.9) so that

$$\begin{aligned}
2A\xi_x\eta_y + b\xi_x\eta_y + \xi_y\eta_x + 2C\xi_y\eta_y &= 0\\
2A\frac{\xi_x}{\zeta_y}\eta_x + B\left(\frac{\xi_x}{\zeta_y}\eta_y + \eta_x\right) + 2C\eta_y &= 0\\
2A\left(-\frac{B}{2A}\right)\eta_x + B\left[\left(-\frac{B}{2A}\right)\eta_y + \eta_x\right] + 2C\eta_y &= 0\\
-B\eta_x - \frac{B^2}{2A}\eta_y + B\eta_x + 2C\eta_y &= 0\\
\left(B^2 - 4AC\right)\eta_y &= 0
\end{aligned}$$

since $B^2 - 4AC = 0$ for a parabolic PDF, η_y could be an arbitrary function of (x, y) and consequently the transformation variable η can be chosen arbitrary, as long as the change of coordinates formulas define a non-degenerate transfoemation. lf the corfficients A,B and C are constants,it is easy to integrate equation (1.37)to obtain the expressions for change of variables formulas for reducing a parabolic PDE to the canonical foem. thus integration of (1.37) produces

$$y = \frac{B}{2A}x + C1 \tag{1.38}$$

or

$$y - \frac{B}{2A}x = C1 \tag{1.39}$$

sincr the families of curves ζ = const are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{B}{2A}x \tag{1.40}$$

$$\eta = x \tag{1.41}$$

where we have set $\eta = x$.the tacobian of this transformation is

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -B/2A & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

naw, we have from(1.9)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\ &= 2A\left(-\frac{B}{2A}\right) + B + 0 = 0 \end{aligned}$$

in these new coordinate variables given by (1.39) and (1.40),equation(1.8) reduces to following canonical form:

$$w_{\eta\eta} = \Psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{1.42}$$

where$\psi = \frac{\varphi}{\varsigma}$. As the choice of η is arbitrary the form taken by ψ will depend on the choice of η . we have from(1.9)

$$c = A\eta_x^2 + B\eta_\chi\eta_\gamma + C\eta_\gamma^2 = A. \tag{1.43}$$

Equation (1.8) may also assume the form

$$\omega_{\xi\xi} = \psi(\xi, \eta, \omega_\xi, \omega_\eta), \tag{1.44}$$

if we choose $c = 0$ instead of $a = 0$

1.1.4 Elliptic equations

For an elliptic PDE the discriminant $\Delta = B^2 - 4AC \prec 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ξ and η

to produce $b = 0$ and $a = 0$, or $b = 0$ and $a - c = 0$. then, from (1.9) we abtain the following equation :

$$A(\xi_\chi^2 - \eta_\chi^2) + B(\xi_\chi \xi_\gamma - \eta_\chi \eta_\gamma) + C(\xi_\gamma^2 6 - \eta_\gamma^2) = 0 \tag{1.45}$$

$$2A\xi_x \eta_x + B\left(\xi_x \eta_y + \xi_y \eta_x\right) + 2C\xi_y \eta_y = 0. \tag{1.46}$$

For hyperbolic and parabolic PDE_S, ξ and η are satisfied by equations that are not coupled each other $(see(1.17)$ and $(1.35))$ .However,equation (1.44) are coupled since both unknowns ξ and η appear in both equations. In an attempt to separate them,we add the first of these equation to complex number i times the second to give

$$A\left(\xi_x + i\eta_x^2\right) + B\left(\xi_x + i\eta_y\right)\left(\xi_x + i\eta_y\right) + C\left(\xi_y + i\eta_y\right)^2 = 0.$$

Dividing the above equation throughout by $\left(\xi_y + i\eta_y\right)^2$to obtain

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi^x + i\eta_y}{\xi^x + i\eta_y}\right) + C = 0 \tag{1.47}$$

This equation can be solved for two possible values of the ratio

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{-B \pm \sqrt{B - 4AC}}{2A} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{1.48}$$

Clearly, these twe roots are complex conjugates and are given by

$$\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}, \tag{1.49}$$

$$\frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A}, \tag{1.50}$$

where $\beta(x, y)$ is the complex conjugate of $\alpha(x, y)$ .They are given by

$$\alpha(x, y) = \xi(x, y) + i\eta(x, y), \tag{1.51}$$

$$\beta(x, y) = \xi(x, y) - i\eta(x, y). \tag{1.52}$$

We will now proceed in a purely formal fashion .As the total derivative of α along the coordinate line $\alpha(x, y)$ =constant , $d\alpha = 0$,it follows that

$$d\alpha = \alpha_x dx + \alpha_y dy = 0,$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = \frac{\alpha_x}{\alpha_y},$$

we also have a similar result along coordinate line $\beta\left(x,y\right)$ =constant,i.e,

$$\frac{dy}{dx}=-\frac{\beta_x}{\beta_y}.$$

From the forgoing discussion it follows that :

$$\frac{dy}{dx} = \lambda_1=\frac{B-i\sqrt{4AC-B^2}}{2A}, \tag{1.53}$$

$$\frac{dy}{dx} = \lambda_2=\frac{B+i\sqrt{4AC-B^2}}{2A}. \tag{1.54}$$

Equations (1.52) and (1.53) are called the characteristic equation of the PDE (1.3).Clearly, the solution of this differential eqution are necessarily complex-valued and as consequence there are no real characteristic axist for an elliptic EDP.The complex variables α and β are determined by the respective solution of the two ordinary differential equtions (1.52) and (1.53). Integration of equation (1.52) leads to the family of curvilinear coordinates $\alpha\left(x,y\right)=c_1$, where the integration of(1.53) gives another fa- mily of curvilinear coordinates $\beta\left(x,y\right)=c_1$ where $c_1$and c_2 are complex constants of integration . Since α and β are complex function the characteristic curves of the elliptic equation (1.3) are not real . Now the real and imaginary parts of α and β give the required trans formation variables ξ and η . Thus, we have

$$\xi=\frac{\alpha+\beta}{2}, \qquad \eta=\frac{\alpha-\beta}{2} \tag{1.55}$$

With the choice of coordinate variables (1.54),equation (1.8) reduces to following canonical form.

$$\omega_{\xi\xi}+\omega_{\eta\eta}=\psi\left(\xi,\eta,\omega,\omega_{\xi},\omega_{\eta}\right), \tag{1.56}$$

where $\psi=\frac{\phi}{\alpha}$

$NOTE$: It may be noted that the quasilinear second-order equation in two independent variables can also be classified in a similar way according to rule analogous to those developed above for semilinear equations. However , since A, B, and C are now functions of u_x, u_y and u its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.Definition and theorems of mathematics In this section , we state some results and Lemmas important for our problem .

Définition 1.1 The Faedo-Galerkin methode with de compctness argument yields a powerful method which allows us to deal with some nonlinear evoiution equation.

CHAPTER 2

Definitions and theorems importants

Définition 2.1 We denote by $L^P([a,b];X)$, the space of L^P functions from $[a,b]$ into X.It is a Banach space for the norm

$$\|u\|_{L^P([a,b];X)} = \left(\int_a^b \|u(t)\|_X^p \ dt \right)^{\frac{1}{p}}.$$

Définition 2.2 $L^\infty([a,b];X)$ is the space of measurable functions from $[a,b]$ into X being essentially bounded. It is a banach spacr of the norm

$$\|u\|_{L^\infty([a,b];X)} = \sup_{t\in[a,b]} \|u(t)\|_X .$$

let B be a banach space . Usually we will

Définition 2.3 encounter the following three different concepts of convergence.(i) Strong convergence .Let $u_n \in B, u \in B$ such that as $n \to +\infty$,

$$\|u_n - u\|_B \to 0$$

Then u_n is said to strongly converge to u. (ii) Weak convergence:Let $u_n, u \in B$ such that for any $f \in B'$ as $n \to +\infty$

$$f(u_n) \to f(u)$$

Then u_n is said to weakly converge to u (iii) Weakly star convergence:Let $u_n, u \in B$,and let B be the dual space of another banach space B^*, $i.e, B = (B^*)'$.

If for any $f \in B^*$, as $n \to +\infty$

$$u_n(f) \to u(f)$$

Then u_n is said to weakly star.

Remarque 2.1 It is well known that strong convergence implies weak convergence, and weak convergence implies weakly star convergence .When B is a reflixive banach

space, weak convergence is equivalent to weakly star convergence .Where B is a reflixive banach space,convergencr is equivalent to weakly slar convergence

In this section, we stste some results and lemmas important for our problem.

Lemme 2.1 *Let S be a number with $2 \leq S \leq M \prec +\infty$ then there is a constant C depending on Ω and S such that*

$$\|u\|_s \leq C \|\nabla u\|_m , \qquad u \in H_0^1(\Omega).$$

Lemme 2.2 *Any bounded set in reflexive banach space is weakly compact i.e any sequence in a bounded set has a weakly converging subsequence.*

Exercice 2.1 *Since for $1 \prec P \prec \infty, L^\infty(\Omega)$ is a reflixive banach space and $L^\infty(\Omega) = (L^1(\Omega))$,anY bounded set in $L^\infty(\Omega)$ with $1 \prec P \leq \infty$ is weakly star compact .In particular, any*

bounded set in $L^\infty(\Omega)$ is weakly star compact .

Lemme 2.3 *Let B_0, B, B_1 be three banach space .Suppose that B_0 is continuously imbedded into B, wich is also continuously imbedded into B_1, and imbedding from B_0*

into B is compat.Then for any $\eta \succ 0$, there is a positive constant C_η depending only on η such that for any $v \in B_0$,the following holds.

$$\|v\|_B \leq \eta \|v\|_{B_0} + C_\eta \|v\|_{B_1}.$$

Théorème 2.1 *(LOCAL EXISTENCE) Suppose that $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, then there exists $T \succ 0$ such that problem (1.1) _ (1.3) has unique solution u stisfiying:*

$$u \in L^\infty\left([0,T], H_1^0(\Omega)\right) \cap L^2\left([0,T], L^2(\Omega)\right) \; u' \in L^\infty\left([0,T], L^2(\Omega)\right).$$

Remarque 2.2 Since for $1 \leq p \prec \infty, L^P(\Omega)$ is a reflexive banach

space and $L^\infty(\Omega) = (L^1(\Omega))'$,any bounded set in $L^P(\Omega)$ with $1 \prec p \leq \infty$ is weakly star compact . In particular ,any bounded set in $L^\infty(\Omega)$ is weakly star compact.

Théorème 2.2 *Let B_0, B, B_1 be three banach space B_0, B_1 are reflexive .Suppose that $B_0$is continuously imbedded into B , which is also continuously imbedded into B_1,* *and imbedding from* B_0 *into* B *is compact .For any given* P_0, P_1 *with* $1 \prec p_0, p_1 \prec \infty$, *let*

$$W = \left\{ v \diagup v \in L^{P_0}\left([O,T], B_0\right), v_t \in L^{p_1}\left[0,T\right], B_1 \right\}.$$

Then the imbedding from W *into* $L^{P_0}\left([0,T], B\right)$ *is compact.*

Remarque 2.3 It can be seen from the proof that if the assumption of reflexivity of $B_0, B_1$is replaced by the assumption that $B_0, B_1$are the dual space of reflexive banach spaces B_0^*, B_1^*,respectively,then the conclusion of Theorem [2.1] still holds.

Lemme 2.4 *Suppose that* Ω *is a bounded domain in* R^n. *Let* $u_n(x), u(x)$ *be real function in* $L^P(\Omega), (1 \leq p \prec \infty)$ *such that* u_n *strongly converges to* u *in* $L^p(\Omega)$. *Then if* $1 \leq p \prec \infty$, *has a subsequence almost everywhere converging to* u *; if* $p = \infty$, *then* u_n *it self almost everywhere converges to* u.

Lemme 2.5 *Suppose that* Ω *is a bounded domain in* R^n. *Let* $u_n(x)$ *be bounded sequence in* $L^p(\Omega)$ *, and* u_n *weakly converges in* $L^p(\Omega)$ *to* u *.*

Remarque 2.4 When $p = \infty$, then the conclusion becomes that u_n weakly star converges to u.

Lemme 2.6 *Let* B *be a banach space , and* $B = (B^*)'$ *with* B^* *being another banach space . Suppose that for* $1 \prec p \leq \infty$,

$$\begin{aligned} u_n &\rightarrow u \ \text{ weakly star in } L^p\left([0,T], B\right), \\ u_n' &\rightarrow u' \ \text{ weakly star in } L^p\left([0,T], B\right). \end{aligned}$$

Then

$$u_n$$

CHAPTER 3

Global existence and general decay of solution for a Quasi-linear parabolic system with a weak vescoelastic term

In this chapter, we deal with the following quasi-linear parabolic system with a weak-viscoelastic term

$$\begin{cases} A(t)\left|u_t^{m-2}u_t + \Delta^2 u - \alpha(t)\left(g * \Delta^2 u\right) = |u|^{p-2}u\right| & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial v} = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(0), \qquad x \in \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with su ciently smooth boundary $\partial\Omega$, ν represents the unit outer normal to $\partial\Omega$. The values of u are taken in $\mathbb{R}^n (n \geq 1)$ and $A \in C(\mathbb{R}^+)$ is a bounded square matrix satisfying

$$(A(t)v, v) \geq c_{0|v|^2.} \quad \forall t \in \mathbb{R}^+, \tag{3.2}$$

where (.,.) is the inner product in $\mathbb{R}^n$ and $c_0 > 0$. The parameter $m \geq 2$ and p satisfies

$$2 \leq p < \frac{2(n-2)}{n-4} \qquad if n \geq 5, 2 < p < \infty \qquad if n \leq 4, \tag{3.3}$$

The term $(g * \Delta^2 u)$ is defined by

$$\left(g * \Delta^2 u\right)(x,t) = \int_0^t g(t-s)\, \Delta^2 u(x,s)\, ds$$

The rest of this chapter is organized as follows. In Section 2, we give some materials to be used for the main results. In Section 3, we prove the global existence of weak solutions by introducing a suitable functional to obtain the potential well. Finally, general decay result for the global solutions of the problem (3.1) has been proved in Section 4.

3.1 Preliminaries

We give some materials that will be needed in the proof of our results. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms.

We introduce the Sobolev's embedding inequality: assume p is a constant which satisfies (3.3), then $H_0^2(\Omega) \to L^p(\Omega)$continuously, and

$$\|u\|_p \leq C_p \|\Delta u\|_2, \quad \text{for } u \in H_0^2(\Omega) \tag{3.4}$$

where C_p is the optimal embedding constant.

For the relaxation function g and the potential α, we assume

(G_1) $g, \alpha \colon \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing dierentiable functions satisfying

$$g(0>0), \quad \alpha(t)>0, \quad 1-\alpha(t)\int_0^t g(s)\,ds \geq \ell > 0, \quad \int_0^{+\infty} g(s)\,ds < \frac{(p-2)\ell}{2p\alpha(0)}$$

In addition, we assume that there exists a positive constant α_0 such that $\alpha(t) \geq \alpha_0$

(G_2) There exists a nonincreasing diffierentiable function $\xi \colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\xi(t)>0, \qquad g'(t) \leq -\xi(t)\, g(t) \quad for\ t \geq 0, \quad \lim_{t\to+\infty} \frac{-\alpha'(t)}{\xi(t)\,\alpha(t)} = 0$$

Remarque 3.1 Note that $(G1)$ and $(G2)$ imply that $\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)} .= 0$.

We introduce the functional

$$J(u(t)) = \frac{1}{2}\left(1-\alpha(t)\int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \Delta u)(t) - \frac{1}{p}\|u\|_{p'}^p \tag{3.5}$$

as the energy functional associated to problem (3.1) where

$$(g \circ \Delta u)(t) = \int_0^t g(t-s)\|\Delta u(t) - \Delta u(s)\|_2^2\,ds$$

Lemme 3.1 *Suppose* (3.2) *and assumption (G1) hold. Let u be the solution of* (3.1). *Then, the energy satisfies*

$$J\left(u\left(t\right)\right)=\int_{0}^{t}\left(-\int_{\Omega}A\left(\tau\right)\left|u_{\tau}\right|^{m}dx+\frac{1}{2}\alpha'\left(\tau\right)\left(g\circ\Delta u\right)\left(\tau\right)+\frac{1}{2}\alpha\left(\tau\right)\left(g'\circ\Delta u\right)\left(\tau\right)\right.$$

$$\left.-\frac{1}{2}\alpha\left(\tau\right)g\left(\tau\right)\left\|\Delta u\right\|_{2}^{2}+\frac{1}{2}\left(-\alpha'\left(\tau\right)\int_{0}^{\tau}g\left(s\right)ds\right)\left\|\Delta u\right\|_{2}^{2}\right)d\tau$$

$$+J\left(u\left(0\right)\right)\leq+\int_{0}^{t}\left(c_{0}\left\|u_{\tau}\right\|_{m}^{m}-\frac{1}{2}\alpha\left(\tau\right)\left(g'\circ\Delta u\right)\left(\tau\right)+\frac{1}{2}\left(\alpha'\left(\tau\right)\int_{0}^{\tau}\int_{0}^{t}g\left(s\right)ds\right)\left\|\Delta u\right\|_{2}^{2}\right)d\tau+Ju\left(0\right) \tag{3.6}$$

Preuve. . By multiplying the equation (3.1) by ut, integrating over$\Omega\times$ $(0;t)$ we get (3.6), after some manipulations. ■

To illustrate the main results of this chapter, we introduce the definition of weak solutions.

Définition 3.1 Let $u_0\in H_0^2\left(\Omega\right)$ A function u is called a weak solution of the problem (3.1) defined on $[0;T)$ if

$$u\in C\left([0;T);\left[H_0^2\left(\Omega\right)\right]^n\right)\cap C^1\left([0;T);\left[L^m\left(\Omega\right)\right]^n\right)$$

satisfies

$$\left(A\left(t\right)\left|u_t\right|^{m-2}u_t,\right)$$

for all $t\in[0;T)$and $\phi\in C\left([0;T);\left[H_0^2\left(\Omega\right)\right]^n\right)$.

Let$Tmax := supf\left\{T>0\right\}$. The problem (3.1) admits weak solution on $[0;T)$. If $Tmax<\infty$, then u is called

a local weak solution; if $Tmax=\infty$, u is called a global weak solution of problem (3.1) for $0\leq t<\infty$.

Remarque 3.2 Similar to [[17]], we assume the existence of solution. For the linear case $(m=2)$, one can easily establish the existence of a weak solution by the Galerkin method. In the one-dimensional case $(n=1)$, the existence is established, in a more general setting, by Yin [[19]].

3.2 Global Existence

We defne the following functional in order to obtain the potential well.

$$I(u(t)) = \left(1 - \alpha(t)\int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 - \|u\|_p^p \tag{3.7}$$

From (3.5) and (3.7) it tells us that

$$J(u(t)) = \frac{1}{p}I(u(t)) + \frac{p-2}{2p}\left(1 - \alpha(t)\int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \Delta u)(t) \tag{3.8}$$

Lemme 3.2 *Assume p satisfes (3.3) and assumption $(G1)$ holds. Let $u_0 \in H_0^2(\Omega)$ satisfy*

$$C_p^p\left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{p-2}{2}} \leq \ell; I(u(t)) > 0. \tag{3.9}$$

Then,$I(u(t)) > 0$ for all $t_2[0; Tmax)$, where C_p is the optimal constant of the embedding $H_0^2(\Omega) \to L^p(\Omega)$,

$$J(u(0)) = \frac{1}{p}I(u(0)) + \frac{p-2}{2p}\|\Delta u\|_2^2, \quad I(u(0)) = \|\Delta u\|_2^2 - \|u\|_p^p.$$

Preuve. Let u be a weak solution of problem(3.1). By definition of functional $I(u(t))$ in(3.7) we know $I \in C[0; Tmax)$. Suppose that there exists $t_0 \in [0; Tmax)$ such that $I(u(t_0)) < 0$ if the conclusion is not true,so there exists an interval $[t_0; t_1] \in [0; Tmax)$ such that $I(u(t_0) = 0$ and $I(u(t)) < 0$ for all $t \in (t_0; t_1]$. From first inequality of (3.9) then we obtain that for $\varepsilon > 0$ sufficiently small such that ■

$$C_p^p\left(\frac{4p}{(p-2)\ell}(J(u(0)) + \varepsilon)\right)^{\frac{p-2}{2}} \leq \ell, \tag{3.10}$$

there exists $t \in (t_0; t_1]$ satisfying

$$I(u(t)) = -p\varepsilon < 0. \tag{3.11}$$

Therefore, from assumption (G1), using (3.6), (3.8) and (3.11), we have

$$\begin{aligned} J(u(t^{\sim})) &\leq J(u(0)) + \frac{\alpha(0)}{2}\left(\int_0^{+\infty} g(s)\,ds\right)\|\Delta u(t^{\sim})\|_2^2 \\ &\leq J(u(0)) + \frac{(p-2)\ell}{4p}\|\Delta u(t^{\sim})\|_2^2, \end{aligned} \tag{3.12}$$

and

$$J(u(t)) \geq -\varepsilon + \left(\frac{(p-2)\ell}{2p}\right)\|\Delta u(t)\|_2^2 \tag{3.13}$$

Combining (3.12) with (3.13), we arrive at

$$\|\Delta u\,(t)\|_2^2 \leq \frac{4p}{(p-2)\ell}\left(J\,(u\,(0)) + \varepsilon\right) \tag{3.14}$$

Therefore, it follows from(3.4) and (3.14) that

$$\|u\,(t)\|_p^p \leq \frac{C_p^p}{\ell}\,\|\Delta u\,(t)\|_2^{p-2}\,\ell\,\|\Delta u\,(t)\|_2^2 \leq \frac{C_p^p}{\ell}\left(\frac{4p}{(p-2)\ell}\left(J\,(u\,(0)) + \varepsilon\right)\right)^{\frac{p-2}{2}} \ell\,\|\Delta u\,(t)\|_2^2\,. \tag{3.15}$$

Then, by the definition of functional (3.7), using (3.10) and (3.15), we get

$$I(u(t)) = \left(1 - \alpha\,(t)\int_0^t g\,(s)\,ds\right)\|\Delta u\|_2^2 - \|u\|_p^p \geq \ell\,\|\Delta u\,(t)\|_2^2 - \ell\,\|\Delta u\,(t)\|_2^2 = 0,$$

which contradicts(3.11). Then the conclusion of Lemma is true.

Théorème 3.1 *Assume p satisfies*(3.3) *and assumption* (G_1) *holds. Let* $u_0 \in H_0^2(\Omega)$ *Then the local weak solution u exists globally*

Preuve. To show $T_{\max} = \infty$, it suffices to show there exists a constant $C > 0$ such that
■

$$\sup_{t\in[0,T_{\max})} \|\Delta u\|_2^2 \leq C \tag{3.16}$$

By virtue of 3.2 and assumption $(G1)$, using(3.6) and (3.8), we get

$$J\,(u\,(0)) + \frac{(p-2)\,\ell}{4p}\,\|\Delta u\|_2^2 \geq \frac{1}{p}I(u(t)) + \frac{p-2}{2p}\left(1 - \alpha\,(t)\int_0^t g\,(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\alpha\,(t)\,(g\circ\Delta u)\,(t)$$

$$\geq \frac{(p-2)\,\ell}{2p}\,\|\Delta u\,(t)\|_2^2$$

Therefore,

$$\|\Delta u\|_2^2 \leq \frac{4p}{(p-2)\ell}J\,(u\,(0))$$

So let

$$C = \frac{4p}{(p-2)\ell}J\,(u\,(0))$$

then(3.16) holds and consequently the solution is global.

3.3 General Decay

In this section, we prove a general decay result for the solution energy. Our main result in this section reads in the following theorem.

Théorème 3.2 *Given $u_0 \in H_0^2(\Omega)$ Assume that(3.2)and $(G1),(G2)$ hold. Then, there exist two positive constants k and K, depending only on the initial data that the solution of(3.1) satisfes*

$$J\left(u\left(t\right)\right) \leq Ke^{-k\int_0^t \alpha(s)\zeta(s)ds}, \quad \forall t \geq 0 \tag{3.17}$$

To prove above theorem, we need the following technical Lemmas. First, we state an important Lemma by Martinez [13].

Lemme 3.3 *Let $E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non-increasing function and ψ :$\mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a C^2-increasing function with $\psi(0) = 0$ and $\lim_{t\rightarrow\infty} \psi\left(t\right) = +\infty$ Assume that there exist $c > 0$ for which*

$$\int_a^{+\infty} E\left(t\right)\psi'\left(t\right)dt \leq cE\left(a\right)\forall a \geq 0.$$

Then,

$$E\left(t\right) \leq \gamma e^{-\omega\psi(t)}, \forall t \geq 0$$

where γ and ω are positive constants.

Lemme 3.4 *Suppose that $(G1)$ and $(G2)$ hold and the initial data $u_0 \in H_0^2(\Omega)$Then $J(u(t))$ is a nonincreasing function satisfying*

$$J\left(u\left(t\right)\right) \leq \int_0^t \left(-\int_\Omega A\left(t\right)\left|u_t\right|^m dx + \frac{1}{2}\alpha\left(\tau\right)\left(g' \circ \Delta u\right)\left(\tau\right)\right) d\tau + J\left(u\left(0\right)\right) \leq J\left(u\left(0\right)\right) \tag{3.18}$$

Preuve. . Taking(3.6) in $(0, t^{\sim})$ for $|t - t^{\sim}| < \delta_0$ and using assumption $(G2)$, we get for δ_0 small enough ■

$$J\left(u\left(t\right)\right) \leq \int_0^{t^{\sim}} \alpha\left(t\right)\left(-\frac{1}{\alpha\left(t\right)}\int_\Omega A\left(t\right)\left|u_{m_{dxt}}\right| + \left(g' \circ \Delta u\right)\left(t\right) + \frac{1}{2}\left(\frac{-\alpha'\left(t\right)}{\alpha\left(t\right)}\int_0^t g\left(s\right)ds\right)\left\|\Delta u\right\|_2^2\right)dt$$
$$+J(u(0)).$$

From the fact that $\lim_{t\to\infty}\frac{-\alpha'(t)}{\alpha(t)}=0$ for $t^{\sim}$belonging to some small neighborhood of $t=0$, that is (3.18) obtained for $t>t^{\sim}>0$.

Proof of Theorem 2. By multiplying the equation in(3.1) by $\alpha(t)\xi(t)u$ and integrating over $\Omega\times(a,T\max)$ and using the boundary data, we get

$$\int\limits_a^{T\max}\alpha(t)\xi(t)\left(\int\limits_\Omega A(t)|u_t|^{m-2}u_tudx+\left(1-\alpha(t)\int\limits_0^t g(s)\,ds\right)\|\Delta u\|_2^2-\|u\|_p^p\right)dt$$

$$=\int\limits_a^{T\max}\alpha(t)\xi(t)\left(\alpha(t)\int\limits_\Omega\Delta u(t)\int\limits_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right)dsdx\right)dt \tag{3.19}$$

Adding $\alpha(t)\ (g\circ\Delta u)(t)$to both sides of(3.19) and taking(3.5) into account, we get

$$2\int\limits_a^{T\max}\alpha(t)\xi(t)J(u(t))\,dt$$

$$\begin{aligned}\leq&\int\limits_a^{T\max}\alpha(t)\xi(t)\alpha(t)(g\circ\Delta u)(t)\,dt+\frac{p-2}{p}\int\limits_a^{T\max}\alpha(t)\xi(t)\|u\|_p^p\,dt\\&-\int\limits_a^{T\max}\alpha(t)\xi(t)\int\limits_\Omega A(t)|u_t|^{m-2}u_tudxdt\\&+\int\limits_a^{T\max}\alpha(t)\xi(t)\left(\alpha(t)\int\limits_\Omega\Delta u(t)\int\limits_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right)dsdx\right)dt\end{aligned} \tag{3.20}$$

Thanks to the Young's inequality,(3.2), assumptions (G1); (G2), the boundedness of matrix A and using the fact that, $\alpha'(t)\leq 0$ for all$t>0$, we get for $\delta>0$

$$\left|-\int\limits_\Omega A(t)|u_t|^{m-2}u_tudx\right|\leq\delta\|u\|_m^m+C_\delta\|u_t\|_m^m, \tag{3.21}$$

and

$$\alpha(t)\int_\Omega\Delta u(t)\int_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right)dsdx\leq\delta\alpha(0)\|\Delta u\|_2^2+\frac{\int_0^{+\infty}g(s)\,ds}{4\delta}\alpha(t)(g\circ\Delta u)(t)$$

$$\leq\delta\alpha(0)\|\Delta u\|_2^2+\frac{(p-2)\ell}{8p\delta\alpha(0)}\alpha(t)(g\circ\Delta u)(t) \tag{3.22}$$

By combining(3.20)–(3.22) and using

$$\xi(t)\,\alpha(t)\,(g\circ\Delta u)(t)\leq-\alpha(t)\,(g\circ\Delta u)(t)\,,$$

we arrive at

$$\begin{aligned}2\int_a^{T_{\max}}\alpha(t)\,\xi(t)\,J(u(t))\,dt &\leq \alpha(0)\,\xi(0)\,C_\delta\int_a^{T_{\max}}\|u_t\|_m^m\,dt\\ &-\alpha(0)\left(1+\frac{(p-2)\,\ell}{8p\delta\alpha(0)}\right)\int_a^{T_{\max}}\alpha(t)\,(g\circ\Delta u)(t)\,dt\\ &+\delta\int_a^{T_{\max}}\alpha(t)\,\xi(t)\,\|u\|_m^m\,dt+\frac{p-2}{p}\int_a^{T_{\max}}\alpha(t)\,\xi(t)\,\|u\|_p^p\,dt\\ &+\delta\alpha(0)\int_a^{T_{\max}}\alpha(t)\,\xi(t)\,\|\Delta u\|_2^2\,dt. \qquad (3.23)\end{aligned}$$

By recalling(3.6), we deduce from the first two integrals on the right-hand side of (3.23) that

$$\begin{aligned}&\alpha(0)\,\xi(0)\,C_\delta\int_a^{T\max}\|u_t\|_m^m\,dt-\alpha(0)\left(1+\frac{(p-2)\,\ell}{8p\delta\alpha(0)}\right)\int_a^{T_{\max}}\alpha(t)\,(g'\circ\Delta u)(t)\,dt\\ &\leq\left(\frac{C_\delta}{2c_0}+\alpha(0)\left(1+\frac{(p-2)\,\ell}{8p\delta\alpha(0)}\right)\alpha(t)\,\xi(t)\right)\int_a^{T_{\max}}\left(\frac{\alpha'(t)}{\xi(t)\,\alpha(t)}\int_0^t g(s)\,ds\right)dt\\ &-\gamma_0\int_a^{T_{\max}}J'(u(t))\,dt, \qquad (3.24)\end{aligned}$$

where

$$\gamma_0:\ =\frac{\alpha(0)\,\xi(0)\,C_\delta}{c_0}+2\alpha(0)\left(1+\frac{(p-2)\,\ell}{8p\delta\alpha(0)}\right)$$

By exploiting 3.4 and using $limt_{t\to+\infty}\frac{-\alpha'(t)}{\xi(t)\alpha(t)}=0$ to choose$t>t^{\sim}>0$, (3.24) takes the form

$$\alpha(0)\,\xi(0)\,C_\delta\int_a^{T\max}\|u_t\|_m^m\,dt-\alpha(0)\left(1+\frac{(p-2)\,\ell}{8p\delta\alpha(0)}\right)\int_a^{T_{\max}}\alpha(t)\,(g'\circ\Delta u)(t)\,dt\leq\gamma_0J(u(a))$$

.Therefore, (3.23) yields that

$$\int_a^{T_{\max}}\alpha(t)\,\xi(t)\,J(u(t))\,dt-\gamma_0J(u(a))$$

$$\leq \delta \int_a^{T_{\max}} \alpha(t)\,\xi(t)\,\|u\|_m^m\,dt + \frac{p-2}{p}\int_a^{T_{\max}} \alpha(t)\,\xi(t)\,\|u\|_p^p\,dt$$
$$+\,\delta\alpha(0)\int_a^{T_{\max}} \alpha(t)\,\xi(t)\,\|\Delta u\|_2^2\,dt \tag{3.25}$$

Since g is positive, we have, for any $t_0 > 0$

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds := g_0 > 0, \quad \forall t \geq t_0 \tag{3.26}$$

To estimate the last integrals in the right-hand side of (3.25), we use (3.26), assumption $(G1)$, (3.4), (3.8), (3.9) and repeatedly embedding inequalities as follows

$$\begin{aligned} J(u(t)) &= \frac{1}{p}I(u(t)) + \frac{p-2}{2p}\left(1-\alpha(t)\int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)\,(g\circ\Delta u)(t) \\ &\geq \frac{p-2}{2p}\left(1-\alpha(t)\int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 \\ &\geq \frac{(p-2)\,\ell}{2p}\|\Delta u\|_2^2, \end{aligned} \tag{3.27}$$

$$\begin{aligned} \|u\|_p^p &\leq C_p^p\,\|\Delta u\|_2^{p-2}\,\|\Delta u\|_2^2 \\ &\leq \frac{C_p^p}{\ell}\left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{p-2}{2}}\ell\,\|\Delta u\|_2^2 \\ &\leq \left(1-\alpha(t)\int_0^t g(s)\,ds\right)\|\Delta u\|_2^2 \\ &\leq (1-\alpha_0 g_0)\,\|\Delta u\|_2^2 \\ &\leq \frac{2p(1-\alpha_0 g_0)}{(p-2)\ell}J(u(t)) \end{aligned} \tag{3.28}$$

where C_p is the optimal constant of the embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$, and

$$\begin{aligned} \|u\|_m^m &\leq \frac{C_m^m}{\ell}\,\|\Delta u\|_2^{m-2}\,\|\Delta u\|_2^2 \\ &\leq C_m^m\left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{m-2}{2}}\|\Delta u\|_2^2 \end{aligned}$$

$$\leq \gamma_1 J\left(u\left(t\right)\right) \tag{3.29}$$

where Cm is the optimal constant of the embedding $H_0^2\left(\Omega\right) \hookrightarrow L^m\left(\Omega\right)$, and

$$\gamma_1 = \frac{2pC_m^m}{(p-2)\ell}\left(\frac{4p}{(p-2)\ell} J\left(u\left(0\right)\right)\right)^{\frac{m-2}{2}}$$

Inserting (3.7)–(3.29) in (3.25) the estimate

$$\left(1 - \frac{\delta\gamma_1}{2} - \frac{\delta p\alpha\left(0\right)}{(p-2)\ell} - \frac{1-\alpha_0 g_0}{\ell}\right)\int_a^{T_{\max}} \alpha\left(t\right)\xi\left(t\right) J\left(u\left(t\right)\right) dt \leq \frac{\gamma_0}{2} J\left(u\left(a\right)\right)$$

is established.

At this point, we pick $\alpha\left(0 > \frac{1-\ell}{g_0}\right)$ and choose δ small enough, to have

$$\lambda :\ = 1 - \frac{\delta\gamma 1}{2} - \frac{\delta p\alpha\left(0\right)}{(p-2)\ell} - \frac{1-\alpha_0 g_0}{\ell} > 0.$$

Thanks to the 3.3, by taking $\psi(t) = \int\limits_0^t \alpha(s)\xi(s)ds$ and letting $T_{\max}$goes to infinity, we obtain the desired result in(3.17) and the proof is completed.

CHAPTER 4

Blow Up in a Nonlinearly Damped Wave Equation

In this chapter we consider the following initial boundary value problem.

$$\begin{aligned} u_{tt} - \Delta u + a u_t \left| u_t \right|^{m-2} &= bu \left| u \right|^{p-2}, x \in \Omega, t > 0, \\ u(x,t) &= 0, x \in \partial\Omega, t \geq 0, \\ u(x,0) &= u_0(x), u_t(x,0) = u_1(x), x \in \Omega \end{aligned} \tag{4.1}$$

where $a, b > 0$,$p, m > 2$, and Ω is a a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$ For $b = 0$.

In this chapter, we prove the same result of [23] without imposing the condition that the initial energy is sufficiently negative. In other words, we show that any solution of (4.1) with negative initial energy — however close to zero is — blows up in finite time.

In addition to ommitting the condition of large "negative" initial data, our technique of proof is simpler than the ones in [23] and [29]. We first state a local result established in [23].

Théorème 4.1 . *Suppose that $m > 2, p > 2$, and*

$$p \leq 2\frac{n-1}{n-2}, n \geq 3 \tag{4.2}$$

Assume further that

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega). \tag{4.3}$$

Then the problem(4.1) has a unique local solution

$$u \in C\left([0,T);H_0^1(\Omega)\right), u_t \in C\left([0,T);L^2(\Omega)\right) \cap L^m(\Omega \times (0,T)), \tag{4.4}$$

T is small.

Théorème 4.2 . *The condition on p, in* (4.2)*, is needed to establish the local existence result (see[23]). In fact under this condition, the nonlinearity is Lipschitz from* $H^1(\Omega)$ *to* $L^2(\Omega)$.

4.1 Main result

In this section we show that the solution(4.4) blows up in finite time if $p > m$ and $E(0) < 0$, where

$$E(t) := \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2\right](x,t)\,dx - \frac{b}{p}\int_\Omega |u(x,t)|^p\,dx. \tag{4.5}$$

Théorème 4.3 Lemme 4.1 . *Suppose that*(4.2) *holds. Then there exists a positive constant* $C > 1$*depending on* Ω *only such that*

$$||u||_p^s \leq C||\nabla u||_2^2 + ||u||_p^p \tag{4.6}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

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Preuve. If $||u||p \leq 1$ then $||u||_p^s \leq ||u||_p^2 \leq C||\nabla u||_2^2$ by Sobolev embedding theorems. ■

If $||u||_p > 1$ then $||u||_p^s \leq ||u||_p^p$. Therefore (4.2) follows.

We set

$$H(t) := -E(t)$$

and use, throughout this paper,C to denote a generic positive constant depending on Ω only. As a result of(4.5),(4.6), we have

Corollary 4.1 . *Let the assumptions of the lemma hold. Then we have*

$$||u||_p^s \leq C|H(t)| + ||ut||_2^2 + ||u||_p^p \tag{4.7}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Théorème 4.4 . *Let the conditions of the 4.1 be fulfilled. Assume further that* $p > m$ *and*

$$E(0) < 0. \tag{4.8}$$

Then the solution(4.4)blows up in finite time.

Théorème 4.5 . *Note that contrary to [23], no condition on the size of the initial data has been done. The blow up takes place for any initial data satisfying*(4.8).

Preuve. . We multiply Equation(4.1) by u_t and integrate over Ω to get ■

$$E'(t) = -a \int_\Omega |u_t(x,t)|^m \, dx, \tag{4.9}$$

for almost every t in $[0,T)$ since $E\prime(t)$is absolutely continuous (see [23]); hence $H'(t) \geq 0$ So we have

$$0 < H(0) \leq H(t) \leq \frac{b}{p} \|u\|_p^p, \tag{4.10}$$

for every t in $[0,T)$, by virtue of(4.8). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t) dx \tag{4.11}$$

for ε small to be chosen later and

$$0 < \alpha \leq min \left\{ \frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)} \right\}. \tag{4.12}$$

By taking a derivative of(4.11) and using Equation (4.1) we obtain

$$\begin{aligned} L\prime(t) \ : \ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 + |\nabla u|^2\right](x,t)\, dx \\ &+ \varepsilon b \int_\Omega |u(x,t)|^p dx - \alpha\varepsilon \int_\Omega |ut|^{m-2} u_t u(x,t) dx. \end{aligned} \tag{4.13}$$

We then exploit Young's inequality

$$XY \leq \frac{\delta^r}{r} X^r + \frac{\delta^{-q}}{q} Y^q, \quad X, Y \geq 0, \quad for\ all \delta > 0, \frac{1}{r} + \frac{1}{q} = 1$$

with $r = m$ and $q = m/(m-1)$ to estimate the last term in (4.13) as follows

$$\int_\Omega |u_t|^{m-1} |u| dx \leq \frac{\delta^m}{m} ||u||_m^m + \frac{m-1}{m} \delta^{-m/(m-1)} ||u_t||_m^m$$

which yields, by substitution in(4.13),

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha) H^{-\alpha}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right] H'(t) \\ &+\varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx + \varepsilon\left[pH(t) + \frac{p}{2}\int_\Omega \left[u_t^2 + |\nabla u^2|\right](x,t)\,dx\right] \end{aligned} \quad (4.14)$$

$-\varepsilon\delta\frac{\delta^m}{m}||u_t||_m^m$, for all $\delta > 0$

Of course(4.14) remains valid even if δ is time dependant since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in(4.14) we arrive at

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)dx \\ &-\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega |\nabla u|^2(x,t)dx + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\alpha H^{\alpha(m-1)}(t)\,\|u\|_m^m\right] \end{aligned} \quad (4.15)$$

By exploiting(4.10) and the inequality $\|u\|_m^m \leq C||u||_p^m$,we obtain

$$H^{\alpha(m-1)}(t)||u||_m^m \leq \left(\frac{b}{p}\right)^{\alpha(m-1)} C||u|_p^{m+\alpha p(m-1)}$$

hence(4.15) yields

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)dx \\ &+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)dx \\ &+\varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\alpha\left(\frac{b}{p}\right)^{\alpha(m-1)} C||u|_p^{m+\alpha p(m-1)}\right] \end{aligned} \quad (4.16)$$

We then use 4.1 and(4.12), for $s = m + \alpha p(m-1) \leq p$, to deduce from(4.16)

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)dx \\ &+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)dx \\ &+\varepsilon\left[pH(t) - C_1K^{1-m}\left\{H(t) + \|u_t\|_2^2 + \|u\|_p^p\right\}\right] \end{aligned} \quad (4.17)$$

,Messaoudi, Blow Up in a Nonlinearly Damped Wave Equation where $C1 = \left(\frac{b}{p}\right)^{\alpha(m-1)} C/m$. By noting that

$$H(t) = \frac{b}{p}||u||_p^p - \frac{1}{2}||u_t||_2^2 - \frac{1}{2}||\nabla u||_2^2$$

and writing $p = (p+2)/2 + (p-2)/2$,(4.17) yields

$$\begin{aligned} L'(t) \quad \geq \quad & \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p-2}{4}\right)\|\nabla u\|_2^2 \\ & +\varepsilon\left[\left(\frac{p+2}{2} - C_1K^{1-m}\right)H(t) + \left(\frac{p-2}{2p}b - C_1K^{1-m}\right)\|u\|_p^p + \left(\frac{p+6}{4} - C_1K^{1-m}\right)\|u_t\|_2^2\right] \end{aligned} \tag{4.18}$$

At this point, we choose k large enough so that the coefficients of $H(t), ||u_t||_2^2$, and $||u||_p^p$ in(4.18) are strictly positive; hence we get

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\gamma\left[H(t) + ||u_t||_2^2 + ||u||_p^p\right] \tag{4.19}$$

where $\gamma > 0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha) - \varepsilon k(m-1)/m \geq 0$ and

$$L(0) = H1 - \alpha(0) + \varepsilon\int_\Omega u_0u_1(x)dx > 0$$

Therefore (4.19) takes the form

$$L'(t) \geq \gamma\varepsilon\left[H(t) + ||ut||_2^2 + ||u||_p^p.\right] \tag{4.20}$$

Consequently we have

$$L(t) \geq L(0) > 0, \text{for all } t \geq 0.$$

Next we would like to show that

$$L'(t) \geq \Gamma L^{1/(1-\alpha)}(t), \text{for all } t \geq 0, \tag{4.21}$$

where Γ is a positive constant depending on $\varepsilon\gamma$ and C (the constant of 4.1).

Once (4.21) is established, we obtain in a standard way the finite time blow up of $L(t)$,hence of u (see [22] for instance).

To prove (4.21), we first estime

$$\left|\int_\Omega uu_t(x,t)\,dx\right| \leq ||u||_2 + ||u_t||_2 \leq C||u||_p + ||u_t||_2$$

which implies

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C||u||_p^{1/(1-\alpha)} + ||u_t||_2^{1/(1-\alpha)}$$

Again Young's inequality gives us

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C||u||_p^{\mu/(1-\alpha)} + ||u_t||_2^{\theta/(1-\alpha)} \tag{4.22}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by (4.12). Therefore(4.22) becomes

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C||u||_p^s + ||u_t||_2^2$$

where$s = 2/(1-2\alpha) \leq p$. By using 4.1 we obtain

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[H\,(t)\,||u||_p^p + ||u_t||_2^2\right], \text{for all} \quad t \geq 0. \tag{4.23}$$

Finally by noting that

$$L^{1/(1-\alpha)}(t) = \left(H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t(x,t)dx\right)^{1/(1-\alpha)}$$

$$\leq 2^{1/(1-\alpha)}\left(H(t) + \left|\int_{\Omega} uut(x,t)dx\right|^{1/(1-\alpha)}\right)$$

and combining it with(4.20) and(4.23), the inequality (4.21) is established. This completes the proof

4.2 CONCLUSION AND PERSPECTIVE

In this memory, we have solved two problems related to differential equations of different classes. We tackled both system of parabolic equations and hyperbolic equation using various methods. Specifically, we utilized the Nehari methods for system of parabolic equation and second method using a Lyaponov function based on the energy total of system. We observed that the solution of the first problem general decay, and for the second problem the solution blow up in finite time. For future work is attempting to study the same previous problems using alternative methods, for instance, the method of semi-groups, which is considered an additional method due to some potentially advantageous features for dealing with such cases.

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