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# Thesis

*submitted for the degree of*

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Option: ***Numerical Analysis, PDE and Applications***

# Periodic solutions of some classes of differential equations with delays depending on time and state

Presented by:

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# Abstract in English

**Periodic solutions of some classes of**
**differential equations with delays depending on time and state**

In this work, two classes of nonlinear functional differential equations with multiple delays depending on the time and the state are investigated. Using as a main tool a hybrid approach that combines fixed point theorems in cones and the Green's functions method, we provide some sufficient conditions that guarantee the existence of multiple positive periodic solutions.

The main idea consists to define the Banach space and the cone that facilitate our study on the one hand, and on the other hand they ensure some desired requirements before transforming the problem into an equivalent integral equation whose kernel is a Green's function, and hence applying a fixed point theorem in cone or Leggett–Williams fixed point theorem.

**Keywords**: Existence, fixed point theorem in cone, *Green*'s function, Leggett–Williams fixed point theorem, time and state delay differential equation.

# Abstract in French


**Solutions périodiques de certaines classes d'équations différentielles à retard dépendant du temps et de l'état**

Dans ce travail, deux classes d'équations différentielles fonctionnelles non linéaires avec plusieurs retards dépendant du temps et de l'état sont étudiées. En utilisant une approche hybride combinant les théorèmes du point fixe dans les cônes et la méthode des fonctions de Green comme outil principal, nous établissons des conditions suffisantes qui guarantissent l'existence de solutions multiples périodiques et positives. L'idée principale consiste à définir un espace de Banach et un cône pour faciliter notre étude d'une part et d'autre part, pour assurer certaines exigences souhaitées avant de transformer le problème en une équation intégrale équivalente dont le noyau est une fonction de *Green*, et donc appliquer le théorème de points fixes multiples.



**Mots-clés**: Existence, théorème de point fixe dans le cone, fonction de Green, théorème de point fixe de Leggett–Williams, équation différentielle à retard dependant du temps et de l'état.

# Abstract in Arabic

**الحلول الدورية لبعض الفئات**
**من المعادلات التفاضلية ذات تأخيرات معتمدة على الزمن والحالة**

في هذا العمل ، تمت دراسة فئتين من المعادلات التفاضلية الدالية غير الخطية بعدة تأخيرات تعتمد على الزمن والحالة.
باستخدام طريقة هجينة تجمع بين نظريات النقطة الثابتة في المخاريط وطريقة دوال غرين كأداة رئيسية ، تم إنشاء بعض الشروط الكافية لضمان وجود حلول دورية و موجبـة متعددة.
تتمثل الفكرة الرئيسية في تحديد فضاء بناخي ومخروط لتسهيل الدراسة من ناحية ، ومن ناحية أخرى ، لضمان متطلبات معينة مرغوبة قبل تحويل المسألة إلى معادلة تكاملية مكافئة نواتها دالة غرين، وبالتالي تطبيق نظرية النقطة الثابتة في المخروط أو نظرية النقطة الثابتة للوجيت- ويليامز.

**كلمات مفتاحيه**: وجود ، نظرية النقطة الثابتة في المخروط ، دالة غرين ، نظرية النقطة الثابتة للوجيت - ويليامز، معادلة تفاضلية بتأخير تعتمد على الزمن والحالة.

# Dedication

I dedicate this master thesis

- To my parents, my dear mother "**RAZIKA"** and my dear father **"RAMDANE**, who always encouraged me, and who let me dream and draw my dreams.

- To my dear brother **OUSSAMA** who supported me in my difficult times

- To my dear sisters, **YASMINE, CHAIMA** and **MERIEM** owner of the words that helped me achieve success.

- To all my best friends: **HALLA**, **NOUR** and **MOHAMED**.

- To my classmates **KHAOULA** and **BOUTHEINA**.

- To all my classmates in the department of Mathematics of Skikda university

- To every person who lit my way and made an effort for me.

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# Acronyms

| Abbreviation | Meaning |
| --- | --- |
| DDE | Delayed differential equation |
| SDDE | State delay differential equation |
| i.e. | that is |

# List of Symbols

**Sets and numbers**

| | |
|---|---|
| $\mathbb{R}$ | : the set of real numbers (1-dimensional real Euclidean space). |
| $\mathbb{R}^+$ | : the set of positive real numbers |
| $[c, d]$ | : the interval of numbers between $c$ and $d$, including $c$ and $d$ |
| $(c, d)$ | : an open interval |
| $[0, +\infty)$ | : left-closed and right–unbounded interval |
| $\mathcal{C}(\mathbb{X})$ | $:= \mathcal{C}(\mathbb{X}, \mathbb{X})$ is the space of continuous functions from $\mathbb{X}$ into itself |
| $T$ | : a period |
| $\mathbb{X}$ | : a *Banach* space |
| $\mathbb{K}$ | : an ordered cone |
| $\mathring{\Omega}$ | : the interior of $\Omega$ |
| $\overline{\Omega}$ | : the closure of $\Omega$ |
| $[c, d]$ | : a closed interval |
| $[c, s)$ | : left closed and right opened interval |
| $(s, d]$ | : right opened and left closed interval |
| $\mathcal{C}(X, Y)$ | : the set of all continuous functions from $X$ into $Y$ |

## Functions

| | |
|---|---|
| $\lvert\cdot\rvert$ | : the absolute value |
| $\lVert . \rVert_X$ | : a norm on $X$ |
| $\lVert f \rVert_\infty$ | : the supremum norm defined by $\sup \lvert f(x) \rvert$ |
| $\lim\limits_{\lvert u \rvert \longrightarrow x_0}$ | : limit as $\lvert u \rvert$ approaches $x_0$ |
| $x'(t)$ | $:= \dfrac{dx(t)}{dt}$ the first derivative of the function $x(t)$ with respect to $t$ |
| sup | : the supremum |
| max | : the maximum |
| min | : the minimum |
| inf | :the infimum |
| $\exp x$ | : the exponential function of $x$ |
| $G(t,x)$ | : the *Green's* function |
| $x^{(n)}(t)$ | $:= \dfrac{d^n x(t)}{dt^n}$ the $n-th$ derivative of the function $x(t)$ with respect to $t$ |

# General introduction

Delay differential equations (DDEs) where the state variable appears with delayed argument, have been successfully used for describing a wide spectrum of nonlinear phenomena in our life, especially since the international congress of mathematicians was held in Rome in 1908. Due to the fact that many processes depend on past history, retarded problems where the evolution of the system at a certain time instant depends on the past history or memory, create more realistic models and give accurate results. The memory or the time-delay can be constant, time varying, state dependent or even distributed where its meaning differs from one model to another. For instance, it can be related to the incubation period of an infectious disease in epidemiology, the time between initiation of cellular production in the bone marrow and release of mature cells into the blood in the production of blood cells in hematology, the transit time or the duration of a cellular transformation in the dynamics of cell populations, the time of gestation, development, the juvenile phase, life cycle or the period of maturation in population dynamics of certain human, animal and plant species, a time lag that often arises in feedback loops involving sensors and actuators in engineering and also an information lag in economic dynamics, to name a few.

To the best of our knowledge, the history of retarded differential equations goes back to the 18th century by virtue of the works of J. Bernoulli, L. Euler, J.L. Lagrange, P. Laplace, S. Poisson that can be considered as the first flame of the passion for research on the concept of the delay in real phenomena and their experiences have supplied the crucial stimulus needed to meet the challenges in investigating these equations. But unfortunately the delay has been ignored at that time because on one hand there was insufficient theory to investigate such models in depth and on the other hand the introduction of such lags in the model may degrade the performance of control systems or even worse lead to system instability.

In fact, this topic knew its hour of glory in the 20th century. The first step towards its revival was taken only in 1908 when Picard underlined in the international conference of mathematicians in Rome, the importance of considering hereditary effects in the modeling of physical phenomena.

Since then, this subject gained considerable momentum, which consequently led to publish a great amount of works especially in the fifties that saw an explosion of scientific activities in this direction where the delay became part of the vocabulary of all scholars working on viscoelasticity, ecological problems, mechanical problems, nuclear reactors, neural networks, epidemiological or physiological models,...,etc. Hereby, the literature of these equations is now extensive.

An important type of delay functional differential equations that captivated the attention of the majority of the scientific community is the class of differential equations with state-dependent delayed arguments (SDDEs). Despite that the origins of this type of equations are traced back to at least 1806, when Poisson investigated them in his celebrated paper [21], their theory is still emergent and not well developed. During the past eight decades, there has been an increasing activity. One of the most prominent models is the two-body problem of classical electrodynamics which, unfortunately, have remained in limbo until the sixties of the last century when R. D. Driver's [8, 9, 10, 11, 12] published his series of extremely interesting papers on delay-differential equations and their applications in classical electrodynamics. His works with Norris have been crowned with the development of a fundamental theory where they proved some existence and uniqueness results for state dependent delay differential equations having Lipschitz continuous initial functions.

Recently, equations of this kind have attracted considerable attention of scholars and now occupy a more central place in all fields of engineering and science due to their widespread applications in many fields of science, ranging from biology and population dynamics to physics and engineering. Some recent contributions to their theory can be seen in [1, 3, 5, 16, 18]

Many authors think that delays appearing in many models are dependent on both the time and the state variable; in other word, the delay function takes the form $\tau(t, x(t))$. But dealing with such models seems to require extra care, perhaps simply because of a lack of a basic theory and the sufficient experience to handle them.

In this work, we present some results on the existence of positive periodic solutions for some classes of nonlinear functional differential equations with multiple delays depending on the time and the state of the following form:

$$x'(t) = \pm a(t, x(t))\, x(t) \pm \lambda f(t, x(t - \tau_1(t, x(t))), \ldots, x(t - \tau_m(t, x(t)))),$$

where $\lambda > 0$ is a parameter.

The variation of the environment plays a crucial role in many dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. For this, in order to incorporate the periodicity of the environment, we assume that the parameters of the equations are periodic with a common period. Moreover, since only positive solutions are meaningful in many applications, we are interested in finding positive solutions. In this situation, the main outcomes can be applied to many biological, ecological, and population models such as Mackey-Glass models, Lasota-Wazewska model and Nicholson's blowflies model.

The specific questions this work aims at answering are:

$(i)$ What is the effect of state dependence in a delay differential equation?

$(ii)$ Does the problem have solutions?

## Objectives

The main goal of this work is to show the efficiency of the technique that combines the fixed point theory and other approaches to establish certain existence results on positive periodic solutions for certain classes of nonlinear differential equations with delays depending on time and state. More precisely, the key object of this manuscript lies in establishing a set of sufficient criteria to ensure the existence, boundedness and positivity of periodic solutions for certain classes of delayed differential functional equations that can describe several biological phenomena.

## Methodology

Over the course of several years, one can find several different approaches that dealt with problems with complicated delays. Methodology used here utilizes two fundamental aspects: the Green functions method and the fixed point theory.

This technique boils down to the conversion of the problem at hand into a fixed point one. The first step of it is to construct an appropriate Banach space and a suitable cone of it before transforming the proposed problem into an equivalent integral equation with a Green's kernel. Next, from this integral equation, some functional analysis tools and certain useful properties of the obtained kernel, one can construct an integral operator that fulfill the requirments of the chosen fixed point theorem. Finally, we apply this fixed point theorem which ensures the existence of fixed points of the constructed operator and which is in turn equivalent to the existence of solutions of the proposed problem.

## Thesis Overview

This manuscript has been divided into three chapters. Its plan is as follows: In the first chapter, we will state some basic definitions and preliminary materials that will be used in the sequel.

Chapter 2 is devoted to present some results on the existence of multiple positive periodic solutions to the following two classes of first order differential equations with multiple time and state-dependent delays:

$$x'(t) = -a(t, x(t))\, x(t) + f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))),$$

and

$$x'(t) = a(t, x(t))\, x(t) - f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))),$$

where two fixed point theorems in cones are the key tools in obtaining the required outcomes.

Chapter 3 gives conditions that guarantee the existence of three positive periodic solutions for the following classes of first order differential equations with many delays depending on both the time and the state variable:

$$x'(t) = -a(t, x(t))\, x(t) + \lambda f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))),$$

and

$$x'(t) = a(t, x(t))\, x(t) - \lambda f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))).$$

These equations are similar to the previous ones, except that here they depend on the parameter $\lambda$. So, with almost similar arguments, various existence and multiplicity results for positive solutions are derived depending on different values of the parameter $\lambda$. The existence of solutions are shown using a hybrid technique that combines the application of the multiple fixed point theorem of Leggett-Williams with the use of certain useful properties of an obtained Green's kernel. The key step lies in converting the problem at hand into an equivalent integral equation whose kernel is a Green's function before using certain of its useful properties together with the Ascoli-Arzelà theorem to pave the way for the application of the aforementioned fixed point theorem.

Finally, the conclusion is drawn to end this manuscript.

# CHAPTER 1

# Primary concepts

## Contents



In this chapter, we would like to introduce some notations, definitions and preliminary results that are used in the remainder of the thesis.

## 1.1 Green's functions

Green's functions are named after the British mathematician George Green, who first developed the concept in the 1820$s$. The Green's function are not special functions such as Bessel functions, Weber function, Airy function...etc. but Green's functions technique is a method that can be used to solve nonhomogeneous linear differential equations of the form $Ly(t) = f(t)$ with specified initial conditions or boundary conditions or to convert them into integral equations where the Green's function is referred to as the kernel of the integral operator $L^{-1}$.

### 1.1.1 Green's functions for a non homogeneous linear differential equations

Consider the following $n-th$ order differential equation:

$$a_0(t)\,y^{(n)}(t) + a_1(t)\,y^{(n-1)}(t) + ... + a_n(t)\,y(t) = f(t)\,,\ \forall t \in [c,d]\,, \tag{1.1}$$

where $a_0$, $a_1$, ..., $a_n$, $f$ are continuous functions on $[c,d]$, $a_0(t) \neq 0$ on $[c,d]$ with the following boundary conditions:

$$\begin{cases} \alpha_1 y(c) + \alpha_1^{(1)} y'(c) + ... + \alpha_1^{(n-1)} y^{(n-1)}(c) + \beta_1 y(d) + ... + \beta_1^{(n-1)} y^{(n-1)}(d) = 0 \\ \alpha_2 y(c) + \alpha_2^{(1)} y'(c) + ... + \alpha_2^{(n-1)} y^{(n-1)}(c) + \beta_2 y(d) + ... + \beta_2^{(n-1)} y^{(n-1)}(d) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_n y(c) + \alpha_n^{(1)} y'(c) + ... + \alpha_n^{(n-1)} y^{(n-1)}(c) + \beta_n y(d) + ... + \beta_n^{(n-1)} y^{(n-1)}(d) = 0. \end{cases} \tag{1.2}$$

The homogeneous equation of (1.1) is

$$a_0(t)\,y^{n}(t) + a_1(t)\,y^{n-1}(t) + ... + a_n(t)\,y(t) = 0,\ \forall t \in [c,d]\,. \tag{1.3}$$

We suppose that the rank of the below array $\Delta$ is $n$:

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_1^{(1)} & \dots & \beta_1 & \dots & \beta_1^{(n-1)} \\ \alpha_2 & \alpha_2^{(1)} & \dots & \beta_2 & \dots & \beta_2^{(n-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_n & \alpha_n^{(1)} & \dots & \beta_n & \dots & \beta_n^{(n-1)} \end{vmatrix}.$$

**Theorem 1.1** *[19]*

**a** Suppose that $a_0(t) \neq 0$. If the homogeneous equation of (1.1) with the boundary conditions (1.2) has only the trivial solution, then the Green's function $G(t,s)$ for this boundary value problem exists and it is unique.

**b** Suppose that the aforementioned conditions are satisfied, then the nonhomogeneous equation (1.1) admits a unique solution given by the following integral expression:

$$y(t) = \int_c^d G(t,s) f(s)\, ds. \tag{1.4}$$

### 1.1.2 Finding Green functions

We will solve $Ly(t) = f(t)$, a differential equation with homogeneous boundary conditions, by finding an inverse operator $L^{-1}$, so that $y(t) = L^{-1}(f(t)) = \int G(t,s) f(s)\, ds$. Assume that the homogeneous equation of (1.1) with the boundary conditions (1.2) has only the trivial solution $y(t) = 0$ which ensures the existence of a unique Green's function $G(t,s)$. The problem now is how to construct $G(t,s)$. Actually there are four necessary ingredients.

**Definition 1.1** [15] We called the Green's function of equation (1.1) with the boundary conditions (1.2), the function $G(t,s)$ that can be quickly constructed via the following properties:
1− $G(t,s)$ satisfies the homogeneous differential equation for any $t \neq s$.
2− For all fixed $s$ in $]c,d[$, the function $G(t,s)$ satisfies the boundary conditions of the problem.
3− The *Green*'s function $G(t,s)$ must be continuous.
4− For all fixed $s$ in $]c,d[$, $\frac{\partial G}{\partial t}(s^+,s) - \frac{\partial G}{\partial t}(s^-,s) = \frac{1}{a_0(s)}$ (jump in derivative or jump discontinuity of $\frac{\partial G}{\partial t}$ at $t = s$).

## 1.2 Bounded, closed and compact subsets in a normed vector space

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$.

**Definition 1.2** [22] A subset $\mathbb{M}$ of $\mathbb{X}$ is said to be bounded if there exists $C > 0$ such that

$$\|x\|_{\mathbb{X}} \leq C, \forall x \in \mathbb{M}.$$

**Theorem 1.2** *[22] A set $\mathbb{M} \subseteq \mathbb{X}$ is closed if and only if, whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{M}$ which converges to an element $x \in \mathbb{X}$, then $x \in \mathbb{M}$.*

**Definition 1.3** [22] A set $\mathbb{M} \subseteq \mathbb{X}$ is called compact if every sequence in $\mathbb{M}$ has a subsequence that converges to a point in $\mathbb{M}$.

**Definition 1.4** [22] The closure of a set $\mathbb{M} \subseteq \mathbb{X}$ (denoted by $\overline{\mathbb{M}}$) is the smallest closed set that contains $\mathbb{M}$.

**Definition 1.5** [22] A set $\mathbb{M} \subseteq \mathbb{X}$ is called relatively compact if its closure $\overline{\mathbb{M}}$ is compact.

**Corollary 1.1** *[22] A set $\mathbb{M} \subseteq \mathbb{X}$ is relatively compact if and only if every sequence in $\mathbb{M}$ has a subsequence that converges to a point in $\mathbb{X}$.*

## 1.3 Continuous, Lipschitz continuous and compact operators

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two normed vector spaces over the same field $\mathbb{F}$.

**Definition 1.6** [6] An operator $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be continuous at a point $x_0 \in \mathbb{X}$ if

$$\lim_{x\longrightarrow x_0} \mathcal{S}x = \mathcal{S}x_0.$$

The continuity at $x_0 \in \mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{X}, \ \ (\|x - x_0\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x - \mathcal{S}x_0\|_{\mathbb{Y}} < \varepsilon).$$

If $\mathcal{S}$ is continuous at every point of $\mathbb{X}$, then $\mathcal{S}$ is said to be continuous on $\mathbb{X}$. The continuity on $\mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \forall x \in \mathbb{X}, \exists \delta > 0, \forall y \in \mathbb{X}, (\|x - y\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} < \varepsilon).$$

**Definition 1.7** [23] A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is called *Lipschitz continuous* if there is a positive constant $C$ such that

$$\forall x, y \in \mathbb{X} : \|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} \leq C \|x - y\|_{\mathbb{X}} .$$

If $C \in [0, 1[$, $\mathcal{S}$ is called a contraction mapping.

**Remark 1.1** If $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ then

$\mathcal{S}$ is a contraction implies that $\mathcal{S}$ is Lipschitz continuous and hence $\mathcal{S}$ is continuous.

**Theorem 1.3** *[22] A continuous function on a closed bounded interval is bounded and attains its bounds.*

**Remark 1.2** The above theorem is hidden in the proof of many theorems and lemmas in the rest of this work where we integrate a continuous function over a compact interval.

**Definition 1.8** A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be compact if and only if $\mathcal{S}$ maps bounded sets into relatively compact sets, i.e.,

$$[\mathcal{S} \text{ compact}] \Longleftrightarrow \left[\forall M \subset E, \ \ (M \text{ bounded}) \Longrightarrow \left(\overline{\mathcal{S}(M)} \text{ compact}\right)\right].$$

Equivalently, $\mathcal{S}$ is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, the sequence $(\mathcal{S}x_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}$.

**Remark 1.3** A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be completely continuous if and only if $\mathcal{S}$ is continuous and compact.

### 1.3.1 Arzelà-Ascoli theorem

Let $\mathbb{X}$ be a compact subset of a normed vector space over $\mathbb{F}$ and let $\mathcal{C}(\mathbb{X})$ be the normed vector space of real valued continuous functions on $\mathbb{X}$ with the *sup*-norm

$$\|f\|_{\infty} = \sup_{x\in\mathbb{X}} |f(x)| .$$

Let $\mathcal{F}$ be a collection of functions in $\mathcal{C}(\mathbb{X})$.

**Definition 1.9** [2] The collection $\mathcal{F}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and $x, y \in \mathbb{X}$ with $\|x - y\|_{\mathbb{X}} < \delta$ we have $|f(x) - f(y)| < \varepsilon$, i.e.,

$$\forall \varepsilon > 0, \ \forall x \in \mathbb{X}, \exists \delta > 0, \ \forall y \in \mathbb{X}, [\|x - y\|_{\mathbb{X}} < \delta] \Longrightarrow [\forall f \in \mathcal{F}, \ \ |f(x) - f(y)| < \varepsilon].$$

**Definition 1.10** [2] The collection $\mathcal{F}$ is said to be uniformly bounded if there is an $M \geq 0$ so that $\|f\|_{\infty} = \sup_{x \in \mathbb{X}} |f(x)| \leq M$ for all $f \in \mathcal{F}$, i.e.,

$$\exists M \geq 0 : \|f\|_{\infty} = \sup_{x \in \mathbb{X}} |f(x)| \leq M, \ \forall f \in \mathcal{F}.$$

**Theorem 1.4** *[2] If $\mathcal{F}$ is a collection of uniformly bounded and equicontinuous functions in $\mathcal{C}(\mathbb{X})$, then $\mathcal{F}$ is relatively compact in $\mathcal{C}(\mathbb{X})$.*

## 1.4 Fixed point theorems in cones

In the following, we list some results for ready reference. To be more precise, we introduce the concepts of an ordered cone and a concave function as well as three fixed point theorem in cones that will be used to show the existence of at least one and at least three positive periodic solutions of the studied delay functional differential equations.

### Cones

**Definition 1.11** [20] Let $\mathbb{X}$ be a real Banach space. A closed convex and nonempty set $\mathbb{K} \subset \mathbb{X}$ is called a (positive) cone if the following conditions are satisfied:
$(i)$ If $x \in \mathbb{K}$, then $\lambda x \in \mathbb{K}$ for $\lambda \geq 0$.
$(ii)$ If $x \in \mathbb{K}$ and $-x \in \mathbb{K}$, then $x = 0_{\mathbb{X}}$.

**Remark 1.4** The condition $(i)$ in the definition (1.11) implies that $0_{\mathbb{X}} \in \mathbb{K}$.

**Definition 1.12** A nonempty subset $\mathbb{K}$ of a real Banach space $\mathbb{X}$ is called a cone if

$$x \in \mathbb{K} \text{ and } \lambda > 0 \text{ implies that } \lambda x \in \mathbb{K}.$$

**Definition 1.13** [20] Let $\mathbb{K}$ be a cone on the Banach space $\mathbb{X}$. Then, we define on $\mathbb{X}$ a partial order relation $\leq$ by

$$\forall x, y \in \mathbb{X} : x \leq y \Longleftrightarrow y - x \in \mathbb{K}.$$

**Definition 1.14** A *Banach space* $\mathbb{X}$ is called ordered if it contains a cone $\mathbb{K}$.

**Definition 1.15** [20] A continuous map $\psi : \mathbb{K} \longrightarrow [0, \infty)$ is said to be a continuous concave positive functional on $\mathbb{K}$ if

$$\psi(\mu x + (1-\mu)y) \geq \mu\psi(x) + (1-\mu)\psi(y), \text{ for } \mu \in [0,1] \text{ and } x, y \in \mathbb{K}.$$

**Example 1.1** Let $\psi : \mathbb{K} \longrightarrow [0, \infty)$ and $x_0 \in \overset{\circ}{\mathbb{K}}$ ( $\overset{\circ}{\mathbb{K}}$ is the interior of $\mathbb{K}$). if

$$\psi(x) = \max\{\gamma : \gamma x_0 \leq x\},$$

then $\psi$ is a concave positive functional on $\mathbb{K}$.

Consider the following subsets:

$$\mathbb{K}(\psi, b, c) = \{x \in \mathbb{K}, \ \psi(x) \geq b, \|x\| \leq c\}, \ b, c \in ]0, +\infty[,$$

where $\psi$ is a continuous concave positive functional on the cone $\mathbb{K}$. Also, for $0 < a \leq +\infty$, we define a set $\mathbb{K}_a$ as follows:

$$\mathbb{K}_a = \{x \in \mathbb{K} : \|x\| < a\}, \text{ if } 0 < a < +\infty \text{ and } \mathbb{K}_\infty = \mathbb{K}.$$

Therefore

$$\overline{\mathbb{K}}_a = \{x \in \mathbb{K} : \|x\| \leq a\}, \text{ if } 0 < c < +\infty \text{ and } \mathbb{K}_\infty = \mathbb{K}.$$

### 1.4.1 Fixed point theorem in cones

**Lemma 1.1** *[7] Let $\mathbb{X} = (\mathbb{X}, \|.\|)$ be a Banach space, let $\mathbb{K}$ be a cone in $\mathbb{X}$, and let $r_1$ and $r_2$ be constant such that $0 < r_1 < r_2$. Suppose that $\phi : \overline{\Omega}_{r_2} \cap \mathbb{K} \to \mathbb{K}$, where*

$$\Omega_{r_2} = \{x : x \in \mathbb{X}, \|x\| < r_2\},$$

*is a completely continuous operator satisfying the following conditions:*

(1) *$x \neq \lambda\phi x$ for $x \in \mathbb{K} \cap \Omega_{r_1}$ and $\lambda \in [0,1]$.*

(2) *there exists $\psi \in \mathbb{K} \backslash \{0\}$ such that $x \neq \phi x + \eta\psi$ for $x \in \mathbb{K} \cap \partial\Omega_{r_2}$ and $\eta \geq 0$.*

*Then $\phi$ has a fixed point in $\mathbb{K} \cap \{x \in \mathbb{X} : r_1 < \|x\| < r_2\}$.*

**Lemma 1.2** *[14] The operator $\phi$ has a fixed point in $\mathbb{K} \cap \{x \in \mathbb{X} : r_1 < \|x\| < r_2\}$ if conditions* (1) *and* (2) *in Lemma 1.1 are replaced by the following conditions:*

(1) *$x \neq \lambda\phi x$ for $x \in \mathbb{K} \cap \Omega_{r_2}$ and $\lambda \in [0, 1]$.*

(2) *there exists $\psi \in \mathbb{K}\backslash \{0\}$ such that $x \neq \phi x + \eta\psi$ for $x \in \mathbb{K} \cap \partial\Omega_{r_1}$ and $\eta \geq 0$.*

### 1.4.2 *Leggett-Williams fixed point theorem*

**Theorem 1.5** ***(Leggett-Williams)*** *[17] Let $(\mathbb{X}, \|.\|)$ be a Banach space and $\mathbb{K} \subset \mathbb{X}$ be a cone, and $c_4 > 0$ be a constant. Suppose there exists a continuous concave nonnegative function $\psi$ on $\mathbb{K}$ with $\psi(x) \leq \|x\|$ for $x \in \overline{\mathbb{K}}_{c_4}$ and let $A : \overline{\mathbb{K}}_{c_4} \longrightarrow \overline{\mathbb{K}}_{c_4}$ be a continuous compact map. Assume that there are numbers $c_1$, $c_2$ and $c_3$ with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(1) *$\{x \in \mathbb{K}(\psi, c_{2,}c_3) : \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ for all $x \in \mathbb{K}(\psi, c_2, c_3)$;*

(2) *$\|Ax\| < c_1$ for all $x \in \overline{\mathbb{K}}_{c_1}$;*

(3) *$\psi(Ax) > c_2$ for all $x \in \mathbb{K}(\psi, c_2, c_4)$ with $\|Ax\| > c_3$.*

*Then $A$ has at least three fixed points $x_1$, $x_2$ and $x_3$ in $\overline{\mathbb{K}}_{c_4}$. Furthermore, we have $x_1 \in \overline{\mathbb{K}}_{c_1}$, $x_2 \in \{x \in \mathbb{K}(\psi, c_2, c_4) : \psi(x) > c_2\}$ and $x_3 \in \overline{\mathbb{K}}_{c_4}\backslash \left\{\mathbb{K}(\psi, c_2, c_4) \cup \overline{\mathbb{K}}_{c_1}\right\}$.*

# CHAPTER 2

# Positive periodic solutions for a first order differential equation with delays depending on time and state

## Contents



In this chapter, we are interested in investigating two classes of first order differential equations with time and state-dependent delays by means of two fixed point theorems in cones.

In this chapter, we study the following two time and state-dependent delay differential equations:

$$x'(t) = -a(t, x(t))\, x(t) + f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))), \qquad (2.1)$$

and

$$x'(t) = a(t, x(t))\, x(t) - f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))), \qquad (2.2)$$

where $\tau_i, a \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$, $f \in C(\mathbb{R} \times [\mathbb{R}^+]^m, \mathbb{R}^+)$,

$$f(t + T, x_1, ..., x_m) = f(t, x_1, ..., x_m) \text{ and } \tau_i(t + T, x) = \tau_i(t, x),$$

for any $x \in \mathbb{R}^+$, $t \in \mathbb{R}$, $\mathbb{R}^+ = [0, \infty)$, $i = 1, 2, ..., m$, and $T > 0$ is a constant and also

$$a(t + T, x) = a(t, x), \;\; a_1(t) \leq a(t, x) \leq a_2(t),$$

for any $(t, x) \in \mathbb{R} \times \mathbb{R}^+$, where $a_1$, $a_2$ are nonnegative $T-$periodic continuous functions on $\mathbb{R}$ and $\int_0^T a_1(s)\, ds > 0$.

**Remark 2.1** In what follows, we only discuss the existence of positive periodic solutions of equation (2.1), though similar results can be obtained for equation (2.2).

For the sake of simplicity, we will adopt the following notations:

$$M_1 = \inf_{0 \leq t \leq s \leq T} e^{\int_t^s a_1(\theta) d\theta}, \;\; M_2 = \sup_{0 \leq t \leq s \leq T} e^{\int_t^s a_2(\theta) d\theta}$$

$$K_1 = e^{\int_0^T a_1(\theta) d\theta}, \;\; K_2 = e^{\int_0^T a_2(\theta) d\theta}, \;\; \sigma = \frac{M_1(K_1 - 1)}{M_2(K_2 - 1)},$$

$$\gamma = \frac{K_2 - 1}{K_1 - 1}.$$

Throughout this chapter, we impose the following hypotheses which will be used in the sequel:

$(H_1)$ For

$$|u| = \max\{u_1, ..., u_m\},$$

we have

$$\lim \inf_{|u| \to 0^+} \frac{f(t, u_1, ..., u_m)}{|u|} > \gamma a_1(t),$$

and

$$\lim_{|u|\to+\infty} \inf \frac{f(t,u_1,...,u_m)}{|u|} > \gamma a_1(t),$$

uniformly for $t \in \mathbb{R}$.

$(H_2)$

$$\lim_{|u|\to0^+} \sup \frac{f(t,u_1,...,u_m)}{|u|} < \frac{a_2(t)}{\gamma},$$

and

$$\lim_{|u|\to+\infty} \sup \frac{f(t,u_1,...,u_m)}{|u|} < \frac{a_2(t)}{\gamma},$$

uniformly for $t \in \mathbb{R}$.

$(H_3)$ there exists a number $p > 0$ such that the inequality $\sigma p \leq |u| \leq p$ yields

$$f(t,u_1,...,u_m) < a_2(t)\frac{p}{\gamma},$$

for $t \in [0,T]$.

$(H_4)$ there exists a $p > 0$ such that the inequality $\sigma p \leq |u| \leq p$ yields

$$f(t,u_1,...,u_m) > a_1(t)p\gamma,$$

for $t \in [0,T]$.

**Remark 2.2** It is evident that

$$\sigma \in (0,1] \text{ and } \gamma \geq 1.$$

Let

$$\mathbb{X} = \{x(t) \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t),\ t \in \mathbb{R}\},$$

endowed with the norm

$$\|x\| = \max_{x\in[0,T]} |x(t)|.$$

Then $(\mathbb{X}, \|\cdot\|)$ is a Banach space. We also define a cone $\Sigma$ in $\mathbb{X}$ by

$$\Sigma = \{x : x \in \mathbb{X},\ x(t) \geq \sigma \|x\|\}.$$

## 2.1 Conversion of the delay differential equation (2.1) into an integral equation

The next lemma shows the equivalence between equation (2.1) and an integral equation.

**Lemma 2.1** *$x \in C^1(\mathbb{R}, \mathbb{R}) \cap \mathbb{X}$ is a $T$-periodic solution of equation (2.1) if and only if $x \in \mathbb{X}$ is a $T$-periodic solution of the following integral equation:*

$$x(t) = \int_t^{t+T} G(t,s) f(s, x(s - \tau_1(s, x(s))), ..., x(s - \tau_m(s, x(s))))\, ds, \tag{2.3}$$

*where*

$$G(t,s) = \frac{\exp\left(\int_t^s a(\theta, x(\theta))\, d\theta\right)}{\exp\left(\int_0^T a(\theta, x(\theta))\, d\theta\right) - 1}. \tag{2.4}$$

**Proof.** Let $x \in C^1(\mathbb{R}, \mathbb{R}) \cap \mathbb{X}$ be a solution of equation (2.1). By multiplying the two members of (2.1) by $e^{\int_0^t a(\theta, x(\theta))ds}$ and integrating from $t$ to $t+T$ we obtain

$$(x'(t) + a(t, x(t))\, x(t))\, e^{\int_0^t a(\theta, x(\theta))d\theta} = \frac{d}{dt}\left(x(t)\, e^{\int_0^t a(\theta, x(\theta))d\theta}\right)$$

$$= (f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))))\, e^{\int_0^t a(\theta, x(\theta))d\theta}.$$

So

$$\int_t^{t+T} \left(x(s)\, e^{\int_0^s a(\theta, x(\theta))d\theta}\right)' ds$$

$$= \int_t^{t+T} (f(s, x(s - \tau_1(t, x(s))), ..., x(s - \tau_m(s, x(s)))))\, e^{\int_0^s a(\theta, x(\theta))d\theta} ds.$$

From the periodicity properties we get

$$x(t+T)\, e^{\int_0^{t+T} a(\theta, x(\theta))d\theta} - x(t)\, e^{\int_0^t a(\theta, x(\theta))d\theta} = x(t)\, e^{\int_0^t a(\theta, x(\theta))d\theta}\left(e^{\int_t^{t+T} a(\theta, x(\theta))d\theta} - 1\right)$$

$$= \int_t^{t+T} (f(s, x(s - \tau_1(t, x(s))), ..., x(s - \tau_m(s, x(s)))))\, e^{\int_0^s a(\theta, x(\theta))d\theta} ds\ .$$

This gives

$$x(t) = \int_t^{t+T} \left(f\left(s, x\left(s-\tau_1\left(t, x\left(s\right)\right)\right), ..., x\left(s-\tau_m\left(s, x\left(s\right)\right)\right)\right)\right) \times \frac{e^{\int_0^s a(\theta,x(\theta))d\theta} e^{-\int_0^t a(\theta,x(\theta))d\theta}}{e^{\int_t^{t+T} a(\theta,x(\theta))d\theta} - 1} ds.$$

We have

$$\int_t^{t+T} a\left(\theta, x\left(\theta\right)\right) d\theta = \int_t^{T} a\left(\theta, x\left(\theta\right)\right) d\theta + \int_T^{t+T} a\left(\theta, x\left(\theta\right)\right) d\theta.$$

If we use the change of variable $v = \theta - T$ we get

$$\begin{aligned} \int_T^{t+T} a\left(\theta, x\left(\theta\right)\right) d\theta &= \int_0^t a\left(v+T, x\left(v+T\right)\right) dv \\ &= \int_0^t a\left(v, x\left(v\right)\right) dv = \int_0^t a\left(\theta, x\left(\theta\right)\right) d\theta. \end{aligned}$$

So

$$\int_t^{t+T} a\left(\theta, x\left(\theta\right)\right) d\theta = \int_t^{T} a\left(\theta, x\left(\theta\right)\right) d\theta + \int_0^{t} a\left(\theta, x\left(\theta\right)\right) d\theta = \int_0^{T} a\left(\theta, x\left(\theta\right)\right) d\theta.$$

Since

$$e^{\int_t^{t+T} a(\theta,x(\theta))d\theta} = e^{\int_0^{T} a(\theta,x(\theta))d\theta}, \tag{2.5}$$

then

$$\begin{aligned} x(t) &= \int_t^{t+T} f\left(s, x\left(s-\tau_1\left(t, x\left(s\right)\right)\right), ..., x\left(s-\tau_m\left(s, x\left(s\right)\right)\right)\right) \\ &\quad \times \frac{e^{\int_t^s a(\theta,x(\theta))d\theta}}{e^{\int_0^T a(\theta,x(\theta))d\theta} -1} ds \\ &= \int_t^{t+T} G\left(t,s\right) f\left(s, x\left(s-\tau_1\left(t, x\left(s\right)\right)\right), ..., x\left(s-\tau_m\left(s, x\left(s\right)\right)\right)\right) ds. \end{aligned}$$

Conversely, let $x \in \mathbb{X}$ be a solution of the integral equation (2.3). The derivation of the integral equation gives

$$\begin{aligned} x'(t) &= f\left(t+T, x\left(t+T-\tau_1\left(t, x\left(t+T\right)\right)\right), ..., x\left(s-\tau_m\left(s, x\left(t+T\right)\right)\right)\right) \\ &\quad \times G\left(t, t+T\right) \\ &\quad - G\left(t,t\right) f\left(t, x\left(t-\tau_1\left(t, x\left(t\right)\right)\right), ..., x\left(s-\tau_m\left(s, x\left(t\right)\right)\right)\right) \\ &\quad + \int_t^{t+T} \frac{d}{dt} G\left(t,s\right) f\left(s, x\left(s-\tau_1\left(t, x\left(s\right)\right)\right), ..., x\left(s-\tau_m\left(s, x\left(s\right)\right)\right)\right) ds. \end{aligned}$$

From the periodicity properties we get

$$\begin{aligned} x'(t) &= (G(t,t+T)-G(t,t)) \\ &\quad \times f(t,x(t-\tau_1(t,x(t))),...,x(s-\tau_m(s,x(t)))) \\ &\quad + \int_t^{t+T} f(s,x(s-\tau_1(t,x(s))),...,x(s-\tau_m(s,x(s))))\frac{d}{dt}G(t,s)\,ds. \end{aligned}$$

Since

$$\begin{aligned} & G(t,t+T)-G(t,t) \\ &= \frac{\exp\left(\int_t^{t+T} a(\theta,x(\theta))\,d\theta\right)}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} - \frac{\exp\left(\int_t^{t} a(\theta,x(\theta))\,d\theta\right)}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} \\ &= \frac{\exp\left(\int_0^{T} a(\theta,x(\theta))\,d\theta\right)}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} - \frac{1}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} \\ &= \frac{\exp\left(\int_0^{T} a(\theta,x(\theta))\,d\theta\right)-1}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} = 1, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}G(t,s) &= \frac{d}{dt}\frac{\exp\left(\int_t^{s} a(\theta,x(\theta))\,d\theta\right)}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} \\ &= -a(t,x(t))\frac{\exp\left(\int_t^{s} a(\theta,x(\theta))\,d\theta\right)}{\exp\left(\int_0^T a(\theta,x(\theta))\,d\theta\right)-1} \\ &= -a(t,x(t))\,G(t,s). \end{aligned}$$

then

$$\begin{aligned} & x'(t) = f(t,x(t-\tau_1(t,x(t))),...,x(s-\tau_m(s,x(t)))) \\ & -a(t,x(t))\int_t^{t+T} G(t,s)\,f(s,x(s-\tau_1(t,x(s))),...,x(s-\tau_m(s,x(s))))\,ds. \end{aligned}$$

From the integral equation (2.3) we obtain

$$x'(t) = -a(t,x(t))\,x(t) + f(t,x(t-\tau_1(t,x(t))),...,x(t-\tau_m(t,x(t)))).$$

Thus we complete the proof. ■

Now, in view of Lemma (2.1) we define an operator $S$ on $\mathbb{X}$ as follows:

$$(Sx)(t) = \int_t^{t+T} G(t,s)\,f(s,x(s-\tau_1(s,x(s))),...,x(s-\tau_m(s,x(s))))\,ds,$$

for all $x \in \mathbb{X}$ and $t \in \mathbb{R}$.

**Remark 2.3** We have

$$\begin{aligned}
&(Sx)(t+T) \\
&= \int_{t+T}^{t+2T} G(t+T,s)\, f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\, ds \\
&= \int_{t}^{t+T} G(t,s)\, f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\, ds \\
&= (Sx)(t) \text{ for } t \in \mathbb{R},\ x \in \mathbb{X}.
\end{aligned}$$

Hence $S : \mathbb{X} \longrightarrow \mathbb{X}$ and $x$ is a periodic solution of equation (2.1) if and only if $x$ is a fixed point of the operator $S$ in $\mathbb{X}$.

**Lemma 2.2** *Operator $S$ maps $\Sigma$ into itself.*

**Proof.** For any $x \in \Sigma$, $t \in \mathbb{R}$, $s \in [t, t+T]$, we have

$$\frac{M_1}{K_2 - 1} \leq \frac{e^{\int_t^s a_1(\theta)d\theta}}{e^{\int_0^T a_2(\theta)d\theta} - 1} \leq G(t,s) \leq \frac{e^{\int_t^s a_2(\theta)d\theta}}{e^{\int_0^T a_1(\theta)d\theta} - 1} \leq \frac{M_2}{K_1 - 1}.$$

Hence, for any $x \in \mathbb{X}$, we have

$$\|Sx\| \leq \frac{M_2}{K_1 - 1} \int_0^T f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\, ds,$$

and

$$\begin{aligned}
(Sx)(t) &\geq \frac{M_1}{K_2 - 1} \int_0^T f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\, ds \\
&\geq \frac{M_1 (K_1 - 1)}{M_2 (K_2 - 1)} \|Sx\| = \sigma \|Sx\|.
\end{aligned}$$

Therefore $Sx \in \Sigma$ for any $x \in \mathbb{X}$. This completes the proof. ∎

## 2.2 Existence of positive periodic solutions

**Lemma 2.3** *Let $\Omega$ be an arbitrary open bounded set in $\Sigma$. $S : \overline{\Omega} \cap \Sigma \rightarrow \Sigma$ is completely continuous.*

**Proof.**

**Step 1:** We first show that $S$ is continuous. Let $\Omega$ be an arbitrary open bounded set in $\Sigma$. There exists a number $M > 0$ such that $\|x\| \leq M$ for any $x \in \overline{\Omega}$.

$f(t, u_1, ..., u_m)$ is uniformly continuous on $[0, T] \times [0, M]^m$ due to the continuity of $f(t, u_1, ..., u_m)$ and periodicity of $f$ respect to $t$. Therefore for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(t, u_1, ..., u_m) - f(t, v_1, ..., v_m)| < \epsilon,$$

for any $t \in \mathbb{R}$ and $u_i, v_i \in [0, M]$ such that $|u_i - v_i| < \delta$, $i = 1, 2, ..., m$.

Choose an arbitrary points $x_0 \in \Omega$. Since $x_0(t)$ is continuous and periodic on $\mathbb{R}$, we have that $x_0(t)$ is uniformly continuous. Then there exists $\delta_1 > 0$ (choose $\delta_1 < \delta$) such that

$$|x_0(t_1) - x_0(t_2)| < \frac{\delta}{4}. \tag{2.6}$$

for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_1$.

With a similar deduction we have that $\tau_i(t, y)$, $i = 1, 2, ..., m$, are uniformly continuous on $[0, T] \times [0, M]$ due to the continuity respect to $t$. Hence there exists a number $\delta_2 > 0 \left( \text{choose } \delta_2 < \frac{\delta_1}{2} \right)$ such that

$$|\tau_i(t, u_1) - \tau_i(t, u_2)| < \delta_1, \tag{2.7}$$

for $t \in \mathbb{R}$ and $u_1, u_2 \in [0, M]$ with $|u_1 - u_2| < \delta_2, i = 1, 2, ..., m$.

Hence in view of (2.7) we have

$$|\tau_i(t, x_0(t)) - \tau_i(t, y(t))| < \delta_1 \quad \text{for } t \in \mathbb{R}, \ i = 1, 2, ..., m, \tag{2.8}$$

for $\|x_0 - y\| < \delta_2$ and $y \in \Omega$.

It follows from (2.6) − (2.8) that

$$\begin{aligned}
&|x_0(t - \tau_i(t, x_0(t))) - y(t - \tau_i(t, y(t)))| \\
&\leq |x_0(t - \tau_i(t, x_0(t))) - x_0(t - \tau_i(t, y(t)))| \\
&+ |x_0(t - \tau_i(t, y(t))) - y(t - \tau_i(t, y(t)))| \\
&\leq \frac{\delta}{4} + \delta_2 < \frac{\delta}{4} + \frac{\delta_2}{2} < \delta, \quad i = 1, 2, ..., m.
\end{aligned}$$

for $t \in \mathbb{R}$, $\|x_0 - y\| < \delta_2$ and $y \in \Omega$. Therefore, for any $y \in \Omega$, if $\|x_0 - y\| < \delta_2$ then

$$\begin{aligned}
&\|f(t, x_0(t - \tau_1(t, x_0(t))), ..., x_0(t - \tau_m(t, x_0(t)))) \\
&- f(t, x_0(t - \tau_1(t, x_0(t))), ..., x_0(t - \tau_m(t, x_0(t))))\| < \epsilon,
\end{aligned}$$

for any $t \in \mathbb{R}$.

Hence, if $t \in \mathbb{R}$, $y \in \Omega$ and $\|x_0 - y\| < \delta_2$, we have

$$\begin{aligned}
&|(Sx_0)(t) - (Sy)(t)| \\
&= \left| \int_t^{t+T} G(t,s) f(s, x_0(s - \tau_1(s, x_0(s))), ..., x(s - \tau_m(s, x_0(s)))) \right. \\
&\left. - f(s, y(s - \tau_1(s, y(s))), ..., y(s - \tau_m(s, y(s))))) ds \right| \\
&\leq \frac{M_2 T}{K_1 - 1}\epsilon.
\end{aligned}$$

Therefore,

$$\|Sx_0 - Sy\| \leq \frac{M_2 T}{K_1 - 1}\epsilon,$$

i.e., the operator $S$ is continuous at $x_0$. Thus $S$ is continuous in $\overline{\Omega}$ due to the arbitrariness of $x_0$ in $\overline{\Omega}$.

**Step 02:** Next we show that $\{Sx : x \in \overline{\Omega}\}$ is a family of uniformly bounded and equicontinuous functions on $[0, T)$. Since $f(t, u_1, ..., u_m)$ is bounded on $\mathbb{R} \times [0, M]^m$, there exists a number $M_3 > 0$ such that

$$\|f(t, u_1, ..., u_m)\| \leq M_3 \text{ for } t \in \mathbb{R}, u_i \in [0, M], i = 1, 2, ...m. \tag{2.9}$$

For any $x \in \overline{\Omega}$, we have $\|x\| \leq M$ and

$$\begin{aligned}
|(Sx)(t)| &= \left| \int_t^{t+T} G(t,s) f(s, x(s - \tau_1(s, x(s))), ..., x(s - \tau_m(s, x(s)))) ds \right| \\
&\leq \frac{M_2}{K_1 - 1} T M_3 \text{ for } t \in \mathbb{R}.
\end{aligned}$$

Hence

$$\|Sx\| \leq \frac{M_2}{K_1 - 1} T M_3. \tag{2.10}$$

Finally, for any $t \in \mathbb{R}$, we have

$$\frac{d(Sx)(t)}{dt} = a(t, x(t))(Sx)(t) + f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))). \tag{2.11}$$

In view of $(2.9) - (2.11)$, we obtain that

$$\left| \frac{d(Sx)(t)}{dt} \right| \leq a_2^* \frac{M_2 T M_3}{K_1 - 1} + M_3 =: M_4 \tag{2.12}$$

where

$$a_2^* = \max_{t \in [0,T]} |a_2(t)|.$$

Since $M_1$, $M_2$ are independent of $x$, we obtain that $\left\{Tx : x \in \overline{\Omega}\right\}$ is a family of uniformly bounded and equicontinuous functions on $[0,T]$. By theorem of Ascoli-Arzela,the operator $S$ is completely continuous. This proves the lemma. ■

**Theorem 2.1** *Assume that hypotheses* $(H_0)$, $(H_1)$ *and* $(H_3)$ *are true .Then equation* $(2.1)$ *has at least two positive* $T$*-periodic solutions* $x_1$, $x_2$ *such that*

$$0 < \|x_1\| < p < \|x_2\| .$$

**Proof.** According to the first inequality of $(H_1)$, i.e.,

$$\lim_{|u|\to 0^+}\inf \frac{f\left(t, u_1, ..., u_m\right)}{|u|} > \gamma a_1\left(t\right)$$

uniformly for $t \in \mathbb{R}$, we can find a sufficiently small number $\epsilon > 0$ and a number $r_1$ with

$$0 < r_1 < p$$

such that

$$f\left(t, u_1, ..., u_m\right) > \gamma a_1\left(t\right)\left(1+\epsilon\right)|u| \text{ for } 0 < |u| < r_1, \ t \in [0,T] . \tag{2.13}$$

Hence for $x \in \Sigma$ and $\|x\| = r_1$, we have

$$r_1 \geq x\left(t\right) \geq \sigma\|x\| = \sigma r_1 > 0.$$

Put $\psi \equiv 1$. We now prove that

$$x \neq Sx + \eta\psi \text{ for } x \in \Sigma \cap \partial\Omega_{r_1} \text{ and } \eta \geq 0, \tag{2.14}$$

where

$$\Omega_{r_1} = \left\{x \in \mathbb{X} : \|x\| < r_1\right\}.$$

If not, then there exists $x_0 \in \Sigma \cap \partial\Omega_{r_1}$ and $\eta_0 \geq 0$ such that

$$x_0 = Sx_0 + \eta_0. \tag{2.15}$$

Let

$$\alpha = \min_{t\in[0,T]} x_0\left(t\right),$$

then $\alpha > 0$. So, for $t \in \mathbb{R}$, from $(2.13) - (2.15)$, we have

$$\begin{aligned}
& x_0(t) = (Sx_0)(t) + \eta_0 \\
& = \int_t^{t+T} G(t,s) f(s, x_0(s - \tau_1(s, x_0(s))), ..., x(s - \tau_m(s, x_0(s)))) \, ds + \eta_0 \\
& > \int_t^{t+T} G(t,s) a_1(s) (1+\epsilon) \gamma \max_{1 \le i \le m} |x_0(s - \tau_i(s, x(s)))| \, ds \\
& \ge (1+\varepsilon) \alpha\gamma \int_t^{t+T} \frac{e^{\int_t^s a_1(\theta) d\theta}}{e^{\int_0^T a(\theta, x_0(\theta)) d\theta} - 1} a_1(s) \, ds \\
& \ge (1+\varepsilon) \alpha\gamma \frac{e^{\int_0^T a_1(\theta) d\theta} - 1}{e^{\int_0^T a_2(\theta) d\theta} - 1} \\
& = (1+\varepsilon) \alpha,
\end{aligned}$$

which yields

$$\alpha > (1+\varepsilon) \alpha.$$

It is a contradiction, therefore $(2.14)$ is valid.

Next, by using the inequality in $(H_3)$ we prove that

$$x \ne \lambda Sx \text{ for any } x \in \Sigma \cap \partial\Omega_p, \ \lambda \in [0,1] \tag{2.16}$$

where

$$\Omega_p = \{x \in \mathbb{X} : \|x\| < p\}.$$

If not, then there exist $x_0 \in K \cap \partial\Omega_p$ and $\lambda_0 \in [0,1]$ such that

$$x_0 = \lambda_0 S x_0. \tag{2.17}$$

Clearly, $\lambda_0 \ne 0$. If not, we would have $x_0 \equiv 0$, which contradicts that $x_0 \in \Sigma \cap \partial\Omega_P$. Thus $\|x_0\| = p$ and $\sigma p \le x_0(t) \le p$ for $t \in \mathbb{R}$. By condition $(H_3)$, we have

$$f(t, x_0(t - \tau_1(t; x_0))), ..., x_0(t - \tau_m(t, x_0(t)))) < \frac{a_2(t)\, p}{\gamma} \text{ for } t \in \mathbb{R}. \tag{2.18}$$

Then from (2.17) and (2.18), for $t \in \mathbb{R}$, we have

$$
\begin{aligned}
x_0(t) &= \lambda_0 (Sx_0)(t) \\
&= \lambda_0 \int_t^{t+T} G(t,s) f(s, x_0(s-\tau_1(s,x_0(s))), ..., x(s-\tau_m(s,x_0(s))))\, ds \\
&< \int_t^{t+T} G(t,s) a_2(s) \frac{p}{\gamma} ds \leq \frac{p}{\gamma} \int_t^{t+T} \frac{e^{\int_t^s a_2(\theta)d\theta}}{e^{\int_0^T a_1(\theta)d\theta} - 1} a_2(s)\, ds. \\
&= \frac{p}{\gamma} \frac{e^{\int_t^s a_2(\theta)d\theta}}{e^{\int_0^T a_1(\theta)d\theta} - 1} = \frac{p}{\gamma}\gamma = p,
\end{aligned}
$$

which yields $\|x_0\| = p < p$. we arrive at a contradiction. Therefore (2.16) is valid. In view of (2.14), (2.16) and Lemma 1.2 we have that $S$ has a fixed point

$$x_1 \in \Sigma \cap \{x \mid r_1 < \|x\| < p\},$$

and

$$x_1(t) \geq \sigma r_1 > 0.$$

Therefore $x_1(t)$ is a periodic positive solution of equation (2.1).

By the second inequality in $(H_1)$, i.e.,

$$\lim_{|u| \to +\infty} \inf \frac{f(t, u_1, ..., u_m)}{|u|} > \gamma a_1(t),$$

uniformly for $t \in \mathbb{R}$, we can find a sufficiently small number $\epsilon > 0$ and a number $r_2 > p$ such that

$$f(t, u_1, ..., u_m) \geq a_1(t)(1+\epsilon)|u|, \tag{2.19}$$

for $|u| \geq r_2$.

Set $\psi \equiv 1$. we show that

$$x \neq Sx + \eta\psi, \tag{2.20}$$

for $x \in \Sigma \cap \partial\Omega_{r_2}$ and $\eta \geq 0$, where

$$\Omega_{r_2} = \{x \in \mathbb{X} : \|x\| < r_2\}.$$

If not, then there exist $x_0 \in \Sigma \cap \partial\Omega_{r_2}$ and $\eta_0 \geq 0$ such that

$$x_0 = \phi x_0 + \eta_0 \psi. \tag{2.21}$$

Let

$$\beta = \min_{0 \leq t \leq T} x_0 (t),$$

then $\beta > 0$. So, for $t \in \mathbb{R}$, from (2.19) and (2.21), we have

$$\begin{aligned}
& x_0 (t) = (Sx_0) (t) + \eta_0 \\
& = \int_t^{t+T} G (t, s) f (s, x_0 (s - \tau_1 (s, x_0 (s))) , ..., x (s - \tau_m (s, x_0 (s)))) ds + \eta_0 \\
& \geq (1 + \varepsilon) \beta\gamma \int_t^{t+T} \frac{e^{\int_t^s a_1(\theta)d\theta}}{e^{\int_0^T a_2(\theta)d\theta} - 1} a_1 (s) ds \\
& = (1 + \epsilon) \beta \frac{\gamma}{\gamma} = (1 + \epsilon) \beta.
\end{aligned}$$

Hence $\beta \geq (1 + \epsilon) \beta$, which yields a contradiction. Thus (2.20) is valid.

In view of (2.16), (2.20) and Lemma 1.1, we obtain that $S$ has a fixed point

$$x_2 \in \Sigma \cap \{x : p < \|x\| < r_2\}$$

and

$$x_2 (t) \geq \sigma p > 0.$$

Thus, $x_2 (t)$ is a positive $T-$periodic solution of equation (2.1). Therefore equation (2.1) has at least tow positive periodic solutions. The proof is completed. ■

**Corollary 2.1** *The conclusion of Theorem* 2.1, *remains valid if conditions* $(H_0)$, $(H_3)$ *are true and* $(H_1)$ *is replaced by the following conditions:*

$(H_1^*)$

$$\lim_{|u| \to 0^+} \min_{t \in [0,T]} \frac{f (t, u_1, ..., u_m)}{|u|} = +\infty,$$

*and*

$$\lim_{|u| \to +\infty} \min_{t \in [0,T]} \frac{f (t, u_1, ..., u_m)}{|u|} = +\infty,$$

*uniformly for* $t \in \mathbb{R}$.

**Theorem 2.2** *Suppose that conditions* $(H_0)$, $(H_2)$ *and* $(H_4)$ *are satisfied. Then equation* $(2.1)$ *has at least two periodic positive solutions* $x_1$ *and* $x_2$ *such that*

$$0 < \|x_1\| < p < \|x_2\| .$$

**Proof.** By the first inequality in $(H_2)$, that is

$$\lim_{|u|\to 0^+} \sup \frac{f(t, u_1, ..., u_m)}{|u|} < \frac{a_2(t)}{\gamma},$$

uniformly for $t \in \mathbb{R}$, we can obtain a sufficient small number $\epsilon > 0$ and number $r_1$ with $0 < r_1 < p$ such that

$$f(t, u_1, ..., u_m) < \frac{a_2(t)}{\gamma}(1-\epsilon)|u|, \tag{2.22}$$

for $|u| \le r_1$, $t \in [0, T]$. Hence, $r_1 \ge x(t) \ge \sigma r_1$ for $x \in \Sigma \cap \partial\Omega_{r_1}$, where

$$\Omega_{r_1} = \{x \in \mathbb{X} : \|x\| < r_1\}.$$

Next we show that

$$x \ne \lambda S x, \tag{2.23}$$

for $x \in \Sigma \cap \partial\Omega_{r_1}$, and $\lambda \in [0, 1]$.

If not, there exist $x_0 \in \Sigma \cap \partial\Omega_{r_1}$ and $\lambda_0 \in [0, 1]$ such that

$$x_0 = \lambda_0 S x_0. \tag{2.24}$$

Clearly, $\lambda_0 \ne 0$. If not, we would have $x_0 \equiv 0$, which contradicts that $x_0 \in \Sigma \cap \partial\Omega_{r_1}$.

Let

$$\alpha = \max_{t\in[0,T]} x_0(t),$$

then $\alpha > 0$. Since $\sigma r_1 \le x_0(t) \le r_1$, then for $t \in \mathbb{R}$, from (2.22) and (2.24), we have

$$\begin{aligned}
x_0(t) &= \lambda_0 (Sx_0)(t) \\
&= \lambda_0 \int_t^{t+T} G(t,s) f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s)))) \\
&< \int_t^{t+T} G(t,s) \frac{a_2(s)}{\gamma}(1-\epsilon) \max_{1\le i\le m} |x_0(s-\tau_i(s,x_0(s)))| \, ds \\
&< \frac{\alpha(1-\epsilon)}{\gamma} \int_t^{t+T} \frac{e^{\int_t^s a_2(\theta)d\theta}}{e^{\int_0^T a_1(\theta)d\theta}-1} a_2(s) \, ds \\
&= \alpha(1-\epsilon)\frac{\gamma}{\gamma} = \alpha(1-\epsilon).
\end{aligned}$$

So we obtain $\alpha < \alpha(1-\epsilon)$, which is a contradiction. Therefore (2.23) is valid.

By utilizing the inequality in $(H_4)$, we now prove that

$$x \neq Sx + \eta\psi, \tag{2.25}$$

for $x \in \Sigma \cap \partial\Omega_p$ and $\eta \geq 0$, where

$$\Omega_p = \{x \in \mathbb{X} : \|x\| < p\}.$$

Set $\psi \equiv 1$. If (2.25) is not satisfied then there would exist $x_0 \in K \cap \partial\Omega_p$ and $\eta_0 \geq 0$ such that

$$x_0 = Sx_0 + \eta\psi. \tag{2.26}$$

Since $x_0 \in \Sigma \cap \partial\Omega_p$, then $\|x\|_0 = p$, $p \geq x_0(t) \geq \sigma\|x_0\| = \sigma p$. By $(H_4)$, for $t \in \mathbb{R}$, we have

$$f(t, x_0(t - \tau_1(t, x_0(t))), ..., x(t - \tau_m(t, x_0(t)))) > a_1(t)\, p\gamma. \tag{2.27}$$

Therefore, for $t \in [0, T)$, from (2.26) and (2.27), we have

$$\begin{aligned} x_0(t) &= (Sx_0(t)) + \eta_0 \\ &\geq \int_t^{t+T} G(t,s) f(s, x_0(s - \tau_1(s, x_0(s))), ..., x(s - \tau_m(s, x_0(s))))\, ds \\ &> \int_t^{t+T} G(t,s)\, a_1(t)\, p\gamma ds = p\gamma \int_t^{t+T} \frac{e^{\int_t^s a_1(\theta)d\theta}}{e^{\int_0^T a_2(\theta)d\theta} - 1} a_1(s)\, ds = p\gamma \frac{e^{\int_0^T a_1(\theta)d\theta} - 1}{e^{\int_0^T a_2(\theta)d\theta} - 1} = p. \end{aligned}$$

Thus we obtain $p > p$, which is a contradiction. Therefore (2.25) is valid.

In view of (2.23), (2.25) and Lemma 1.1, we obtain that $S$ has fixed point $x_1 \in \Sigma$ such that $r_1 < \|x_1\| < p$ and $x_1(t) \geq \sigma r_1 > 0$. Thus $x_1$ is a periodic positive solution of equation (2.1).

Secondly, the second inequality $(H_2)$, i.e., $\limsup_{|u| \to +\infty} \dfrac{f(t, u_1, ..., u_m)}{|u|} < \dfrac{a_2(t)}{\gamma}$ uniformly for $t \in \mathbb{R}$, we can find a sufficiently small number $\epsilon > 0$ and a number $r_2 > p$ such that

$$f(t, u_1, ..., u_m) < \frac{a_2(t)}{\gamma}(1-\epsilon)|u|, \tag{2.28}$$

for $|u| \geq r_2$, $t \in \mathbb{R}$.

Set $r_3 = r_2/\sigma$. Hence $x(t) \geq \sigma\|x\| = \sigma r_2$ for any $x \in K \cap \partial\Omega_{r_3}$, where

$$\Omega_{r_3} = \{x \in X : \|x\| < r_3\}.$$

We now show that

$$x \neq \lambda Sx, \tag{2.29}$$

for $x \in \Sigma \cap \partial\Omega_{r_3}$, and $\lambda \in [0,1]$.

If not, there exist $x_0 \in \Sigma \cap \partial\Omega_{r_3}$ and $\lambda_0 \in [0,1]$ such that

$$x_0 = \lambda_0 Sx_0. \tag{2.30}$$

Evidently, $\lambda_0 \neq 0$. If not, then $x_0 \equiv 0$, which contracts that $x_0 \in \Sigma \cap \partial\Omega_{r_3}$. Thus for $t \in \mathbb{R}$, from (2.28) and (2.30), we have

$$\begin{aligned}
x_0(t) &= \lambda_0 (Sx_0)(t) \\
&= \lambda_0 \int_t^{t+T} G(t,s) f(s, x_0(s-\tau_1(s,x_0(s))), ..., x(s-\tau_m(s,x_0(s)))) ds \\
&\leq \frac{(1-\epsilon)}{\gamma} r_3 \int_t^{t+T} \frac{e^{\int_t^s a_2(\theta)d\theta}}{e^{\int_0^T a_1(\theta)d\theta}-1} a_2(s) ds \\
&= \frac{(1-\epsilon)}{\gamma} \gamma r_3 = (1-\epsilon) r_3.
\end{aligned}$$

Therefore we have $r_3 \leq (1-\epsilon) r_3$, which is contradiction. Thus (2.29) is valid

In view of (2.25), (2.29) and Lemma 1.2, we have that the operator $S$ has fixed point

$$x_2 \in \Sigma \cap \{x \in X : p < \|x\| < r_3\},$$

and $x_2(t) \geq \sigma r_3 > 0$. Therefore $x_2$ is a positive $T-$periodic solution of equation $(2.1)$.

■

**Corollary 2.2** *Theorem* (2.2) *is valid if conditions* $(H_0)$ *and* $(H_4)$ *are true and* $(H_2)$ *replaced by the following condition:*

$(H_2^*)$

$$\lim_{|u|\to 0^+} \max_{t\in[0,T]} \frac{f(t,u_1,...,u_m)}{|u|} = 0,$$

*and*

$$\lim_{|u|\to +\infty} \max_{t\in[0,T]} \frac{f(t,u_1,...,u_m)}{|u|} = 0.$$

# CHAPTER 3

# Positive periodic solutions for a first order differential equation with a parameter and delays depending on time and state

## Contents



This chapter is devoted to study of the existence of three positive periodic solutions for two classes of first order differential equations with a parameter and time and state-dependent delays via the *Leggett-Williams fixed point theorem.*

Consider the following first order differential equations with a parameter and delays depending on both the time and the state variable:

$$x'(t) = -a(t, x(t))\, x(t) + \lambda f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))), \quad (3.1)$$

where $\lambda > 0$ is a parameter and functions $a(t, x(t))$, $f$ and $\tau_i$, $i = \overline{1, m}$ are defined as in the second chapter.

**Remark 3.1**

1) We wish to point out that method applied in this chapter can also be used to obtain similar results for state-dependent delay differential equations of the form:

$$x'(t) = a(t, x(t))\, x(t) - \lambda f(t, x(t - \tau_1(t, x(t))), ..., x(t - \tau_m(t, x(t)))). \quad (3.2)$$

2) If $a(t, x) = a(t)\, g(x(t))$ and $\tau_i(t, x(t)) = \tau_i(t)$, $i = 1, 2, ..., n$, where $g \in C([0, \infty), [0, \infty))$, then equations (3.1) and (3.2) take the forms

$$x'(t) = -a(t)\, g(x(t))\, x(t) + \lambda f(t, x(t - \tau_1(t)), x(t - \tau_2(t)), ..., x(t - \tau_m(t))), \quad (3.3)$$

and

$$x'(t) = a(t)\, g(x(t))\, x(t) - \lambda f t, x(t - \tau_1(t)), x(t - \tau_2(t)), ..., x(t - \tau_m(t))), \quad (3.4)$$

respectively.

We consider the same Banach space $\mathbb{X}$ as in the previous chapter and a cone $\Lambda$ on the space $\mathbb{X}$ given by

$$\Lambda = \left\{ x : x \in \mathbb{X},\ x(t) \geq \frac{1}{\delta} \|x\| \right\}.$$

We will use in the sequel the following notations:

$$f^{\theta} = \lim_{|x| \to \theta} \sup \max_{0 \leq t \leq T} \frac{f(t, x)}{a_2(t)\, |x|},$$

and

$$\widetilde{f}^{h} = \lim_{|x| \to h} \sup \max_{0 \leq t \leq T} \frac{f(t, x)}{a(t)\, |x|},$$

where

$$|x| = \max_{1 \leq i \leq m} \{x_1, x_2, ..., x_m\}.$$

## 3.1 Conversion of the delay differential equation (3.1) into an integral equation

It's not hard to prove the following lemma through which we can transform equation (3.1) into an equivalent integral equation.

**Lemma 3.1** *$x \in C^1(\mathbb{R},\mathbb{R}) \cap \mathbb{X}$ is a $T$-periodic solution of equation (3.1) if and only if $x \in \mathbb{X}$ is a $T$-periodic solution of the following integral equation:*

$$x(t) = \lambda \int_t^{t+T} G(t,s) f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\, ds, \tag{3.5}$$

*where $G(t,s)$ is the Green's function that is given by expression (2.4).*

Now, we will need to construct an operator $A_\lambda$ as follows:

$$(A_\lambda x)(t) = \lambda \int_t^{t+T} G(t,s) f(s, x(s-\tau_1(s,(x(s)))), ..., x(s-\tau_m(s,x(s)))))\, ds, \tag{3.6}$$

for every $x \in \mathbb{X}$ and $t \in \mathbb{R}$. Thereby fixed points of $A_\lambda$ are solutions of (3.1) and vice versa.

**Remark 3.2**

1) $A_\lambda x(t+T) = A_\lambda x(t)$.

2) $A_\lambda : \mathbb{X} \longrightarrow \mathbb{X}$.

3) If

$$\alpha = \frac{1}{K_2 - 1} \text{ and } \beta = \frac{K_2}{K_1 - 1}, \tag{3.7}$$

then the Green's kernel $G(t,s)$ satisfies the following inequality:

$$\alpha \leq G(t,s) \leq \beta, \tag{3.8}$$

for every $0 \leq t \leq s \leq t+T$.

4) We have

$$K_1 \leq K_2 \text{ and } \delta = \frac{\beta}{\alpha} = \frac{K_2(K_2-1)}{(K_1-1)} > 1. \tag{3.9}$$

**Remark 3.3** For any $x \in \mathbb{X}$, then $A_\lambda : \Lambda \to \Lambda$. Indeed, we have

$$\| A_\lambda x \| \leq \frac{\lambda K_2}{K_1 - 1} \int_0^T f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\, ds, \tag{3.10}$$

and

$$\begin{aligned}(A_\lambda x)(t) &\geq \frac{\lambda}{K_2-1}\int_0^T f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\,ds \\ &\geq \frac{(K_1-1)}{K_2(K_2-1)}\|A_\lambda x\| = \frac{1}{\delta}\|A_\lambda x\|. \end{aligned} \tag{3.11}$$

## 3.2 Existence of positive periodic solutions

**Lemma 3.2** *Operator $A_\lambda : \Lambda \to \Lambda$ is completely continuous.*

**Proof.** The proof of this lemma is similar to the one of Lemma 2.3. ■

Now, we state and prove the existence theorem.

**Theorem 3.1** *Let $f^0 < T$, $f^\infty < T$, and assume that there exists a constant $c_2 > 0$ such that*

$$f(t, x_1, x_2, ..., x_m) \geq 2TK_1\left(\frac{K_2-1}{K_1-1}\right)^2 a_1(t)\,c_2, \tag{3.12}$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation 3.1 has at least three positive $T$-periodic solutions for*

$$\frac{1}{2T}\frac{K_1-1}{K_2-1} \leq \lambda \leq \frac{1}{T}\frac{K_1-1}{K_2-1}.$$

**Proof.**

$A_\lambda : \overline{\Lambda}_{c_4} \to \overline{\Lambda}_{c_4}$: Suppose that $f^\infty < T$. Then there exist $0 < \varepsilon < T$ and $c_3 = \delta c_2 > c_2$ such that

$$f(t, x_1, x_2, ..., x_m) < a_2(t)(T-\varepsilon)|x|.$$

for $|x| > c_3$ and $t \in \mathbb{R}$.

Set $c_4 = \delta c_3$. Clearly

$$c_4 \geq \|x\| \geq x \geq \frac{1}{\delta}\|x\| \text{ for } x \in \Lambda \cap \overline{\Lambda}_{c_4}.$$

For $x \in \overline{\Lambda}_{c_4}$, we have

$$\begin{aligned}\|A_\lambda x\| &= \lambda\int_t^T G(t,s) f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s))))\,ds \\ &\leq \lambda(T-\varepsilon)\int_t^{t+T} G(t,s)\,a_2(s)\max_{1\leq i\leq m}|x(s-\tau_i(s,x(s)))|\,ds \\ &\leq \lambda(T-\varepsilon)\,c_4\int_t^{t+T}\frac{\exp\left(\int_t^s a_2(\theta)\,d\theta\right)}{\exp\left(\int_0^T a_1(\theta)\,d\theta\right)-1}a_2(s)\,ds,\end{aligned}$$

which implies that

$$\|A_\lambda x\| \leq \lambda T c_4 \frac{K_2 - 1}{K_1 - 1} \leq c_4.$$

Since $A_\lambda : \Lambda \to \Lambda$, then, in addition to the above, it follows that $A_\lambda : \overline{\Lambda}_{c_4} \to \overline{\Lambda}_{c_4}$.

$\{x \in \Lambda(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi$: Define a nonnegative continuos function $\psi$ on $\Lambda$ by

$$\psi(x) = \min_{t\in[0,T]} |x(t)|.$$

Then $\psi(t) \leq \|x\|$.

Let $\phi_0(t) = \phi_0$, where $\phi_0$ is any given number satisfying $c_2 < \phi_0 < c_3$.Then, $\phi_0 \in \{x \in \Lambda(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi$.

$\psi(A_\lambda x) \geq c_2$ for $x \in \Lambda(\psi, c_2, c_3)$: Let $x \in \Lambda(\psi, c_2, c_3)$, using 3.12 we obtain

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0\leq t\leq T} \lambda \int_t^{t+T} G(t,x) f(s, x(s - \tau_1(s, x(s))), ..., x(s - \tau_m(s, x(s)))) ds \\
&\geq 2TK_1 \left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \lambda \int_t^{t+T} \frac{1}{\exp\left(\int_0^T a_2(\theta) d\theta\right) - 1} a_1(s) ds \\
&\geq 2TK_1 \left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \lambda \int_t^{t+T} \frac{\exp\left(\int_t^s a_1(\theta) d\theta\right) a_1(s)}{\exp\left(\int_t^s a_1(\theta) d\theta\right)\left(\exp\left(\int_0^T a_2(\theta) d\theta\right) - 1\right)} ds \\
&\geq 2TK_1 \left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \lambda \int_t^{t+T} \frac{1}{\exp\left(\int_0^T a_1(\theta) d\theta\right)} \int_t^{t+T} \frac{\exp\left(\int_t^s a_1(\theta) d\theta\right) a_1(s)}{\exp\left(\int_0^T a_2(\theta) d\theta\right) - 1} ds \\
&\geq 2TK_1 \left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \lambda \frac{1}{K_1} . \frac{K_1 - 1}{K_2 - 1} \geq c_2.
\end{aligned}$$

Hence, the first condition of the Leggett-Williams fixed point theorem is satisfied.

$\|Ax\| < c_1$ for all $x \in \overline{\Lambda}_{c_1}$ : Now $f^0 < T$ implies there are $\epsilon_1 > 0$ and $c_1 < c_2$ such that

$$f(t, x_1, x_2, ..., x_m) < c_1 (T - \epsilon_1) a_2(t) |x| \quad \text{for } 0 < |x| < c_1.$$

Since $c_1 \geq \|x\| \geq x(t) \geq \frac{1}{\delta}\|x\|$ for any $x \in \Lambda \cap \overline{\Lambda}_{c_1}$, then for any $x \in \overline{\Lambda}_{c_1}$,we have

$$\begin{aligned}
\|A_\lambda x\| &\leq \lambda \int_t^{t+T} G(t,s)(T - \epsilon_1) a_2(s) \max_{0\leq i\leq m} |x(s - \tau_i(s, x(s)))| ds \\
&\leq \lambda (T - \epsilon_1) c_1 \int_t^{t+T} \frac{\exp\left(\int_t^s a_2(\theta) d\theta\right) a_2(s)}{\exp\left(\int_0^T a_1(\theta) d\theta\right) - 1} ds \\
&\leq \lambda T c_1 \left(\frac{K_2 - 1}{K_1 - 1}\right) \leq c_1.
\end{aligned}$$

Thus, we infer that the second condition of Leggett-Williams fixed point theorem is fulfilled.

$\psi(Ax) > c_2$ for all $x \in \Lambda(\psi, c_2, c_4)$ with $\|Ax\| > c_3$ : Finally, for any $x \in \Lambda(\psi, c_2, c_4)$ and $\|A_\lambda x\| > c_3$, we see that

$$c_3 \leq \|A_\lambda x\| \leq \beta\lambda \int_t^{t+T} f(s, x(s - \tau_1(s, x(s))), ..., x(s - \tau_m(s, x(s)))) \, ds,$$

and it follows that

$$\begin{aligned} \psi(A_\lambda x) &\geq \alpha\lambda \int_t^T f(s, x(s - \tau_1(s, x(s))), ..., x(s - \tau_m(s, x(s)))) \, ds \\ &\geq \frac{\alpha}{\beta} c_3 = \frac{c_3}{\delta} = c_2. \end{aligned}$$

Hence, by Theorem 1.5, equation (3.1) has at least three positive $T$-periodic solutions. This completes the proof of the theorem. ■

The following theorem follows from the proof of Theorem 3.1

**Theorem 3.2** *Let $f^0 < 1$, $f^\infty < 1$, and assume three exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, ..., x_m) \geq 2K_1 \left(\frac{K_2 - 1}{K_1 - 1}\right)^2 a_1(t) c_2,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.1) has at least three positive $T$-periodic solutions for*

$$\frac{1}{2} \frac{K_1 - 1}{K_2 - 1} \leq \lambda \leq \frac{K_1 - 1}{K_2 - 1}.$$

**Theorem 3.3** *Let $f^0 < T$, $f^\infty < T$, and assume three exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, ..., x_m) \geq \frac{\beta}{\alpha} T K_1 \left(\frac{K_2 - 1}{K_1 - 1}\right)^2 a_1(t) c_2,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.1) has at least three positive $T$-periodic solutions for*

$$\frac{\alpha}{\beta T} \frac{K_1 - 1}{K_2 - 1} \leq \lambda \leq \frac{K_1 - 1}{K_2 - 1}.$$

**Proof.** Choose $c_3$ and $c_4$ as in the proof of Theorem 3.1. Proceeding along the lines of that proof, we can show that $A_\lambda : \overline{\Lambda}_{c_4} \to \overline{\Lambda}_{c_4}$ and condition (2) and (3) of the Theorem

1.5 hold.

In order to complete the proof of the theorem, we need to verify condition (1) of Theorem 1.5. Let $\phi_0(t) = \phi_0$, where $\phi_0$ is any number satisfying $c_2 < \phi_0 < c_3$. Then,

$$\phi_0 \in \{x \in \Lambda(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi.$$

For $x \in \Lambda(\psi, c_2, c_3)$ we have

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0<t<T} \int_t^T G(t,s) f(s, x(s-\tau_1(s,x(s))), ..., x(s-\tau_m(s,x(s)))) \, ds \\
&\geq \lambda T K_1 \left(\frac{\beta}{\alpha}\frac{K_2-1}{K_1-1}\right)^2 c_2 \int_t^{t+T} \frac{1}{\exp\left(\int_0^T a_2(\theta)\, d\theta\right)-1} a_1(s)\, ds \\
&\geq \lambda T K_1 \frac{\beta}{\alpha}\left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \int_t^{t+T} \frac{\exp\left(\int_t^s a_1(\theta)\, d\theta\right) a_1(s)}{\exp\left(\int_t^s a_1(\theta)\, d\theta\right)\left(\exp\left(\int_0^T a_2(\theta)\, d\theta\right)-1\right)} ds \\
&\geq \lambda T K_1 \frac{\beta}{\alpha}\left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \frac{1}{K_1} \int_t^{t+T} \frac{\exp\left(\int_t^s a_1(\theta)\, d\theta\right) a_1(s)}{\exp\left(\int_0^T a_2(\theta)\, d\theta\right)-1} ds \\
&\geq \lambda T \frac{\beta}{\alpha}\left(\frac{K_2-1}{K_1-1}\right)^2 c_2 \frac{K_1-1}{K_2-1} \geq c_2.
\end{aligned}$$

Hence, by Theorem 1.5, equation (3.1) has at least three positive $T$-periodic solutions, and this proves the theorem. ■

The following theorem follows from the proof of Theorem 3.3.

**Theorem 3.4** *Let $f^0 < 1$, $f^\infty < 1$, and assume three exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, ..., x_m) \geq \frac{\beta}{\alpha} K_1 \left(\frac{K_2-1}{K_1-1}\right)^2 a_1(t)\, c_2 \ \text{for } x \in \Lambda \text{ and } c_2 \leq |x| \leq \delta c_2.$$

*Then equation (3.1) has at least three positive $T$-periodic solutions for*

$$\frac{\alpha}{\beta}\frac{K_1-1}{K_2-1} \leq \lambda \leq \frac{K_1-1}{K_2-1}.$$

**Theorem 3.5** *Let $f^0 < T$, $f^\infty < T$, and assume three exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, ..., x_m) \geq T \frac{\beta^2}{\alpha^2} c_2 \int_0^T a_2(s)\, ds,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.1) has at least three positive $T$-periodic solutions for*

$$\frac{\alpha}{\beta^2 T \int_0^T a_2(s)\, ds} \leq \lambda \leq \frac{1}{\beta T \int_0^T a_2(s)\, ds}.$$

**Proof.** Let $c_3$ and $c_4$ be as in the proof of Theorem 3.1. Following the proof of the proof of Theorem 3.1 with small modifications, it can be shown that $A_\lambda : \overline{\Lambda}_{c_4} \to \overline{\Lambda}_{c_4}$ and conditions (2) and (3) of Theorem 1.5 hold. Let $\phi_0(t) = \phi_0$, where $\phi_0$ is any number satisfying $c_2 < \phi_0 < c_3$. Then

$$\phi_0 \in \{x \in \Lambda(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi.$$

In order to apply Theorem 1.5, we only need to show that $\psi(A_\lambda x) > c_2$ for all $x \in \Lambda(\psi, c_2, c_3)$. Now for $x \in \Lambda(\psi, c_2, c_3)$,

$$\begin{aligned} \psi(A_\lambda x) &\geq \lambda\alpha \int_t^T f(s, x(s - \tau_1(s, x(s))), \ldots, x(s - \tau_m(s, x(s))))\, ds \\ &\geq \lambda\alpha \frac{\beta^2}{\alpha^2} T c_2 \int_0^T a_2(s)\, ds \\ &\geq \frac{\alpha}{\beta^2 T \int_0^T a_2(s)\, ds} \alpha \frac{\beta^2}{\alpha^2} T c_2 \int_0^T a_2(s)\, ds = c_2. \end{aligned}$$

By Theorem 1.5, equation (3.1) has at least three positive $T$-periodic solution. The proof of the theorem is complete. ■

The proof of the following theorem should now be clear.

**Theorem 3.6** *Let* $f^0 < 0$, $f^\infty < 1$, *and*

$$f(t, x_1, x_2, \ldots, x_m) \geq \frac{\beta^2}{\alpha^2} c_2 \int_0^T a_2(s)\, ds \text{ for } x \in \Lambda \text{ and } c_2 \leq |x| \leq \delta c_2.$$

*Then equation* (3.1) *has at least three positive* $T$*-periodic solutions for*

$$\frac{\alpha}{\beta^2 \int_0^T a_2(s)\, ds} \leq \lambda \leq \frac{1}{\beta \int_0^T a_2(s)\, ds}.$$

Now, we shall apply the previous theorems to delay differential equations with a parameter of the form (3.3). Similar results can be obtained for (3.4).

We assume that $g \in C([0,\infty), [0,\infty))$ and there are constants $0 < l < L$ such that $l \leq g(x) \leq L$ for $x \geq 0$.

Set

$$\sigma = \exp\left(\int_0^T a(\theta) d\theta\right), \; K_1 = \sigma^l \text{ and } K_2 = \sigma^L.$$

We have the following relation:

$$\alpha = \frac{1}{\sigma^L - 1}, \; \beta = \frac{\sigma^L}{\sigma^l - 1}, \; \delta = \frac{\beta}{\alpha} = \sigma^L \frac{\sigma^L - 1}{\sigma^l - 1} \text{ and } \frac{K_2 - 1}{K_1 - 1} = \frac{\sigma^L - 1}{\sigma^l - 1}.$$

Applying Theorems 3.1-3.6 to equation (3.3), we obtain the following results .

**Theorem 3.7** *Let* $\widetilde{f}^0 < LT$, $\widetilde{f}^\infty < LT$, *and assume there is a constant* $c_2 > 0$ *such that*

$$f(t, x_1, x_2, ..., x_m) \geq 2Tl\sigma^l \left(\frac{\sigma^L - 1}{\sigma^l - 1}\right)^2 a(t)c_2,$$

*for* $x \in \Lambda$ *and* $c_2 \leq |x| \leq \delta c_2$. *Then equation* (3.3) *has at least three positive* $T$*-periodic solutions for*

$$\frac{1}{2T}\frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{1}{T}\frac{\sigma^l - 1}{\sigma^L - 1}.$$

**Theorem 3.8** *Let* $\widetilde{f}^0 < L$, $\widetilde{f}^\infty < L$, *and assume there is a constant* $c_2 > 0$ *such that*

$$f(t, x_1, x_2, ..., x_m) \geq 2l\sigma^l \left(\frac{\sigma^L - 1}{\sigma^l - 1}\right)^2 a(t)c_2,$$

*for* $x \in \Lambda$ *and* $c_2 \leq |x| \leq \delta c_2$. *hen equation* (3.3) *has at least three positive* $T$*-periodic solutions for*

$$\frac{1}{2}\frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{\sigma^l - 1}{\sigma^L - 1}.$$

**Theorem 3.9** *Let* $\widetilde{f}^0 < LT$, $\widetilde{f}^\infty < LT$, *and assume there is a constant* $c_2 > 0$ *such that*

$$f(t, x_1, x_2, ..., x_m) \geq \sigma^{l+L}T \left(\frac{\sigma^L - 1}{\sigma^l - 1}\right)^3 la(t)c_2,$$

*for* $x \in \Lambda$ *and* $c_2 \leq |x| \leq \delta c_2$. *hen equation* (3.3) *has at least three positive* $T$*-periodic solutions for*

$$\frac{1}{T\sigma^{L-l}}\frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{1}{T}\frac{\sigma^l - 1}{\sigma^L - 1}.$$

**Theorem 3.10** *Let* $\widetilde{f}^0 < L$, $\widetilde{f}^\infty < L$, *and assume there is a constant* $c_2 > 0$ *such that*

$$f(t, x_1, x_2, ..., x_m) \geq \sigma^{l+L}T \left(\frac{\sigma^L - 1}{\sigma^l - 1}\right)^3 la(t)c_2,$$

*for* $x \in \Lambda$ *and* $c_2 \leq |x| \leq \delta c_2$. *Then equation* (3.3) *has at least three positive* $T$*-periodic solutions for*

$$\frac{1}{\sigma^{L-l}}\frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{\sigma^l - 1}{\sigma^L - 1}.$$

**Theorem 3.11** *Let $\widetilde{f}^0 < LT$, $\widetilde{f}^\infty < LT$,and assume there is a constant $c_2 > 0$ such that*

$$f\left(t, x_1, x_2, ..., x_m\right) \geq \sigma^{2l} c_2 T \left(\frac{\sigma^L - 1}{\sigma^l - 1}\right)^2 L \int_0^T a(s)ds,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$ Then equation (3.3) has at least three positive $T$-periodic solutions for*

$$\frac{\left(e^l - 1\right)^2}{T\left(e^L - 1\right) e^{2L} L \int_0^T a(\theta)d\theta} \leq \lambda \leq \frac{\left(e^l - 1\right)}{T e^L L \int_0^T a(\theta)d\theta}.$$

**Theorem 3.12** *Let $\widetilde{f}^0 < L$, $\widetilde{f}^\infty < L$, and assume there a positive constant $c_2 > 0$ such that*

$$f\left(t, x_1, x_2, ..., x_m\right) \geq \sigma^{2l} c_2 \left(\frac{\sigma^L - 1}{\sigma^l - 1}\right)^2 L \int_0^T a(s)\ ds,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.3) has at least three positive $T$-periodic solutions for*

$$\frac{\left(e^l - 1\right)^2}{\left(e^L - 1\right) e^{2L} L \int_0^T a(\theta)d\theta} \leq \lambda \leq \frac{\left(e^l - 1\right)}{e^L L \int_0^T a(\theta)d\theta}.$$

Now we direct our attention to the particular case when $a(t) \equiv a$ is a constant and $g(x) \equiv 1$ with $l = L = 1$,Then $\delta = \sigma = e^{aT}$ and $K_1 = K_2$. Let

$$f^{*\theta} = \frac{1}{a} \lim_{|x| \to \theta} \sup \max_{0 \leq t \leq T} \frac{f(t; x)}{|x|}.$$

Applying Theorems 3.7-3.12 to the equation

$$x^{'}\left(t\right) = -ax\left(t\right) + \lambda f\left(t, x\left(t - \tau\right)\right), \tag{3.13}$$

we obtain the following interesting results:

**Theorem 3.13** *Let $f^{*0} < aT$, $f^{*\infty} < aT$, and assume there exists a constant $c_2 > 0$ such that*

$$f(t, x) = 2a\delta c_2 T \text{ for } x \in \Lambda \text{ and } c_2 \leq |x| \leq \delta c_2.$$

*Then equation (3.13) has at least three positive $T$-periodic solutions for*

$$\frac{1}{2T} \leq \lambda \leq \frac{1}{T}.$$

**Theorem 3.14** *Let $f^{*0} < a$, $f^{*\infty} < a$, and assume there exists a constant $c_2 > 0$ such that*

$$f(t, x) \geq 2a\delta c_2,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.13) has at least three positive $T$-periodic solutions for*

$$\frac{1}{2} \leq \lambda \leq 1.$$

**Theorem 3.15** *Let $f^{*0} < aT$, $f^{*\infty} < aT$, and assume there exists a constant $c_2 > 0$ such that*

$$f(t, x) \geq a\delta^2 c_2,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.13) has at least three positive $T$-periodic solutions for $\lambda = \frac{1}{T}$.*

In particular, for $\lambda = 1$, the next theorem follows form Theorem 3.15 .

**Theorem 3.16** *Let $f^{*0} < a$, $f^{*\infty} < a$, and assume there exists a constant $c_2 > 0$ such that*

$$f(t, x) \geq aT\delta^2 c_2,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.13) has at least three positive $T$-periodic solutions for $\lambda = 1$.*

**Theorem 3.17** *Let $f^{*0} < aT$, $f^{*\infty} < aT$, and assume there exists a constant $c_2 > 0$ such that*

$$f(t, x) \geq aT^2\delta^2 c_2,$$

*for $x \in \Lambda$ and $c_2 \leq |x| \leq \delta c_2$. Then equation (3.13) has at least three positive $T$-periodic solutions for*

$$\frac{e-1}{e^2 aT^2} \leq \lambda \leq \frac{e-1}{eaT^2}.$$

**Theorem 3.18** *Let* $f^{*0} < a$, $f^{*\infty} < a$, *and assume there exists a constant* $c_2 > 0$ *such that*

$$f(t, x) \geq aT\delta^2 c_2.$$

*for* $x \in \Lambda$ *and* $c_2 \leq |x| \leq \delta c_2$. *Then equation* (3.13) *has at least three positive* $T$*-periodic solutions for*

$$\frac{e-1}{e^2 aT} \leq \lambda \leq \frac{e-1}{eaT}.$$

# General conclusion and perspectives

During the past decades, many researchers paid much attention to functional differential equations with delays depending on time and state. In the present thesis, we have studied the existence of multiple positive periodic solutions for two nonlinear first order differential equations with delays depending on time and state.

After the introduction that has given an overview and some insight into the concept of differential equations with delays depending on state, we have introduced some definitions, tools and necessary preliminary results that allow us to build a better understanding of the other chapters. In the second chapter, by means of two fixed point theorems in cones, we have derived some suitable criteria under which the existence of two positive periodic solutions for two classes of differential equations with state-dependent delays was established. While the main task of the third chapter was employing Leggett-Williams fixed point theorem to prove that other two classes of differential equations with a parameter and time and state-dependent delays had at least three positive periodic solutions.

For tackling these classes of equations, we have employed a hybrid technique that combines the fixed point theory with the Green functions method where the key steps were as follows:

$(i)$ First of all, we have defined an appropriate Banach space and two cones of it to pave the way for employing the fixed point theorems and also ensure a priori that the sought results will be more realistic and credible.

$(ii)$ Secondly, we have established an equivalence between the studied equations and integral ones whose kernels are Green's functions.

(*iii*) Thirdly and lastly, from the obtained integral equations, we have constructed integral operators and where emphasis is placed on the use of some properties of the obtained kernel along with the fixed point theorems to establish the desired results.

This technique will provide a good reference to deal with other type of functional differential equations. Let us mention here two of many open questions.

– Do state delay differential equations with other assumptions have a unique solution?

– Does the unique solution, if it exists, depends on model parameters?

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